

Chapter P: Preparing for Calculus

P.1: Functions and Their Graphs

Concepts and Vocabulary

1. If f is a function defined by $y = f(x)$, then x is called the **independent** variable and y is the **dependent** variable.
2. **True**. The independent variable is sometimes referred to as the argument of the function.
3. **False**. If no domain is specified for a function f , then the domain of f is taken to be the largest set of real numbers for which the value $f(x)$ is defined and is a real number.
4. **False**. The domain of the function $f(x) = \frac{3(x^2 - 1)}{x - 1}$ is the set $\{x|x \neq 1\}$.
5. **False**. A function can have at most one y -intercept; otherwise, there would be more than one y -value corresponding to $x = 0$.
6. A set of points in the xy -plane is the graph of a function if and only if every **vertical** line intersects the graph in at most one point.
7. If the point $(5, -3)$ is on the graph of f , then $f(\boxed{5}) = \boxed{-3}$.
8. Let $f(x) = ax^2 + 4$. For the point $(-1, 2)$ to be on the graph of f , we must have $f(-1) = 2$. But $f(-1) = a(-1)^2 + 4 = a + 4$, so we must have $a + 4 = 2$, or $\boxed{a = -2}$.
9. A function f is **(a) increasing** on an interval I if, for any choice of x_1 and x_2 in I , with $x_1 < x_2$, then $f(x_1) < f(x_2)$.
10. A function is **(a) even** if for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$. A function f is **(b) odd** if for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$.
11. **False**. Even functions have graphs that are symmetric with respect to the y -axis. Odd functions have graphs that are symmetric with respect to the origin.
12. The average rate of change of $f(x) = 2x^3 - 3$ from 0 to 2 is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{2(2)^3 - 3 - (2(0)^3 - 3)}{2} = \frac{13 - (-3)}{2} = \frac{16}{2} = \boxed{8}.$$

Practice Problems

13. Let $f(x) = 3x^2 + 2x - 4$. Then
 - (a) $f(0) = 3(0)^2 + 2(0) - 4 = \boxed{-4}$.
 - (b) $f(-x) = 3(-x)^2 + 2(-x) - 4 = \boxed{3x^2 - 2x - 4}$.
 - (c) $-f(x) = -(3x^2 + 2x - 4) = \boxed{-3x^2 - 2x + 4}$.
 - (d) $f(x + 1) = 3(x + 1)^2 + 2(x + 1) - 4 = 3(x^2 + 2x + 1) + 2x + 2 - 4 = \boxed{3x^2 + 8x + 1}$.

$$(e) f(x+h) = 3(x+h)^2 + 2(x+h) - 4 = 3(x^2 + 2xh + h^2) + 2x + 2h - 4 = \boxed{3x^2 + 6xh + 3h^2 + 2x + 2h - 4}.$$

14. Let $f(x) = \frac{x}{x^2 + 1}$. Then

$$(a) f(0) = \frac{0}{0^2 + 1} = \boxed{0}.$$

$$(b) f(-x) = \frac{-x}{(-x)^2 + 1} = \boxed{-\frac{x}{x^2 + 1}}.$$

$$(c) -f(x) = \boxed{-\frac{x}{x^2 + 1}}.$$

$$(d) f(x+1) = \frac{x+1}{(x+1)^2 + 1} = \boxed{\frac{x+1}{x^2 + 2x + 2}}.$$

$$(e) f(x+h) = \frac{x+h}{(x+h)^2 + 1} = \boxed{\frac{x+h}{x^2 + 2xh + h^2 + 1}}.$$

15. Let $f(x) = |x| + 4$. Then

$$(a) f(0) = |0| + 4 = \boxed{4}.$$

$$(b) f(-x) = |-x| + 4 = \boxed{|x| + 4}.$$

$$(c) -f(x) = -(|x| + 4) = \boxed{-|x| - 4}.$$

$$(d) f(x+1) = \boxed{|x+1| + 4}.$$

$$(e) f(x+h) = \boxed{|x+h| + 4}.$$

16. Let $f(x) = \sqrt{3-x}$. Then

$$(a) f(0) = \sqrt{3-0} = \boxed{\sqrt{3}}.$$

$$(b) f(-x) = \sqrt{3-(-x)} = \boxed{\sqrt{3+x}}.$$

$$(c) -f(x) = \boxed{-\sqrt{3-x}}.$$

$$(d) f(x+1) = \sqrt{3-(x+1)} = \boxed{\sqrt{2-x}}.$$

$$(e) f(x+h) = \sqrt{3-(x+h)} = \boxed{\sqrt{3-x-h}}.$$

17. Because $f(x) = x^3 - 1$ is defined for any real number x , the domain of f is the set of $\boxed{\text{all real numbers}}$, or in interval notation $\boxed{(-\infty, \infty)}$.

18. Because $x^2 + 1$ is never equal to zero for any real number x , $f(x) = \frac{x}{x^2 + 1}$ is defined for any real number x . The domain of f is therefore the set of $\boxed{\text{all real numbers}}$, or in interval notation $\boxed{(-\infty, \infty)}$.

19. Because the square root of a negative number is not a real number, the value of $t^2 - 9$ must be nonnegative. The solution of the inequality $t^2 - 9 \geq 0$ is $\{t|t \leq -3\} \cup \{t|t \geq 3\}$, so the domain of v is the set of real numbers $\boxed{\{t|t \leq -3\} \cup \{t|t \geq 3\}}$, or in interval notation $\boxed{(-\infty, -3] \cup [3, \infty)}$.

20. Because the expression $x - 1$ appears under the square root and in the denominator, the value of $x - 1$ must be positive; that is, $x - 1 > 0$. The solution of this inequality is $\{x|x > 1\}$, so the domain of g is the set of real numbers $\boxed{\{x|x > 1\}}$, or in interval notation $\boxed{(1, \infty)}$.

21. Because division by zero is not defined, $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$ cannot be zero; that is, $x \neq 0$, $x \neq 2$, and $x \neq -2$. Thus, the domain of h is the set of real numbers $\boxed{\{x|x \neq -2, x \neq 0, x \neq 2\}}$.

22. Because the square root of a negative number is not a real number, the value of $t + 1$ must be nonnegative. The solution of the inequality $t + 1 \geq 0$ is $\{t|t \geq -1\}$. Moreover, because division by zero is not defined, $t - 5$ cannot be 0; that is, t cannot be equal to 5. The intersection of the sets $\{t|t \geq -1\}$ and $\{t|t \neq 5\}$ is the set $\{t|-1 \leq t < 5\} \cup \{t|t > 5\}$. The domain of s is therefore the set of real numbers $\boxed{\{t|-1 \leq t < 5\} \cup \{t|t > 5\}}$, or in interval notation $\boxed{[-1, 5) \cup (5, \infty)}$.

23. Let $f(x) = -3x + 1$. Then

$$\frac{f(x+h) - f(x)}{h} = \frac{-3(x+h) + 1 - (-3x+1)}{h} = \frac{-3x - 3h + 1 + 3x - 1}{h} = \frac{-3h}{h} = \boxed{-3}.$$

24. Let $f(x) = \frac{1}{x+3}$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \cdot \frac{(x+h+3)(x+3)}{(x+h+3)(x+3)} \\ &= \frac{x+3 - (x+h+3)}{h(x+h+3)(x+3)} = \frac{-h}{h(x+h+3)(x+3)} = \boxed{-\frac{1}{(x+h+3)(x+3)}}. \end{aligned}$$

25. Let $f(x) = \sqrt{x+7}$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h+7} - \sqrt{x+7}}{h} \cdot \frac{\sqrt{x+h+7} + \sqrt{x+7}}{\sqrt{x+h+7} + \sqrt{x+7}} \\ &= \frac{x+h+7 - (x+7)}{h(\sqrt{x+h+7} + \sqrt{x+7})} = \frac{h}{h(\sqrt{x+h+7} + \sqrt{x+7})} = \boxed{\frac{1}{\sqrt{x+h+7} + \sqrt{x+7}}}. \end{aligned}$$

26. Let $f(x) = \frac{2}{\sqrt{x+7}}$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\frac{2}{\sqrt{x+h+7}} - \frac{2}{\sqrt{x+7}}}{h} \cdot \frac{\sqrt{x+h+7}\sqrt{x+7}}{\sqrt{x+h+7}\sqrt{x+7}} \\ &= \frac{2\sqrt{x+7} - 2\sqrt{x+h+7}}{h\sqrt{x+h+7}\sqrt{x+7}} \cdot \frac{\sqrt{x+7} + \sqrt{x+h+7}}{\sqrt{x+7} + \sqrt{x+h+7}} \\ &= \frac{2(x+7) - 2(x+h+7)}{h\sqrt{x+h+7}\sqrt{x+7}(\sqrt{x+7} + \sqrt{x+h+7})} \\ &= \frac{-2h}{h\sqrt{x+h+7}\sqrt{x+7}(\sqrt{x+7} + \sqrt{x+h+7})} \\ &= \boxed{-\frac{2}{\sqrt{x+h+7}\sqrt{x+7}(\sqrt{x+7} + \sqrt{x+h+7})}}. \end{aligned}$$

27. Let $f(x) = x^2 + 2x$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} = \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\ &= \frac{2xh + h^2 + 2h}{h} = \frac{h(2x + h + 2)}{h} = \boxed{2x + h + 2}. \end{aligned}$$

28. Let $f(x) = (2x + 3)^2$. Then

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(2(x+h) + 3)^2 - (2x + 3)^2}{h} = \frac{4(x+h)^2 + 12(x+h) + 9 - (4x^2 + 12x + 9)}{h} \\ &= \frac{4x^2 + 8xh + 4h^2 + 12x + 12h + 9 - 4x^2 - 12x - 9}{h} \\ &= \frac{8xh + 4h^2 + 12h}{h} = \frac{h(8x + 4h + 12)}{h} = \boxed{8x + 4h + 12}.\end{aligned}$$

29. This graph fails the vertical line test (for example, the vertical line $x = 2$ intersects the graph in two points), so it **does not represent a function**.

30. This graph passes the vertical line test, and so it **does represent a function**.

(a) From the graph, the domain of the function is the set of **all real numbers** or the interval $(-\infty, \infty)$, and the range of the function is the set of **all positive real numbers** or the interval $(0, \infty)$.

(b) The graph has no x -intercepts, and the y -intercept is 1. Thus, the only intercept is $(0, 1)$.

(c) The graph is not symmetric with respect to the x -axis, the y -axis, or the origin.

31. This graph passes the vertical line test, and so it **does represent a function**.

(a) From the graph, the domain of the function is the set $\{x \mid -\pi \leq x \leq \pi\}$ or the closed interval $[-\pi, \pi]$, and the range of the function is the set $\{y \mid -1 \leq y \leq 1\}$ or the closed interval $[-1, 1]$.

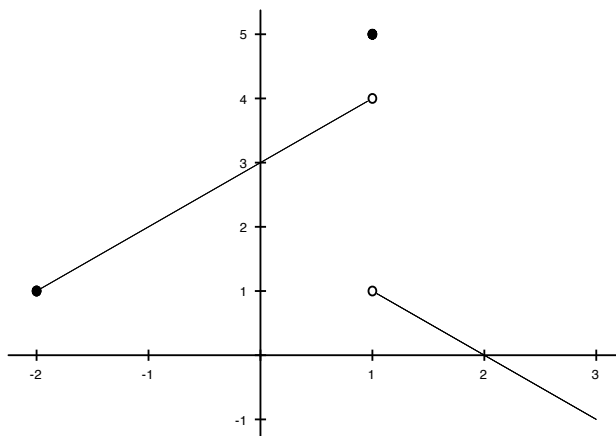
(b) The graph has x -intercepts of $\pm \frac{\pi}{2}$ and a y -intercept of 1, so the intercepts are $(\pm \frac{\pi}{2}, 0)$ and $(0, 1)$.

(c) The graph is symmetric with respect to the y -axis but not symmetric with respect to the x -axis or the origin.

32. This graph fails the vertical line test (for example, the vertical line $x = 2$ intersects the graph in two points), so it **does not represent a function**.

33. (a) $\boxed{f(-1) = -1 + 3 = 2; f(0) = 0 + 3 = 3; f(1) = 5; f(8) = -8 + 2 = -6}$

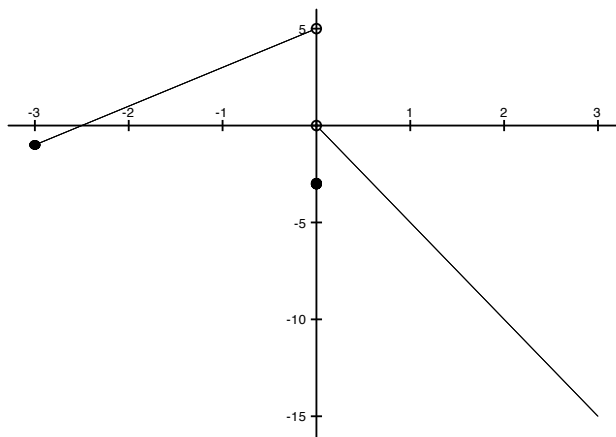
(b) The graph of f consists of three pieces corresponding to each equation in the definition. On the interval $[-2, 1)$, the graph is the line $y = x + 3$, and on the interval $(1, \infty)$, the graph is the line $y = -x + 2$. The graph also contains the point $(1, 5)$.



- (c) The individual components of f have domains of $\{x | -2 \leq x < 1\}$, $\{x | x = 1\}$, and $\{x | x > 1\}$. The domain of f is the union of these three sets; that is, $\{x | x \geq -2\}$ in set notation or $[-2, \infty)$ in interval notation. Moreover, the individual components of f have ranges of $\{y | 1 \leq y < 4\}$, $\{y | y = 5\}$, and $\{y | y < 1\}$. The range of f is the union of these three sets; that is, $\{y | y = 5 \text{ or } y < 4\}$ in set notation or $(-\infty, 4) \cup \{5\}$ in interval notation. The x -intercept is 2, and the y -intercept is 3, so the intercepts are $(2, 0)$ and $(0, 3)$.

34. (a) $f(-1) = 2(-1) + 5 = 3; f(0) = -3; f(1) = -5(1) = -5; f(8) = -5(8) = -40$

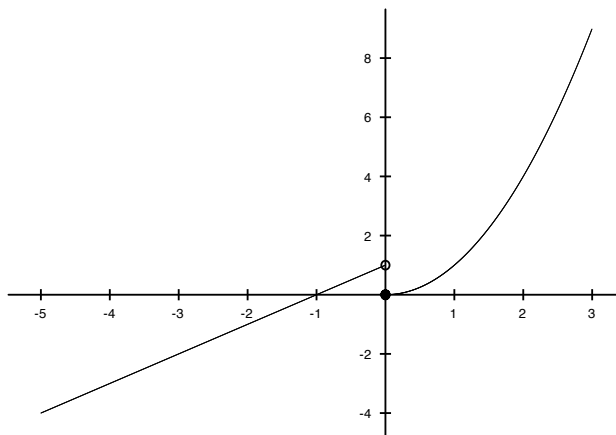
- (b) The graph of f consists of three pieces corresponding to each equation in the definition. On the interval $[-3, 0)$, the graph is the line $y = 2x + 5$, and on the interval $(0, \infty)$, the graph is the line $y = -5x$. The graph also contains the point $(0, -3)$.



- (c) The individual components of f have domains of $\{x | -3 \leq x < 0\}$, $\{x | x = 0\}$, and $\{x | x > 0\}$. The domain of f is the union of these three sets; that is, $\{x | x \geq -3\}$ in set notation or $[-3, \infty)$ in interval notation. Moreover, the individual components of f have ranges of $\{y | -1 \leq y < 5\}$, $\{y | y = -3\}$, and $\{y | y < 0\}$. The range of f is the union of these three sets; that is, $\{y | y < 5\}$ in set notation or $(-\infty, 5)$ in interval notation. The x -intercept is $-\frac{5}{2}$, and the y -intercept is -3 , so the intercepts are $(-\frac{5}{2}, 0)$ and $(0, -3)$.

35. (a) $f(-1) = 1 + (-1) = 0; f(0) = 0^2 = 0; f(1) = 1^2 = 1; f(8) = 8^2 = 64$

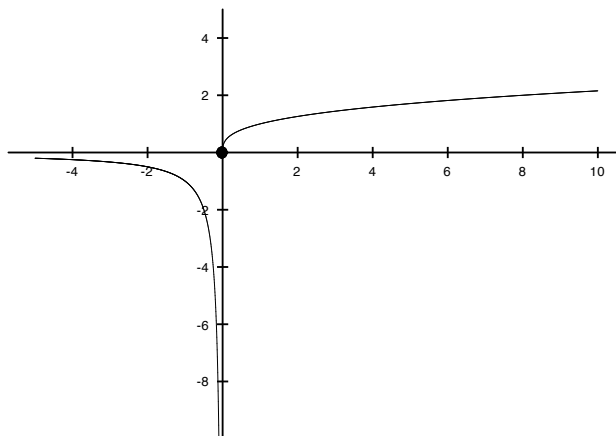
- (b) The graph of f consists of two pieces corresponding to each equation in the definition. On the interval $(-\infty, 0)$, the graph is the line $y = 1 + x$, and on the interval $[0, \infty)$, the graph is the parabola $y = x^2$.



- (c) The individual components of f have domains of $\{x|x < 0\}$ and $\{x|x \geq 0\}$. The domain of f is the union of these two sets; that is, all real numbers or the interval $(-\infty, \infty)$. Moreover, the individual components of f have ranges of $\{y|y < 1\}$ and $\{y|y \geq 0\}$. The range of f is the union of these two sets; that is, all real numbers or the interval $(-\infty, \infty)$. The x -intercepts are -1 and 0 , and the y -intercept is 0 , so the intercepts are $(-1, 0)$ and $(0, 0)$.

36. (a) $f(-1) = \frac{1}{-1} = -1$; $f(0) = \sqrt[3]{0} = 0$; $f(1) = \sqrt[3]{1} = 1$; $f(8) = \sqrt[3]{8} = 2$

- (b) The graph of f consists of two pieces corresponding to each equation in the definition. On the interval $(-\infty, 0)$, the graph is $y = \frac{1}{x}$, and on the interval $[0, \infty)$, the graph is $y = \sqrt[3]{x}$.



- (c) The individual components of f have domains of $\{x|x < 0\}$ and $\{x|x \geq 0\}$. The domain of f is the union of these two sets; that is, all real numbers or the interval $(-\infty, \infty)$. Moreover, the individual components of f have ranges of $\{y|y < 0\}$ and $\{y|y \geq 0\}$. The range of f is the union of these two sets; that is, all real numbers or the interval $(-\infty, \infty)$. The x -intercept is 0 , and the y -intercept is 0 , so the only intercept is $(0, 0)$.

37. Because the graph of f includes the point $(0, 3)$, $f(0) = 3$; because the graph also includes the point $(-6, -3)$, $f(-6) = -3$.

38. Because the graph of f lies above the x -axis at $x = 3$, $f(3)$ is positive.

39. Because the graph of f lies below the x -axis at $x = -4$, $f(-4)$ is negative.
40. Because the graph of f includes the points $(-3, 0)$, $(6, 0)$, and $(10, 0)$, $f(x) = 0$ for $x = -3$, $x = 6$, and $x = 10$.
41. Because the graph of f lies above the x -axis for $-3 < x < 6$ and for $10 < x \leq 11$, $f(x) > 0$ for $-3 < x < 6$ and for $10 < x \leq 11$. In interval notation, this can be written as $(-3, 6) \cup (10, 11]$.
42. The points on the graph of f have x -coordinates between -6 and 11 inclusive. The domain of f is therefore the set of real numbers $\{x | -6 \leq x \leq 11\}$ or the closed interval $[-6, 11]$.
43. The points on the graph of f have y -coordinates between -3 and 3 inclusive. The range of f is therefore the set of real numbers $\{y | -3 \leq y \leq 3\}$ or the closed interval $[-3, 3]$.
44. Because the graph of f includes the points $(-3, 0)$, $(6, 0)$, and $(10, 0)$, the x -intercepts are -3 , 6 , and 10 .
45. Because the graph of f includes the point $(0, 3)$, the y -intercept is 3 .
46. The graph of the horizontal line $y = \frac{1}{2}$ will intersect the graph of f three times.
47. The graph of the vertical line $x = 5$ will intersect the graph of f once.
48. Because the graph of f includes the point $(x, 3)$ for $0 \leq x \leq 4$, $f(x) = 3$ for $0 \leq x \leq 4$.
49. Because the graph of f includes the points $(-5, -2)$ and $(8, -2)$, $f(x) = -2$ for $x = -5$ and for $x = 8$.
50. The function f is increasing on the intervals $(-6, 0)$ and $(8, 11)$.
51. The function f is decreasing on the interval $(4, 8)$.
52. The function f is constant on the interval $(0, 4)$.
53. The function f is nonincreasing (that is, the function is constant or decreasing) on the interval $(0, 8)$.
54. The function f is nondecreasing (that is, the function is constant or increasing) on the intervals $(-6, 4)$ and $(8, 11)$.
55. Because division by zero is not defined, $x - 6$ cannot be equal to 0 . The domain of g is therefore the set $\{x | x \neq 6\}$.
56. Note that $g(3) = \frac{3+2}{3-6} = -\frac{5}{3}$. Because $g(3) \neq 14$, the point $(3, 14)$ is not on the graph of g .
57. If $x = 4$, then $g(x) = g(4) = \frac{4+2}{4-6} = -3$; therefore, the point $(4, -3)$ is on the graph of g .
58. If $g(x) = 2$, then $\frac{x+2}{x-6} = 2$. Multiplying both sides of this equation by $x - 6$ yields

$$x + 2 = 2(x - 6) = 2x - 12,$$

so that $x = 14$. Accordingly, the point $(14, 2)$ is on the graph of g .

59. The x -intercepts occur when $g(x) = 0$. The value of g can only be zero when its numerator is zero; therefore, $g(x) = 0$ only when $x + 2 = 0$. Thus, the graph of g has only one x -intercept, and this intercept is $\boxed{-2}$.

60. Because $g(0) = \frac{0+2}{0-6} = -\frac{1}{3}$, the y -intercept is $\boxed{-\frac{1}{3}}$.

61. The domain of h is $\{x|x \neq \pm 1\}$, so for every number x in the domain of h , the number $-x$ is also in the domain. Moreover,

$$h(-x) = \frac{-x}{(-x)^2 - 1} = -\frac{x}{x^2 - 1} = -h(x),$$

so that the function h is $\boxed{\text{odd}}$. Because h is an odd function, its graph is symmetric with respect to the $\boxed{\text{origin}}$.

62. The domain of f is all real numbers, so for every number x in the domain of f , the number $-x$ is also in the domain. Moreover,

$$f(-x) = \sqrt[3]{3(-x)^2 + 1} = \sqrt[3]{3x^2 + 1} = f(x),$$

so that the function f is $\boxed{\text{even}}$. Because f is an even function, its graph is symmetric with respect to the $\boxed{y\text{-axis}}$.

63. The domain of G is $\{x|x \geq 0\}$. The number $x = 1$ is in the domain of G , but $x = -1$ is not; therefore, the function G is $\boxed{\text{neither even nor odd}}$. Because G is neither an even nor an odd function, its graph is $\boxed{\text{not symmetric with respect to either the } y\text{-axis or the origin}}$.

64. The domain of F is $\{x|x \neq 0\}$, so for every number x in the domain of F , the number $-x$ is also in the domain. Moreover,

$$F(-x) = \frac{2(-x)}{|-x|} = -\frac{2x}{|x|} = -F(x),$$

so that the function F is $\boxed{\text{odd}}$. Because F is an odd function, its graph is symmetric with respect to the $\boxed{\text{origin}}$.

65. Let $f(x) = -2x^2 + 4$.

- (a) The average rate of change of f from 1 to 2 is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{-4 - 2}{2 - 1} = \frac{-6}{1} = \boxed{-6}.$$

- (b) The average rate of change of f from 1 to 3 is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{-14 - 2}{3 - 1} = \frac{-16}{2} = \boxed{-8}.$$

- (c) The average rate of change of f from 1 to 4 is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{-28 - 2}{4 - 1} = \frac{-30}{3} = \boxed{-10}.$$

- (d) The average rate of change of f from 1 to x for $x \neq 1$ is

$$\frac{f(x) - f(1)}{x - 1} = \frac{-2x^2 + 4 - 2}{x - 1} = \frac{2(1 - x^2)}{x - 1} = \frac{2(1 - x)(1 + x)}{x - 1} = \boxed{-2(1 + x)}.$$

66. Let $s(t) = 20 - 0.8t^2$.

(a) The average rate of change of s from 1 to 4 is

$$\frac{s(4) - s(1)}{4 - 1} = \frac{7.2 - 19.2}{4 - 1} = \frac{-12}{3} = \boxed{-4}.$$

(b) The average rate of change of s from 1 to 3 is

$$\frac{s(3) - s(1)}{3 - 1} = \frac{12.8 - 19.2}{3 - 1} = \frac{-6.4}{2} = \boxed{-3.2}.$$

(c) The average rate of change of s from 1 to 2 is

$$\frac{s(2) - s(1)}{2 - 1} = \frac{16.8 - 19.2}{2 - 1} = \frac{-2.4}{1} = \boxed{-2.4}.$$

(d) The average rate of change of s from 1 to t for $t \neq 1$ is

$$\frac{s(t) - s(1)}{t - 1} = \frac{20 - 0.8t^2 - 19.2}{t - 1} = \frac{0.8(1 - t^2)}{t - 1} = \frac{0.8(1 - t)(1 + t)}{t - 1} = \boxed{-0.8(1 + t)}.$$

67. For $-1 \leq x < 0$, the graph is a line through the points $(-1, 1)$ and $(0, 0)$. The slope of this line is $m = -1$, and the y -intercept is 0; therefore, the equation for this component of the function is $y = -x$. For $0 \leq x \leq 2$, the graph is a line through the points $(0, 0)$ and $(2, 1)$. The slope of this line is $m = \frac{1}{2}$, and the y -intercept is 0; therefore, the equation for this component of the function is $y = \frac{1}{2}x$. Combining these two equations, the definition for this piecewise function is

$$f(x) = \begin{cases} -x, & \text{if } -1 \leq x < 0 \\ \frac{1}{2}x, & \text{if } 0 \leq x \leq 2. \end{cases}$$

The domain of f is the set $\{x | -1 \leq x \leq 2\}$ or the closed interval $[-1, 2]$, and the range is the set $\{y | 0 \leq y \leq 1\}$ or the closed interval $[0, 1]$.

68. For $x \leq 0$, the graph is a line with slope 1 through the origin, and for $0 < x \leq 2$, the graph is a horizontal line at height $y = 1$. The definition for this piecewise function is therefore

$$f(x) = \begin{cases} x, & \text{if } x \leq 0 \\ 1, & \text{if } 0 < x \leq 2. \end{cases}$$

The domain of f is $\{x | x \leq 2\}$ in set notation or $(-\infty, 2]$ in interval notation, and the range is $\{y | y = 1 \text{ or } y \leq 0\}$ in set notation or $(-\infty, 0] \cup \{1\}$ in interval notation.

69. For $x < 1$, the graph is a horizontal line at height $y = -1$; at $x = 1$, the value of the function is 0; and for $1 < x \leq 2$, the graph is a line with slope -1 and x -intercept 2. The definition for this piecewise function is therefore

$$f(x) = \begin{cases} -1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ 2 - x, & \text{if } 1 < x \leq 2. \end{cases}$$

The domain of f is $\{x | x \leq 2\}$ in set notation or $(-\infty, 2]$ in interval notation, and the range is $\{y | y = -1 \text{ or } 0 \leq y < 1\}$ in set notation or $\{-1\} \cup [0, 1)$ in interval notation.

70. For $x < -1$, the graph is a horizontal line at height $y = 2$; at $x = -1$, the value of the function is 0; and for $-1 < x \leq 1$, the graph is a line with slope -2 and y -intercept -1 . The definition for this piecewise function is therefore

$$f(x) = \begin{cases} 2, & \text{if } x < -1 \\ 0, & \text{if } x = -1 \\ -2x - 1, & \text{if } -1 < x \leq 1. \end{cases}$$

The domain of f is $\{x|x \leq 1\}$ in set notation or $(-\infty, 1]$ in interval notation, and the range is $\{y|y = 2 \text{ or } -3 \leq y < 1\}$ in set notation of $[-3, 1) \cup \{2\}$ in interval notation.

71. The definition for this piecewise function is

$$f(x) = \begin{cases} -x^3, & \text{if } -2 < x < 1 \\ 0, & \text{if } x = 1 \\ x^2, & \text{if } 1 < x \leq 3. \end{cases}$$

The domain of f is the set $\{x|-2 < x \leq 3\}$ or the interval $(-2, 3]$, and the range is the set $\{y|-1 < y \leq 9\}$ or the interval $(-1, 9]$.

72. The definition for this piecewise function is

$$f(x) = \begin{cases} x^2 - 1, & \text{if } -3 \leq x \leq 0 \\ x^3, & \text{if } 0 < x \leq 2. \end{cases}$$

The domain of f is the set $\{x|-3 \leq x \leq 2\}$ or the closed interval $[-3, 2]$, and the range is the set $\{y|-1 \leq y \leq 8\}$ or the closed interval $[-1, 8]$.

73. (a) The cost of manufacturing 100 road bikes is

$$C(100) = 0.004(100)^3 - 0.6(100)^2 + 250(100) + 100,500 = 123,500;$$

the cost of manufacturing 101 road bikes is

$$C(101) = 0.004(101)^3 - 0.6(101)^2 + 250(101) + 100,500 \approx 123,750.60.$$

The average rate of change of C from 100 to 101 road bikes is therefore

$$\frac{C(101) - C(100)}{101 - 100} \approx \frac{123,750.60 - 123,500}{1} = \boxed{\$250.60 \text{ per bike}}.$$

- (b) The cost of manufacturing 500 road bikes is

$$C(500) = 0.004(500)^3 - 0.6(500)^2 + 250(500) + 100,500 = 575,500;$$

the cost of manufacturing 501 road bikes is

$$C(501) = 0.004(501)^3 - 0.6(501)^2 + 250(501) + 100,500 \approx 578,155.40.$$

The average rate of change of C from 500 to 501 road bikes is therefore

$$\frac{C(501) - C(500)}{501 - 500} \approx \frac{578,155.40 - 575,500}{1} = \boxed{\$2655.40 \text{ per bike}}.$$

- (c) Answers will vary. One possible interpretation is that the unit cost per road bike increases as the number of road bikes manufactured increases.

74. (a) The cost of producing 500 tons of steel is

$$C(500) = \frac{1}{10}(500)^2 + 5(500) + 1500 = 29000;$$

the cost of producing 501 tons of steel is

$$C(501) = \frac{1}{10}(501)^2 + 5(501) + 1500 = 29105.1;$$

The average rate of change of C from 500 to 501 tons of steel is therefore

$$\frac{C(501) - C(500)}{501 - 500} = \frac{29105.1 - 29000}{1} = \boxed{\$105.10 \text{ per ton}}.$$

- (b) The cost of producing 1000 tons of steel is

$$C(1000) = \frac{1}{10}(1000)^2 + 5(1000) + 1500 = 106,500;$$

the cost of producing 1001 tons of steel is

$$C(1001) = \frac{1}{10}(1001)^2 + 5(1001) + 1500 = 106,705.1;$$

The average rate of change of C from 1000 to 1001 tons of steel is therefore

$$\frac{C(1001) - C(1000)}{1001 - 1000} = \frac{106,705.1 - 106,500}{1} = \boxed{\$205.10 \text{ per ton}}.$$

- (c) Answers will vary. One possible interpretation is that the unit cost per ton of steel increases as the number of tons produced increases.

P.2: Library of Functions; Mathematical Modeling

Concepts and Vocabulary

1. The function $f(x) = x^2$ is **(a) increasing** on the interval $(0, \infty)$.
2. **True**. The floor function $f(x) = \lfloor x \rfloor$ is an example of a step function.
3. **True**. The cube function is odd and is increasing on the interval $(-\infty, \infty)$.
4. **False**. The cube root function is odd, but is increasing on the interval $(-\infty, \infty)$.
5. **False**. The domain and range of the reciprocal function are all real nonzero numbers.
6. A number r for which $f(r) = 0$ is called a **zero** of the function f .
7. If r is a zero of even multiplicity of a function f , the graph of f **(b) touches** the x -axis at r .
8. **True**. The x -intercepts of the graph of a polynomial function are called zeros of the function.
9. **True**. The function $f(x) = [x + \sqrt[3]{x^2 - \pi}]^{2/3}$ is an algebraic function.
10. **False**. The domain of every rational function is the set of all real numbers except those for which the value of the denominator polynomial is zero.

Practice Problems

11. This is the graph of a **cube function (D)**.
12. This is the graph of an **identity function (B)**.
13. This is the graph of a **reciprocal function (F)**.
14. This is the graph of a **constant function (A)**.
15. This is the graph of a **square function (C)**.
16. This is the graph of a **square root function (E)**.
17. This is the graph of an **absolute value function (G)**.
18. This is the graph of a **cube root function (H)**.
19. Let $f(x) = \lfloor 2x \rfloor$. Then:
 - (a) $f(1.2) = \lfloor 2(1.2) \rfloor = \lfloor 2.4 \rfloor = \mathbf{2}$.
 - (b) $f(1.6) = \lfloor 2(1.6) \rfloor = \lfloor 3.2 \rfloor = \mathbf{3}$.
 - (c) $f(-1.8) = \lfloor 2(-1.8) \rfloor = \lfloor -3.6 \rfloor = \mathbf{-4}$.
20. Let $f(x) = \left\lceil \frac{x}{2} \right\rceil$. Then:
 - (a) $f(1.2) = \left\lceil \frac{1.2}{2} \right\rceil = \lceil 0.6 \rceil = \mathbf{1}$.

(b) $f(1.6) = \left\lceil \frac{1.6}{2} \right\rceil = \lceil 0.8 \rceil = \boxed{1}$.

(c) $f(-1.8) = \left\lceil \frac{-1.8}{2} \right\rceil = \lceil -0.9 \rceil = \boxed{0}$.

21. Let $f(x) = 3(x - 7)(x + 4)^3$.

(a) The zeros of f are: $x = 7$ with multiplicity one and $x = -4$ with multiplicity three.

(b) The x -intercepts of the graph of f are the zeros 7 and -4 . The y -intercept is

$$f(0) = 3(0 - 7)(0 + 4)^3 = \boxed{-1344}.$$

(c) Because both x -intercepts are zeros with odd multiplicity, the graph of f crosses the x -axis at each x -intercept.

22. Let $f(x) = 4x(x^2 + 1)(x - 2)^3$.

(a) The zeros of f are: $x = 0$ with multiplicity one and $x = 2$ with multiplicity three. Note that the factor $x^2 + 1$ is never equal to zero for any real number x .

(b) The x -intercepts of the graph of f are the zeros 0 and 2. The y -intercept is

$$f(0) = 4(0)(0^2 + 1)(0 - 2)^3 = \boxed{0}.$$

(c) Because both x -intercepts are zeros with odd multiplicity, the graph of f crosses the x -axis at each x -intercept.

23. The graph of f has x -intercepts of 0, 1, and 2, so the function must have factors of x , $x - 1$, and $x - 2$, which is true for all of the listed functions. Because the graph crosses the x -axis at each x -intercept, the zeros must each have odd multiplicity. This rules out functions **(b)** and **(d)**. Finally, f becomes unbounded in the positive direction as x becomes unbounded in the positive direction, and f becomes unbounded in the negative direction as x becomes unbounded in the negative direction. This rules out function **(a)**. Thus, functions **(c)**, **(e)**, and **(f)** might have the given graph.

24. The graph of f has x -intercepts of 0, 1, and 2, so the function must have factors of x , $x - 1$, and $x - 2$, which is true for all of the listed functions. Because the graph crosses the x -axis at $x = 0$ and at $x = 2$ but just touches the x -axis at $x = 1$, the zeros 0 and 2 must have odd multiplicity while the zero 1 must have even multiplicity. This rules out functions **(a)**, **(b)**, and **(d)**. Finally, f becomes unbounded in the positive direction as x becomes unbounded in either direction. Each of the remaining functions satisfies this condition. Thus, functions **(c)**, **(e)**, and **(f)** might have the given graph.

25. The domain of R is the set $\{x | x \neq -3\}$. Because 0 is in the domain of R and $R(0) = 0$, the y -intercept is 0. The zeros of R are solutions of the equation $5x^2 = 0$, or $x = 0$. Because 0 is in the domain of R , the x -intercept is 0. Thus, the only intercept is $(0, 0)$.

26. The domain of H is the set $\{x | x \neq 2, x \neq -4\}$. Because 0 is in the domain of H and $H(0) = 0$, the y -intercept is 0. The zeros of H are solutions of the equation $-4x^2 = 0$, or $x = 0$. Because 0 is in the domain of H , the x -intercept is 0. Thus, the only intercept is $(0, 0)$.

27. Because the denominator $x^2 + 4$ is never equal to zero for any real number x , the domain of R is the set of all real numbers or the interval $(-\infty, \infty)$. Because 0 is in the domain of R and $R(0) = 0$, the y -intercept is 0. The zeros of R are solutions of the equation $3x^2 - x = x(3x - 1) = 0$, or $x = 0$ and $x = \frac{1}{3}$. Because both 0 and $\frac{1}{3}$ are in the domain of R , the x -intercepts are 0 and $\frac{1}{3}$. Thus, the intercepts are $(0, 0)$ and $\left(\frac{1}{3}, 0\right)$.

28. The domain of R is the set $\{x|x \neq \pm 3\}$. Because 0 is in the domain of R and $R(0) = \frac{1}{2}$, the y -intercept is $\frac{1}{2}$. The zeros of R are solutions of the equation $3(x^2 - x - 6) = 3(x - 3)(x + 2) = 0$, or $x = 3$ and $x = -2$. However, $x = 3$ is not in the domain of R , so the only x -intercept is -2 . Thus, the intercepts

are $(-2, 0)$ and $(0, \frac{1}{2})$

29. (a) Let $A(x)$ denote the area of the rectangles as a function of x . From the diagram, each rectangle has length x and height y . Because the point (x, y) lies on the graph of $y = 16 - x^2$, it follows that the height of each rectangle is $16 - x^2$. Therefore,

$$A(x) = xy = x(16 - x^2) = 16x - x^3.$$

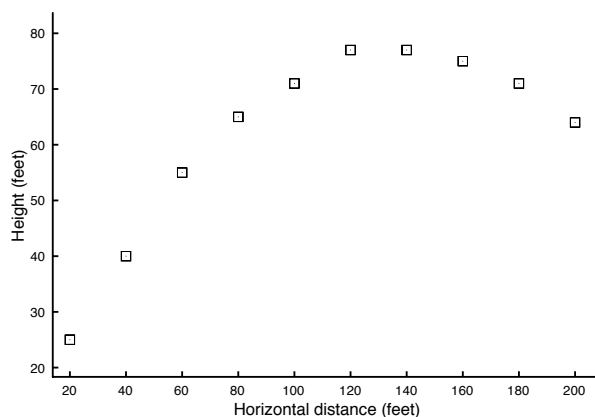
- (b) Because the point (x, y) must lie in the first quadrant, $x \geq 0$. As y must also be greater than or equal to 0, x can only be as large as the positive zero of $16 - x^2$. Thus, the domain of A is the set $\{x|0 \leq x \leq 4\}$ or the closed interval $[0, 4]$.
30. (a) Let $A(x)$ denote the area of the rectangles as a function of x . From the diagram, each rectangle has length $2x$ and height y . Because the point (x, y) is in the first quadrant and lies on the graph of a circle of radius 2 centered at the origin, whose equation is $x^2 + y^2 = 4$, it follows that the height of each rectangle is $y = \sqrt{4 - x^2}$. Therefore,

$$A(x) = 2xy = 2x\sqrt{4 - x^2}.$$

- (b) Let $p(x)$ denote the perimeter of the rectangles as a function of x . From the diagram, each rectangle has length $2x$ and height y . Because the point (x, y) is in the first quadrant and lies on the graph of a circle of radius 2 centered at the origin, whose equation is $x^2 + y^2 = 4$, it follows that the height of each rectangle is $y = \sqrt{4 - x^2}$. Therefore,

$$p(x) = 2(2x) + 2y = 4x + 2\sqrt{4 - x^2}.$$

31. (a) A scatter plot of the data is shown below. The pattern in the points suggests a quadratic relationship between the height of the ball and the horizontal distance traveled.



- (b) Using the QuadReg function on a TI-84 Plus calculator yields

$$h(x) = -0.0037x^2 + 1.03x + 5.7$$

as the quadratic function that best fits these data. The ball will hit the ground when $h(x) = 0$. Using the quadratic formula, this will happen when

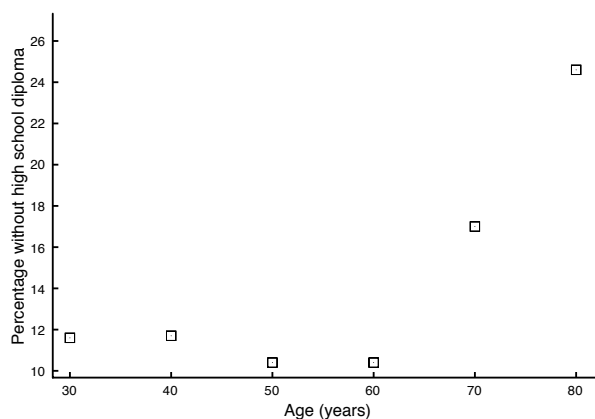
$$\begin{aligned} x &= \frac{-1.03 \pm \sqrt{(1.03)^2 - 4(-0.0037)(5.7)}}{2(-0.0037)} = \frac{-1.03 \pm 1.07}{-0.0074} \\ &\approx 283.8, -5.4 \end{aligned}$$

The value $x = -5.4$ may be disregarded. The ball will therefore hit the ground after traveling a horizontal distance of approximately 283.8 feet.

- (c) When the ball has traveled a horizontal distance of 10 feet, the height will be approximately

$$h(10) = -0.0037(10)^2 + 1.03(10) + 5.7 = 15.63 \text{ feet}.$$

32. (a) A scatter plot of the data is shown below. The pattern in the points suggests a cubic relationship between the percentage of the population without a high school diploma and age.



- (b) Using the CubicReg function on a TI-84 Plus calculator yields

$$P(x) = 0.00026x^3 - 0.0303x^2 + 1.0877x - 0.5071$$

as the cubic function that best fits these data.

- (c) Based on these data, the approximate percentage of 35-year olds who do not have a high school diploma is

$$P(35) = 0.00026(35)^3 - 0.0303(35)^2 + 1.0877(35) - 0.5071 \approx 11.59.$$

P.3: Operations on Functions; Graphing Techniques

Concepts and Vocabulary

1. If the domain of function f is $\{x|0 \leq x \leq 7\}$ and the domain of function g is $\{x|-2 \leq x \leq 5\}$, then the domain of the sum function $f + g$ is $\{x|0 \leq x \leq 5\}$; that is, the intersection of the domains of f and g .
2. **False**. If f and g are functions, then the domain of $\frac{f}{g}$ consists of all numbers x that are in the domains of both f and g , but excluding the numbers x for which $g(x) = 0$.
3. **True**. The domain of $(f \cdot g)(x)$ consists of the numbers x that are in the domains of both f and g .
4. **False**. The domain of the composite function $(f \circ g)(x)$ is the set of all numbers x in the domain of g for which $g(x)$ is in the domain of f .
5. **True**. The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x -axis.
6. **False**. To obtain the graph of $y = f(x + 2) - 3$, shift the graph of $y = f(x)$ horizontally to the left 2 units and vertically down 3 units.
7. **True**. Suppose the x -intercepts of the graph of the function f are -3 and 2 . Then the x -intercepts of the graph of $y = 2f(x)$ are -3 and 2 .
8. Suppose that the graph of a function f is known. Then the graph of the function $y = f(x - 2)$ can be obtained by a **horizontal** shift of the graph of f to the **right** a distance of 2 units.
9. Suppose that the graph of a function f is known. Then the graph of the function $y = f(-x)$ can be obtained by a reflection about the **y** -axis of the graph of f .
10. Suppose the x -intercepts of the graph of the function f are -2 , 1 , and 5 . The x -intercepts of $y = f(x + 3)$ are -5 , -2 , and 2 .

Practice Problems

11. Let $f(x) = 3x + 4$ and $g(x) = 2x - 3$. Note the domain of both f and g is the set of all real numbers.
 - (a) $(f + g)(x) = f(x) + g(x) = 3x + 4 + 2x - 3 = 5x + 1$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
 - (b) $(f - g)(x) = f(x) - g(x) = 3x + 4 - (2x - 3) = x + 7$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
 - (c) $(f \cdot g)(x) = f(x)g(x) = (3x + 4)(2x - 3) = 6x^2 - x - 12$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
 - (d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x + 4}{2x - 3}$. The domain of this function is the set $\{x|x \neq \frac{3}{2}\}$.
12. Let $f(x) = 1 + \frac{1}{x}$ and $g(x) = \frac{1}{x}$. Note the domain of both f and g is the set $\{x|x \neq 0\}$.
 - (a) $(f + g)(x) = f(x) + g(x) = 1 + \frac{1}{x} + \frac{1}{x} = 1 + \frac{2}{x}$. The domain of this function is the set $\{x|x \neq 0\}$.

- (b) $(f - g)(x) = f(x) - g(x) = 1 + \frac{1}{x} - \frac{1}{x} = \boxed{1}$. The domain of this function is the set $\boxed{\{x|x \neq 0\}}$.
- (c) $(f \cdot g)(x) = f(x)g(x) = \left(1 + \frac{1}{x}\right) \frac{1}{x} = \boxed{\frac{1}{x} + \frac{1}{x^2}}$. The domain of this function is the set $\boxed{\{x|x \neq 0\}}$.
- (d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1 + \frac{1}{x}}{\frac{1}{x}} = \boxed{x + 1}$. The domain of this function is the set $\boxed{\{x|x \neq 0\}}$.
13. Let $f(x) = \sqrt{x+1}$ and $g(x) = \frac{2}{x}$. Note the domain of f is the set $\{x|x \geq -1\}$ and the domain of g is the set $\{x|x \neq 0\}$.

- (a) $(f + g)(x) = f(x) + g(x) = \boxed{\sqrt{x+1} + \frac{2}{x}}$. The domain of this function is the set $\boxed{\{x|-1 \leq x < 0\} \cup \{x|x > 0\}}$.
- (b) $(f - g)(x) = f(x) - g(x) = \boxed{\sqrt{x+1} - \frac{2}{x}}$. The domain of this function is the set $\boxed{\{x|-1 \leq x < 0\} \cup \{x|x > 0\}}$.
- (c) $(f \cdot g)(x) = f(x)g(x) = \sqrt{x+1} \cdot \frac{2}{x} = \boxed{\frac{2\sqrt{x+1}}{x}}$. The domain of this function is the set $\boxed{\{x|-1 \leq x < 0\} \cup \{x|x > 0\}}$.
- (d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\frac{2}{x}} = \boxed{\frac{x}{2}\sqrt{x+1}}$. The domain of this function is the set $\boxed{\{x|-1 \leq x < 0\} \cup \{x|x > 0\}}$.

14. Let $f(x) = |x|$ and $g(x) = x$. Note the domain of both f and g is the set of all real numbers.

- (a) $(f + g)(x) = f(x) + g(x) = \boxed{|x| + x}$. The domain of this function is the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.
- (b) $(f - g)(x) = f(x) - g(x) = \boxed{|x| - x}$. The domain of this function is the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.
- (c) $(f \cdot g)(x) = f(x)g(x) = \boxed{|x|x}$. The domain of this function is the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.
- (d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{|x|}{x}}$. The domain of this function is the set $\boxed{\{x|x \neq 0\}}$.

15. Let $f(x) = 2x$ and $g(x) = 3x^2 + 1$.

- (a) $(f \circ g)(4) = f(g(4)) = f(49) = \boxed{98}$.
- (b) $(g \circ f)(2) = g(f(2)) = g(4) = \boxed{49}$.
- (c) $(f \circ f)(1) = f(f(1)) = f(2) = \boxed{4}$.
- (d) $(g \circ g)(0) = g(g(0)) = g(1) = \boxed{4}$.

16. Let $f(x) = \frac{3}{x+1}$ and $g(x) = \sqrt{x}$.

- (a) $(f \circ g)(4) = f(g(4)) = f(2) = \boxed{1}$.

- (b) $(g \circ f)(2) = g(f(2)) = g(1) = \boxed{1}$.
- (c) $(f \circ f)(1) = f(f(1)) = f\left(\frac{3}{2}\right) = \boxed{\frac{6}{5}}$.
- (d) $(g \circ g)(0) = g(g(0)) = g(0) = \boxed{0}$.
17. (a) $(f \circ g)(1) = f(g(1)) = f(0) = \boxed{-1}$.
- (b) $(f \circ g)(-1) = f(g(-1)) = f(0) = \boxed{-1}$.
- (c) $(g \circ f)(-1) = g(f(-1)) = g(-3) = \boxed{8}$.
- (d) $(g \circ f)(1) = g(f(1)) = g(3) = \boxed{8}$.
- (e) $(g \circ g)(-2) = g(g(-2)) = g(3) = \boxed{8}$.
- (f) $(f \circ f)(-1) = f(f(-1)) = f(-3) = \boxed{-7}$.
18. (a) $(f \circ g)(1) = f(g(1)) = f(0) = \boxed{5}$.
- (b) $(f \circ g)(2) = f(g(2)) = f(-3) = \boxed{11}$.
- (c) $(g \circ f)(2) = g(f(2)) = g(1) = \boxed{0}$.
- (d) $(g \circ f)(3) = g(f(3)) = g(-1) = \boxed{0}$.
- (e) $(g \circ g)(1) = g(g(1)) = g(0) = \boxed{1}$.
- (f) $(f \circ f)(3) = f(f(3)) = f(-1) = \boxed{7}$.
19. (a) $(g \circ f)(-1) = g(f(-1)) = g(1) = \boxed{4}$.
- (b) $(g \circ f)(6) = g(f(6)) = g(2) = \boxed{2}$.
- (c) $(f \circ g)(6) = f(g(6)) = f(5) = \boxed{1}$.
- (d) $(f \circ g)(4) = f(g(4)) = f(2) = \boxed{-2}$.
20. (a) $(g \circ f)(1) = g(f(1)) = g(-1) = \boxed{3}$.
- (b) $(g \circ f)(5) = g(f(5)) = g(1) = \boxed{4}$.
- (c) $(f \circ g)(7) = f(g(7)) = f(5) = \boxed{1}$.
- (d) $(f \circ g)(2) = f(g(2)) = f(2) = \boxed{-2}$.
21. Let $f(x) = 3x + 1$ and $g(x) = 8x$. Note the domain of both f and g is the set of all real numbers.
- (a) $(f \circ g)(x) = f(g(x)) = f(8x) = 3(8x) + 1 = \boxed{24x + 1}$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (b) $(g \circ f)(x) = g(f(x)) = g(3x + 1) = 8(3x + 1) = \boxed{24x + 8}$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (c) $(f \circ f)(x) = f(f(x)) = f(3x + 1) = 3(3x + 1) + 1 = \boxed{9x + 4}$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) $(g \circ g)(x) = g(g(x)) = g(8x) = 8(8x) = \boxed{64x}$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
22. Let $f(x) = -x$ and $g(x) = 2x - 4$. Note the domain of both f and g is the set of all real numbers.

- (a) $(f \circ g)(x) = f(g(x)) = f(2x - 4) = -(2x - 4) = 4 - 2x$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (b) $(g \circ f)(x) = g(f(x)) = g(-x) = 2(-x) - 4 = -2x - 4$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (c) $(f \circ f)(x) = f(f(x)) = f(-x) = -(-x) = x$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) $(g \circ g)(x) = g(g(x)) = g(2x - 4) = 2(2x - 4) - 4 = 4x - 12$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
23. Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 1}$. Note the domain of f is the set of all real numbers and the domain of g is the set $\{x|x \geq 1\}$.
- (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x - 1}) = (\sqrt{x - 1})^2 + 1 = x$. The domain of $f \circ g$ begins with the domain of g : $\{x|x \geq 1\}$. Because the domain of f is the set of all real numbers, any value of $g(x)$ will be in the domain of f . Therefore, the domain of $f \circ g$ is the set $\{x|x \geq 1\}$.
- (b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \sqrt{x^2 + 1 - 1} = |x|$. The domain of $g \circ f$ begins with the domain of f : the set of all real numbers. Because the domain of g is the set $\{x|x \geq 1\}$ and the value of $f(x)$ is always a number greater than or equal to one, it follows that the value of $f(x)$ is always in the domain of g . Therefore, the domain of $g \circ f$ is the set of all real numbers or the interval $(-\infty, \infty)$.
- (c) $(f \circ f)(x) = f(f(x)) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) $(g \circ g)(x) = g(g(x)) = g(\sqrt{x - 1}) = \sqrt{\sqrt{x - 1} - 1}$. The domain of $g \circ g$ begins with the domain of g : $\{x|x \geq 1\}$. Next, the value of $g(x) = \sqrt{x - 1}$ must be in the domain of g ; that is, the inequality $\sqrt{x - 1} \geq 1$ must be satisfied. Solving this inequality yields $x - 1 \geq 1$, or $x \geq 2$. Therefore, the domain of $g \circ g$ is the set $\{x|x \geq 2\}$.
24. Let $f(x) = 2x + 3$ and $g(x) = \sqrt{x}$. Note the domain of f is the set of all real numbers and the domain of g is the set $\{x|x \geq 0\}$.
- (a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2\sqrt{x} + 3$. The domain of $f \circ g$ begins with the domain of g : $\{x|x \geq 0\}$. Because the domain of f is the set of all real numbers, any value of $g(x)$ will be in the domain of f . Therefore, the domain of $f \circ g$ is the set $\{x|x \geq 0\}$.
- (b) $(g \circ f)(x) = g(f(x)) = g(2x + 3) = \sqrt{2x + 3}$. The domain of $g \circ f$ begins with the domain of f : the set of all real numbers. Next, the value of $f(x) = 2x + 3$ must be in the domain of g ; that is, the inequality $2x + 3 \geq 0$ must be satisfied. Solving this inequality yields $x \geq -\frac{3}{2}$. Therefore, the domain of $g \circ f$ is the set $\{x|x \geq -\frac{3}{2}\}$.
- (c) $(f \circ f)(x) = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 10$. The domain of this function is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) $(g \circ g)(x) = g(g(x)) = g(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. The domain of $g \circ g$ begins with the domain of g : $\{x|x \geq 0\}$. Next, the value of $g(x) = \sqrt{x}$ must be in the domain of g ; that is, the inequality $\sqrt{x} \geq 0$ must be satisfied. Solving this inequality yields $x \geq 0$. Therefore, the domain of $g \circ g$ is the set $\{x|x \geq 0\}$.

25. Let $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{2}{x}$. Note the domain of f is the set $\{x|x \neq 1\}$ and the domain of g is the set $\{x|x \neq 0\}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x}\right) = \frac{\frac{2}{x}}{\frac{2}{x}-1} = \boxed{\frac{2}{2-x}}$. The domain of $f \circ g$ begins with the domain of g : $\{x|x \neq 0\}$. Next, the value of $g(x)$ must be in the domain of f ; that is, the value of $\frac{2}{x}$ cannot be equal to 1. This means that x cannot be equal to 2. The domain of $f \circ g$ is therefore the set $\boxed{\{x|x \neq 0, x \neq 2\}}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x-1}\right) = \frac{2}{\frac{x}{x-1}} = \boxed{\frac{2(x-1)}{x}}$. The domain of $g \circ f$ begins with the domain of f : $\{x|x \neq 1\}$. Next, the value of $f(x)$ must be in the domain of g ; that is, the value of $\frac{x}{x-1}$ cannot be equal to 0. This means that x cannot be equal to 0. The domain of $g \circ f$ is therefore the set $\boxed{\{x|x \neq 0, x \neq 1\}}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = \boxed{x}$. The domain of $f \circ f$ begins with the domain of f : $\{x|x \neq 1\}$. Next, the value of $f(x)$ must be in the domain of f ; that is, the value of $\frac{x}{x-1}$ cannot be equal to 1. The equation

$$\frac{x}{x-1} = 1,$$

however, has no solution. The domain of $f \circ f$ is therefore the set $\boxed{\{x|x \neq 1\}}$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\frac{2}{x}\right) = \frac{2}{\frac{2}{x}} = \boxed{x}$. The domain of $g \circ g$ begins with the domain of g : $\{x|x \neq 0\}$. Next, the value of $g(x)$ must be in the domain of g ; that is, the value of $\frac{2}{x}$ cannot be equal to 0. The equation

$$\frac{2}{x} = 0,$$

however, has no solution. The domain of $g \circ g$ is therefore the set $\boxed{\{x|x \neq 0\}}$.

26. Let $f(x) = \frac{1}{x+3}$ and $g(x) = -\frac{2}{x}$. Note the domain of f is the set $\{x|x \neq -3\}$ and the domain of g is the set $\{x|x \neq 0\}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(-\frac{2}{x}\right) = \frac{1}{-\frac{2}{x}+3} = \boxed{\frac{x}{-2+3x}}$. The domain of $f \circ g$ begins with the domain of g : $\{x|x \neq 0\}$. Next, the value of $g(x)$ must be in the domain of f ; that is, the value of $-\frac{2}{x}$ cannot be equal to -3. This means that x cannot be equal to $\frac{2}{3}$. The domain of $f \circ g$ is therefore the set $\boxed{\{x|x \neq 0, x \neq \frac{2}{3}\}}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x+3}\right) = -\frac{2}{\frac{1}{x+3}} = \boxed{-2(x+3)}$. The domain of $g \circ f$ begins with the domain of f : $\{x|x \neq -3\}$. Next, the value of $f(x)$ must be in the domain of g ; that is, the value of $\frac{1}{x+3}$ cannot be equal to 0. The equation

$$\frac{1}{x+3} = 0,$$

however, has no solution. The domain of $g \circ f$ is therefore the set $\boxed{\{x|x \neq -3\}}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+3}\right) = \frac{1}{\frac{1}{x+3} + 3} = \frac{x+3}{1+3(x+3)} = \frac{x+3}{3x+10}$. The domain of $f \circ f$ begins with the domain of f : $\{x|x \neq -3\}$. Next, the value of $f(x)$ must be in the domain of f ; that is, the value of $\frac{1}{x+3}$ cannot be equal to -3 . This means that x cannot be equal to $-\frac{10}{3}$. The domain of $f \circ f$ is therefore the set $\{x|x \neq -3, x \neq -\frac{10}{3}\}$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(-\frac{2}{x}\right) = -\frac{2}{-\frac{2}{x}} = x$. The domain of $g \circ g$ begins with the domain of g : $\{x|x \neq 0\}$. Next, the value of $g(x)$ must be in the domain of g ; that is, the value of $-\frac{2}{x}$ cannot be equal to 0. The equation

$$-\frac{2}{x} = 0,$$

however, has no solution. The domain of $g \circ g$ is therefore the set $\{x|x \neq 0\}$.

27. Let $f(x) = x^4$ and $g(x) = 2x + 3$. Then

$$(f \circ g)(x) = f(g(x)) = f(2x + 3) = (2x + 3)^4 = F(x).$$

28. Let $f(x) = x^3$ and $g(x) = 1 + x^2$. Then

$$(f \circ g)(x) = f(g(x)) = f(1 + x^2) = (1 + x^2)^3 = F(x).$$

29. Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1} = F(x).$$

30. Let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Then

$$(f \circ g)(x) = f(g(x)) = f(1 - x^2) = \sqrt{1 - x^2} = F(x).$$

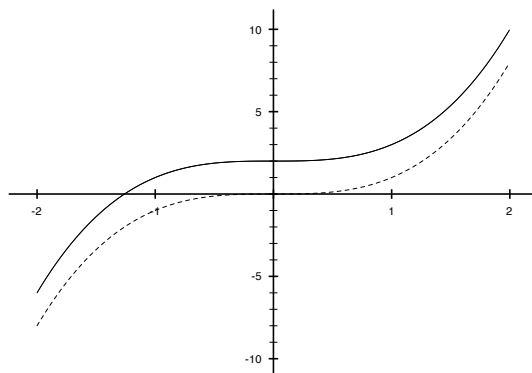
31. Let $f(x) = |x|$ and $g(x) = 2x + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(2x + 1) = |2x + 1| = F(x).$$

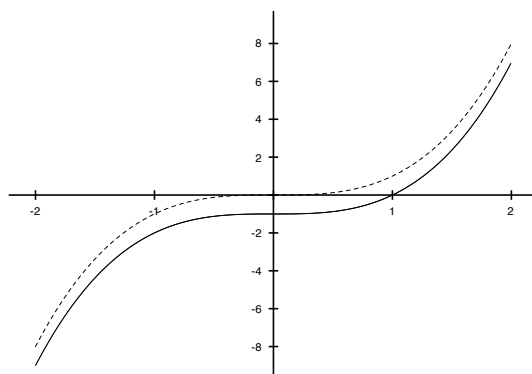
32. Let $f(x) = |x|$ and $g(x) = 2x^2 + 3$. Then

$$(f \circ g)(x) = f(g(x)) = f(2x^2 + 3) = |2x^2 + 3| = F(x).$$

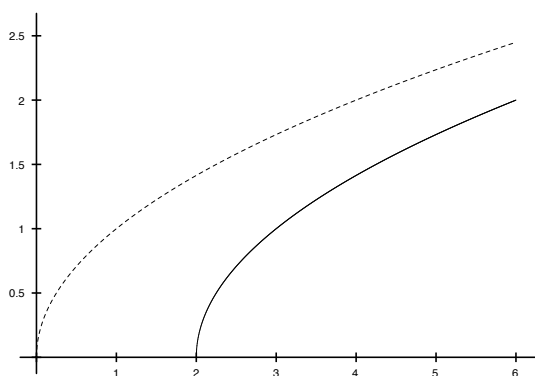
33. The graph of $f(x) = x^3 + 2$ can be obtained from the graph of $y = x^3$ by shifting vertically up 2 units. In the figure below, the graph of $y = x^3$ is represented by the dashed curve, and the graph of $y = x^3 + 2$ is represented by the solid curve.



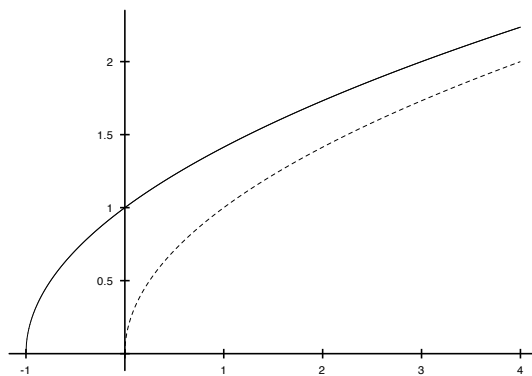
34. The graph of $g(x) = x^3 - 1$ can be obtained from the graph of $y = x^3$ by shifting vertically down 1 unit. In the figure below, the graph of $y = x^3$ is represented by the dashed curve, and the graph of $y = x^3 - 1$ is represented by the solid curve.



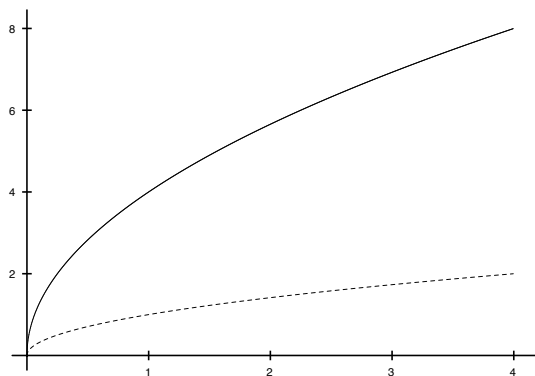
35. The graph of $h(x) = \sqrt{x-2}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting horizontally right 2 units. In the figure below, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = \sqrt{x-2}$ is represented by the solid curve.



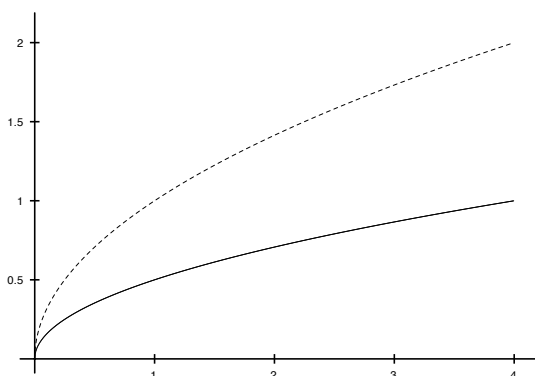
36. The graph of $f(x) = \sqrt{x+1}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting horizontally left 1 unit. In the figure below, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = \sqrt{x+1}$ is represented by the solid curve.



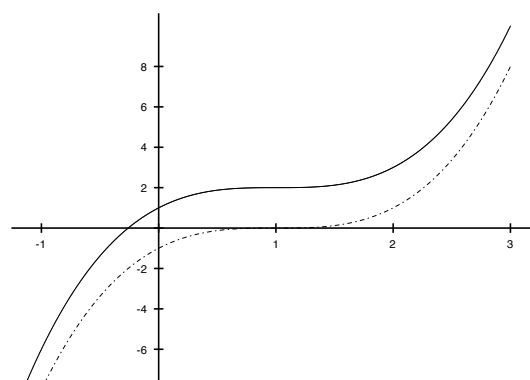
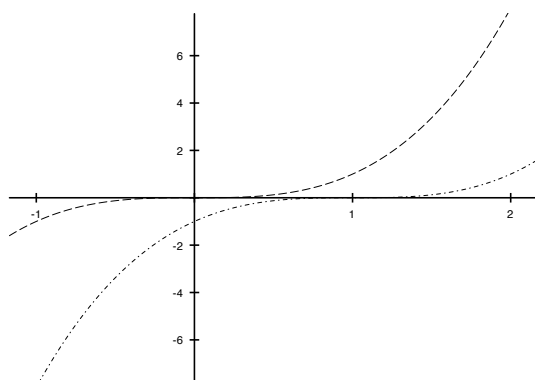
37. The graph of $g(x) = 4\sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by stretching vertically by a factor of 4. In the figure below, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = 4\sqrt{x}$ is represented by the solid curve.



38. The graph of $f(x) = \frac{1}{2}\sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by compressing vertically by a factor of $\frac{1}{2}$. In the figure below, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = \frac{1}{2}\sqrt{x}$ is represented by the solid curve.

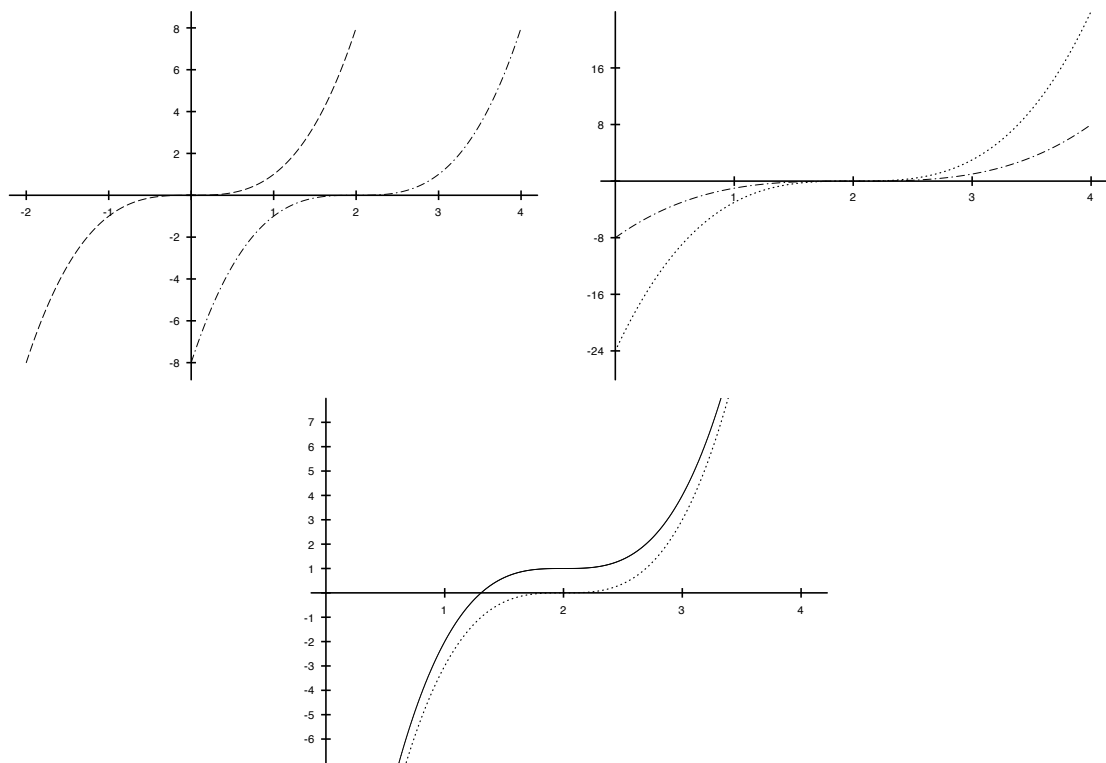


39. The graph of $f(x) = (x - 1)^3 + 2$ can be obtained from the graph of $y = x^3$ by shifting horizontally right 1 unit and then shifting vertically up 2 units. In the figure below at the left, the graph of $y = x^3$ is represented by the dashed curve, and the graph of $y = (x - 1)^3$ is represented by the dash-dot curve. In the figure below at the right, the graph of $y = (x - 1)^3$ is represented by the dash-dot curve, and the graph of $y = (x - 1)^3 + 2$ is represented by the solid curve.

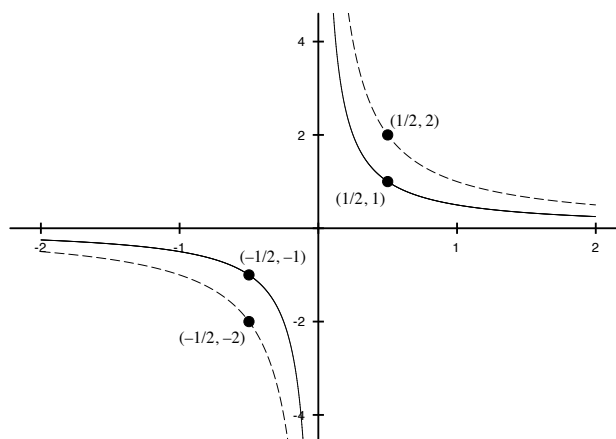


40. The graph of $g(x) = 3(x - 2)^3 + 1$ can be obtained from the graph of $y = x^3$ by shifting horizontally right 2 units, then stretching vertically by a factor of 3 and finally shifting vertically up 1 unit. In the figure below at the top left, the graph of $y = x^3$ is represented

by the dashed curve, and the graph of $y = (x - 2)^3$ is represented by the dash-dot curve. In the figure below at the top right, the graph of $y = (x - 2)^3$ is represented by the dash-dot curve, and the graph of $y = 3(x - 2)^3$ is represented by the dotted curve. In the bottom figure, the graph of $y = 3(x - 2)^3$ is represented by the dotted curve, and the graph of $y = 3(x - 2)^3 + 1$ is represented by the solid curve.

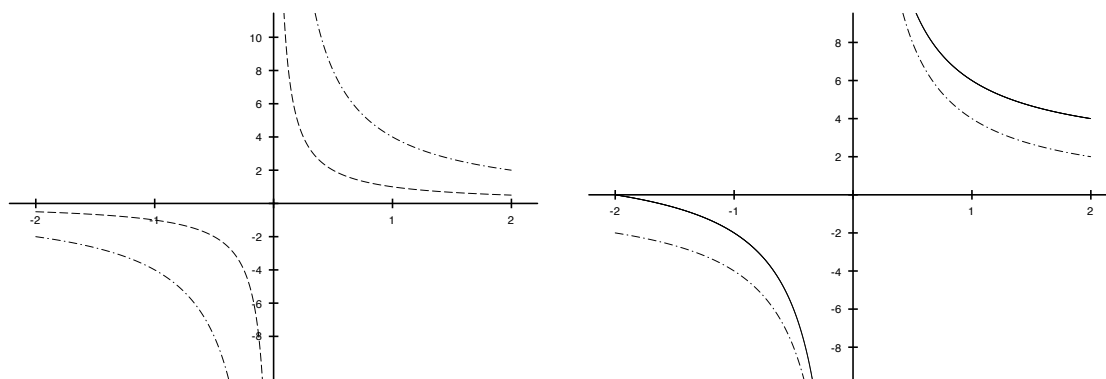


41. The graph of $h(x) = \frac{1}{2x}$ can be obtained from the graph of $y = \frac{1}{x}$ by compressing vertically by a factor of $\frac{1}{2}$. In the figure below, the graph of $y = \frac{1}{x}$ is represented by the dashed curve, and the graph of $y = \frac{1}{2x}$ is represented by the solid curve.

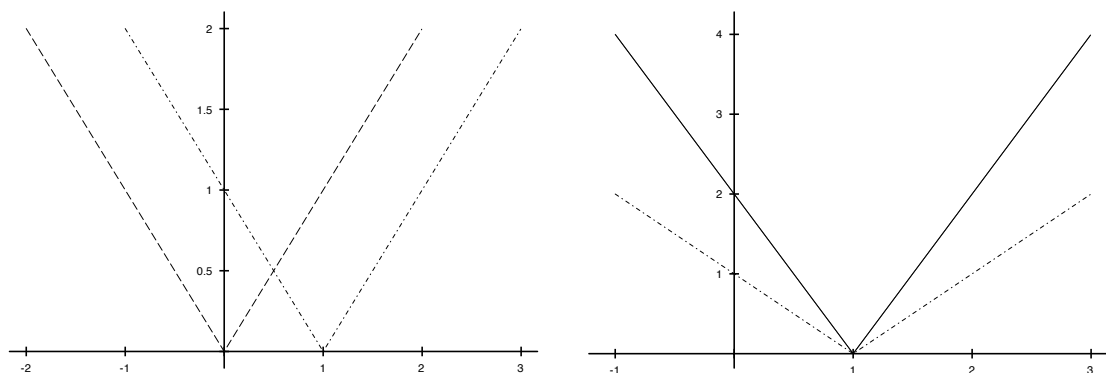


42. The graph of $f(x) = \frac{4}{x} + 2$ can be obtained from the graph of $y = \frac{1}{x}$ by stretching vertically by a factor of 4 and then shifting vertically up 2 units. In the figure below at the left, the graph of $y = \frac{1}{x}$ is represented by the dashed curve, and the graph of $y = \frac{4}{x}$ is represented

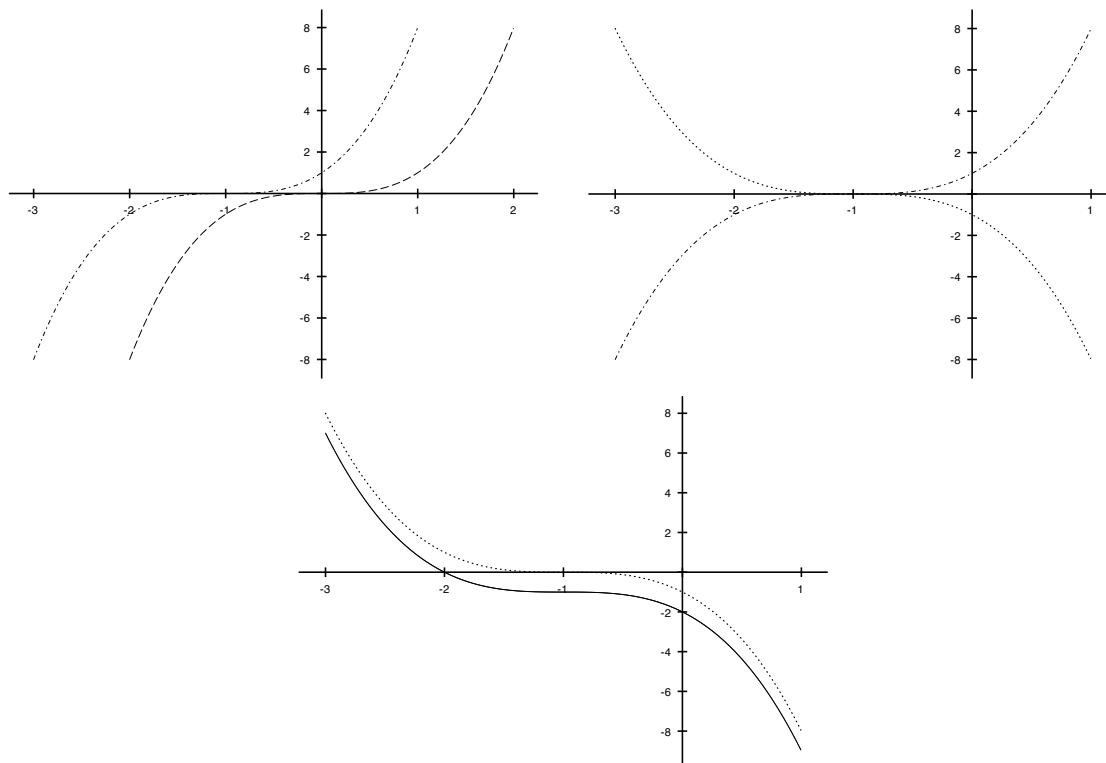
by the dash-dot curve. In the figure below at the right, the graph of $y = \frac{4}{x}$ is represented by the dash-dot curve, and the graph of $y = \frac{4}{x} + 2$ is represented by the solid curve.



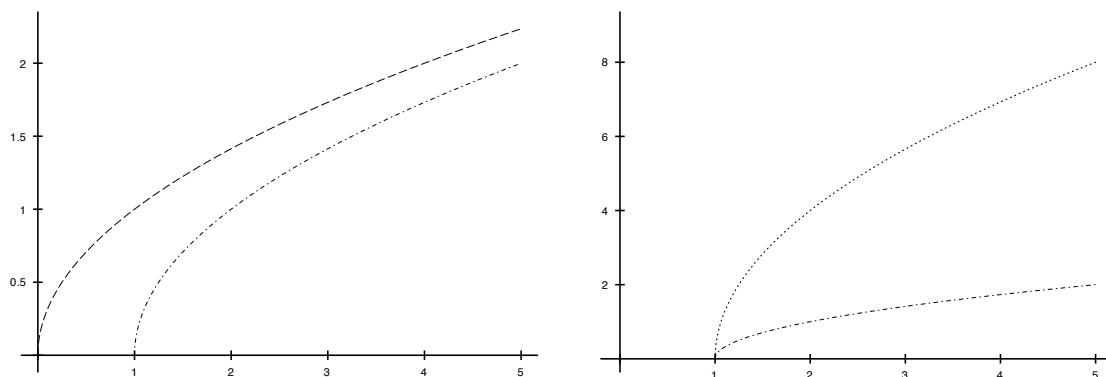
43. First note that because $|-t| = |t|$, the function G can be written as $G(x) = 2|-(x-1)| = 2|x-1|$. The graph of G can therefore be obtained from the graph of $y = |x|$ by shifting horizontally right 1 unit and then stretching vertically by a factor of 2. In the figure below at the left, the graph of $y = |x|$ is represented by the dashed curve, and the graph of $y = |x-1|$ is represented by the dash-dot curve. In the figure below at the right, the graph of $y = |x-1|$ is represented by the dash-dot curve, and the graph of $y = 2|x-1|$ is represented by the solid curve.

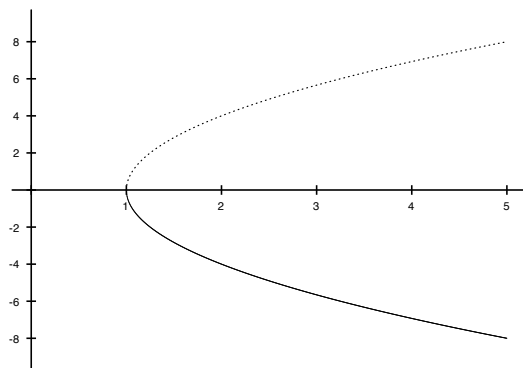


44. The graph of $g(x) = -(x+1)^3 - 1$ can be obtained from the graph of $y = x^3$ by shifting horizontally left 1 unit, then reflecting about the x -axis and finally shifting vertically down 1 unit. In the figure below at the top left, the graph of $y = x^3$ is represented by the dashed curve, and the graph of $y = (x+1)^3$ is represented by the dash-dot curve. In the figure below at the top right, the graph of $y = (x+1)^3$ is represented by the dash-dot curve, and the graph of $y = -(x+1)^3$ is represented by the dotted curve. In the bottom figure, the graph of $y = -(x+1)^3$ is represented by the dotted curve, and the graph of $y = -(x+1)^3 - 1$ is represented by the solid curve.

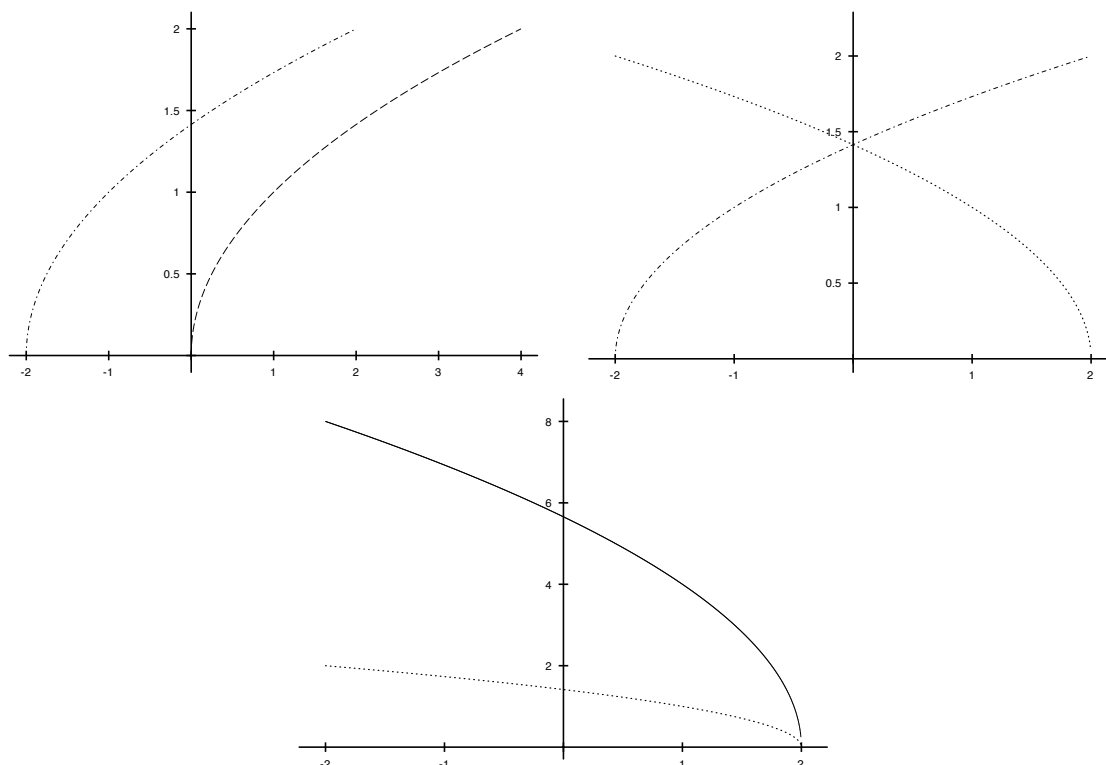


45. The graph of $g(x) = -4\sqrt{x-1}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting horizontally right 1 unit, then stretching vertically by a factor of 4, and finally reflecting about the x -axis. In the figure below at the top left, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = \sqrt{x-1}$ is represented by the dash-dot curve. In the figure below at the top right, the graph of $y = \sqrt{x-1}$ is represented by the dash-dot curve, and the graph of $y = 4\sqrt{x-1}$ is represented by the dotted curve. In the bottom figure, the graph of $y = 4\sqrt{x-1}$ is represented by the dotted curve, and the graph of $y = -4\sqrt{x-1}$ is represented by the solid curve.

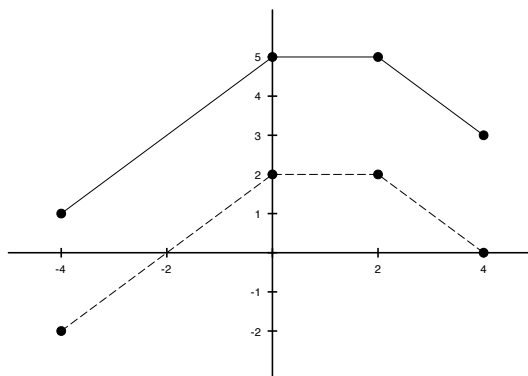




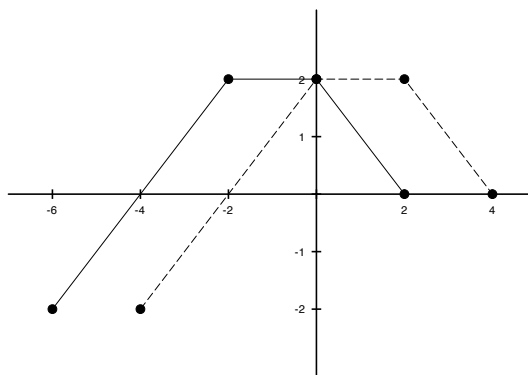
46. The graph of $f(x) = 4\sqrt{2-x} = 4\sqrt{-x+2}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting horizontally left 2 units, then reflecting about the y -axis, and finally stretching vertically by a factor of 4. In the figure below at the top left, the graph of $y = \sqrt{x}$ is represented by the dashed curve, and the graph of $y = \sqrt{x+2}$ is represented by the dash-dot curve. In the figure below at the top right, the graph of $y = \sqrt{x+2}$ is represented by the dash-dot curve, and the graph of $y = \sqrt{-x+2}$ is represented by the dotted curve. In the bottom figure, the graph of $y = \sqrt{-x+2}$ is represented by the dotted curve, and the graph of $y = 4\sqrt{-x+2}$ is represented by the solid curve.



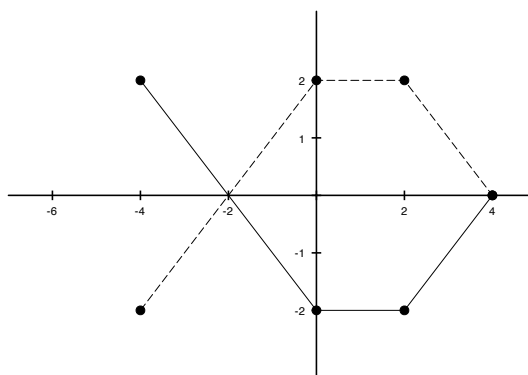
47. (a) The graph of $F(x) = f(x) + 3$ can be obtained from the graph of f by shifting vertically up 3 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of F is represented by the solid curve.



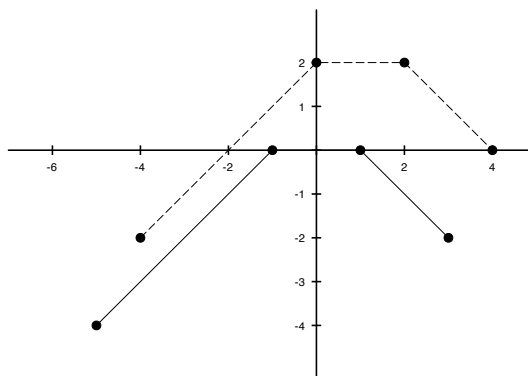
- (b) The graph of $G(x) = f(x + 2)$ can be obtained from the graph of f by shifting horizontally left 2 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of G is represented by the solid curve.



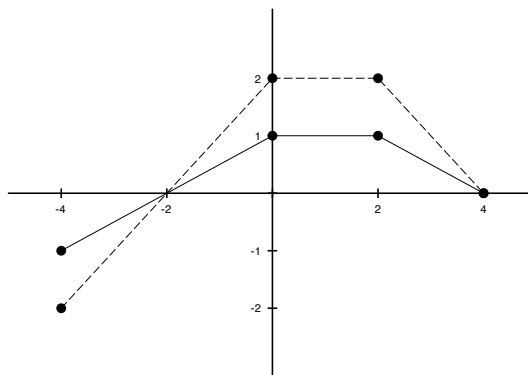
- (c) The graph of $P(x) = -f(x)$ can be obtained from the graph of f by reflecting about the x -axis. In the figure below, the graph of f is represented by the dashed curve, and the graph of P is represented by the solid curve.



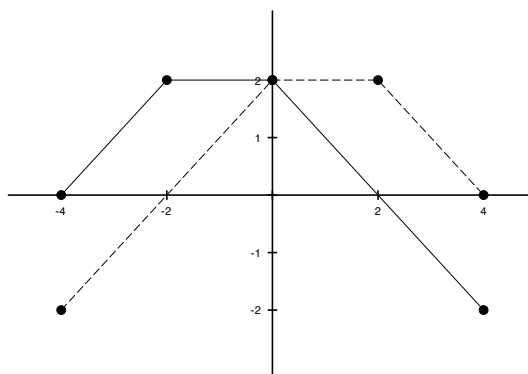
- (d) The graph of $H(x) = f(x + 1) - 2$ can be obtained from the graph of f by shifting horizontally left 1 unit and vertically down 2 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of H is represented by the solid curve.



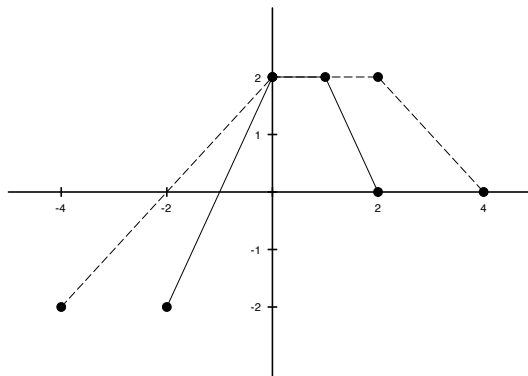
- (e) The graph of $Q(x) = \frac{1}{2}f(x)$ can be obtained from the graph of f by compressing vertically by a factor of $\frac{1}{2}$. In the figure below, the graph of f is represented by the dashed curve, and the graph of Q is represented by the solid curve.



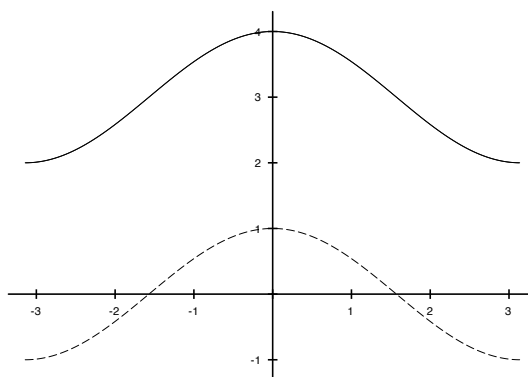
- (f) The graph of $g(x) = f(-x)$ can be obtained from the graph of f by reflecting about the y -axis. In the figure below, the graph of f is represented by the dashed curve, and the graph of g is represented by the solid curve.



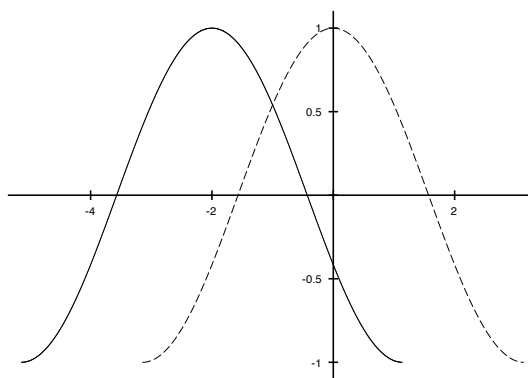
- (g) The graph of $h(x) = f(2x)$ can be obtained from the graph of f by compressing horizontally by a factor of $\frac{1}{2}$. In the figure below, the graph of f is represented by the dashed curve, and the graph of h is represented by the solid curve.



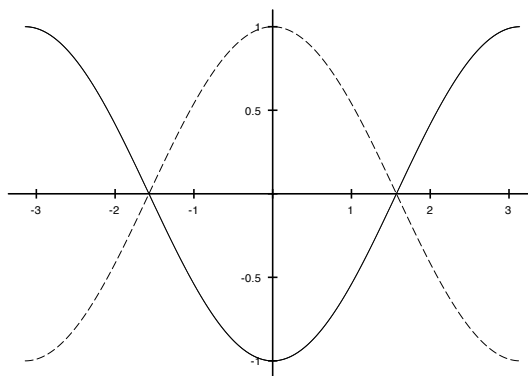
48. (a) The graph of $F(x) = f(x) + 3$ can be obtained from the graph of f by shifting vertically up 3 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of F is represented by the solid curve.



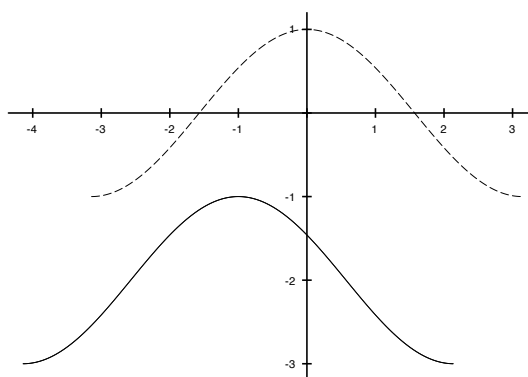
- (b) The graph of $G(x) = f(x + 2)$ can be obtained from the graph of f by shifting horizontally left 2 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of G is represented by the solid curve.



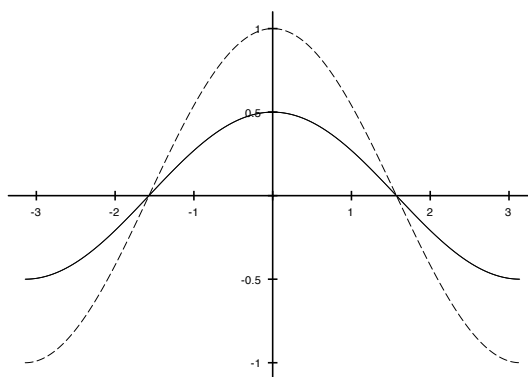
- (c) The graph of $P(x) = -f(x)$ can be obtained from the graph of f by reflecting about the x -axis. In the figure below, the graph of f is represented by the dashed curve, and the graph of P is represented by the solid curve.



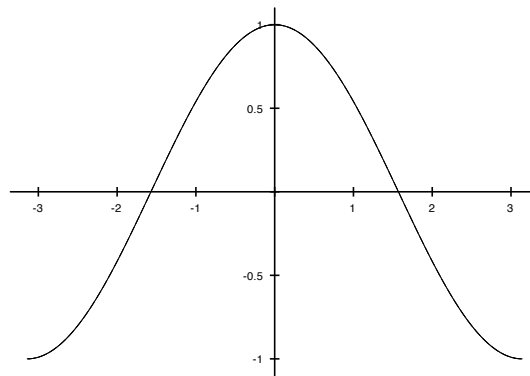
- (d) The graph of $H(x) = f(x + 1) - 2$ can be obtained from the graph of f by shifting horizontally left 1 unit and vertically down 2 units. In the figure below, the graph of f is represented by the dashed curve, and the graph of H is represented by the solid curve.



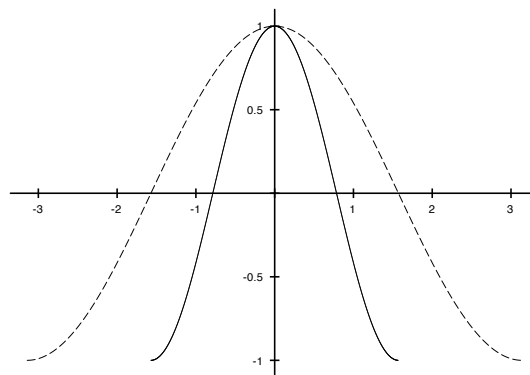
- (e) The graph of $Q(x) = \frac{1}{2}f(x)$ can be obtained from the graph of f by compressing vertically by a factor of $\frac{1}{2}$. In the figure below, the graph of f is represented by the dashed curve, and the graph of Q is represented by the solid curve.



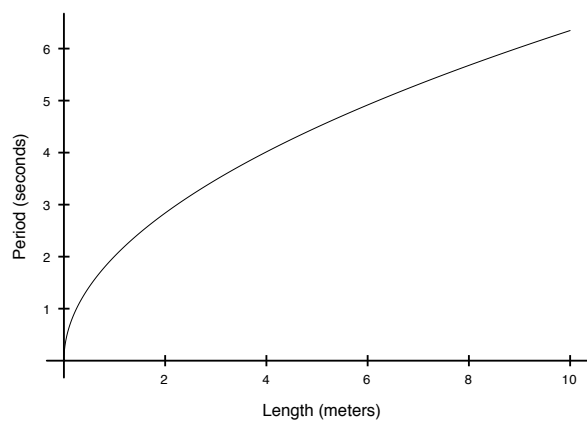
- (f) The graph of $g(x) = f(-x)$ can be obtained from the graph of f by reflecting about the y -axis. Because the graph of f is symmetric with respect to the y -axis, the graphs of f and g coincide.



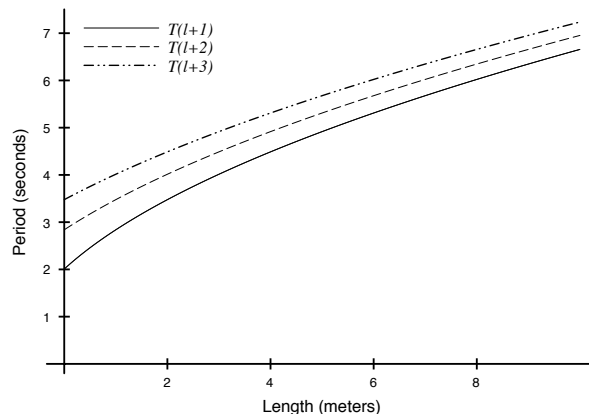
- (g) The graph of $h(x) = f(2x)$ can be obtained from the graph of f by compressing horizontally by a factor of $\frac{1}{2}$. In the figure below, the graph of f is represented by the dashed curve, and the graph of h is represented by the solid curve.



49. (a) The graph of $T(l)$ is shown below:



- (b) The graphs of $T(l + 1)$, $T(l + 2)$, and $T(l + 3)$ are shown below:



- (c) Answers will vary, but one interpretation follows. As seen in the graphs for parts (a) and (b), the period T is an increasing function of the length l . Adding to the length of the pendulum therefore increases the period. Algebraically, observe that

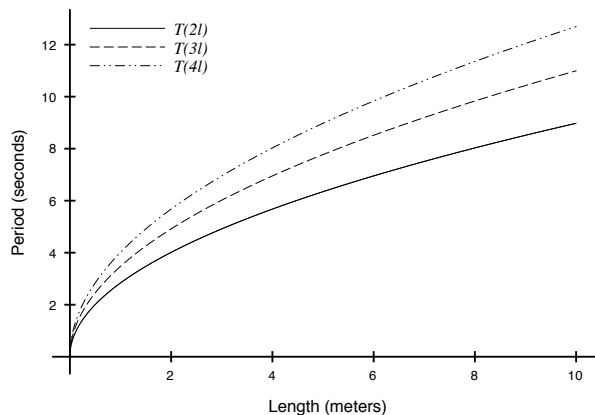
$$l + 1 = l \left(1 + \frac{1}{l} \right),$$

so that

$$T(l+1) = 2\pi \sqrt{\frac{l+1}{g}} = \sqrt{1 + \frac{1}{l}} \cdot 2\pi \sqrt{\frac{l}{g}} = \sqrt{1 + \frac{1}{l}} T(l).$$

The factor $\sqrt{1 + \frac{1}{l}}$ is always larger than 1 (indicating that the period increases) but becomes smaller as l becomes larger.

- (d) The graphs of $T(2l)$, $T(3l)$, and $T(4l)$ are shown below:



- (e) Answers will vary, but one interpretation follows. Note that

$$T(2l) = 2\pi \sqrt{\frac{2l}{g}} = \sqrt{2} \cdot 2\pi \sqrt{\frac{l}{g}} = \sqrt{2} T(l).$$

Similarly, $T(3l) = \sqrt{3}T(l)$, and $T(4l) = \sqrt{4}T(l) = 2T(l)$. Thus, multiplying the length of the pendulum by 2, 3, and 4, multiplies the period of the pendulum by $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{4} = 2$, respectively.

50. (a) The graph of $y = g(x + 3) - 5$ can be obtained from the graph of $y = g(x)$ by shifting horizontally left 3 units and vertically down 5 units. The graph of $y = g(x + 3) - 5$ therefore contains the point $(1 - 3, 3 - 5) = \boxed{(-2, -2)}$.
- (b) The graph of $y = -2g(x - 2) + 1$ can be obtained from the graph of $y = g(x)$ by shifting horizontally right 2 units, stretching vertically by a factor of 2, reflecting about the x -axis, and finally shifting vertically up 1 unit. The graph of $y = -2g(x - 2) + 1$ therefore contains the point $(1 + 2, -2 \cdot 3 + 1) = \boxed{(3, -5)}$.
- (c) The graph of $y = g(2x + 3)$ can be obtained from the graph of $y = g(x)$ by shifting horizontally left 3 units and then compressing horizontally by a factor of $\frac{1}{2}$. The graph of $y = g(2x + 3)$ therefore contains the point $\left(\frac{1}{2}(1 - 3), 3\right) = \boxed{(-1, 3)}$.

P.4: Inverse Functions

Concepts and Vocabulary

1. False. If every **horizontal** line intersects the graph of a function f at no more than one point, f is a one-to-one function.
2. If the domain of a one-to-one function f is $[4, \infty)$, the range of its inverse f^{-1} is $[4, \infty)$.
3. False. If f and g are inverse functions, the domain of f **does not need to be** the same as the domain of g . However, the domain of f will always be the same as the **range** of g .
4. True. If f and g are inverse functions, their graphs are symmetric with respect to the line $y = x$.
5. False. If f and g are inverse functions, then $(f \circ g)(x) = (g \circ f)(x) = x$; that is, $f(g(x)) = g(f(x)) = x$.
6. False. If a function f is one-to-one, then $f(f^{-1}(x)) = x$, where x is in the domain of f^{-1} .
7. First, examine the x -coordinates of the ordered pairs. If each value of x appears in one and only one ordered pair, then the collection of points represents a function $y = f(x)$; otherwise, the collection of points is not a function. If the collection of points does represent a function, next check the y -coordinates. If each value of y appears in one and only one ordered pair, then the function represented by the collection of points is one-to-one; otherwise, the collection of points represents a function that is not one-to-one.
8. Given the graph of the one-to-one function $y = f(x)$, the graph of the inverse function f^{-1} can be obtained by reflecting the graph of f about the line $y = x$.

Practice Problems

9. The graph of f passes the horizontal line test, so this function is one-to-one.
10. The graph of f passes the horizontal line test, so this function is one-to-one.
11. The graph of f does not pass the horizontal line test (for example, the horizontal line $y = 2$ intersects the graph of f in two points), so this function is not one-to-one.
12. The graph of f does not pass the horizontal line test (for example, the horizontal line $y = 2$ intersects the graph of f in two points), so this function is not one-to-one.
13. The graph of f passes the horizontal line test, so this function is one-to-one.
14. The graph of f does not pass the horizontal line test (the horizontal line $y = 2$ intersects the graph in infinitely many points), so this function is not one-to-one.
15. Let $f(x) = 3x + 4$ and $g(x) = \frac{1}{3}(x - 4)$. Then

$$f(g(x)) = f\left(\frac{1}{3}(x - 4)\right) = 3 \cdot \frac{1}{3}(x - 4) + 4 = x - 4 + 4 = x,$$

and

$$g(f(x)) = g(3x + 4) = \frac{1}{3}(3x + 4 - 4) = \frac{1}{3}(3x) = x,$$

so that f and g are inverses of each other.

16. Let $f(x) = x^3 - 8$ and $g(x) = \sqrt[3]{x+8}$. Then

$$f(g(x)) = f(\sqrt[3]{x+8}) = (\sqrt[3]{x+8})^3 - 8 = x + 8 - 8 = x,$$

and

$$g(f(x)) = g(x^3 - 8) = \sqrt[3]{x^3 - 8 + 8} = \sqrt[3]{x^3} = x,$$

so that f and g are inverses of each other.

17. Let $f(x) = g(x) = \frac{1}{x}$. Then

$$f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x,$$

and

$$g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x,$$

so that f and g are inverses of each other.

18. Let $f(x) = \frac{2x+3}{x+4}$ and $g(x) = \frac{4x-3}{2-x}$. Then

$$f(g(x)) = f\left(\frac{4x-3}{2-x}\right) = \frac{2\frac{4x-3}{2-x} + 3}{\frac{4x-3}{2-x} + 4} = \frac{2(4x-3) + 3(2-x)}{4x-3 + 4(2-x)} = \frac{8x-6+6-3x}{4x-3+8-4x} = \frac{5x}{5} = x,$$

and

$$g(f(x)) = g\left(\frac{2x+3}{x+4}\right) = \frac{4\frac{2x+3}{x+4} - 3}{2 - \frac{2x+3}{x+4}} = \frac{4(2x+3) - 3(x+4)}{2(x+4) - (2x+3)} = \frac{8x+12-3x-12}{2x+8-2x-3} = \frac{5x}{5} = x,$$

so that f and g are inverses of each other.

19. (a) Because each value of the second coordinate appears in one and only one ordered pair, this is a one-to-one function.
 (b) Because the function is one-to-one, the function has an inverse. To obtain the inverse, interchange the coordinates in each ordered pair; thus, the inverse f^{-1} is the set

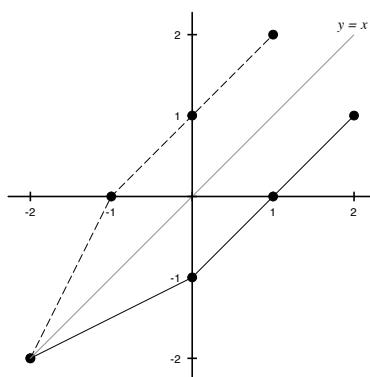
$$\{(5, -3), (9, -2), (2, -1), (11, 0), (-5, 1)\}.$$

- (c) The domain of f is the set $\{-3, -2, -1, 0, 1\}$, and the range of f is the set $\{-5, 2, 5, 9, 11\}$. The domain of f^{-1} is the set $\{-5, 2, 5, 9, 11\}$, and the range of f^{-1} is the set $\{-3, -2, -1, 0, 1\}$.
20. (a) Because the number 8 appears as the second coordinate in two ordered pairs with different first coordinates, $(0, 8)$ and $(2, 8)$, this is not a one-to-one function.
 (b) Because the function is not one-to-one, the function does not have an inverse.
 (c) The domain of f is the set $\{-2, -1, 0, 1, 2\}$, and the range of f is the set $\{-3, 2, 6, 8\}$. There is no inverse function.
21. (a) Because the number 1 appears as the second coordinate in two ordered pairs with different first coordinates, $(-2, 1)$ and $(2, 1)$, this is not a one-to-one function.
 (b) Because the function is not one-to-one, the function does not have an inverse.
 (c) The domain of f is the set $\{-10, -3, -2, 1, 2\}$, and the range of f is the set $\{0, 1, 2, 9\}$. There is no inverse function.

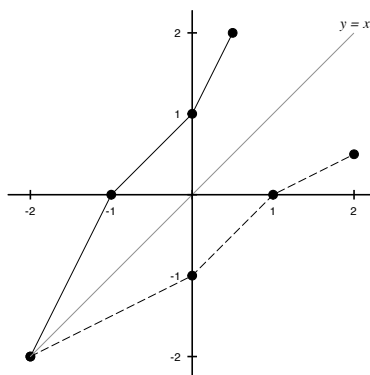
22. (a) Because each value of the second coordinate appears in one and only one ordered pair, this is a one-to-one function.
- (b) Because the function is one-to-one, the function has an inverse. To obtain the inverse, interchange the coordinates in each ordered pair; thus, the inverse f^{-1} is the set

$$\{(-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2)\}.$$

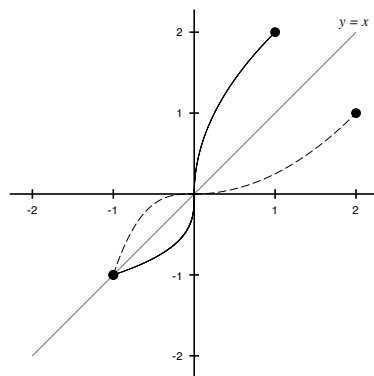
- (c) The domain of f is the set $\{-2, -1, 0, 1, 2\}$, and the range of f is the set $\{-8, -1, 0, 1, 8\}$. The domain of f^{-1} is the set $\{-8, -1, 0, 1, 8\}$, and the range of f^{-1} is the set $\{-2, -1, 0, 1, 2\}$.
23. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



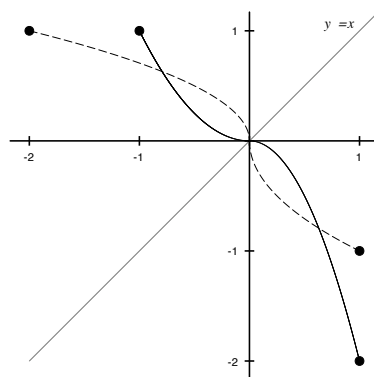
24. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



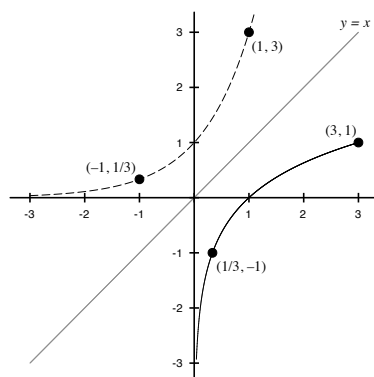
25. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



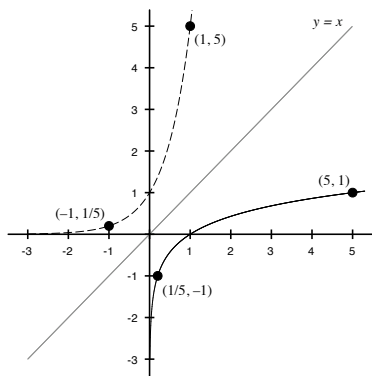
26. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



27. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



28. In the figure below, the dashed curve is the graph of $y = f(x)$ and the solid curve is the graph of $y = f^{-1}(x)$. The graph of $y = x$ is included for clarity. Observe how the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



29. (a) Start by writing $y = f(x) = 4x + 2$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = 4y + 2$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$4y = x - 2, \quad \text{so that} \quad y = \frac{1}{4}(x - 2).$$

Thus, $\boxed{f^{-1}(x) = \frac{1}{4}(x - 2)}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{1}{4}(x - 2)\right) = 4\left(\frac{1}{4}(x - 2)\right) + 2 = x - 2 + 2 = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}(4x + 2) = \frac{1}{4}(4x + 2 - 2) = \frac{1}{4}(4x) = \boxed{x}. \end{aligned}$$

- (b) Both the domain and range of f are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$. The same is true for the inverse function. Both the domain and range of f^{-1} are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.
30. (a) Start by writing $y = f(x) = 1 - 3x$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = 1 - 3y$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$3y = 1 - x, \quad \text{so that} \quad y = \frac{1}{3}(1 - x).$$

Thus, $\boxed{f^{-1}(x) = \frac{1}{3}(1 - x)}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{1}{3}(1 - x)\right) = 1 - 3\left(\frac{1}{3}(1 - x)\right) = 1 - (1 - x) = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}(1 - 3x) = \frac{1}{3}(1 - (1 - 3x)) = \frac{1}{3}(3x) = \boxed{x}. \end{aligned}$$

- (b) Both the domain and range of f are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$. The same is true for the inverse function. Both the domain and range of f^{-1} are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.
31. (a) Start by writing $y = f(x) = \sqrt[3]{x + 10}$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = \sqrt[3]{y + 10}$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$y + 10 = x^3, \quad \text{so that} \quad y = x^3 - 10.$$

Thus, $\boxed{f^{-1}(x) = x^3 - 10}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f(x^3 - 10) = \sqrt[3]{(x^3 - 10) + 10} = \sqrt[3]{x^3} = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x + 10}) = (\sqrt[3]{x + 10})^3 - 10 = x + 10 - 10 = \boxed{x}. \end{aligned}$$

- (b) Both the domain and range of f are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$. The same is true for the inverse function. Both the domain and range of f^{-1} are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.

32. (a) Start by writing $y = f(x) = 2x^3 + 4$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = 2y^3 + 4$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$2y^3 = x - 4, \quad \text{so that} \quad y = \sqrt[3]{\frac{x - 4}{2}}.$$

Thus, $\boxed{f^{-1}(x) = \sqrt[3]{\frac{x - 4}{2}}}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\sqrt[3]{\frac{x - 4}{2}}\right) = 2\left(\sqrt[3]{\frac{x - 4}{2}}\right)^3 + 4 = 2\left(\frac{x - 4}{2}\right) + 4 = x - 4 + 4 = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}(2x^3 + 4) = \sqrt[3]{\frac{2x^3 + 4 - 4}{2}} = \sqrt[3]{x^3} = \boxed{x}. \end{aligned}$$

- (b) Both the domain and range of f are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$. The same is true for the inverse function. Both the domain and range of f^{-1} are the set of $\boxed{\text{all real numbers}}$ or the interval $\boxed{(-\infty, \infty)}$.

33. (a) Start by writing $y = f(x) = \frac{1}{x - 2}$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = \frac{1}{y - 2}$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$y - 2 = \frac{1}{x}, \quad \text{so that} \quad y = \frac{1}{x} + 2.$$

Thus, $\boxed{f^{-1}(x) = \frac{1}{x} + 2}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{1}{x} + 2\right) = \frac{1}{\frac{1}{x} + 2 - 2} = \frac{1}{\frac{1}{x}} = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x - 2}\right) = \frac{1}{\frac{1}{x - 2}} + 2 = x - 2 + 2 = \boxed{x}. \end{aligned}$$

- (b) The domain of f is the set $\boxed{\{x|x \neq 2\}}$, and the range of f is the set $\boxed{\{y|y \neq 0\}}$. The domain of f^{-1} is the set $\boxed{\{x|x \neq 0\}}$, and the range of f^{-1} is the set $\boxed{\{y|y \neq 2\}}$.

34. (a) Start by writing $y = f(x) = \frac{2x}{3x-1}$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = \frac{2y}{3y-1}$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$\begin{aligned} x(3y-1) &= 2y \\ 3xy-x &= 2y \\ 3xy-2y &= x \\ y(3x-2) &= x \\ y &= \boxed{\frac{x}{3x-2} = f^{-1}(x)}. \end{aligned}$$

To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{x}{3x-2}\right) = \frac{2\frac{x}{3x-2}}{3\frac{x}{3x-2}-1} = \frac{2x}{3x-(3x-2)} = \frac{2x}{2} = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}\left(\frac{2x}{3x-1}\right) = \frac{\frac{2x}{3x-1}}{3\frac{2x}{3x-1}-2} = \frac{2x}{6x-2(3x-1)} = \frac{2x}{2} = \boxed{x}. \end{aligned}$$

- (b) The domain of f is the set $\boxed{\{x|x \neq \frac{1}{3}\}}$, and the range of f is the set $\boxed{\{y|y \neq \frac{2}{3}\}}$. The domain of f^{-1} is the set $\boxed{\{x|x \neq \frac{2}{3}\}}$, and the range of f^{-1} is the set $\boxed{\{y|y \neq \frac{1}{3}\}}$.
35. (a) Start by writing $y = f(x) = \frac{2x+3}{x+2}$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = \frac{2y+3}{y+2}$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$\begin{aligned} x(y+2) &= 2y+3 \\ xy+2x &= 2y+3 \\ xy-2y &= 3-2x \\ y(x-2) &= 3-2x \\ y &= \boxed{\frac{3-2x}{x-2} = f^{-1}(x)}. \end{aligned}$$

To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{3-2x}{x-2}\right) = \frac{2\frac{3-2x}{x-2}+3}{\frac{3-2x}{x-2}+2} = \frac{2(3-2x)+3(x-2)}{3-2x+2(x-2)} = \frac{-x}{-1} = \boxed{x}, \\ f^{-1}(f(x)) &= f^{-1}\left(\frac{2x+3}{x+2}\right) = \frac{3-2\frac{2x+3}{x+2}}{\frac{2x+3}{x+2}-2} = \frac{3(x+2)-2(2x+3)}{2x+3-2(x+2)} = \frac{-x}{-1} = \boxed{x}. \end{aligned}$$

- (b) The domain of f is the set $\boxed{\{x|x \neq -2\}}$, and the range of f is the set $\boxed{\{y|y \neq 2\}}$. The domain of f^{-1} is the set $\boxed{\{x|x \neq 2\}}$, and the range of f^{-1} is the set $\boxed{\{y|y \neq -2\}}$.
36. (a) Start by writing $y = f(x) = \frac{-3x-4}{x-2}$. Next, interchange the x and y variables to obtain an implicit definition of the inverse function: $x = \frac{-3y-4}{y-2}$. To obtain the inverse function explicitly,

solve the implicit equation for y :

$$\begin{aligned}x(y-2) &= -3y-4 \\xy-2x &= -3y-4 \\xy+3y &= 2x-4 \\y(x+3) &= 2x-4 \\y &= \boxed{\frac{2x-4}{x+3} = f^{-1}(x)}.\end{aligned}$$

To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned}f(f^{-1}(x)) &= f\left(\frac{2x-4}{x+3}\right) = \frac{-3\frac{2x-4}{x+3}-4}{\frac{2x-4}{x+3}-2} = \frac{-3(2x-4)-4(x+3)}{2x-4-2(x+3)} = \frac{-10x}{-10} = \boxed{x}, \\f^{-1}(f(x)) &= f^{-1}\left(\frac{-3x-4}{x-2}\right) = \frac{2\frac{-3x-4}{x-2}-4}{\frac{-3x-4}{x-2}+3} = \frac{2(-3x-4)-4(x-2)}{-3x-4+3(x-2)} = \frac{-10x}{-10} = \boxed{x}.\end{aligned}$$

- (b) The domain of f is the set $\boxed{\{x|x \neq 2\}}$, and the range of f is the set $\boxed{\{y|y \neq -3\}}$. The domain of f^{-1} is the set $\boxed{\{x|x \neq -3\}}$, and the range of f^{-1} is the set $\boxed{\{y|y \neq 2\}}$.
37. (a) Start by writing $y = f(x) = x^2 + 4$, where $x \geq 0$. Next interchange the x and y variables to obtain an implicit definition of the inverse function: $x = y^2 + 4$, where $y \geq 0$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$y^2 = x - 4, \quad \text{so that} \quad y = \pm\sqrt{x-4}.$$

Because $y \geq 0$, only the positive square root is appropriate; thus, $\boxed{f^{-1}(x) = \sqrt{x-4}}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned}f(f^{-1}(x)) &= f(\sqrt{x-4}) = (\sqrt{x-4})^2 + 4 = x - 4 + 4 = \boxed{x}, \\f^{-1}(f(x)) &= f^{-1}(x^2 + 4) = \sqrt{x^2 + 4 - 4} = \sqrt{x^2} = |x| = \boxed{x},\end{aligned}$$

where, in both calculations, attention is restricted to $x \geq 0$.

- (b) The domain of f is the set $\boxed{\{x|x \geq 0\}}$, and the range of f is the set $\boxed{\{y|y \geq 4\}}$. The domain of f^{-1} is the set $\boxed{\{x|x \geq 4\}}$, and the range of f^{-1} is the set $\boxed{\{y|y \geq 0\}}$.
38. (a) Start by writing $y = f(x) = (x-2)^2 + 4$, where $x \leq 2$. Next interchange the x and y variables to obtain an implicit definition of the inverse function: $x = (y-2)^2 + 4$, where $y \leq 2$. To obtain the inverse function explicitly, solve the implicit equation for y :

$$(y-2)^2 = x - 4, \quad \text{so that} \quad y = 2 \pm \sqrt{x-4}.$$

Because $y \leq 2$, only the negative square root is appropriate; thus, $\boxed{f^{-1}(x) = 2 - \sqrt{x-4}}$. To check that this is the correct formula for the inverse function, show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$:

$$\begin{aligned}f(f^{-1}(x)) &= f(2 - \sqrt{x-4}) = (2 - \sqrt{x-4} - 2)^2 + 4 \\&= (-\sqrt{x-4})^2 + 4 = x - 4 + 4 = \boxed{x}, \\f^{-1}(f(x)) &= f^{-1}((x-2)^2 + 4) = 2 - \sqrt{(x-2)^2 + 4 - 4} \\&= 2 - \sqrt{(x-2)^2} = 2 - |x-2| = 2 - (2-x) = \boxed{x},\end{aligned}$$

where, in both calculations, attention is restricted to $x \leq 2$.

- (b) The domain of f is the set $\{x|x \leq 2\}$, and the range of f is the set $\{y|y \geq 4\}$. The domain of f^{-1} is the set $\{x|x \geq 4\}$, and the range of f^{-1} is the set $\{y|y \leq 2\}$.

P.5: Exponential and Logarithmic Functions

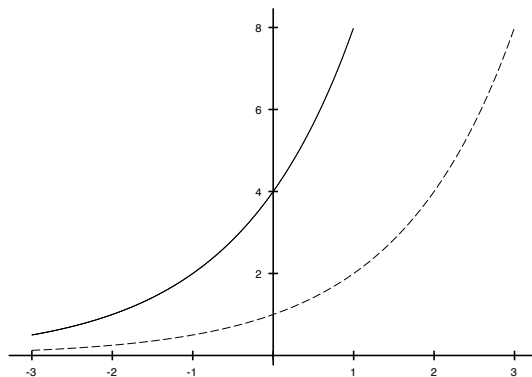
Concepts and Vocabulary

- The graph of every exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$, passes through three points: $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
- False**. The graph of the exponential function $f(x) = \left(\frac{3}{2}\right)^x$ is **increasing**, because the base $a = \frac{3}{2} > 1$.
- If $3^x = 3^4$, then $x = 4$.
- If $4^x = 8^2$, then $x = 3$.
- False**. The graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x = 3^{-x}$ are symmetric with respect to the y -axis. The graphs of $y = 3^x$ and its inverse function $y = \log_3 x$ are symmetric with respect to the line $y = x$.
- False**. The range of the exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$, is the set of **positive** real numbers.
- The number e is defined as the base of the exponential function f whose tangent line to the graph of f at the point $(0, 1)$ has slope **1**.
- The domain of the logarithmic function $f(x) = \log_a x$ is the set $\{x|x > 0\}$.
- The graph of every logarithmic function $f(x) = \log_a x$, $a > 0$ and $a \neq 1$, passes through three points: $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.
- The graph of $f(x) = \log_2 x$ is **(a) increasing**, because the base $a = 2 > 1$.
- False**. If $y = \log_a x$, then $x = a^y$.
- True**. The graph of $f(x) = \log_a x$, $a > 0$ and $a \neq 1$, has an x -intercept equal to 1 and no y -intercept.
- True**. $\ln e^x = x$ for all real numbers.
- $\ln e = 1$.
- The number e is defined as the base of the exponential function f whose tangent line to the graph of f at the point $(0, 1)$ has slope 1.
- The x -intercept of the function $f(x) = \ln x$ is 1. Because the graph of $h(x) = \ln(x+1)$ can be obtained from the graph of $f(x) = \ln x$ by shifting horizontally left 1 unit, the x -intercept of the function $h(x) = \ln(x+1)$ is $1 - 1 = 0$.

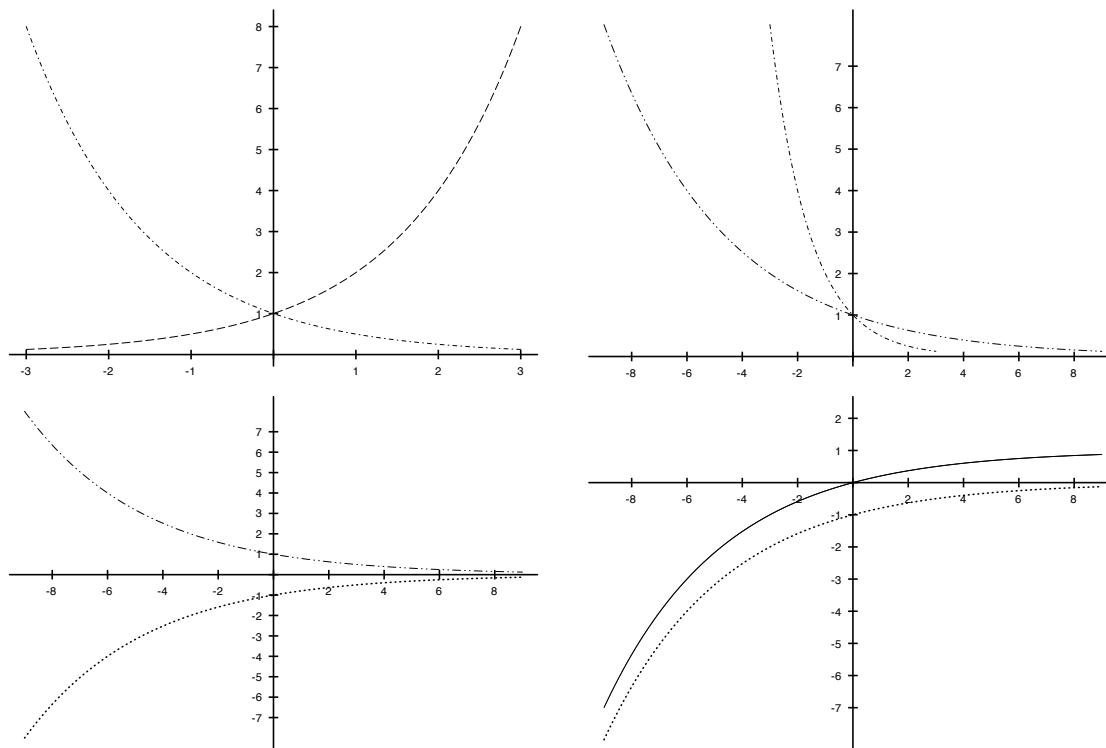
Practice Problems

- Let $g(x) = 4^x + 2$.
 - $g(-1) = 4^{-1} + 2 = \frac{1}{4} + 2 = \frac{9}{4}$. The corresponding point on the graph of g is $\left(-1, \frac{9}{4}\right)$.
 - If $g(x) = 4^x + 2 = 66$, then $4^x = 64 = 4^3$. It follows that $x = 3$. The corresponding point on the graph of g is $(3, 66)$.
- Let $g(x) = 5^x - 3$.
 - $g(-1) = 5^{-1} - 3 = \frac{1}{5} - 3 = -\frac{14}{5}$. The corresponding point on the graph of g is $\left(-1, -\frac{14}{5}\right)$.

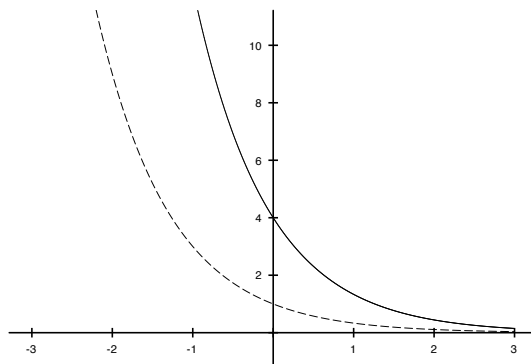
- (b) If $g(x) = 5^x - 3 = 122$, then $5^x = 125 = 5^3$. It follows that $x = 3$. The corresponding point on the graph of g is $(3, 122)$.
19. The graph appears to be the reflection of $y = 3^x$ about the y -axis. The function would therefore be **(a):** $y = 3^{-x}$.
20. The graph appears to be $y = 3^x$ shifted horizontally right 1 unit. The function would therefore be **(e):** $y = 3^{x-1}$.
21. The graph appears to be the reflection of $y = 3^x$ about both the x -axis and the y -axis. The function would therefore be **(c):** $y = -3^{-x}$.
22. The graph appears to be the reflection of $y = 3^x$ about the x -axis and then shifted vertically up 1 unit. The function would therefore be **(f):** $y = 1 - 3^x$.
23. The graph appears to be the reflection of $y = 3^x$ about the x -axis. The function would therefore be **(b):** $y = -3^x$.
24. The graph appears to be $y = 3^x$ shifted vertically down 1 unit. The function would therefore be **(d):** $y = 3^x - 1$.
25. The graph of $y = 2^{x+2}$ can be obtained from the graph of $y = 2^x$ by shifting horizontally left 2 units. In the figure below, the dashed curve represents the graph of $y = 2^x$, and the solid curve represents the graph of $y = 2^{x+2}$. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y > 0\}$ or the interval $(0, \infty)$.



26. The graph of $y = 1 - 2^{-x/3}$ can be obtained from the graph of $y = 2^x$ through the following sequence of transformations: reflect about the y -axis, stretch horizontally by a factor of 3, reflect about the x -axis, and shift vertically up 1 unit. In the figures below, the dashed curve represents the graph of $y = 2^x$, the dash-dot curve represents the graph of $y = 2^{-x}$, the dash-double dot curve represents the graph of $y = 2^{-x/3}$, the dotted curve represents the graph of $y = -2^{-x/3}$, and the solid curve represents the graph of $y = 1 - 2^{-x/3}$. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y < 1\}$ or the interval $(-\infty, 1)$.

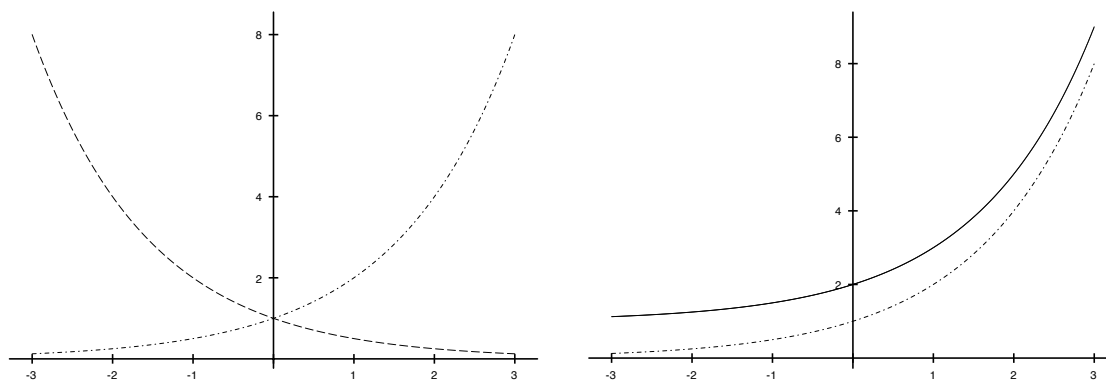


27. The graph of $y = 4\left(\frac{1}{3}\right)^x$ can be obtained from the graph of $y = \left(\frac{1}{3}\right)^x$ by stretching vertically by a factor of 4. In the figure below, the dashed curve represents the graph of $y = \left(\frac{1}{3}\right)^x$, and the solid curve represents the graph of $y = 4\left(\frac{1}{3}\right)^x$. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y > 0\}$ or the interval $(0, \infty)$.

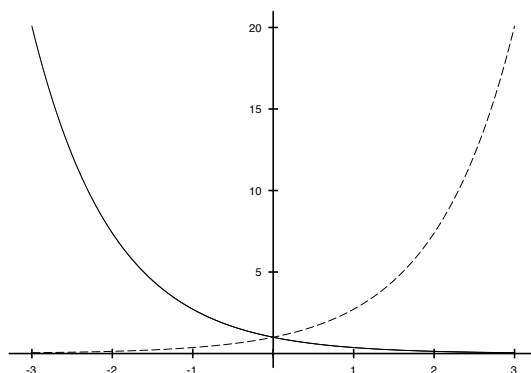


28. The graph of $f(x) = \left(\frac{1}{2}\right)^{-x} + 1$ can be obtained from the graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ by reflecting about the y -axis and then shifting vertically up 1 unit. In the figure below at the left, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ is represented by the dashed curve, and the graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is

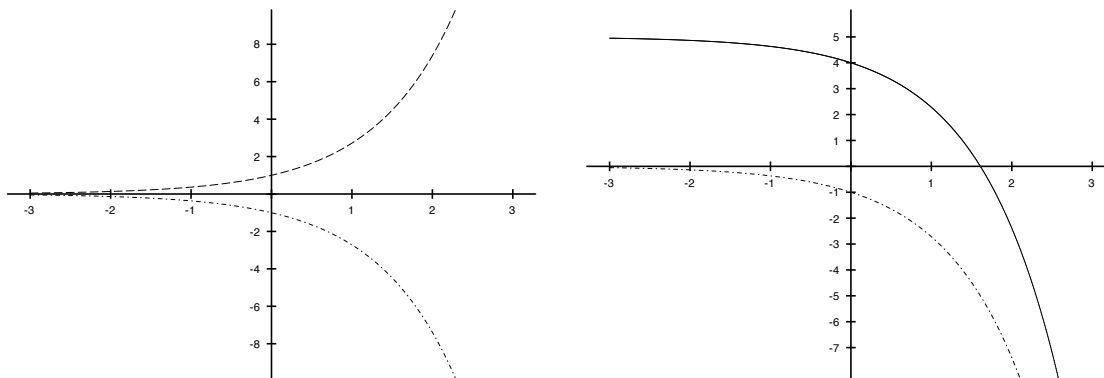
represented by the dash-dot curve. In the figure below at the right, the graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is represented by the dash-dot curve, and the graph of $f(x) = \left(\frac{1}{2}\right)^{-x} + 1$ is represented by the solid curve. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y > 1\}$ or the interval $(1, \infty)$.



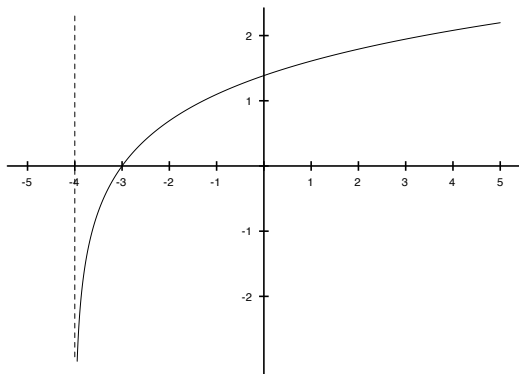
29. The graph of $y = e^{-x}$ can be obtained from the graph of $y = e^x$ by reflecting about the y -axiss. In the figure below, the dashed curve represents the graph of $y = e^x$, and the solid curve represents the graph of $y = e^{-x}$. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y > 0\}$ or the interval $(0, \infty)$.



30. The graph of $y = 5 - e^x$ can be obtained from the graph of $y = e^x$ by reflecting about the x -axis and then shifting vertically up 5 units. In the figure below at the left, the graph of $y = e^x$ is represented by the dashed curve, and the graph of $y = -e^x$ is represented by the dash-dot curve. In the figure below at the right, the graph of $y = -e^x$ is represented by the dash-dot curve, and the graph of $y = 5 - e^x$ is represented by the solid curve. The domain of f is the set of all real numbers or the interval $(-\infty, \infty)$, and the range is the set $\{y|y < 5\}$ or the interval $(-\infty, 5)$.



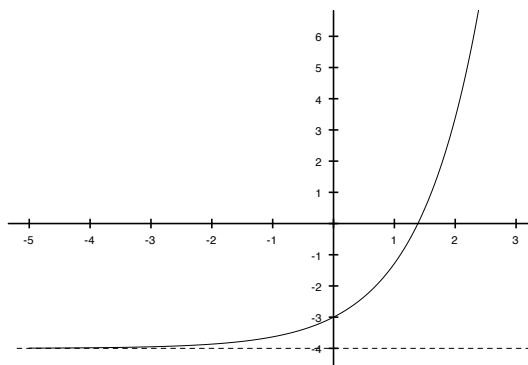
31. The argument of a logarithmic function must be positive, so the domain of $F(x) = \log_2 x^2$ is the solution of the inequality $x^2 > 0$. The domain of F is therefore the set $\{x|x \neq 0\}$.
32. The argument of a logarithmic function must be positive, so the domain of $g(x) = 8 + 5 \ln(2x + 3)$ is the solution of the inequality $2x + 3 > 0$. The domain of g is therefore the set $\{x|x > -\frac{3}{2}\}$.
33. The argument of a logarithmic function must be positive, so the domain of $f(x) = \ln(x - 1)$ is the solution of the inequality $x - 1 > 0$. The domain of f is therefore the set $\{x|x > 1\}$.
34. The square root function is only defined for nonnegative real numbers, so the domain of $g(x) = \sqrt{\ln x}$ is the solution of the inequality $\ln x \geq 0$. Because $\ln 1 = 0$ and the natural logarithm function is increasing for all real numbers, it follows that the domain of g is the set $\{x|x \geq 1\}$.
35. The graph appears to be the reflection of $y = \log_3 x$ about the y -axis. The function would therefore be **(b)**: $y = \log_3(-x)$.
36. The graph appears to be $y = \log_3 x$ shifted horizontally right 1 unit. The function would therefore be **(e)**: $y = \log_3(x - 1)$.
37. The graph appears to be the reflection of $y = \log_3 x$ about the x -axis and then shifted vertically up 1 unit. The function would therefore be **(f)**: $y = 1 - \log_3 x$.
38. The graph appears to be $y = \log_3 x$. The function would therefore be **(a)**: $y = \log_3 x$.
39. The graph appears to be the reflection of $y = \log_3 x$ about the x -axis. The function would therefore be **(c)**: $y = -\log_3 x$.
40. The graph appears to be $y = \log_3 x$ shifted vertically down 1 unit. The function would therefore be **(d)**: $y = \log_3 x - 1$.
41. (a) The argument of a logarithmic function must be positive, so the domain of $f(x) = \ln(x + 4)$ is the solution of the inequality $x + 4 > 0$. The domain of f is therefore the set $\{x|x > -4\}$.
- (b) The graph of $f(x) = \ln(x + 4)$ is shown below:



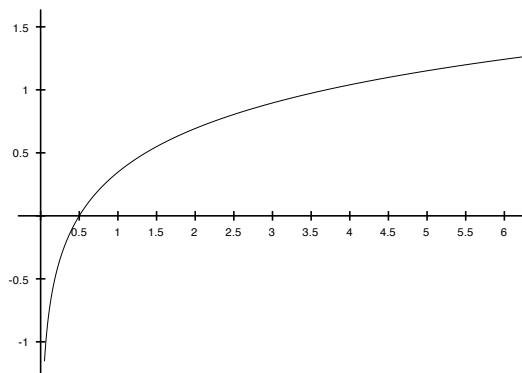
- (c) From the graph of f , the range of f is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) Start by writing $y = f(x) = \ln(x + 4)$. Next, interchange the variables x and y to obtain $x = \ln(y + 4)$. Finally, solve for y :

$$y + 4 = e^x, \quad \text{so that} \quad y = \boxed{f^{-1}(x) = e^x - 4}.$$

- (e) Recall that the range of f is the domain of its inverse function f^{-1} . Because the domain of f^{-1} is the set of all real numbers, the range of f is the set of all real numbers or the interval $(-\infty, \infty)$.
- (f) The graph of $f^{-1}(x) = e^x - 4$ is shown below:



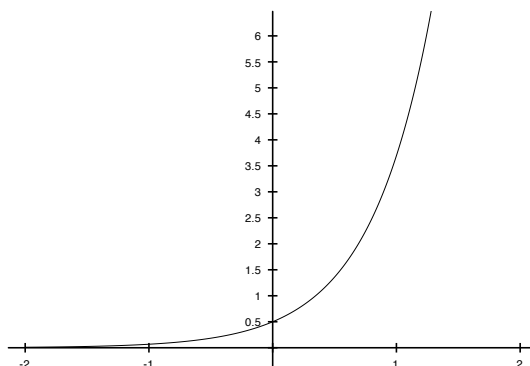
42. (a) The argument of a logarithmic function must be positive, so the domain of $f(x) = \frac{1}{2} \ln(2x)$ is the solution of the inequality $2x > 0$. The domain of f is therefore the set $\{x | x > 0\}$.
- (b) The graph of $f(x) = \frac{1}{2} \ln(2x)$ is shown below:



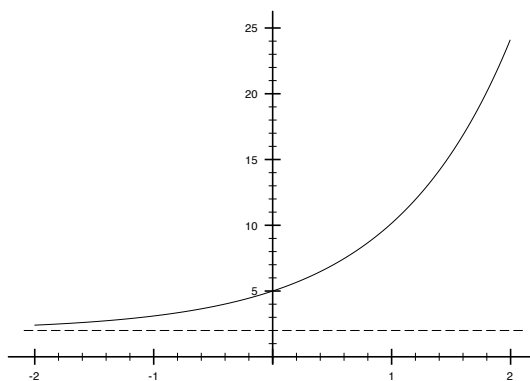
- (c) From the graph of f , the range of f is the set of all real numbers or the interval $(-\infty, \infty)$.
- (d) Start by writing $y = f(x) = \frac{1}{2} \ln(2x)$. Next, interchange the variables x and y to obtain $x = \frac{1}{2} \ln(2y)$. Finally, solve for y :

$$2y = e^{2x}, \quad \text{so that} \quad y = \boxed{f^{-1}(x) = \frac{1}{2}e^{2x}}.$$

- (e) Recall that the range of f is the domain of its inverse function f^{-1} . Because the domain of f^{-1} is the set of all real numbers, the range of f is the set of all real numbers or the interval $(-\infty, \infty)$.
- (f) The graph of $f^{-1}(x) = \frac{1}{2}e^{2x}$ is shown below:



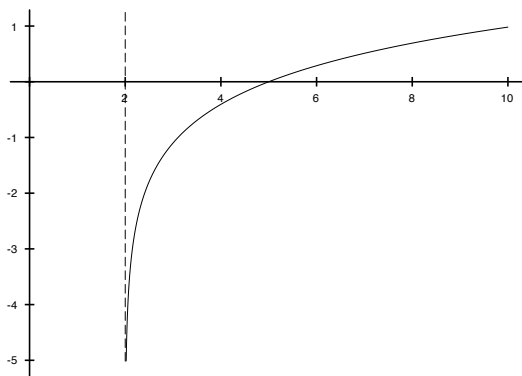
43. (a) The domain of $f(x) = 3e^x + 2$ is the set of all real numbers or the interval $(-\infty, \infty)$.
- (b) The graph of $f(x) = 3e^x + 2$ is shown below:



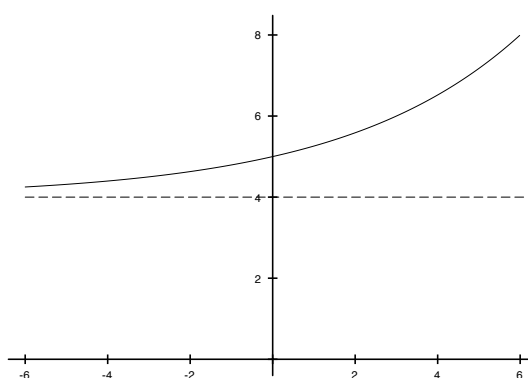
- (c) From the graph of f , the range of f is the set $\{y|y > 2\}$.
- (d) Start by writing $y = f(x) = 3e^x + 2$. Next, interchange the variables x and y to obtain $x = 3e^y + 2$. Finally, solve for y :

$$e^y = \frac{1}{3}(x - 2), \quad \text{so that} \quad y = \boxed{f^{-1}(x) = \ln\left(\frac{x-2}{3}\right)}.$$

- (e) Recall that the range of f is the domain of its inverse function f^{-1} . Because the domain of f^{-1} is the set $\{x|x > 2\}$, the range of f is the set $\{y|y > 2\}$.
- (f) The graph of $f^{-1}(x) = \ln\left(\frac{x-2}{3}\right)$ is shown below:



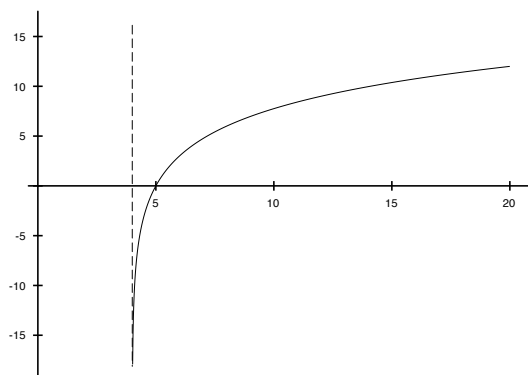
44. (a) The domain of $f(x) = 2^{x/3} + 4$ is the set of all real numbers or the interval $(-\infty, \infty)$.
 (b) The graph of $f(x) = 2^{x/3} + 4$ is shown below:



- (c) From the graph of f , the range of f is the set $\{y|y > 4\}$.
 (d) Start by writing $y = f(x) = 2^{x/3} + 4$. Next, interchange the variables x and y to obtain $x = 2^{y/3} + 4$. Finally, solve for y :

$$2^{y/3} = x - 4, \quad \text{so that} \quad y = \boxed{f^{-1}(x) = 3 \log_2(x - 4)}.$$

- (e) Recall that the range of f is the domain of its inverse function f^{-1} . Because the domain of f^{-1} is the set $\{x|x > 4\}$, the range of f is the set $\{y|y > 4\}$.
 (f) The graph of $f^{-1}(x) = 3 \log_2(x - 4)$ is shown below:



45. The transformation $y = \ln(x + c)$, $c > 0$, shifts the graph of $f(x) = \ln x$ horizontally left by c units. This will move the x -intercept c units to the left.

46. The transformation $y = e^{cx}$, $c > 0$, horizontally stretches (if $0 < c < 1$) or compresses (if $c > 1$) the graph of $f(x) = e^x$. Because a horizontal transformation has no effect on the y -intercept of a graph, the y -intercept of $y = e^{cx}$ is the same as the y -intercept of $y = e^x$.

47. Because $9 = 3^2$, the equation can be written as

$$3^{x^2} = (3^2)^x = 3^{2x}.$$

By the one-to-one property of exponential functions, it follows that

$$x^2 = 2x, \quad \text{or} \quad x^2 - 2x = x(x - 2) = 0.$$

Thus, $x = 0$ or $x = 2$, and the solution set is $\{0, 2\}$.

48. Because $125 = 5^3$, the equation can be written as

$$5^{x^2+8} = (5^3)^{2x} = 5^{6x}.$$

By the one-to-one property of exponential functions, it follows that

$$x^2 + 8 = 6x, \quad \text{or} \quad x^2 - 6x + 8 = (x - 4)(x - 2) = 0.$$

Thus, $x = 4$ or $x = 2$, and the solution set is $\{2, 4\}$.

49. First, rewrite the equation as $e^{3x} = e^{2-x}$. Then, by the one-to-one property of exponential functions,

$$3x = 2 - x, \quad \text{or} \quad 4x = 2,$$

so that $x = \frac{1}{2}$.

50. First, rewrite the equation as $e^{4x+x^2} = e^{12}$. Then, by the one-to-one property of exponential functions,

$$4x + x^2 = 12, \quad \text{or} \quad x^2 + 4x - 12 = (x + 6)(x - 2) = 0.$$

Thus, $x = -6$ or $x = 2$, and the solution set is $\{-6, 2\}$.

51. The numbers e and 4 cannot be expressed with the same base, so apply the natural logarithm function to both sides of the equation. This yields

$$\begin{aligned} \ln(e^{1-2x}) &= \ln 4 \\ 1 - 2x &= \ln 4 \\ 2x &= 1 - \ln 4 \\ x &= \frac{1}{2}(1 - \ln 4). \end{aligned}$$

52. The numbers e and 5 cannot be expressed with the same base, so apply the natural logarithm function to both sides of the equation. This yields

$$\begin{aligned} \ln(e^{1-x}) &= \ln 5 \\ 1 - x &= \ln 5 \\ x &= 1 - \ln 5. \end{aligned}$$

53. First, rewrite the equation as $2^{3x} = \frac{9}{5}$, and then apply the natural logarithm function to both sides. This yields

$$\begin{aligned} \ln(2^{3x}) &= \ln\left(\frac{9}{5}\right) \\ 3x \ln 2 &= \ln\left(\frac{9}{5}\right) \\ x &= \frac{\ln(\frac{9}{5})}{3 \ln 2}. \end{aligned}$$

54. First, rewrite the equation as $4^{0.2x} = \frac{0.2}{0.3} = \frac{2}{3}$, and then apply the natural logarithm function to both sides. This yields

$$\begin{aligned}\ln(4^{0.2x}) &= \ln\left(\frac{2}{3}\right) \\ 0.2x \ln 4 &= \ln\left(\frac{2}{3}\right) \\ x &= \frac{\ln(\frac{2}{3})}{0.2 \ln 4} = \boxed{\frac{5 \ln(\frac{2}{3})}{\ln 4}}.\end{aligned}$$

55. Because the numbers 3 and 4 cannot be expressed with the same base, apply the natural logarithm function to both sides of the equation. This yields

$$\begin{aligned}\ln(3^{1-2x}) &= \ln(4^x) \\ (1-2x) \ln 3 &= x \ln 4 \\ \ln 3 - 2x \ln 3 &= x \ln 4 \\ -2x \ln 3 - x \ln 4 &= -\ln 3 \\ x(2 \ln 3 + \ln 4) &= \ln 3 \\ x &= \frac{\ln 3}{2 \ln 3 + \ln 4}.\end{aligned}$$

Using properties of logarithms, the answer can be simplified further as

$$x = \frac{\ln 3}{2 \ln 3 + \ln 4} = \frac{\ln 3}{\ln 3^2 + \ln 4} = \frac{\ln 3}{\ln(3^2 \cdot 4)} = \boxed{\frac{\ln 3}{\ln 36}}.$$

56. Because the numbers 2 and 5 cannot be expressed with the same base, apply the natural logarithm function to both sides of the equation. This yields

$$\begin{aligned}\ln(2^{x+1}) &= \ln(5^{1-2x}) \\ (x+1) \ln 2 &= (1-2x) \ln 5 \\ x \ln 2 + \ln 2 &= \ln 5 - 2x \ln 5 \\ x \ln 2 + 2x \ln 5 &= \ln 5 - \ln 2 \\ x(\ln 2 + 2 \ln 5) &= \ln 5 - \ln 2 \\ x &= \frac{\ln 5 - \ln 2}{\ln 2 + 2 \ln 5}.\end{aligned}$$

Using properties of logarithms, the answer can be simplified further as

$$x = \frac{\ln 5 - \ln 2}{\ln 2 + 2 \ln 5} = \frac{\ln(\frac{5}{2})}{\ln 2 + \ln 5^2} = \frac{\ln(\frac{5}{2})}{\ln(2 \cdot 5^2)} = \boxed{\frac{\ln(\frac{5}{2})}{\ln 50}}.$$

57. Change the logarithmic equation $\log_2(2x+1) = 3$ into the equivalent exponential equation $2x+1 = 2^3 = 8$. Then $2x = 7$, or $\boxed{x = \frac{7}{2}}$.

58. Change the logarithmic equation $\log_3(3x-2) = 2$ into the equivalent exponential equation $3x-2 = 3^2 = 9$. Then $3x = 11$, or $\boxed{x = \frac{11}{3}}$.

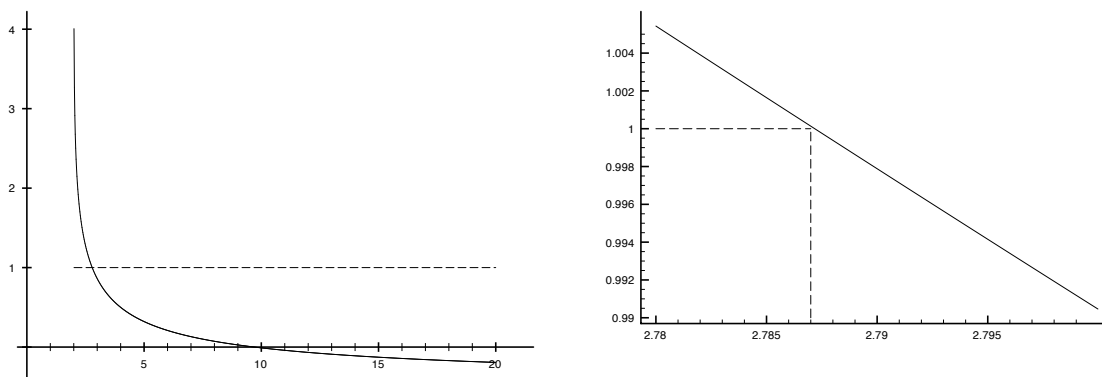
59. Change the logarithmic equation $\log_x\left(\frac{1}{8}\right) = 3$ into the equivalent exponential equation $x^3 = \frac{1}{8}$. Then

$$x = \sqrt[3]{\frac{1}{8}} = \boxed{\frac{1}{2}}.$$

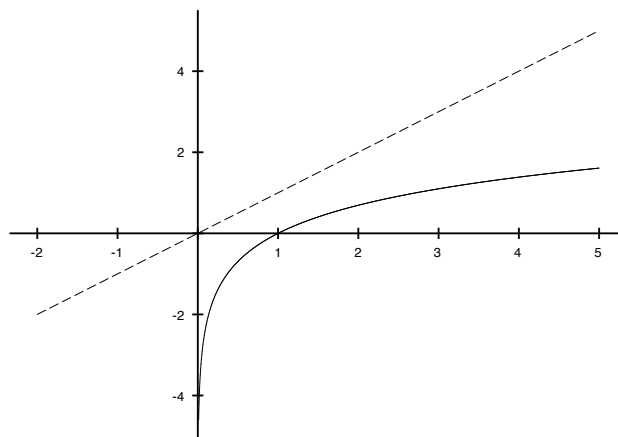
60. Change the logarithmic equation $\log_x 64 = -3$ into the equivalent exponential equation $x^{-3} = 64$. Then

$$x^3 = \frac{1}{64}, \quad \text{or} \quad x = \sqrt[3]{\frac{1}{64}} = \boxed{\frac{1}{4}}.$$

61. Using the properties of logarithmic functions, $2 \ln 3 = \ln 3^2 = \ln 9$, so the equation can be written as $\ln(2x + 3) = \ln 9$. By the one-to-one property of logarithmic functions, it follows that $2x + 3 = 9$, so that $2x = 6$, or $\boxed{x = 3}$.
62. Using the properties of logarithmic functions, $\frac{1}{2} \log_3 x = \log_3 \sqrt{x}$ and $2 \log_3 2 = \log_3 2^2 = \log_3 4$, so the equation can be written as $\log_3 \sqrt{x} = \log_3 4$. By the one-to-one property of logarithmic functions, it follows that $\sqrt{x} = 4$, so that $\boxed{x = 16}$.
63. In the figure below at the left, the graph of $y = \log_5(x + 1) - \log_4(x - 2)$ is shown as the solid curve, and the graph of $y = 1$ is shown as the dashed curve. The two curves intersect once for x roughly between 2.5 and 3. Repeatedly zooming in until the x -coordinate of the point of intersection could be determined to three decimal places (see the figure below at the right), it follows that $x \approx 2.787$. Thus, to three decimal places the solution to the equation $\log_5(x + 1) - \log_4(x - 2) = 1$ is $\boxed{2.787}$.

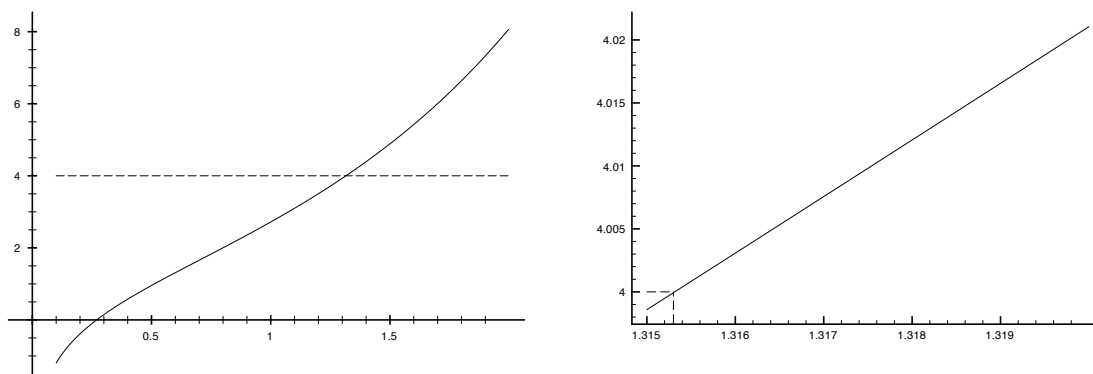


64. In the figure below at the left, the graph of $y = \ln x$ is shown as the solid curve, and the graph of $y = x$ is shown as the dashed curve. As the two curves do not intersect, the equation $\ln x = x$ has $\boxed{\text{no solution}}$.

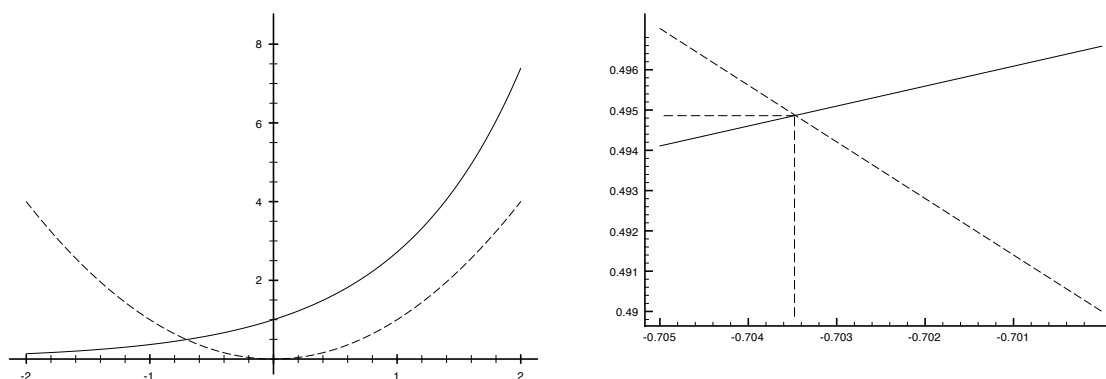


65. In the figure below at the left, the graph of $y = e^x + \ln x$ is shown as the solid curve, and the graph of $y = 4$ is shown as the dashed curve. The two curves intersect once for x roughly between 1.3 and 1.4. Repeatedly zooming in until the x -coordinate of the point of intersection could be determined to three

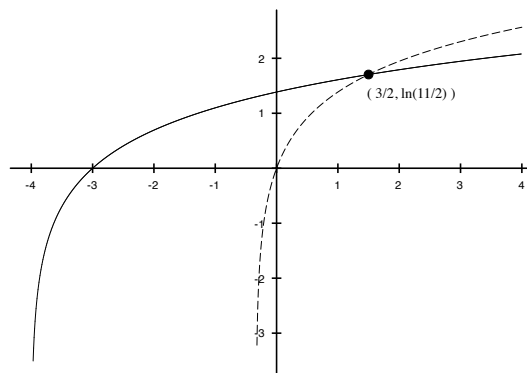
decimal places (see the figure below at the right), it follows that $x \approx 1.315$. Thus, to three decimal places the solution to the equation $e^x + \ln x = 4$ is $\boxed{1.315}$.



66. In the figure below at the left, the graph of $y = e^x$ is shown as the solid curve, and the graph of $y = x^2$ is shown as the dashed curve. The two curves intersect once for x between -0.8 and -0.6 . Repeatedly zooming in until the x -coordinate of the point of intersection could be determined to three decimal places (see the figure below at the right), it follows that $x \approx -0.703$. Thus, to three decimal places the solution to the equation $e^x = x^2$ is $\boxed{-0.703}$.



67. (a) The graphs of $f(x) = \ln(x + 4)$ and $g(x) = \ln(3x + 1)$ are shown in the figure below. The solid curve is the graph of f , and the dashed curve is the graph of g .



- (b) Form the equation $\ln(x + 4) = \ln(3x + 1)$. By the one-to-one property of the natural logarithm function, it follows that $x + 4 = 3x + 1$, so that $2x = 3$, or $x = \frac{3}{2}$. Thus, the graphs of

$f(x) = \ln(x + 4)$ and $g(x) = \ln(3x + 1)$ intersect in one point:

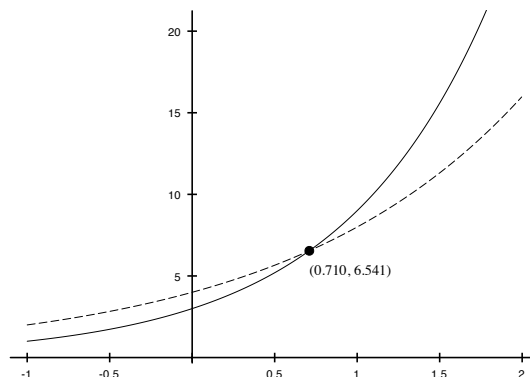
$$\left(\frac{3}{2}, \ln \frac{11}{2} \right).$$

This intersection point is marked and labeled in the figure above.

- (c) Based on the graph in part (a), the solution of the inequality $f(x) > g(x)$ is the set $\left\{ x \mid -\frac{1}{3} < x < \frac{3}{2} \right\}$.

The inequality $-\frac{1}{3} < x$ arises from the fact that the domain of the function g is $x > -\frac{1}{3}$.

68. (a) The graphs of $f(x) = 3^{x+1}$ and $g(x) = 2^{x+2}$ are shown in the figure below. The solid curve is the graph of f , and the dashed curve is the graph of g .



- (b) Form the equation $3^{x+1} = 2^{x+2}$. Because 2 and 3 cannot be expressed in terms of the same base, apply the natural logarithm function to both sides of the equation. This yields

$$\begin{aligned} \ln(3^{x+1}) &= \ln(2^{x+2}) \\ (x+1)\ln 3 &= (x+2)\ln 2 \\ x\ln 3 + \ln 3 &= x\ln 2 + 2\ln 2 \\ x(\ln 3 - \ln 2) &= 2\ln 2 - \ln 3 \\ x &= \frac{2\ln 2 - \ln 3}{\ln 3 - \ln 2} \approx 0.710. \end{aligned}$$

Thus, the graphs of $f(x) = 3^{x+1}$ and $g(x) = 2^{x+2}$ intersect in one point, given approximately by $(0.710, 6.541)$. This intersection point is marked and labeled in the figure above.

- (c) Based on the graph in part (a), the solution of the inequality $f(x) > g(x)$ is the set $\{x \mid x > 0.710\}$.

P.6: Trigonometric Functions

Concepts and Vocabulary

1. The sine, cosine, cosecant, and secant functions have period $\boxed{2\pi}$; the tangent and cotangent functions have period $\boxed{\pi}$.
2. The domain of the tangent function $f(x) = \tan x$ is $\boxed{\text{the set of all real numbers except odd multiples of } \frac{\pi}{2}}$.
3. The range of the sine function $f(x) = \sin x$ is $\boxed{\{y \mid -1 \leq y \leq 1\}}$.
4. The tangent function $f(x) = \tan x$ is periodic with period π . Therefore,

$$\tan \frac{\pi}{4} = \tan \left(\frac{\pi}{4} + \pi \right) = \tan \left(\frac{\pi}{4} + 2\pi \right).$$
5. $\boxed{\text{False}}$. The range of the secant function is the set $\{y \mid |y| \geq 1\}$, which is not all positive real numbers.
6. The function $f(x) = 3 \cos(6x)$ has amplitude $\boxed{3}$ and period $\frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$.
7. $\boxed{\text{True}}$. The graphs of $y = \sin x$ and $y = \cos x$ are identical except for a horizontal shift. For example, the graph of $y = \cos x$ shifted horizontally right by $\frac{\pi}{2}$ units coincides with the graph of $y = \sin x$.
8. $\boxed{\text{False}}$. The amplitude of the function $f(x) = 2 \sin(\pi x)$ is 2 but its period is $\frac{2\pi}{\pi} = 2$.
9. $\boxed{\text{True}}$. The graph of the sine function has infinitely many x -intercepts. In particular, any x of the form $n\pi$ where n is an integer is an x -intercept of the sine function.
10. The graph of $y = \tan x$ is symmetric with respect to the $\boxed{\text{origin}}$.
11. The graph of $y = \sec x$ is symmetric with respect to the $\boxed{y\text{-axis}}$.
12. Answers will vary. One possibility is that a function is periodic if its values repeat over intervals of equal length.

Practice Problems

13. $\tan \left(-\frac{\pi}{4} \right) = -\tan \frac{\pi}{4} = \boxed{-1}$.
14. $\sin \left(-\frac{3\pi}{2} \right) = -\sin \frac{3\pi}{2} = -(-1) = \boxed{1}$.
15. $\csc \left(-\frac{\pi}{3} \right) = -\csc \frac{\pi}{3} = -\frac{2}{\sqrt{3}} = \boxed{-\frac{2\sqrt{3}}{3}}$.
16. $\cos \left(-\frac{\pi}{6} \right) = \cos \frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2}}$.
17. Because $f(0) = \tan 0 = 0$, the y -intercept of the tangent function is $\boxed{0}$.
18. In general, the x -intercepts of the function $f(x) = \sin x$ are the x values of the form $n\pi$ where n is an integer. Thus, the x -intercepts of the function $f(x) = \sin x$ on the closed interval $[-2\pi, 2\pi]$ are $\boxed{-2\pi, -\pi, 0, \pi, \text{ and } 2\pi}$.

19. The smallest value of the function $f(x) = \cos x$ is $\boxed{-1}$.

20. For $-2\pi \leq x \leq 2\pi$, $\sin x = 1$ when $\boxed{x = -\frac{3\pi}{2} \text{ and when } x = \frac{\pi}{2}}$. On the same interval, $\sin x = -1$ when $\boxed{x = -\frac{\pi}{2} \text{ and when } x = \frac{3\pi}{2}}$.

21. The graph appears to be a sine function, reflected about the x -axis, with an amplitude of 2 and a period of 4π . It follows that

$$A = -2 \quad \text{and} \quad \omega = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Thus, the function is $\boxed{\text{(f): } y = -2 \sin\left(\frac{1}{2}x\right)}$.

22. The graph appears to be a cosine function with an amplitude of 2 and a period of 4. It follows that

$$A = 2 \quad \text{and} \quad \omega = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Thus, the function is $\boxed{\text{(b): } y = 2 \cos\left(\frac{\pi}{2}x\right)}$.

23. The graph appears to be a sine function with an amplitude of 2 and a period of 4. It follows that

$$A = 2 \quad \text{and} \quad \omega = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Thus, the function is $\boxed{\text{(a): } y = 2 \sin\left(\frac{\pi}{2}x\right)}$.

24. The graph appears to be a cosine function, reflected about the x -axis, with an amplitude of 2 and a period of 4. It follows that

$$A = -2 \quad \text{and} \quad \omega = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Thus, the function is $\boxed{\text{(e): } y = -2 \cos\left(\frac{\pi}{2}x\right)}$.

25. The graph appears to be a sine function, reflected about the x -axis, with an amplitude of 3 and a period of π . It follows that

$$A = -3 \quad \text{and} \quad \omega = \frac{2\pi}{\pi} = 2.$$

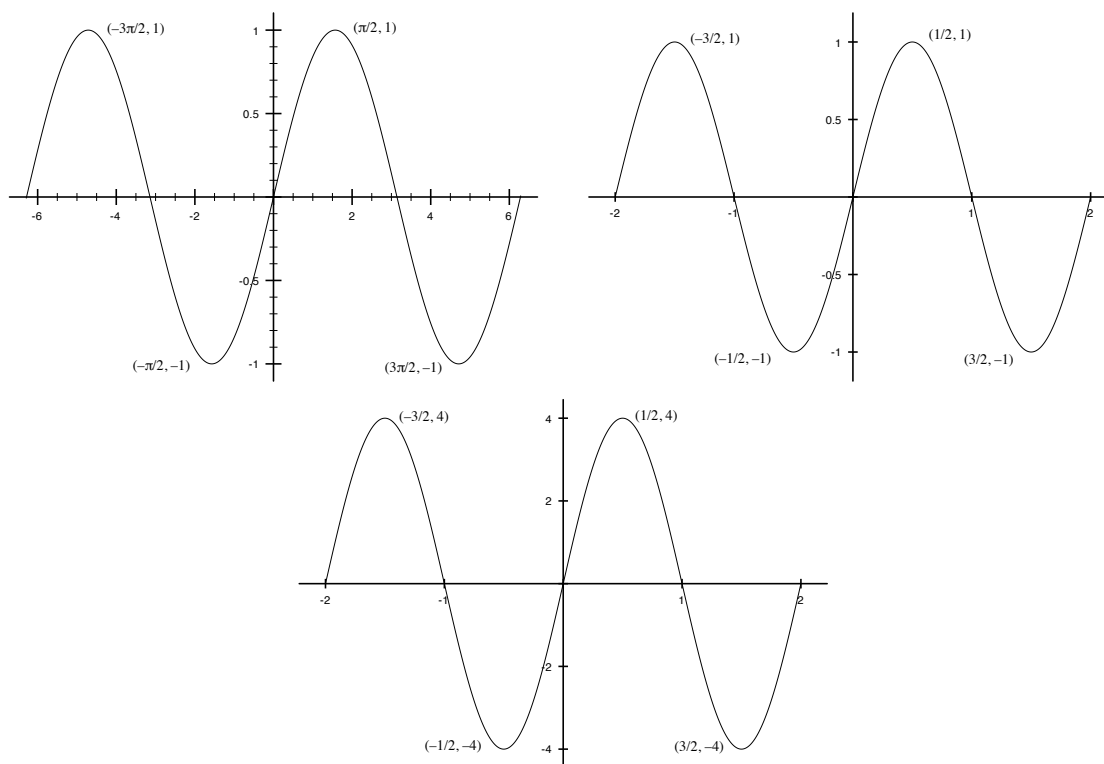
Thus, the function is $\boxed{\text{(d): } y = -3 \sin(2x)}$.

26. The graph appears to be a cosine function with an amplitude of 3 and a period of π . It follows that

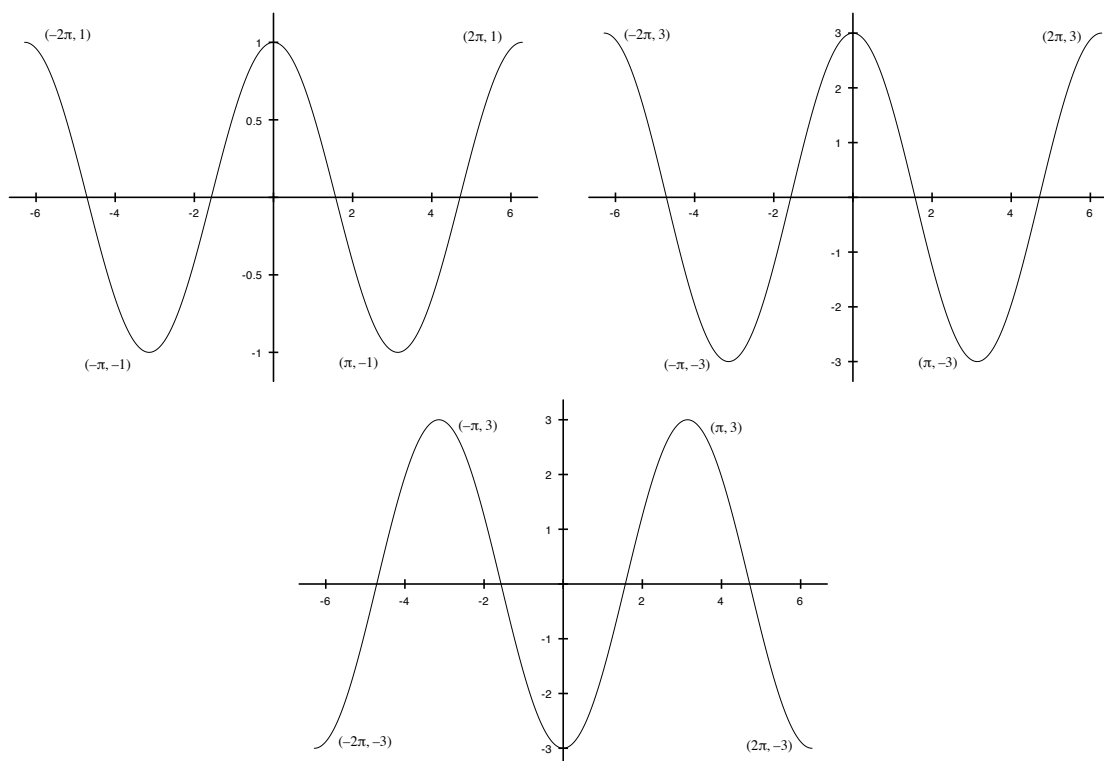
$$A = 3 \quad \text{and} \quad \omega = \frac{2\pi}{\pi} = 2.$$

Thus, the function is $\boxed{\text{(c): } y = 3 \cos(2x)}$.

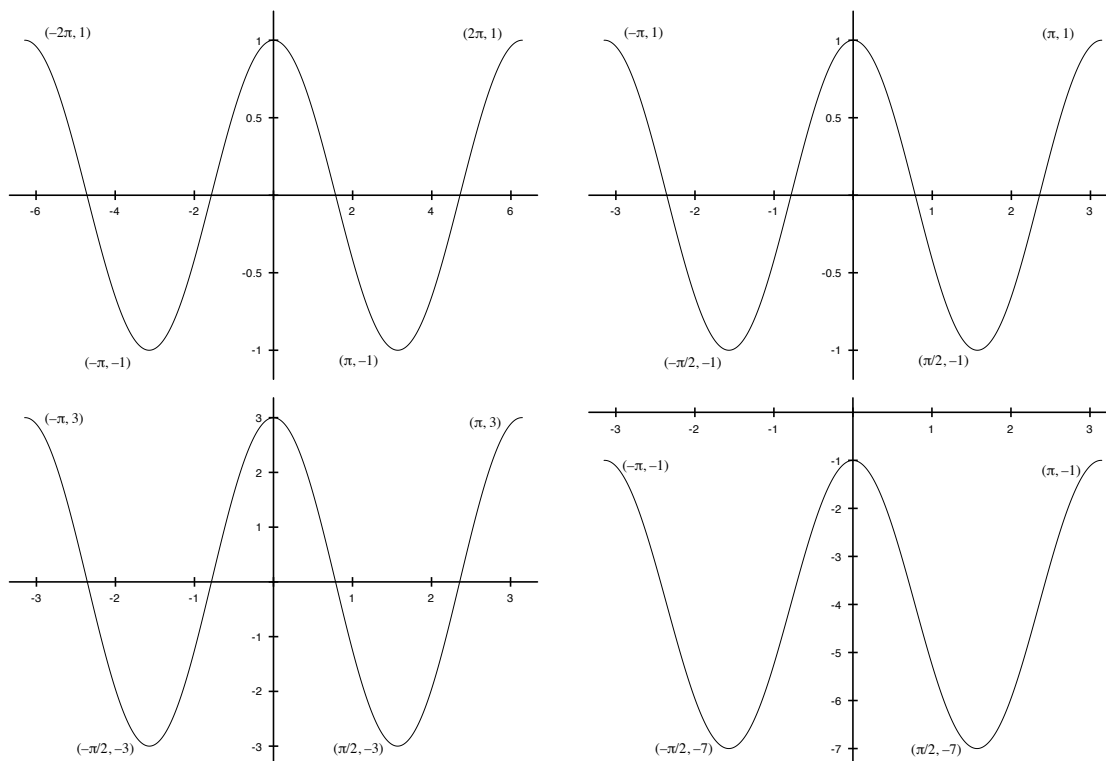
27. The graph of $f(x) = 4 \sin(\pi x)$ can be obtained from the graph of $y = \sin x$ by $\boxed{\text{compressing horizontally by a factor of } \frac{1}{\pi} \text{ and stretching vertically by a factor of } 4}$. The resulting amplitude will be 4, and the resulting period will be $\frac{2\pi}{\pi} = 2$. The graphs of $y = \sin x$, $y = \sin(\pi x)$, and $y = 4 \sin(\pi x)$ are shown below top left, below top right, and below bottom, respectively.



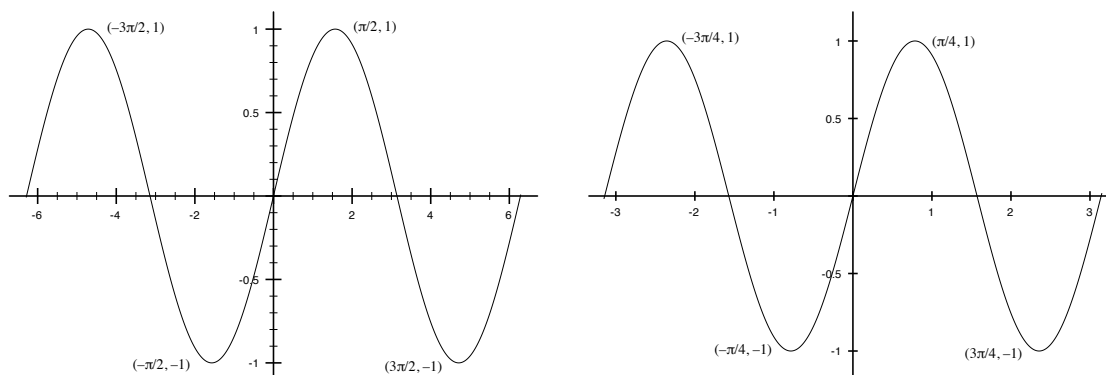
28. The graph of $f(x) = -3\cos x$ can be obtained from the graph of $y = \cos x$ by stretching vertically by a factor of 3 and then reflecting about the x -axis. The resulting amplitude will be $|-3| = 3$. The graphs of $y = \cos x$, $y = 3\cos x$, and $y = -3\cos x$ are shown below top left, below top right, and below bottom, respectively.

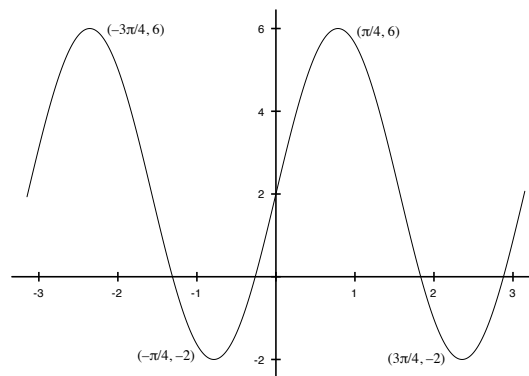
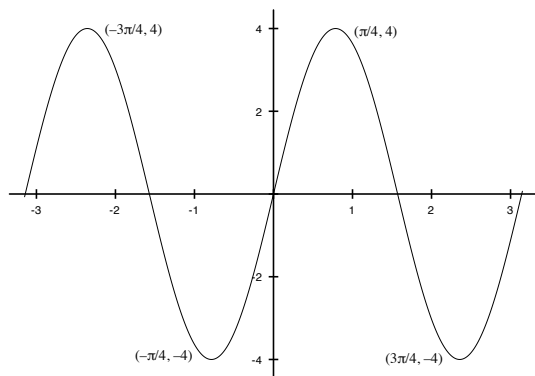


29. The graph of $f(x) = 3\cos(2x) - 4$ can be obtained from the graph of $y = \cos x$ by compressing horizontally by a factor of $\frac{1}{2}$, stretching vertically by a factor of 3, and then shifting vertically down 4 units. The resulting amplitude will be 3, and the resulting period will be $\frac{2\pi}{2} = \pi$. The graphs of $y = \cos x$, $y = \cos(2x)$, $y = 3\cos(2x)$, and $y = 3\cos(2x) - 4$ are shown below top left, below top right, below bottom left, and below bottom right, respectively.

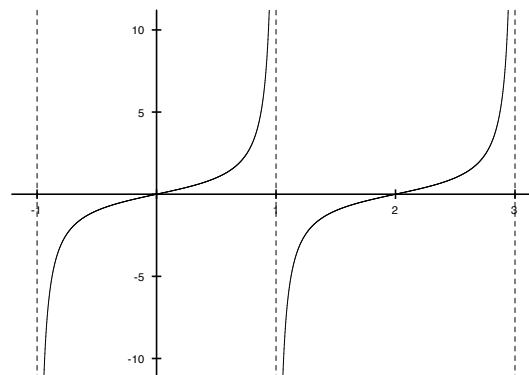
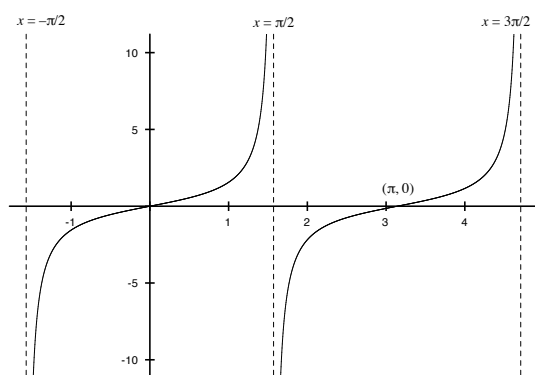


30. The graph of $f(x) = 4\sin(2x) + 2$ can be obtained from the graph of $y = \sin x$ by compressing horizontally by a factor of $\frac{1}{2}$, stretching vertically by a factor of 4, and then shifting vertically up 2 units. The resulting amplitude will be 4, and the resulting period will be $\frac{2\pi}{2} = \pi$. The graphs of $y = \sin x$, $y = \sin(2x)$, $y = 4\sin(2x)$, and $y = 4\sin(2x) + 2$ are shown below top left, below top right, below bottom left, and below bottom right, respectively.

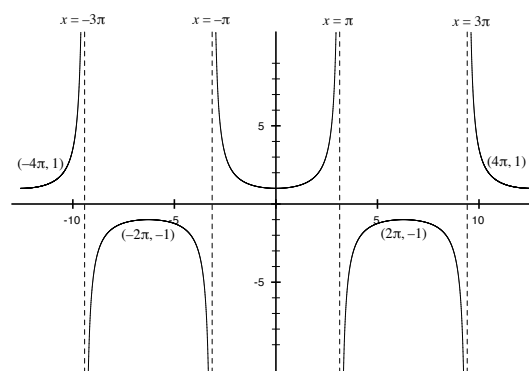
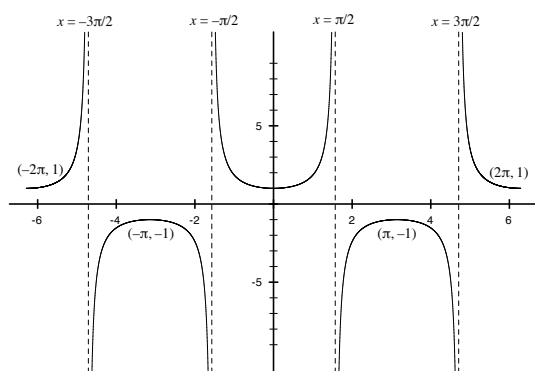


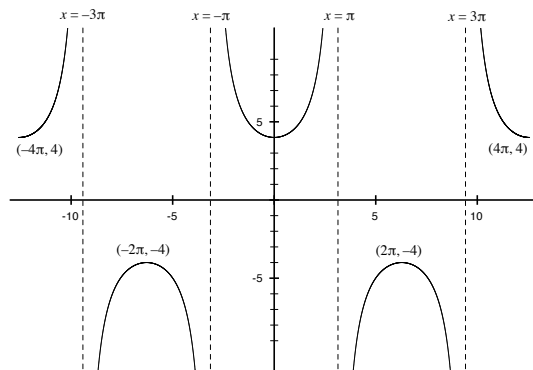


31. The graph of $f(x) = \tan\left(\frac{\pi}{2}x\right)$ can be obtained from the graph of $y = \tan x$ by compressing horizontally by a factor of $\frac{2}{\pi}$. The resulting period will be $\frac{\pi}{\pi/2} = 2$. The graphs of $y = \tan x$ and $y = \tan\left(\frac{\pi}{2}x\right)$ are shown below left and below right, respectively.



32. The graph of $f(x) = 4\sec\left(\frac{1}{2}x\right)$ can be obtained from the graph of $y = \sec x$ by stretching horizontally by a factor of 2 and stretching vertically by a factor of 4. The resulting period is $\frac{2\pi}{1/2} = 4\pi$. The graphs of $y = \sec x$, $y = \sec\left(\frac{1}{2}x\right)$, and $y = 4\sec\left(\frac{1}{2}x\right)$ are shown below top left, below top right, and below bottom, respectively.





33. Matching the function $g(x) = \frac{1}{2} \cos(\pi x)$ to the standard form $A \cos(\omega x)$, it follows that $A = \frac{1}{2}$ and $\omega = \pi$. Thus, the amplitude of the function g is $|A| = \frac{1}{2}$, and the period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2.$$

34. Matching the function $f(x) = \sin(2x)$ to the standard form $A \sin(\omega x)$, it follows that $A = 1$ and $\omega = 2$. Thus, the amplitude of the function f is $|A| = 1$, and the period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$$

35. Matching the function $g(x) = 3 \sin x$ to the standard form $A \sin(\omega x)$, it follows that $A = 3$ and $\omega = 1$. Thus, the amplitude of the function g is $|A| = 3$, and the period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi.$$

36. Matching the function $f(x) = -2 \cos\left(\frac{3}{2}x\right)$ to the standard form $A \cos(\omega x)$, it follows that $A = -2$ and $\omega = \frac{3}{2}$. Thus, the amplitude of the function f is $|A| = 2$, and the period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}.$$

37. With an amplitude of 2 and a period of π , $A = 2$,

$$\omega = \frac{2\pi}{\pi} = 2,$$

and the function is $f(x) = 2 \sin(2x)$.

38. With an amplitude of $\frac{1}{3}$ and a period of 2, $A = \frac{1}{3}$,

$$\omega = \frac{2\pi}{2} = \pi,$$

and the function is $f(x) = \frac{1}{3} \sin(\pi x)$.

39. With an amplitude of $\frac{1}{2}$ and a period of π , $A = \frac{1}{2}$,

$$\omega = \frac{2\pi}{\pi} = 2,$$

and the function is $f(x) = \frac{1}{2} \cos(2x)$.

40. With an amplitude of 3 and a period of 4π , $A = 3$,

$$\omega = \frac{2\pi}{4\pi} = \frac{1}{2},$$

and the function is $\boxed{f(x) = 3 \cos\left(\frac{1}{2}x\right)}$.

41. The graph appears to be a sine function with an amplitude of 1 and a period of $\frac{4\pi}{3}$ that has been reflected about the x -axis. Thus, $A = -1$,

$$\omega = \frac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2},$$

and $\boxed{f(x) = -\sin\left(\frac{3}{2}x\right)}$.

42. The graph appears to be a cosine function with an amplitude of π and a period of 2π that has been reflected about the x -axis. Thus, $A = -\pi$,

$$\omega = \frac{2\pi}{2\pi} = 1,$$

and $\boxed{f(x) = -\pi \cos x}$.

43. The graph appears to be a cosine function with an amplitude of 1 and a period of $\frac{3}{2}$ that has been reflected about the x -axis and shifted vertically up 1 unit. Thus, $A = -1$,

$$\omega = \frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3},$$

and $\boxed{f(x) = 1 - \cos\left(\frac{4\pi}{3}x\right)}$.

44. The graph appears to be a sine function with an amplitude of $\frac{1}{2}$ and a period of $\frac{4\pi}{3}$ that has been reflected about the x -axis and shifted vertically down 1 unit. Thus, $A = -\frac{1}{2}$,

$$\omega = \frac{2\pi}{\frac{4\pi}{3}} = \frac{3}{2},$$

and $\boxed{f(x) = -1 - \frac{1}{2} \sin\left(\frac{3}{2}x\right)}$.

45. The graph appears to be just $\boxed{f(x) = \cot x}$.

46. The graph appears to be just $\boxed{f(x) = \tan x}$.

47. The graph appears to be a tangent function that has been shifted horizontally right $\frac{\pi}{2}$ units. Thus,

$$\boxed{f(x) = \tan\left(x - \frac{\pi}{2}\right)}.$$

48. The graph appears to be a cotangent function that has been shifted horizontally right $\frac{\pi}{2}$ units. Thus,

$$\boxed{f(x) = \cot\left(x - \frac{\pi}{2}\right)}.$$

P.7: Inverse Trigonometric Functions

Concepts and Vocabulary

1. $y = \sin^{-1} x$ if and only if $\boxed{x = \sin y}$, where $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
2. $\boxed{\text{False}}$. $\sin^{-1}(\sin x) = x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. $\sin(\sin^{-1} x) = x$, where $-1 \leq x \leq 1$.
3. $\boxed{\text{False}}$. The domain of $f^{-1}(x) = \sin^{-1} x$ is $-1 \leq x \leq 1$. The range of the inverse sine function is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\boxed{\text{True}}$. $\sin(\sin^{-1} 0) = \sin 0 = 0$.
5. $\boxed{\text{False}}$. $f^{-1}(x) = \tan^{-1} x$ means $x = \tan y$, where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
6. $\boxed{\text{True}}$. The domain of the inverse tangent function is the set of all real numbers.
7. $\boxed{\text{True}}$. $\sec^{-1} 0.5$ is not defined. The domain of the inverse secant function is the set $\{x \mid |x| \geq 1\}$.
8. $\boxed{\text{True}}$. Trigonometric equations can have multiple solutions.

Practice Problems

9. The angle from the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Thus, $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$.
10. The angle from the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$ is $-\frac{\pi}{6}$. Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{6}}$.
11. The angle from $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ whose secant is 2 is $\frac{\pi}{3}$. Thus, $\sec^{-1}(2) = \boxed{\frac{\pi}{3}}$.
12. The angle from the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $\sqrt{3}$ is $\frac{\pi}{3}$. Thus, $\tan^{-1}(\sqrt{3}) = \boxed{\frac{\pi}{3}}$.
13. The angle from the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is -1 is $-\frac{\pi}{4}$. Thus, $\tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$.
14. The angle from $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ whose secant is $\frac{2\sqrt{3}}{3}$ is $\frac{\pi}{3}$. Thus, $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \boxed{\frac{\pi}{3}}$.
15. Because $\frac{\pi}{4}$ is in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1}\left(\tan \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}$.
16. The angle from the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 0 is 0. Thus, $\tan^{-1}(\sin 0) = \tan^{-1} 0 = \boxed{0}$.
17. $\sin\left(\sin^{-1} \frac{3}{5}\right) = \boxed{\frac{3}{5}}$.
18. Let $\theta = \sec^{-1}(-3)$. The problem then becomes: evaluate $\tan \theta$, where $\pi \leq \theta < \frac{3\pi}{2}$ and $\sec \theta = -3$. From the identity $1 + \tan^2 x = \sec^2 x$, it follows that $\tan x = \pm\sqrt{\sec^2 x - 1}$. Because θ is a third quadrant angle and the tangent function is positive for all third quadrant angles,

$$\tan[\sec^{-1}(-3)] = \tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{(-3)^2 - 1} = \sqrt{8} = \boxed{2\sqrt{2}}.$$

19. Because $\frac{4\pi}{5}$ is not in the range of the inverse sine function, an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ must be found for which the sine function has the same value as $\sin \frac{4\pi}{5}$. Using the identity $\sin x = \sin(\pi - x)$, it follows that

$$\sin \frac{4\pi}{5} = \sin \left(\pi - \frac{4\pi}{5} \right) = \sin \frac{\pi}{5},$$

so

$$\sin^{-1} \left(\sin \frac{4\pi}{5} \right) = \sin^{-1} \left(\sin \frac{\pi}{5} \right) = \boxed{\frac{\pi}{5}}.$$

20. $\sec(\sec^{-1} 3) = \boxed{3}$.

21. Let $\theta = \sin^{-1} u$. The problem then becomes: evaluate $\cos \theta$ in term of u , where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\sin \theta = u$. From the identity $\sin^2 x + \cos^2 x = 1$, it follows that $\cos x = \pm \sqrt{1 - \sin^2 x}$. Because θ is a first or fourth quadrant angle and the cosine function is positive for first and fourth quadrant angles,

$$\cos(\sin^{-1} u) = \cos \theta = \sqrt{1 - \sin^2 \theta} = \boxed{\sqrt{1 - u^2}}.$$

22. First, note that

$$\tan(\sin^{-1} u) = \frac{\sin(\sin^{-1} u)}{\cos(\sin^{-1} u)}.$$

Now, $\sin(\sin^{-1} u) = u$, and $\cos(\sin^{-1} u) = \sqrt{1 - u^2}$ by Problem 21, so

$$\tan(\sin^{-1} u) = \boxed{\frac{u}{\sqrt{1 - u^2}}}.$$

23. Let $\theta = \tan^{-1} u$. The problem then becomes: evaluate $\sec \theta$ in term of u , where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan \theta = u$. From the identity $1 + \tan^2 x = \sec^2 x$, it follows that $\sec x = \pm \sqrt{1 + \tan^2 x}$. Because θ is a first or fourth quadrant angle and the secant function is positive for first and fourth quadrant angles,

$$\sec(\tan^{-1} u) = \sec \theta = \sqrt{1 + \tan^2 \theta} = \boxed{\sqrt{1 + u^2}}.$$

24. Let x be in the closed interval $[-1, 1]$; that is, in the domain of the inverse sine function. It follows that $-x$ is also in the domain of the inverse sine function. Now, suppose that $\sin^{-1} x = \theta$ for some angle θ in the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $x = \sin \theta$. Multiplying both sides of this equation by -1 and using the fact that the sine function is odd yields

$$-x = -\sin \theta = \sin(-\theta).$$

Because $-\theta$ is also an angle in the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\sin^{-1}(-x) = -\theta = -\sin^{-1} x,$$

and the inverse sine function is odd.

25. Let x be any real number; that is, in the domain of the inverse tangent function. It follows that $-x$ is also in the domain of the inverse tangent function. Now, suppose that $\tan^{-1} x = \theta$ for some angle θ in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then $x = \tan \theta$. Multiplying both sides of this equation by -1 and using the fact that the tangent function is odd yields

$$-x = -\tan \theta = \tan(-\theta).$$

Because $-\theta$ is also an angle in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\tan^{-1}(-x) = -\theta = -\tan^{-1} x,$$

and the inverse tangent function is odd.

26. Let $x = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. Then

$$\begin{aligned}\cos x &= \cos \frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2}}, \\ \tan x &= \tan \frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{3}}, \\ \cot x &= \cot \frac{\pi}{6} = \boxed{\sqrt{3}}, \\ \sec x &= \sec \frac{\pi}{6} = \boxed{\frac{2\sqrt{3}}{3}}, \\ \csc x &= \csc \frac{\pi}{6} = \boxed{2}.\end{aligned}$$

27. The angles whose tangent is $-\frac{\sqrt{3}}{3}$ are of the form $-\frac{\pi}{6} + k\pi$, where k is any integer. Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$. The solution set is therefore $\boxed{\left\{\frac{5\pi}{6}, \frac{11\pi}{6}\right\}}$.

28. The angles whose secant is -2 are of the form $\frac{2\pi}{3} + 2k\pi$ and $\frac{4\pi}{3} + 2k\pi$, where k is any integer. Thus

$$\frac{3\theta}{2} = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad \frac{3\theta}{2} = \frac{4\pi}{3} + 2k\pi,$$

so that

$$\theta = \frac{4\pi}{9} + \frac{4k\pi}{3} \quad \text{or} \quad \theta = \frac{8\pi}{9} + \frac{4k\pi}{3}.$$

Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{4\pi}{9}$, $\frac{8\pi}{9}$, and $\frac{16\pi}{9}$. The solution set is therefore $\boxed{\left\{\frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}\right\}}$.

29. Solving the given equation for $\sin \theta$ yields $\sin \theta = -\frac{1}{2}$. The angles whose sine is $-\frac{1}{2}$ are of the form $-\frac{\pi}{6} + 2k\pi$ and $-\frac{5\pi}{6} + 2k\pi$, where k is any integer. Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The solution set is therefore $\boxed{\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$.

30. Solving the given equation for $\cos \theta$ yields $\cos \theta = \frac{1}{2}$. The angles whose cosine is $\frac{1}{2}$ are of the form $\pm\frac{\pi}{3} + 2k\pi$, where k is any integer. Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. The solution set is therefore $\boxed{\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}}$.

31. The angles whose sine is -1 are of the form $\frac{3\pi}{2} + 2k\pi$, where k is any integer. Thus,

$$3\theta = \frac{3\pi}{2} + 2k\pi, \quad \text{so that} \quad \theta = \frac{\pi}{2} + \frac{2k\pi}{3}.$$

Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{2}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. The solution set is therefore

$$\boxed{\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}.$$

32. The angles whose cosine is $\frac{1}{2}$ are of the form $\pm\frac{\pi}{3} + 2k\pi$, where k is any integer. Thus,

$$2\theta = \pm\frac{\pi}{3} + 2k\pi, \quad \text{so that} \quad \theta = \pm\frac{\pi}{6} + k\pi.$$

Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. The solution set is

therefore $\boxed{\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}.$

33. Solving the given equation for $\cos \theta$ yields

$$\cos \theta = \pm\frac{1}{2}.$$

The angles that satisfy this equation are of the form $\pm\frac{\pi}{3} + 2k\pi$ and $\pm\frac{2\pi}{3} + 2k\pi$, where k is any integer.

Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$. The solution set is

therefore $\boxed{\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}}.$

34. Solving the given equation for $\sin \theta$ yields

$$\sin \theta = \pm\frac{\sqrt{2}}{2}.$$

The angles that satisfy this equation are of the form $\frac{\pi}{4} + \frac{k\pi}{2}$, where k is any integer. Among these, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$. The solution set is there-

fore $\boxed{\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}}.$

35. Factor the given equation as

$$(2\sin \theta - 3)(\sin \theta - 1) = 0.$$

Then either $\sin \theta = \frac{3}{2}$ or $\sin \theta = 1$. The former equation has no solution, while the angles that satisfy the latter equation are of the form $\frac{\pi}{2} + 2k\pi$, where k is any integer. Among these, the only value that

is in the interval $[0, 2\pi)$ is $\frac{\pi}{2}$. The solution set is therefore $\boxed{\left\{\frac{\pi}{2}\right\}}.$

36. Factor the given equation as

$$(2\cos \theta + 1)(\cos \theta - 4) = 0.$$

Then either $\cos \theta = -\frac{1}{2}$ or $\cos \theta = 4$. The latter equation has no solution, while the angles that satisfy the former equation are of the form $\pm\frac{2\pi}{3} + 2k\pi$, where k is any integer. Among these, the ones that

are in the interval $[0, 2\pi)$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. The solution set is therefore $\boxed{\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}}.$

37. Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ to write $\cos^2 \theta = 1 - \sin^2 \theta$. The original equation then becomes

$$1 + \sin \theta = 2 - 2\sin^2 \theta, \quad \text{or} \quad 2\sin^2 \theta + \sin \theta - 1 = 0.$$

Factoring the left-hand side yields

$$(2\sin \theta - 1)(\sin \theta + 1) = 0,$$

so that either $\sin \theta = \frac{1}{2}$ or $\sin \theta = -1$. The angles that satisfy the former equation are of the form $\frac{\pi}{6} + 2k\pi$ and $\frac{5\pi}{6} + 2k\pi$, while the angles that satisfy the latter equation are of the form $\frac{3\pi}{2} + 2k\pi$, where k is any integer. Among all of these angles, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$. The solution set is therefore $\boxed{\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \right\}}$.

38. Use the identity $1 + \tan^2 \theta = \sec^2 \theta$ to write the original equation as

$$\tan^2 \theta + \tan \theta + 1 = 0.$$

Completing the square on the left-hand side then yields

$$\left(\tan \theta + \frac{1}{2} \right)^2 + \frac{3}{4} = 0.$$

This latter equation has no solution, so the original equation also has no solution.

39. First, multiply both sides of the original equation by $\frac{\sqrt{2}}{2}$ to obtain

$$\frac{\sqrt{2}}{2} \sin \theta + \frac{\sqrt{2}}{2} \cos \theta = 0.$$

Next, note that

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

so that

$$\frac{\sqrt{2}}{2} \sin \theta + \frac{\sqrt{2}}{2} \cos \theta = \sin \frac{\pi}{4} \sin \theta + \cos \frac{\pi}{4} \cos \theta = \cos \left(\theta - \frac{\pi}{4} \right).$$

The equation to be solved has therefore been reduced to

$$\cos \left(\theta - \frac{\pi}{4} \right) = 0.$$

The angles that satisfy this last expression are of the form $(2k+1)\frac{\pi}{2}$, where k is any integer, so

$$\theta - \frac{\pi}{4} = (2k+1)\frac{\pi}{2} \quad \text{or} \quad \theta = (2k+1)\frac{\pi}{2} + \frac{\pi}{4},$$

where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi)$ are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

The solution set is therefore $\boxed{\left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}}$.

40. Multiplying both sides of the original equation by $\tan \theta$ and using the identity $\tan \theta \cot \theta = 1$ yields

$$\tan^2 \theta = 1 \quad \text{so that} \quad \tan \theta = \pm 1.$$

The angles that satisfy the equation $\tan \theta = 1$ are of the form $\frac{\pi}{4} + k\pi$, while the angles that satisfy the equation $\tan \theta = -1$ are of the form $\frac{3\pi}{4} + k\pi$, where k is any integer. Among all of these angles, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$. The solution set is therefore $\boxed{\left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}}$.

41. Use the identity $\cos(2\theta) = 1 - 2\sin^2\theta$ to write the original equation as

$$1 - 2\sin^2\theta + 6\sin^2\theta = 4.$$

Thus

$$\sin^2\theta = \frac{3}{4}, \quad \text{or} \quad \sin\theta = \pm\frac{\sqrt{3}}{2}.$$

The angles whose sine is $\pm\frac{\sqrt{3}}{2}$ are of the form $\pm\frac{\pi}{3} + 2k\pi$ and $\pm\frac{2\pi}{3} + 2k\pi$, where k is any integer.

Among these angles, the ones that are in the interval $[0, 2\pi)$ are $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$. The solution set

is therefore $\left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$.

42. Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to write the original equation as

$$2\cos^2\theta - \cos\theta - 1 = 0.$$

Factoring the left-hand side yields

$$(2\cos\theta + 1)(\cos\theta - 1) = 0,$$

so that $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$. The angles that satisfy the former equation are of the form $\pm\frac{2\pi}{3} + 2k\pi$, and the angles that satisfy the latter equation are of the form $2k\pi$, where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi)$ are 0 , $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$. The solution set is therefore

$$\left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}.$$

43. Use the identity $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$ to write the original equation as

$$\sin(2\theta) + 2\sin(2\theta)\cos(2\theta) = \sin(2\theta)(1 + 2\cos(2\theta)) = 0.$$

Thus, $\sin(2\theta) = 0$ or $\cos(2\theta) = -\frac{1}{2}$. The angles whose sine is 0 are of the form $k\pi$, and the angles whose cosine is $-\frac{1}{2}$ are of the form $\pm\frac{2\pi}{3} + 2k\pi$, where k is any integer. It follows that

$$2\theta = k\pi, \quad \text{or} \quad 2\theta = \pm\frac{2\pi}{3} + 2k\pi,$$

so that

$$\theta = \frac{k\pi}{2}, \quad \text{or} \quad \theta = \pm\frac{\pi}{3} + k\pi.$$

Among these angles, the ones that are in the interval $[0, 2\pi)$ are

$$0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \text{ and } \frac{5\pi}{3}.$$

The solution set is therefore

$$\left\{ 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3} \right\}.$$

44. Using the identities for the cosine of a sum and a difference of two angles,

$$\cos(4\theta) = \cos(5\theta - \theta) = \cos(5\theta)\cos\theta + \sin(5\theta)\sin\theta$$

and

$$\cos(6\theta) = \cos(5\theta + \theta) = \cos(5\theta)\cos\theta - \sin(5\theta)\sin\theta.$$

Subtracting the latter equation from the former yields

$$\cos(4\theta) - \cos(6\theta) = 2 \sin(5\theta) \sin \theta,$$

so that the original equation is equivalent to $2 \sin(5\theta) \sin \theta = 0$. Thus $\sin(5\theta) = 0$ or $\sin \theta = 0$. The angles whose sine is 0 are of the form $k\pi$, where k is any integer. Thus,

$$\theta = \frac{k\pi}{5} \quad \text{or} \quad \theta = k\pi.$$

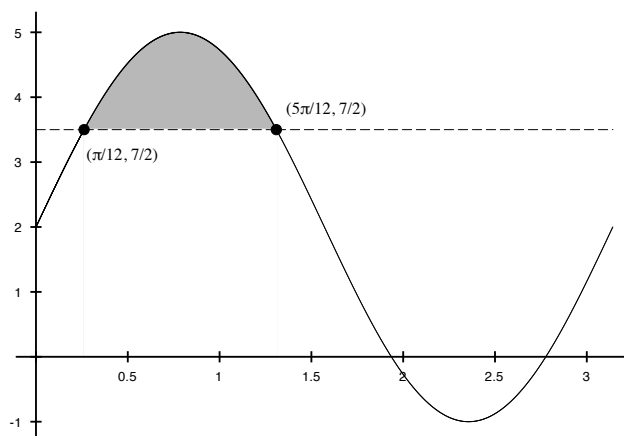
Among these angles, the ones that are in the interval $[0, 2\pi)$ are

$$0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \text{ and } \frac{9\pi}{5}.$$

The solution set is therefore

$$\left\{ 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5} \right\}.$$

45. To three decimal places, $\tan^{-1} 5 = 1.373$. The angles whose tangent is 5 are therefore of the form $1.373 + k\pi$, where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi)$ are 1.373 and $1.373 + \pi = 4.515$. The solution set is then $\{1.373, 4.515\}$.
46. To three decimal places, $\cos^{-1} 0.6 = 0.927$. The angles whose cosine is 0.6 are therefore of the form $\pm 0.927 + 2k\pi$, where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi)$ are 0.927 and $-0.927 + 2\pi = 5.356$. The solution set is then $\{0.927, 5.356\}$.
47. The equation $2 + 3 \sin \theta = 0$ is equivalent to $\sin \theta = -\frac{2}{3}$. To three decimal places, $\sin^{-1}(-\frac{2}{3}) = -0.730$. The angles whose sine is $-\frac{2}{3}$ are therefore of the form $-0.730 + 2k\pi$ or $\pi - (-0.730) + 2k\pi$, where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi)$ are $\pi - (-0.730) = 3.871$ and $-0.730 + 2\pi = 5.553$. The solution set is then $\{3.871, 5.553\}$.
48. The equation $4 + \sec \theta = 0$ is equivalent to $\sec \theta = -4$, or $\cos \theta = -0.25$. To three decimal places, $\cos^{-1}(-0.25) = 1.823$. The angles whose cosine is -0.25 are therefore of the form $\pm 1.823 + 2k\pi$. Among these angles, the ones that are in the interval $[0, 2\pi)$ are 1.823 and $-1.823 + 2\pi = 4.460$. The solution set is then $\{1.823, 4.460\}$.
49. (a) In the figure below, the solid curve represents the graph of $f(x) = 3 \sin(2x) + 2$, and the dashed curve represents the graph of $g(x) = \frac{7}{2}$.



- (b) Solving $3\sin(2x) + 2 = \frac{7}{2}$ for $\sin(2x)$ yields $\sin(2x) = \frac{1}{2}$. The angles whose sine is $\frac{1}{2}$ are of the form $\frac{\pi}{6} + 2k\pi$ and $\frac{5\pi}{6} + 2k\pi$, where k is any integer. Thus,

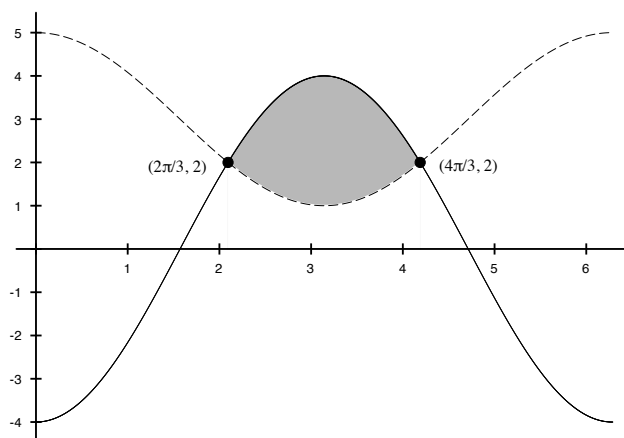
$$x = \frac{\pi}{12} + k\pi \quad \text{or} \quad x = \frac{5\pi}{12} + k\pi.$$

Among these angles, the ones that are in the interval $[0, \pi]$ are $\frac{\pi}{12}$ and $\frac{5\pi}{12}$. The points of intersection,

$$\left(\frac{\pi}{12}, \frac{7}{2}\right) \quad \text{and} \quad \left(\frac{5\pi}{12}, \frac{7}{2}\right)$$

are labeled on the graph in part (a).

- (c) The region bounded by the graphs of $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ between the points of intersection determined in part (b) is shaded in the graph drawn in part (a).
- (d) Based on the shaded region in the graph in part (a), on the interval $[0, \pi]$, the solution set for the inequality $f(x) > g(x)$ is $\left\{x \mid \frac{\pi}{12} < x < \frac{5\pi}{12}\right\}$.
50. (a) In the figure below, the solid curve represents the graph of $f(x) = -4\cos x$, and the dashed curve represents the graph of $g(x) = 2\cos x + 3$.



- (b) Solving $-4\cos x = 2\cos x + 3$ for $\cos x$ yields $\cos x = -\frac{1}{2}$. The angles whose cosine is $-\frac{1}{2}$ are of the form $\pm\frac{2\pi}{3} + 2k\pi$, where k is any integer. Among these angles, the ones that are in the interval $[0, 2\pi]$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. The points of intersection,

$$\left(\frac{2\pi}{3}, 2\right) \quad \text{and} \quad \left(\frac{4\pi}{3}, 2\right)$$

are labeled on the graph in part (a).

- (c) The region bounded by the graphs of $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ between the points of intersection determined in part (b) is shaded in the graph drawn in part (a).
- (d) Based on the shaded region in the graph in part (a), on the interval $[0, 2\pi]$, the solution set for the inequality $f(x) > g(x)$ is $\left\{x \mid \frac{2\pi}{3} < x < \frac{4\pi}{3}\right\}$.

Chapter 1: Limits and Continuity

1.1: Limits of Functions Using Numerical and Graphical Techniques

Concepts and Vocabulary

1. The limit as x approaches c of a function f is written symbolically as $\boxed{(c) \lim_{x \rightarrow c} f(x)}$.
2. $\boxed{\text{True}}$. The tangent line to the graph of f at a point $P = (c, f(c))$ is the limiting position of the secant lines passing through P and a point $(x, f(x))$, $x \neq c$, as x moves closer to c .
3. $\boxed{\text{False}}$. If f is not defined at $x = c$, the $\lim_{x \rightarrow c} f(x)$ **may** exist.
4. $\boxed{\text{False}}$. The limit L of a function $y = f(x)$ as x approaches the number c **does not** depend on the value of f at c .
5. If $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists, it equals the $\boxed{\text{slope}}$ of the tangent line to the graph of f at the point $(c, f(c))$.
6. $\boxed{\text{False}}$. The limit of a function $y = f(x)$ as x approaches a number c equals L **if and only if both** of the one-sided limits as x approaches c equal L .

Skill Building

7. The values in the table below suggest that the value of $f(x) = 2x$ can be made “as close as we please” to 2 by choosing x “sufficiently close” to 1. It therefore appears that

$$\lim_{x \rightarrow 1} (2x) = \boxed{2}.$$

x	0.9	0.99	0.999	$\rightarrow 1 \leftarrow$	1.001	1.01	1.1
$f(x) = 2x$	1.8	1.98	1.998	$f(x)$ approaches 2	2.002	2.02	2.2

8. The values in the table below suggest that the value of $f(x) = x + 3$ can be made “as close as we please” to 5 by choosing x “sufficiently close” to 2. It therefore appears that

$$\lim_{x \rightarrow 2} (x + 3) = \boxed{5}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x) = x + 3$	4.9	4.99	4.999	$f(x)$ approaches 5	5.001	5.01	5.1

9. The values in the table below suggest that the value of $f(x) = x^2 + 2$ can be made “as close as we please” to 2 by choosing x “sufficiently close” to 0. It therefore appears that

$$\lim_{x \rightarrow 0} (x^2 + 2) = \boxed{2}.$$

x	-0.1	-0.01	-0.001	$\rightarrow 0 \leftarrow$	0.001	0.01	0.1
$f(x) = x^2 + 2$	2.01	2.0001	2.000001	$f(x)$ approaches 2	2.000001	2.0001	2.01

10. The values in the table below suggest that the value of $f(x) = x^2 - 2$ can be made “as close as we please” to -1 by choosing x “sufficiently close” to -1 . It therefore appears that

$$\lim_{x \rightarrow -1} (x^2 - 2) = \boxed{-1}.$$

x	-1.1	-1.01	-1.001	$\rightarrow -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = x^2 - 2$	-0.79	-0.9799	-0.997999	$f(x)$ approaches -1	-1.001999	-1.0199	-1.19

11. The values in the table below suggest that the value of $f(x) = \frac{x^2 - 9}{x + 3}$ can be made “as close as we please” to -6 by choosing x “sufficiently close” to -3 . It therefore appears that

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = \boxed{-6}.$$

x	-3.5	-3.1	-3.01	$\rightarrow -3 \leftarrow$	-2.99	-2.9	-2.5
$f(x) = \frac{x^2 - 9}{x + 3}$	-6.5	-6.1	-6.01	$f(x)$ approaches -6	-5.99	-5.9	-5.5

12. The values in the table below suggest that the value of $f(x) = \frac{x^3 + 1}{x + 1}$ can be made “as close as we please” to 3 by choosing x “sufficiently close” to -1 . It therefore appears that

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} = \boxed{3}.$$

x	-1.1	-1.01	-1.001	$\rightarrow -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \frac{x^3 + 1}{x + 1}$	3.31	3.0301	3.003001	$f(x)$ approaches 3	2.997001	2.9701	2.71

13. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \frac{2 - 2e^x}{x}$ can be made “as close as we please” to -2 by choosing x “sufficiently close” to 0 . It therefore appears that

$$\lim_{x \rightarrow 0} \frac{2 - 2e^x}{x} = \boxed{-2}.$$

x	-0.2	-0.1	-0.01	$\rightarrow 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \frac{2 - 2e^x}{x}$	-1.81269	-1.90325	-1.99003	$f(x)$ approaches -2	-2.01003	-2.10342	-2.21403

14. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \frac{\ln x}{x - 1}$ can be made “as close as we please” to 1 by choosing x “sufficiently close” to 1 . It therefore appears that

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \boxed{1}.$$

x	0.9	0.99	0.999	$\rightarrow 1 \leftarrow$	1.001	1.01	1.1
$f(x) = \frac{\ln x}{x - 1}$	1.05361	1.00503	1.00050	$f(x)$ approaches 1	0.99950	0.99503	0.95310

15. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \frac{1 - \cos x}{x}$ can be made “as close as we please” to 0 by choosing x “sufficiently close” to 0. It therefore appears that

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \boxed{0}.$$

x	-0.2	-0.1	-0.01	$\rightarrow 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \frac{1 - \cos x}{x}$	-0.09967	-0.04996	-0.00500	$f(x)$ approaches 0	0.00500	0.04996	0.09967

16. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \frac{\sin x}{1 + \tan x}$ can be made “as close as we please” to 0 by choosing x “sufficiently close” to 0. It therefore appears that

$$\lim_{x \rightarrow 0} \frac{\sin x}{1 + \tan x} = \boxed{0}.$$

x	-0.2	-0.1	-0.01	$\rightarrow 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \frac{\sin x}{1 + \tan x}$	-0.24918	-0.11097	-0.01010	$f(x)$ approaches 0	0.00990	0.09073	0.16518

17. The graph suggests that the value of f approaches 2 as x approaches 2 from the left and as x approaches 2 from the right. Thus,

(a) $\lim_{x \rightarrow 2^-} f(x) = \boxed{2};$

(b) $\lim_{x \rightarrow 2^+} f(x) = \boxed{2};$ and

(c) $\lim_{x \rightarrow 2} f(x) = \boxed{2}.$

18. The graph suggests that the value of f approaches 4 as x approaches 2 from the left and as x approaches 2 from the right. Thus,

(a) $\lim_{x \rightarrow 2^-} f(x) = \boxed{4};$

(b) $\lim_{x \rightarrow 2^+} f(x) = \boxed{4};$ and

(c) $\lim_{x \rightarrow 2} f(x) = \boxed{4}.$

19. The graph suggests that the value of f approaches 3 as x approaches 2 from the left but the value of f approaches 6 as x approaches 2 from the right. Thus,

(a) $\lim_{x \rightarrow 2^-} f(x) = \boxed{3};$

(b) $\lim_{x \rightarrow 2^+} f(x) = \boxed{6};$ and

(c) $\lim_{x \rightarrow 2} f(x)$ does not exist because there is no single number that the values of f approach when x is close to 2.

20. The graph suggests that the value of f approaches 4 as x approaches 2 from the left but the value of f approaches 2 as x approaches 2 from the right. Thus,

(a) $\lim_{x \rightarrow 2^-} f(x) = \boxed{4};$

(b) $\lim_{x \rightarrow 2^+} f(x) = \boxed{2}$; and

(c) $\lim_{x \rightarrow 2} f(x)$ does not exist because there is no single number that the values of f approach when x is close to 2.

21. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow c} f(x) = \boxed{1}.$$

22. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow c} f(x) = \boxed{1}.$$

23. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow c} f(x) = \boxed{1}.$$

24. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 2.$$

Because the two one-sided limits are not equal (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \rightarrow c} f(x) \text{ does not exist }.$$

25. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = -1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 1.$$

Because the two one-sided limits are not equal (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \rightarrow c} f(x) \text{ does not exist.}$$

26. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 3.$$

Because the two one-sided limits are not equal (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \rightarrow c} f(x) \text{ does not exist.}$$

27. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 2,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 1.$$

Because the two one-sided limits are not equal (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \rightarrow c} f(x) \text{ does not exist.}$$

28. The graph suggests that, as x approaches c from the left,

$$\lim_{x \rightarrow c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \rightarrow c^+} f(x) = 2.$$

Because the two one-sided limits are not equal (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \rightarrow c} f(x) \text{ does not exist.}$$

29. The graph of f shown below suggests that, as x approaches 2 from the left,

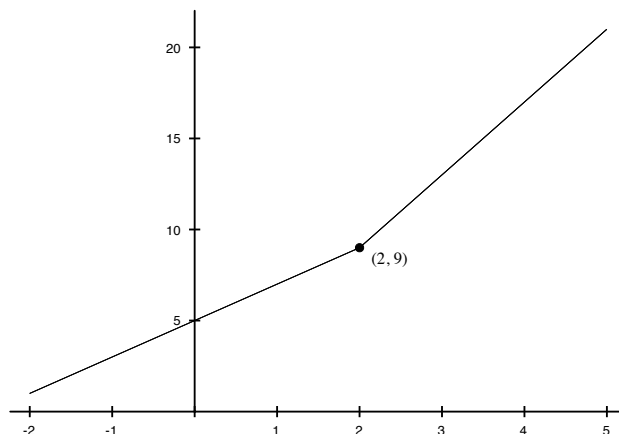
$$\lim_{x \rightarrow 2^-} f(x) = 9,$$

while, as x approaches 2 from the right,

$$\lim_{x \rightarrow 2^+} f(x) = 9.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow 2} f(x) = \boxed{9}.$$



30. The graph of f shown below suggests that, as x approaches 0 from the left,

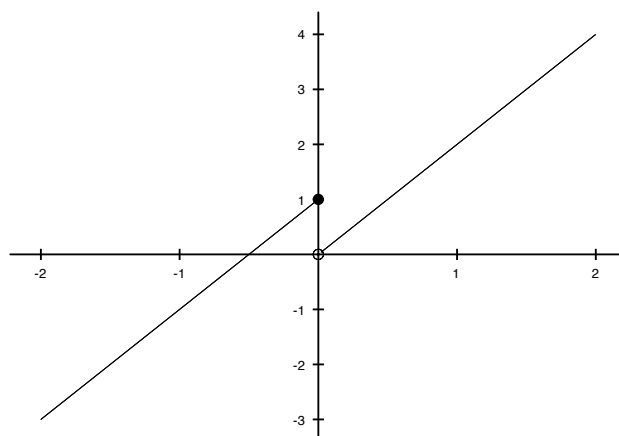
$$\lim_{x \rightarrow 0^-} f(x) = 1,$$

while, as x approaches 0 from the right,

$$\lim_{x \rightarrow 0^+} f(x) = 0.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist}.$$



31. The graph of f shown below suggests that, as x approaches 1 from the left,

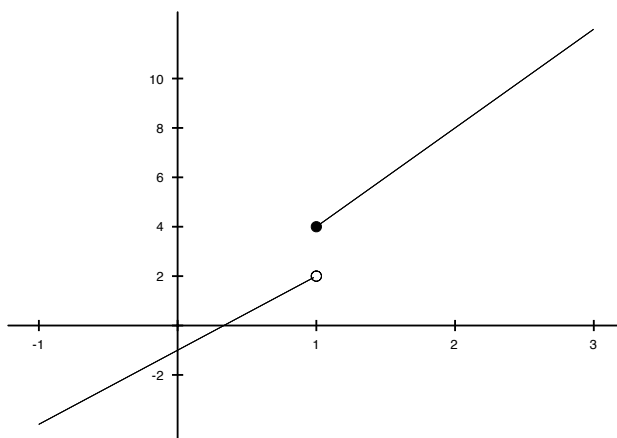
$$\lim_{x \rightarrow 1^-} f(x) = 2,$$

while, as x approaches 1 from the right,

$$\lim_{x \rightarrow 1^+} f(x) = 4.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$



32. The graph of f shown below suggests that, as x approaches 2 from the left,

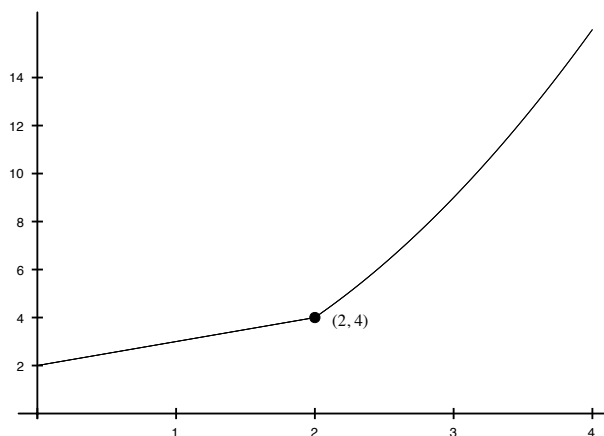
$$\lim_{x \rightarrow 2^-} f(x) = 4,$$

while, as x approaches 2 from the right,

$$\lim_{x \rightarrow 2^+} f(x) = 4.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow 2} f(x) = 4.$$



33. The graph of f shown below suggests that, as x approaches 1 from the left,

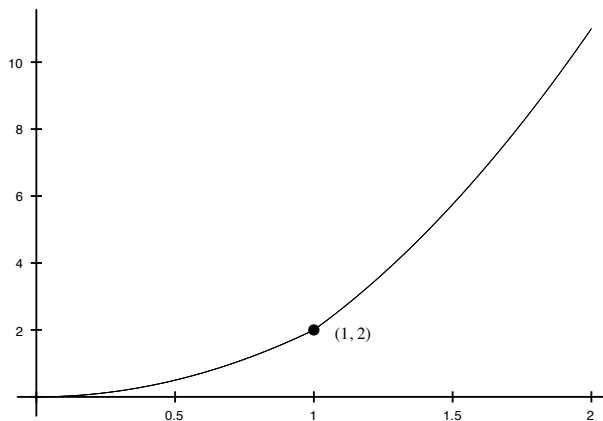
$$\lim_{x \rightarrow 1^-} f(x) = 2,$$

while, as x approaches 1 from the right,

$$\lim_{x \rightarrow 1^+} f(x) = 2.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \rightarrow 1} f(x) = \boxed{2}.$$



34. The graph of f shown below suggests that, as x approaches -1 from the left,

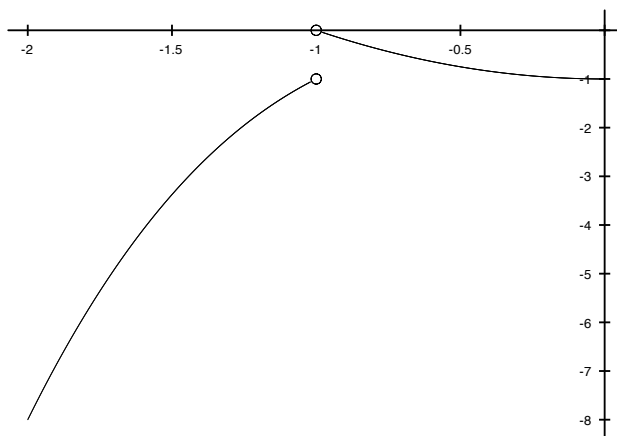
$$\lim_{x \rightarrow -1^-} f(x) = -1,$$

while, as x approaches -1 from the right,

$$\lim_{x \rightarrow -1^+} f(x) = 0.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \rightarrow -1} f(x) \text{ does not exist}.$$



35. The graph of f shown below suggests that, as x approaches 0 from the left,

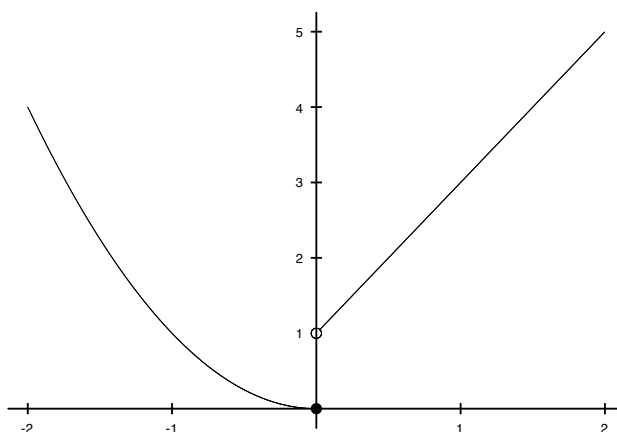
$$\lim_{x \rightarrow 0^-} f(x) = 0,$$

while, as x approaches 0 from the right,

$$\lim_{x \rightarrow 0^+} f(x) = 1.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$



36. The graph of f shown below suggests that, as x approaches 1 from the left,

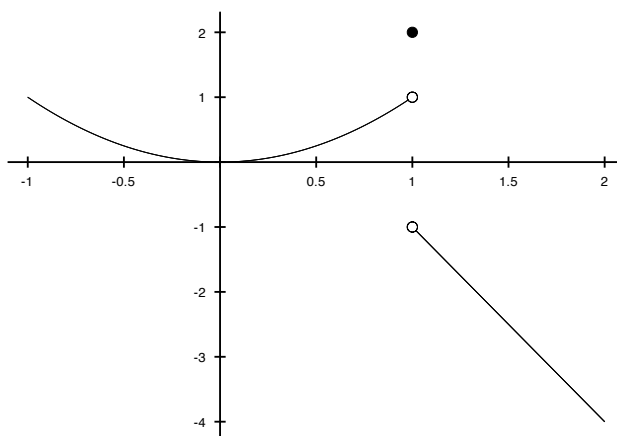
$$\lim_{x \rightarrow 1^-} f(x) = 1,$$

while, as x approaches 1 from the right,

$$\lim_{x \rightarrow 1^+} f(x) = -1.$$

Because the two one-sided limits are not equal, it follows that

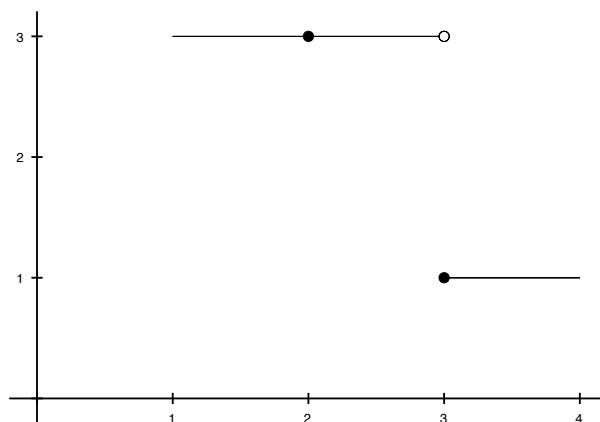
$$\lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$



Applications and Extensions

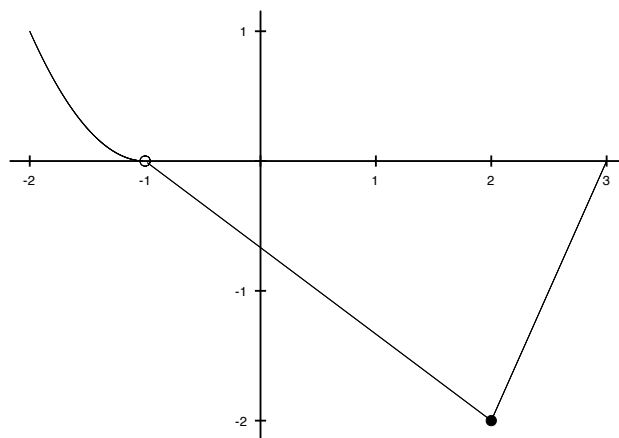
37. Answers will vary. Below is the graph of a function f for which

$$\lim_{x \rightarrow 2} f(x) = 3; \quad \lim_{x \rightarrow 3^-} f(x) = 3; \quad \lim_{x \rightarrow 3^+} f(x) = 1; \quad f(2) = 3; \quad f(3) = 1.$$



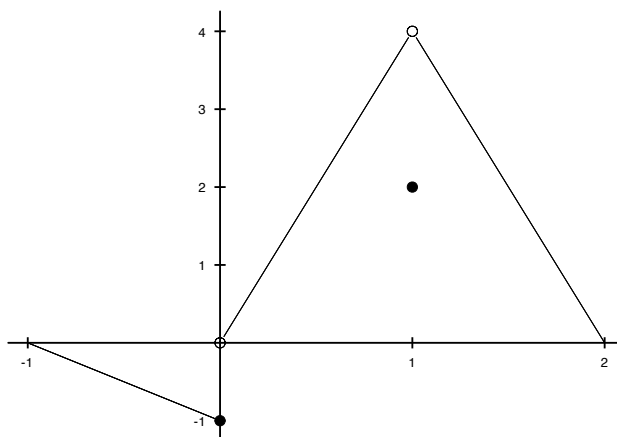
38. Answers will vary. Below is the graph of a function f for which

$$\lim_{x \rightarrow -1} f(x) = 0; \quad \lim_{x \rightarrow 2^-} f(x) = -2; \quad \lim_{x \rightarrow 2^+} f(x) = -2; \quad f(-1) \text{ is not defined}; \quad f(2) = -2.$$



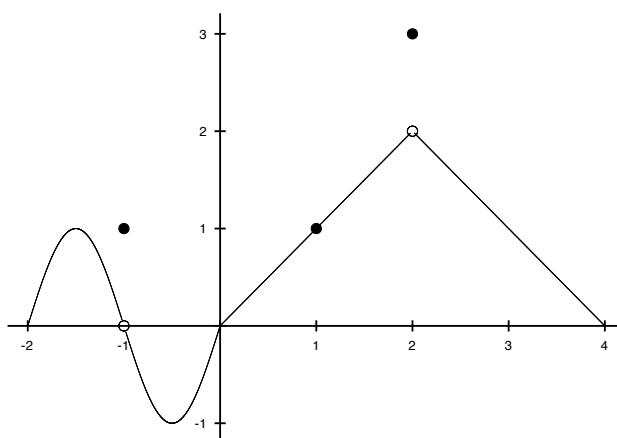
39. Answers will vary. Below is the graph of a function f for which

$$\lim_{x \rightarrow 1} f(x) = 4; \quad \lim_{x \rightarrow 0^-} f(x) = -1; \quad \lim_{x \rightarrow 0^+} f(x) = 0; \quad f(0) = -1; \quad f(1) = 2.$$



40. Answers will vary. Below is the graph of a function f for which

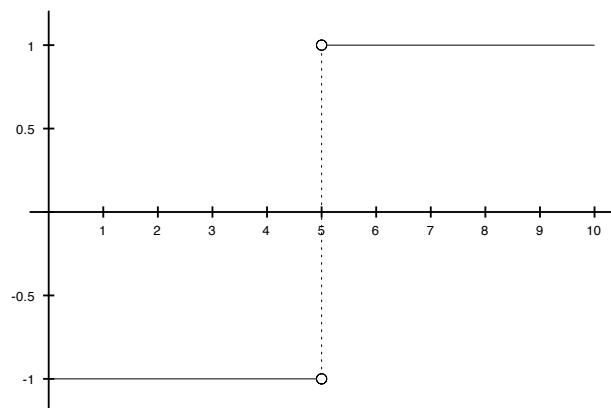
$$\lim_{x \rightarrow 2} f(x) = 2; \quad \lim_{x \rightarrow -1} f(x) = 0; \quad \lim_{x \rightarrow 1} f(x) = 1; \quad f(-1) = 1; \quad f(2) = 3.$$



41. The table of values below suggests $\lim_{x \rightarrow 5^+} \frac{|x-5|}{x-5} = \boxed{1}$.

x	$5 \leftarrow$	5.001	5.01	5.1
$f(x) = \frac{ x-5 }{x-5}$	$f(x)$ approaches 1	1	1	1

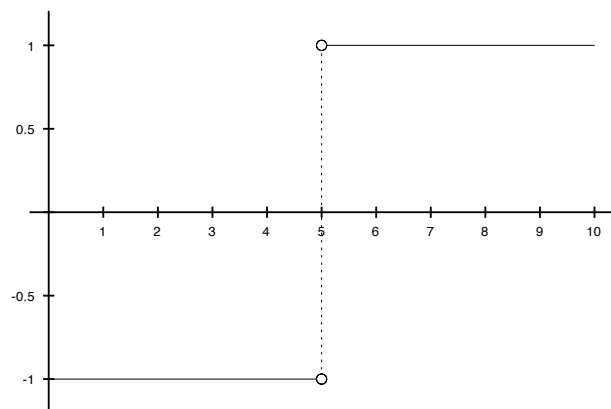
Alternately, the graph below suggests $\lim_{x \rightarrow 5^+} \frac{|x-5|}{x-5} = \boxed{1}$.



42. The table of values below suggests $\lim_{x \rightarrow 5^-} \frac{|x-5|}{x-5} = \boxed{-1}$.

x	4.9	4.99	4.999	$\rightarrow 5$
$f(x) = \frac{ x-5 }{x-5}$	-1	-1	-1	$f(x)$ approaches -1

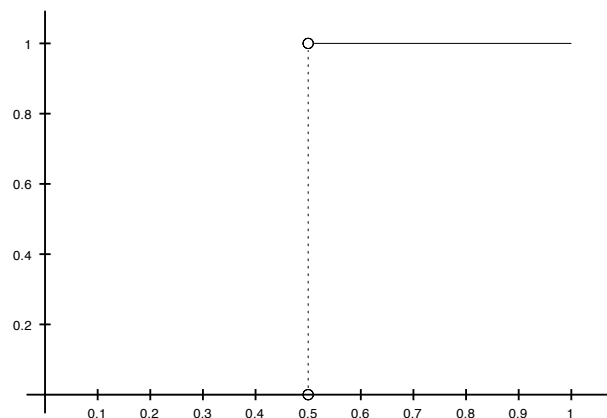
Alternately, the graph below suggests $\lim_{x \rightarrow 5^-} \frac{|x-5|}{x-5} = \boxed{-1}$.



43. The table of values below suggests $\lim_{x \rightarrow (\frac{1}{2})^-} \lfloor 2x \rfloor = \boxed{0}$.

x	0.4	0.49	0.499	$\rightarrow \frac{1}{2}$
$f(x) = \lfloor 2x \rfloor$	0	0	0	$f(x)$ approaches 0

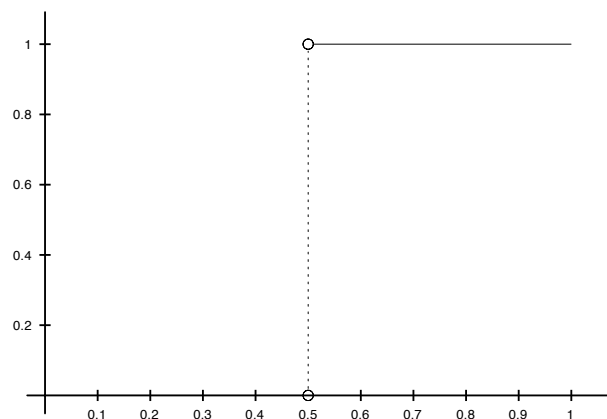
Alternately, the graph below suggests $\lim_{x \rightarrow (\frac{1}{2})^-} \lfloor 2x \rfloor = \boxed{0}$.



44. The table of values below suggests $\lim_{x \rightarrow (\frac{1}{2})^+} \lfloor 2x \rfloor = \boxed{1}$.

x	$\frac{1}{2} \leftarrow$	0.501	0.51	0.6
$f(x) = \lfloor 2x \rfloor$	$f(x)$ approaches 1	1	1	1

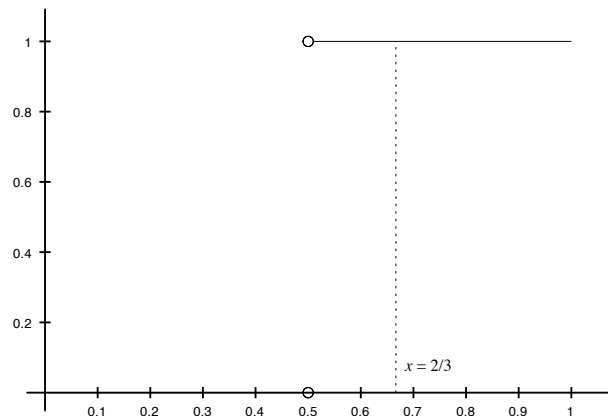
Alternately, the graph below suggests $\lim_{x \rightarrow (\frac{1}{2})^+} \lfloor 2x \rfloor = \boxed{1}$.



45. The table of values below suggests $\lim_{x \rightarrow (\frac{2}{3})^-} \lfloor 2x \rfloor = \boxed{1}$.

x	0.6	0.66	0.666	$\rightarrow \frac{2}{3}$
$f(x) = \lfloor 2x \rfloor$	1	1	1	$f(x)$ approaches 1

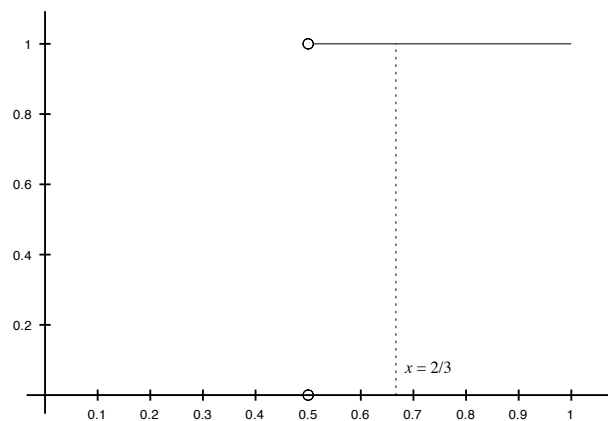
Alternately, the graph below suggests $\lim_{x \rightarrow (\frac{2}{3})^-} \lfloor 2x \rfloor = \boxed{1}$.



46. The table of values below suggests $\lim_{x \rightarrow (\frac{2}{3})^+} [2x] = \boxed{1}$.

x	$\frac{2}{3} \leftarrow$	0.667	0.67	0.7
$f(x) = [2x]$	$f(x)$ approaches 1	1	1	1

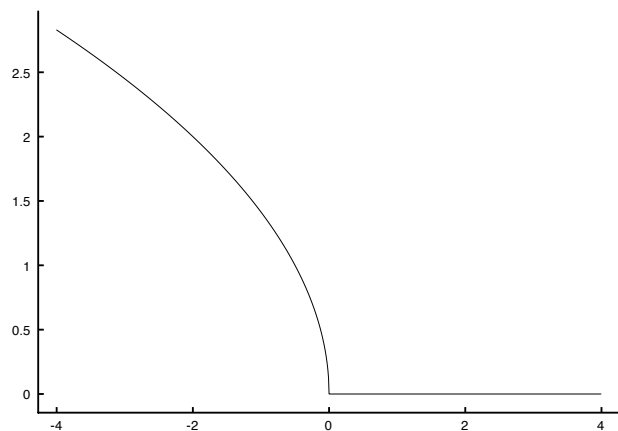
Alternately, the graph below suggests $\lim_{x \rightarrow (\frac{2}{3})^+} [2x] = \boxed{1}$.



47. The table of values below suggests $\lim_{x \rightarrow 2^+} \sqrt{|x| - x} = \boxed{0}$.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \sqrt{ x - x}$	$f(x)$ approaches 0	0	0	0

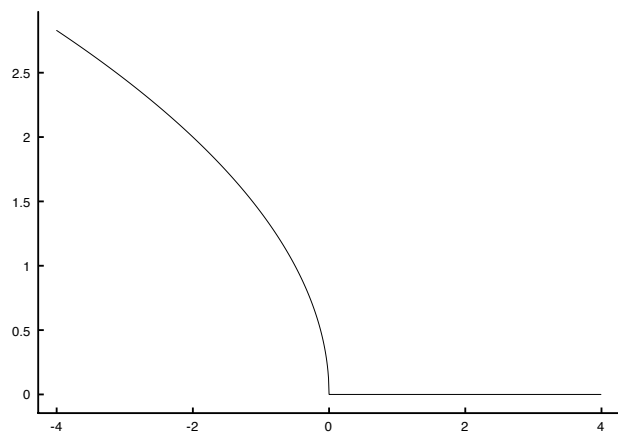
Alternately, the graph below suggests $\lim_{x \rightarrow 2^+} \sqrt{|x| - x} = \boxed{0}$.



48. The table of values below suggests $\lim_{x \rightarrow 2^-} \sqrt{|x| - x} = \boxed{0}$.

x	1.9	1.99	1.999	$\rightarrow 2$
$f(x) = \sqrt{ x - x}$	0	0	0	$f(x)$ approaches 0

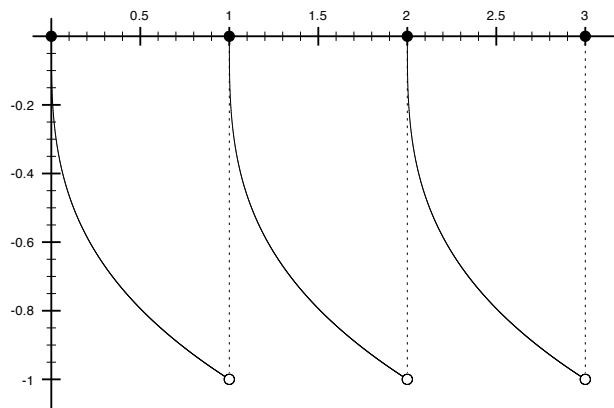
Alternately, the graph below suggests $\lim_{x \rightarrow 2^-} \sqrt{|x| - x} = \boxed{0}$.



49. The table of values below suggests $\lim_{x \rightarrow 2^+} \sqrt[3]{[x] - x} = \boxed{0}$.

x	$2 \leftarrow$	2.000000001	2.000001	2.001
$f(x) = \sqrt[3]{[x] - x}$	$f(x)$ approaches 0	-0.001	-0.01	-0.1

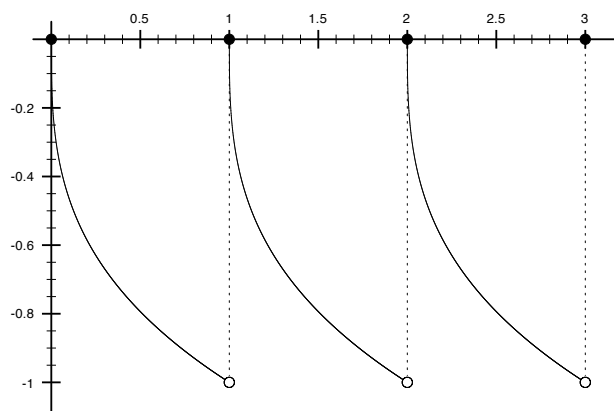
Alternately, the graph below suggests $\lim_{x \rightarrow 2^+} \sqrt[3]{[x] - x} = \boxed{0}$.



50. The table of values below, in which the function values have been rounded to five decimal places for display purposes, suggests $\lim_{x \rightarrow 2^-} \sqrt[3]{[x]} - x = \boxed{-1}$.

x	1.9	1.99	1.999	$\rightarrow 2$
$f(x) = \sqrt[3]{[x]} - x$	-0.96549	-0.99666	-0.99967	$f(x)$ approaches -1

Alternately, the graph below suggests $\lim_{x \rightarrow 2^-} \sqrt[3]{[x]} - x = \boxed{-1}$.



51. (a) The secant line containing the points (2, 12) and (3, 27) has a slope of

$$m_{\text{sec}} = \frac{27 - 12}{3 - 2} = \frac{15}{1} = \boxed{15}.$$

- (b) The secant line containing the points (2, 12) and $(x, f(x))$ for $x \neq 2$ has a slope of

$$m_{\text{sec}} = \frac{3x^2 - 12}{x - 2} = \frac{3(x - 2)(x + 2)}{x - 2} = \boxed{3(x + 2)}.$$

- (c) The values in the table below suggest that the slope of the tangent line to the graph of f at 2 is

$$\lim_{x \rightarrow 2} m_{\text{sec}} = \boxed{12}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
m_{sec}	11.7	11.97	11.997	m_{sec} approaches 12	12.003	12.03	12.3

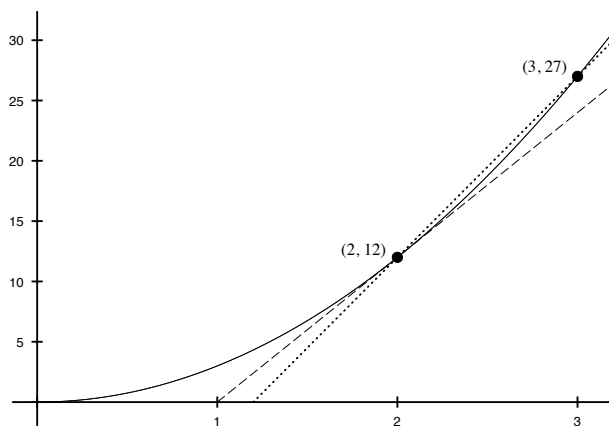
- (d) The secant line from part (a) has slope 15 and passes through the point $(2, 12)$. The equation of this secant line is therefore

$$y - 12 = 15(x - 2) \quad \text{or} \quad y = 15x - 18.$$

The tangent line at $x = 2$ has slope 12 and also passes through the point $(2, 12)$; the equation of this line is

$$y - 12 = 12(x - 2) \quad \text{or} \quad y = 12x - 12.$$

The figure below displays the graph of f as the solid curve, the graph of the tangent line as the dashed curve, and the graph of the secant line as the dotted curve.



52. (a) The secant line containing the points $(2, 8)$ and $(3, 27)$ has a slope of

$$m_{\text{sec}} = \frac{27 - 8}{3 - 2} = \frac{19}{1} = \boxed{19}.$$

- (b) The secant line containing the points $(2, 8)$ and $(x, f(x))$ for $x \neq 2$ has a slope of

$$m_{\text{sec}} = \frac{x^3 - 8}{x - 2} = \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = \boxed{x^2 + 2x + 4}.$$

- (c) The values in the table below suggest that the slope of the tangent line to the graph of f at 2 is

$$\lim_{x \rightarrow 2} m_{\text{sec}} = \boxed{12}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
m_{sec}	11.41	11.9401	11.994001	m_{sec} approaches 12	12.006001	12.0601	12.61

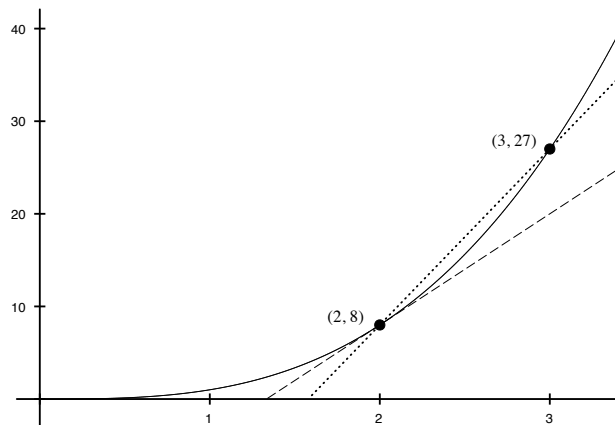
- (d) The secant line from part (a) has slope 19 and passes through the point $(2, 8)$. The equation of this secant line is therefore

$$y - 8 = 19(x - 2) \quad \text{or} \quad y = 19x - 30.$$

The tangent line at $x = 2$ has slope 12 and also passes through the point $(2, 8)$; the equation of this line is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

The figure below displays the graph of f as the solid curve, the graph of the tangent line as the dashed curve, and the graph of the secant line as the dotted curve.



53. (a) Let $f(x) = \frac{1}{2}x^2 - 1$. The slope of the secant line containing the points $P = (2, f(2))$ and $Q = (2 + h, f(2 + h))$ is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{\frac{1}{2}(2+h)^2 - 1 - (\frac{1}{2}2^2 - 1)}{h} \\ &= \frac{\frac{1}{2}(4 + 4h + h^2) - 1 - \frac{1}{2}2^2 + 1}{h} = \frac{2 + 2h + \frac{1}{2}h^2 - 2}{h} = \frac{2h + \frac{1}{2}h^2}{h} = \boxed{2 + \frac{1}{2}h}, \end{aligned}$$

provided $h \neq 0$.

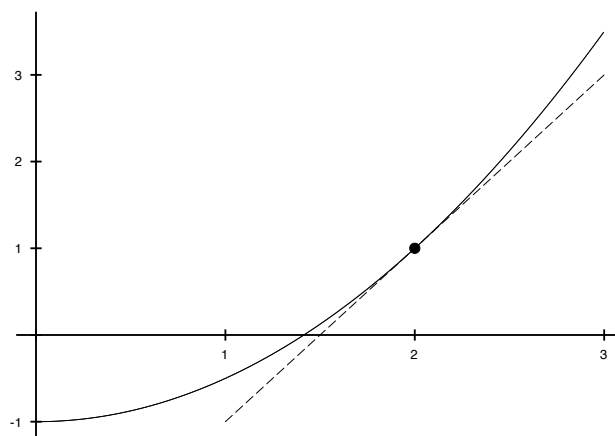
- (b) Using the result from part (a),

h	-0.5	-0.1	-0.001	0.001	0.1	0.5
m_{sec}	1.75	1.95	1.9995	2.0005	2.05	2.25

- (c) The table from part (b) suggests that $\lim_{h \rightarrow 0} m_{\text{sec}} = \boxed{2}$.
- (d) Because the limit of the slope of the secant line is 2, the slope of the line tangent to the graph of f at the point $P = (2, f(2))$ is $\boxed{2}$.
- (e) The tangent line to f at the point $P = (2, f(2))$ has slope 2 and contains the point $(2, f(2)) = (2, 1)$. The equation of the tangent line is therefore

$$y - 1 = 2(x - 2) \quad \text{or} \quad y = 2x - 3.$$

The figure below displays the graph of f as the solid curve and the graph of the tangent line as the dashed curve.



54. (a) Let $f(x) = x^2 - 1$. The slope of the secant line containing the points $P = (-1, f(-1))$ and $Q = (-1 + h, f(-1 + h))$ is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(-1 + h) - f(-1)}{(-1 + h) - (-1)} = \frac{(-1 + h)^2 - 1 - ((-1)^2 - 1)}{h} \\ &= \frac{1 - 2h + h^2 - 1 - 1 + 1}{h} = \frac{-2h + h^2}{h} = \boxed{-2 + h}, \end{aligned}$$

provided $h \neq 0$.

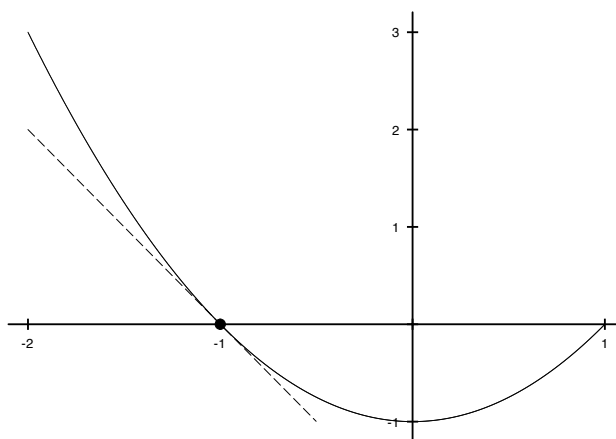
- (b) Using the result from part (a),

h	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
m_{sec}	-2.1	-2.01	-2.001	-2.0001	-1.9999	-1.999	-1.99	-1.9

- (c) The table from part (b) suggests that $\lim_{h \rightarrow 0} m_{\text{sec}} = \boxed{-2}$.
- (d) Because the limit of the slope of the secant line is -2 , the slope of the line tangent to the graph of f at the point $P = (-1, f(-1))$ is $\boxed{-2}$.
- (e) The tangent line to f at the point $P = (-1, f(-1))$ has slope -2 and contains the point $(-1, f(-1)) = (-1, 0)$. The equation of the tangent line is therefore

$$y - 0 = -2(x + 1) \quad \text{or} \quad y = -2x - 2.$$

The figure below displays the graph of f as the solid curve and the graph of the tangent line as the dashed curve.



55. (a) The values in the table below suggest that

$$\lim_{x \rightarrow 0} \cos \frac{\pi}{x} = \boxed{1}.$$

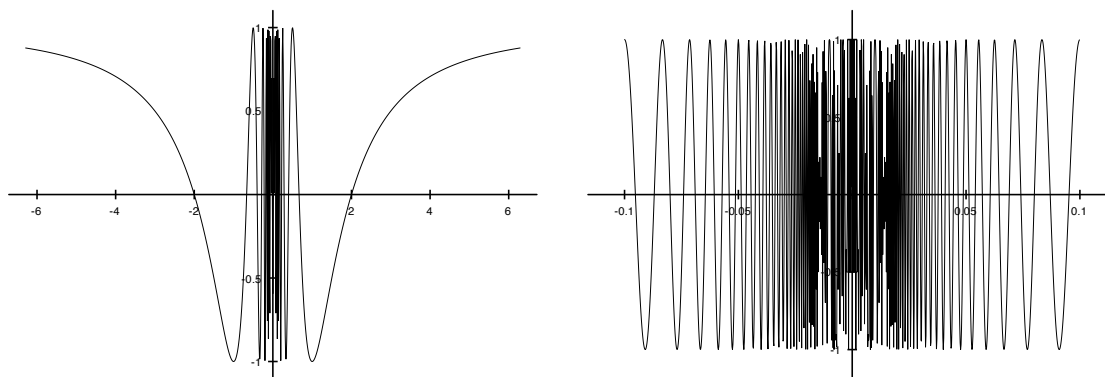
x	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$\rightarrow 0 \leftarrow$	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$f(x) = \cos \frac{\pi}{x}$	1	1	1	1	1	$f(x)$ approaches 1	1	1	1	1	1

- (b) The values in the table below suggest that

$$\lim_{x \rightarrow 0} \cos \frac{\pi}{x} = \boxed{-1}.$$

x	-1	$-\frac{1}{3}$	$-\frac{1}{5}$	$-\frac{1}{7}$	$-\frac{1}{9}$	$\rightarrow 0 \leftarrow$	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$	1
$f(x) = \cos \frac{\pi}{x}$	-1	-1	-1	-1	-1	f approaches	-1	-1	-1	-1	-1

- (c) Because the values obtained in parts (a) and (b) are not equal, we conclude the limit does not exist. As x gets closer to 0, the argument to the cosine function, $\frac{\pi}{x}$, becomes unbounded. Consequently, the cosine function oscillates repeatedly between -1 and 1 and never approaches a single value. A table of values can be a useful tool for investigating a limit, but should only be viewed as providing evidence of a possible value for a limit. A final conclusion regarding a limit should be based on the properties of limits that will be developed in subsequent sections of this chapter.
- (d) The figure below left displays the graph of f with an x -window of $(-2\pi, 2\pi)$, and the figure below right displays the graph of f with an x -window of $(-0.1, 0.1)$. Using either graph, it appears that $\lim_{x \rightarrow 0} f(x)$ does not exist because the function value does not approach a single number. Instead, the function seems to oscillate more rapidly between -1 and 1 as x gets closer to 0.



56. (a) The values in the table below suggest that

$$\lim_{x \rightarrow 0} \cos \frac{\pi}{x^2} = \boxed{1}.$$

x	-0.1	-0.01	-0.001	-0.0001	$\rightarrow 0 \leftarrow$	0.0001	0.001	0.01	0.1
$f(x) = \cos \frac{\pi}{x^2}$	1	1	1	1	$f(x)$ approaches	1	1	1	1

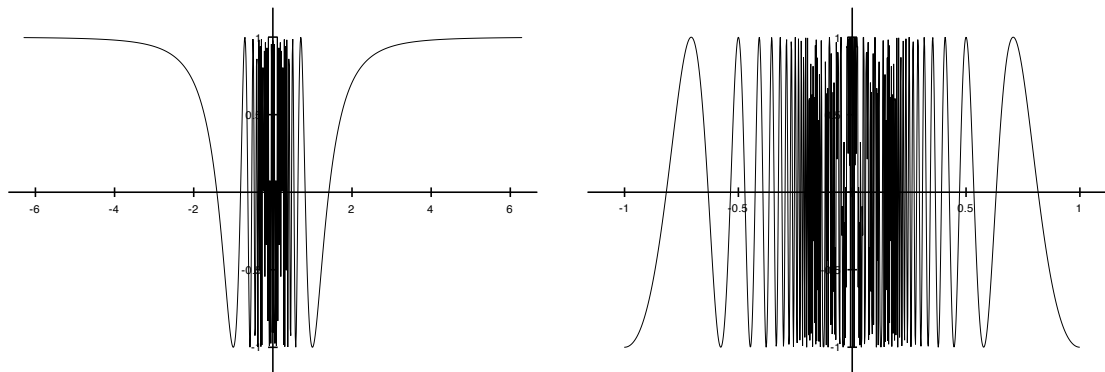
- (b) The values in the table below suggest that

$$\lim_{x \rightarrow 0} \cos \frac{\pi}{x^2} = \boxed{\frac{\sqrt{2}}{2}}.$$

x	$-\frac{2}{3}$	$-\frac{2}{5}$	$-\frac{2}{7}$	$-\frac{2}{9}$	$\rightarrow 0 \leftarrow$	$\frac{2}{9}$	$\frac{2}{7}$	$\frac{2}{5}$	$\frac{2}{3}$
$f(x) = \cos \frac{\pi}{x^2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$f(x)$ approaches	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

- (c) Because the values obtained in parts (a) and (b) are not equal, we conclude the limit does not exist. As x gets closer to 0, the argument to the cosine function, $\frac{\pi}{x^2}$, becomes unbounded. Consequently, the cosine function oscillates repeatedly between -1 and 1 and never approaches a single value. A table of values can be a useful tool for investigating a limit, but should only be viewed as providing evidence of a possible value for a limit. A final conclusion regarding a limit should be based on the properties of limits that will be developed in subsequent sections of this chapter.

- (d) The figure below left displays the graph of f with an x -window of $(-2\pi, 2\pi)$, and the figure below right displays the graph of f with an x -window of $(-1, 1)$. Using either graph, it appears that $\lim_{x \rightarrow 0} f(x)$ does not exist because the function value does not approach a single number. Instead, the function seems to oscillate more rapidly between -1 and 1 as x gets closer to 0 .



57. (a) The values in the table below suggest that

$$\lim_{x \rightarrow 2} \frac{x-8}{2} = \boxed{-3}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x) = \frac{x-8}{2}$	-3.05	-3.005	-3.0005	$f(x)$ approaches -3	-2.9995	-2.995	-2.95

- (b) The function $f(x) = \frac{x-8}{2}$ is within 0.1 of -3 provided $|f(x) - (-3)| \leq 0.1$; that is,

$$\begin{aligned} \left| \frac{x-8}{2} + 3 \right| &\leq 0.1 \\ |(x-8) + 6| &\leq 0.2 \\ |x-2| &\leq 0.2. \end{aligned}$$

Thus, if $\boxed{1.8 \leq x \leq 2.2}$, then $f(x)$ is within 0.1 of -3 .

- (c) The function $f(x) = \frac{x-8}{2}$ is within 0.01 of -3 provided $|f(x) - (-3)| \leq 0.01$; that is,

$$\begin{aligned} \left| \frac{x-8}{2} + 3 \right| &\leq 0.01 \\ |(x-8) + 6| &\leq 0.02 \\ |x-2| &\leq 0.02. \end{aligned}$$

Thus, if $\boxed{1.98 \leq x \leq 2.02}$, then $f(x)$ is within 0.01 of -3 .

58. (a) The values in the table below suggest that

$$\lim_{x \rightarrow 2} (5 - 2x) = \boxed{1}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x) = 5 - 2x$	1.2	1.02	1.002	$f(x)$ approaches 1	0.998	0.98	0.8

- (b) The function $f(x) = 5 - 2x$ is within 0.1 of 1 provided $|f(x) - 1| \leq 0.1$; that is,

$$\begin{aligned} |(5 - 2x) - 1| &\leq 0.1 \\ |4 - 2x| &\leq 0.1 \\ |-2(x - 2)| &\leq 0.1 \\ |x - 2| &\leq 0.05. \end{aligned}$$

Thus, if $\boxed{1.95 \leq x \leq 2.05}$, then $f(x)$ is within 0.1 of 1.

- (c) The function $f(x) = 5 - 2x$ is within 0.01 of 1 provided $|f(x) - 1| \leq 0.01$; that is,

$$\begin{aligned} |(5 - 2x) - 1| &\leq 0.01 \\ |4 - 2x| &\leq 0.01 \\ |-2(x - 2)| &\leq 0.01 \\ |x - 2| &\leq 0.005. \end{aligned}$$

Thus, if $\boxed{1.995 \leq x \leq 2.005}$, then $f(x)$ is within 0.01 of 1.

59. (a) Because the Postal Service rounds the weight of the letter up to the next whole number of ounces, the first-class postage charged is

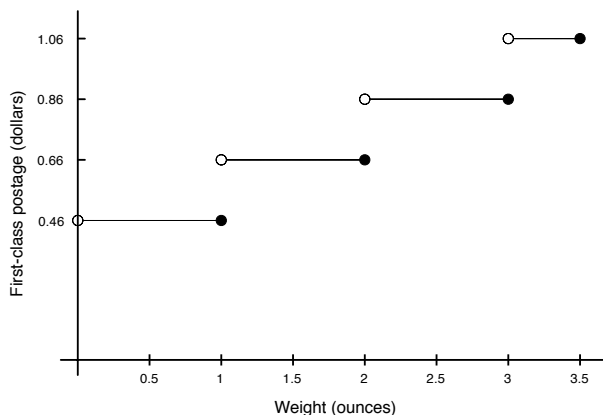
$$C(w) = \begin{cases} 0.46, & \text{if } 0 < w \leq 1 \\ 0.66, & \text{if } 1 < w \leq 2 \\ 0.86, & \text{if } 2 < w \leq 3 \\ 1.06, & \text{if } 3 < w \leq 3.5. \end{cases}$$

where postage is measured in dollars and weight is measured in ounces. This can be written compactly in terms of the ceiling function as

$$C(w) = 0.46 + 0.20\lceil w - 1 \rceil.$$

- (b) The domain of C is the set $\boxed{\{w | 0 < w \leq 3.5\}}$.

- (c) The graph of the postage function C is shown below.



- (d) The graph of C suggests that

$$\boxed{\lim_{w \rightarrow 2^-} C(w) = 0.66 \quad \text{and} \quad \lim_{w \rightarrow 2^+} C(w) = 0.86}.$$

Because these two one-sided limits are not equal, this suggests that $\boxed{\lim_{w \rightarrow 2} C(w) \text{ does not exist}}.$

(e) The graph of C suggests that $\lim_{w \rightarrow 0^+} C(w) = 0.46$.

(f) The graph of C suggests that $\lim_{w \rightarrow 3.5^-} C(w) = 1.06$.

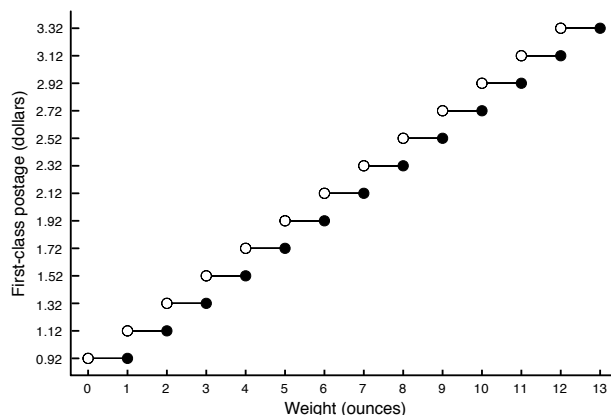
60. (a) Because the Postal Service rounds the weight of the letter up to the next whole number of ounces, the first-class postage charged is

$$C(w) = 0.92 + 0.20 \lceil w - 1 \rceil,$$

for $0 < w \leq 13$, where postage is measured in dollars and weight is measured in ounces.

(b) The domain of C is the set $\{w | 0 < w \leq 13\}$.

(c) The graph of the postage function C is shown below.



(d) The graph of C suggests that

$$\lim_{w \rightarrow 1^-} C(w) = 0.92 \quad \text{and} \quad \lim_{w \rightarrow 1^+} C(w) = 1.12.$$

Because these two one-sided limits are not equal, this suggests that $\lim_{w \rightarrow 1} C(w)$ does not exist.

(e) The graph of C suggests that

$$\lim_{w \rightarrow 12^-} C(w) = 3.12 \quad \text{and} \quad \lim_{w \rightarrow 12^+} C(w) = 3.32.$$

Because these two one-sided limits are not equal, this suggests that $\lim_{w \rightarrow 12} C(w)$ does not exist.

(f) The graph of C suggests that $\lim_{w \rightarrow 0^+} C(w) = 0.92$.

(g) The graph of C suggests that $\lim_{w \rightarrow 13^-} C(w) = 3.32$.

61. Let $S(t)$ denote a student's final exam score as a function of the time t that the student studies.

(a) Professor Smith's claim can be written symbolically as

$$\lim_{t \rightarrow 7} S(t) = 100.$$

(b) Using the ϵ - δ definition of a limit, Professor Smith's can be written as:
 given any $\epsilon > 0$, there is a number $\delta > 0$ so that whenever $0 < |t - 7| < \delta$ then $|S(t) - 100| < \epsilon$.

62. If $h = x - c$, then $x = h + c$ and h approaches zero as x approaches c . Thus,

$$m_{\tan} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{(c + h) - c} = \boxed{\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h}}.$$

63. No, the value of the function f **at** $x = 2$ has no bearing on $\lim_{x \rightarrow 2} f(x)$. The limit examines the value of f as x **approaches** 2 but is not equal to 2.

64. No, $\lim_{x \rightarrow 2} f(x)$ conveys no information about the value of f **at** $x = 2$. The limit examines the value of f as x **approaches** 2 but is not equal to 2.

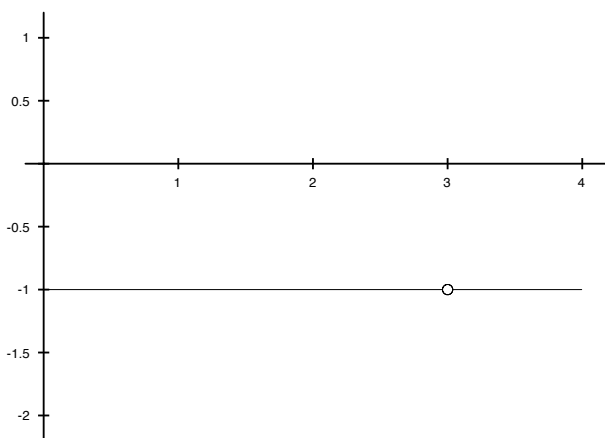
65. (a) Because

$$\frac{x - 3}{3 - x} = \frac{-(3 - x)}{3 - x} = -1$$

provided $x \neq 3$, the graph of $f(x)$ is the horizontal line $y = -1$ excluding the point $(3, -1)$.

- (b) The graph of f (see below) suggests that

$$\boxed{\lim_{x \rightarrow 3^-} f(x) = -1 \quad \text{and} \quad \lim_{x \rightarrow 3^+} f(x) = -1}.$$



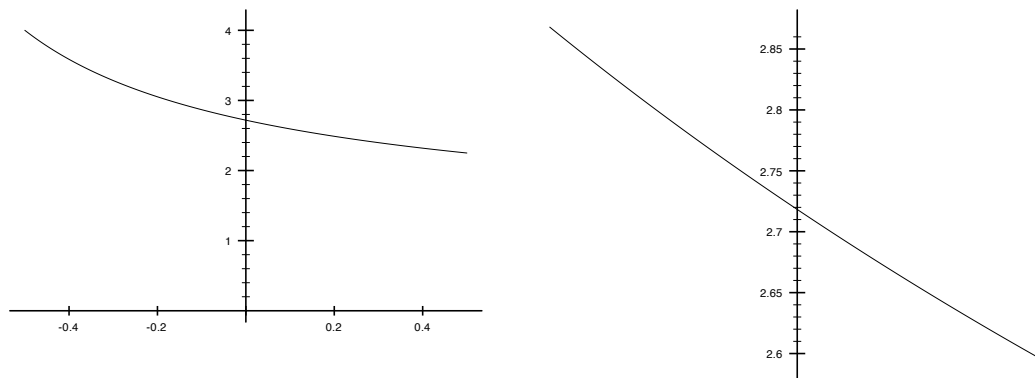
- (c) Because the two one-sided limits are equal, the graph suggests that $\lim_{x \rightarrow 3} f(x)$ exists and is equal to -1 .

66. (a) The values in the table below, which have been rounded to five decimal places for display purposes, suggest that

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} \approx \boxed{2.72}.$$

x	-0.01	-0.001	-0.0001	$\rightarrow 0 \leftarrow$	0.0001	0.001	0.01
$f(x) = (1 + x)^{1/x}$	2.731999	2.719642	2.718418		2.718146	2.716924	2.704814

- (b) The figure below left displays the graph of g for $-0.5 \leq x \leq 0.5$. The figure below right displays a closer view of the graph of g , focusing on the y -intercept.



(c) The table in part (a) and the graphs in part (b) all suggest that

$$\lim_{x \rightarrow 0} (1+x)^{1/x} \approx \boxed{2.72}.$$

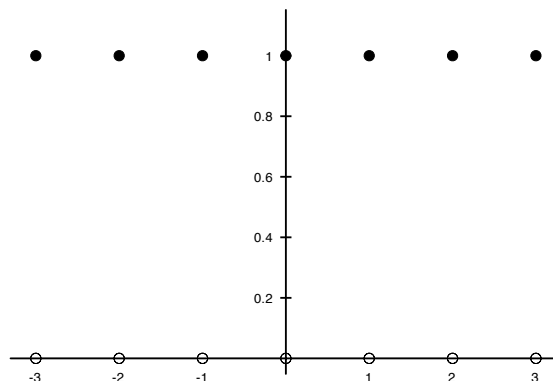
(d) Using the computer algebra system *Mathematica*,

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = \boxed{e \approx 2.71828}.$$

Challenge Problems

67. The graph of the function f is shown below. Note that except when x is an integer, the graph of f coincides with the x -axis. The graph suggests that

$$\lim_{x \rightarrow 2} f(x) = \boxed{0}.$$



68. The graph of f (see Problem 67) suggests

$$\lim_{x \rightarrow 1/2} f(x) = \boxed{0}.$$

69. The graph of f (see Problem 67) suggests

$$\lim_{x \rightarrow 3} f(x) = \boxed{0}.$$

70. The graph of f (see Problem 67) suggests

$$\lim_{x \rightarrow 0} f(x) = \boxed{0}.$$

1.2: Limits of Functions Using Properties of Limits

Concepts and Vocabulary

- $\lim_{x \rightarrow 4} (-3) = \boxed{-3}$.
 - $\lim_{x \rightarrow 0} \pi = \boxed{\pi}$.
- If $\lim_{x \rightarrow c} f(x) = 3$, then $\lim_{x \rightarrow c} [f(x)]^5 = 3^5 = \boxed{243}$.
- If $\lim_{x \rightarrow c} f(x) = 64$, then $\lim_{x \rightarrow c} \sqrt[3]{f(x)} = \sqrt[3]{64} = \boxed{4}$.
- $\lim_{x \rightarrow -1} x = \boxed{-1}$.
 - $\lim_{x \rightarrow e} x = \boxed{e}$.
- $\lim_{x \rightarrow 0} (x - 2) = 0 - 2 = \boxed{-2}$.
 - $\lim_{x \rightarrow \frac{1}{2}} (3 + x) = 3 + \frac{1}{2} = \boxed{\frac{7}{2}}$.
- $\lim_{x \rightarrow 2} (-3x) = -3(2) = \boxed{-6}$.
 - $\lim_{x \rightarrow 0} (3x) = 3(0) = \boxed{0}$.
- $\boxed{\text{True}}$. If p is a polynomial, then $\lim_{x \rightarrow 5} p(x) = p(5)$.
- If the domain of a rational function R is $\{x | x \neq 0\}$, then $\lim_{x \rightarrow 2} R(x) = R(\boxed{2})$.
- $\boxed{\text{False}}$. Properties of limits **can** be used for one-sided limits.
- $\boxed{\text{True}}$. If $f(x) = \frac{(x+1)(x+2)}{x+1}$ and $g(x) = x+2$, then $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = 1$.

Skill Building

$$11. \lim_{x \rightarrow 3} [2(x+4)] = 2 \lim_{x \rightarrow 3} (x+4) = 2 \left[\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 4 \right] = 2(3+4) = \boxed{14}.$$

$$12. \lim_{x \rightarrow -2} [3(x+1)] = 3 \lim_{x \rightarrow -2} (x+1) = 3 \left[\lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \right] = 3(-2+1) = \boxed{-3}.$$

13.

$$\begin{aligned} \lim_{x \rightarrow -2} [x(3x-1)(x+2)] &= \lim_{x \rightarrow -2} x \cdot \lim_{x \rightarrow -2} (3x-1) \cdot \lim_{x \rightarrow -2} (x+2) \\ &= \lim_{x \rightarrow -2} x \cdot \left(3 \lim_{x \rightarrow -2} x - \lim_{x \rightarrow -2} 1 \right) \cdot \left(\lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 2 \right) \\ &= -2[3(-2) - 1][-2 + 2] = -2(-7)(0) = \boxed{0}. \end{aligned}$$

14.

$$\begin{aligned} \lim_{x \rightarrow -1} [x(x-1)(x+10)] &= \lim_{x \rightarrow -1} x \cdot \lim_{x \rightarrow -1} (x-1) \cdot \lim_{x \rightarrow -1} (x+10) \\ &= \lim_{x \rightarrow -1} x \cdot \left(\lim_{x \rightarrow -1} x - \lim_{x \rightarrow -1} 1 \right) \cdot \left(\lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 10 \right) \\ &= -1(-1-1)(-1+10) = -1(-2)(9) = \boxed{18}. \end{aligned}$$

$$15. \lim_{t \rightarrow 1} (3t - 2)^3 = \left[\lim_{t \rightarrow 1} (3t - 2) \right]^3 = [3(1) - 2]^3 = 1^3 = \boxed{1}.$$

$$16. \lim_{x \rightarrow 0} (-3x + 1)^2 = \left[\lim_{x \rightarrow 0} (-3x + 1) \right]^2 = [-3(0) + 1]^2 = 1^2 = \boxed{1}.$$

$$17. \lim_{x \rightarrow 4} 3\sqrt{x} = 3\sqrt{\lim_{x \rightarrow 4} x} = 3\sqrt{4} = \boxed{6}.$$

$$18. \lim_{x \rightarrow 8} \left(\frac{1}{4} \sqrt[3]{x} \right) = \frac{1}{4} \sqrt[3]{\lim_{x \rightarrow 8} x} = \frac{1}{4} \sqrt[3]{8} = \frac{1}{4} \cdot 2 = \boxed{\frac{1}{2}}.$$

$$19. \lim_{x \rightarrow 3} \sqrt{5x - 4} = \sqrt{\lim_{x \rightarrow 3} (5x - 4)} = \sqrt{5(3) - 4} = \boxed{\sqrt{11}}.$$

$$20. \lim_{t \rightarrow 2} \sqrt{3t + 4} = \sqrt{\lim_{t \rightarrow 2} (3t + 4)} = \sqrt{3(2) + 4} = \boxed{\sqrt{10}}.$$

21.

$$\begin{aligned} \lim_{t \rightarrow 2} [t \sqrt{(5t + 3)(t + 4)}] &= \lim_{t \rightarrow 2} t \cdot \lim_{t \rightarrow 2} \sqrt{(5t + 3)(t + 4)} = 2 \sqrt{\lim_{t \rightarrow 2} [(5t + 3)(t + 4)]} \\ &= 2 \sqrt{\lim_{t \rightarrow 2} (5t + 3) \cdot \lim_{t \rightarrow 2} (t + 4)} = 2 \sqrt{(5(2) + 3)(2 + 4)} = \boxed{2\sqrt{78}}. \end{aligned}$$

22.

$$\begin{aligned} \lim_{t \rightarrow -1} [t \sqrt[3]{(t + 1)(2t - 1)}] &= \lim_{t \rightarrow -1} t \cdot \lim_{t \rightarrow -1} \sqrt[3]{(t + 1)(2t - 1)} = -1 \sqrt[3]{\lim_{t \rightarrow -1} [(t + 1)(2t - 1)]} \\ &= - \sqrt[3]{\lim_{t \rightarrow -1} (t + 1) \cdot \lim_{t \rightarrow -1} (2t - 1)} = - \sqrt[3]{(-1 + 1)(2(-1) - 1)} = - \sqrt[3]{0} = \boxed{0}. \end{aligned}$$

23.

$$\begin{aligned} \lim_{x \rightarrow 3} (\sqrt{x} + x + 4)^{1/2} &= \left[\lim_{x \rightarrow 3} (\sqrt{x} + x + 4) \right]^{1/2} = \left[\lim_{x \rightarrow 3} \sqrt{x} + \lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 4 \right]^{1/2} \\ &= \left[\sqrt{\lim_{x \rightarrow 3} x} + 3 + 4 \right]^{1/2} = \boxed{(\sqrt{3} + 7)^{1/2}}. \end{aligned}$$

24.

$$\begin{aligned} \lim_{t \rightarrow 2} (t\sqrt{2t} + 4)^{1/3} &= \left[\lim_{t \rightarrow 2} (t\sqrt{2t} + 4) \right]^{1/3} = \left[\lim_{t \rightarrow 2} t\sqrt{2t} + \lim_{t \rightarrow 2} 4 \right]^{1/3} = \left[\lim_{t \rightarrow 2} t \cdot \lim_{t \rightarrow 2} \sqrt{2t} + 4 \right]^{1/3} \\ &= \left[2 \cdot \sqrt{\lim_{t \rightarrow 2} (2t)} + 4 \right]^{1/3} = \left[2\sqrt{2(2)} + 4 \right]^{1/3} = 8^{1/3} = \boxed{2}. \end{aligned}$$

25.

$$\begin{aligned} \lim_{t \rightarrow -1} [4t(t + 1)]^{2/3} &= \left[\lim_{t \rightarrow -1} 4t(t + 1) \right]^{2/3} = \left[\lim_{t \rightarrow -1} (4t) \cdot \lim_{t \rightarrow -1} (t + 1) \right]^{2/3} \\ &= \left[4 \lim_{t \rightarrow -1} t \cdot \left(\lim_{t \rightarrow -1} t + \lim_{t \rightarrow -1} 1 \right) \right]^{2/3} = [4(-1)(-1 + 1)]^{2/3} = 0^{2/3} = \boxed{0}. \end{aligned}$$

$$26. \lim_{x \rightarrow 0} (x^2 - 2x)^{3/5} = \left[\lim_{x \rightarrow 0} (x^2 - 2x) \right]^{3/5} = \left[\left(\lim_{x \rightarrow 0} x \right)^2 - 2 \lim_{x \rightarrow 0} x \right]^{3/5} = (0^2 - 2(0))^{3/5} = 0^{3/5} = \boxed{0}.$$

$$27. \lim_{t \rightarrow 1} (3t^2 - 2t + 4) = 3(1)^2 - 2(1) + 4 = \boxed{5}.$$

$$28. \lim_{x \rightarrow 0} (-3x^4 + 2x + 1) = -3(0)^4 + 2(0) + 1 = \boxed{1}.$$

$$29. \lim_{x \rightarrow \frac{1}{2}} (2x^4 - 8x^3 + 4x - 5) = 2 \left(\frac{1}{2} \right)^4 - 8 \left(\frac{1}{2} \right)^3 + 4 \left(\frac{1}{2} \right) - 5 = \frac{1}{8} - 1 + 2 - 5 = \boxed{-\frac{31}{8}}.$$

$$30. \lim_{x \rightarrow -\frac{1}{3}} (27x^3 + 9x + 1) = 27 \left(-\frac{1}{3} \right)^3 + 9 \left(-\frac{1}{3} \right) + 1 = -1 - 3 + 1 = \boxed{-3}.$$

31. Because the limit of the denominator $\lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2 \neq 0$,

$$\lim_{x \rightarrow 4} \frac{x^2 + 4}{\sqrt{x}} = \frac{\lim_{x \rightarrow 4} (x^2 + 4)}{\lim_{x \rightarrow 4} \sqrt{x}} = \frac{4^2 + 4}{\sqrt{\lim_{x \rightarrow 4} x}} = \frac{20}{\sqrt{4}} = \frac{20}{2} = \boxed{10}.$$

32. Because the limit of the denominator $\lim_{x \rightarrow 3} \sqrt{3x} = \sqrt{9} = 3 \neq 0$,

$$\lim_{x \rightarrow 3} \frac{x^2 + 5}{\sqrt{3x}} = \frac{\lim_{x \rightarrow 3} (x^2 + 5)}{\lim_{x \rightarrow 3} \sqrt{3x}} = \frac{3^2 + 5}{\sqrt{\lim_{x \rightarrow 3} 3x}} = \frac{14}{\sqrt{9}} = \boxed{\frac{14}{3}}.$$

33. Because $x = -2$ is in the domain of the rational function $R(x) = \frac{2x^3 + 5x}{3x - 2}$,

$$\lim_{x \rightarrow -2} R(x) = R(-2) = \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} = \frac{-26}{-8} = \boxed{\frac{13}{4}}.$$

34. Because $x = 1$ is in the domain of the rational function $R(x) = \frac{2x^4 - 1}{3x^3 + 2}$,

$$\lim_{x \rightarrow 1} R(x) = R(1) = \frac{2(1)^4 - 1}{3(1)^3 + 2} = \boxed{\frac{1}{5}}.$$

35. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = \boxed{4}.$$

36. Observe that the limit of the denominator is equal to zero. Factoring the denominator and simplifying yields

$$\lim_{x \rightarrow -2} \frac{x + 2}{x^2 - 4} = \lim_{x \rightarrow -2} \frac{x + 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow -2} \frac{1}{x - 2} = \boxed{-\frac{1}{4}}.$$

37. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} = \lim_{x \rightarrow -1} \frac{x(x - 1)(x + 1)}{x + 1} = \lim_{x \rightarrow -1} [x(x - 1)] = (-1)(-2) = \boxed{2}.$$

38. Observe that the limit of the denominator is equal to zero. Factoring both the numerator and the denominator and simplifying yields

$$\lim_{x \rightarrow -1} \frac{x^3 + x^2}{x^2 - 1} = \lim_{x \rightarrow -1} \frac{x^2(x + 1)}{(x - 1)(x + 1)} = \lim_{x \rightarrow -1} \frac{x^2}{x - 1} = \frac{(-1)^2}{-2} = \boxed{-\frac{1}{2}}.$$

39. Observe that

$$\frac{2x}{x+8} + \frac{16}{x+8} = \frac{2x+16}{x+8} = \frac{2(x+8)}{x+8} = 2,$$

provided that $x \neq -8$. Therefore,

$$\lim_{x \rightarrow -8} \left(\frac{2x}{x+8} + \frac{16}{x+8} \right) = \lim_{x \rightarrow -8} 2 = \boxed{2}.$$

40. Observe that

$$\frac{3x}{x-2} - \frac{6}{x-2} = \frac{3x-6}{x-2} = \frac{3(x-2)}{x-2} = 3,$$

provided that $x \neq 2$. Therefore,

$$\lim_{x \rightarrow 2} \left(\frac{3x}{x-2} - \frac{6}{x-2} \right) = \lim_{x \rightarrow 2} 3 = \boxed{3}.$$

41. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x} + \sqrt{2}$, the conjugate of $\sqrt{x} - \sqrt{2}$, yields

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} &= \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}. \end{aligned}$$

42. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x} + \sqrt{3}$, the conjugate of $\sqrt{x} - \sqrt{3}$, yields

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} = \lim_{x \rightarrow 3} \frac{x - 3}{(x - 3)(\sqrt{x} + \sqrt{3})} \\ &= \lim_{x \rightarrow 3} \frac{1}{\sqrt{x} + \sqrt{3}} = \frac{1}{\sqrt{3} + \sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}. \end{aligned}$$

43. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x+5} + 3$, the conjugate of $\sqrt{x+5} - 3$, yields

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{(x-4)(x+1)} &= \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{(x-4)(x+1)} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} = \lim_{x \rightarrow 4} \frac{(x+5) - 9}{(x-4)(x+1)(\sqrt{x+5} + 3)} \\ &= \lim_{x \rightarrow 4} \frac{x - 4}{(x-4)(x+1)(\sqrt{x+5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{(x+1)(\sqrt{x+5} + 3)} \\ &= \frac{1}{(4+1)(\sqrt{4+5} + 3)} = \boxed{\frac{1}{30}}. \end{aligned}$$

44. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x+1} + 2$, the conjugate of $\sqrt{x+1} - 2$, yields

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x(x-3)} &= \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x(x-3)} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \rightarrow 3} \frac{(x+1) - 4}{x(x-3)(\sqrt{x+1} + 2)} \\ &= \lim_{x \rightarrow 3} \frac{x - 3}{x(x-3)(\sqrt{x+1} + 2)} = \lim_{x \rightarrow 3} \frac{1}{x(\sqrt{x+1} + 2)} = \frac{1}{3(\sqrt{3+1} + 2)} = \boxed{\frac{1}{12}}. \end{aligned}$$

45. $\lim_{x \rightarrow 3^-} (x^2 - 4) = (3)^2 - 4 = \boxed{5}.$

46. $\lim_{x \rightarrow 2^+} (3x^2 + x) = 3(2)^2 + 2 = \boxed{14}$.

47. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x \rightarrow 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3^-} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^-} (x + 3) = 3 + 3 = \boxed{6}.$$

48. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3^+} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^+} (x + 3) = 3 + 3 = \boxed{6}.$$

49. $\lim_{x \rightarrow 3^-} (\sqrt{9 - x^2} + x)^2 = \left(\lim_{x \rightarrow 3^-} (\sqrt{9 - x^2} + x) \right)^2 = (\sqrt{9 - 3^2} + 3)^2 = \boxed{9}$.

50. $\lim_{x \rightarrow 2^+} (2\sqrt{x^2 - 4} + 3x) = 2\sqrt{2^2 - 4} + 3(2) = \boxed{6}$.

51. $\lim_{x \rightarrow c} [f(x) - 3g(x)] = \lim_{x \rightarrow c} f(x) - 3 \lim_{x \rightarrow c} g(x) = 5 - 3(2) = \boxed{-1}$.

52. $\lim_{x \rightarrow c} [5f(x)] = 5 \lim_{x \rightarrow c} f(x) = 5(5) = \boxed{25}$.

53. $\lim_{x \rightarrow c} [g(x)]^3 = \left[\lim_{x \rightarrow c} g(x) \right]^3 = 2^3 = \boxed{8}$.

54. Because the limit of the denominator is not equal to zero,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x) - h(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} (g(x) - h(x))} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x) - \lim_{x \rightarrow c} h(x)} = \frac{5}{2 - 0} = \boxed{\frac{5}{2}}.$$

55. Because the limit of the denominator is not equal to zero,

$$\lim_{x \rightarrow c} \frac{h(x)}{g(x)} = \frac{\lim_{x \rightarrow c} h(x)}{\lim_{x \rightarrow c} g(x)} = \frac{0}{2} = \boxed{0}.$$

56. $\lim_{x \rightarrow c} [4f(x) \cdot g(x)] = 4 \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = 4(5)(2) = \boxed{40}$.

57. Because the limit of the denominator is not equal to zero,

$$\lim_{x \rightarrow c} \left[\frac{1}{g(x)} \right]^2 = \left[\lim_{x \rightarrow c} \frac{1}{g(x)} \right]^2 = \left[\frac{\lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} g(x)} \right]^2 = \left[\frac{1}{2} \right]^2 = \boxed{\frac{1}{4}}.$$

58. $\lim_{x \rightarrow c} \sqrt[3]{5g(x) - 3} = \sqrt[3]{\lim_{x \rightarrow c} (5g(x) - 3)} = \sqrt[3]{5 \lim_{x \rightarrow c} g(x) - \lim_{x \rightarrow c} 3} = \sqrt[3]{5(2) - 3} = \boxed{\sqrt[3]{7}}$.

59. (a) $\lim_{x \rightarrow 4} [f(x) + g(x)] = \lim_{x \rightarrow 4} f(x) + \lim_{x \rightarrow 4} g(x) = 8 + (-2) = \boxed{6}$.

(b) $\lim_{x \rightarrow 4} \{f(x)[g(x) - h(x)]\} = \lim_{x \rightarrow 4} f(x) \left(\lim_{x \rightarrow 4} g(x) - \lim_{x \rightarrow 4} h(x) \right) = 8(-2 + 0) = \boxed{-16}$.

(c) $\lim_{x \rightarrow 4} [f(x) \cdot g(x)] = \lim_{x \rightarrow 4} f(x) \cdot \lim_{x \rightarrow 4} g(x) = 8(-2) = \boxed{-16}$.

(d) $\lim_{x \rightarrow 4} [2h(x)] = 2 \lim_{x \rightarrow 4} h(x) = 2(0) = \boxed{0}$.

(e) $\lim_{x \rightarrow 4} \frac{g(x)}{f(x)} = \frac{\lim_{x \rightarrow 4} g(x)}{\lim_{x \rightarrow 4} f(x)} = \frac{-2}{8} = \boxed{-\frac{1}{4}}$.

$$(f) \lim_{x \rightarrow 4} \frac{h(x)}{f(x)} = \frac{\lim_{x \rightarrow 4} h(x)}{\lim_{x \rightarrow 4} f(x)} = \frac{0}{8} = \boxed{0}.$$

$$60. (a) \lim_{x \rightarrow 3} \{2[f(x) + h(x)]\} = 2 \left(\lim_{x \rightarrow 3} f(x) + \lim_{x \rightarrow 3} h(x) \right) = 2(0 + (-2)) = \boxed{-4}.$$

$$(b) \lim_{x \rightarrow 3^-} [g(x) + h(x)] = \lim_{x \rightarrow 3^-} g(x) + \lim_{x \rightarrow 3^-} h(x) = 6 + (-2) = \boxed{4}.$$

$$(c) \lim_{x \rightarrow 3} \sqrt[3]{h(x)} = \sqrt[3]{\lim_{x \rightarrow 3} h(x)} = \sqrt[3]{-2} = \boxed{-\sqrt[3]{2}}.$$

$$(d) \lim_{x \rightarrow 3} \frac{f(x)}{h(x)} = \frac{\lim_{x \rightarrow 3} f(x)}{\lim_{x \rightarrow 3} h(x)} = \frac{0}{-2} = \boxed{0}.$$

$$(e) \lim_{x \rightarrow 3} [h(x)]^3 = \left(\lim_{x \rightarrow 3} h(x) \right)^3 = (-2)^3 = \boxed{-8}.$$

(f) Because

$$\lim_{x \rightarrow 3} [f(x) - 2h(x)] = \lim_{x \rightarrow 3} f(x) - 2 \lim_{x \rightarrow 3} h(x) = 0 - 2(-2) = 4,$$

it follows that

$$\lim_{x \rightarrow 3} [f(x) - 2h(x)]^{3/2} = \left[\lim_{x \rightarrow 3} (f(x) - 2h(x)) \right]^{3/2} = 4^{3/2} = \boxed{8}.$$

61. Let $f(x) = 3x^2$ and $c = 1$. Then

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{3x^2 - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{3(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} 3(x + 1) = 3(1 + 1) = \boxed{6}. \end{aligned}$$

62. Let $f(x) = 8x^3$ and $c = 2$. Then

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{8x^3 - 64}{x - 2} = \lim_{x \rightarrow 2} \frac{8(x - 2)(x^2 + 2x + 4)}{x - 2} \\ &= 8 \lim_{x \rightarrow 2} (x^2 + 2x + 4) = 8(2^2 + 2(2) + 4) = \boxed{96}. \end{aligned}$$

63. Let $f(x) = -2x^2 + 4$ and $c = 1$. Then

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{-2x^2 + 4 - 2}{x - 1} = \lim_{x \rightarrow 1} \frac{-2x^2 + 2}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{-2(x - 1)(x + 1)}{x - 1} = -2 \lim_{x \rightarrow 1} (x + 1) = -2(1 + 1) = \boxed{-4}. \end{aligned}$$

64. Let $f(x) = 20 - 0.8x^2$ and $c = 3$. Then

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{20 - 0.8x^2 - 12.8}{x - 3} = \lim_{x \rightarrow 3} \frac{-0.8x^2 + 7.2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{-0.8(x - 3)(x + 3)}{x - 3} = -0.8 \lim_{x \rightarrow 3} (x + 3) = -0.8(3 + 3) = \boxed{-4.8}. \end{aligned}$$

65. Let $f(x) = \sqrt{x}$ and $c = 1$. Then

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\ &= \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{1 + 1} = \boxed{\frac{1}{2}}. \end{aligned}$$

66. Let $f(x) = \sqrt{2x}$ and $c = 5$. Then

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \rightarrow 5} \frac{\sqrt{2x} - \sqrt{10}}{x - 5} = \lim_{x \rightarrow 5} \frac{\sqrt{2x} - \sqrt{10}}{x - 5} \cdot \frac{\sqrt{2x} + \sqrt{10}}{\sqrt{2x} + \sqrt{10}} \\ &= \lim_{x \rightarrow 5} \frac{2x - 10}{(x - 5)(\sqrt{2x} + \sqrt{10})} = \lim_{x \rightarrow 5} \frac{2}{\sqrt{2x} + \sqrt{10}} = \frac{2}{\sqrt{10} + \sqrt{10}} = \boxed{\frac{1}{\sqrt{10}}}.\end{aligned}$$

67. Let $f(x) = 4x - 3$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{4(x+h) - 3 - (4x - 3)}{h} = \lim_{h \rightarrow 0} \frac{4x + 4h - 3 - 4x + 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{4h}{h} = \lim_{h \rightarrow 0} 4 = \boxed{4}.\end{aligned}$$

68. Let $f(x) = 3x + 5$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{3(x+h) + 5 - (3x + 5)}{h} = \lim_{h \rightarrow 0} \frac{3x + 3h + 5 - 3x - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h} = \lim_{h \rightarrow 0} 3 = \boxed{3}.\end{aligned}$$

69. Let $f(x) = 3x^2 + 4x + 1$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 + 4(x+h) + 1 - (3x^2 + 4x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 4x + 4h + 1 - 3x^2 - 4x - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 4h}{h} = \lim_{h \rightarrow 0} (6x + 3h + 4) = 6x + 3(0) + 4 = \boxed{6x + 4}.\end{aligned}$$

70. Let $f(x) = 2x^2 + x$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h \rightarrow 0} (4x + 2h + 1) = 4x + 2(0) + 1 = \boxed{4x + 1}.\end{aligned}$$

71. Let $f(x) = \frac{2}{x}$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} = \lim_{h \rightarrow 0} \frac{2x - 2(x+h)}{hx(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{hx(x+h)} = -\lim_{h \rightarrow 0} \frac{2}{x(x+h)} = -\frac{2}{x(x+0)} = \boxed{-\frac{2}{x^2}}.\end{aligned}$$

72. Let $f(x) = \frac{3}{x^2}$. Then

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h} \cdot \frac{x^2(x+h)^2}{x^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 - 3(x+h)^2}{hx^2(x+h)^2} = \lim_{h \rightarrow 0} \frac{3x^2 - 3x^2 - 6xh - 3h^2}{hx^2(x+h)^2} \\ &= -\lim_{h \rightarrow 0} \frac{h(6x + 3h)}{hx^2(x+h)^2} = -\lim_{h \rightarrow 0} \frac{6x + 3h}{x^2(x+h)^2} = -\frac{6x + 3(0)}{x^2(x+0)^2} = -\frac{6x}{x^4} = \boxed{-\frac{6}{x^3}}.\end{aligned}$$

73. For $x < 1$, $f(x) = 2x - 3$, so that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x - 3) = 2(1) - 3 = \boxed{-1},$$

while for $x > 1$, $f(x) = 3 - x$, so that

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3 - x) = 3 - 1 = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are not equal, $\boxed{\lim_{x \rightarrow 1} f(x) \text{ does not exist}}.$

74. For $x < 2$, $f(x) = 5x + 2$, so that

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (5x + 2) = 5(2) + 2 = \boxed{12},$$

while for $x > 2$, $f(x) = 1 + 3x$, so that

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (1 + 3x) = 1 + 3(2) = \boxed{7}.$$

Because the two one-sided limits as x approaches 2 are not equal, $\boxed{\lim_{x \rightarrow 2} f(x) \text{ does not exist}}.$

75. For $x < 1$, $f(x) = 3x - 1$, so that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 1) = 3(1) - 1 = \boxed{2},$$

while for $x > 1$, $f(x) = 2x$, so that

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2(1) = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are equal to 2, $\boxed{\lim_{x \rightarrow 1} f(x) = 2}.$

76. For $x < 1$, $f(x) = 3x - 1$, so that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 1) = 3(1) - 1 = \boxed{2},$$

while for $x > 1$, $f(x) = 2x$, so that

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2(1) = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are equal to 2, $\boxed{\lim_{x \rightarrow 1} f(x) = 2}.$

77. For $x < 1$, $f(x) = x - 1$, so that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x - 1) = 1 - 1 = \boxed{0},$$

while for $x > 1$, $f(x) = \sqrt{x - 1}$, so that

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x - 1} = \sqrt{1 - 1} = \boxed{0}.$$

Because the two one-sided limits as x approaches 1 are equal to 0, $\boxed{\lim_{x \rightarrow 1} f(x) = 0}.$

78. For $x < 3$, $f(x) = \sqrt{9 - x^2}$, so that

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \sqrt{9 - x^2} = \sqrt{9 - 3^2} = \boxed{0},$$

while for $x > 3$, $f(x) = \sqrt{x^2 - 9}$, so that

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \sqrt{x^2 - 9} = \sqrt{3^2 - 9} = \boxed{0}.$$

Because the two one-sided limits as x approaches 3 are equal to 0, $\boxed{\lim_{x \rightarrow 3} f(x) = 0}$.

79. For $x \neq 3$, $f(x) = \frac{x^2 - 9}{x - 3}$, so that

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3^-} \frac{(x + 3)(x - 3)}{x - 3} = \lim_{x \rightarrow 3^-} (x + 3) = 3 + 3 = \boxed{6},$$

and

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3^+} \frac{(x + 3)(x - 3)}{x - 3} = \lim_{x \rightarrow 3^+} (x + 3) = 3 + 3 = \boxed{6}.$$

Because the two one-sided limits as x approaches 3 are equal to 6, $\boxed{\lim_{x \rightarrow 3} f(x) = 6}$.

80. For $x \neq 2$, $f(x) = \frac{x - 2}{x^2 - 4}$, so that

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2^-} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2^-} \frac{1}{x + 2} = \frac{1}{2 + 2} = \boxed{\frac{1}{4}},$$

and

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2^+} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2^+} \frac{1}{x + 2} = \frac{1}{2 + 2} = \boxed{\frac{1}{4}}.$$

Because the two one-sided limits as x approaches 2 are equal to $\frac{1}{4}$, $\boxed{\lim_{x \rightarrow 2} f(x) = \frac{1}{4}}$.

Applications and Extensions

81. For $t < 1$,

$$\lim_{t \rightarrow 1^-} u_1(t) = \lim_{t \rightarrow 1^-} 0 = 0,$$

while for $t > 1$,

$$\lim_{t \rightarrow 1^+} u_1(t) = \lim_{t \rightarrow 1^+} 1 = 1.$$

Because the two one-sided limits as x approaches 1 are not equal, $\boxed{\lim_{t \rightarrow 1} u_1(t) \text{ does not exist}}$.

82. For $t < 3$,

$$\lim_{t \rightarrow 3^-} u_3(t) = \lim_{t \rightarrow 3^-} 0 = 0,$$

while for $t > 3$,

$$\lim_{t \rightarrow 3^+} u_3(t) = \lim_{t \rightarrow 3^+} 1 = 1.$$

Because the two one-sided limits as x approaches 3 are not equal, $\boxed{\lim_{t \rightarrow 3} u_3(t) \text{ does not exist}}$.

$$83. \lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} (2x + h) = \boxed{2x}.$$

84.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}.\end{aligned}$$

85.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{hx(x+h)} = -\lim_{h \rightarrow 0} \frac{h}{hx(x+h)} \\ &= -\lim_{h \rightarrow 0} \frac{1}{x(x+h)} = \boxed{-\frac{1}{x^2}}.\end{aligned}$$

86.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h} \cdot \frac{x^3(x+h)^3}{x^3(x+h)^3} = \lim_{h \rightarrow 0} \frac{x^3 - (x+h)^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{hx^3(x+h)^3} = -\lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{hx^3(x+h)^3} \\ &= -\lim_{h \rightarrow 0} \frac{3x^2 + 3xh + h^2}{x^3(x+h)^3} = -\frac{3x^2}{x^6} = \boxed{-\frac{3}{x^4}}.\end{aligned}$$

87. Observe that

$$\frac{1}{4+x} - \frac{1}{4} = \frac{4 - (4+x)}{4(4+x)} = -\frac{x}{4(4+x)},$$

so that

$$\lim_{x \rightarrow 0} \left[\frac{1}{x} \left(\frac{1}{4+x} - \frac{1}{4} \right) \right] = \lim_{x \rightarrow 0} \left[\frac{1}{x} \left(-\frac{x}{4(4+x)} \right) \right] = -\lim_{x \rightarrow 0} \frac{1}{4(4+x)} = \boxed{-\frac{1}{16}}.$$

88. Observe that

$$\frac{1}{3} - \frac{1}{x+4} = \frac{(x+4) - 3}{3(x+4)} = \frac{x+1}{3(x+4)},$$

so that

$$\lim_{x \rightarrow -1} \left[\frac{2}{x+1} \left(\frac{1}{3} - \frac{1}{x+4} \right) \right] = \lim_{x \rightarrow -1} \left[\frac{2}{x+1} \left(\frac{x+1}{3(x+4)} \right) \right] = \lim_{x \rightarrow -1} \frac{2}{3(x+4)} = \boxed{\frac{2}{9}}.$$

89.

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{x-7}{\sqrt{x+2}-3} &= \lim_{x \rightarrow 7} \frac{x-7}{\sqrt{x+2}-3} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x \rightarrow 7} \frac{(x-7)(\sqrt{x+2}+3)}{(x+2)-9} \\ &= \lim_{x \rightarrow 7} \frac{(x-7)(\sqrt{x+2}+3)}{x-7} = \lim_{x \rightarrow 7} (\sqrt{x+2}+3) = \sqrt{9}+3 = \boxed{6}.\end{aligned}$$

90.

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2}-2} &= \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2}-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2}+2)}{(x+2)-4} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x+2}+2)}{x-2} = \lim_{x \rightarrow 2} (\sqrt{x+2}+2) = \sqrt{4}+2 = \boxed{4}.\end{aligned}$$

$$91. \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^2 - 2x + 1} = \lim_{x \rightarrow 1} \frac{(x-1)^3}{(x-1)^2} = \lim_{x \rightarrow 1} (x-1) = 1 - 1 = \boxed{0}.$$

$$92. \lim_{x \rightarrow -3} \frac{x^3 + 7x^2 + 15x + 9}{x^2 + 6x + 9} = \lim_{x \rightarrow -3} \frac{(x+1)(x^2 + 6x + 9)}{x^2 + 6x + 9} = \lim_{x \rightarrow -3} (x+1) = -3 + 1 = \boxed{-2}.$$

93. (a) Using the rate schedule provided,

$$C(x) = \begin{cases} 9.00, & \text{if } 0 \leq x \leq 10 \\ 9.00 + 0.95(x - 10), & \text{if } 10 < x \leq 30 \\ 28.00 + 1.65(x - 30), & \text{if } 30 < x \leq 100 \\ 143.50 + 2.20(x - 100), & \text{if } x > 100. \end{cases}$$

(b) The domain of C is the set $\{x | x \geq 0\}$.

(c) Start with the one-sided limits:

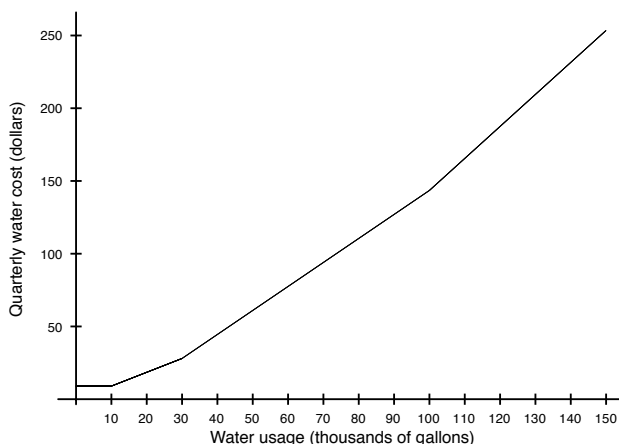
$$\begin{aligned} \lim_{x \rightarrow 5^-} C(x) &= \lim_{x \rightarrow 5^-} 9.00 = 9.00 \\ \lim_{x \rightarrow 5^+} C(x) &= \lim_{x \rightarrow 5^+} 9.00 = 9.00 \\ \lim_{x \rightarrow 10^-} C(x) &= \lim_{x \rightarrow 10^-} 9.00 = 9.00 \\ \lim_{x \rightarrow 10^+} C(x) &= \lim_{x \rightarrow 10^+} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(10 - 10) = 9.00 \\ \lim_{x \rightarrow 30^-} C(x) &= \lim_{x \rightarrow 30^-} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(30 - 10) = 28.00 \\ \lim_{x \rightarrow 30^+} C(x) &= \lim_{x \rightarrow 30^+} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(30 - 30) = 28.00 \\ \lim_{x \rightarrow 100^-} C(x) &= \lim_{x \rightarrow 100^-} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(100 - 30) = 143.50 \\ \lim_{x \rightarrow 100^+} C(x) &= \lim_{x \rightarrow 100^+} [143.50 + 2.20(x - 100)] = 143.50 + 2.20(100 - 100) = 143.50 \end{aligned}$$

Based on these one-sided limits, it follows that

$$\boxed{\lim_{x \rightarrow 5} C(x) = 9.00, \quad \lim_{x \rightarrow 10} C(x) = 9.00, \quad \lim_{x \rightarrow 30} C(x) = 28.00, \quad \text{and} \quad \lim_{x \rightarrow 100} C(x) = 143.50}.$$

$$(d) \lim_{x \rightarrow 0^+} C(x) = \lim_{x \rightarrow 0^+} 9.00 = \boxed{9.00}.$$

(e) The graph of C is shown below.



94. (a) Let x denote the monthly usage of natural gas, measured in therms, and let $C(x)$ denote the monthly cost. If $x \leq 50$, the cost consists of the monthly customer charge of \$22.13, the distribution charge of \$0.25963 per therm, and the gas charge of \$0.3631 per therm. Thus,

$$C(x) = 22.13 + 0.25963x + 0.3631x = 22.13 + 0.62273x.$$

On the other hand, if $x > 50$, the cost consists of the monthly customer charge of \$22.13, a charge of \$12.98 for the first 50 therms, the distribution charge of \$0.11806 for each therm over 50, and the gas charge of \$0.3631 per therm. Thus,

$$C(x) = 22.13 + 12.98 + 0.11806(x - 50) + 0.3631x = 29.207 + 0.48116x.$$

Bringing these two pieces together, the complete cost function is

$$C(x) = \begin{cases} 22.13 + 0.62273x, & \text{if } 0 \leq x \leq 50 \\ 29.207 + 0.48116x, & \text{if } x > 50. \end{cases}$$

- (b) The domain of C is the set $\{x|x \geq 0\}$.

- (c) Start with the one-sided limits:

$$\lim_{x \rightarrow 50^-} C(x) = \lim_{x \rightarrow 50^-} (22.13 + 0.62273x) = 53.2665,$$

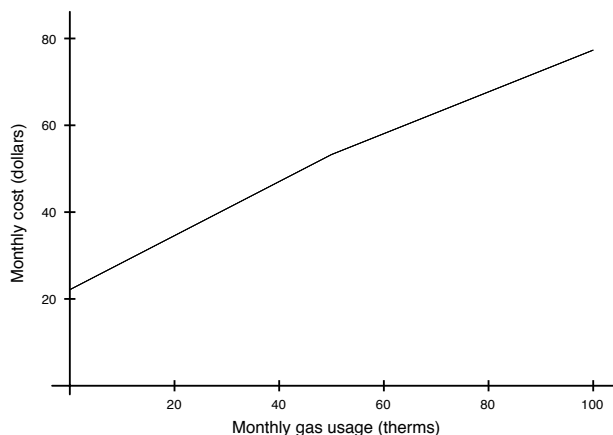
and

$$\lim_{x \rightarrow 50^+} C(x) = \lim_{x \rightarrow 50^+} (29.207 + 0.48116x) = 53.265.$$

From a purely mathematical standpoint, because the two one-sided limits as x approaches 50 are not equal, $\lim_{x \rightarrow 50} C(x)$ does not exist. However, from the practical cost standpoint, the two one-sided limits correspond to the same cost (\$53.27); so, in this sense, $\lim_{x \rightarrow 50} C(x)$ does exist.

- (d) $\lim_{x \rightarrow 0^+} C(x) = \lim_{x \rightarrow 0^+} (22.13 + 0.62273x) = 22.13$.

- (e) The graph of C is shown below.



95. (a) Solving the equation $\frac{1}{2}mv^2 = \frac{3}{2}kT$ for $v(T)$ yields $v(T) = \sqrt{\frac{3kT}{m}}$. Thus

$$\lim_{T \rightarrow 0} v(T) = \lim_{T \rightarrow 0} \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3k}{m}} \sqrt{\lim_{T \rightarrow 0} T} = 0.$$

- (b) Answers will vary. One interpretation is that as the temperature of a gas approaches absolute zero, the average speed of the molecules in the gas also approaches zero. In other words, the molecules in the gas stop moving.

96. (a) For $h < 0$, $2 + h < 2$, so that $f(2 + h) = 3(2 + h) + 5 = 3h + 11$. Thus,

$$\lim_{h \rightarrow 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{3h + 11 - 11}{h} = \lim_{h \rightarrow 0^-} \frac{3h}{h} = \lim_{h \rightarrow 0^-} 3 = \boxed{3}.$$

- (b) For $h > 0$, $2 + h > 2$, so that $f(2 + h) = 13 - (2 + h) = 11 - h$. Thus,

$$\lim_{h \rightarrow 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0^+} \frac{11 - h - 11}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^+} -1 = \boxed{-1}.$$

- (c) Because the two one-sided limits as h approaches 0 are not equal, $\lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h}$
does not exist.

97. Consider the one-sided limits:

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

and

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0.$$

Because the two one-sided limits as x approaches 0 are equal to 0, $\lim_{x \rightarrow 0} |x| = 0$.

98. $\lim_{x \rightarrow 0} |x| = \lim_{x \rightarrow 0} \sqrt{x^2} = \sqrt{\lim_{x \rightarrow 0} x^2} = \sqrt{0} = 0$.

99. Answers will vary. One possibility is the following. Let

$$f(x) = \begin{cases} 1, & \text{if } x < 2 \\ 0, & \text{if } x \geq 2 \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1, & \text{if } x \geq 2 \end{cases},$$

and $c = 2$. Then

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 1 = 1 \text{ and } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 0 = 0$$

so that $\lim_{x \rightarrow 2} f(x)$ does not exist, and

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} 0 = 0 \text{ and } \lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} 1 = 1$$

so that $\lim_{x \rightarrow 2} g(x)$ does not exist. However, $(f + g)(x) = 1$ for all x so that $\lim_{x \rightarrow 2} [f(x) + g(x)] = 1$.

100. Answers will vary. One possibility is the following. Let

$$f(x) = \begin{cases} 1, & \text{if } x < 2 \\ 0, & \text{if } x \geq 2 \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1, & \text{if } x \geq 2 \end{cases},$$

and $c = 2$. Then

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 1 = 1 \text{ and } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 0 = 0$$

so that $\lim_{x \rightarrow 2} f(x)$ does not exist, and

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} 0 = 0 \text{ and } \lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} 1 = 1$$

so that $\lim_{x \rightarrow 2} g(x)$ does not exist. However, $(f \cdot g)(x) = 0$ for all x so that $\lim_{x \rightarrow 2} [f(x)g(x)] = 0$.

101. Answers will vary. One possibility is the following. Let

$$f(x) = \begin{cases} 2, & \text{if } x < 1 \\ -2, & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} 1, & \text{if } x < 1 \\ -1, & \text{if } x \geq 1 \end{cases},$$

and $c = 1$. Then

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2 = 2 \text{ and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} -2 = -2$$

so that $\lim_{x \rightarrow 1} f(x)$ does not exist, and

$$\lim_{x \rightarrow 1^-} g(x) = \lim_{x \rightarrow 1^-} 1 = 1 \text{ and } \lim_{x \rightarrow 1^+} g(x) = \lim_{x \rightarrow 1^+} -1 = -1$$

so that $\lim_{x \rightarrow 1} g(x)$ does not exist. However, $\left(\frac{f}{g}\right)(x) = 2$ for all x so that $\lim_{x \rightarrow 1} \left[\frac{f(x)}{g(x)}\right] = 2$.

102. Answers will vary. One possibility is the following. Let

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

and $c = 0$. Then

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 1 = 1 \text{ and } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} -1 = -1$$

so that $\lim_{x \rightarrow 0} f(x)$ does not exist. However, $|f(x)| = 1$ for all x so that $\lim_{x \rightarrow 0} |f(x)| = 1$.

103. Let $f(x) = k$, where k is any real number, and let g be a function for which $\lim_{x \rightarrow c} g(x)$ exists. Then $\lim_{x \rightarrow c} f(x)$ exists and is equal to k , so that by the Limit of a Product Theorem, $\lim_{x \rightarrow c} [f(x)g(x)]$ exists and $\lim_{x \rightarrow c} [f(x)g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$. It then follows that

$$\lim_{x \rightarrow c} [kg(x)] = \lim_{x \rightarrow c} [f(x)g(x)]$$

exists, and

$$\lim_{x \rightarrow c} [kg(x)] = \lim_{x \rightarrow c} [f(x)g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = k \lim_{x \rightarrow c} g(x).$$

104. Let c be a number in the domain of the rational function $R(x) = \frac{p(x)}{q(x)}$, where p and q are both polynomial functions. Because c is in the domain of R , $q(c) \neq 0$. Moreover, because q is a polynomial function, $\lim_{x \rightarrow c} q(x) = q(c) \neq 0$. Therefore, by the Limit of a Quotient Theorem and the Limit of a Polynomial Theorem,

$$\lim_{x \rightarrow c} R(x) = \lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{\lim_{x \rightarrow c} p(x)}{\lim_{x \rightarrow c} q(x)} = \frac{p(c)}{q(c)} = R(c).$$

Challenge Problems

105. When n is a positive integer,

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),$$

so that

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x - a} \\ &= \lim_{x \rightarrow a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}) \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-2}a + a^{n-1} = \boxed{na^{n-1}}. \end{aligned}$$

106. If n is an even positive integer, then

$$\lim_{x \rightarrow -a} (x^n + a^n) = (-a)^n + a^n = 2a^n \quad \text{and} \quad \lim_{x \rightarrow -a} (x + a) = -a + a = 0,$$

so that

$$\lim_{x \rightarrow -a} \frac{x^n + a^n}{x + a}$$

does not exist. On the other hand, if n is an odd positive integer, then

$$\begin{aligned} x^n + a^n &= (x + a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1}) \\ &= (x + a)(x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1}), \end{aligned}$$

so that

$$\begin{aligned} \lim_{x \rightarrow -a} \frac{x^n + a^n}{x + a} &= \lim_{x \rightarrow -a} \frac{(x + a)(x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1})}{x + a} \\ &= \lim_{x \rightarrow -a} (x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1}) \\ &= (-a)^{n-1} + (-a)(-a)^{n-2} + (-a)^2(-a)^{n-3} + \cdots + (-a)^{n-2}(-a) + (-a)^{n-1} \\ &= n(-a)^{n-1} = \boxed{na^{n-1}}. \end{aligned}$$

107. When m and n are positive integers,

$$x^m - 1 = (x - 1)(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1)$$

and

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1).$$

Thus

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1)}{(x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1)} \\ &= \lim_{x \rightarrow 1} \frac{x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1}{x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1} \\ &= \frac{\overbrace{1 + 1 + 1 + \cdots + 1}^{m \text{ terms}}}{\underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}}} = \boxed{\frac{m}{n}}. \end{aligned}$$

108.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1}{x} \cdot \frac{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1}{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1} = \lim_{x \rightarrow 0} \frac{(1+x) - 1}{x(\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1} \\ &= \frac{1}{\sqrt[3]{1} + \sqrt[3]{1} + 1} = \boxed{\frac{1}{3}}. \end{aligned}$$

109.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{(1+ax)(1+bx)} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{(1+ax)(1+bx)} - 1}{x} \cdot \frac{\sqrt{(1+ax)(1+bx)} + 1}{\sqrt{(1+ax)(1+bx)} + 1} \\ &= \lim_{x \rightarrow 0} \frac{(1+ax)(1+bx) - 1}{x(\sqrt{(1+ax)(1+bx)} + 1)} = \lim_{x \rightarrow 0} \frac{1 + (a+b)x + abx^2 - 1}{x(\sqrt{(1+ax)(1+bx)} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{x[(a+b) + abx]}{x(\sqrt{(1+ax)(1+bx)} + 1)} = \lim_{x \rightarrow 0} \frac{(a+b) + abx}{\sqrt{(1+ax)(1+bx)} + 1} \\ &= \frac{a + b + ab(0)}{\sqrt{(1+0)(1+0)} + 1} = \boxed{\frac{a+b}{2}}. \end{aligned}$$

110. Note that

$$\begin{aligned}(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx) &= 1 + (a_1 + a_2 + \cdots + a_n)x \\ &\quad + \text{terms containing } x^m \text{ where } m \geq 2,\end{aligned}$$

so that

$$(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx) - 1 = (a_1 + a_2 + \cdots + a_n)x + \text{terms containing } x^m \text{ where } m \geq 2.$$

Thus,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} - 1}{x} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} - 1}{x} \cdot \frac{\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} + 1}{\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} + 1} \\&= \lim_{x \rightarrow 0} \frac{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx) - 1}{x(\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} + 1)} \\&= \lim_{x \rightarrow 0} \frac{(a_1 + a_2 + \cdots + a_n)x + \text{terms containing } x^m \text{ where } m \geq 2}{x(\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} + 1)} \\&= \lim_{x \rightarrow 0} \frac{(a_1 + a_2 + \cdots + a_n) + \text{terms containing } x^m \text{ where } m \geq 1}{\sqrt{(1 + a_1x)(1 + a_2x) \cdots (1 + a_nx)} + 1} \\&= \frac{a_1 + a_2 + \cdots + a_n + 0}{\sqrt{1} + 1} = \boxed{\frac{a_1 + a_2 + \cdots + a_n}{2}}.\end{aligned}$$

111. Let $f(x) = x|x|$. For $x < 0$, $f(x) = x(-x) = -x^2$. Thus,

$$\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-h^2 - 0}{h} = \lim_{h \rightarrow 0^-} (-h) = 0.$$

For $x \geq 0$, $f(x) = x(x) = x^2$. Thus,

$$\lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^2 - 0}{h} = \lim_{h \rightarrow 0^+} h = 0.$$

Because the two one-sided limits as h approaches 0 are equal to 0,

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \boxed{0}.$$

1.3: Continuity

Concepts and Vocabulary

1. True. A polynomial function is continuous at every real number.
2. False. Piecewise-defined functions **may not be** continuous at numbers where the function changes equations.
3. The three conditions necessary for a function f to be continuous at a number c are $f(c)$ is defined, $\lim_{x \rightarrow c} f(x)$ exists, and $\lim_{x \rightarrow c} f(x) = f(c)$.
4. True. If $f(x)$ is continuous at 0, then $g(x) = \frac{1}{4}f(x)$ is continuous at 0.
5. False. If f is a function defined everywhere in an open interval containing c , except possibly at c , then the number c is called a removable discontinuity of f if the function f is not continuous at c but $\lim_{x \rightarrow c} f(x)$ exists.
6. False. If a function f is discontinuous at a number c , then it might be the case that $\lim_{x \rightarrow c} f(x)$ does not exist. However, the function could be discontinuous at c because $\lim_{x \rightarrow c} f(x)$ exists but either $f(c)$ is not defined or $\lim_{x \rightarrow c} f(x) \neq f(c)$.
7. False. If a function f is continuous on an open interval (a, b) , then it is continuous on the closed interval $[a, b]$ only if the function is also continuous from the right at the number a and continuous from the left at the number b .
8. True. If a function f is continuous on the closed interval $[a, b]$, then f is continuous on the open interval (a, b) .
9. This function is discontinuous. When the ball lands on the ground and stops, there will be a jump discontinuity as the velocity abruptly changes from a nonzero value to zero.
10. This function is continuous. Though the temperature might change rapidly when the oven is first turned on and then again after the oven is turned off, there will be no abrupt jumps in the temperature.
11. True. If a function f is continuous on a closed interval $[a, b]$, then the Intermediate Value Theorem guarantees that the function takes on every value y between $f(a)$ and $f(b)$.
12. False. If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, but both $f(a) > 0$ and $f(b) > 0$, then f **may have** a zero on the open interval (a, b) , but the Intermediate Value Theorem cannot guarantee that a zero exists. The Intermediate Value Theorem can never guarantee that a function does not have a zero on a particular interval.

Skill Building

13. (a) The function f is not continuous at $c = -3$.
 (b) Although f is defined at $c = -3$ with $f(-3) = 1$ and $\lim_{x \rightarrow -3} f(x) = -2$, $\lim_{x \rightarrow -3} f(x) \neq f(-3)$.
 (c) The discontinuity at $c = -3$ is removable because $\lim_{x \rightarrow -3} f(x)$ exists.
 (d) Redefine f at $c = -3$ so that $f(-3) = -2 = \lim_{x \rightarrow -3} f(x)$. The resulting function will then be continuous at $c = -3$.

14. (a) Because f is defined at $c = 0$ with $f(0) = 2$, $\lim_{x \rightarrow 0} f(x) = 2$, and $\lim_{x \rightarrow 0} f(x) = f(0)$, the function f is continuous at $c = 0$.
15. (a) The function f is not continuous at $c = 2$.
- (b) Although f is defined at $c = 2$ with $f(2) = 3$, $\lim_{x \rightarrow 2} f(x)$ does not exist.
- (c) The discontinuity at $c = 2$ is not removable because $\lim_{x \rightarrow 2} f(x)$ does not exist.
16. (a) The function f is not continuous at $c = 3$.
- (b) Though f is defined at $c = 3$ with $f(3) = -1$, $\lim_{x \rightarrow 3} f(x)$ does not exist.
- (c) The discontinuity at $c = 3$ is not removable because $\lim_{x \rightarrow 3} f(x)$ does not exist.
17. (a) Because f is defined at $c = 4$ with $f(4) = 0$, $\lim_{x \rightarrow 4} f(x) = 0$, and $\lim_{x \rightarrow 4} f(x) = f(4)$, the function f is continuous at $c = 4$.
18. (a) The function f is not continuous at $c = 5$.
- (b) The function f is discontinuous at $c = 5$ because f is not defined at $c = 5$ and because $\lim_{x \rightarrow 5^-} f(x) = -3$ and $\lim_{x \rightarrow 5^+} f(x) = 1$ so that $\lim_{x \rightarrow 5} f(x)$ does not exist.
- (c) The discontinuity at $c = 5$ is not removable, because $\lim_{x \rightarrow 5} f(x)$ does not exist.
19. The domain of the function $f(x) = x^2 + 1$ is the set of all real numbers, so f is defined at $c = -1$ with $f(-1) = 2$. Next,
- $$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x^2 + 1) = (-1)^2 + 1 = 2,$$
- so that $\lim_{x \rightarrow -1} f(x)$ exists. Finally, $\lim_{x \rightarrow -1} f(x) = f(-1)$. Because all three conditions of the definition of continuity at $c = -1$ are satisfied, the function f is continuous at $c = -1$.
20. The domain of the function $f(x) = x^3 - 5$ is the set of all real numbers, so f is defined at $c = 5$ with $f(5) = 120$. Next,
- $$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (x^3 - 5) = 5^3 - 5 = 120,$$
- so that $\lim_{x \rightarrow 5} f(x)$ exists. Finally, $\lim_{x \rightarrow 5} f(x) = f(5)$. Because all three conditions of the definition of continuity at $c = 5$ are satisfied, the function f is continuous at $c = 5$.
21. Because $x^2 + 4$ is never equal to zero for any real number x , the domain of the function $f(x) = \frac{x}{x^2 + 4}$ is the set of all real numbers. Thus, f is defined at $c = -2$, with $f(-2) = -\frac{1}{4}$. Next,
- $$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{x}{x^2 + 4} = \frac{-2}{(-2)^2 + 4} = -\frac{1}{4},$$
- so that $\lim_{x \rightarrow -2} f(x)$ exists. Finally, $\lim_{x \rightarrow -2} f(x) = f(-2)$. Because all three conditions of the definition of continuity at $c = -2$ are satisfied, the function f is continuous at $c = -2$.
22. The domain of the function $f(x) = \frac{x}{x-2}$ is the set $\{x | x \neq 2\}$. Because f is not defined at $c = 2$, the function f is not continuous at $c = 2$.

23. The domain of the function f is the set of all real numbers, so f is defined at $c = 2$ with $f(2) = 2(2) + 5 = 9$. Next,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 5) = 9 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4x + 1) = 9,$$

so that $\lim_{x \rightarrow 2} f(x)$ exists and is equal to 9. Finally, $\lim_{x \rightarrow 2} f(x) = f(2)$. Because all three conditions of the definition of continuity at $c = 2$ are satisfied, the function f is continuous at $c = 2$.

24. The domain of the function f is the set of all real numbers, so f is defined at $c = 0$ with $f(0) = 2(0) + 1 = 2$. Next,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x + 1) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x = 0,$$

so that $\lim_{x \rightarrow 0} f(x)$ does not exist. Therefore, the function f is not continuous at $c = 0$.

25. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 4$. Next,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 1) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2,$$

so that $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 2. However, $\lim_{x \rightarrow 1} f(x) \neq f(1)$, so the function f is not continuous at $c = 1$.

26. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 1) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2,$$

so that $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 2. Finally, $\lim_{x \rightarrow 1} f(x) = f(1)$. Because all three conditions of the definition of continuity at $c = 1$ are satisfied, the function f is continuous at $c = 1$.

27. The domain of the function f is the set $\{x | x \neq 1\}$. Because f is not defined at $c = 1$, the function f is not continuous at $c = 1$.

28. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 1) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 3x = 3,$$

so that $\lim_{x \rightarrow 1} f(x)$ does not exist. Therefore, the function f is not continuous at $c = 1$.

29. The domain of the function f is the set of all real numbers, so f is defined at $c = 0$ with $f(0) = 0$. Next,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x = 0,$$

so that $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 0. Finally, $\lim_{x \rightarrow 0} f(x) = f(0)$. Because all three conditions of the definition of continuity at $c = 0$ are satisfied, the function f is continuous at $c = 0$.

30. The domain of the function f is the set of all real numbers, so f is defined at $c = -1$ with $f(-1) = 2$. Next,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^2 = 1 \quad \text{and} \quad \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (-3x + 2) = 5,$$

so that $\lim_{x \rightarrow -1} f(x)$ does not exist. Therefore, the function f is not continuous at $c = -1$.

31. The domain of the function f is the set $\{x|x < 4\}$, so f is defined at $c = 0$ with $f(0) = 4$. Next,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (4 - 3x^2) = 4 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{\frac{16 - x^2}{4 - x}} = 2,$$

so that $\lim_{x \rightarrow 0} f(x)$ does not exist. Therefore, the function f is not continuous at $c = 0$.

32. The domain of the function f is the set $\{x|x \geq -4\}$, so f is defined at $c = 4$ with $f(4) = \sqrt{8} = 2\sqrt{2}$. Next,

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} \sqrt{4 + x} = 2\sqrt{2}$$

and

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \sqrt{\frac{x^2 - 3x - 4}{x - 4}} = \lim_{x \rightarrow 4^+} \sqrt{\frac{(x - 4)(x + 1)}{x - 4}} = \lim_{x \rightarrow 4^+} \sqrt{x + 1} = \sqrt{5},$$

so that $\lim_{x \rightarrow 4} f(x)$ does not exist. Therefore, the function f is not continuous at $c = 4$.

33. Because

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4,$$

$f(2)$ should be assigned the value 4. Then $\lim_{x \rightarrow 2} f(x) = f(2)$, and the resulting function will be continuous at $c = 2$.

34. Because

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 + x - 12}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 4)}{x - 3} = \lim_{x \rightarrow 3} (x + 4) = 7,$$

$f(3)$ should be assigned the value 7. Then $\lim_{x \rightarrow 3} f(x) = f(3)$, and the resulting function will be continuous at $c = 3$.

35. Because

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1 + x) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2,$$

it follows that $\lim_{x \rightarrow 1} f(x) = 2$, so $f(1)$ should be assigned the value 2. Then $\lim_{x \rightarrow 1} f(x) = f(1)$, and the resulting function will be continuous at $c = 1$.

36. Because

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (x^2 + 5x) = -4 \quad \text{and} \quad \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (x - 3) = -4,$$

it follows that $\lim_{x \rightarrow -1} f(x) = -4$, so $f(-1)$ should be assigned the value -4. Then $\lim_{x \rightarrow -1} f(x) = f(-1)$, and the resulting function will be continuous at $c = -1$.

37. The domain of the function $f(x) = \frac{x^2 - 9}{x - 3}$ is the set $\{x|x \neq 3\}$, so that f is defined on the interval $[-3, 3)$. Let c be any number in the open interval $(-3, 3)$. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{x^2 - 9}{x - 3} = \frac{c^2 - 9}{c - 3} = f(c),$$

so f is continuous on the open interval $(-3, 3)$. Finally,

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{x - 3} = 0 = f(-3),$$

so f is also continuous from the right at $c = -3$. Therefore, the function f is continuous on the interval $[-3, 3)$.

38. The domain of the function $f(x) = 1 + \frac{1}{x}$ is the set $\{x|x \neq 0\}$, so that f is defined on the interval $[-1, 0)$. Let c be any number in the open interval $(-1, 0)$. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(1 + \frac{1}{x}\right) = 1 + \frac{1}{c} = f(c),$$

so f is continuous on the open interval $(-1, 0)$. Finally,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \left(1 + \frac{1}{x}\right) = 0 = f(-1),$$

so f is also continuous from the right at $c = -1$. Therefore, the function f is continuous on the interval $[-1, 0)$.

39. The domain of $f(x) = \frac{1}{\sqrt{x^2 - 9}}$ is the set $\{x| |x| > 3\}$, so that f is not defined anywhere on the closed interval $[-3, 3]$. Therefore, the function f is not continuous anywhere on the closed interval $[-3, 3]$. However, let c be any number in the domain of f . Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{1}{\sqrt{x^2 - 9}} = \frac{1}{\sqrt{c^2 - 9}} = f(c),$$

so that the function f is continuous on its domain, the set $\{x|x < -3\} \cup \{x|x > 3\}$.

40. The domain of the function $f(x) = \sqrt{9 - x^2}$ is the set $\{x| -3 \leq x \leq 3\}$, so that f is defined on the closed interval $[-3, 3]$. Let c be any number in the open interval $(-3, 3)$. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \sqrt{9 - x^2} = \sqrt{9 - c^2} = f(c),$$

so f is continuous on the open interval $(-3, 3)$. At $c = -3$,

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \sqrt{9 - x^2} = 0 = f(-3),$$

so f is also continuous from the right at $c = -3$. Finally, at $c = 3$,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \sqrt{9 - x^2} = 0 = f(3),$$

and f is continuous from the left at $c = 3$. Therefore, the function f is continuous on the closed interval $[-3, 3]$.

41. Let $g(x) = 2x^2 + 5x$ and $h(x) = \frac{1}{x}$. The domain of the polynomial function g is the set of all real numbers, and the domain of the rational function h is the set $\{x|x \neq 0\}$. Each function is continuous on its domain. Because the function f is the difference of the functions g and h , the domain of f is the intersection of the domains of g and h ; that is, the domain of f is the set $\{x|x \neq 0\}$. The function f is continuous for all values x at which both g and h are continuous, so that f is continuous on the set $\{x|x \neq 0\}$.
42. Let $g(x) = x + 1$ and $h(x) = \frac{2x}{x^2 + 5}$. The domain of the polynomial function g is the set of all real numbers, and the domain of the rational function h is also the set of all real numbers (because $x^2 + 5$ is never equal to zero for any real number x). Each function is continuous on its domain. Because the function f is the sum of the functions g and h , the domain of f is the intersection of the domains of g and h ; that is, the domain of f is the set of all real numbers. The function f is continuous for all values x at which both g and h are continuous, so that f is continuous for all real numbers.

43. Let $g(x) = x - 1$ and $h(x) = x^2 + x + 1$. The domain of the polynomial function g is the set of all real numbers, and the domain of the polynomial function h is also the set of all real numbers. Each function is continuous on its domain. Because the function f is the product of the functions g and h , the domain of f is the intersection of the domains of g and h ; that is, the domain of f is the set of all real numbers. The function f is continuous for all values x at which both g and h are continuous, so that f is continuous for all real numbers.
44. Let $g(x) = \sqrt{x}$ and $h(x) = x^3 - 5$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the polynomial function h is the set of all real numbers. Each function is continuous on its domain. Because the function f is the product of the functions g and h , the domain of f is the intersection of the domains of g and h ; that is, the domain of f is the set $\{x|x \geq 0\}$. The function f is continuous at all values x at which both g and h are continuous, so that f is continuous on the set $\{x|x \geq 0\}$.
45. Let $g(x) = x - 9$ and $h(x) = \sqrt{x} - 3$. The domain of the polynomial function g is the set of all real numbers, and the domain of the function h is the set $\{x|x \geq 0\}$. Each function is continuous on its domain. Because the function f is the quotient of the functions g and h , the domain of f is the intersection of the domains of g and h excluding any x for which $h(x) = 0$; that is, the domain of f is the set $\{x|x \geq 0, x \neq 9\}$. The function f is continuous for all values x at which both g and h are continuous, excluding any x for which $h(x) = 0$, so that f is continuous on the set $\{x|x \geq 0, x \neq 9\}$.
46. Let $g(x) = x - 4$ and $h(x) = \sqrt{x} - 2$. The domain of the polynomial function g is the set of all real numbers, and the domain of the function h is the set $\{x|x \geq 0\}$. Each function is continuous on its domain. Because the function f is the quotient of the functions g and h , the domain of f is the intersection of the domains of g and h excluding any x for which $h(x) = 0$; that is, the domain of f is the set $\{x|x \geq 0, x \neq 4\}$. The function f is continuous for all values x at which both g and h are continuous, excluding any x for which $h(x) = 0$, so that f is continuous on the set $\{x|x \geq 0, x \neq 4\}$.
47. Let $g(x) = \sqrt{x}$ and $h(x) = \frac{x^2 + 1}{2 - x}$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the rational function h is the set $\{x|x \neq 2\}$. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x|x < 2\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $x < 2$, it follows that the domain of f is the set $\{x|x < 2\}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is continuous on the set $\{x|x < 2\}$.
48. Let $g(x) = \sqrt{x}$ and $h(x) = \frac{4}{x^2 - 1}$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the rational function h is the set $\{x|x \neq \pm 1\}$. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x| |x| > 1\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $|x| > 1$, it follows that the domain of f is the set $\{x| |x| > 1\}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is continuous on the set $\{x| |x| > 1\}$.
49. Let $g(x) = x^{2/3}$ and $h(x) = 2x^2 + 5x - 3$. The domain of the function g is the set of all real numbers, and the domain of the polynomial function h is also the set of all real numbers. Each function is continuous on its domain. Because f is the composition $g(h(x))$ and $h(x)$ is always in the domain of g , it follows that the domain of f is the set of all real numbers. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is continuous on the set of all real numbers.

50. Let $g(x) = x^{1/2}$ and $h(x) = x + 2$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the polynomial function h is the set of all real numbers. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x|x \geq -2\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $x \geq -2$, it follows that the domain of f is the set $\{x|x \geq -2\}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is continuous on the set $\{x|x \geq -2\}$.

51. Because f is defined at $c = 0$ with $f(0) = \sqrt{15}$ and

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sqrt{15 - 3x} = \sqrt{15} = f(0),$$

the function f is continuous at $c = 0$.

52. Because

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} \lfloor x - 2 \rfloor = 1 \quad \text{but} \quad \lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \lfloor x - 2 \rfloor = 2,$$

it follows that $\lim_{x \rightarrow 4} f(x)$ does not exist and f is not continuous at $c = 4$.

53. Because

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (9 - x^2) = 0 \quad \text{but} \quad \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \lfloor x - 2 \rfloor = 1,$$

it follows that $\lim_{x \rightarrow 3} f(x)$ does not exist and f is not continuous at $c = 3$.

54. Because

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \sqrt{15 - 3x} = 3 \quad \text{but} \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (9 - x^2) = 5,$$

it follows that $\lim_{x \rightarrow 2} f(x)$ does not exist and f is not continuous at $c = 2$.

55. Because f is defined at $c = 1$ with $f(1) = \sqrt{12} = 2\sqrt{3}$ and

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \sqrt{15 - 3x} = \sqrt{12} = 2\sqrt{3} = f(1),$$

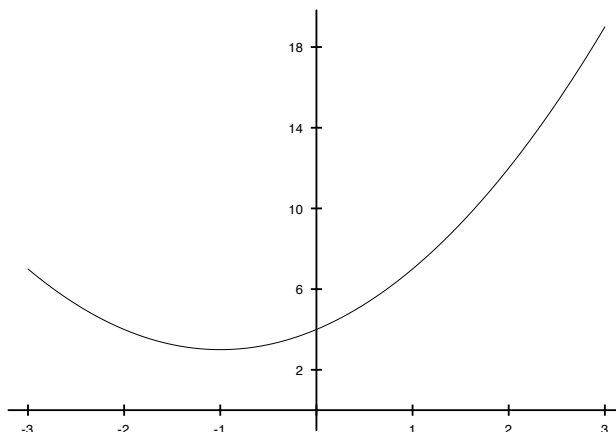
the function f is continuous at $c = 1$.

56. Because f is defined at $c = 2.5$ with $f(2.5) = 2.75$ and

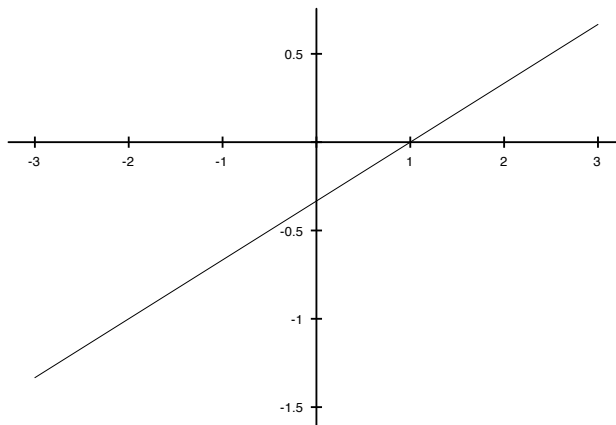
$$\lim_{x \rightarrow 2.5} f(x) = \lim_{x \rightarrow 2.5} (9 - x^2) = 9 - 2.5^2 = 2.75 = f(2.5),$$

the function f is continuous at $c = 2.5$.

57. (a) A graph of the function f is shown below



- (b) Based on the graph from part (a), the function f appears to be continuous on the set of all real numbers.
- (c) The polynomial functions $x^3 - 8$ and $x - 2$ are continuous on the set of all real numbers. Because the function f is the quotient of these two polynomials, f is continuous on the set of all real numbers excluding any values for x at which $x - 2 = 0$. Thus, f is continuous on the set $\{x | x \neq 2\}$.
- (d) Answers will vary. One possible response is that graphing technology can be a useful tool to suggest where a function is continuous, but “conclusions” drawn from graphing technology should always be confirmed using some basic analysis.
58. (a) A graph of the function f is shown below



- (b) Based on the graph from part (a), the function f appears to be continuous on the set of all real numbers.
- (c) The polynomial functions $x^2 - 3x + 2$ and $3x - 6$ are continuous on the set of all real numbers. Because the function f is the quotient of these two polynomials, f is continuous on the set of all real numbers excluding any values for x at which $3x - 6 = 0$. Thus, f is continuous on the set $\{x | x \neq 2\}$.
- (d) Answers will vary. One possible response is that graphing technology can be a useful tool to suggest where a function is continuous, but “conclusions” drawn from graphing technology should always be confirmed using some basic analysis.
59. The polynomial function $f(x) = x^3 - 3x$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^3 - 3(-2) = -2 < 0$ and $f(2) = 2^3 - 3(2) = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-2, 2)$.
60. The polynomial function $f(x) = x^4 - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^4 - 1 = 15 > 0$ and $f(2) = 2^4 - 1 = 15 > 0$, the Intermediate Value Theorem gives no information about the presence of a zero of f on the interval $(-2, 2)$.
61. The domain of the function $f(x) = \frac{x}{(x+1)^2} - 1$ is the set $\{x | x \neq -1\}$. Because f is the difference of a rational function and a polynomial function, it is continuous on its domain. It follows that f is continuous on the closed interval $[10, 20]$. Now,
- $$f(10) = \frac{10}{11^2} - 1 = -\frac{111}{121} < 0 \quad \text{and} \quad f(20) = \frac{20}{21^2} - 1 = -\frac{421}{441} < 0,$$
- so the Intermediate Value Theorem gives no information about the presence of a zero of f on the interval $(10, 20)$.

62. The polynomial function $f(x) = x^3 - 2x^2 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[3, 4]$. Because $f(3) = 3^3 - 2(3)^2 - 3 + 2 = 8 > 0$ and $f(4) = 4^3 - 2(4)^2 - 4 + 2 = 30 > 0$, the Intermediate Value Theorem gives no information about the presence of a zero of f on the interval $(3, 4)$.
63. The domain of the function $f(x) = \frac{x^3 - 1}{x - 1}$ is the set $\{x | x \neq 1\}$. Because the closed interval $[0, 2]$ contains $x = 1$, f is not continuous on this interval, so the Intermediate Value Theorem does not apply. Therefore, the Intermediate Value Theorem gives no information about the presence of a zero of f on the interval $(0, 2)$.
64. The domain of the function $f(x) = \frac{x^2 + 3x + 2}{x^2 - 1}$ is the set $\{x | x \neq \pm 1\}$. Because the closed interval $[-3, 0]$ contains $x = -1$, f is not continuous on this interval, so the Intermediate Value Theorem does not apply. Therefore, the Intermediate Value Theorem gives no information about the presence of a zero of f on the interval $(-3, 0)$.
65. The polynomial function $f(x) = x^3 + 3x - 5$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1^3 + 3(1) - 5 = -1 < 0$ and $f(2) = 2^3 + 3(2) - 5 = 9 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.1) = -0.369 < 0$ and $f(1.2) = 0.328 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.1, 1.2)$. Repeating the process by subdividing the interval $[1.1, 1.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.15, 1.16)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $[1.154]$, correct to three decimal places.

[1, 2]		[1.1, 1.2]		[1.15, 1.16]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	-1.000	1.10	-0.36900	1.150	-0.02913
1.1	-0.369	1.11	-0.30237	1.151	-0.02215
1.2	0.328	1.12	-0.23507	1.152	-0.01518
1.3	1.097	1.13	-0.16710	1.153	-0.00819
1.4	1.944	1.14	-0.09846	1.154	-0.00120
1.5	2.875	1.15	-0.02913	1.155	0.00580
1.6	3.896	1.16	0.04090	1.156	0.01280
1.7	5.013	1.17	0.11161	1.157	0.01982
1.8	6.232	1.18	0.18303	1.158	0.02684
1.9	7.559	1.19	0.25516	1.159	0.03386
2.0	9.000	1.20	0.32800	1.160	0.04090

66. The polynomial function $f(x) = x^3 - 4x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1^3 - 4(1) + 2 = -1 < 0$ and $f(2) = 2^3 - 4(2) + 2 = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.6) = -0.304 < 0$ and $f(1.7) = 0.113 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.6, 1.7)$. Repeating the process by subdividing the interval $[1.6, 1.7]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.67, 1.68)$. Repeating the subdivision process once more,

the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.675}$, correct to three decimal places.

[1, 2]		[1.6, 1.7]		[1.67, 1.68]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	-1.000	1.60	-0.30400	1.670	-0.02254
1.1	-1.069	1.61	-0.26672	1.671	-0.01817
1.2	-1.072	1.62	-0.22847	1.672	-0.01378
1.3	-1.003	1.63	-0.18925	1.673	-0.00939
1.4	-0.856	1.64	-0.14906	1.674	-0.00499
1.5	-0.625	1.65	-0.10788	1.675	-0.00058
1.6	-0.304	1.66	-0.06570	1.676	0.00384
1.7	0.133	1.67	-0.02254	1.677	0.00828
1.8	0.632	1.68	0.02163	1.678	0.01272
1.9	1.259	1.69	0.06681	1.679	0.01717
2.0	2.000	1.70	0.11300	1.680	0.02163

67. The polynomial function $f(x) = 2x^3 + 3x^2 + 4x - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -1 < 0$ and $f(1) = 2 + 3 + 4 - 1 = 8 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$. To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(0.2) = -0.064 < 0$ and $f(0.3) = 0.524 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(0.2, 0.3)$. Repeating the process by subdividing the interval $[0.2, 0.3]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(0.21, 0.22)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.211}$, correct to three decimal places.

[0, 1]		[0.2, 0.3]		[0.21, 0.22]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	-1.000	0.20	-0.06400	0.210	-0.00918
0.1	-0.568	0.21	-0.00918	0.211	-0.00365
0.2	-0.064	0.22	0.04650	0.212	0.00189
0.3	0.524	0.23	0.10303	0.213	0.00743
0.4	1.208	0.24	0.16045	0.214	0.01299
0.5	2.000	0.25	0.21875	0.215	0.01855
0.6	2.912	0.26	0.27795	0.216	0.02412
0.7	3.956	0.27	0.33807	0.217	0.02970
0.8	5.144	0.28	0.39910	0.218	0.03529
0.9	6.488	0.29	0.46108	0.219	0.04089
1.0	8.000	0.30	0.52400	0.220	0.04650

68. The polynomial function $f(x) = x^3 - x^2 - 2x + 1$ is continuous for all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = 1 > 0$ and $f(1) = 1 - 1 - 2 + 1 = -1 < 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$. To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(0.4) = 0.104 > 0$ and $f(0.5) = -0.125 < 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(0.4, 0.5)$. Repeating the process by subdividing the interval $[0.4, 0.5]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to

five decimal places for display purposes. The zero has now been bracketed in the interval $(0.44, 0.45)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.445}$, correct to three decimal places.

$[0, 1]$		$[0.4, 0.5]$		$[0.44, 0.45]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	1.000	0.40	0.10400	0.440	0.01158
0.1	0.791	0.41	0.08082	0.441	0.00929
0.2	0.568	0.42	0.05769	0.442	0.00699
0.3	0.337	0.43	0.03461	0.443	0.00469
0.4	0.104	0.44	0.01158	0.444	0.00239
0.5	-0.125	0.45	-0.01138	0.445	0.00010
0.6	-0.344	0.46	-0.03426	0.446	-0.00220
0.7	-0.547	0.47	-0.05708	0.447	-0.00449
0.8	-0.728	0.48	-0.07981	0.448	-0.00679
0.9	-0.881	0.49	-0.10245	0.449	-0.00908
1.0	-1.000	0.50	-0.12500	0.450	-0.01138

69. The polynomial function $f(x) = x^3 - 6x - 12$ is continuous for all real numbers, so it is continuous on the closed interval $[3, 4]$. Because $f(3) = 3^3 - 6(3) - 12 = -3 < 0$ and $f(4) = 4^3 - 6(4) - 12 = 28 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(3, 4)$. To approximate this zero, subdivide the interval $[3, 4]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(3.1) = -0.809 < 0$ and $f(3.2) = 1.568 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(3.1, 3.2)$. Repeating the process by subdividing the interval $[3.1, 3.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(3.13, 3.14)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{3.134}$, correct to three decimal places.

$[3, 4]$		$[3.1, 3.2]$		$[3.13, 3.14]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
3.0	-3.000	3.10	-0.80900	3.130	-0.11570
3.1	-0.809	3.11	-0.57977	3.131	-0.09230
3.2	1.568	3.12	-0.34867	3.132	-0.06888
3.3	4.137	3.13	-0.11570	3.133	-0.04545
3.4	6.904	3.14	0.11914	3.134	-0.02199
3.5	9.875	3.15	0.35587	3.135	0.00149
3.6	13.056	3.16	0.59450	3.136	0.02498
3.7	16.453	3.17	0.83501	3.137	0.04849
3.8	20.072	3.18	1.07743	3.138	0.07202
3.9	23.919	3.19	1.32176	3.139	0.09557
4.0	28.000	3.20	1.56800	3.140	0.11914

70. The polynomial function $f(x) = 3x^3 + 5x - 40$ is continuous for all real numbers, so it is continuous on the closed interval $[2, 3]$. Because $f(2) = 3(2)^3 + 5(2) - 40 = -6 < 0$ and $f(3) = 3(3)^3 + 5(3) - 40 = 56 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(2, 3)$. To approximate this zero, subdivide the interval $[2, 3]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(2.1) = -1.717 < 0$ and $f(2.2) = 2.2944 > 0$, the Intermediate Value

Theorem guarantees the zero lies in the interval $(2.1, 2.2)$. Repeating the process by subdividing the interval $[2.1, 2.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(2.13, 2.14)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{2.137}$, correct to three decimal places.

$[2, 3]$		$[2.1, 2.2]$		$[2.13, 2.14]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
2.0	-6.000	2.10	-1.71700	2.130	-0.35921
2.1	-1.717	2.11	-1.26281	2.131	-0.31336
2.2	2.944	2.12	-0.81562	2.132	-0.26747
2.3	8.001	2.13	-0.35921	2.133	-0.22154
2.4	13.472	2.14	0.10103	2.134	-0.17557
2.5	19.375	2.15	0.56512	2.135	-0.12957
2.6	25.728	2.16	1.03309	2.136	-0.08353
2.7	32.549	2.17	1.50494	2.137	-0.03744
2.8	39.856	2.18	1.98070	2.138	0.00868
2.9	47.667	2.19	2.46038	2.139	0.05483
3.0	56.000	2.20	2.94400	2.140	0.10103

71. The polynomial function $f(x) = x^4 - 2x^3 + 21x - 23$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 2 + 21 - 23 = -3 < 0$ and $f(2) = 2^4 - 2(2)^3 + 21(2) - 23 = 19 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.1) = -1.0979 < 0$ and $f(1.2) = 0.8176 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.1, 1.2)$. Repeating the process by subdividing the interval $[1.1, 1.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.15, 1.16)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.157}$, correct to three decimal places.

$[1, 2]$		$[1.1, 1.2]$		$[1.15, 1.16]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	-3.0000	1.10	-1.09790	1.150	-0.14274
1.1	-1.0979	1.11	-0.90719	1.151	-0.12359
1.2	0.8176	1.12	-0.71634	1.152	-0.10444
1.3	2.7621	1.13	-0.52532	1.153	-0.08529
1.4	4.7536	1.14	-0.33413	1.154	-0.06613
1.5	6.8125	1.15	-0.14274	1.155	-0.04698
1.6	8.9616	1.16	0.04885	1.156	-0.02781
1.7	11.2261	1.17	0.24066	1.157	-0.00865
1.8	13.6336	1.18	0.43271	1.158	0.01051
1.9	16.2141	1.19	0.62502	1.159	0.02968
2.0	19.0000	1.20	0.81760	1.160	0.04885

72. The polynomial function $f(x) = x^4 - x^3 + x - 2$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 1 + 1 - 2 = -1 < 0$ and $f(2) = 2^4 - 2^3 + 2 - 2 = 8 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals,

each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.3) = -0.0409 < 0$ and $f(1.4) = 0.4976 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.3, 1.4)$. Repeating the process by subdividing the interval $[1.3, 1.4]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.30, 1.31)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.308}$, correct to three decimal places.

[1, 2]		[1.3, 1.4]		[1.30, 1.31]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	-1.0000	1.30	-0.04090	1.300	-0.04090
1.1	-0.7669	1.31	0.00691	1.301	-0.03618
1.2	-0.4544	1.32	0.05599	1.302	-0.03144
1.3	-0.0409	1.33	0.10637	1.303	-0.02669
1.4	0.4976	1.34	0.15808	1.304	-0.02193
1.5	1.1875	1.35	0.21113	1.305	-0.01715
1.6	2.0576	1.36	0.26556	1.306	-0.01237
1.7	3.1391	1.37	0.32140	1.307	-0.00757
1.8	4.4656	1.38	0.37867	1.308	-0.00275
1.9	6.0731	1.39	0.43739	1.309	0.00207
2.0	8.0000	1.40	0.49760	1.310	0.00691

73. (a) The polynomial function $x^2 + 4x$ is continuous on the set of all real numbers and is non-negative on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$. The function $f(x) = \sqrt{x^2 + 4x} - 2$ is therefore continuous on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$, which contains the closed interval $[0, 1]$. Because $f(0) = \sqrt{0} - 2 = -2 < 0$ and $f(1) = \sqrt{5} - 2 \approx 0.236 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$.
- (b) Using the `FindRoot` command in the computer algebra system *Mathematica* produces the zero $\boxed{x \approx 0.828}$, rounded to three decimal places.
74. (a) The polynomial function $f(x) = x^3 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 0]$. Because $f(-2) = (-2)^3 - (-2) + 2 = -4 < 0$ and $f(0) = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-2, 0)$.
- (b) Using the `FindRoot` command in the computer algebra system *Mathematica* produces the zero $\boxed{x \approx -1.521}$, rounded to three decimal places.

Applications and Extensions

75. The function $u_1(t)$ is defined at $c = 1$ with $u_1(1) = 1$. Next,

$$\lim_{t \rightarrow 1^-} u_1(t) = \lim_{t \rightarrow 1^-} 0 = 0 \quad \text{and} \quad \lim_{t \rightarrow 1^+} u_1(t) = \lim_{t \rightarrow 1^+} 1 = 1.$$

Because the two one-sided limits at t approaches 1 are not equal, $\lim_{t \rightarrow 1} u_1(t)$ does not exist. Therefore, the function $u_1(t)$ is not continuous at $c = 1$.

76. The function $u_3(t)$ is defined at $c = 3$ with $u_3(3) = 1$. Next,

$$\lim_{t \rightarrow 3^-} u_3(t) = \lim_{t \rightarrow 3^-} 0 = 0 \quad \text{and} \quad \lim_{t \rightarrow 3^+} u_3(t) = \lim_{t \rightarrow 3^+} 1 = 1.$$

Because the two one-sided limits at t approaches 3 are not equal, $\lim_{t \rightarrow 3} u_3(t)$ does not exist. Therefore, the function $u_3(t)$ is not continuous at $c = 3$.

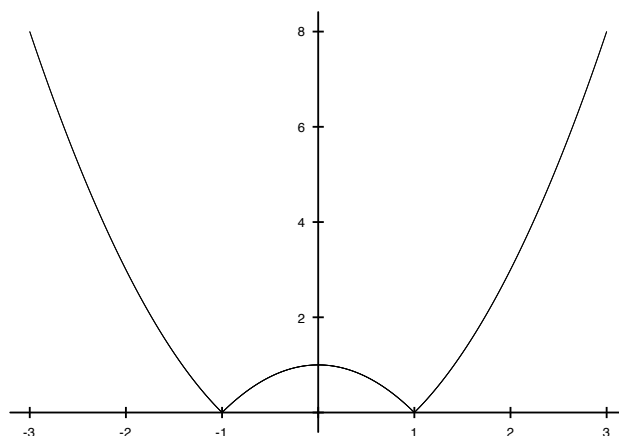
77. The function f is continuous on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ because the function is a polynomial on each of these intervals. At $x = -1$,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (x^2 - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (1 - x^2) = 0,$$

so that $\lim_{x \rightarrow -1} f(x)$ exists and is equal to 0. As $f(-1) = 0$, it follows that f is continuous at $x = -1$. Similarly, at $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1 - x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 - 1) = 0,$$

so that $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 0. As $f(1) = 0$, it follows that f is continuous at $x = 1$. Hence, f is continuous on the set of all real numbers. A graph of f is shown below.



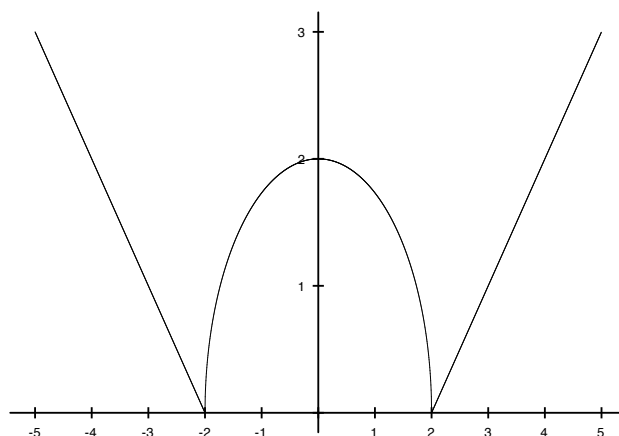
78. The function f is continuous on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ because the component functions are continuous on each of these intervals. At $x = -2$,

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (|x| - 2) = 0 \quad \text{and} \quad \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0,$$

so that $\lim_{x \rightarrow -2} f(x)$ exists and is equal to 0. As $f(-2) = 0$, it follows that f is continuous at $x = -2$. Similarly, at $x = 2$,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (|x| - 2) = 0,$$

so that $\lim_{x \rightarrow 2} f(x)$ exists and is equal to 0. As $f(2) = 0$, it follows that f is continuous at $x = 2$. Hence, f is continuous on the set of all real numbers. A graph of f is shown below.



79. (a) Because the Postal Service rounds the weight of the letter up to next whole number of ounces, the first-class postage charged is

$$C(w) = \begin{cases} 0.46, & \text{if } 0 < w \leq 1 \\ 0.66, & \text{if } 1 < w \leq 2 \\ 0.86, & \text{if } 2 < w \leq 3 \\ 1.06, & \text{if } 3 < w \leq 3.5. \end{cases}$$

where postage is measured in dollars and weight is measured in ounces. This can be written compactly in terms of the ceiling function as

$$C(w) = 0.46 + 0.20[w - 1].$$

- (b) The domain of C is the set $\{w | 0 < w \leq 3.5\}$.
- (c) The function C is continuous on the intervals $(0, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 3.5)$ because the function is a constant (polynomial) on each of these intervals. At $w = 1$,

$$\lim_{w \rightarrow 1^-} C(w) = \lim_{w \rightarrow 1^-} 0.46 = 0.46 \quad \text{and} \quad \lim_{w \rightarrow 1^+} = \lim_{w \rightarrow 1^+} 0.66 = 0.66,$$

so that $\lim_{w \rightarrow 1} C(w)$ does not exist. Similarly, at $w = 2$ and $w = 3$,

$$\lim_{w \rightarrow 2^-} C(w) = \lim_{w \rightarrow 2^-} 0.66 = 0.66 \quad \text{and} \quad \lim_{w \rightarrow 2^+} = \lim_{w \rightarrow 2^+} 0.86 = 0.86,$$

and

$$\lim_{w \rightarrow 3^-} C(w) = \lim_{w \rightarrow 3^-} 0.86 = 0.86 \quad \text{and} \quad \lim_{w \rightarrow 3^+} = \lim_{w \rightarrow 3^+} 1.06 = 1.06,$$

respectively, so that $\lim_{w \rightarrow 2} C(w)$ does not exist and $\lim_{w \rightarrow 3} C(w)$ does not exist. Therefore, C is not continuous at $w = 1$, $w = 2$, or $w = 3$. However,

$$\lim_{w \rightarrow 1^-} C(w) = 0.46 = C(1), \quad \lim_{w \rightarrow 2^-} C(w) = 0.66 = C(2), \quad \text{and} \quad \lim_{w \rightarrow 3^-} C(w) = 0.86 = C(3),$$

so C is continuous from the left at $w = 1$, $w = 2$, and $w = 3$. Additionally,

$$\lim_{w \rightarrow 3.5^-} C(w) = \lim_{w \rightarrow 3.5^-} 1.06 = 1.06 = C(3.5),$$

so C is continuous from the left at $w = 3.5$. Thus, C is continuous on the intervals $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 3.5]$.

- (d) At each number where C is not continuous ($w = 1$, $w = 2$, and $w = 3$), the two one-sided limits exist but are not equal, so each discontinuity is a **jump discontinuity**.
- (e) Answers will vary. One possible response is that because any fraction of an ounce results in a charge for a full ounce, it is in the consumer's best interest to have letters weigh as close as possible to a whole number of ounces, without going over.
80. (a) Because the Postal Service rounds the weight of the letter up to next whole number of ounces, the first-class postage charged is

$$C(w) = 0.92 + 0.20[w - 1],$$

for $0 < w \leq 13$, where postage is measured in dollars and weight is measured in ounces.

- (b) The domain of C is the set $\{w | 0 < w \leq 13\}$.

- (c) Let c be any whole number between 1 and 12, inclusive. Then C is continuous on the interval $(c-1, c)$, because the function is a constant (polynomial) on this interval. The function C is continuous on the interval $(12, 13)$ for the same reason. Now, at $w = c$,

$$\lim_{w \rightarrow c^-} C(w) = \lim_{w \rightarrow c^-} (0.92 + 0.20\lceil w - 1 \rceil) = 0.92 + 0.20(c-1)$$

and

$$\lim_{w \rightarrow c^+} C(w) = \lim_{w \rightarrow c^+} (0.92 + 0.20\lceil w - 1 \rceil) = 0.92 + 0.20c,$$

so that $\lim_{w \rightarrow c} C(w)$ does not exist and C is not continuous at $w = c$. However

$$\lim_{w \rightarrow c^-} C(w) = 0.92 + 0.20(c-1) = C(c),$$

so C is continuous from the left at $w = c$. Additionally,

$$\begin{aligned} \lim_{w \rightarrow 13^-} C(w) &= \lim_{w \rightarrow 13^-} (0.92 + 0.20\lceil w - 1 \rceil) = 0.92 + 0.20(12) \\ &= 3.32 = C(13), \end{aligned}$$

so C is continuous from the left at $w = 13$. Finally, C is continuous on the intervals

$(0, 1],$	$(1, 2],$	$(2, 3],$	$(3, 4],$	$(4, 5],$	$(5, 6],$	$(6, 7],$
$(7, 8],$	$(8, 9],$	$(9, 10],$	$(10, 11],$	$(11, 12],$	$(12, 13].$	

- (d) At each number where C is not continuous, the two one-sided limits exist but are not equal, so each discontinuity is a jump discontinuity.
- (e) Answers will vary. One possible response is that because any fraction of an ounce results in a charge for a full ounce, it is in the consumer's best interest to have retail flats weigh as close as possible to a whole number of ounces, without going over.
81. (a) Let x denote the monthly usage of natural gas, measured in therms, and let $C(x)$ denote the monthly cost. If $x \leq 50$, the cost consists of the monthly customer charge of \$22.13, the distribution charge of \$0.25963 per therm, and the gas charge of \$0.3631 per therm. Thus,

$$C(x) = 22.13 + 0.25963x + 0.3631x = 22.13 + 0.62273x.$$

On the other hand, if $x > 50$, the cost consists of the monthly customer charge of \$22.13, a charge of \$12.98 for the first 50 therms, the distribution charge of \$0.11806 for each therm over 50, and the gas charge of \$0.3631 per therm. Thus,

$$C(x) = 22.13 + 12.98 + 0.11806(x - 50) + 0.3631x = 29.207 + 0.48116x.$$

Bringing these two pieces together, the complete cost function is

$C(x) = \begin{cases} 22.13 + 0.62273x, & \text{if } 0 \leq x \leq 50 \\ 29.207 + 0.48116x, & \text{if } x > 50. \end{cases}$

- (b) The domain of C is the set $\{x | x \geq 0\}$.
- (c) The function C is continuous on the intervals $(0, 50)$ and $(50, \infty)$ because it is a polynomial on each of these intervals. At $x = 50$,

$$\lim_{x \rightarrow 50^-} C(x) = \lim_{x \rightarrow 50^-} (22.13 + 0.62273x) = \$53.27$$

and

$$\lim_{x \rightarrow 50^+} C(x) = \lim_{x \rightarrow 50^+} (29.207 + 0.48116x) = \$53.27,$$

so that $\lim_{x \rightarrow 50} C(x)$ exists and is equal to $C(50) = \$53.27$. Therefore, C is continuous at $x = 50$. Because

$$\lim_{x \rightarrow 0^+} C(x) = \lim_{x \rightarrow 0^+} (22.13 + 0.62273x) = 22.13 = C(0),$$

C is continuous from the right at $x = 0$. Therefore, C is continuous on the interval $[0, \infty)$.

- (d) The function C is continuous on its domain.
- (e) Answers will vary. One possible response is that from a purely mathematical standpoint, the two one-sided limits as x approaches 50 are not equal; however, from a practical cost standpoint, the two one-sided limits correspond to the same cost (\$53.27). Thus, from a practical standpoint, the cost function is continuous at $x = 50$.
82. (a) Using the rate schedule provided,

$$C(x) = \begin{cases} 9.00, & \text{if } 0 \leq x \leq 10 \\ 9.00 + 0.95(x - 10), & \text{if } 10 < x \leq 30 \\ 28.00 + 1.65(x - 30), & \text{if } 30 < x \leq 100 \\ 143.50 + 2.20(x - 100), & \text{if } x > 100. \end{cases}$$

- (b) The domain of C is the set $\{x | x \geq 0\}$.
- (c) The function C is continuous on the intervals $(0, 10)$, $(10, 30)$, $(30, 100)$, and $(100, \infty)$, because it is a polynomial on each of these intervals. At $x = 10$,

$$\lim_{x \rightarrow 10^-} C(x) = \lim_{x \rightarrow 10^-} 9.00 = 9.00$$

and

$$\lim_{x \rightarrow 10^+} C(x) = \lim_{x \rightarrow 10^+} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(10 - 10) = 9.00,$$

so that $\lim_{x \rightarrow 10} C(x)$ exists and is equal to 9.00. As $C(10) = 9.00$, it follows that C is continuous at $x = 10$. Similarly,

$$\lim_{x \rightarrow 30^-} C(x) = \lim_{x \rightarrow 30^-} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(30 - 10) = 28.00$$

and

$$\lim_{x \rightarrow 30^+} C(x) = \lim_{x \rightarrow 30^+} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(30 - 30) = 28.00,$$

so that $\lim_{x \rightarrow 30} C(x) = 28.00 = C(30)$ and C is continuous at $x = 30$. At $w = 100$,

$$\lim_{x \rightarrow 100^-} C(x) = \lim_{x \rightarrow 100^-} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(100 - 30) = 143.50$$

and

$$\lim_{x \rightarrow 100^+} C(x) = \lim_{x \rightarrow 100^+} [143.50 + 2.20(x - 100)] = 143.50 + 2.20(100 - 100) = 143.50,$$

so that $\lim_{x \rightarrow 100} C(x)$ exists and is equal to $143.50 = C(100)$. Thus, C is continuous at $x = 100$. Finally,

$$\lim_{x \rightarrow 0^+} C(x) = \lim_{x \rightarrow 0^+} 9.00 = 9.00 = C(0),$$

so C is continuous from the right at $x = 0$. Thus, C is continuous on $[0, \infty)$.

- (d) The function C is continuous on its domain.

- (e) Answers will vary. One possible response is that there is no “penalty” to the consumer who goes just a little over 10,000 or 30,000 or 100,000 gallons rather than trying to keep consumption at or a little below these amounts.

83. (a) Because

$$\lim_{r \rightarrow R^-} g(r) = \lim_{r \rightarrow R^-} \frac{Gm}{R^3} r = \frac{Gm}{R^2}$$

and

$$\lim_{r \rightarrow R^+} g(r) = \lim_{r \rightarrow R^+} \frac{Gm}{r^2} = \frac{Gm}{R^2}$$

are equal, $\boxed{g(R) \text{ must equal } \frac{Gm}{R^2}}$ in order for the gravitational field of Europa to be continuous at its surface.

- (b) With $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, $m = 4.8 \times 10^{22} \text{ kg}$ and $R = 1.569 \times 10^6 \text{ m}$,

$$g(R) = \frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \cdot 4.8 \times 10^{22} \text{ kg}}{(1.569 \times 10^6 \text{ m})^2} \approx \boxed{1.3 \text{ m/s}^2}.$$

- (c) Europa’s gravity is $\boxed{\text{less than}}$ that on Earth.

84. The function f is continuous on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$, because it is a polynomial on each of these intervals. At $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x - 1)^2 = 1$$

and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (A - x)^2 = A^2 = f(0).$$

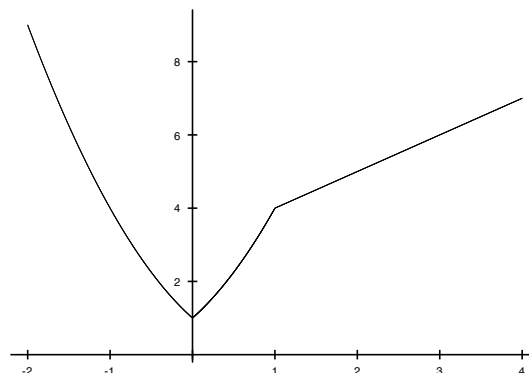
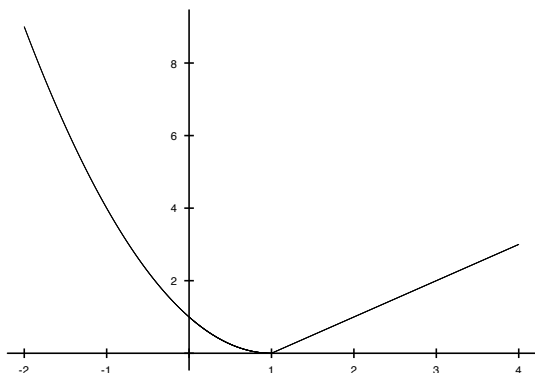
For f to be continuous at $x = 0$, the constant A must satisfy $A^2 = 1$, or $A = \pm 1$. At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (A - x)^2 = (A - 1)^2$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + B) = 1 + B = f(1).$$

For f to be continuous at $x = 1$, the constant B must satisfy $1 + B = (A - 1)^2$, or $B = A^2 - 2A$. There are therefore two sets of values for A and B for which the function f is continuous for all x : $\boxed{\{A = 1, B = -1\} \text{ and } \{A = -1, B = 3\}}$. The figure below left displays the graph with $A = 1$ and $B = -1$; the figure below right displays the graph with $A = -1$ and $B = 3$.



85. The function f is continuous on the intervals $(-\infty, 4)$, $(4, 9)$, and $(9, \infty)$, because it is a polynomial on each of these intervals. At $x = 4$,

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (x + A) = 4 + A$$

and

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} (x - 1)^2 = 9 = f(4).$$

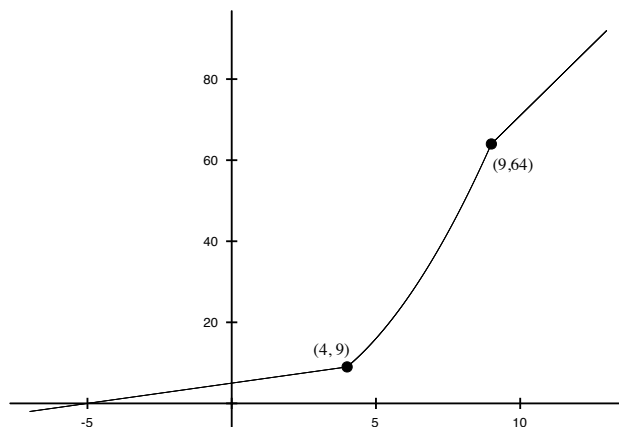
For f to be continuous at $x = 4$, the constant A must satisfy $4 + A = 9$, or $A = 5$. At $x = 9$,

$$\lim_{x \rightarrow 9^-} f(x) = \lim_{x \rightarrow 9^-} (x - 1)^2 = 64 = f(9)$$

and

$$\lim_{x \rightarrow 9^+} f(x) = \lim_{x \rightarrow 9^+} (Bx + 1) = 9B + 1.$$

For f to be continuous at $x = 9$, the constant B must satisfy $9B + 1 = 64$, or $B = 7$. Therefore, the function f will be continuous for all x provided $A = 5$ and $B = 7$. The graph of the resulting function is shown below



86. In order to make f continuous at $x = 2$, k should be set equal to

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \cdot \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} \\ &= \lim_{x \rightarrow 2} \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = \frac{1}{\sqrt{9} + \sqrt{9}} = \boxed{\frac{1}{6}}. \end{aligned}$$

87. Let

$$f(x) = \frac{x^2 - 6x - 16}{(x^2 - 7x - 8)\sqrt{x^2 - 4}} = \frac{(x-8)(x+2)}{(x-8)(x+1)\sqrt{x^2 - 4}}.$$

- (a) The function f is defined for all values x for which the denominator is not equal to zero and $x^2 - 4 > 0$. Thus, the domain of f is the set $\{x|x < -2\} \cup \{x|x > 2, x \neq 8\}$. Note that the condition $x \neq -1$ need not be explicitly included because -1 does not have an absolute value greater than 2, and so is already eliminated by virtue of this condition.
- (b) Because the function f is the product, quotient and composition of functions that are continuous on their domains, f is continuous on its domain. Thus, f is discontinuous at $x = 8$ and on the interval $[-2, 2]$.

(c) Because

$$\lim_{x \rightarrow 8} f(x) = \lim_{x \rightarrow 8} \frac{(x-8)(x+2)}{(x-8)(x+1)\sqrt{x^2-4}} = \lim_{x \rightarrow 8} \frac{(x+2)}{(x+1)\sqrt{x^2-4}} = \frac{10}{9\sqrt{60}} = \frac{5}{9\sqrt{15}}$$

exists, the discontinuity at $x = 8$ is removable.

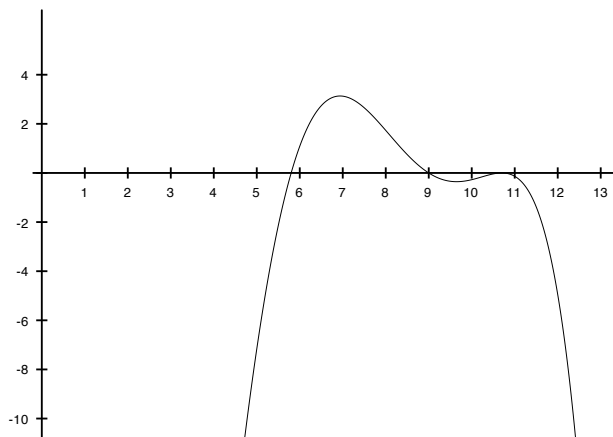
88. (a) Because the function $f(x) = \sin x + x - 3$ is the sum of the sine function and a polynomial function, both of which are continuous on the set of all real numbers, f is continuous on the set of all real numbers and is therefore continuous on the closed interval $[0, \pi]$. Now, $f(0) = \sin 0 + 0 - 3 = -3 < 0$ and $f(\pi) = \sin \pi + \pi - 3 = \pi - 3 > 0$, so the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, \pi)$.

(b) Using a TI-84 Plus calculator, the zero is $x \approx 2.180$, rounded to three decimal places.

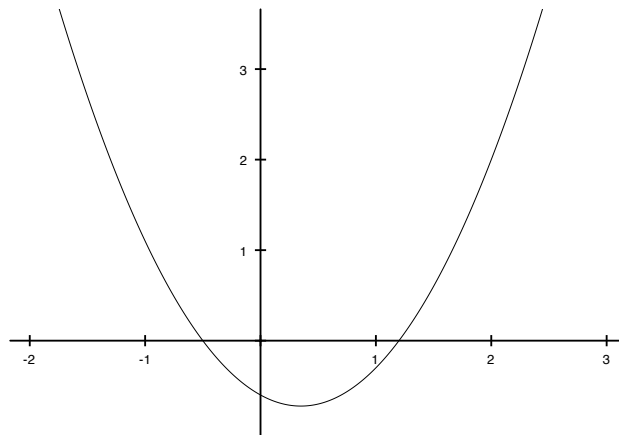
89. (a) Because the function $f(x) = e^x + x - 2$ is the sum of an exponential function and a polynomial function, both of which are continuous on the set of all real numbers, f is continuous on the set of all real numbers and is therefore continuous on the closed interval $[0, 2]$. Now, $f(0) = e^0 + 0 - 2 = -1 < 0$ and $f(2) = e^2 + 2 - 2 = e^2 > 0$, so the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 2)$.

(b) Using a TI-84 Plus calculator, the zero is $x \approx 0.443$, rounded to three decimal places.

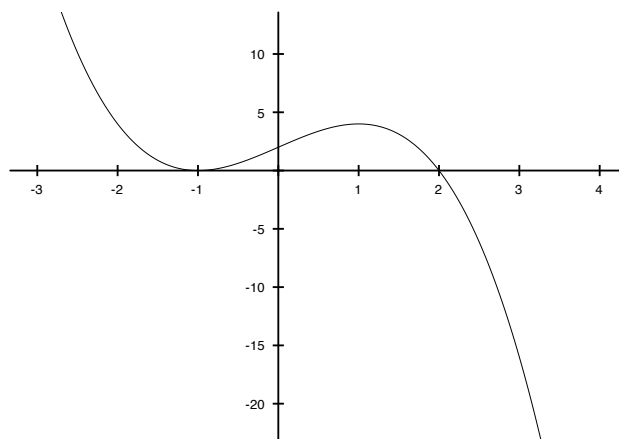
90. Answers will vary. The figure below displays the graph of a function that is continuous on $[5, 12]$, that is negative at both endpoints, and has exactly three zeros in the interval. This does not contradict the Intermediate Value Theorem, because the theorem provides no information about zeros of a function when the endpoint values are the same sign.



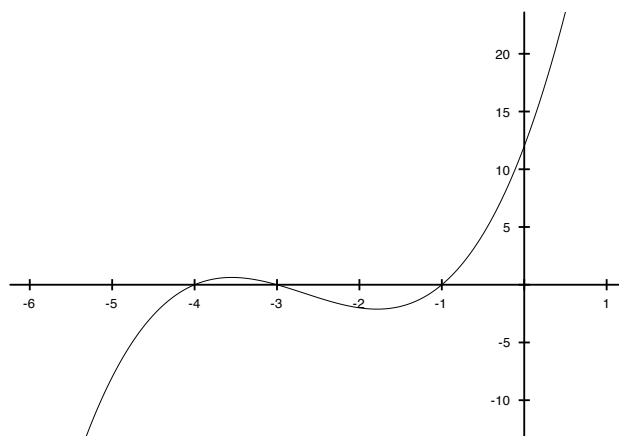
91. Answers will vary. The figure below displays the graph of a function that is continuous on $[-1, 2]$, that is positive at both endpoints, and has exactly two zeros in the interval. This does not contradict the Intermediate Value Theorem, because the theorem provides no information about zeros of a function when the endpoint values are the same sign.



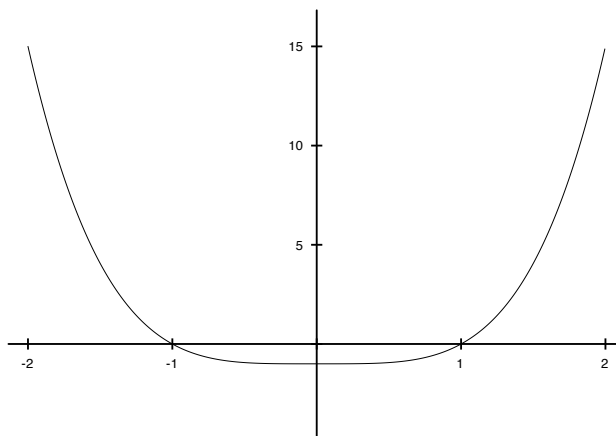
92. Answers will vary. The figure below displays the graph of a function that is continuous on $[-2, 3]$, that is positive at -2 and negative at 3 , and has exactly two zeros in the interval. This does not contradict the Intermediate Value Theorem, because the theorem guarantees that the function has **at least** one zero on the interval $(-2, 3)$.



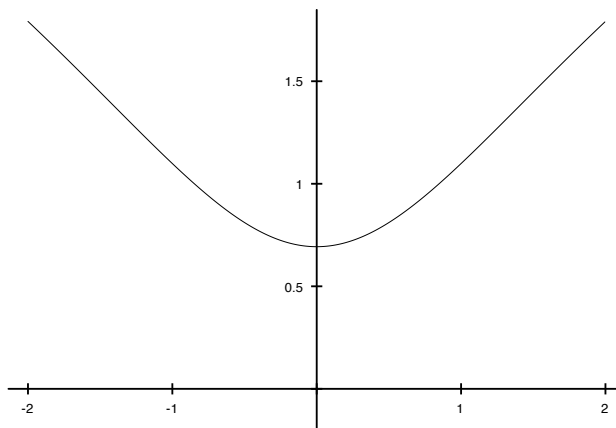
93. Answers will vary. The figure below displays the graph of a function that is continuous on $[-5, 0]$, that is negative at -5 and positive at 0 , and has exactly three zeros in the interval. This does not contradict the Intermediate Value Theorem, because the theorem guarantees that the function has **at least** one zero on the interval $(-5, 0)$.



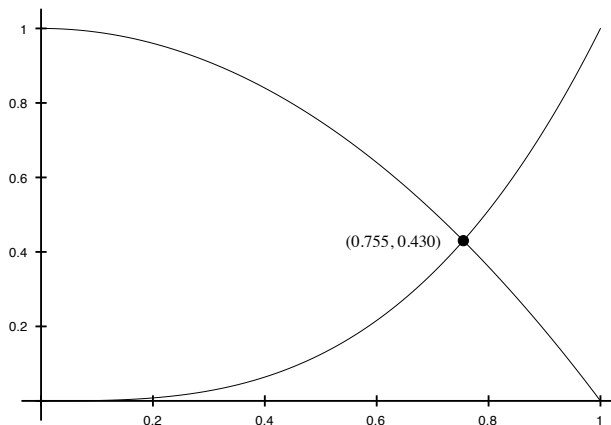
94. (a) Although the function $f(x) = x^4 - 1$ is continuous on the closed interval $[-2, 2]$, $f(-2) = (-2)^4 - 1 = 15 > 0$ and $f(2) = 2^4 - 1 = 15 > 0$. Because the function has the same sign at both endpoints, the Intermediate Value Theorem gives no information about the zeros of f on the interval $(-2, 2)$.
- (b) The graph of f shown below indicates that f has two zeros on the interval $(-2, 2)$: one at $x = -1$, the other at $x = 1$.



95. (a) Although the function $f(x) = \ln(x^2 + 2)$ is continuous on the closed interval $[-2, 2]$, $f(-2) = \ln 6 > 0$ and $f(2) = \ln 6 > 0$. Because the function has the same sign at both endpoints, the Intermediate Value Theorem gives no information about the zeros of f on the interval $(-2, 2)$.
- (b) The graph of f shown below indicates that f has no zero on the interval $[-2, 2]$.



96. (a) If the graphs of the functions $y = x^3$ and $y = 1 - x^2$ intersect, then the x -coordinate of the point of intersection must be a solution of the equation $x^3 = 1 - x^2$, or $x^3 + x^2 - 1 = 0$. Let $f(x) = x^3 + x^2 - 1$. This function is continuous on the closed interval $[0, 1]$ with $f(0) = -1 < 0$ and $f(1) = 1 > 0$. The Intermediate Value Theorem therefore guarantees that f has a zero on the interval $(0, 1)$. Hence, the graphs of the functions $y = x^3$ and $y = 1 - x^2$ do intersect somewhere between $x = 0$ and $x = 1$.
- (b) Using a TI-84 Plus calculator, the point of intersection, rounded to three decimal places, is $(0.755, 0.430)$.
- (c) The figure below displays the graphs of both functions with the point of intersection labeled to three decimal places.



97. Let $v(t)$ denote the speed of the airplane as a function of time. Further, let t_1 denote a time when the speed of the airplane was 620 miles per hour, $t_2 > t_1$ denote a time when the speed of the airplane had slowed to 608 miles per hour, and $t_3 > t_2$ denote a time when the speed of the airplane had increased to 614 miles per hour. Now, consider the function $f(t) = v(t) - 610$. Assuming $v(t)$ is continuous for all t , f is also continuous for all t . Because

$$f(t_1) = v(t_1) - 610 = 620 - 610 = 10 > 0$$

and

$$f(t_2) = v(t_2) - 610 = 608 - 610 = -2 < 0,$$

the Intermediate Value Theorem guarantees that $f(t) = 0$, or $v(t) = 610$, for some time between t_1 and t_2 . Similarly,

$$f(t_2) = v(t_2) - 610 = 608 - 610 = -2 < 0$$

and

$$f(t_3) = v(t_3) - 610 = 614 - 610 = 4 > 0,$$

so the Intermediate Value Theorem guarantees that $f(t) = 0$, or $v(t) = 610$, for some time between t_2 and t_3 . Thus, the airplane's speed is 610 miles per hour on at least two different occasions during the flight.

98. Let f be a function that is defined and continuous on the closed interval $[a, b]$. The function

$$h(x) = \frac{1}{f(x)}$$

will therefore be continuous on the closed interval $[a, b]$, except for those values x at which $f(x) = 0$. Thus, if f is never zero on the closed interval $[a, b]$, then h will be continuous on the closed interval $[a, b]$.

99. (a) Factoring, $f(x) = x^3 - 3x^2 - 4x + 12 = (x - 3)(x^2 - 4) = (x - 3)(x - 2)(x + 2)$. Thus, the zeros of the function f are $x = -2$, $x = 2$, and $x = 3$.
- (b) The function h will be continuous at $x = 3$ provided $p = \lim_{x \rightarrow 3} h(x)$. Now,

$$\lim_{x \rightarrow 3} h(x) = \lim_{x \rightarrow 3} \frac{(x - 3)(x^2 - 4)}{x - 3} = \lim_{x \rightarrow 3} (x^2 - 4) = 5.$$

Therefore, h is continuous at $x = 3$ when $p = 5$.

- (c) With $p = 5$, the function h reduces to $x^2 - 4$ for all x . Because

$$h(-x) = (-x)^2 - 4 = x^2 - 4 = h(x),$$

$h(x)$ is an even function.

100. Consider the one-sided limits as x approaches 0:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} -1 = -1$$

and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1.$$

Because the two one-sided limits as x approaches 0 are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist. Therefore, the discontinuity at $x = 0$ is not removable; e.g., it is impossible to define $f(0)$ so that f is continuous at $x = 0$.

101. Answers will vary. One possible response is the following. The polynomial functions $f(x) = x^2 - 1$ and $g(x) = x - 3$ are continuous at $c = 3$; however, because $g(3) = 0$, the function $\frac{f}{g}$ is not continuous at $c = 3$.

102. Answers will vary. One possible response is the following. A discontinuity at $x = c$ is removable when the limit as x approaches c exists but that limit is not equal to the function value at $x = c$; a discontinuity at $x = c$ is nonremovable when the limit as x approaches c does not exist. An example of a removable discontinuity is $x = 3$ for the function $f(x) = \frac{x^2 - 9}{x - 3}$. Because

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 6,$$

the discontinuity at $x = 3$ can be removed by defining $f(3) = 6$. An example of a nonremovable discontinuity is $x = 3$ for the function

$$g(x) = \begin{cases} x + 3, & x \leq 3 \\ 9 - x^2, & x > 3. \end{cases}$$

Here,

$$\lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3^-} (x + 3) = 6 \quad \text{but} \quad \lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^+} (9 - x^2) = 0.$$

Because the two one-sided limits as x approaches 3 are not equal, $\lim_{x \rightarrow 3} g(x)$ does not exist.

103. Let $f(x) = x^3 + 3x - 5$, and note that $f(1) = -1 < 0$ and $f(2) = 9 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 2.875 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) = 0.703125 > 0$. The sign of $f(1.25)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.25)$. Finally, set $m_3 = 1.125$, the midpoint of the interval $(1, 1.25)$, and calculate $f(m_3) = f(1.125) \approx -0.201172 < 0$. The sign of $f(1.125)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.125, 1.25)$ and is given approximately by the midpoint of this subinterval $m_4 = 1.1875$.
104. Let $f(x) = x^3 - 4x + 2$, and note that $f(1) = -1 < 0$ and $f(2) = 2 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = -0.625 < 0$. The sign of $f(1.5)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(1.5, 2)$. Now repeat the process and set $m_2 = 1.75$, the midpoint of the interval $(1.5, 2)$. Next, calculate $f(m_2) = f(1.75) = 0.359375 > 0$. The sign of $f(1.75)$ is opposite that of $f(1.5)$, so the zero lies in the subinterval $(1.5, 1.75)$. Finally, set $m_3 = 1.625$, the midpoint of the interval $(1.5, 1.75)$, and calculate $f(m_3) = f(1.625) \approx -0.208984 < 0$. The sign of $f(1.625)$ is opposite that of $f(1.75)$, so the zero lies in the subinterval $(1.625, 1.75)$ and is given approximately by the midpoint of this subinterval $m_4 = 1.6875$.

105. Let $f(x) = 2x^3 + 3x^2 + 4x - 1$, and note that $f(0) = -1 < 0$ and $f(1) = 8 > 0$. Set $m_1 = 0.5$, the midpoint of the interval $(0, 1)$, and then calculate $f(m_1) = f(0.5) = 2 > 0$. The sign of $f(0.5)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.5)$. Now repeat the process and set $m_2 = 0.25$, the midpoint of the interval $(0, 0.5)$. Next, calculate $f(m_2) = f(0.25) = 0.21875 > 0$. The sign of $f(0.25)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.25)$. Finally, set $m_3 = 0.125$, the midpoint of the interval $(0, 0.25)$, and calculate $f(m_3) = f(0.125) \approx -0.449219 < 0$. The sign of $f(0.125)$ is opposite that of $f(0.25)$, so the zero lies in the subinterval $(0.125, 0.25)$ and is given approximately by the midpoint of this subinterval $m_4 = 0.1875$.
106. Let $f(x) = x^3 - x^2 - 2x + 1$, and note that $f(0) = 1 > 0$ and $f(1) = -1 < 0$. Set $m_1 = 0.5$, the midpoint of the interval $(0, 1)$, and then calculate $f(m_1) = f(0.5) = -0.125 < 0$. The sign of $f(0.5)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.5)$. Now repeat the process and set $m_2 = 0.25$, the midpoint of the interval $(0, 0.5)$. Next, calculate $f(m_2) = f(0.25) = 0.453125 > 0$. The sign of $f(0.25)$ is opposite that of $f(0.5)$, so the zero lies in the subinterval $(0.25, 0.5)$. Finally, set $m_3 = 0.375$, the midpoint of the interval $(0.25, 0.5)$, and calculate $f(m_3) = f(0.375) \approx 0.162109 > 0$. The sign of $f(0.375)$ is opposite that of $f(0.5)$, so the zero lies in the subinterval $(0.375, 0.5)$ and is given approximately by the midpoint of this subinterval $m_4 = 0.4375$.
107. Let $f(x) = x^3 - 6x - 12$, and note that $f(3) = -3 < 0$ and $f(4) = 28 > 0$. Set $m_1 = 3.5$, the midpoint of the interval $(3, 4)$, and then calculate $f(m_1) = f(3.5) = 9.875 > 0$. The sign of $f(3.5)$ is opposite that of $f(3)$, so the zero lies in the subinterval $(3, 3.5)$. Now repeat the process and set $m_2 = 3.25$, the midpoint of the interval $(3, 3.5)$. Next, calculate $f(m_2) = f(3.25) = 2.828125 > 0$. The sign of $f(3.25)$ is opposite that of $f(3)$, so the zero lies in the subinterval $(3, 3.25)$. Finally, set $m_3 = 3.125$, the midpoint of the interval $(3, 3.25)$, and calculate $f(m_3) = f(3.125) \approx -0.232422 < 0$. The sign of $f(3.125)$ is opposite that of $f(3.25)$, so the zero lies in the subinterval $(3.125, 3.25)$ and is given approximately by the midpoint of this subinterval $m_4 = 3.1875$.
108. Let $f(x) = 3x^3 + 5x - 40$, and note that $f(2) = -6 < 0$ and $f(3) = 56 > 0$. Set $m_1 = 2.5$, the midpoint of the interval $(2, 3)$, and then calculate $f(m_1) = f(2.5) = 19.375 > 0$. The sign of $f(2.5)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(2, 2.5)$. Now repeat the process and set $m_2 = 2.25$, the midpoint of the interval $(2, 2.5)$. Next, calculate $f(m_2) = f(2.25) = 5.421875 > 0$. The sign of $f(2.25)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(2, 2.25)$. Finally, set $m_3 = 2.125$, the midpoint of the interval $(2, 2.25)$, and calculate $f(m_3) = f(2.125) \approx -0.587891 < 0$. The sign of $f(2.125)$ is opposite that of $f(2.25)$, so the zero lies in the subinterval $(2.125, 2.25)$ and is given approximately by the midpoint of this subinterval $m_4 = 2.1875$.
109. Let $f(x) = x^4 - 2x^3 + 21x - 23$, and note that $f(1) = -3 < 0$ and $f(2) = 19 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 6.8125 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) \approx 1.785156 > 0$. The sign of $f(1.25)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.25)$. Finally, set $m_3 = 1.125$, the midpoint of the interval $(1, 1.25)$, and calculate $f(m_3) = f(1.125) \approx -0.620850 < 0$. The sign of $f(1.125)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.125, 1.25)$ and is given approximately by the midpoint of this subinterval $m_4 = 1.1875$.
110. Let $f(x) = x^4 - x^3 + x - 2$, and note that $f(1) = -1 < 0$ and $f(2) = 8 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 1.1875 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) \approx -0.261719 < 0$. The sign of $f(1.25)$ is opposite that of $f(1.5)$, so the zero lies in the subinterval $(1.25, 1.5)$. Finally, set $m_3 = 1.375$, the midpoint of the interval $(1.25, 1.5)$, and calculate $f(m_3) = f(1.375) \approx 0.349854 > 0$. The sign of $f(1.375)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.25, 1.375)$ and is given approximately by the midpoint of this subinterval $m_4 = 1.3125$.

111. The polynomial function $x^2 + 4x$ is continuous on the set of all real numbers and is non-negative on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$. The function $f(x) = \sqrt{x^2 + 4x} - 2$ is therefore continuous on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$, which contains the closed interval $[0, 1]$. Because $f(0) = \sqrt{0} - 2 = -2 < 0$ and $f(1) = \sqrt{5} - 2 \approx 0.236 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$.

To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint, looking for two successive function values with opposite signs. This yields $f(0.8) \approx -0.040408$ and $f(0.9) = 0.1$, indicating that the zero lies in the subinterval $(0.8, 0.9)$. Thus, correct to one decimal place, the zero is $\boxed{x = 0.8}$.

112. The polynomial function $f(x) = x^3 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 0]$. Because $f(-2) = (-2)^3 - (-2) + 2 = -4 < 0$ and $f(0) = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-2, 0)$.

To approximate this zero, subdivide the interval $[-2, 0]$ into 20 subintervals, each of length 0.1, and evaluate f at each endpoint, looking for two successive function values with opposite signs. This yields $f(-1.6) = -0.496$ and $f(-1.5) = 0.125$, indicating that the zero lies in the subinterval $(-1.6, -1.5)$. Next, subdivide the interval $[-1.6, -1.5]$ into 10 subintervals, each of length 0.01. The two successive function values that are of opposite sign are $f(-1.53) = -0.051577$ and $f(-1.52) = 0.008192$, so the zero has now been isolated to the interval $(-1.53, -1.52)$. Thus, correct to two decimal places, the zero is $\boxed{x = -1.52}$.

113. Let f and g be functions that are continuous at c . Then,

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = g(c).$$

To prove that $f + g$ is continuous at c , it must be shown that $\lim_{x \rightarrow c} [f(x) + g(x)] = f(c) + g(c)$. Using the Limit of a Sum Property, it follows that

$$\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = f(c) + g(c),$$

as required.

114. Let f and g be functions that are continuous on the closed interval $[a, b]$, with $f(a) < g(a)$ and $f(b) > g(b)$. Define the function $h(x) = f(x) - g(x)$. Because f and g are both continuous on $[a, b]$, it follows that h is also continuous on $[a, b]$. Now,

$$h(a) = f(a) - g(a) < 0 \quad \text{and} \quad h(b) = f(b) - g(b) > 0.$$

Thus, by the Intermediate Value Theorem, there is a number c between a and b such that $h(c) = f(c) - g(c) = 0$, or $f(c) = g(c)$. This implies that the graphs of $y = f(x)$ and $y = g(x)$ intersect at $x = c$; that is, the graphs intersect somewhere between $x = a$ and $x = b$.

Challenge Problems

115. Let $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$. Because f is continuous on the interval $(1, 2)$, it is continuous on any closed interval contained within $(1, 2)$, say $[1.1, 1.9]$. With

$$f(1.1) = \frac{1}{1.1-1} + \frac{1}{1.1-2} = 10 - \frac{10}{9} = \frac{80}{9} > 0$$

and

$$f(1.9) = \frac{1}{1.9-1} + \frac{1}{1.9-2} = \frac{10}{9} - 10 = -\frac{80}{9} < 0,$$

the Intermediate value Theorem guarantees there exists a number c between 1.1 and 1.9, and hence between 1 and 2, such that $f(c) = 0$.

116. Let $f(x) = x^2 - 7$. This polynomial function is continuous on the closed interval $[2.64, 2.65]$. With

$$f(2.64) = 2.64^2 - 7 = -0.0304 < 0 \quad \text{and} \quad f(2.65) = 2.65^2 - 7 = 0.0225 > 0,$$

the Intermediate Value Theorem guarantees there exists a number c between 2.64 and 2.65 such that $f(c) = c^2 - 7 = 0$, or $c^2 = 7$.

117. Let f be a function for which $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists. Now, let $x = a + h$. Then, as h approaches 0, x approaches a , and

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

To prove that f is continuous at $x = a$, it must be shown that $\lim_{x \rightarrow a} f(x) = f(a)$. Because $f(a)$ is a constant, $\lim_{x \rightarrow a} f(a) = f(a)$ and

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \text{is equivalent to} \quad \lim_{x \rightarrow a} [f(x) - f(a)] = 0.$$

Proceeding with this last limit, we find

$$\lim_{x \rightarrow a} [f(x) - f(a)] = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (x - a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot 0 = 0,$$

where, in going from the first line to the second line, we have used the fact that $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists so the Limit of a Product property applies.

118. The rational function $\frac{x^2 + x - 2}{x - 1}$ is continuous on the set of all real numbers except $x = 1$, so f is continuous on the interval $(-\infty, 1)$. The function f is also continuous on the intervals $(1, 4)$ and $(4, \infty)$ because it is a polynomial on each of these intervals. At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x^2 + x - 2}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(x - 1)(x + 2)}{x - 1} = \lim_{x \rightarrow 1^-} (x + 2) = 3$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} B(x - C)^2 = B(1 - C)^2.$$

Then, at $x = 4$,

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} B(x - C)^2 = B(4 - C)^2$$

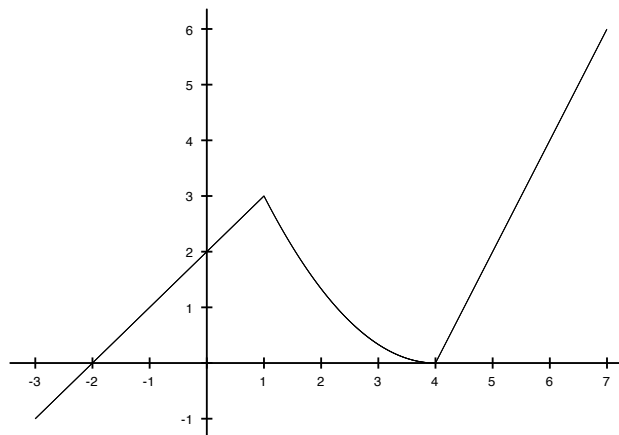
and

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} (2x - 8) = 0.$$

Therefore, for f to be continuous at $x = 1$ and at $x = 4$, the constants A , B , C , and D must satisfy the equations

$$A = 3, \quad B(1 - C)^2 = 3, \quad B(4 - C)^2 = 0, \quad D = 0.$$

The solution of these equations is $A = 3, B = \frac{1}{3}, C = 4$, and $D = 0$. The figure below displays a graph of f with these values for the constants.



119. Let f be a function that is continuous on the closed interval $[0, 1]$ and for which $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Define the function $g(x) = x - f(x)$. Because g is the difference between two functions that are continuous on $[0, 1]$, g is also continuous on $[0, 1]$. Now,

$$g(0) = 0 - f(0) \leq 0 \quad \text{and} \quad g(1) = 1 - f(1) \geq 0.$$

If either $g(0) = 0$ or $g(1) = 0$, then either $f(0) = 0$ or $f(1) = 1$ and a c in $[0, 1]$ has been found such that $f(c) = c$. Otherwise, $g(0) < 0$ and $g(1) > 0$, so that the Intermediate Value Theorem guarantees there exists a c in $(0, 1)$ such that $g(c) = 0$, or $f(c) = c$.

1.4: Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions

Concepts and Vocabulary

- $\lim_{x \rightarrow 0} \sin x = \sin 0 = \boxed{0}$.
- $\boxed{\text{False}}$. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$.
- The Squeeze Theorem states that if functions f , g , and h have the property $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing c , except possibly at c , and if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} g(x) = \boxed{L}$.
- $\boxed{\text{False}}$. $f(x) = \csc x$ is continuous for all real numbers except $x = k\pi$, where k is any integer.

Skill Building

- Because $-x^2 + 1 \leq g(x) \leq x^2 + 1$ for all x in an open interval containing 0 and

$$\lim_{x \rightarrow 0} (-x^2 + 1) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (x^2 + 1) = 1,$$

it follows from the Squeeze Theorem that $\lim_{x \rightarrow 0} g(x) = \boxed{1}$.

- Because $-(x - 2)^2 - 3 \leq g(x) \leq (x - 2)^2 - 3$ for all x in an open interval containing 2 and

$$\lim_{x \rightarrow 2} [-(x - 2)^2 - 3] = -3 \quad \text{and} \quad \lim_{x \rightarrow 2} [(x - 2)^2 - 3] = -3,$$

it follows from the Squeeze Theorem that $\lim_{x \rightarrow 2} g(x) = \boxed{-3}$.

- Because $\cos x \leq g(x) \leq 1$ for all x in an open interval containing 0 and

$$\lim_{x \rightarrow 0} \cos x = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} 1 = 1,$$

it follows from the Squeeze Theorem that $\lim_{x \rightarrow 0} g(x) = \boxed{1}$.

- Because $-x^2 + 1 \leq g(x) \leq \sec x$ for all x in an open interval containing 0 and

$$\lim_{x \rightarrow 0} (-x^2 + 1) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \sec x = 1,$$

it follows from the Squeeze Theorem that $\lim_{x \rightarrow 0} g(x) = \boxed{1}$.

- $\lim_{x \rightarrow 0} (x^3 + \sin x) = 0^3 + \sin 0 = 0 + 0 = \boxed{0}$.
- $\lim_{x \rightarrow 0} (x^2 - \cos x) = 0^2 - \cos 0 = 0 - 1 = \boxed{-1}$.
- $\lim_{x \rightarrow \pi/3} (\cos x + \sin x) = \cos \frac{\pi}{3} + \sin \frac{\pi}{3} = \boxed{\frac{\sqrt{3}}{2} + \frac{1}{2}}$.
- $\lim_{x \rightarrow \pi/3} (\sin x - \cos x) = \sin \frac{\pi}{3} - \cos \frac{\pi}{3} = \boxed{\frac{1}{2} - \frac{\sqrt{3}}{2}}$.

$$13. \lim_{x \rightarrow 0} \frac{\cos x}{1 + \sin x} = \frac{\cos 0}{1 + \sin 0} = \frac{1}{1 + 0} = \boxed{1}.$$

$$14. \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = \frac{\sin 0}{1 + \cos 0} = \frac{0}{1 + 1} = \boxed{0}.$$

$$15. \lim_{x \rightarrow 0} \frac{3}{1 + e^x} = \frac{3}{1 + e^0} = \frac{3}{1 + 1} = \boxed{\frac{3}{2}}.$$

$$16. \lim_{x \rightarrow 0} \frac{e^x - 1}{1 + e^x} = \frac{e^0 - 1}{1 + e^0} = \frac{1 - 1}{1 + 1} = \boxed{0}.$$

$$17. \lim_{x \rightarrow 0} (e^x \sin x) = e^0 \sin 0 = 1(0) = \boxed{0}.$$

$$18. \lim_{x \rightarrow 0} (e^{-x} \tan x) = e^{-0} \tan 0 = 1(0) = \boxed{0}.$$

$$19. \lim_{x \rightarrow 1} \ln \left(\frac{e^x}{x} \right) = \ln \left(\frac{e^1}{1} \right) = \ln e = \boxed{1}.$$

$$20. \lim_{x \rightarrow 1} \ln \left(\frac{x}{e^x} \right) = \ln \left(\frac{1}{e^1} \right) = \ln e^{-1} = \boxed{-1}.$$

$$21. \lim_{x \rightarrow 0} \frac{e^{2x}}{1 + e^x} = \frac{e^{2(0)}}{1 + e^0} = \frac{1}{1 + 1} = \boxed{\frac{1}{2}}.$$

$$22. \lim_{x \rightarrow 0} \frac{1 - e^x}{1 - e^{2x}} = \lim_{x \rightarrow 0} \frac{1 - e^x}{(1 - e^x)(1 + e^x)} = \lim_{x \rightarrow 0} \frac{1}{1 + e^x} = \frac{1}{1 + e^0} = \frac{1}{1 + 1} = \boxed{\frac{1}{2}}.$$

$$23. \lim_{x \rightarrow 0} \frac{\sin(7x)}{x} = \lim_{x \rightarrow 0} \frac{7 \sin(7x)}{7x} = 7 \lim_{x \rightarrow 0} \frac{\sin(7x)}{7x} = 7(1) = \boxed{7}.$$

$$24. \lim_{x \rightarrow 0} \frac{\sin(\frac{x}{3})}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{3} \sin(\frac{x}{3})}{\frac{x}{3}} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin(\frac{x}{3})}{\frac{x}{3}} = \frac{1}{3}(1) = \boxed{\frac{1}{3}}.$$

$$25. \lim_{\theta \rightarrow 0} \frac{\theta + 3 \sin \theta}{2\theta} = \lim_{\theta \rightarrow 0} \frac{\theta}{2\theta} + \frac{3}{2} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{1}{2} + \frac{3}{2} = \boxed{2}.$$

$$26. \lim_{x \rightarrow 0} \frac{2x - 5 \sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{2x}{x} - 5 \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} 2 - 5 \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 2 - 15 = \boxed{-13}.$$

27. First note that

$$\lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} = \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{1}{1} = 1.$$

Then

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\theta}{\sin \theta} + \sec \theta} = \frac{1}{1 + 1} = \boxed{\frac{1}{2}}.$$

$$28. \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta \cdot \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} = 1 \cdot \frac{1}{1} = \boxed{1}.$$

$$29. \lim_{\theta \rightarrow 0} \frac{5}{\theta \cdot \csc \theta} = 5 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 5(1) = \boxed{5}.$$

$$30. \lim_{\theta \rightarrow 0} \frac{\sin(3\theta)}{\sin(2\theta)} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin(3\theta)}{\theta}}{\frac{\sin(2\theta)}{\theta}} = \frac{3 \lim_{\theta \rightarrow 0} \frac{\sin(3\theta)}{3\theta}}{2 \lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{2\theta}} = \frac{3(1)}{2(1)} = \boxed{\frac{3}{2}}.$$

$$31. \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \sin \theta = 1(0) = \boxed{0}.$$

$$32. \lim_{\theta \rightarrow 0} \frac{\cos(4\theta) - 1}{2\theta} = \lim_{\theta \rightarrow 0} \frac{2[\cos(4\theta) - 1]}{4\theta} = 2 \lim_{\theta \rightarrow 0} \frac{\cos(4\theta) - 1}{4\theta} = 2(0) = \boxed{0}.$$

$$33. \lim_{\theta \rightarrow 0} (\theta \cdot \cot \theta) = \lim_{\theta \rightarrow 0} \frac{\theta \cdot \cos \theta}{\sin \theta} = \frac{\lim_{\theta \rightarrow 0} \cos \theta}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{1}{1} = \boxed{1}.$$

34. Because

$$\sin \theta \left(\frac{\cot \theta - \csc \theta}{\theta} \right) = \frac{\cos \theta - 1}{\theta},$$

it follows that

$$\lim_{\theta \rightarrow 0} \left[\sin \theta \left(\frac{\cot \theta - \csc \theta}{\theta} \right) \right] = \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = \boxed{0}.$$

35. First, f is defined at $c = 0$ with $f(0) = 3$. Next,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (3 \cos x) = 3(1) = 3 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 3) = 3.$$

Because the two one-sided limits as x approaches 0 are equal to 3, it follows that $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 3. Finally, $\lim_{x \rightarrow 0} f(x) = f(0)$, so $\boxed{f \text{ is continuous at } c = 0}$.

36. First, f is defined at $c = 0$ with $f(0) = 0$. Next,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \cos x = \cos 0 = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^x = e^0 = 1.$$

Because the two one-sided limits as x approaches 0 are equal to 1, it follows that $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 1. However, $\lim_{x \rightarrow 0} f(x) \neq f(0)$, so $\boxed{f \text{ is not continuous at } c = 0}$.

37. First, f is defined at $c = \frac{\pi}{4}$ with $f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Next,

$$\lim_{x \rightarrow \pi/4^-} f(x) = \lim_{x \rightarrow \pi/4^-} \sin x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{x \rightarrow \pi/4^+} f(x) = \lim_{x \rightarrow \pi/4^+} \cos x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Because the two one-sided limits as x approaches $\pi/4$ are equal to $\frac{\sqrt{2}}{2}$, it follows that $\lim_{x \rightarrow \pi/4} f(x)$ exists and is equal to $\frac{\sqrt{2}}{2}$. Finally, $\lim_{x \rightarrow \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$, so $\boxed{f \text{ is continuous at } c = \frac{\pi}{4}}$.

38. First, f is defined at $c = 1$ with $f(1) = \ln 1 = 0$. Next,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \tan^{-1} x = \tan^{-1} 1 = \frac{\pi}{4}$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \ln x = \ln 1 = 0.$$

Because the two one-sided limits as x approaches 0 are not equal, it follows that $\lim_{x \rightarrow 1} f(x)$ does not exist. Therefore, $\boxed{f \text{ is not continuous at } c = 1}$.

39. Let $g(x) = \sin x$ and $h(x) = \frac{x^2 - 4x}{x - 4}$. The trigonometric function g is continuous on the set of all real numbers, and the rational function h is continuous on the set $\{x|x \neq 4\}$. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x at which h is continuous, it follows that f is continuous on the set $\{x|x \neq 4\}$.
40. Let $g(x) = \cos x$ and $h(x) = \frac{x^2 - 5x + 1}{2x}$. The trigonometric function g is continuous on the set of all real numbers, and the rational function h is continuous on the set $\{x|x \neq 0\}$. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x at which h is continuous, it follows that f is continuous on the set $\{x|x \neq 0\}$.
41. The constant function 1 and the trigonometric function $\sin \theta$ are continuous on the set of all real numbers, so the sum of these functions, $1 + \sin \theta$, is also continuous on the set of all real numbers. Now, $1 + \sin \theta = 0$ when $\sin \theta = -1$. This happens for $\theta = \frac{3\pi}{2} + 2k\pi$, where k is any integer. Because f is the quotient of the constant function 1 and the function $1 + \sin \theta$, it follows that f is continuous on the set $\left\{x|x \neq \frac{3\pi}{2} + 2k\pi\right\}$, where k is any integer.
42. The constant function 1 and the trigonometric function $\cos \theta$ are continuous on the set of all real numbers. The function $\cos^2 \theta = \cos \theta \cdot \cos \theta$, being the product of continuous functions, and the function $1 + \cos^2 \theta$, being the sum of continuous functions, are then also continuous on the set of all real numbers. Finally, because $1 + \cos^2 \theta$ is never equal to zero for any real number θ and f is the quotient of the constant function 1 and the function $1 + \cos^2 \theta$, it follows that f is continuous on the set of all real numbers.
43. Let $g(x) = \ln x$ and $h(x) = x - 3$. The logarithmic function g is continuous on the set $\{x|x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As f is the quotient of the functions g and h and the only value x for which $h(x) = 0$ is $x = 3$, it follows that the function f is continuous on the set $\{x|x > 0, x \neq 3\}$.
44. Let $g(x) = \ln x$ and $h(x) = x^2 + 1$. The logarithmic function g is continuous on the set $\{x|x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x because $x^2 + 1 \geq 1 > 0$ for any real number x , it follows that f is continuous on the set of all real numbers.
45. Let $g(x) = e^{-x}$ and $h(x) = \sin x$. The exponential function g and the trigonometric function h are both continuous on the set of all real numbers. As f is the product of g and h , it follows that f is also continuous on the set of all real numbers.
46. The exponential function e^x , the constant function 1, and the trigonometric function $\sin x$ are all continuous on the set of all real numbers. The function $\sin^2 x = \sin x \cdot \sin x$, being the product of continuous functions, and the function $1 + \sin^2 x$, being the sum of continuous functions, are then also continuous on the set of all real numbers. Finally, because $1 + \sin^2 x$ is never equal to zero for any real number x and f is the quotient of the exponential function e^x and the function $1 + \sin^2 x$, it follows that f is continuous on the set of all real numbers.

Applications and Extensions

47. Start from the compound inequality

$$-1 \leq \sin \left(\frac{1}{x} \right) \leq 1,$$

which holds for all $x \neq 0$. Multiplying by x^2 , which is non-negative for all x , then yields

$$-x^2 \leq x^2 \sin \left(\frac{1}{x} \right) \leq x^2.$$

As

$$\lim_{x \rightarrow 0} (-x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = \boxed{0}.$$

48. Start from the compound inequality

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Then

$$0 \leq 1 - \cos\left(\frac{1}{x}\right) \leq 2$$

and

$$\left|1 - \cos\left(\frac{1}{x}\right)\right| \leq 2.$$

Multiplying by $|x|$, which is non-negative for all x , then yields

$$|x| \left|1 - \cos\left(\frac{1}{x}\right)\right| = \left|x \left(1 - \cos\left(\frac{1}{x}\right)\right)\right| \leq 2|x|,$$

or

$$-2|x| \leq x \left(1 - \cos\left(\frac{1}{x}\right)\right) \leq 2|x|.$$

As

$$\lim_{x \rightarrow 0} (-2|x|) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (2|x|) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} \left[x \left(1 - \cos\left(\frac{1}{x}\right)\right) \right] = \boxed{0}.$$

49. Start from the compound inequality

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Then

$$0 \leq 1 - \cos\left(\frac{1}{x}\right) \leq 2.$$

Multiplying by x^2 , which is non-negative for all x , then yields

$$0 \leq x^2 \left(1 - \cos\left(\frac{1}{x}\right)\right) \leq 2x^2.$$

As

$$\lim_{x \rightarrow 0} 0 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (2x^2) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} \left[x^2 \left(1 - \cos\left(\frac{1}{x}\right)\right) \right] = \boxed{0}.$$

50. First note that $x^3 + 3x^2 = x^2(x + 3)$ is non-negative for $x \geq -3$. Thus, as x approaches 0, $\sqrt{x^3 + 3x^2}$ is defined. Now, consider the compound inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Multiplying by $\sqrt{x^3 + 3x^2}$, which is non-negative for all x , then yields

$$-\sqrt{x^3 + 3x^2} \leq \sqrt{x^3 + 3x^2} \sin\left(\frac{1}{x}\right) \leq \sqrt{x^3 + 3x^2}.$$

As

$$\lim_{x \rightarrow 0} (-\sqrt{x^3 + 3x^2}) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sqrt{x^3 + 3x^2} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + 3x^2} \sin\left(\frac{1}{x}\right) = \boxed{0}.$$

$$51. \lim_{x \rightarrow 0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x \rightarrow 0} \frac{\frac{\sin(ax)}{ax}}{\frac{\sin(bx)}{bx}} = \lim_{x \rightarrow 0} \frac{\frac{a \sin(ax)}{ax}}{\frac{b \sin(bx)}{bx}} = \frac{a}{b} \lim_{x \rightarrow 0} \frac{\frac{\sin(ax)}{ax}}{\frac{\sin(bx)}{bx}} = \frac{a}{b} \cdot \frac{1}{1} = \boxed{\frac{a}{b}}.$$

$$52. \lim_{x \rightarrow 0} \frac{\cos(ax)}{\cos(bx)} = \frac{\lim_{x \rightarrow 0} \cos(ax)}{\lim_{x \rightarrow 0} \cos(bx)} = \frac{1}{1} = \boxed{1}.$$

$$53. \lim_{x \rightarrow 0} \frac{\sin(ax)}{bx} = \lim_{x \rightarrow 0} \frac{a \sin(ax)}{abx} = \frac{a}{b} \lim_{x \rightarrow 0} \frac{\sin(ax)}{ax} = \frac{a}{b} \cdot 1 = \boxed{\frac{a}{b}}.$$

$$54. \lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{bx} = \lim_{x \rightarrow 0} \frac{a(1 - \cos(ax))}{abx} = \frac{a}{b} \lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{ax} = \frac{a}{b} \cdot 0 = \boxed{0}.$$

55. (a) Using the given functions $x(\theta)$ and $y(\theta)$,

$$\lim_{\theta \rightarrow \frac{\pi}{4}^+} x(\theta) = \lim_{\theta \rightarrow \frac{\pi}{4}^+} [(10 \cos \theta)t] = \left(10 \cos \frac{\pi}{4}\right)t = \boxed{5\sqrt{2}t},$$

and

$$\lim_{\theta \rightarrow \frac{\pi}{4}^+} y(\theta) = \lim_{\theta \rightarrow \frac{\pi}{4}^+} [-16t^2 + (10 \sin \theta)t] = -16t^2 + \left(10 \sin \frac{\pi}{4}\right)t = \boxed{-16t^2 + 5\sqrt{2}t}.$$

- (b) The limits from part (a) indicate that the object would immediately begin moving downward into the inclined plane. This is not consistent with what would be expected for the motion of an object propelled at an angle close to $\frac{\pi}{4}$; that is, propelled along the inclined plane.
- (c) Using the given functions $x(\theta)$ and $y(\theta)$,

$$\lim_{\theta \rightarrow \frac{\pi}{2}^-} x(\theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} [(10 \cos \theta)t] = \left(10 \cos \frac{\pi}{2}\right)t = \boxed{0},$$

and

$$\lim_{\theta \rightarrow \frac{\pi}{2}^-} y(\theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} [-16t^2 + (10 \sin \theta)t] = -16t^2 + \left(10 \sin \frac{\pi}{2}\right)t = \boxed{-16t^2 + 10t}.$$

- (d) The limits in part (c) indicate that the horizontal position remains at 0. The vertical position, given by $y = -16t^2 + 10t$, indicates that the object reaches a maximum height at $t = \frac{5}{16}$ seconds and returns to the ground after $t = \frac{5}{8}$ seconds. These conclusions are consistent with what would be expected for the motion of an object propelled close to the vertical.

56.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} = 1 \cdot 1 \cdot \frac{1}{1 + 1} = \frac{1}{2}.\end{aligned}$$

57. Let f be a function for which $0 \leq f(x) \leq 1$ for every x . Multiplying by x^2 , which is non-negative for all x , then yields $0 \leq x^2 f(x) \leq x^2$ for every x . As

$$\lim_{x \rightarrow 0} 0 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} [x^2 f(x)] = 0.$$

58. Let f be a function for which $0 \leq f(x) \leq M$ for every x . Multiplying by x^2 , which is non-negative for all x , then yields $0 \leq x^2 f(x) \leq Mx^2$ for every x . As

$$\lim_{x \rightarrow 0} 0 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (Mx^2) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} [x^2 f(x)] = 0.$$

59. To make f continuous at 0, $f(0)$ should be defined equal to the value of $\lim_{x \rightarrow 0} f(x)$, provided this limit exists. Here,

$$\lim_{x \rightarrow 0} \frac{\sin(\pi x)}{x} = \lim_{x \rightarrow 0} \frac{\pi \sin(\pi x)}{\pi x} = \pi \lim_{x \rightarrow 0} \frac{\sin(\pi x)}{\pi x} = \pi \cdot 1 = \pi.$$

Thus, $f(0)$ should be set equal to π to make f continuous at 0.

60. The functions $\sin(\pi x)$ and $\frac{1}{x(1-x)}$ are both continuous on the open interval $(0, 1)$; consequently, the function $f(x) = \frac{\sin(\pi x)}{x(1-x)}$ is also continuous on the open interval $(0, 1)$. Now,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sin(\pi x)}{x(1-x)} = \lim_{x \rightarrow 0^+} \frac{\pi \sin(\pi x)}{\pi x(1-x)} = \pi \lim_{x \rightarrow 0^+} \frac{\sin(\pi x)}{\pi x} \cdot \lim_{x \rightarrow 0^+} \frac{1}{1-x} = \pi \cdot 1 \cdot \frac{1}{1-0} = \pi,$$

and, using the identity $\sin \theta = \sin(\pi - \theta)$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{\sin(\pi x)}{x(1-x)} = \lim_{x \rightarrow 1^-} \frac{\pi \sin(\pi(1-x))}{\pi x(1-x)} = \pi \lim_{x \rightarrow 1^-} \frac{\sin(\pi(1-x))}{\pi(1-x)} \cdot \lim_{x \rightarrow 1^-} \frac{1}{x} = \pi \cdot 1 \cdot \frac{1}{1} = \pi.$$

Thus, defining $f(0) = f(1) = \pi$, f will be continuous on the closed interval $[0, 1]$.

61. Because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 = f(0),$$

f is continuous at 0.

62. Because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 = f(0),$$

f is continuous at 0.

63. Let n be a positive integer, and start from the inequality

$$\left| \sin \left(\frac{1}{x} \right) \right| \leq 1,$$

which holds for all $x \neq 0$. Multiplying by $|x^n|$, which is non-negative for all x , then yields

$$|x^n| \cdot \left| \sin \left(\frac{1}{x} \right) \right| = \left| x^n \sin \left(\frac{1}{x} \right) \right| \leq |x^n|,$$

or

$$-|x^n| \leq x^n \sin \left(\frac{1}{x} \right) \leq |x^n|.$$

As

$$\lim_{x \rightarrow 0} (-|x^n|) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} |x^n| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} \left[x^n \sin \left(\frac{1}{x} \right) \right] = 0.$$

64. Let $0 < \theta < \frac{\pi}{2}$. Consider the diagram presented in the problem statement. The length of the segment $\overline{PB}$ is shorter than the length of the segment $\overline{PA}$ because $\overline{PA}$ is the hypotenuse of the right triangle PBA for which $\overline{PB}$ is one of the legs. Moreover, the length of the segment $\overline{PA}$ is shorter than the length of the arc PA because the shortest distance between any two distinct points is the length of the line segment connecting the two points. Thus, the segment joining the points P and B is shorter than the length of the arc AP along the circle. As the length of the segment joining P and B is $\sin \theta$ while the length of the arc AP is θ , it follows that $\sin \theta \leq \theta$. Because θ is a first quadrant angle, it is also true that $\sin \theta \geq 0$. Thus

$$0 \leq \sin \theta \leq \theta.$$

With

$$\lim_{\theta \rightarrow 0^+} 0 = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0^+} \theta = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0.$$

If, instead, $-\frac{\pi}{2} < \theta < 0$, then the length of the segment joining P and B is $-\sin \theta$, so that

$$0 \leq -\sin \theta \leq \theta \quad \text{or} \quad -\theta \leq \sin \theta \leq 0.$$

With

$$\lim_{\theta \rightarrow 0^-} (-\theta) = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0^-} 0 = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{\theta \rightarrow 0^-} \sin \theta = 0.$$

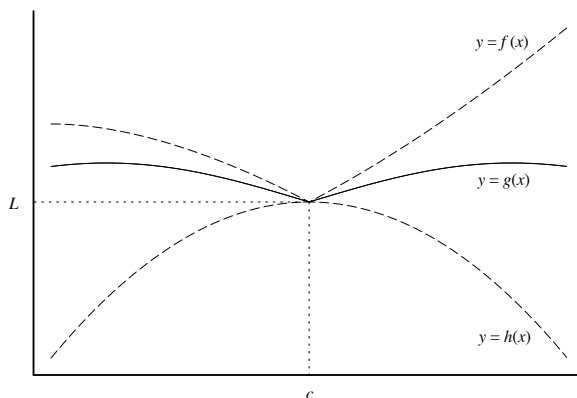
Finally, because the two one-sided limits are equal,

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$

65. From the Pythagorean identity $\cos^2 \theta + \sin^2 \theta = 1$ it follows that $\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$. In the limit as θ approaches 0, θ will eventually lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, so that $\cos \theta$ will be positive; hence, $\cos \theta = \sqrt{1 - \sin^2 \theta}$. Therefore,

$$\lim_{\theta \rightarrow 0} \cos \theta = \lim_{\theta \rightarrow 0} \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 0^2} = 1.$$

66. The function $\cos \theta$ is continuous on the set of all real numbers, and the polynomial function $5x^3 + 2x^2 - 8x + 1$ is also continuous on the set of all real numbers. As $f(x) = \cos(5x^3 + 2x^2 - 8x + 1)$ is the composition of the two previously mentioned functions, it follows that f is continuous on the set of all real numbers.
67. Answers will vary. One possible response is the following. Suppose we wish to evaluate $\lim_{x \rightarrow c} g(x)$ for some function g . If two functions f and h can be found such that $f(x) \leq g(x) \leq h(x)$ for all x in a neighborhood of c , except possibly at c , and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ for some real number L , then $\lim_{x \rightarrow c} g(x) = L$. In plain terms, the functions f and h squeeze the value of g toward L . A graph to illustrate the process is shown below.



Challenge Problems

68. To show that the sine function is continuous on its domain, it must be established that $\lim_{\theta \rightarrow c} \sin \theta = \sin c$ for any real number c . So let c be any real number. Then

$$\begin{aligned} \lim_{\theta \rightarrow c} \sin \theta &= \lim_{x \rightarrow 0} \sin(x + c) = \lim_{x \rightarrow 0} [\sin x \cos c + \cos x \sin c] \\ &= \cos c \lim_{x \rightarrow 0} \sin x + \sin c \lim_{x \rightarrow 0} \cos x = \cos c \cdot 0 + \sin c \cdot 1 = \sin c. \end{aligned}$$

Similarly, to show that the cosine function is continuous on its domain, it must be established that $\lim_{\theta \rightarrow c} \cos \theta = \cos c$ for any real number c . So let c be any real number. Then

$$\begin{aligned} \lim_{\theta \rightarrow c} \cos \theta &= \lim_{x \rightarrow 0} \cos(x + c) = \lim_{x \rightarrow 0} [\cos x \cos c - \sin x \sin c] \\ &= \cos c \lim_{x \rightarrow 0} \cos x - \sin c \lim_{x \rightarrow 0} \sin x = \cos c \cdot 1 - \sin c \cdot 0 = \cos c. \end{aligned}$$

69. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \frac{x \sin x^2}{x^2} = \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 0(1) = \boxed{0}.$

70. As defined, $0 \leq f(x) \leq 1$ for all x , so that $|f(x)| \leq 1$. Multiplying this last inequality by $|x|$, which is non-negative for all x , yields

$$|x| \cdot |f(x)| = |xf(x)| \leq |x|, \quad \text{or} \quad -|x| \leq xf(x) \leq |x|.$$

Because

$$\lim_{x \rightarrow 0} (-|x|) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} |x| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} [xf(x)] = 0.$$

71. Using the diagram in the problem statement, let d denote the x -coordinate of the point D . Applying right angle trigonometry to the triangle ACD yields $d \tan \theta$ as the length of the segment $\overline{CD}$, while applying right angle trigonometry to the triangle BCD yields $(1 - d) \tan(n\theta)$ as the length of the segment $\overline{CD}$. Thus,

$$d \tan \theta = (1 - d) \tan(n\theta),$$

or

$$d = \frac{\tan(n\theta)}{\tan \theta + \tan(n\theta)} = \frac{\frac{\tan(n\theta)}{\tan \theta}}{1 + \frac{\tan(n\theta)}{\tan \theta}}.$$

Now,

$$\lim_{\theta \rightarrow 0} \frac{\tan(n\theta)}{\tan \theta} = \lim_{\theta \rightarrow 0} \frac{\sin(n\theta)}{\sin \theta} \cdot \frac{\cos \theta}{\cos(n\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin(n\theta)}{\sin \theta} \cdot \lim_{\theta \rightarrow 0} \frac{\cos \theta}{\cos(n\theta)} = n(1) = n,$$

using the results of Problems 51 and 52. Thus, as θ approaches 0, $d \rightarrow \frac{n}{n+1}$, and the limiting position

of D is the point $\boxed{\left(\frac{n}{n+1}, 0\right)}$.

1.5: Infinite Limits; Limits at Infinity; Asymptotes

Concepts and Vocabulary

1. False. ∞ is **not** a number; ∞ is a symbol to represent the concept of becoming unbounded.
2. (a) $\lim_{x \rightarrow 0^-} \frac{1}{x} = \boxed{-\infty}$.
 (b) $\lim_{x \rightarrow 0^+} \frac{1}{x} = \boxed{\infty}$.
 (c) $\lim_{x \rightarrow 0^+} \ln x = \boxed{-\infty}$.
3. False. The graph of a rational function **may have** a vertical asymptote at **a** number x at which the function is undefined. The graph will have a vertical asymptote provided that at least one of the one-sided limits at that number is infinite. If neither of the one-sided limits is infinite, the graph will have a hole.
4. If $\lim_{x \rightarrow 4} f(x) = \infty$, then the line $x = 4$ is a vertical asymptote of the graph of f .
5. (a) $\lim_{x \rightarrow \infty} \frac{1}{x} = \boxed{0}$.
 (b) $\lim_{x \rightarrow \infty} \frac{1}{x^2} = \boxed{0}$.
 (c) $\lim_{x \rightarrow \infty} \ln x = \boxed{\infty}$.
6. False. $\lim_{x \rightarrow -\infty} 5 = 5$.
7. (a) $\lim_{x \rightarrow -\infty} e^x = \boxed{0}$.
 (b) $\lim_{x \rightarrow \infty} e^x = \boxed{\infty}$.
 (c) $\lim_{x \rightarrow \infty} e^{-x} = \boxed{0}$.
8. True. The graph of a function can have at most two horizontal asymptotes.

Skill Building

9. As x becomes unbounded in the positive direction, the graph of f approaches the line $y = 2$. Thus,
 $\lim_{x \rightarrow \infty} f(x) = 2$.
10. As x becomes unbounded in the negative direction, the graph of f approaches the line $y = 0$. Thus,
 $\lim_{x \rightarrow -\infty} f(x) = 0$.
11. As x approaches -1 from the left, the graph of f becomes unbounded in the positive direction. Thus,
 $\lim_{x \rightarrow -1^-} f(x) = \infty$.
12. As x approaches -1 from the right, the graph of f becomes unbounded in the positive direction. Thus,
 $\lim_{x \rightarrow -1^+} f(x) = \infty$.
13. As x approaches 3 from the left, the graph of f becomes unbounded in the positive direction. Thus,
 $\lim_{x \rightarrow 3^-} f(x) = \infty$.

14. As x approaches 3 from the right, the graph of f becomes unbounded in the positive direction. Thus,

$$\lim_{x \rightarrow 3^+} f(x) = \infty.$$

15. The graph of f has two vertical asymptotes: $x = -1$ and $x = 3$.

16. The graph of f has two horizontal asymptotes: $y = 0$ and $y = 2$.

17. As x becomes unbounded in the positive direction, the graph of f approaches the line $y = -3$. Thus,

$$\lim_{x \rightarrow \infty} f(x) = -3.$$

18. As x becomes unbounded in the negative direction, the graph of f approaches the line $y = 0$. Thus,

$$\lim_{x \rightarrow -\infty} f(x) = 0.$$

19. As x approaches -3 from the left, the graph of f becomes unbounded in the positive direction. Thus,

$$\lim_{x \rightarrow -3^-} f(x) = \infty.$$

20. As x approaches -3 from the right, the graph of f becomes unbounded in the negative direction. Thus,

$$\lim_{x \rightarrow -3^+} f(x) = -\infty.$$

21. As x approaches 0 from the left, the graph of f approaches the origin. Thus, $\lim_{x \rightarrow 0^-} f(x) = 0$.

22. As x approaches 0 from the right, the graph of f becomes unbounded in the positive direction. Thus,

$$\lim_{x \rightarrow 0^+} f(x) = \infty.$$

23. As x approaches 4 from the left, the graph of f becomes unbounded in the positive direction. Thus,

$$\lim_{x \rightarrow 4^-} f(x) = \infty.$$

24. As x approaches 4 from the right, the graph of f becomes unbounded in the positive direction. Thus,

$$\lim_{x \rightarrow 4^+} f(x) = \infty.$$

25. The graph of f has three vertical asymptotes: $x = -3$, $x = 0$, and $x = 4$.

26. The graph of f has two horizontal asymptotes: $y = 0$ and $y = -3$.

27. As x approaches 2 from the left, $3x$ approaches 6 and $x - 2$ approaches 0 from the left. Therefore, the ratio $\frac{3x}{x-2}$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow 2^-} \frac{3x}{x-2} = -\infty.$$

The values in the table below support this conclusion.

x	1.9	1.99	1.999	$\rightarrow 2$
$f(x) = \frac{3x}{x-2}$	-57	-597	-5997	$f(x)$ approaches $-\infty$

28. As x approaches -4 from the right, $2x + 1$ approaches -7 and $x + 4$ approaches 0 from the right. Therefore, the ratio $\frac{2x+1}{x+4}$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow -4^+} \frac{2x+1}{x+4} = -\infty.$$

The values in the table below support this conclusion.

x	$-4 \leftarrow$	-3.999	-3.99	-3.9
$f(x) = \frac{2x+1}{x+4}$	$f(x)$ approaches $-\infty$	-6998	-698	-68

29. As x approaches 2 from the right, 5 approaches 5 and $x^2 - 4$ approaches 0 from the right. Therefore, the ratio $\frac{5}{x^2-4}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow 2^+} \frac{5}{x^2 - 4} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \frac{5}{x^2-4}$	$f(x)$ approaches ∞	1249.69	124.69	12.20

30. As x approaches 1 from the left, $2x$ approaches 2 and $x^3 - 1$ approaches 0 from the left. Therefore, the ratio $\frac{2x}{x^3-1}$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow 1^-} \frac{2x}{x^3 - 1} = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	0.9	0.99	0.999	$\rightarrow 1$
$f(x) = \frac{2x}{x^3-1}$	-6.64	-66.66	-666.67	$f(x)$ approaches $-\infty$

31. As x approaches -1 from the right, $5x + 3$ approaches -2 and $x(x + 1)$ approaches 0 from the left. Therefore, the ratio $\frac{5x+3}{x(x+1)}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow -1^+} \frac{5x + 3}{x(x + 1)} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \frac{5x+3}{x(x+1)}$	$f(x)$ approaches ∞	1997.00	196.97	16.67

32. As x approaches 0 from the left, $5x + 3$ approaches 3 and $5x(x - 1)$ approaches 0 from the right. Therefore, the ratio $\frac{5x+3}{5x(x-1)}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow 0^-} \frac{5x + 3}{5x(x - 1)} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x) = \frac{5x+3}{5x(x-1)}$	4.55	58.42	598.40	$f(x)$ approaches ∞

33. As x approaches -3 from the left, 1 approaches 1 and $x^2 - 9$ approaches 0 from the right. Therefore, the ratio $\frac{1}{x^2-9}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow -3^-} \frac{1}{x^2 - 9} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	-3.1	-3.01	-3.001	$\rightarrow -3$
$f(x) = \frac{1}{x^2 - 9}$	1.64	16.64	166.64	$f(x)$ approaches ∞

34. As x approaches 2 from the right, x approaches 2 and $x^2 - 4$ approaches 0 from the right. Therefore, the ratio $\frac{x}{x^2-4}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \frac{x}{x^2 - 4}$	$f(x)$ approaches ∞	500.12	50.12	5.12

35. As x approaches 3 , $1 - x$ approaches -2 and $(3 - x)^2$ approaches 0 from the right. Therefore, the ratio $\frac{1-x}{(3-x)^2}$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow 3} \frac{1 - x}{(3 - x)^2} = \boxed{-\infty}.$$

The values in the table below support this conclusion.

x	2.9	2.99	2.999	$\rightarrow 3 \leftarrow$	3.001	3.01	3.1
$f(x) = \frac{1-x}{(3-x)^2}$	-190	-19900	-1999000	$f(x)$ approaches $-\infty$	-2001000	-20100	-210

36. As x approaches -1 , $x + 2$ approaches 1 and $(x + 1)^2$ approaches 0 from the right. Therefore, the ratio $\frac{x+2}{(x+1)^2}$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow -1} \frac{x + 2}{(x + 1)^2} = \boxed{\infty}.$$

The values in the table below support this conclusion.

x	-1.1	-1.01	-1.001	$\rightarrow -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \frac{x+2}{(x+1)^2}$	90	9900	999000	$f(x)$ approaches ∞	1001000	10100	110

37. As x approaches π from the left, $\cos x$ approaches -1 and $\sin x$ approaches 0 from the right. Therefore, the ratio $\frac{\cos x}{\sin x} = \cot x$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow \pi^-} \cot x = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$\pi - 0.1$	$\pi - 0.01$	$\pi - 0.001$	$\rightarrow \pi$
$f(x) = \cot x$	-9.97	-100.00	-1000.00	$f(x)$ approaches $-\infty$

38. As x approaches $-\pi/2$ from the left, $\sin x$ approaches -1 and $\cos x$ approaches 0 from the left. Therefore, the ratio $\frac{\sin x}{\cos x} = \tan x$ becomes unbounded in the positive direction, so

$$\lim_{x \rightarrow -\pi/2^-} \tan x = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-\frac{\pi}{2} - 0.1$	$-\frac{\pi}{2} - 0.01$	$-\frac{\pi}{2} - 0.001$	$\rightarrow -\pi/2$
$f(x) = \tan x$	9.97	100.00	1000.00	$f(x)$ approaches ∞

39. As x approaches $\pi/2$ from the right, $2x$ approaches π from the right, so $\sin(2x)$ approaches 0 from the left. Therefore, the ratio $\frac{1}{\sin(2x)} = \csc(2x)$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow \pi/2^+} \csc(2x) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$\frac{\pi}{2} \leftarrow$	$\frac{\pi}{2} + 0.001$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.1$
$f(x) = \csc(2x)$	$f(x)$ approaches $-\infty$	-500.00	-50.00	-5.03

40. As x approaches $-\pi/2$ from the left, $\cos x$ approaches 0 from the left. Therefore, the ratio $\frac{1}{\cos x} = \sec x$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow -\pi/2^-} \sec x = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-\frac{\pi}{2} - 0.1$	$-\frac{\pi}{2} - 0.01$	$-\frac{\pi}{2} - 0.001$	$\rightarrow -\pi/2$
$f(x) = \sec x$	-10.02	-100.00	-1000.00	$f(x)$ approaches $-\infty$

41. As x approaches -1 from the right, $x + 1$ approaches 0 from the right. Therefore, $\ln(x + 1)$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow -1^+} \ln(x + 1) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-1 \leftarrow$	$-1 + 10^{-6}$	$-1 + 10^{-4}$	$-1 + 10^{-2}$
$f(x) = \ln(x + 1)$	$f(x)$ approaches $-\infty$	-13.82	-9.21	-4.61

42. As x approaches 1 from the right, $x - 1$ approaches 0 from the right. Therefore, $\ln(x - 1)$ becomes unbounded in the negative direction, so

$$\lim_{x \rightarrow 1^+} \ln(x - 1) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$1 \leftarrow$	$1 + 10^{-6}$	$1 + 10^{-4}$	$1 + 10^{-2}$
$f(x) = \ln(x - 1)$	$f(x)$ approaches $-\infty$	-13.82	-9.21	-4.61

$$43. \lim_{x \rightarrow \infty} \frac{5}{x^2 + 4} = \lim_{x \rightarrow \infty} \frac{\frac{5}{x^2}}{\frac{x^2 + 4}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{5}{x^2}}{1 + \frac{4}{x^2}} = \frac{0}{1 + 0} = \boxed{0}.$$

$$44. \lim_{x \rightarrow -\infty} \frac{1}{x^2 - 9} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}}{\frac{x^2 - 9}{x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}}{1 - \frac{9}{x^2}} = \frac{0}{1 - 0} = \boxed{0}.$$

$$45. \lim_{x \rightarrow \infty} \frac{2x + 4}{5x} = \lim_{x \rightarrow \infty} \frac{\frac{2x + 4}{5x}}{\frac{5x}{5x}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{5} + \frac{4}{5x}}{1} = \frac{\frac{2}{5} + 0}{1} = \boxed{\frac{2}{5}}.$$

$$46. \lim_{x \rightarrow \infty} \frac{x + 1}{x} = \lim_{x \rightarrow \infty} \frac{\frac{x + 1}{x}}{\frac{x}{x}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{1} = \frac{1 + 0}{1} = \boxed{1}.$$

$$47. \lim_{x \rightarrow \infty} \frac{x^3 + x^2 + 2x - 1}{x^3 + x + 1} = \lim_{x \rightarrow \infty} \frac{\frac{x^3 + x^2 + 2x - 1}{x^3}}{\frac{x^3 + x + 1}{x^3}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \frac{1 + 0 + 0 - 0}{1 + 0 + 0} = \boxed{1}.$$

$$48. \lim_{x \rightarrow \infty} \frac{2x^2 - 5x + 2}{5x^2 + 7x - 1} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2 - 5x + 2}{5x^2}}{\frac{5x^2 + 7x - 1}{5x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{5} - \frac{1}{x} + \frac{2}{5x^2}}{1 + \frac{7}{5x} - \frac{1}{5x^2}} = \frac{\frac{2}{5} - 0 + 0}{1 + 0 - 0} = \boxed{\frac{2}{5}}.$$

$$49. \lim_{x \rightarrow -\infty} \frac{x^2 + 1}{x^3 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{x^2 + 1}{x^3}}{\frac{x^3 - 1}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{1}{x^3}}{1 - \frac{1}{x^3}} = \frac{0 + 0}{1 - 0} = \boxed{0}.$$

$$50. \lim_{x \rightarrow \infty} \frac{x^2 - 2x + 1}{x^3 + 5x + 4} = \lim_{x \rightarrow \infty} \frac{\frac{x^2 - 2x + 1}{x^3}}{\frac{x^3 + 5x + 4}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}}{1 + \frac{5}{x^2} + \frac{4}{x^3}} = \frac{0 - 0 + 0}{1 + 0 + 0} = \boxed{0}.$$

51.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left[\frac{3x}{2x + 5} - \frac{x^2 + 1}{4x^2 + 8} \right] &= \lim_{x \rightarrow \infty} \frac{3x}{2x + 5} - \lim_{x \rightarrow \infty} \frac{x^2 + 1}{4x^2 + 8} = \lim_{x \rightarrow \infty} \frac{\frac{3x}{2x + 5}}{\frac{2x + 5}{2x}} - \lim_{x \rightarrow \infty} \frac{\frac{x^2 + 1}{4x^2 + 8}}{\frac{4x^2 + 8}{4x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{2}}{1 + \frac{5}{2x}} - \lim_{x \rightarrow \infty} \frac{\frac{1}{4} + \frac{1}{4x^2}}{1 + \frac{2}{x^2}} = \frac{\frac{3}{2}}{1 + 0} - \frac{\frac{1}{4} + 0}{1 + 0} = \frac{3}{2} - \frac{1}{4} = \boxed{\frac{5}{4}}. \end{aligned}$$

52.

$$\begin{aligned}
\lim_{x \rightarrow \infty} \left[\frac{1}{x^2 + x + 4} - \frac{x+1}{3x-1} \right] &= \lim_{x \rightarrow \infty} \frac{1}{x^2 + x + 4} - \lim_{x \rightarrow \infty} \frac{x+1}{3x-1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{x^2 + x + 4}{x^2}} - \lim_{x \rightarrow \infty} \frac{\frac{x+1}{3x}}{\frac{3x-1}{3x}} \\
&= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{1 + \frac{1}{x} + \frac{4}{x^2}} - \lim_{x \rightarrow \infty} \frac{\frac{1}{3} + \frac{1}{3x}}{1 - \frac{1}{3x}} \\
&= \frac{0}{1+0+0} - \frac{\frac{1}{3}+0}{1-0} = 0 - \frac{1}{3} = \boxed{-\frac{1}{3}}.
\end{aligned}$$

53.

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \left[2e^x \left(\frac{5x+1}{3x} \right) \right] &= \lim_{x \rightarrow -\infty} (2e^x) \cdot \lim_{x \rightarrow -\infty} \frac{5x+1}{3x} = \lim_{x \rightarrow -\infty} (2e^x) \cdot \left(\lim_{x \rightarrow -\infty} \frac{\frac{5x+1}{3x}}{\frac{3x}{3x}} \right) \\
&= \lim_{x \rightarrow -\infty} (2e^x) \cdot \lim_{x \rightarrow -\infty} \frac{\frac{5}{3} + \frac{1}{3x}}{1} = 0 \cdot \frac{\frac{5}{3}+0}{1} = \boxed{0}.
\end{aligned}$$

54.

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \left[e^x \left(\frac{x^2 + x - 3}{2x^3 - x^2} \right) \right] &= \lim_{x \rightarrow -\infty} e^x \cdot \lim_{x \rightarrow -\infty} \frac{x^2 + x - 3}{2x^3 - x^2} = \lim_{x \rightarrow -\infty} e^x \cdot \left(\lim_{x \rightarrow -\infty} \frac{\frac{x^2 + x - 3}{2x^3}}{\frac{2x^3 - x^2}{2x^3}} \right) \\
&= \lim_{x \rightarrow -\infty} e^x \cdot \lim_{x \rightarrow -\infty} \frac{\frac{1}{2x} + \frac{1}{2x^2} - \frac{3}{2x^3}}{1 - \frac{1}{2x}} = 0 \cdot \frac{0+0-0}{1-0} = \boxed{0}.
\end{aligned}$$

$$55. \lim_{x \rightarrow \infty} \frac{\sqrt{x} + 2}{3x - 4} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{x} + 2}{3x}}{\frac{3x - 4}{3x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{3\sqrt{x}} + \frac{2}{3x}}{1 - \frac{4}{3x}} = \frac{0+0}{1-0} = \boxed{0}.$$

$$56. \lim_{x \rightarrow \infty} \frac{\sqrt{3x^3} + 2}{x^2 + 6} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{3x^3} + 2}{x^2}}{\frac{x^2 + 6}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt{3}}{\sqrt{x}} + \frac{2}{x^2}}{1 + \frac{6}{x^2}} = \frac{0+0}{1+0} = \boxed{0}.$$

57. Because

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 1}{x^2 + 4} = \lim_{x \rightarrow \infty} \frac{\frac{3x^2 - 1}{x^2}}{\frac{x^2 + 4}{x^2}} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x^2}}{1 + \frac{4}{x^2}} = \frac{3-0}{1+0} = 3,$$

it follows that

$$\lim_{x \rightarrow \infty} \sqrt{\frac{3x^2 - 1}{x^2 + 4}} = \sqrt{\lim_{x \rightarrow \infty} \frac{3x^2 - 1}{x^2 + 4}} = \boxed{\sqrt{3}}.$$

58. Because

$$\lim_{x \rightarrow \infty} \frac{16x^3 + 2x + 1}{2x^3 + 3x} = \lim_{x \rightarrow \infty} \frac{\frac{16x^3 + 2x + 1}{2x^3}}{\frac{2x^3 + 3x}{2x^3}} = \lim_{x \rightarrow \infty} \frac{8 + \frac{1}{x^2} + \frac{1}{2x^3}}{1 + \frac{3}{2x^2}} = \frac{8+0+0}{1+0} = 8,$$

it follows that

$$\lim_{x \rightarrow \infty} \left(\frac{16x^3 + 2x + 1}{2x^3 + 3x} \right)^{2/3} = \left(\lim_{x \rightarrow \infty} \frac{16x^3 + 2x + 1}{2x^3 + 3x} \right)^{2/3} = 8^{2/3} = \boxed{4}.$$

$$59. \lim_{x \rightarrow -\infty} \frac{5x^3}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{\frac{5x^3}{x^2}}{\frac{x^2 + 1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{5x}{1 + \frac{1}{x^2}} = \boxed{-\infty}.$$

$$60. \lim_{x \rightarrow -\infty} \frac{x^4}{x - 2} = \lim_{x \rightarrow -\infty} \frac{\frac{x^4}{x}}{\frac{x - 2}{x}} = \lim_{x \rightarrow -\infty} \frac{x^3}{1 - \frac{2}{x}} = \boxed{-\infty}.$$

61. The domain of $f(x) = 3 + \frac{1}{x}$ is the set $\{x|x \neq 0\}$. The one-sided limits as x approaches 0 are

$$\lim_{x \rightarrow 0^-} \left(3 + \frac{1}{x} \right) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \left(3 + \frac{1}{x} \right) = \infty,$$

so $\boxed{x = 0 \text{ is a vertical asymptote of the graph of } f}$. Because

$$\lim_{x \rightarrow -\infty} \left(3 + \frac{1}{x} \right) = 3 + 0 = 3 \quad \text{and} \quad \lim_{x \rightarrow \infty} \left(3 + \frac{1}{x} \right) = 3 + 0 = 3,$$

$\boxed{y = 3 \text{ is a horizontal asymptote of the graph of } f}$.

62. The domain of $f(x) = 2 - \frac{1}{x^2}$ is the set $\{x|x \neq 0\}$. The one-sided limits as x approaches 0 are

$$\lim_{x \rightarrow 0^-} \left(2 - \frac{1}{x^2} \right) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \left(2 - \frac{1}{x^2} \right) = -\infty,$$

so $\boxed{x = 0 \text{ is a vertical asymptote of the graph of } f}$. Because

$$\lim_{x \rightarrow -\infty} \left(2 - \frac{1}{x^2} \right) = 2 - 0 = 2 \quad \text{and} \quad \lim_{x \rightarrow \infty} \left(2 - \frac{1}{x^2} \right) = 2 - 0 = 2,$$

$\boxed{y = 2 \text{ is a horizontal asymptote of the graph of } f}$.

63. The domain of the function $f(x) = \frac{x^2}{x^2 - 1}$ is the set $\{x|x \neq \pm 1\}$. The one-sided limits as x approaches -1 are

$$\lim_{x \rightarrow -1^-} \frac{x^2}{x^2 - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{x^2}{x^2 - 1} = -\infty,$$

so $\boxed{x = -1 \text{ is a vertical asymptote of the graph of } f}$. The one-sided limits as x approaches 1 are

$$\lim_{x \rightarrow 1^-} \frac{x^2}{x^2 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^2}{x^2 - 1} = \infty,$$

so $\boxed{x = 1 \text{ is also a vertical asymptote of the graph of } f}$. Because

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{1}{1 - \frac{1}{x^2}} = \frac{1}{1 - 0} = 1,$$

and

$$\lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{1}{x^2}} = \frac{1}{1 - 0} = 1,$$

$y = 1$ is a horizontal asymptote of the graph of f .

64. The domain of the function $f(x) = \frac{2x^2 - 1}{x^2 - 1}$ is the set $\{x | x \neq \pm 1\}$. The one-sided limits as x approaches -1 are

$$\lim_{x \rightarrow -1^-} \frac{2x^2 - 1}{x^2 - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{2x^2 - 1}{x^2 - 1} = -\infty,$$

so $x = -1$ is a vertical asymptote of the graph of f . The one-sided limits as x approaches 1 are

$$\lim_{x \rightarrow 1^-} \frac{2x^2 - 1}{x^2 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{2x^2 - 1}{x^2 - 1} = \infty,$$

so $x = 1$ is also a vertical asymptote of the graph of f . Because

$$\lim_{x \rightarrow -\infty} \frac{2x^2 - 1}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{2x^2 - 1}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{2 - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{2 - 0}{1 - 0} = 2,$$

and

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 1}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2 - 1}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{2 - 0}{1 - 0} = 2,$$

$y = 2$ is a horizontal asymptote of the graph of f .

65. By completing the square, we find

$$2x^2 - x + 10 = 2 \left(x^2 - \frac{1}{2}x + \frac{1}{16} \right) + 10 - \frac{1}{8} = 2 \left(x - \frac{1}{4} \right)^2 + \frac{79}{8} \geq \frac{79}{8} > 0$$

for all x . Therefore, $\sqrt{2x^2 - x + 10}$ is defined for all x , and the domain of $f(x) = \frac{\sqrt{2x^2 - x + 10}}{2x - 3}$ is the set $\{x | x \neq \frac{3}{2}\}$. The one-sided limits as x approaches $\frac{3}{2}$ are

$$\lim_{x \rightarrow 3/2^-} \frac{\sqrt{2x^2 - x + 10}}{2x - 3} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 3/2^+} \frac{\sqrt{2x^2 - x + 10}}{2x - 3} = \infty,$$

so $x = \frac{3}{2}$ is a vertical asymptote of the graph of f . To examine the limits at infinity, note that

$$\frac{\sqrt{2x^2 - x + 10}}{2x - 3} = \frac{|x| \sqrt{2 - \frac{1}{x} + \frac{10}{x^2}}}{2x - 3} = \frac{|x|}{x} \cdot \frac{\sqrt{2 - \frac{1}{x} + \frac{10}{x^2}}}{2 - \frac{3}{x}}.$$

Note that $|x|/x$ is -1 as x approaches $-\infty$ and $+1$ as x approaches ∞ ; therefore,

$$\lim_{x \rightarrow -\infty} R(x) = -\frac{\sqrt{2 - 0 + 0}}{2 - 0} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{x \rightarrow \infty} R(x) = \frac{\sqrt{2 - 0 + 0}}{2 - 0} = \frac{\sqrt{2}}{2},$$

so that $y = -\frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ are horizontal asymptotes of the graph of f .

66. The domain of the function $f(x) = \frac{\sqrt[3]{x^2 + 5x}}{x - 6}$ is the set $\{x | x \neq 6\}$. The one-sided limits as x approaches 6 are

$$\lim_{x \rightarrow 6^-} \frac{\sqrt[3]{x^2 + 5x}}{x - 6} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 6^+} \frac{\sqrt[3]{x^2 + 5x}}{x - 6} = \infty,$$

so $x = 6$ is a vertical asymptote of the graph of f . Because

$$\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^2 + 5x}}{x - 6} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^2 + 5x}}{x}}{\frac{x - 6}{x}} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{\frac{1}{x} + \frac{5}{x^2}}}{1 - \frac{6}{x}} = \frac{\sqrt[3]{0 + 0}}{1 - 0} = 0,$$

and

$$\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2 + 5x}}{x - 6} = \lim_{x \rightarrow \infty} \frac{\frac{\sqrt[3]{x^2 + 5x}}{x}}{\frac{x - 6}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt[3]{\frac{1}{x} + \frac{5}{x^2}}}{1 - \frac{6}{x}} = \frac{\sqrt[3]{0 + 0}}{1 - 0} = 0,$$

$y = 0$ is a horizontal asymptote of the graph of f .

67. (a) Factoring the denominator of R yields

$$2x^3 + 4x^2 = 2x^2(x + 2),$$

so the domain of R is the set $\{x | x \neq -2, x \neq 0\}$.

- (b) Because

$$\lim_{x \rightarrow -\infty} \frac{-2x^2 + 1}{2x^3 + 4x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{-2x^2 + 1}{2x^3}}{\frac{2x^3 + 4x^2}{2x^3}} = \lim_{x \rightarrow -\infty} \frac{-\frac{1}{x} + \frac{1}{2x^3}}{1 + \frac{2}{x}} = \frac{0 + 0}{1 + 0} = 0$$

and

$$\lim_{x \rightarrow \infty} \frac{-2x^2 + 1}{2x^3 + 4x^2} = \lim_{x \rightarrow \infty} \frac{\frac{-2x^2 + 1}{2x^3}}{\frac{2x^3 + 4x^2}{2x^3}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x} + \frac{1}{2x^3}}{1 + \frac{2}{x}} = \frac{0 + 0}{1 + 0} = 0,$$

$y = 0$ is a horizontal asymptote of the graph of R .

- (c) The one-sided limits as x approaches -2 are

$$\lim_{x \rightarrow -2^-} \frac{-2x^2 + 1}{2x^3 + 4x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{-2x^2 + 1}{2x^3 + 4x^2} = -\infty,$$

so $x = -2$ is a vertical asymptote of the graph of R . The one-sided limits as x approaches 0 are

$$\lim_{x \rightarrow 0^-} \frac{-2x^2 + 1}{2x^3 + 4x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{-2x^2 + 1}{2x^3 + 4x^2} = \infty,$$

so $x = 0$ is also a vertical asymptote of the graph of R .

- (d) The only numbers where R is not defined are $x = -2$ and $x = 0$. Using the limits from part (c), it follows that as x approaches -2 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches -2 from the right, the graph of R becomes unbounded in the negative direction. As x approaches 0 from either direction, the graph of R becomes unbounded in the positive direction.

68. (a) Factoring the denominator of R yields

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1),$$

so the domain of R is the set $\{x | x \neq \pm 1\}$.

- (b) Because

$$\lim_{x \rightarrow -\infty} \frac{x^3}{x^4 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{x^3}{x^4}}{\frac{x^4 - 1}{x^4}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^4}} = \frac{0}{1 - 0} = 0$$

and

$$\lim_{x \rightarrow \infty} \frac{x^3}{x^4 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{x^3}{x^4}}{\frac{x^4 - 1}{x^4}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^4}} = \frac{0}{1 - 0} = 0,$$

$y = 0$ is a horizontal asymptote of the graph of R .

- (c) The one-sided limits as x approaches -1 are

$$\lim_{x \rightarrow -1^-} \frac{x^3}{x^4 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{x^3}{x^4 - 1} = \infty,$$

so $x = -1$ is a vertical asymptote of the graph of R . The one-sided limits as x approaches 1 are

$$\lim_{x \rightarrow 1^-} \frac{x^3}{x^4 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x^3}{x^4 - 1} = \infty,$$

so $x = 1$ is also a vertical asymptote of the graph of R .

- (d) The only numbers where R is not defined are $x = \pm 1$. Using the limits from part (c), it follows that as x approaches -1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches -1 from the right, the graph of R becomes unbounded in the positive direction. As x approaches 1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches 1 from the right, the graph of R becomes unbounded in the positive direction.

69. (a) Factoring the denominator of R yields

$$2x^2 - 7x + 6 = (2x - 3)(x - 2),$$

so the domain of R is the set $\{x | x \neq \frac{3}{2}, x \neq 2\}$.

- (b) Because

$$\lim_{x \rightarrow -\infty} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x \rightarrow -\infty} \frac{\frac{x^2 + 3x - 10}{2x^2}}{\frac{2x^2 - 7x + 6}{2x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{2} + \frac{3}{2x} - \frac{5}{x^2}}{1 - \frac{7}{2x} + \frac{3}{x^2}} = \frac{\frac{1}{2} + 0 - 0}{1 - 0 + 0} = \frac{1}{2}$$

and

$$\lim_{x \rightarrow \infty} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x \rightarrow \infty} \frac{\frac{x^2 + 3x - 10}{2x^2}}{\frac{2x^2 - 7x + 6}{2x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2} + \frac{3}{2x} - \frac{5}{x^2}}{1 - \frac{7}{2x} + \frac{3}{x^2}} = \frac{\frac{1}{2} + 0 - 0}{1 - 0 + 0} = \frac{1}{2},$$

$y = \frac{1}{2}$ is a horizontal asymptote of the graph of R .

- (c) The one-sided limits as x approaches $\frac{3}{2}$ are

$$\lim_{x \rightarrow 3/2^-} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x \rightarrow 3/2^-} \frac{(x-2)(x+5)}{(2x-3)(x-2)} = -\infty$$

and

$$\lim_{x \rightarrow 3/2^+} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x \rightarrow 3/2^+} \frac{(x-2)(x+5)}{(2x-3)(x-2)} = \infty,$$

so $x = \frac{3}{2}$ is a vertical asymptote of the graph of R . Because

$$\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x+5)}{(2x-3)(x-2)} = \lim_{x \rightarrow 2} \frac{x+5}{2x-3} = 7,$$

$x = 2$ is not a vertical asymptote of the graph of R . Rather, the graph of R has a hole at the point $(2, 7)$.

- (d) The only numbers where R is not defined are $x = \frac{3}{2}$ and $x = 2$. Using the limits from part (c), it follows that as x approaches $\frac{3}{2}$ from the left, the graph of R becomes unbounded in the negative direction, while as x approaches $\frac{3}{2}$ from the right, the graph of R becomes unbounded in the positive direction. The graph of R has a hole at the point $(2, 7)$ and approaches that point as x approaches 2 from either direction.
70. (a) The domain of R is the set $\{x | x \neq -3\}$.
- (b) Because

$$\lim_{x \rightarrow -\infty} \frac{x(x-1)^2}{(x+3)^3} = \lim_{x \rightarrow -\infty} \frac{\frac{x(x-1)^2}{x^3}}{\frac{(x+3)^3}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\left(1 - \frac{1}{x}\right)^2}{\left(1 + \frac{3}{x}\right)^3} = \frac{(1-0)^2}{(1+0)^3} = 1$$

and

$$\lim_{x \rightarrow \infty} \frac{x(x-1)^2}{(x+3)^3} = \lim_{x \rightarrow \infty} \frac{\frac{x(x-1)^2}{x^3}}{\frac{(x+3)^3}{x^3}} = \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{1}{x}\right)^2}{\left(1 + \frac{3}{x}\right)^3} = \frac{(1-0)^2}{(1+0)^3} = 1,$$

$y = 1$ is a horizontal asymptote of the graph of R .

- (c) The one-sided limits as x approaches -3 are

$$\lim_{x \rightarrow -3^-} \frac{x(x-1)^2}{(x+3)^3} = \infty \quad \text{and} \quad \lim_{x \rightarrow -3^+} \frac{x(x-1)^2}{(x+3)^3} = -\infty,$$

so $x = -3$ is a vertical asymptote of the graph of R .

- (d) The only number where R is not defined is $x = -3$. Using the limits from part (c), it follows that as x approaches -3 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches -3 from the right, the graph of R becomes unbounded in the negative direction.
71. (a) Factoring the denominator of R yields

$$x - x^2 = x(1 - x),$$

so the domain of R is the set $\{x | x \neq 0, x \neq 1\}$.

(b) Because

$$\lim_{x \rightarrow -\infty} \frac{x^3 - 1}{x - x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{x^3 - 1}{-x^2}}{\frac{x - x^2}{-x^2}} = \lim_{x \rightarrow -\infty} \frac{-x + \frac{1}{x^2}}{-\frac{1}{x} + 1} = \infty$$

and

$$\lim_{x \rightarrow \infty} \frac{x^3 - 1}{x - x^2} = \lim_{x \rightarrow \infty} \frac{\frac{x^3 - 1}{-x^2}}{\frac{x - x^2}{-x^2}} = \lim_{x \rightarrow \infty} \frac{-x + \frac{1}{x^2}}{-\frac{1}{x} + 1} = -\infty,$$

the graph of R has no horizontal asymptotes.

(c) The one-sided limits as x approaches 0 are

$$\lim_{x \rightarrow 0^-} \frac{x^3 - 1}{x - x^2} = \lim_{x \rightarrow 0^-} \frac{(x - 1)(x^2 + x + 1)}{x(1 - x)} = \infty$$

and

$$\lim_{x \rightarrow 0^+} \frac{x^3 - 1}{x - x^2} = \lim_{x \rightarrow 0^+} \frac{(x - 1)(x^2 + x + 1)}{x(1 - x)} = -\infty,$$

so $x = 0$ is a vertical asymptote of the graph of R . As x approaches 1,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - x^2} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{x(1 - x)} \\ &= -\lim_{x \rightarrow 1} \frac{x^2 + x + 1}{x} = -\frac{1 + 1 + 1}{1} = -3; \end{aligned}$$

therefore, $x = 1$ is not a vertical asymptote of the graph of R . Rather, the graph of R has a hole at the point $(1, -3)$.

(d) The only numbers where R is not defined are $x = 0$ and $x = 1$. Using the limits from part (c), it follows that as x approaches 0 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches 0 from the right, the graph of R becomes unbounded in the negative direction. The graph of R has a hole at the point $(1, -3)$ and approaches that point as x approaches 1 from either direction.

72. (a) Factoring the denominator of R yields

$$x^3 - 1 = (x - 1)(x^2 + x + 1),$$

so the domain of R is the set $\{x | x \neq 1\}$.

(b) Because

$$\lim_{x \rightarrow -\infty} \frac{4x^5}{x^3 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{4x^5}{x^3}}{\frac{x^3 - 1}{x^3}} = \lim_{x \rightarrow -\infty} \frac{4x^2}{1 - \frac{1}{x^3}} = \infty$$

and

$$\lim_{x \rightarrow \infty} \frac{4x^5}{x^3 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{4x^5}{x^3}}{\frac{x^3 - 1}{x^3}} = \lim_{x \rightarrow \infty} \frac{4x^2}{1 - \frac{1}{x^3}} = \infty,$$

the graph of R has no horizontal asymptotes.

- (c) The one-sided limits as x approaches 1 are

$$\lim_{x \rightarrow 1^-} \frac{4x^5}{x^3 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{4x^5}{x^3 - 1} = \infty,$$

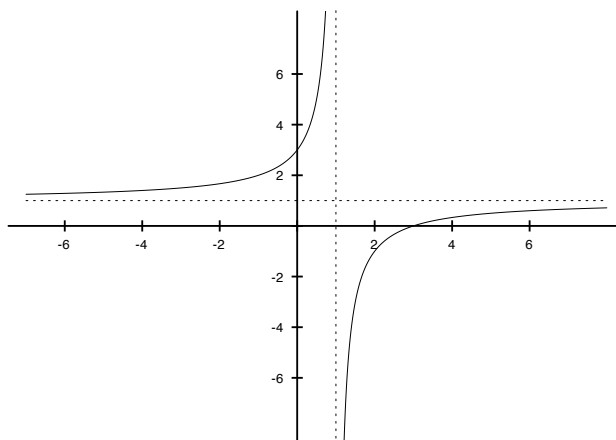
so $x = 1$ is a vertical asymptote of the graph of R .

- (d) The only number where R is not defined is $x = 1$. Using the limits from part (c), it follows that as x approaches 1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches 1 from the right, the graph of R becomes unbounded in the positive direction.

Applications and Extensions

73. (a) Answers will vary. The figure below displays the graph of a function f with the properties

$$f(3) = 0, \quad \lim_{x \rightarrow \infty} f(x) = 1, \quad \lim_{x \rightarrow -\infty} f(x) = 1, \quad \lim_{x \rightarrow 1^-} f(x) = \infty, \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = -\infty.$$

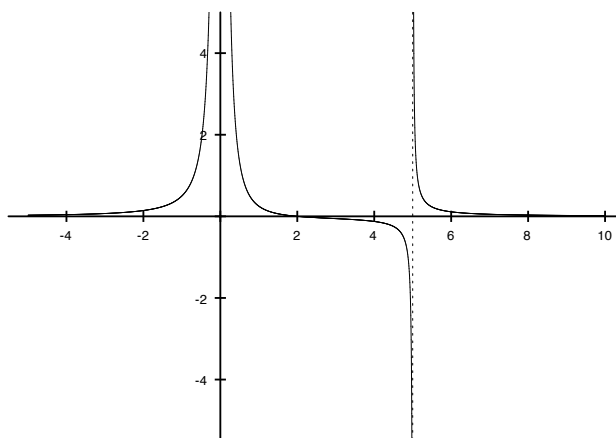


- (b) Answers will vary. The function shown above is $f(x) = \frac{x-3}{x-1}$.

74. (a) Answers will vary. The figure below displays the graph of a function f with the properties

$$f(2) = 0, \quad \lim_{x \rightarrow \infty} f(x) = 0, \quad \lim_{x \rightarrow -\infty} f(x) = 0, \quad \lim_{x \rightarrow 0} f(x) = \infty,$$

$$\lim_{x \rightarrow 5^-} f(x) = -\infty, \quad \text{and} \quad \lim_{x \rightarrow 5^+} f(x) = \infty.$$



- (b) Answers will vary. The function shown above is $f(x) = \frac{x-2}{x^2(x-5)}$.

75. (a) Because $k < 0$,

$$\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} [(u_0 - T)e^{kt} + T] = (u_0 - T) \lim_{t \rightarrow \infty} e^{kt} + \lim_{t \rightarrow \infty} T = (u_0 - T)(0) + T = T.$$

As time increases, one would expect the object to lose heat to its surroundings until the temperature of the object had decreased to the temperature of the surroundings, so this limit value is in line with expectations.

- (b)

$$\begin{aligned} \lim_{t \rightarrow 0^+} u(t) &= \lim_{t \rightarrow 0^+} [(u_0 - T)e^{kt} + T] = (u_0 - T) \lim_{t \rightarrow 0^+} e^{kt} + \lim_{t \rightarrow 0^+} T \\ &= (u_0 - T)e^{k(0)} + T = (u_0 - T) + T = u_0. \end{aligned}$$

This value is again in line with expectations because as t approaches 0 from the right, the temperature should approach the temperature of the object when it was placed into the lower temperature medium.

76. (a) To remove 45% of the pollutants, $p = 45$, and

$$C(45) = \frac{70,000(45)}{100 - 45} \approx 57,272.73.$$

The cost to remove 45% of the pollutants is therefore approximately \$57,272.73.

- (b) To remove 90% of the pollutants, $p = 90$, and

$$C(90) = \frac{70,000(90)}{100 - 90} = 630,000.$$

The cost to remove 90% of the pollutants is therefore \$630,000.

(c) $\lim_{p \rightarrow 100^-} C(p) = \lim_{p \rightarrow 100^-} \frac{70,000p}{100 - p} = \infty.$

- (d) As the percentage of the pollutants removed from the air increases toward 100, the cost of removing those pollutants increases without bound.

77. $\lim_{x \rightarrow 100^-} C(x) = \lim_{x \rightarrow 100^-} \frac{5x}{100 - x} = \infty.$ As the percentage of the pollutants removed from the lake increases toward 100, the cost of removing those pollutants increases without bound.

78. (a) The projected size of the colony after 1 year (365 days) is

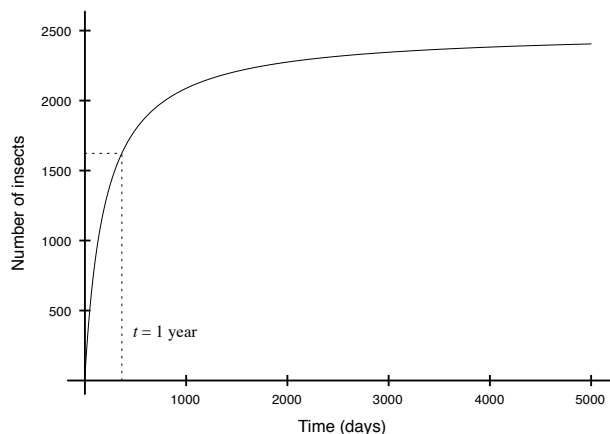
$$P(365) = \frac{50(1 + 0.5 \cdot 365)}{2 + 0.01 \cdot 365} = \frac{9175}{5.65} \approx 1623.89,$$

or 1624 insects.

- (b) The largest population that the protected area can sustain is

$$\begin{aligned} \lim_{t \rightarrow \infty} P(t) &= \lim_{t \rightarrow \infty} \frac{50(1 + 0.5t)}{2 + 0.01t} = \lim_{t \rightarrow \infty} \frac{50(1 + 0.5t)}{2 + 0.01t} \cdot \frac{\frac{1}{t}}{\frac{1}{t}} \\ &= \lim_{t \rightarrow \infty} \frac{50\left(\frac{1}{t} + 0.5\right)}{\frac{2}{t} + 0.01} = \frac{50(0 + 0.5)}{0 + 0.01} = 2500 \text{ insects}. \end{aligned}$$

- (c) The figure below displays the graph of the population P as a function of time t .

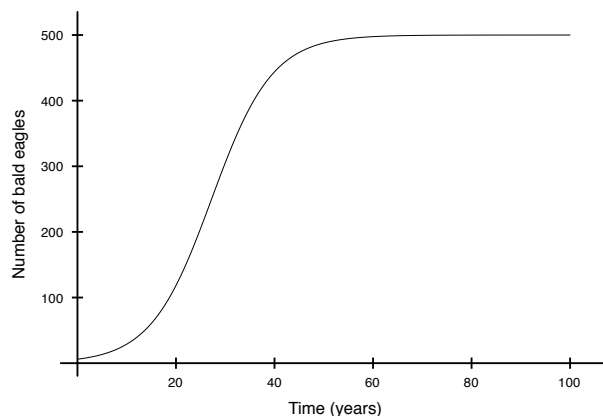


- (d) Within the first year, the population increases rapidly from the initial size of 25 insects to over 1600 insects. After that the population grows more slowly, eventually leveling off at 2500 insects, which is consistent with the result from part (b).

79. (a) If the model is correct, the environment can sustain

$$\begin{aligned}\lim_{t \rightarrow \infty} P(t) &= \lim_{t \rightarrow \infty} \frac{500}{1 + 82.3e^{-0.162t}} = \frac{\lim_{t \rightarrow \infty} 500}{\lim_{t \rightarrow \infty} [1 + 82.3e^{-0.162t}]} \\ &= \frac{500}{1 + 82.3 \lim_{t \rightarrow \infty} e^{-0.162t}} = \frac{500}{1 + 82.3(0)} = \boxed{500 \text{ bald eagles}}.\end{aligned}$$

- (b) The figure below displays the graph of the population P as a function of time t .



- (c) The number of bald eagles increases slowly for roughly the first ten years, after which the rate of population growth accelerates rapidly. This phase of growth continues until roughly year 35, after which the rate of growth slows dramatically. Eventually, the number of bald eagles levels off at 500, which is consistent with the result from part (a).

80. (a) The limiting speed of the hailstone is

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left[\frac{mg}{k} (1 - e^{-kt/m}) \right] = \frac{mg}{k} \left(1 - \lim_{t \rightarrow \infty} e^{-kt/m} \right) = \frac{mg}{k} (1 - 0) = \frac{mg}{k}.$$

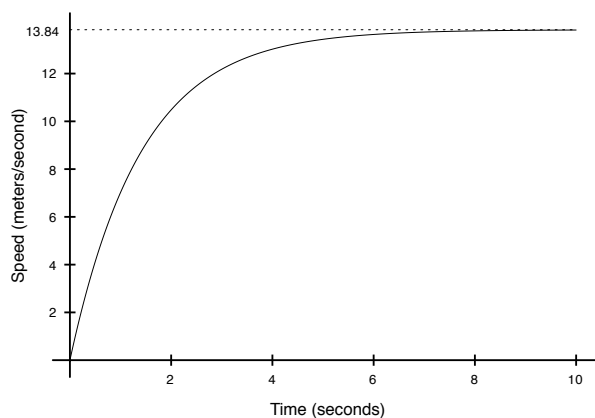
With $m = 4.8 \times 10^{-4}$ kg, $g = 9.8$ m/s², and $k = 3.4 \times 10^{-4}$ kg/s, the limiting speed is

$$\frac{4.8 \times 10^{-4} \text{ kg} \cdot 9.8 \text{ m/s}^2}{3.4 \times 10^{-4} \text{ kg/s}} \approx \boxed{13.84 \text{ m/s}}.$$

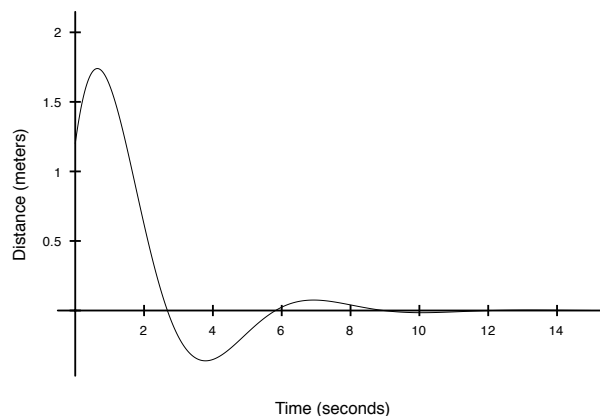
In miles per hour, this is

$$13.84 \text{ m/s} \cdot \frac{1 \text{ mi/h}}{0.447 \text{ m/s}} \approx \boxed{30.96 \text{ mi/h}}.$$

- (b) The figure below displays a graph of $v(t)$. The speed appears to approach 13.84 m/s, consistent with the result from part (a).



81. (a) The figure below displays the graph of $y = x(t)$. The graph suggests that $\lim_{t \rightarrow \infty} x(t) = 0$.



- (b) Note that $-1 \leq \cos t \leq 1$ and $-1 \leq \sin t \leq 1$ for all t . Thus,

$$-1.2e^{-t/2} \leq 1.2e^{-t/2} \cos t \leq 1.2e^{-t/2}$$

and

$$-2.4e^{-t/2} \leq 2.4e^{-t/2} \sin t \leq 2.4e^{-t/2}.$$

Because $\lim_{t \rightarrow \infty} e^{-t/2} = 0$, it follows that

$$\lim_{t \rightarrow \infty} (-1.2e^{-t/2}) = \lim_{t \rightarrow \infty} (1.2e^{-t/2}) = \lim_{t \rightarrow \infty} (-2.4e^{-t/2}) = \lim_{t \rightarrow \infty} (2.4e^{-t/2}) = 0,$$

so the Squeeze Theorem guarantees

$$\lim_{t \rightarrow \infty} (1.2e^{-t/2} \cos t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} (2.4e^{-t/2} \sin t) = 0.$$

Therefore,

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} (1.2e^{-t/2} \cos t) + \lim_{t \rightarrow \infty} (2.4e^{-t/2} \sin t) = 0 + 0 = \boxed{0}.$$

- (c) The answer from part (b) is supported by the graph in part (a).

82. (a) First determine the value of the rate constant k . With $C(0) = 2.5$ ppm, $C(t) = 2.5e^{kt}$. Given that the amount of free chlorine after 24 hours is 2.2 ppm, it follows that

$$2.2 = 2.5e^{24k} \quad \text{or} \quad e^{24k} = \frac{2.2}{2.5},$$

so that

$$k = \frac{1}{24} \ln \left(\frac{2.2}{2.5} \right) \approx -0.005326 \text{ hours}^{-1}.$$

Thus, after 72 hours,

$$C(72) \approx 2.5e^{-0.005326(72)} \approx 1.7,$$

so the amount of free chlorine is approximately 1.7 ppm.

- (b) The approximate time when the amount of free chlorine reaches 1.0 ppm is the solution of the equation

$$1.0 = 2.5e^{-0.005326t},$$

which is

$$t = \frac{1}{-0.005326} \ln \left(\frac{1.0}{2.5} \right) \approx 172 \text{ hours}.$$

Therefore, Ben can go roughly 172 hours (a little more than one week) before he must shock the pool again.

- (c) $\lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} (2.5e^{-0.005326t}) = \boxed{0}$ ppm.
- (d) In the long run, all of the free chlorine in the pool will decompose.
83. (a) First determine the value of the rate constant k . With $A(0) = 0.40$ moles, $A(t) = 0.40e^{kt}$. Given that the amount of sucrose present after 30 minutes is 0.36 moles, it follows that

$$0.36 = 0.40e^{30k} \quad \text{or} \quad e^{30k} = \frac{0.36}{0.40},$$

so that

$$k = \frac{1}{30} \ln \left(\frac{0.36}{0.40} \right) \approx -0.003512 \text{ minutes}^{-1}.$$

Thus, after 2 hours (120 minutes),

$$A(120) \approx 0.40e^{-0.003512(120)} \approx 0.26,$$

so the amount of sucrose present is approximately 0.26 moles.

- (b) The approximate time when the amount of sucrose remaining will be 0.10 moles is the solution of the equation

$$0.10 = 0.40e^{-0.003512t},$$

which is

$$t = \frac{1}{-0.003512} \ln \left(\frac{0.10}{0.40} \right) \approx \boxed{395 \text{ minutes}}.$$

(c) $\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} (0.40e^{-0.003512t}) = \boxed{0}$ moles.

(d) In the long run, all of the sucrose will decompose into glucose and fructose.

84. (a) Solving the thin film equation for q yields

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{pf} \quad \text{or} \quad q = \frac{pf}{p-f}.$$

Note that division by $p-f$ is permitted here because the problem statement indicates that $p > f$, so that $p-f$ will never be equal to zero. Thus,

$$\lim_{p \rightarrow f^+} q = \lim_{p \rightarrow f^+} \frac{pf}{p-f} = \infty.$$

Therefore, the distance q of the image from the lens is not continuous as the distance of the object approaches the focal length of the lens.

(b) A camera cannot focus on an object placed close to its focal length because the distance of the image from the lens becomes unbounded.

85. If $c \neq 0$, then

$$\lim_{x \rightarrow \infty} \frac{ax^3 + b}{cx^4 + d} = \lim_{x \rightarrow \infty} \frac{ax^3 + b}{cx^4 + d} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x \rightarrow \infty} \frac{\frac{a}{x} + \frac{b}{x^4}}{c + \frac{d}{x^4}} = \frac{0+0}{c+0} = 0,$$

so that $y = 0$ would be a horizontal asymptote of the graph of f . Thus, for the graph of f to have no horizontal asymptotes, c must be zero. It then follows that d cannot be zero, otherwise the denominator of f would be zero. Now, with $c = 0$ and $d \neq 0$, the function f reduces to a polynomial, and therefore does not have vertical asymptotes. Next, note that if $a = 0$, then $f(x) = \frac{b}{d}$, so that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{b}{d} = \frac{b}{d},$$

and $y = \frac{b}{d}$ would be a horizontal asymptote of the graph of f . Thus, a cannot be zero. Finally, for the graph of f to have no horizontal or vertical asymptotes, we must have $a \neq 0$, $c = 0$, $d \neq 0$, and b can be any real number.

86. If $c \neq 0$, then

$$\lim_{x \rightarrow \infty} \frac{ax + b}{cx + d} = \lim_{x \rightarrow \infty} \frac{ax + b}{cx + d} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x}}{c + \frac{d}{x}} = \frac{a+0}{c+0} = \frac{a}{c},$$

so that $y = \frac{a}{c}$ would be a horizontal asymptote of the graph of f . Thus, for the graph of f to have no horizontal asymptotes, c must be zero. It then follows that d cannot be zero, otherwise the denominator of f would be zero. Now, with $c = 0$ and $d \neq 0$, the function f reduces to a polynomial, and therefore does not have vertical asymptotes. Next, note that if $a = 0$, then $f(x) = \frac{b}{d}$, so that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{b}{d} = \frac{b}{d},$$

and $y = \frac{b}{d}$ would be a horizontal asymptote of the graph of f . Thus, a cannot be zero. Finally, for the graph of f to have no horizontal or vertical asymptotes, we must have $a \neq 0$, $c = 0$, $d \neq 0$, and b can be any real number.

87. (a) Let n be an even positive integer. Then, as x approaches c , $(x-c)^n$ approaches 0 from the right and $\frac{1}{(x-c)^n}$ becomes unbounded in the positive direction; that is, $\lim_{x \rightarrow c} \frac{1}{(x-c)^n} = \infty$. Answers will vary, but one example is $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

- (b) Let n be an odd positive integer. Then, as x approaches c from the left, $(x-c)^n$ approaches 0 from the left and $\frac{1}{(x-c)^n}$ becomes unbounded in the negative direction; that is, $\lim_{x \rightarrow c^-} \frac{1}{(x-c)^n} = -\infty$.

Answers will vary, but one example is $\lim_{x \rightarrow 4^-} \frac{1}{x-4} = -\infty$.

- (c) Let n be an odd positive integer. Then, as x approaches c from the right, $(x-c)^n$ approaches 0 from the right and $\frac{1}{(x-c)^n}$ becomes unbounded in the positive direction; that is, $\lim_{x \rightarrow c^+} \frac{1}{(x-c)^n} = \infty$.

Answers will vary, but one example is $\lim_{x \rightarrow -2^+} \frac{1}{(x+2)^3} = \infty$.

88. Let R be a rational function whose numerator and denominator have no common zeros, and let c be a point of discontinuity of R . Then c must be a zero of the denominator of R . Because the numerator and denominator of R have no common zeros, c is not a zero of the numerator, so that, as x approaches c , the numerator of R approaches a non-zero number, while the denominator approaches 0. It follows that the value of R must become unbounded as x approaches c and that the graph of R must have a vertical asymptote at $x = c$.
89. Let p be a polynomial function of degree 1 or higher. Because all polynomial functions are defined for all real numbers, the graph of p will have no vertical asymptotes. Moreover, because p contains at least one term of the form ax^k where $a \neq 0$ and k is an integer greater than or equal to 1 (as otherwise p would not be a polynomial function of degree 1 or higher), $p(x)$ will become unbounded as x becomes unbounded in either direction. Thus, the graph of f will also have no horizontal asymptotes.
90. Let P and Q be polynomial functions of degree m and n , respectively. In particular, suppose

$$P(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$

and

$$Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0,$$

where the a_j ($j = 0, 1, 2, \dots, m$) and the b_k ($k = 0, 1, 2, \dots, n$) are real numbers with $a_m \neq 0$ and $b_n \neq 0$.

- (a) Suppose $m > n$. Multiplying the numerator and denominator of $\frac{P(x)}{Q(x)}$ by $\frac{1}{x^n}$ then yields an expression of the form

$$\frac{a_m x^{m-n} + a_{m-1} x^{m-n-1} + \cdots + \frac{a_0}{x^n}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_0}{x^n}}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \begin{cases} \infty, & \text{if } a_m > 0 \\ -\infty, & \text{if } a_m < 0. \end{cases}$$

- (b) Suppose $m = n$. Multiplying the numerator and denominator of $\frac{P(x)}{Q(x)}$ by $\frac{1}{x^n}$ then yields an expression of the form

$$\frac{a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_0}{x^n}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_0}{x^n}}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \frac{a_m + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \boxed{\frac{a_m}{b_n}}.$$

In other words, when $m = n$, the limit is the ratio of the leading coefficients of the two polynomial functions.

- (c) Suppose $m < n$. Multiplying the numerator and denominator of $\frac{P(x)}{Q(x)}$ by $\frac{1}{x^n}$ then yields an expression of the form

$$\frac{\frac{a_m}{x^{n-m}} + \frac{a_{m-1}}{x^{n-m+1}} + \cdots + \frac{a_0}{x^n}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_0}{x^n}}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \frac{0 + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \boxed{0}.$$

91. (a) The table of values below, which have been rounded to six decimal places, suggests

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \approx \boxed{2.718282}.$$

x	100	10,000	1,000,000	100,000,000	$\rightarrow \infty$
$f(x) = \left(1 + \frac{1}{x}\right)^x$	2.704814	2.718146	2.718280	2.718282	$f(x)$ approaches 2.718282

- (b) Using the computer algebra system *Mathematica*,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \boxed{e \approx 2.718281828}.$$

- (c) Answers will vary. One possible response is that the results from parts (a) and (b) agree to five decimal places. Though it is possible to achieve accuracy to as many decimal places as desired by calculating $\left(1 + \frac{1}{x}\right)^x$ for larger and larger x , it is impossible to determine every digit of this limit. The number e , like the number π , has a nonrepeating, nonterminating decimal expansion.

Challenge Problems

92. In $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$, the exponent is the variable x . The property $\lim_{x \rightarrow \infty} [f(x)]^n = \left[\lim_{x \rightarrow \infty} f(x)\right]^n$ requires the exponent to be a constant, independent of the variable x .
93. (a) As v approaches c from the left, $\frac{v^2}{c^2}$ approaches 1 and $1 - \frac{v^2}{c^2}$ approaches 0. Therefore

$$\lim_{v \rightarrow c^-} K_{\text{gen}}(v) = mc^2 \lim_{v \rightarrow c^-} \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = \boxed{\infty}.$$

- (b) Because it is not possible to have infinite kinetic energy, the result from part (a) suggests that it is not possible to reach the speed of light.

1.6: The ϵ - δ Definition of a Limit

Concepts and Vocabulary

1. False. The limit of a function as x approaches c **does not depend** on the value of the function at c .
2. True. In the ϵ - δ definition of a limit, we require $0 < |x - c|$ to ensure that $x \neq c$.
3. True. In an ϵ - δ proof of a limit, the size of δ usually depends on the size of ϵ .
4. False. When proving $\lim_{x \rightarrow c} f(x) = L$ using the ϵ - δ definition, you try to find a connection between $|f(x) - L|$ and $|x - c|$.
5. True. Given any $\epsilon > 0$, suppose there is a $\delta > 0$, so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Then $\lim_{x \rightarrow c} f(x) = L$.
6. False. A function f has a limit L at infinity, if for any given $\epsilon > 0$, there is a positive number M so that whenever $x > M$, $|f(x) - L| < \epsilon$.

Skill Building

7. Here, $f(x) = 2x$, $c = 1$, and $L = 2$. To make

$$|f(x) - L| = |2x - 2| = 2|x - 1| < 0.01$$

requires

$$|x - 1| < \frac{0.01}{2} = 0.005.$$

Thus, the largest δ that “works” for $\epsilon = 0.01$ is $\delta = 0.005$.

8. Here, $f(x) = -3x$, $c = 2$, and $L = -6$. To make

$$|f(x) - L| = |-3x - (-6)| = |-3x + 6| = |-3||x - 2| = 3|x - 2| < 0.01$$

requires

$$|x - 2| < \frac{0.01}{3} = \frac{1}{300}.$$

Thus, the largest δ that “works” for $\epsilon = 0.01$ is $\delta = \frac{1}{300}$.

9. Here $f(x) = 6x - 1$, $c = 2$, and $L = 11$. To make

$$|f(x) - L| = |(6x - 1) - 11| = |6x - 12| = 6|x - 2| < \frac{1}{2}$$

requires

$$|x - 2| < \frac{1}{12}.$$

Thus, the largest δ that “works” for $\epsilon = \frac{1}{2}$ is $\delta = \frac{1}{12}$.

10. Here $f(x) = 2 - 3x$, $c = -3$, and $L = 11$. To make

$$|f(x) - L| = |(2 - 3x) - 11| = |-3x - 9| = |-3||x + 3| = 3|x + 3| < \frac{1}{3}$$

requires

$$|x + 3| < \frac{1}{9}.$$

Thus, the largest δ that “works” for $\epsilon = \frac{1}{3}$ is $\boxed{\delta = \frac{1}{9}}$.

11. Here $f(x) = -\frac{1}{2}x + 5$, $c = 2$, and $L = 4$. To make

$$|f(x) - L| = \left| \left(-\frac{1}{2}x + 5 \right) - 4 \right| = \left| -\frac{1}{2}x + 1 \right| = \left| -\frac{1}{2} \right| |x - 2| = \frac{1}{2}|x - 2| < 0.01$$

requires

$$|x - 2| < 2(0.01) = 0.02.$$

Thus, the largest δ that “works” for $\epsilon = 0.01$ is $\boxed{\delta = 0.02}$.

12. Here $f(x) = 3x + \frac{1}{2}$, $c = \frac{5}{6}$, and $L = 3$. To make

$$|f(x) - L| = \left| \left(3x + \frac{1}{2} \right) - 3 \right| = \left| 3x - \frac{5}{2} \right| = 3 \left| x - \frac{5}{6} \right| < 0.3$$

requires

$$\left| x - \frac{5}{6} \right| < \frac{0.3}{3} = 0.1.$$

Thus, the largest δ that “works” for $\epsilon = 0.3$ is $\boxed{\delta = 0.1}$.

13. To make

$$|(4x - 1) - 11| = |4x - 12| = 4|x - 3| < \epsilon$$

requires

$$|x - 3| < \frac{\epsilon}{4}.$$

Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{4}$.

(a) For $\epsilon = 0.1$, we can choose any $\boxed{\delta \leq \frac{0.1}{4} = 0.025}$.

(b) For $\epsilon = 0.01$, we can choose any $\boxed{\delta \leq \frac{0.01}{4} = 0.0025}$.

(c) For $\epsilon = 0.001$, we can choose any $\boxed{\delta \leq \frac{0.001}{4} = 0.00025}$.

(d) For arbitrary $\epsilon > 0$, we can choose any $\boxed{\delta \leq \frac{\epsilon}{4}}$.

14. To make

$$|(2 - 5x) - 12| = |-5x - 10| = |-5| |x + 2| = 5|x + 2| < \epsilon$$

requires

$$|x + 2| < \frac{\epsilon}{5}.$$

Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{5}$.

(a) For $\epsilon = 0.2$, we can choose any $\boxed{\delta \leq \frac{0.2}{5} = 0.04}$.

(b) For $\epsilon = 0.02$, we can choose any $\delta \leq \frac{0.02}{5} = 0.004$.

(c) For $\epsilon = 0.002$, we can choose any $\delta \leq \frac{0.002}{5} = 0.0004$.

(d) For arbitrary $\epsilon > 0$, we can choose any $\delta \leq \frac{\epsilon}{5}$.

15. The inequality $0 < |x + 3|$ guarantees that x cannot be equal to -3 . Now, for $x \neq -3$,

$$\frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = x - 3.$$

Therefore, for $x \neq -3$,

$$\left| \frac{x^2 - 9}{x + 3} - (-6) \right| = |(x - 3) + 6| = |x + 3|.$$

To make this less than ϵ , requires $|x + 3| < \epsilon$, so the largest δ that “works” for an arbitrary ϵ is $\delta = \epsilon$.

(a) For $\epsilon = 0.1$, we can choose any $\delta \leq 0.1$.

(b) For $\epsilon = 0.01$, we can choose any $\delta \leq 0.01$.

(c) For arbitrary $\epsilon > 0$, we can choose any $\delta \leq \epsilon$.

16. The inequality $0 < |x - 2|$ guarantees that x cannot be equal to 2. Now, for $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2.$$

Therefore, for $x \neq 2$,

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| = |(x + 2) - 4| = |x - 2|.$$

To make this less than ϵ , requires $|x - 2| < \epsilon$, so the largest δ that “works” for an arbitrary ϵ is $\delta = \epsilon$.

(a) For $\epsilon = 0.1$, we can choose any $\delta \leq 0.1$.

(b) For $\epsilon = 0.01$, we can choose any $\delta \leq 0.01$.

(c) For arbitrary $\epsilon > 0$, we can choose any $\delta \leq \epsilon$.

17. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 3x$, $c = 2$, and $L = 6$. To make

$$|f(x) - L| = |3x - 6| = 3|x - 2| < \epsilon$$

requires $|x - 2| < \epsilon/3$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{3}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/3$. Whenever $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |3x - 6| = 3|x - 2| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} (3x) = 6$.

18. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 4x$, $c = 3$, and $L = 12$. To make

$$|f(x) - L| = |4x - 12| = 4|x - 3| < \epsilon$$

requires $|x - 3| < \epsilon/4$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{4}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/4$. Whenever $0 < |x - 3| < \delta$, then

$$|f(x) - L| = |4x - 12| = 4|x - 3| < 4\delta = 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 3} (4x) = 12$.

19. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2x + 5$, $c = 0$, and $L = 5$. To make

$$|f(x) - L| = |(2x + 5) - 5| = 2|x| < \epsilon$$

requires $|x| = |x - 0| < \epsilon/2$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{2}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/2$. Whenever $0 < |x - 0| = |x| < \delta$, then

$$|f(x) - L| = |(2x + 5) - 5| = 2|x| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 0} (2x + 5) = 5$.

20. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2 - 3x$, $c = -1$, and $L = 5$. To make

$$|f(x) - L| = |(2 - 3x) - 5| = |-3x - 3| = |-3| |x + 1| = 3|x + 1| < \epsilon$$

requires $|x + 1| < \epsilon/3$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{3}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/3$. Whenever $0 < |x - (-1)| = |x + 1| < \delta$, then

$$|f(x) - L| = |(2 - 3x) - 5| = |-3x - 3| = 3|x + 1| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x \rightarrow -1} (2 - 3x) = 5$.

21. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = -5x + 2$, $c = -3$, and $L = 17$. To make

$$|f(x) - L| = |(-5x + 2) - 17| = |-5x - 15| = |-5| |x + 3| = 5|x + 3| < \epsilon$$

requires $|x + 3| < \epsilon/5$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{5}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/5$. Whenever $0 < |x - (-3)| = |x + 3| < \delta$, then

$$|f(x) - L| = |(-5x + 2) - 17| = |-5x - 15| = 5|x + 3| < 5\delta = 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Therefore, $\lim_{x \rightarrow -3} (-5x + 2) = 17$.

22. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2x - 3$, $c = 2$, and $L = 1$. To make

$$|f(x) - L| = |(2x - 3) - 1| = |2x - 4| = 2|x - 2| < \epsilon$$

requires $|x - 2| < \epsilon/2$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \frac{\epsilon}{2}$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon/2$. Whenever $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |(2x - 3) - 1| = 2|x - 2| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} (2x - 3) = 1$.

23. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2 - 2x$, $c = 2$, and $L = 0$. Then

$$|f(x) - L| = |(x^2 - 2x) - 0| = |x(x - 2)| = |x| \cdot |x - 2|.$$

The factor $|x - 2|$ will be smaller than δ , but what about the factor $|x|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x - 2| < \delta$ guarantees that $|x - 2| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $|x| < 3$ and

$$|f(x) - L| = |x| \cdot |x - 2| < 3|x - 2| < 3\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/3$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{3} \right\}$. Whenever $0 < |x - 2| < \delta$, it follows that $|x| < 3$, and then

$$|f(x) - L| = |(x^2 - 2x) - 0| = |x| \cdot |x - 2| < 3|x - 2| < 3\delta \leq 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} (x^2 - 2x) = 0$.

24. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2 + 3x$, $c = 0$, and $L = 0$. Then

$$|f(x) - L| = |(x^2 + 3x) - 0| = |x(x + 3)| = |x| \cdot |x + 3|.$$

The factor $|x|$ will be smaller than δ , but what about the factor $|x + 3|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x| < \delta$ guarantees that $|x| < 1$, or $-1 < x < 1$. Thus, $2 < x + 3 < 4$, so that $|x + 3| < 4$ and

$$|f(x) - L| = |x| \cdot |x + 3| < 4|x| < 4\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/4$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{4} \right\}$. Whenever $0 < |x| < \delta$, it follows that $|x + 3| < 4$, and then

$$|f(x) - L| = |(x^2 + 3x) - 0| = |x| \cdot |x + 3| < 4|x| < 4\delta \leq 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 0} (x^2 + 3x) = 0$.

25. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \frac{1+2x}{3-x}$, $c = 1$, and $L = \frac{3}{2}$. Then

$$|f(x) - L| = \left| \frac{1+2x}{3-x} - \frac{3}{2} \right| = \left| \frac{7x-7}{2(3-x)} \right| = \frac{7}{2} \frac{|x-1|}{|3-x|}.$$

The factor $|x-1|$ will be smaller than δ , but what about the factor $|3-x|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x-1| < \delta$ guarantees that $|x-1| < 1$. Removing the absolute value from this last inequality yields $-1 < x-1 < 1$, or $0 < x < 2$. Thus, $-2 < -x < 0$ and $1 < 3-x < 3$. Because $3-x > 1 > 0$, $3-x = |3-x|$, so that $1 < |3-x| < 3$. Therefore, $\frac{1}{3} < \frac{1}{|3-x|} < 1$ and

$$|f(x) - L| = \frac{7}{2} \frac{|x-1|}{|3-x|} < \frac{7}{2} |x-1| < \frac{7}{2} \delta.$$

To make this less than ϵ , we can choose $\delta \leq 2\epsilon/7$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{2\epsilon}{7} \right\}$. Whenever $0 < |x-1| < \delta$, it follows that $\frac{1}{|3-x|} < 1$, and then

$$|f(x) - L| = \left| \frac{1+2x}{3-x} - \frac{3}{2} \right| = \left| \frac{7x-7}{2(3-x)} \right| = \frac{7}{2} \frac{|x-1|}{|3-x|} < \frac{7}{2} |x-1| < \frac{7}{2} \delta \leq \frac{7}{2} \cdot \frac{2\epsilon}{7} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 1} \frac{1+2x}{3-x} = \frac{3}{2}$.

26. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \frac{2x}{4+x}$, $c = 2$, and $L = \frac{2}{3}$. Then

$$|f(x) - L| = \left| \frac{2x}{4+x} - \frac{2}{3} \right| = \left| \frac{4x-8}{3(4+x)} \right| = \frac{4}{3} \frac{|x-2|}{|4+x|}.$$

The factor $|x-2|$ will be smaller than δ , but what about the factor $|4+x|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x-2| < \delta$ guarantees that $|x-2| < 1$. Removing the absolute value from this last inequality yields $-1 < x-2 < 1$, or $1 < x < 3$. Thus, $5 < 4+x < 7$. Because $4+x > 5 > 0$, $4+x = |4+x|$, so that $5 < |4+x| < 7$. Therefore, $\frac{1}{7} < \frac{1}{|4+x|} < \frac{1}{5}$ and

$$|f(x) - L| = \frac{4}{3} \frac{|x-2|}{|4+x|} < \frac{4}{15} |x-2| < \frac{4}{15} \delta.$$

To make this less than ϵ , we can choose $\delta \leq 15\epsilon/4$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{15\epsilon}{4} \right\}$. Whenever $0 < |x-2| < \delta$, it follows that $\frac{1}{|4+x|} < \frac{1}{5}$, and then

$$|f(x) - L| = \left| \frac{2x}{4+x} - \frac{2}{3} \right| = \left| \frac{4x-8}{3(4+x)} \right| = \frac{4}{3} \frac{|x-2|}{|4+x|} < \frac{4}{15} |x-2| < \frac{4}{15} \delta \leq \frac{4}{15} \cdot \frac{15\epsilon}{4} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} \frac{2x}{4+x} = \frac{2}{3}$.

27. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \sqrt[3]{x}$, $c = 0$, and $L = 0$. To make

$$|f(x) - L| = |\sqrt[3]{x} - 0| = \sqrt[3]{|x|} < \epsilon$$

requires $|x| = |x - 0| < \epsilon^3$. Thus, the largest δ that “works” for an arbitrary ϵ is $\delta = \epsilon^3$. The ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon^3$. Whenever $0 < |x - 0| = |x| < \delta$, then

$$|f(x) - L| = |\sqrt[3]{x} - 0| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \sqrt[3]{\epsilon^3} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 0} \sqrt[3]{x} = 0$.

28. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \sqrt{2 - x}$, $c = 1$, and $L = 1$. Then

$$|f(x) - L| = |\sqrt{2 - x} - 1| \cdot \frac{|\sqrt{2 - x} + 1|}{|\sqrt{2 - x} + 1|} = \frac{|(2 - x) - 1|}{|\sqrt{2 - x} + 1|} = \frac{|1 - x|}{|\sqrt{2 - x} + 1|} = \frac{|x - 1|}{|\sqrt{2 - x} + 1|}.$$

The factor $|x - 1|$ will be smaller than δ , but what about the factor $|\sqrt{2 - x} + 1|$. Because $\sqrt{2 - x} \geq 0$, $\sqrt{2 - x} + 1 \geq 1$. Thus,

$$0 < \frac{1}{\sqrt{2 - x} + 1} \leq 1 \quad \text{so that} \quad \frac{1}{|\sqrt{2 - x} + 1|} \leq 1,$$

and

$$|f(x) - L| = \frac{|x - 1|}{|\sqrt{2 - x} + 1|} \leq |x - 1|.$$

To make this less than ϵ requires $|x - 1| < \epsilon$. Therefore, the largest δ that “works” for an arbitrary ϵ is $\delta = \epsilon$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \epsilon$. Whenever $0 < |x - 1| < \delta$, then

$$\begin{aligned} |f(x) - L| &= |\sqrt{2 - x} - 1| \cdot \frac{|\sqrt{2 - x} + 1|}{|\sqrt{2 - x} + 1|} = \frac{|(2 - x) - 1|}{|\sqrt{2 - x} + 1|} \\ &= \frac{|1 - x|}{|\sqrt{2 - x} + 1|} = \frac{|x - 1|}{|\sqrt{2 - x} + 1|} \leq |x - 1| < \delta = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} \sqrt{2 - x} = 1$.

29. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2$, $c = -1$, and $L = 1$. Then

$$|f(x) - L| = |x^2 - 1| = |(x - 1)(x + 1)| = |x - 1| \cdot |x + 1|.$$

The factor $|x + 1|$ will be smaller than δ , but what about the factor $|x - 1|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x + 1| < \delta$ guarantees that $|x + 1| < 1$. Removing the absolute value from this last inequality yields $-1 < x + 1 < 1$, or $-2 < x < 0$. Thus, $-3 < x - 1 < -1$, so that $|x - 1| < 3$ and

$$|f(x) - L| = |x - 1| \cdot |x + 1| < 3|x + 1| < 3\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/3$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{3} \right\}$. Whenever $0 < |x + 1| < \delta$, it follows that $|x - 1| < 3$,

and then

$$|f(x) - L| = |x^2 - 1| = |x - 1| \cdot |x + 1| < 3|x + 1| < 3\delta \leq 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x \rightarrow -1} x^2 = 1$.

30. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^3$, $c = 2$, and $L = 8$. Then

$$|f(x) - L| = |x^3 - 8| = |(x - 2)(x^2 + 2x + 4)| = |x^2 + 2x + 4| \cdot |x - 2|.$$

The factor $|x - 2|$ will be smaller than δ , but what about the factor $|x^2 + 2x + 4|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x - 2| < \delta$ guarantees that $|x - 2| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $|x| < 3$, so that $|x^2 + 2x + 4| \leq |x|^2 + 2|x| + 4 < 19$ and

$$|f(x) - L| = |x^2 + 2x + 4| \cdot |x - 2| < 19|x - 2| < 19\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/19$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{19} \right\}$. Whenever $0 < |x - 2| < \delta$, it follows that $|x^2 + 2x + 4| < 19$, and then

$$\begin{aligned} |f(x) - L| &= |x^3 - 8| = |(x - 2)(x^2 + 2x + 4)| = |x^2 + 2x + 4| \cdot |x - 2| \\ &< 19|x - 2| < 19\delta \leq 19 \cdot \frac{\epsilon}{19} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \rightarrow 2} x^3 = 8$.

31. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \frac{1}{x}$, $c = 3$, and $L = \frac{1}{3}$. Then

$$|f(x) - L| = \left| \frac{1}{x} - \frac{1}{3} \right| = \frac{|3 - x|}{3|x|} = \frac{|x - 3|}{3|x|}.$$

The factor $|x - 3|$ will be smaller than δ , but what about the factor $|x|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x - 3| < \delta$ guarantees that $|x - 3| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 3 < 1$, or $2 < x < 4$. Because $x > 2 > 0$, $x = |x|$ so that $2 < |x - 2| < 4$. Therefore, $\frac{1}{4} < \frac{1}{|x|} < \frac{1}{2}$, and

$$|f(x) - L| = \frac{|x - 3|}{3|x|} < \frac{|x - 2|}{6} < \frac{\delta}{6}.$$

To make this less than ϵ , we can choose $\delta \leq 6\epsilon$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \{1, 6\epsilon\}$. Whenever $0 < |x - 3| < \delta$, it follows that $\frac{1}{|x|} < \frac{1}{2}$, and then

$$|f(x) - L| = \left| \frac{1}{x} - \frac{1}{3} \right| = \frac{|3 - x|}{3|x|} = \frac{|x - 3|}{3|x|} < \frac{|x - 2|}{6} < \frac{\delta}{6} \leq \frac{6\epsilon}{6} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 3} \frac{1}{x} = \frac{1}{3}$.

32. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \frac{1}{x^2}$, $c = 2$, and $L = \frac{1}{4}$. Then

$$|f(x) - L| = \left| \frac{1}{x^2} - \frac{1}{4} \right| = \frac{|4 - x^2|}{4x^2} = \frac{|x - 2| \cdot |x + 2|}{4x^2}.$$

The factor $|x - 2|$ will be smaller than δ , but what about the factors $|x + 2|$ and x^2 ? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x - 2| < \delta$ guarantees that $|x - 2| < 1$.

Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $\frac{1}{x^2} < 1$ and $3 < x + 2 < 5$, so that $|x + 2| < 5$ and

$$|f(x) - L| = \frac{|x - 2| \cdot |x + 2|}{4x^2} < \frac{5}{4}|x - 2| < \frac{5}{4}\delta.$$

To make this less than ϵ , we can choose $\delta \leq 4\epsilon/5$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{4\epsilon}{5} \right\}$. Whenever $0 < |x - 2| < \delta$, it follows that $\frac{1}{x^2} < 1$ and $|x + 2| < 5$. Then

$$|f(x) - L| = \left| \frac{1}{x^2} - \frac{1}{4} \right| = \frac{|4 - x^2|}{4x^2} = \frac{|x - 2| \cdot |x + 2|}{4x^2} < \frac{5}{4}|x - 2| < \frac{5}{4}\delta \leq \frac{5}{4} \cdot \frac{4\epsilon}{5} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} \frac{1}{x^2} = \frac{1}{4}$.

33. Negating the ϵ - δ definition of a limit yields the statement: $\lim_{x \rightarrow c} f(x) \neq L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$. To establish that $\lim_{x \rightarrow 3} (3x - 1) \neq 12$, let $\epsilon = 1$ and $\delta > 0$. From the set of x values satisfying $0 < |x - 3| < \delta$ select any x for which $-\delta < x - 3 < 0$, or $3 - \delta < x < 3$. Then

$$-4 - 3\delta < (3x - 1) - 12 < -4 \quad \text{or} \quad |(3x - 1) - 12| > 4 > 1 = \epsilon.$$

Therefore, $\lim_{x \rightarrow 3} (3x - 1) \neq 12$.

34. Negating the ϵ - δ definition of a limit yields the statement: $\lim_{x \rightarrow c} f(x) \neq L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$. To establish that $\lim_{x \rightarrow -2} (4x) \neq -7$, let $\epsilon = \frac{1}{2}$ and $\delta > 0$. From the set of x values satisfying $0 < |x + 2| < \delta$ select any x for which $-\delta < x + 2 < 0$, or $-2 - \delta < x < -2$. Then

$$-1 - 4\delta < 4x + 7 < -1 \quad \text{or} \quad |4x + 7| > 1 > \frac{1}{2} = \epsilon.$$

Therefore, $\lim_{x \rightarrow -2} (4x) \neq -7$.

Applications and Extensions

35. Note that

$$\left| \frac{1}{x^2 + 9} - \frac{1}{18} \right| = \left| \frac{18 - (x^2 + 9)}{18(x^2 + 9)} \right| = \left| \frac{9 - x^2}{18(x^2 + 9)} \right| = \frac{|3 - x| \cdot |3 + x|}{18(x^2 + 9)} = \frac{|x + 3|}{18(x^2 + 9)} |x - 3|.$$

If $2 < x < 4$, then

$$13 < x^2 + 9 < 25 \quad \text{so that} \quad \frac{1}{x^2 + 9} < \frac{1}{13}$$

and

$$5 < x + 3 < 7 \quad \text{so that} \quad |x + 3| < 7;$$

therefore,

$$\left| \frac{1}{x^2 + 9} - \frac{1}{18} \right| = \frac{|x + 3|}{18(x^2 + 9)} |x - 3| < \frac{7}{18(13)} |x - 3| = \frac{7}{234} |x - 3| < \frac{7}{234} \delta.$$

To make this less than ϵ , we can choose $\delta \leq 234\epsilon/7$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $2 < x < 4$, then $-1 < x - 3 < 1$, and

$|x - 3| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{234}{7}\epsilon \right\}$. Whenever $0 < |x - 3| < \delta$, it follows that

$\frac{1}{x^2 + 9} < \frac{1}{13}$ and $|x + 3| < 7$. Then

$$\left| \frac{1}{x^2 + 9} - \frac{1}{18} \right| = \frac{|x + 3|}{18(x^2 + 9)} |x - 3| < \frac{7}{234} |x - 3| < \frac{7}{234} \delta \leq \frac{7}{234} \cdot \frac{234}{7} \epsilon = \epsilon.$$

Therefore, $\lim_{x \rightarrow 3} \frac{1}{x^2 + 9} = \frac{1}{18}$.

36. Note that

$$|(2 + x)^2 - 4| = |4 + 4x + x^2 - 4| = |x(4 + x)| = |x| \cdot |x + 4|.$$

If $-1 < x < 1$, then $3 < x + 4 < 5$ and $|x + 4| < 5$. Thus,

$$|(2 + x)^2 - 4| = |x| \cdot |x + 4| < 5|x| < 5\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/5$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $-1 < x < 1$, then $|x| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{5} \right\}$. Whenever $0 < |x| < \delta$, it follows that $|x + 4| < 5$, and then

$$|(2 + x)^2 - 4| = |4 + 4x + x^2 - 4| = |x(4 + x)| = |x| \cdot |x + 4| < 5|x| < 5\delta \leq 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 0} (2 + x)^2 = 4$.

37. Note that

$$\left| \frac{1}{x^2 + 9} - \frac{1}{13} \right| = \left| \frac{13 - (x^2 + 9)}{13(x^2 + 9)} \right| = \left| \frac{4 - x^2}{13(x^2 + 9)} \right| = \frac{|2 - x| \cdot |2 + x|}{13(x^2 + 9)} = \frac{|x + 2|}{13(x^2 + 9)} |x - 2|.$$

If $1 < x < 3$, then

$$10 < x^2 + 9 < 18 \quad \text{so that} \quad \frac{1}{x^2 + 9} < \frac{1}{10}$$

and

$$3 < x + 2 < 5 \quad \text{so that} \quad |x + 2| < 5.$$

Thus,

$$\left| \frac{1}{x^2 + 9} - \frac{1}{13} \right| = \frac{|x + 2|}{13(x^2 + 9)} |x - 2| < \frac{5}{13(10)} |x - 2| = \frac{1}{26} |x - 2| < \frac{1}{26} \delta.$$

To make this less than ϵ , we can choose $\delta \leq 26\epsilon$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $1 < x < 3$, then $-1 < x - 2 < 1$ and $|x - 2| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \{1, 26\epsilon\}$. Whenever $0 < |x - 2| < \delta$, it follows that

$\frac{1}{x^2 + 9} < \frac{1}{10}$ and $|x + 2| < 5$. Then

$$\left| \frac{1}{x^2 + 9} - \frac{1}{13} \right| = \frac{|x + 2|}{13(x^2 + 9)} |x - 2| < \frac{1}{26} |x - 2| < \frac{1}{26} \delta \leq \frac{1}{26} \cdot 26\epsilon = \epsilon.$$

Therefore, $\lim_{x \rightarrow 2} \frac{1}{x^2 + 9} = \frac{1}{13}$.

38. Negating the ϵ - δ definition of a limit yields the statement: $\lim_{x \rightarrow c} f(x) \neq L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$. To establish that $\lim_{x \rightarrow 1} x^2 \neq 1.31$, let $\epsilon = 0.1$ and $\delta > 0$. If $\delta < 1$, from the set of x values satisfying $0 < |x - 1| < \delta$ select any x for which $-\delta < x - 1 < 0$, or $1 - \delta < x < 1$, and note that $1 - \delta > 0$. On the other hand, if $\delta \geq 1$, then from the set of x values satisfying $0 < |x - 1| < \delta$ select any x for which $-1 < x - 1 < 0$, or $0 < x < 1$. Then, for any $\delta > 0$,

$$x^2 < 1 \quad \text{so that} \quad |x^2 - 1.31| > 0.31 > 0.1 = \epsilon.$$

Therefore, $\lim_{x \rightarrow 1} x^2 \neq 1.31$.

39. $\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \frac{\epsilon}{1 + |m|}$. The absolute value of m appears in the formula for δ to insure that δ will be positive. Moreover, 1 has been added to the denominator to allow for the possibility that m could be zero. Whenever $0 < |x - c| < \delta$, then

$$|(mx + b) - (mc + b)| = |m| |x - c| < |m| \cdot \delta = \frac{|m|}{1 + |m|} \epsilon < \epsilon.$$

Therefore, $\lim_{x \rightarrow c} (mx + b) = mc + b$.

40. Recall that

$$|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2| \cdot |x + 2|.$$

If $|x - 2| < \frac{1}{3}$, then $\frac{5}{3} < x < \frac{7}{3}$, so that

$$\frac{11}{3} < x + 2 < \frac{13}{3} \quad \text{and} \quad |x + 2| < \frac{13}{3}.$$

Thus,

$$|x^2 - 4| = |x - 2| \cdot |x + 2| < \frac{13}{3} |x - 2| < \frac{13}{3} \delta.$$

To make this less than ϵ , we can choose $\delta \leq \frac{3\epsilon}{13}$. Combining the two restrictions placed on δ yields $\delta \leq \min \left\{ \frac{1}{3}, \frac{3\epsilon}{13} \right\}$.

41. For $x \neq 3$, to guarantee that $|(2x - 1) - 5| < 0.1$ requires

$$|2x - 6| = 2|x - 3| < 0.1 \quad \text{or} \quad |x - 3| < 0.05.$$

Thus, x must be within 0.05 of 3 to guarantee that $2x - 1$ differs from 5 by less than 0.1.

42. For $x \neq 0$, to guarantee that $|3^x - 1| < 0.1$ requires

$$-0.1 < 3^x - 1 < 0.1 \quad \text{or} \quad 0.9 < 3^x < 1.1.$$

This last compound inequality yields

$$-0.0959 \approx \frac{\ln 0.9}{\ln 3} < x < \frac{\ln 1.1}{\ln 3} \approx 0.0868.$$

Thus, x must be within approximately 0.087 of 0 to guarantee that 3^x differs from 1 by less than 0.1.

43. Suppose $\lim_{x \rightarrow c} f(x) = L$ where $L < 0$, and let $\epsilon = \frac{|L|}{2} > 0$. Then there is a $\delta > 0$ such that, whenever $0 < |x - c| < \delta$,

$$|f(x) - L| < \frac{|L|}{2} \quad \text{or} \quad -\frac{|L|}{2} < f(x) - L < \frac{|L|}{2}.$$

Because $L < 0$, $|L| = -L$, so the last compound inequality becomes

$$\frac{L}{2} < f(x) - L < -\frac{L}{2} \quad \text{or} \quad \frac{3L}{2} < f(x) < \frac{L}{2} < 0.$$

Thus, everywhere in the open interval $|x - c| < \delta$, except possibly at c , $f(x) < 0$.

44. Given any $\epsilon > 0$, we can choose $N = -\frac{1}{\epsilon}$. Whenever $x < N$, then

$$\left| \frac{1}{x} - 0 \right| = -\frac{1}{x} < -\frac{1}{N} = \epsilon.$$

Therefore, $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

45. Given any $\epsilon > 0$, we can choose $M = \frac{1}{\epsilon^2}$. Whenever $x > M$, then

$$\left| -\frac{1}{\sqrt{x}} - 0 \right| = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{M}} = \epsilon.$$

Therefore, $\lim_{x \rightarrow \infty} \left(-\frac{1}{\sqrt{x}} \right) = 0$.

46. To make

$$\left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < 0.1$$

requires

$$x^2 > 10 \quad \text{or} \quad |x| > \sqrt{10}.$$

As this limit is for x approaching $-\infty$, take $N = -\sqrt{10}$.

47. First, consider $L > 0$. Take $\epsilon = L/2$, let $\delta > 0$, and choose $x = -\delta/2$. Then $0 < |x - 0| = |x| < \delta$ but

$$\left| \frac{1}{x} - L \right| = \left| -\frac{2}{\delta} - L \right| = \left| \frac{2}{\delta} + L \right| > L > \frac{L}{2} = \epsilon,$$

where the inequality $L > L/2$ follows because $L > 0$. Therefore, $\lim_{x \rightarrow 0} \frac{1}{x}$ cannot equal L . Next, consider $L < 0$. Take $\epsilon = -L/2$, let $\delta > 0$, and choose $x = \delta/2$. Then $0 < |x - 0| = |x| < \delta$ but

$$\left| \frac{1}{x} - L \right| = \left| \frac{2}{\delta} + (-L) \right| > -L > -\frac{L}{2} = \epsilon,$$

where the inequality $-L > -L/2$ follows because $-L > 0$. Therefore, $\lim_{x \rightarrow 0} \frac{1}{x}$ cannot equal L . Finally, consider $L = 0$. Take $\epsilon = 1$, let $\delta > 0$, and choose $|x| < \min\{1, \delta\}$. Then $0 < |x - 0| = |x| < \delta$ but

$$\left| \frac{1}{x} - L \right| = \left| \frac{1}{x} \right| > 1 = \epsilon.$$

Therefore, $\lim_{x \rightarrow 0} \frac{1}{x}$ cannot equal 0. In summary, there is no number L such that $\lim_{x \rightarrow 0} \frac{1}{x} = L$.

48. The strict inequality on the left ($0 < |x - c|$) is to remove the function value at $x = c$ from consideration; the strict inequality on the right ($|x - c| < \delta$) is to create an **open** interval containing $x = c$.
49. A limit is supposed to capture the behavior of the value of a function as the value of the independent variable approaches a target value. We do not want the limit to depend on the value of the function at that location or even to depend on whether the function is defined at that location. Including the phrase *except possibly at c* imposes these restrictions.
50. In the ϵ - δ definition of a limit, ϵ measures the “closeness” of the function value $f(x)$ to the value of the limit L , while δ measures the “closeness” of the value of the independent variable x to the target value c . For example, notice the placement of ϵ and δ in the proof that $\lim_{x \rightarrow 3} (2x - 1) = 5$:

Given any $\epsilon > 0$, take $\delta = \epsilon/2$. Then, whenever $0 < |x - 3| < \delta$,

$$|(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim_{x \rightarrow 3} (2x - 1) = 5$.

51. First consider $\lim_{x \rightarrow 0} f(x)$. Any open interval containing 0 will contain both rational numbers and irrational numbers. As the rational numbers approach 0, the function value x^2 will approach 0; as the irrational numbers approach 0, the function value will also approach 0. This suggests that $\lim_{x \rightarrow 0} f(x) = 0$.

This argument can be made precise using $\delta = \sqrt{\epsilon}$ within the ϵ - δ definition.

Next, consider $\lim_{x \rightarrow 1} f(x)$. Any open interval containing 1 will contain both rational numbers and irrational numbers. As the rational numbers approach 1, the function value x^2 will approach 1; as the irrational numbers approach 1, however, the function value will approach 0. This suggests that $\lim_{x \rightarrow 1} f(x)$ does not exist. This argument can be made precise as follows: Suppose the limit does exist and is equal to L . Take $\epsilon = 1/4$. There would then be a δ such that whenever $0 < |x - 1| < \delta$, the function value would be within $1/4$ of L . Considering the rational and irrational values of x separately would require that L be simultaneously within $1/4$ of 1 and 0, which is impossible.

52. Any open interval containing 0 will contain both rational numbers and irrational numbers. As the rational numbers approach 0, the function value x^2 will approach 0; as the irrational numbers approach 0, the function value $\tan x$ will also approach 0. This suggests that $\lim_{x \rightarrow 0} f(x) = 0$.

Challenge Problems

53. Note that

$$|f(x) - L| = |4x^3 + 3x^2 - 24x + 22 - 5| = |4x^3 + 3x^2 - 24x + 17| = |4x^2 + 7x - 17| \cdot |x - 1|.$$

The factor $|x - 1|$ will be smaller than δ , but what about the factor $|4x^2 + 7x - 17|$? If, in addition to any other restrictions placed on δ , we require $\delta \leq 1$, then $|x - 1| < \delta$ guarantees that $|x - 1| < 1$, or $0 < x < 2$. Thus, $|x| < 2$ so that $|4x^2 + 7x - 17| \leq 4|x|^2 + 7|x| + 17 < 47$ and

$$|f(x) - L| = |4x^2 + 7x - 17| \cdot |x - 1| < 47|x - 1| < 47\delta.$$

To make this less than ϵ , we can choose $\delta \leq \epsilon/47$. Combining all of this information, the ϵ - δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min \left\{ 1, \frac{\epsilon}{47} \right\}$. Whenever $0 < |x - 1| < \delta$, it follows that $|4x^2 + 7x - 17| < 47$, and then

$$\begin{aligned} |f(x) - L| &= |4x^3 + 3x^2 - 24x + 22 - 5| = |4x^3 + 3x^2 - 24x + 17| = |4x^2 + 7x - 17| \cdot |x - 1| \\ &< 47|x - 1| < 47\delta \leq 47 \cdot \frac{\epsilon}{47} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} (4x^3 + 3x^2 - 24x + 22) = 5$.

54. Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$. Given any $\epsilon > 0$, there then exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1,$$

and a $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_2.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Whenever $0 < |x - c| < \delta$, then

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$.

55. First note that

$$\sqrt{5 + 4x^2} > \sqrt{4x^2} \quad \text{so that} \quad \frac{1}{\sqrt{5 + 4x^2}} < \frac{1}{\sqrt{4x^2}}$$

for all x . For $x > 2$, $2 - x < 0$ so that upon multiplying the latter inequality above by the negative expression $2 - x$,

$$\frac{2 - x}{\sqrt{5 + 4x^2}} > \frac{2 - x}{\sqrt{4x^2}} = \frac{2 - x}{2x} = \frac{1}{x} - \frac{1}{2} > -\frac{1}{2}.$$

Thus, to be within $\epsilon = 0.01$ of $L = -1/2$, we need

$$\frac{2 - x}{\sqrt{5 + 4x^2}} < -0.49 \quad \text{or} \quad \frac{x - 2}{\sqrt{5 + 4x^2}} > 0.49.$$

Squaring both sides of this last inequality and gathering terms yields

$$0.0396x^2 - 4x + 2.7995 > 0.$$

The roots of the quadratic function are

$$x = \frac{4 \pm \sqrt{16 - 4(0.0396)(2.7995)}}{2(0.0396)} \approx 100.3, 0.7.$$

so

$$\frac{2 - x}{\sqrt{5 + 4x^2}} < -0.49$$

for approximately $x < 0.7$ and $x > 100.3$. In the ϵ - δ definition of a limit at infinity, we can therefore take $\boxed{M = 101}$.

56. To prove that the linear function $f(x) = ax + b$ is continuous everywhere, it must be shown that

$\lim_{x \rightarrow c} f(x) = ac + b$ for any real number c . Now, given any $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{1 + |a|}$. The

absolute value of a appears in the formula for δ to ensure that δ will be positive. Moreover, 1 has been added to the denominator to allow for the possibility that a could be zero. Whenever $0 < |x - c| < \delta$, then

$$|(ax + b) - (ac + b)| = |a| |x - c| < |a| \cdot \delta = \frac{|a|}{1 + |a|} \epsilon < \epsilon.$$

Therefore, $\lim_{x \rightarrow c} f(x) = ac + b$, and the linear function $f(x) = ax + b$ is continuous everywhere.

57. Start by proving continuity at $x = 0$. Because 0 is a rational number, $f(0) = 0$. Now, given any $\epsilon > 0$, choose $\delta = \epsilon$. Whenever $0 < |x - c| < \delta$, then

$$|f(x) - 0| = |f(x)| = |x| < \delta = \epsilon$$

for every rational number x and

$$|f(x) - 0| = |f(x)| = 0 < \delta = \epsilon$$

for every irrational number x . Therefore, $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ and f is continuous at $x = 0$.

Now, let c be any non-zero real number. We will proceed with a proof by contradiction. Toward this end, suppose that f is continuous at $x = c$. Then the limit as x approaches c must exist and be equal to $f(c)$. Based on the definition of f , $\lim_{x \rightarrow c} f(x)$ must therefore be equal to either 0 (if c is a rational number) or c (if c is an irrational number). We will now show that neither of these are the limit. Take $\epsilon = |c|/2$, and choose $\delta > 0$. For any irrational number x such that $0 < |x - c| < \delta$,

$$|f(x) - 0| = |f(x)| = |c| > \frac{|c|}{2} = \epsilon,$$

so $\lim_{x \rightarrow c} f(x) \neq 0$. Moreover, for every rational number x such that $0 < |x - c| < \delta$,

$$|f(x) - c| = |c| > \frac{|c|}{2} = \epsilon,$$

so $\lim_{x \rightarrow c} f(x) \neq c$. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist, and f is not continuous at $x = c$.

58. Let $f(x) = x^3$. Then

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + cx + c^2)| = |x - c| \cdot |x^2 + cx + c^2|.$$

With both x and c in the open interval $(0, 2)$, $|x| < 2$ and $|c| < 2$. Therefore

$$|x^2 + cx + c^2| \leq |x|^2 + |c| |x| + |c|^2 < 2^2 + 2^2 + 2^2 = 12,$$

and

$$|f(x) - f(c)| = |x - c| \cdot |x^2 + cx + c^2| < 12|x - c|.$$

Thus, $\boxed{K = 12}$ is a Lipschitz constant for $f(x) = x^3$ on $(0, 2)$.

Chapter 1 Review Exercises

1. The values in the table below, which have been rounded for display purposes, suggest that the value of $f(x) = \frac{1 - \cos x}{1 + \cos x}$ can be made “as close as we please” to 0 by choosing x “sufficiently close” to 0. It therefore appears that

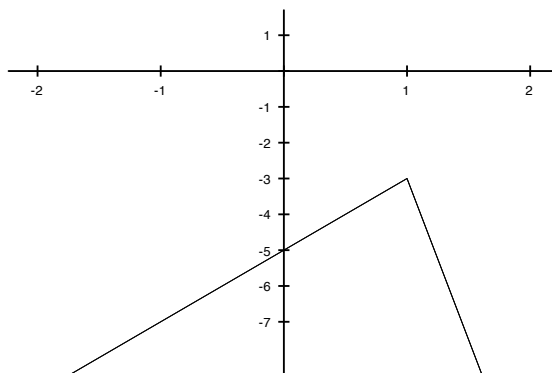
$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + \cos x} = \boxed{0}.$$

x	-0.1	-0.01	-0.001	$\rightarrow 0 \leftarrow$	0.001	0.01	0.1
$f(x) = \frac{1 - \cos x}{1 + \cos x}$	0.002504	0.000025	0.00000025	$f(x)$ approaches 0	0.00000025	0.000025	0.002504

2. The figure below displays a graph of f . Using the graph,

$$\lim_{x \rightarrow 1^-} f(x) = -3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = -3.$$

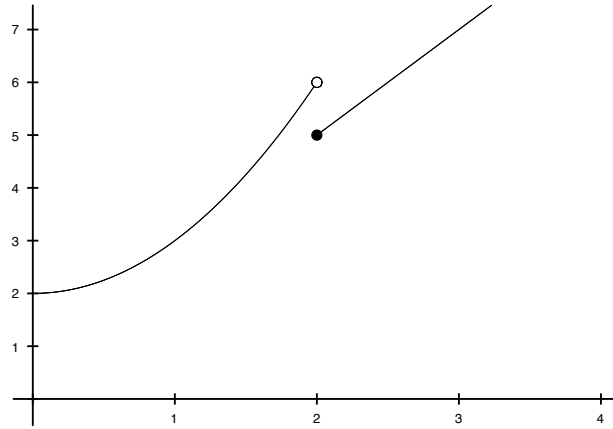
Because the two one-sided limits as x approaches 1 are equal, $\lim_{x \rightarrow 1} f(x)$ exists. Moreover, because the two one-sided limits are equal to -3 , $\boxed{\lim_{x \rightarrow 1} f(x) = -3}$.



3. The figure below displays a graph of f . Using the graph,

$$\lim_{x \rightarrow 2^-} f(x) = 6 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = 5.$$

Because the two one-sided limits as x approaches 2 are not equal, $\boxed{\lim_{x \rightarrow 2} f(x) \text{ does not exist}}$.



4. Let $f(x) = x^2 - 3$.

(a) The slope of the secant line joining $(1, -2)$ and $(2, 1)$ is

$$\frac{1 - (-2)}{2 - 1} = \frac{3}{1} = \boxed{3}.$$

(b) The slope of the tangent line to the graph of f at $(1, -2)$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 3 - (-2)}{h} = \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 - 3 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h + h^2}{h} = \lim_{h \rightarrow 0} (2 + h) = \boxed{2}. \end{aligned}$$

5. Let $f(x) = \frac{3}{x}$. Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)} = \lim_{h \rightarrow 0} \frac{3x - 3(x+h)}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{-3h}{hx(x+h)} \\ &= -3 \lim_{h \rightarrow 0} \frac{1}{x(x+h)} = -3 \cdot \frac{1}{x^2} = \boxed{-\frac{3}{x^2}}. \end{aligned}$$

6. Let $f(x) = 3x^2 + 2x$. Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 + 2(x+h) - (3x^2 + 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 3x^2 - 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 2h}{h} = \lim_{h \rightarrow 0} (6x + 3h + 2) = \boxed{6x + 2}. \end{aligned}$$

7. Because $1 + \sin x \leq f(x) \leq |x| + 1$ for all x in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ containing 0 and

$$\lim_{x \rightarrow 0} (1 + \sin x) = 1 + \sin 0 = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (|x| + 1) = |0| + 1 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} f(x) = \boxed{1}.$$

$$8. \lim_{x \rightarrow 2} \left(2x - \frac{1}{x} \right) = 2(2) - \frac{1}{2} = \boxed{\frac{7}{2}}.$$

$$9. \lim_{x \rightarrow \pi} (x \cos x) = \pi \cos \pi = \boxed{-\pi}.$$

$$10. \lim_{x \rightarrow -1} (x^3 + 3x^2 - x - 1) = (-1)^3 + 3(-1)^2 - (-1) - 1 = \boxed{2}.$$

$$11. \lim_{x \rightarrow 0} \sqrt[3]{x(x+2)^3} = \sqrt[3]{0(2)^3} = \boxed{0}.$$

$$12. \lim_{x \rightarrow 0} [(2x+3)(x^5+5x)] = (0+3)(0^5+0) = \boxed{0}.$$

$$13. \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x \rightarrow 3} (x^2+3x+9) = 3^2 + 3(3) + 9 = \boxed{27}.$$

$$14. \lim_{x \rightarrow 3} \left(\frac{x^2}{x-3} - \frac{3x}{x-3} \right) = \lim_{x \rightarrow 3} \frac{x^2 - 3x}{x-3} = \lim_{x \rightarrow 3} \frac{x(x-3)}{x-3} = \lim_{x \rightarrow 3} x = \boxed{3}.$$

$$15. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 2+2 = \boxed{4}.$$

$$16. \lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x^2 + 4x + 3} = \lim_{x \rightarrow -1} \frac{(x+1)(x+2)}{(x+1)(x+3)} = \lim_{x \rightarrow -1} \frac{x+2}{x+3} = \frac{-1+2}{-1+3} = \boxed{\frac{1}{2}}.$$

$$17. \lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 6x}{x^2 + x - 2} = \lim_{x \rightarrow -2} \frac{x(x+2)(x+3)}{(x+2)(x-1)} = \lim_{x \rightarrow -2} \frac{x(x+3)}{x-1} = \frac{-2(-2+3)}{-2-1} = \boxed{\frac{2}{3}}.$$

$$18. \lim_{x \rightarrow 1} \left(x^2 - 3x + \frac{1}{x} \right)^{15} = \left(1^2 - 3(1) + \frac{1}{1} \right)^{15} = (-1)^{15} = \boxed{-1}.$$

19.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{3 - \sqrt{x^2 + 5}}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{3 - \sqrt{x^2 + 5}}{x^2 - 4} \cdot \frac{3 + \sqrt{x^2 + 5}}{3 + \sqrt{x^2 + 5}} = \lim_{x \rightarrow 2} \frac{9 - (x^2 + 5)}{(x^2 - 4)(3 + \sqrt{x^2 + 5})} \\ &= \lim_{x \rightarrow 2} \frac{4 - x^2}{(x^2 - 4)(3 + \sqrt{x^2 + 5})} = -\lim_{x \rightarrow 2} \frac{1}{3 + \sqrt{x^2 + 5}} = -\frac{1}{3 + \sqrt{2^2 + 5}} = \boxed{-\frac{1}{6}}. \end{aligned}$$

20. Note that

$$\frac{1}{(2+x)^2} - \frac{1}{4} = \frac{4 - (2+x)^2}{4(2+x)^2} = \frac{4 - 4 - 4x - x^2}{4(2+x)^2} = -\frac{x(x+4)}{4(2+x)^2}.$$

Therefore,

$$\lim_{x \rightarrow 0} \left\{ \frac{1}{x} \left[\frac{1}{(2+x)^2} - \frac{1}{4} \right] \right\} = -\lim_{x \rightarrow 0} \left[\frac{1}{x} \cdot \frac{x(x+4)}{4(2+x)^2} \right] = -\lim_{x \rightarrow 0} \frac{x+4}{4(2+x)^2} = -\frac{0+4}{4(2+0)^2} = \boxed{-\frac{1}{4}}.$$

21.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(x+3)^2 - 9}{x} &= \lim_{x \rightarrow 0} \frac{x^2 + 6x + 9 - 9}{x} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + 6x}{x} = \lim_{x \rightarrow 0} \frac{x(x+6)}{x} = \lim_{x \rightarrow 0} (x+6) = 0+6 = \boxed{6}. \end{aligned}$$

$$22. \lim_{x \rightarrow 1} [(x^3 - 3x^2 + 3x - 1)(x+1)^2] = (1^3 - 3(1)^2 + 3(1) - 1)(1+1)^2 = 0(4) = \boxed{0}.$$

$$23. \lim_{x \rightarrow -2^+} \frac{x^2 + 5x + 6}{x + 2} = \lim_{x \rightarrow -2^+} \frac{(x+2)(x+3)}{x+2} = \lim_{x \rightarrow -2^+} (x+3) = -2+3 = \boxed{1}.$$

24. As x approaches 5 from the right, $x - 5 > 0$, so that

$$\lim_{x \rightarrow 5^+} \frac{|x - 5|}{x - 5} = \lim_{x \rightarrow 5^+} \frac{x - 5}{x - 5} = \lim_{x \rightarrow 5^+} 1 = \boxed{1}.$$

25. As x approaches 1 from the left, $x - 1 < 0$, so that

$$\lim_{x \rightarrow 1^-} \frac{|x - 1|}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x - 1)}{x - 1} = \lim_{x \rightarrow 1^-} -1 = \boxed{-1}.$$

26. As x approaches $3/2$ from the right, $2x$ approaches 3 from the right, so that

$$\lim_{x \rightarrow 3/2^+} \lfloor 2x \rfloor = \boxed{3}.$$

$$27. \lim_{x \rightarrow 4^-} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4^-} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \rightarrow 4^-} (x + 4) = 4 + 4 = \boxed{8}.$$

28. As x approaches 1 from the right, $x - 1 > 0$, so that

$$\lim_{x \rightarrow 1^+} \sqrt{x - 1} = \sqrt{1 - 1} = \boxed{0}.$$

29. Because

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 3) = 2(2) + 3 = \boxed{7}$$

and

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (9 - x) = 9 - 2 = \boxed{7}$$

are equal, $\lim_{x \rightarrow 2} f(x)$ exists. Moreover, because both one-sided limits are equal to 7, $\boxed{\lim_{x \rightarrow 2} f(x) = 7}$.

30. Because

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (3x + 1) = 3(3) + 1 = \boxed{10}$$

and

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (4x - 2) = 4(3) - 2 = \boxed{10}$$

are equal, $\lim_{x \rightarrow 3} f(x)$ exists. Moreover, because both one-sided limits are equal to 10, $\boxed{\lim_{x \rightarrow 3} f(x) = 10}$.

31. The function f is defined at $c = 1$ with $f(1) = 5$. Because

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (5x - 2) = 5(1) - 2 = 3$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x + 1) = 2(1) + 1 = 3$$

are equal, $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 3. However $f(1) = 5 \neq 3 = \lim_{x \rightarrow 1} f(x)$, so

$\boxed{f \text{ is not continuous at } c = 1}.$

32. The function f is defined at $c = -1$ with $f(-1) = 2$. Because

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^2 = (-1)^2 = 1$$

and

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (-3x - 2) = -3(-1) - 2 = 1$$

are equal, $\lim_{x \rightarrow -1} f(x)$ exists and is equal to 1. However $f(-1) = 2 \neq 1 = \lim_{x \rightarrow -1} f(x)$, so

$\boxed{f \text{ is not continuous at } c = -1}.$

33. The function f is defined at $c = 0$ with $f(0) = 4$. Because

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (4 - 3x^2) = 4 - 3(0)^2 = 4$$

and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{16 - x^2} = \sqrt{16 - 0^2} = 4$$

are equal, $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 4. Finally, $f(0) = \lim_{x \rightarrow 0} f(x)$, so $\boxed{f \text{ is continuous at } c = 0}$.

34. The function f is defined at $c = 4$ with $f(4) = \sqrt{4 + 4} = 2\sqrt{2}$. Because

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} \sqrt{4 + x} = \sqrt{4 + 4} = 2\sqrt{2}$$

and

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \sqrt{\frac{x^2 - 16}{x - 4}} = \lim_{x \rightarrow 4^+} \sqrt{\frac{(x - 4)(x + 4)}{x - 4}} = \lim_{x \rightarrow 4^+} \sqrt{x + 4} = \sqrt{4 + 4} = 2\sqrt{2}$$

are equal, $\lim_{x \rightarrow 4} f(x)$ exists and is equal to $2\sqrt{2}$. Finally, $f(4) = \lim_{x \rightarrow 4} f(x)$, so $\boxed{f \text{ is continuous at } c = 4}$.

35. The function f is defined at $c = 1/2$ with $f(1/2) = \lfloor 1 \rfloor = 1$. As x approaches $1/2$ from the left, $2x$ approaches 1 from the left, so that

$$\lim_{x \rightarrow 1/2^-} f(x) = \lim_{x \rightarrow 1/2^-} \lfloor 2x \rfloor = 0.$$

On the other hand, as x approaches $1/2$ from the right, $2x$ approaches 1 from the right, so that

$$\lim_{x \rightarrow 1/2^+} f(x) = \lim_{x \rightarrow 1/2^+} \lfloor 2x \rfloor = 1.$$

Because the two one-sided limits as x approaches $1/2$ are not equal $\lim_{x \rightarrow 1/2} f(x)$ does not exist. Therefore,

$\boxed{f \text{ is not continuous at } c = 1/2}$.

36. The function f is defined at $c = 5$ with $f(5) = |5 - 5| = 0$. Because

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} |x - 5| = \lim_{x \rightarrow 5^-} [-(x - 5)] = -(5 - 5) = 0$$

and

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} |x - 5| = \lim_{x \rightarrow 5^+} (x - 5) = 5 - 5 = 0$$

are equal, $\lim_{x \rightarrow 5} f(x)$ exists and is equal to 0. Finally, $f(5) = \lim_{x \rightarrow 5} f(x)$, so $\boxed{f \text{ is continuous at } c = 5}$.

37. Let $f(x) = 2x^2 - 5x$.

(a) The average rate of change of f from 1 to x is

$$\frac{f(x) - f(1)}{x - 1} = \frac{2x^2 - 5x - (-3)}{x - 1} = \frac{2x^2 - 5x + 3}{x - 1} = \frac{(x - 1)(2x - 3)}{x - 1} = \boxed{2x - 3},$$

for $x \neq 1$.

(b) Using the result from part (a),

$$\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} (2x - 3) = 2(1) - 3 = \boxed{-1}.$$

38. The function

$$f(x) = \begin{cases} -1, & \text{if } -1 \leq x \leq 0 \\ 1, & \text{if } 0 < x \leq 1 \end{cases}$$

satisfies the indicated conditions: f is continuous on the interval $[-1, 1]$ except at 0, $f(-1) = -1 < 0$, $f(1) = 1 > 0$, and f has no zeroes. This does not contradict the Intermediate Value Theorem because the function is not continuous on the closed interval $[-1, 1]$.

39. Let
- $g(x) = x$
- and
- $h(x) = x^3 - 27$
- . The polynomial functions
- g
- and
- h
- are both continuous on the set of all real numbers. Because
- f
- is the quotient of
- g
- and
- h
- ,
- f
- is continuous on the set of all real numbers except those for which
- $h(x) = 0$
- . The only real solution to the equation

$$h(x) = x^3 - 27 = (x - 3)(x^2 + 3x + 9) = 0$$

is $x = 3$, so f is continuous on the set $\{x|x \neq 3\}$.

40. Let
- $g(x) = x^2 - 3$
- and
- $h(x) = x^2 + 5x + 6$
- . The polynomial functions
- g
- and
- h
- are both continuous on the set of all real numbers. Because
- f
- is the quotient of
- g
- and
- h
- ,
- f
- is continuous on the set of all real numbers except those for which
- $h(x) = 0$
- . The solutions to the equation
- $h(x) = x^2 + 5x + 6 = (x + 3)(x + 2) = 0$
- are
- $x = -3$
- and
- $x = -2$
- , so
- f is continuous on the set $\{x|x \neq -3, x \neq -2\}$
- .

41. Let
- $g(x) = 2x + 1$
- and
- $h(x) = x^3 + 4x^2 + 4x$
- . The polynomial functions
- g
- and
- h
- are both continuous on the set of all real numbers. Because
- f
- is the quotient of
- g
- and
- h
- ,
- f
- is continuous on the set of all real numbers except those for which
- $h(x) = 0$
- . The solutions to the equation
- $h(x) = x^3 + 4x^2 + 4x = x(x + 2)^2 = 0$
- are
- $x = -2$
- and
- $x = 0$
- , so
- f is continuous on the set $\{x|x \neq -2, x \neq 0\}$
- .

42. Let
- $g(x) = \sqrt{x}$
- and
- $h(x) = x - 1$
- . The function
- g
- is continuous on the set
- $\{x|x \geq 0\}$
- , and the polynomial function
- h
- is the continuous on the set of all real numbers. Moreover, the solution of the inequality
- $h(x) \geq 0$
- is the set
- $\{x|x \geq 1\}$
- . As
- f
- is the composition
- $g(h(x))$
- , the function
- f
- is continuous at
- c
- provided
- h
- is continuous at
- c
- and
- g
- is continuous at
- $h(c)$
- ; thus,
- f is continuous on the set $\{x|x \geq 1\}$
- .

43. Let
- $g(x) = 2^x$
- and
- $h(x) = -x$
- . The exponential function
- g
- and the polynomial function
- h
- are both continuous on the set of all real numbers. As
- f
- is the composition
- $g(h(x))$
- , the function
- f
- is continuous at
- c
- provided
- h
- is continuous at
- c
- and
- g
- is continuous at
- $h(c)$
- ;
- f is continuous on the set of all real numbers
- .

44. Let
- $f(x) = 2x^3 + 3x^2 - 23x - 42$
- . This polynomial function is continuous on the set of all real numbers, so it is continuous on the closed interval
- $[3, 4]$
- . Because

$$f(3) = 2(3)^3 + 3(3)^2 - 23(3) - 42 = -30 < 0$$

and

$$f(4) = 2(4)^3 + 3(4)^2 - 23(4) - 42 = 42 > 0,$$

the Intermediate Value Theorem guarantees there is a number c in the interval $(3, 4)$ such that $f(c) = 0$. Thus, the equation $2x^3 + 3x^2 - 23x - 42 = 0$ does have a solution in the interval $(3, 4)$.

45. The polynomial function
- $f(x) = 8x^4 - 2x^2 + 5x - 1$
- is continuous for all real numbers, so it is continuous on the closed interval
- $[0, 1]$
- . Because
- $f(0) = -1 < 0$
- and
- $f(1) = 8 - 2 + 5 - 1 = 10 > 0$
- , the Intermediate Value Theorem guarantees that
- f
- must have a zero on the interval
- $(0, 1)$
- . To approximate this zero, subdivide the interval
- $[0, 1]$
- into 10 subintervals, each of length 0.1, and evaluate
- f
- at each endpoint. The results are shown in the first two columns of the table below. Because
- $f(0.2) = -0.0672 < 0$
- and
- $f(0.3) = 0.3848 > 0$
- , the Intermediate Value Theorem guarantees the zero lies in the interval
- $(0.2, 0.3)$
- . Repeating the process by subdividing the interval
- $[0.2, 0.3]$
- into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval
- $(0.21, 0.22)$
- . Repeating the subdivision process once more, the results in the last two columns of the

table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.215}$, correct to three decimal places.

$[0, 1]$		$[0.2, 0.3]$		$[0.21, 0.22]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	-1.0000	0.20	-0.06720	0.210	-0.02264
0.1	-0.5192	0.21	-0.02264	0.211	-0.01819
0.2	-0.0672	0.22	0.02194	0.212	-0.01373
0.3	0.3848	0.23	0.06659	0.213	-0.00927
0.4	0.8848	0.24	0.11134	0.214	-0.00481
0.5	1.5000	0.25	0.15625	0.215	-0.00036
0.6	2.3168	0.26	0.20136	0.216	0.00410
0.7	3.4408	0.27	0.24672	0.217	0.00856
0.8	4.9968	0.28	0.29237	0.218	0.01302
0.9	7.1288	0.29	0.33838	0.219	0.01748
1.0	10.0000	0.30	0.38480	0.220	0.02194

46. The polynomial function $f(x) = 3x^3 - 10x + 9$ is continuous for all real numbers, so it is continuous on the closed interval $[-3, -2]$. Because $f(-3) = 3(-3)^3 - 10(-3) + 9 = -42 < 0$ and $f(-2) = 3(-2)^3 - 10(-2) + 9 = 5 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-3, -2)$. To approximate this zero, subdivide the interval $[-3, -2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(-2.2) = -0.944 < 0$ and $f(-2.1) = 2.217 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(-2.2, -2.1)$. Repeating the process by subdividing the interval $[-2.2, -2.1]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(-2.18, -2.17)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{-2.171}$, correct to three decimal places.

$[-3, -2]$		$[-2.2, -2.1]$		$[-2.18, -2.17]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
-3.0	-42.000	-2.20	-0.94400	-2.180	-0.28070
-2.9	-35.167	-2.19	-0.61038	-2.179	-0.24794
-2.8	-28.856	-2.18	-0.28070	-2.178	-0.21523
-2.7	-23.049	-2.17	0.04506	-2.177	-0.18256
-2.6	-17.728	-2.16	0.36691	-2.176	-0.14992
-2.5	-12.875	-2.15	0.68488	-2.175	-0.11733
-2.4	-8.472	-2.14	0.99897	-2.174	-0.08477
-2.3	-4.501	-2.13	1.30921	-2.173	-0.05226
-2.2	-0.944	-2.12	1.61562	-2.172	-0.01978
-2.1	2.217	-2.11	1.91821	-2.171	0.01266
-2.0	5.000	-2.10	2.21700	-2.170	0.04506

47. For $x > 0$, $|x| = x$, so

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x}(1-x) = \lim_{x \rightarrow 0^+} \frac{x}{x}(1-x) = \lim_{x \rightarrow 0^+} (1-x) = 1-0 = \boxed{1}.$$

For $x < 0$, $|x| = -x$, so

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x}(1-x) = \lim_{x \rightarrow 0^-} \frac{-x}{x}(1-x) = \lim_{x \rightarrow 0^-} (x-1) = 0-1 = \boxed{-1}.$$

Because the two one-sided limits as x approaches 0 are not equal, $\lim_{x \rightarrow 0} \frac{|x|}{x}(1-x)$ does not exist.

$$48. \lim_{x \rightarrow 2} \left(\frac{x^2}{x-2} - \frac{2x}{x-2} \right) = \lim_{x \rightarrow 2} \frac{x^2 - 2x}{x-2} = \lim_{x \rightarrow 2} \frac{x(x-2)}{x-2} = \lim_{x \rightarrow 2} x = \boxed{2}.$$

Individually, the functions

$$\frac{x^2}{x-2} \quad \text{and} \quad \frac{2x}{x-2}$$

become unbounded as x approaches 2, so that neither of the individual limits exists. Because the individual limits do not exist, the Limit of a Sum property does not apply, and

$$\lim_{x \rightarrow 2} \left(\frac{x^2}{x-2} - \frac{2x}{x-2} \right) \quad \text{is not given by} \quad \lim_{x \rightarrow 2} \frac{x^2}{x-2} - \lim_{x \rightarrow 2} \frac{2x}{x-2}.$$

49. Let $f(x) = \sqrt{x}$. Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}. \end{aligned}$$

50. To make

$$|(2x+1) - 7| = |2x-6| = 2|x-3| < 0.01,$$

requires that $|x-3| < 0.005$. Thus, the largest δ that “works” for $\epsilon = 0.01$ is $\boxed{\delta = 0.005}$.

$$51. \lim_{x \rightarrow 0} \cos(\tan x) = \cos(\tan 0) = \cos 0 = \boxed{1}.$$

$$52. \lim_{x \rightarrow 0} \frac{\sin \frac{x}{4}}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{4} \sin \frac{x}{4}}{\frac{x}{4}} = \frac{1}{4} \lim_{x \rightarrow 0} \frac{\sin \frac{x}{4}}{\frac{x}{4}} = \frac{1}{4} \cdot 1 = \boxed{\frac{1}{4}}.$$

53.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan(3x)}{\tan(4x)} &= \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)} \cdot \frac{\cos(4x)}{\cos(3x)} = \lim_{x \rightarrow 0} \frac{\frac{3 \sin(3x)}{3x}}{\frac{4 \sin(4x)}{4x}} \cdot \lim_{x \rightarrow 0} \frac{\cos(4x)}{\cos(3x)} \\ &= \frac{3 \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x}}{4 \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x}} \cdot \frac{\cos 0}{\cos 0} = \frac{3}{4} \cdot \frac{1}{1} \cdot \frac{1}{1} = \boxed{\frac{3}{4}}. \end{aligned}$$

$$54. \lim_{x \rightarrow 0} \frac{\cos \frac{x}{3} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{3} (\cos \frac{x}{3} - 1)}{\frac{x}{3}} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\cos \frac{x}{3} - 1}{\frac{x}{3}} = \frac{1}{3} \cdot 0 = \boxed{0}.$$

$$55. \lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \right)^{10} = \left(\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \right)^{10} = 0^{10} = \boxed{0}.$$

56.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{4x} - 1}{e^x - 1} &= \lim_{x \rightarrow 0} \frac{(e^{2x} - 1)(e^{2x} + 1)}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(e^x - 1)(e^x + 1)(e^{2x} + 1)}{e^x - 1} \\ &= \lim_{x \rightarrow 0} [(e^x + 1)(e^{2x} + 1)] = (e^0 + 1)(e^0 + 1) = \boxed{4}. \end{aligned}$$

57. As x approaches $\pi/2$ from the right, $\sin x$ approaches 1 and $\cos x$ approaches 0 from the left. Thus, $\tan x = \frac{\sin x}{\cos x}$ becomes unbounded in the negative direction, so that

$$\lim_{x \rightarrow \pi/2^+} \tan x = \boxed{-\infty}.$$

58. As x approaches -3 , $2 + x$ approaches -1 and $(x + 3)^2$ approaches 0 from the right. Thus, $\frac{2 + x}{(x + 3)^2}$ becomes unbounded in the negative direction, so that

$$\lim_{x \rightarrow -3} \frac{2 + x}{(x + 3)^2} = \boxed{-\infty}.$$

$$59. \lim_{x \rightarrow \infty} \frac{3x^3 - 2x + 1}{x^3 - 8} = \lim_{x \rightarrow \infty} \frac{\frac{3x^3 - 2x + 1}{x^3}}{\frac{x^3 - 8}{x^3}} = \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{1 - \frac{8}{x^3}} = \frac{3 - 0 + 0}{1 - 0} = \boxed{3}.$$

$$60. \lim_{x \rightarrow \infty} \frac{3x^4 + x}{2x^2} = \lim_{x \rightarrow \infty} \left(\frac{3}{2}x^2 + \frac{1}{2x} \right) = \boxed{\infty}.$$

61. The domain of the rational function $f(x) = \frac{4x - 2}{x + 3}$ is the set $\{x | x \neq -3\}$. The one-sided limits as x approaches -3 are

$$\lim_{x \rightarrow -3^-} \frac{4x - 2}{x + 3} = \infty \quad \text{and} \quad \lim_{x \rightarrow -3^+} \frac{4x - 2}{x + 3} = -\infty,$$

so $\boxed{x = -3 \text{ is a vertical asymptote of the graph of } f}$. Moreover, because

$$\lim_{x \rightarrow -\infty} \frac{4x - 2}{x + 3} = \lim_{x \rightarrow -\infty} \frac{\frac{4x - 2}{x}}{\frac{x + 3}{x}} = \lim_{x \rightarrow -\infty} \frac{4 - \frac{2}{x}}{1 + \frac{3}{x}} = \frac{4 - 0}{1 + 0} = 4$$

and

$$\lim_{x \rightarrow \infty} \frac{4x - 2}{x + 3} = \lim_{x \rightarrow \infty} \frac{\frac{4x - 2}{x}}{\frac{x + 3}{x}} = \lim_{x \rightarrow \infty} \frac{4 - \frac{2}{x}}{1 + \frac{3}{x}} = \frac{4 - 0}{1 + 0} = 4,$$

$\boxed{y = 4 \text{ is a horizontal asymptote of the graph of } f}$.

62. The domain of the rational function $f(x) = \frac{2x}{x^2 - 4}$ is the set $\{x | x \neq \pm 2\}$. The one-sided limits as x approaches -2 are

$$\lim_{x \rightarrow -2^-} \frac{2x}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{2x}{x^2 - 4} = \infty,$$

so $\boxed{x = -2 \text{ is a vertical asymptote of the graph of } f}$. The one-sided limits as x approaches 2 are

$$\lim_{x \rightarrow 2^-} \frac{2x}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{2x}{x^2 - 4} = \infty,$$

so $\boxed{x = 2 \text{ is also a vertical asymptote of the graph of } f}$. Moreover, because

$$\lim_{x \rightarrow -\infty} \frac{2x}{x^2 - 4} = \lim_{x \rightarrow -\infty} \frac{\frac{2x}{x^2}}{\frac{x^2 - 4}{x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{2}{x}}{1 - \frac{4}{x^2}} = \frac{0}{1 - 0} = 0$$

and

$$\lim_{x \rightarrow \infty} \frac{2x}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2}}{\frac{x^2 - 4}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{1 - \frac{4}{x^2}} = \frac{0}{1 - 0} = 0,$$

$y = 0$ is a horizontal asymptote of the graph of f .

63. The function f is defined at 0 with $f(0) = 1/2$. Because

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\tan x}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{2} \cdot 1 \cdot \frac{1}{1} = \frac{1}{2}, \end{aligned}$$

it follows that $\lim_{x \rightarrow 0} f(x)$ exists and $f(0) = \lim_{x \rightarrow 0} f(x)$. Therefore, f is continuous at 0.

64. The function f is defined at 0 with $f(0) = 1$. Because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{3 \sin(3x)}{3x} = 3 \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3,$$

it follows that $\lim_{x \rightarrow 0} f(x)$ exists but $f(0) \neq \lim_{x \rightarrow 0} f(x)$. Therefore, f is not continuous at 0.

65. Note that

$$\cos\left(\pi x + \frac{\pi}{2}\right) = \cos(\pi x) \cos \frac{\pi}{2} - \sin(\pi x) \sin \frac{\pi}{2} = \cos(\pi x) \cdot 0 - \sin(\pi x) \cdot 1 = -\sin(\pi x).$$

Thus,

$$\lim_{x \rightarrow 0} f(x) = -\lim_{x \rightarrow 0} \frac{\sin(\pi x)}{x} = -\lim_{x \rightarrow 0} \frac{\pi \sin(\pi x)}{\pi x} = -\pi \lim_{x \rightarrow 0} \frac{\sin(\pi x)}{\pi x} = -\pi \cdot 1 = -\pi.$$

To make f continuous at 0, define $f(0) = -\pi$.

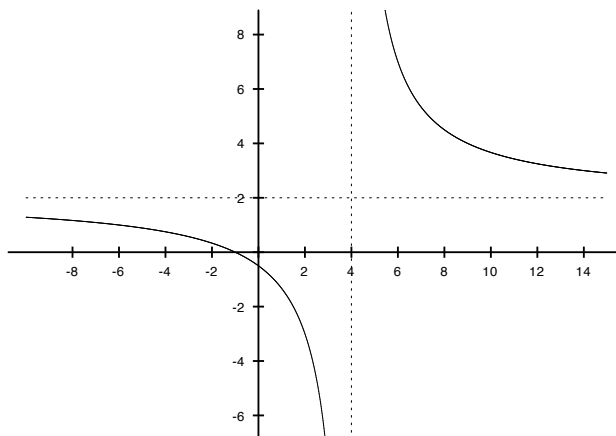
66. Take $\epsilon = 1$, let $\delta > 0$, and choose any x satisfying $0 < |x + 3| < \delta$. Then

$$|(x^2 - 9) - (-18)| = |x^2 + 9| = x^2 + 9 \geq 9 > 1 = \epsilon,$$

so that $\lim_{x \rightarrow -3} (x^2 - 9) \neq -18$.

67. (a) Answers will vary. The figure below displays the graph of a function f with the properties

$$f(-1) = 0, \quad \lim_{x \rightarrow \infty} f(x) = 2, \quad \lim_{x \rightarrow -\infty} f(x) = 2, \quad \lim_{x \rightarrow 4^-} f(x) = -\infty, \quad \text{and} \quad \lim_{x \rightarrow 4^+} f(x) = \infty.$$



(b) Answers will vary. The function shown above is $f(x) = \frac{2x+2}{x-4}$.

68. (a) Let $R(x) = \frac{2x^2 - 5x + 2}{5x^2 - x - 2}$. Factoring the numerator yields

$$2x^2 - 5x + 2 = (2x - 1)(x - 2).$$

Applying the quadratic formula to the equation $5x^2 - x - 2 = 0$ yields

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(5)(-2)}}{2(5)} = \frac{1 \pm \sqrt{41}}{10}.$$

Therefore, the domain of R is the set $\left\{x \mid x \neq \frac{1 \pm \sqrt{41}}{10}\right\}$, the x -intercepts are $\frac{1}{2}$ and 2 , and the y -intercept is $R(0) = -1$.

(b) The one-sided limits as x approaches $\frac{1 - \sqrt{41}}{10} \approx -0.54$ are

$$\lim_{x \rightarrow \frac{1 - \sqrt{41}}{10}^-} R(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \frac{1 - \sqrt{41}}{10}^+} R(x) = -\infty.$$

Thus, the graph of R becomes unbounded in the positive direction as x approaches $\frac{1 - \sqrt{41}}{10}$ from the left and becomes unbounded in the negative direction as x approaches $\frac{1 - \sqrt{41}}{10}$ from the right. The one-sided limits as x approaches $\frac{1 + \sqrt{41}}{10} \approx 0.74$ are

$$\lim_{x \rightarrow \frac{1 + \sqrt{41}}{10}^-} R(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \frac{1 + \sqrt{41}}{10}^+} R(x) = -\infty.$$

Thus, the graph of R becomes unbounded in the positive direction as x approaches $\frac{1 + \sqrt{41}}{10}$ from the left and becomes unbounded in the negative direction as x approaches $\frac{1 + \sqrt{41}}{10}$ from the right.

(c) Based on the limits from part (b), $x = \frac{1 - \sqrt{41}}{10}$ and $x = \frac{1 + \sqrt{41}}{10}$ are both vertical asymptotes of the graph of R . Because

$$\lim_{x \rightarrow -\infty} \frac{2x^2 - 5x + 2}{5x^2 - x - 2} = \lim_{x \rightarrow -\infty} \frac{\frac{2x^2 - 5x + 2}{5x^2}}{\frac{5x^2 - x - 2}{5x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{2}{5} - \frac{1}{x} + \frac{2}{2x^2}}{1 - \frac{1}{5x} - \frac{2}{5x^2}} = \frac{\frac{2}{5} - 0 + 0}{1 - 0 - 0} = \frac{2}{5}$$

and

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 5x + 2}{5x^2 - x - 2} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2 - 5x + 2}{5x^2}}{\frac{5x^2 - x - 2}{5x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{5} - \frac{1}{x} + \frac{2}{2x^2}}{1 - \frac{1}{5x} - \frac{2}{5x^2}} = \frac{\frac{2}{5} - 0 + 0}{1 - 0 - 0} = \frac{2}{5},$$

$y = \frac{2}{5}$ is a horizontal asymptote of the graph of R .

69. Because $1 - x^2 \leq f(x) \leq \cos x$ for all x in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ containing 0 and

$$\lim_{x \rightarrow 0} (1 - x^2) = 1 - 0 = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \cos x = \cos 0 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x \rightarrow 0} f(x) = 1.$$

Chapter 3: More About Derivatives

3.1: The Chain Rule

Concepts and Vocabulary

1. The derivative of a composite function $(f \circ g)(x)$ can be found using the **Chain** Rule.
2. **True**. If $y = f(u)$ and $u = g(x)$ are differentiable functions, then $y = f(g(x))$ is differentiable.
3. **False**. If $y = f(g(x))$ is a differentiable function, then $y' = f'(g(x))g'(x)$, by the Chain Rule.
4. To find the derivative of $y = \tan(1 + \cos x)$, using the Chain Rule, begin with $y = \tan u$ and $u = 1 + \cos x$.
5. If $y = (x^3 + 4x + 1)^{100}$, then

$$y' = 100(x^3 + 4x + 1)^{100-1} \cdot \frac{d}{dx}(x^3 + 4x + 1) = 100(x^3 + 4x + 1)^{99}(3x^2 + 4).$$

6. If $f(x) = e^{3x^2+5}$, then $f'(x) = e^{3x^2+5} \cdot \frac{d}{dx}(3x^2 + 5) = 6xe^{3x^2+5}$.
7. **True**. The Chain Rule can be applied to multiple composite functions.
8. $\frac{d}{dx} \sin x^2 = \cos x^2 \cdot \frac{d}{dx} x^2 = 2x \cos x^2$.

Skill Building

9. Let $y = u^5$ and $u = x^3 + 1$. Then $y = (x^3 + 1)^5$. Moreover,

$$\frac{dy}{du} = \frac{d}{du} u^5 = 5u^4 = 5(x^3 + 1)^4 \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}(x^3 + 1) = 3x^2,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5(x^3 + 1)^4 \cdot 3x^2 = 15x^2(x^3 + 1)^4.$$

10. Let $y = u^3$ and $u = 2x + 5$. Then $y = (2x + 5)^3$. Moreover,

$$\frac{dy}{du} = \frac{d}{du} u^3 = 3u^2 = 3(2x + 5)^2 \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}(2x + 5) = 2,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3(2x + 5)^2 \cdot 2 = 6(2x + 5)^2.$$

11. Let $y = \frac{u}{u+1}$ and $u = x^2 + 1$. Then $y = \frac{x^2 + 1}{(x^2 + 1) + 1} = \frac{x^2 + 1}{x^2 + 2}$. Moreover,

$$\frac{dy}{du} = \frac{d}{du} \left(\frac{u}{u+1} \right) = \frac{(u+1)(1) - u(1)}{(u+1)^2} = \frac{1}{(u+1)^2} = \frac{1}{(x^2 + 2)^2}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 1) = 2x,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{(x^2 + 2)^2} \cdot 2x = \boxed{\frac{2x}{(x^2 + 2)^2}}.$$

12. Let $y = \frac{u-1}{u}$ and $u = x^2 - 1$. Then $y = \frac{(x^2 - 1) - 1}{x^2 - 1} = \frac{x^2 - 2}{x^2 - 1}$. Moreover,

$$\frac{dy}{du} = \frac{d}{du} \left(\frac{u-1}{u} \right) = \frac{u(1) - (u-1)(1)}{u^2} = \frac{1}{u^2} = \frac{1}{(x^2 - 1)^2}$$

and

$$\frac{du}{dx} = \frac{d}{dx}(x^2 - 1) = 2x,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{(x^2 - 1)^2} \cdot 2x = \boxed{\frac{2x}{(x^2 - 1)^2}}.$$

13. Let $y = (u+1)^2$ and $u = \frac{1}{x}$. Then $y = \left(\frac{1}{x} + 1\right)^2$. Moreover,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du}(u+1)^2 = 2(u+1)^{2-1} \cdot \frac{d}{du}(u+1) \\ &= 2(u+1) \cdot 1 = 2(u+1) = 2\left(\frac{1}{x} + 1\right) \end{aligned}$$

and

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2},$$

so, by the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[2\left(\frac{1}{x} + 1\right) \cdot \left(-\frac{1}{x^2}\right) \right].$$

14. Let $y = (u^2 - 1)^3$ and $u = \frac{1}{x+2}$. Then

$$y = \left[\left(\frac{1}{x+2} \right)^2 - 1 \right]^3 = \left(\frac{1}{(x+2)^2} - 1 \right)^3.$$

Moreover,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du}(u^2 - 1)^3 = 3(u^2 - 1)^{3-1} \cdot \frac{d}{du}(u^2 - 1) \\ &= 3(u^2 - 1)^2 \cdot 2u = 6u(u^2 - 1)^2 = \frac{6}{x+2} \left(\frac{1}{(x+2)^2} - 1 \right)^2 \end{aligned}$$

and

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{1}{x+2} \right) = \frac{(x+2)(0) - 1(1)}{(x+2)^2} = -\frac{1}{(x+2)^2},$$

so, by the Chain Rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{6}{x+2} \left(\frac{1}{(x+2)^2} - 1 \right)^2 \cdot \left(-\frac{1}{(x+2)^2} \right) \\ &= \boxed{-\frac{6}{(x+2)^3} \left(\frac{1}{(x+2)^2} - 1 \right)^2}. \end{aligned}$$

15. Let $f(x) = (3x + 5)^2$. Then

$$\frac{df}{dx} = \frac{d}{dx}(3x + 5)^2 = 2(3x + 5)^{2-1} \cdot \frac{d}{dx}(3x + 5) = 2(3x + 5) \cdot 3 = \boxed{6(3x + 5)}.$$

16. Let $f(x) = (2x - 5)^3$. Then

$$\frac{df}{dx} = \frac{d}{dx}(2x - 5)^3 = 3(2x - 5)^{3-1} \cdot \frac{d}{dx}(2x - 5) = 3(2x - 5)^2 \cdot 2 = \boxed{6(2x - 5)^2}.$$

17. Let $f(x) = (6x - 5)^{-3}$. Then

$$\frac{df}{dx} = \frac{d}{dx}(6x - 5)^{-3} = -3(6x - 5)^{-3-1} \cdot \frac{d}{dx}(6x - 5) = -3(6x - 5)^{-4} \cdot 6 = \boxed{-\frac{18}{(6x - 5)^4}}.$$

18. Let $f(t) = (4t + 1)^{-2}$. Then

$$\frac{df}{dt} = \frac{d}{dt}(4t + 1)^{-2} = -2(4t + 1)^{-2-1} \cdot \frac{d}{dt}(4t + 1) = -2(4t + 1)^{-3} \cdot 4 = \boxed{-\frac{8}{(4t + 1)^3}}.$$

19. Let $g(x) = (x^2 + 5)^4$. Then

$$\frac{dg}{dx} = \frac{d}{dx}(x^2 + 5)^4 = 4(x^2 + 5)^{4-1} \cdot \frac{d}{dx}(x^2 + 5) = 4(x^2 + 5)^3 \cdot 2x = \boxed{8x(x^2 + 5)^3}.$$

20. Let $F(x) = (x^3 - 2)^5$. Then

$$\frac{dF}{dx} = \frac{d}{dx}(x^3 - 2)^5 = 5(x^3 - 2)^{5-1} \cdot \frac{d}{dx}(x^3 - 2) = 5(x^3 - 2)^4 \cdot 3x^2 = \boxed{15x^2(x^3 - 2)^4}.$$

21. Let $f(u) = \left(u - \frac{1}{u}\right)^3$. Then

$$\frac{df}{du} = \frac{d}{du} \left(u - \frac{1}{u}\right)^3 = 3 \left(u - \frac{1}{u}\right)^{3-1} \cdot \frac{d}{du} \left(u - \frac{1}{u}\right) = \boxed{3 \left(u - \frac{1}{u}\right)^2 \left(1 + \frac{1}{u^2}\right)}.$$

22. Let $f(x) = \left(x + \frac{1}{x}\right)^3$. Then

$$\frac{df}{dx} = \frac{d}{dx} \left(x + \frac{1}{x}\right)^3 = 3 \left(x + \frac{1}{x}\right)^{3-1} \cdot \frac{d}{dx} \left(x + \frac{1}{x}\right) = \boxed{3 \left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right)}.$$

23. Let $g(x) = (4x + e^x)^3$. Then

$$\frac{dg}{dx} = \frac{d}{dx}(4x + e^x)^3 = 3(4x + e^x)^{3-1} \cdot \frac{d}{dx}(4x + e^x) = \boxed{3(4x + e^x)^2(4 + e^x)}.$$

24. Let $F(x) = (e^x - x^2)^2$. Then

$$\frac{dF}{dx} = \frac{d}{dx}(e^x - x^2)^2 = 2(e^x - x^2)^{2-1} \cdot \frac{d}{dx}(e^x - x^2) = \boxed{2(e^x - x^2)(e^x - 2x)}.$$

25. Let $f(x) = \tan^2 x = (\tan x)^2$. Then

$$\frac{df}{dx} = \frac{d}{dx} \tan^2 x = 2 \tan^{2-1} x \cdot \frac{d}{dx} \tan x = \boxed{2 \tan x \sec^2 x}.$$

26. Let $f(x) = \sec^3 x = (\sec x)^3$. Then

$$\frac{df}{dx} = \frac{d}{dx} \sec^3 x = 3 \sec^{3-1} x \cdot \frac{d}{dx} \sec x = 3 \sec^2 x \cdot \sec x \tan x = \boxed{3 \sec^3 x \tan x}.$$

27. Let $f(z) = (\tan z + \cos z)^2$. Then

$$\begin{aligned} \frac{df}{dz} &= \frac{d}{dz} (\tan z + \cos z)^2 = 2(\tan z + \cos z)^{2-1} \cdot \frac{d}{dz} (\tan z + \cos z) \\ &= \boxed{2(\tan z + \cos z)(\sec^2 z - \sin z)}. \end{aligned}$$

28. Let $f(z) = (e^z + 2 \sin z)^3$. Then

$$\begin{aligned} \frac{df}{dz} &= \frac{d}{dz} (e^z + 2 \sin z)^3 = 3(e^z + 2 \sin z)^{3-1} \cdot \frac{d}{dz} (e^z + 2 \sin z) \\ &= 3(e^z + 2 \sin z)^2 \cdot (e^z + 2 \cos z) = \boxed{3(e^z + 2 \cos z)(e^z + 2 \sin z)^2}. \end{aligned}$$

29. Let $y = (x^2 + 4)^2(2x^3 - 1)^3$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [(x^2 + 4)^2(2x^3 - 1)^3] = (x^2 + 4)^2 \frac{d}{dx} (2x^3 - 1)^3 + (2x^3 - 1)^3 \frac{d}{dx} (x^2 + 4)^2 \\ &= (x^2 + 4)^2 \cdot 3(2x^3 - 1)^{3-1} \cdot \frac{d}{dx} (2x^3 - 1) + (2x^3 - 1)^3 \cdot 2(x^2 + 4)^{2-1} \cdot \frac{d}{dx} (x^2 + 4) \\ &= 3(x^2 + 4)^2 (2x^3 - 1)^2 \cdot 6x^2 + 2(x^2 + 4)(2x^3 - 1)^3 \cdot 2x \\ &= 18x^2(x^2 + 4)^2(2x^3 - 1)^2 + 4x(x^2 + 4)(2x^3 - 1)^3 \\ &= 2x(x^2 + 4)(2x^3 - 1)^2[9x(x^2 + 4) + 2(2x^3 - 1)] \\ &= \boxed{2x(x^2 + 4)(2x^3 - 1)(13x^3 + 36x - 2)}. \end{aligned}$$

30. Let $y = (x^2 - 2)^3(3x^4 + 1)^2$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [(x^2 - 2)^3(3x^4 + 1)^2] = (x^2 - 2)^3 \frac{d}{dx} (3x^4 + 1)^2 + (3x^4 + 1)^2 \frac{d}{dx} (x^2 - 2)^3 \\ &= (x^2 - 2)^3 \cdot 2(3x^4 + 1)^{2-1} \cdot \frac{d}{dx} (3x^4 + 1) + (3x^4 + 1)^2 \cdot 3(x^2 - 2)^{3-1} \cdot \frac{d}{dx} (x^2 - 2) \\ &= 2(x^2 - 2)^3(3x^4 + 1) \cdot 12x^3 + 3(x^2 - 2)^2(3x^4 + 1)^2 \cdot 2x \\ &= 24x^3(x^2 - 2)^3(3x^4 + 1) + 6x(x^2 - 2)^2(3x^4 + 1)^2 \\ &= 2x(x^2 - 2)^2(3x^4 + 1)[12x^2(x^2 - 2) + 3(3x^4 + 1)] \\ &= \boxed{2x(x^2 - 2)^2(3x^4 + 1)(21x^4 - 24x^2 + 3)}. \end{aligned}$$

31. Let $y = \left(\frac{\sin x}{x}\right)^2$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\sin x}{x}\right)^2 = 2 \left(\frac{\sin x}{x}\right)^{2-1} \cdot \frac{d}{dx} \left(\frac{\sin x}{x}\right) \\ &= 2 \frac{\sin x}{x} \cdot \frac{x \cos x - \sin x}{x^2} = \boxed{\frac{2 \sin x (x \cos x - \sin x)}{x^3}}. \end{aligned}$$

32. Let $y = \left(\frac{x + \cos x}{x}\right)^5$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x + \cos x}{x}\right)^5 = 5 \left(\frac{x + \cos x}{x}\right)^{5-1} \cdot \frac{d}{dx} \left(\frac{x + \cos x}{x}\right) \\ &= 5 \left(\frac{x + \cos x}{x}\right)^4 \cdot \frac{x(1 - \sin x) - (x + \cos x)}{x^2} = \boxed{-5 \left(\frac{x + \cos x}{x}\right)^4 \frac{x \sin x + \cos x}{x^2}}. \end{aligned}$$

33. Let $y = \sin(4x)$. Then $y' = \cos(4x) \cdot \frac{d}{dx}(4x) = \cos(4x) \cdot 4 = \boxed{4 \cos(4x)}$.

34. Let $y = \cos(5x)$. Then $y' = -\sin(5x) \cdot \frac{d}{dx}(5x) = -\sin(5x) \cdot 5 = \boxed{-5 \sin(5x)}$.

35. Let $y = 2 \sin(x^2 + 2x - 1)$. Then

$$y' = 2 \cos(x^2 + 2x - 1) \cdot \frac{d}{dx}(x^2 + 2x - 1) = \boxed{2 \cos(x^2 + 2x - 1) \cdot (2x + 2)}.$$

36. Let $y = \frac{1}{2} \cos(x^3 - 2x + 5)$. Then

$$\begin{aligned} y' &= -\frac{1}{2} \sin(x^3 - 2x + 5) \cdot \frac{d}{dx}(x^3 - 2x + 5) \\ &= -\frac{1}{2} \sin(x^3 - 2x + 5) \cdot (3x^2 - 2) = \boxed{-\frac{3x^2 - 2}{2} \sin(x^3 - 2x + 5)}. \end{aligned}$$

37. Let $y = \sin \frac{1}{x}$. Then

$$y' = \cos \frac{1}{x} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) = \boxed{\cos \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right)}.$$

38. Let $y = \sin \frac{3}{x}$. Then

$$y' = \cos \frac{3}{x} \cdot \frac{d}{dx} \left(\frac{3}{x} \right) = \boxed{\cos \frac{3}{x} \cdot \left(-\frac{3}{x^2} \right)}.$$

39. Let $y = \sec(4x)$. Then

$$y' = \sec(4x) \tan(4x) \cdot \frac{d}{dx}(4x) = \sec(4x) \tan(4x) \cdot 4 = \boxed{4 \sec(4x) \tan(4x)}.$$

40. Let $y = \cot(5x)$. Then

$$y' = -\csc^2(5x) \cdot \frac{d}{dx}(5x) = -\csc^2(5x) \cdot 5 = \boxed{-5 \csc^2(5x)}.$$

41. Let $y = e^{1/x}$. Then $y' = e^{1/x} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) = \boxed{e^{1/x} \cdot \left(-\frac{1}{x^2} \right)}$.

42. Let $y = e^{1/x^2}$. Then $y' = e^{1/x^2} \cdot \frac{d}{dx} \left(\frac{1}{x^2} \right) = \boxed{e^{1/x^2} \cdot \left(-\frac{2}{x^3} \right)}$.

43. Let $y = \frac{1}{x^4 - 2x + 1} = (x^4 - 2x + 1)^{-1}$. Then

$$y' = -(x^4 - 2x + 1)^{-1-1} \cdot \frac{d}{dx}(x^4 - 2x + 1) = -(x^4 - 2x + 1)^{-2} \cdot (4x^3 - 2) = \boxed{-\frac{4x^3 - 2}{(x^4 - 2x + 1)^2}}.$$

44. Let $y = \frac{3}{x^5 + 2x^2 - 3} = 3(x^5 + 2x^2 - 3)^{-1}$. Then

$$y' = -3(x^5 + 2x^2 - 3)^{-1-1} \cdot \frac{d}{dx}(x^5 + 2x^2 - 3) = -3(x^5 + 2x^2 - 3)^{-2} \cdot (5x^4 + 4x) = \boxed{-\frac{15x^4 + 12x}{(x^5 + 2x^2 - 3)^2}}.$$

45. Let $y = \frac{100}{1 + 99e^{-x}} = 100(1 + 99e^{-x})^{-1}$. Then

$$\begin{aligned} y' &= -100(1 + 99e^{-x})^{-1-1} \cdot \frac{d}{dx}(1 + 99e^{-x}) = -100(1 + 99e^{-x})^{-2} \cdot 99e^{-x} \frac{d}{dx}(-x) \\ &= -9900e^{-x}(1 + 99e^{-x})^{-2} \cdot (-1) = \boxed{\frac{9900e^{-x}}{(1 + 99e^{-x})^2}}. \end{aligned}$$

46. Let $y = \frac{1}{1 + 2e^{-x}} = (1 + 2e^{-x})^{-1}$. Then

$$\begin{aligned} y' &= -(1 + 2e^{-x})^{-1-1} \cdot \frac{d}{dx}(1 + 2e^{-x}) = -(1 + 2e^{-x})^{-2} \cdot 2e^{-x} \cdot \frac{d}{dx}(-x) \\ &= -2e^{-x}(1 + 2e^{-x})^{-2} \cdot (-1) = \boxed{\frac{2e^{-x}}{(1 + 2e^{-x})^2}}. \end{aligned}$$

47. Let $y = 2^{\sin x}$. Then $y' = 2^{\sin x} \ln 2 \cdot \frac{d}{dx}(\sin x) = \boxed{2^{\sin x} \ln 2 \cdot \cos x}$.

48. Let $y = (\sqrt{3})^{\cos x}$. Then

$$y' = (\sqrt{3})^{\cos x} \ln \sqrt{3} \cdot \frac{d}{dx}(\cos x) = \boxed{-\frac{1}{2}(\sqrt{3})^{\cos x} \ln 3 \cdot \sin x}.$$

49. Let $y = 6^{\sec x}$. Then $y' = 6^{\sec x} \ln 6 \cdot \frac{d}{dx}(\sec x) = \boxed{6^{\sec x} \ln 6 \cdot \sec x \tan x}$.

50. Let $y = 3^{\tan x}$. Then $y' = 3^{\tan x} \ln 3 \cdot \frac{d}{dx}(\tan x) = \boxed{3^{\tan x} \ln 3 \cdot \sec^2 x}$.

51. Let $y = 5xe^{3x}$. Then

$$y' = 5x \frac{d}{dx}e^{3x} + e^{3x} \frac{d}{dx}(5x) = 5xe^{3x} \frac{d}{dx}(3x) + e^{3x}(5) = 5xe^{3x}(3) + 5e^{3x} = \boxed{15xe^{3x} + 5e^{3x}}.$$

52. Let $y = x^3e^{2x}$. Then

$$y' = x^3 \frac{d}{dx}e^{2x} + e^{2x} \frac{d}{dx}x^3 = x^3e^{2x} \frac{d}{dx}(2x) + e^{2x}(3x^2) = x^3e^{2x}(2) + 3x^2e^{2x} = \boxed{(2x^3 + 3x^2)e^{2x}}.$$

53. Let $y = x^2 \sin(4x)$. Then

$$\begin{aligned} y' &= x^2 \frac{d}{dx} \sin(4x) + \sin(4x) \frac{d}{dx}x^2 = x^2 \cos(4x) \frac{d}{dx}(4x) + \sin(4x) \cdot 2x \\ &= x^2 \cos(4x) \cdot 4 + 2x \sin(4x) = \boxed{4x^2 \cos(4x) + 2x \sin(4x)}. \end{aligned}$$

54. Let $y = x^2 \cos(4x)$. Then

$$\begin{aligned} y' &= x^2 \frac{d}{dx} \cos(4x) + \cos(4x) \frac{d}{dx}x^2 = -x^2 \sin(4x) \frac{d}{dx}(4x) + \cos(4x) \cdot 2x \\ &= -x^2 \sin(4x) \cdot 4 + 2x \cos(4x) = \boxed{-4x^2 \sin(4x) + 2x \cos(4x)}. \end{aligned}$$

55. Let $y = e^{-ax} \sin(bx)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{-ax} \sin(bx)] = e^{-ax} \frac{d}{dx} \sin(bx) + \sin(bx) \frac{d}{dx} e^{-ax} \\ &= e^{-ax} \cos(bx) \frac{d}{dx}(bx) + \sin(bx) e^{-ax} \frac{d}{dx}(-ax) = e^{-ax} \cos(bx) \cdot b + \sin(bx) e^{-ax} \cdot (-a) \\ &= \boxed{be^{-ax} \cos(bx) - ae^{-ax} \sin(bx)}.\end{aligned}$$

56. Let $y = e^{ax} \cos(-bx) = e^{ax} \cos(bx)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[e^{ax} \cos(bx)] = e^{ax} \frac{d}{dx} \cos(bx) + \cos(bx) \frac{d}{dx} e^{ax} \\ &= e^{ax} \cdot (-\sin(bx)) \frac{d}{dx}(bx) + \cos(bx) e^{ax} \frac{d}{dx}(ax) = -e^{ax} \sin(bx) \cdot b + \cos(bx) e^{ax} \cdot a \\ &= \boxed{ae^{ax} \cos(bx) - be^{ax} \sin(bx)}.\end{aligned}$$

57. Let $y = \frac{e^{ax} - 1}{e^{ax} + 1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{e^{ax} - 1}{e^{ax} + 1} \right) = \frac{(e^{ax} + 1) \frac{d}{dx}(e^{ax} - 1) - (e^{ax} - 1) \frac{d}{dx}(e^{ax} + 1)}{(e^{ax} + 1)^2} \\ &= \frac{(e^{ax} + 1)e^{ax} \frac{d}{dx}(ax) - (e^{ax} - 1)e^{ax} \frac{d}{dx}(ax)}{(e^{ax} + 1)^2} = \frac{(e^{ax} + 1)e^{ax} \cdot a - (e^{ax} - 1)e^{ax} \cdot a}{(e^{ax} + 1)^2} \\ &= \frac{ae^{2ax} + ae^{ax} - ae^{2ax} + ae^{ax}}{(e^{ax} + 1)^2} = \boxed{\frac{2ae^{ax}}{(e^{ax} + 1)^2}}.\end{aligned}$$

58. Let $y = \frac{e^{-ax} + 1}{e^{bx} - 1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{e^{-ax} + 1}{e^{bx} - 1} \right) = \frac{(e^{bx} - 1) \frac{d}{dx}(e^{-ax} + 1) - (e^{-ax} + 1) \frac{d}{dx}(e^{bx} - 1)}{(e^{bx} - 1)^2} \\ &= \frac{(e^{bx} - 1)e^{-ax} \frac{d}{dx}(-ax) - (e^{-ax} + 1)e^{bx} \frac{d}{dx}(bx)}{(e^{bx} - 1)^2} = \frac{(e^{bx} - 1)e^{-ax} \cdot (-a) - (e^{-ax} + 1)e^{bx} \cdot b}{(e^{bx} - 1)^2} \\ &= \frac{-ae^{(b-a)x} + ae^{-ax} - be^{(b-a)x} - be^{bx}}{(e^{bx} - 1)^2} = \boxed{\frac{ae^{-ax} - (a+b)e^{(b-a)x} - be^{bx}}{(e^{bx} - 1)^2}}.\end{aligned}$$

59. Let $y = u^3$, $u = 3v^2 + 1$, and $v = \frac{4}{x^2}$. Then

$$y = u^3 = (3v^2 + 1)^3 = \left[3 \left(\frac{4}{x^2} \right)^2 + 1 \right]^3 = \boxed{\left(\frac{48}{x^4} + 1 \right)^3}.$$

Moreover,

$$\begin{aligned}\frac{dy}{du} &= \frac{d}{du} u^3 = 3u^2 = 3(3v^2 + 1)^2 = 3 \left[3 \left(\frac{4}{x^2} \right)^2 + 1 \right]^2 = 3 \left(\frac{48}{x^4} + 1 \right)^2 \\ \frac{du}{dv} &= \frac{d}{dv} (3v^2 + 1) = 6v = \frac{24}{x^2} \\ \frac{dv}{dx} &= \frac{d}{dx} \left(\frac{4}{x^2} \right) = -\frac{8}{x^3},\end{aligned}$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = 3 \left(\frac{48}{x^4} + 1 \right)^2 \cdot \frac{24}{x^2} \cdot \left(-\frac{8}{x^3} \right) = \boxed{-\frac{576}{x^5} \left(\frac{48}{x^4} + 1 \right)^2}.$$

60. Let $y = 3u$, $u = 3v^2 - 4$, and $v = \frac{1}{x}$. Then

$$y = 3u = 3(3v^2 - 4) = 3 \left[3 \left(\frac{1}{x} \right)^2 - 4 \right] = \boxed{\frac{9}{x^2} - 12}.$$

Moreover,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du} 3u = 3 \\ \frac{du}{dv} &= \frac{d}{dv} (3v^2 - 4) = 6v = \frac{6}{x} \\ \frac{dv}{dx} &= \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}, \end{aligned}$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = 3 \cdot \frac{6}{x} \cdot \left(-\frac{1}{x^2} \right) = \boxed{-\frac{18}{x^3}}.$$

61. Let $y = u^2 + 1$, $u = \frac{4}{v}$, and $v = x^2$. Then

$$y = u^2 + 1 = \frac{16}{v^2} + 1 = \frac{16}{(x^2)^2} = \boxed{\frac{16}{x^4} + 1}.$$

Moreover,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du} (u^2 + 1) = 2u = \frac{8}{v} = \frac{8}{x^2} \\ \frac{du}{dv} &= \frac{d}{dv} \left(\frac{4}{v} \right) = -\frac{4}{v^2} = -\frac{4}{x^4} \\ \frac{dv}{dx} &= \frac{d}{dx} x^2 = 2x, \end{aligned}$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{8}{x^2} \cdot \left(-\frac{4}{x^4} \right) \cdot 2x = \boxed{-\frac{64}{x^5}}.$$

62. Let $y = u^3 - 1$, $u = -\frac{2}{v}$, and $v = x^3$. Then

$$y = u^3 - 1 = -\frac{8}{v^3} - 1 = -\frac{8}{(x^3)^3} - 1 = \boxed{-\frac{8}{x^9} - 1}.$$

Moreover,

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du} (u^3 - 1) = 3u^2 = \frac{12}{v^2} = \frac{12}{x^6} \\ \frac{du}{dv} &= \frac{d}{dv} \left(-\frac{2}{v} \right) = \frac{2}{v^2} = \frac{2}{x^6} \\ \frac{dv}{dx} &= \frac{d}{dx} x^3 = 3x^2, \end{aligned}$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{12}{x^6} \cdot \frac{2}{x^6} \cdot 3x^2 = \boxed{\frac{72}{x^{10}}}.$$

63. Let $y = e^{-2x} \cos(3x)$. Then

$$\begin{aligned} y' &= \frac{d}{dx}[e^{-2x} \cos(3x)] = e^{-2x} \frac{d}{dx} \cos(3x) + \cos(3x) \frac{d}{dx} e^{-2x} \\ &= e^{-2x} (-\sin(3x)) \frac{d}{dx} (3x) + \cos(3x) e^{-2x} \frac{d}{dx} (-2x) = -e^{-2x} \sin(3x) \cdot 3 + e^{-2x} \cos(3x) \cdot (-2) \\ &= \boxed{-3e^{-2x} \sin(3x) - 2e^{-2x} \cos(3x)}. \end{aligned}$$

64. Let $y = e^{\pi x} \tan(\pi x)$. Then

$$\begin{aligned} y' &= \frac{d}{dx}[e^{\pi x} \tan(\pi x)] = e^{\pi x} \frac{d}{dx} \tan(\pi x) + \tan(\pi x) \frac{d}{dx} e^{\pi x} \\ &= e^{\pi x} \sec^2(\pi x) \frac{d}{dx} (\pi x) + \tan(\pi x) \cdot e^{\pi x} \frac{d}{dx} (\pi x) = e^{\pi x} \sec^2(\pi x) \cdot \pi + e^{\pi x} \tan(\pi x) \cdot \pi \\ &= \boxed{\pi e^{\pi x} \sec^2(\pi x) + \pi e^{\pi x} \tan(\pi x)}. \end{aligned}$$

65. Let $y = \cos(e^{x^2})$. Then

$$y' = \frac{d}{dx} \cos(e^{x^2}) = -\sin(e^{x^2}) \frac{d}{dx} e^{x^2} = -\sin(e^{x^2}) e^{x^2} \frac{d}{dx} x^2 = -e^{x^2} \sin(e^{x^2}) \cdot 2x = \boxed{-2xe^{x^2} \sin(e^{x^2})}.$$

66. Let $y = \tan(e^{x^2})$. Then

$$y' = \frac{d}{dx} \tan(e^{x^2}) = \sec^2(e^{x^2}) \frac{d}{dx} e^{x^2} = \sec^2(e^{x^2}) e^{x^2} \frac{d}{dx} x^2 = e^{x^2} \sec^2(e^{x^2}) \cdot 2x = \boxed{2xe^{x^2} \sec^2(e^{x^2})}.$$

67. Let $y = e^{\cos(4x)}$. Then

$$\begin{aligned} y' &= \frac{d}{dx} e^{\cos(4x)} = e^{\cos(4x)} \frac{d}{dx} \cos(4x) = e^{\cos(4x)} (-\sin(4x)) \frac{d}{dx} (4x) \\ &= \boxed{e^{\cos(4x)} (-4 \sin(4x))}. \end{aligned}$$

68. Let $y = e^{\csc^2 x}$. Then

$$\begin{aligned} y' &= \frac{d}{dx} e^{\csc^2 x} = e^{\csc^2 x} \frac{d}{dx} \csc^2 x = e^{\csc^2 x} \cdot 2 \csc x \cdot \frac{d}{dx} \csc x \\ &= 2e^{\csc^2 x} \csc x \cdot (-\csc x \cot x) = \boxed{-2e^{\csc^2 x} \csc^2 x \cot x}. \end{aligned}$$

69. Let $y = 4 \sin^2(3x)$. Then

$$\begin{aligned} y' &= \frac{d}{dx} [4 \sin^2(3x)] = 8 \sin(3x) \frac{d}{dx} \sin(3x) = 8 \sin(3x) \cos(3x) \frac{d}{dx} (3x) = 8 \sin(3x) \cos(3x) \cdot 3 \\ &= \boxed{24 \sin(3x) \cos(3x)}. \end{aligned}$$

70. Let $y = 2 \cos^2(x^2)$. Then

$$\begin{aligned} y' &= \frac{d}{dx} 2 \cos^2(x^2) = 4 \cos(x^2) \frac{d}{dx} \cos(x^2) = 4 \cos(x^2) (-\sin(x^2)) \frac{d}{dx} x^2 = -4 \sin(x^2) \cos(x^2) \cdot 2x \\ &= \boxed{-8x \sin(x^2) \cos(x^2)}. \end{aligned}$$

71. Let $y = (x^3 + 1)^2$.

(a) Let $y = u^2$ and $u = x^3 + 1$. Then

$$\frac{dy}{du} = \frac{d}{du}u^2 = 2u = 2(x^3 + 1) \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}(x^3 + 1) = 3x^2,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2(x^3 + 1) \cdot 3x^2 = \boxed{6x^2(x^3 + 1)}.$$

(b) By the Power Rule for Functions,

$$\frac{dy}{dx} = 2(x^3 + 1)^{2-1} \frac{d}{dx}(x^3 + 1) = 2(x^3 + 1)(3x^2) = \boxed{6x^2(x^3 + 1)}.$$

(c) Expanding first to obtain

$$y = (x^3 + 1)^2 = x^6 + 2x^3 + 1$$

and then differentiating yields

$$\frac{dy}{dx} = \boxed{6x^5 + 6x^2}.$$

(d) Because $6x^5 + 6x^2 = 6x^2(x^3 + 1)$, we see that the answers from parts (a) - (c) are identical.

72. Let $y = (x^2 - 2)^3$.

(a) Let $y = u^3$ and $u = x^2 - 2$. Then

$$\frac{dy}{du} = \frac{d}{du}u^3 = 3u^2 = 3(x^2 - 2)^2 \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}(x^2 - 2) = 2x,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3(x^2 - 2)^2 \cdot 2x = \boxed{6x(x^2 - 2)^2}.$$

(b) By the Power Rule for Functions,

$$\frac{dy}{dx} = 3(x^2 - 2)^{3-1} \frac{d}{dx}(x^2 - 2) = 3(x^2 - 2)^2(2x) = \boxed{6x(x^2 - 2)^2}.$$

(c) Expanding first to obtain

$$y = (x^2 - 2)^3 = x^6 - 6x^4 + 12x^2 - 8$$

and then differentiating yields

$$\frac{dy}{dx} = \boxed{6x^5 - 24x^3 + 24x}.$$

(d) Because $6x^5 - 24x^3 + 24x = 6x(x^4 - 4x^2 + 4) = 6x(x^2 - 2)^2$, we see that the answers from parts (a) - (c) are identical.

73. Let $f(x) = (x^2 - 2x + 1)^5$.

(a) The slope of the line tangent to the graph of f at the point $(1, 0)$ is given by $f'(1)$. Now,

$$f'(x) = 5(x^2 - 2x + 1)^{5-1} \frac{d}{dx}(x^2 - 2x + 1) = 5(x^2 - 2x + 1)^4(2x - 2),$$

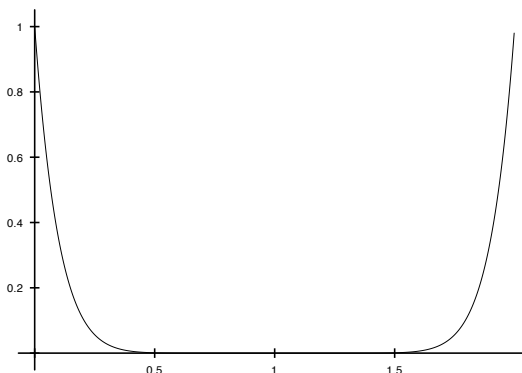
so

$$m_{\text{tan}} = f'(1) = 5(1 - 2 + 1)^4(2 - 2) = 0.$$

Therefore, the equation of the tangent line is

$$y - 0 = 0(x - 1) \quad \text{or} \quad \boxed{y = 0}.$$

- (b) The figure below displays the graph of f and its tangent line at the point $(1, 0)$. Here, the tangent line coincides with the x -axis.



74. Let $f(x) = (x^3 - x^2 + x - 1)^{10}$.

- (a) The slope of the line tangent to the graph of f at the point $(0, 1)$ is given by $f'(0)$. Now,

$$f'(x) = 10(x^3 - x^2 + x - 1)^{10-1} \frac{d}{dx}(x^3 - x^2 + x - 1) = 10(x^3 - x^2 + x - 1)^9(3x^2 - 2x + 1),$$

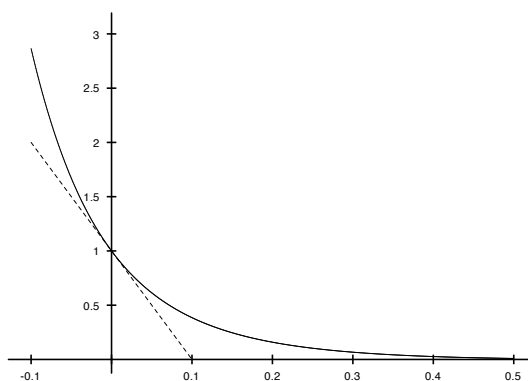
so

$$m_{\text{tan}} = f'(0) = 10(-1)^9(1) = -10.$$

Therefore, the equation of the tangent line is

$$y - 1 = -10(x - 0) \quad \text{or} \quad \boxed{y = -10x + 1}.$$

- (b) The figure below displays the graph of f (the solid curve) and its tangent line (the dashed curve) at the point $(0, 1)$.



75. Let $f(x) = \frac{x}{(x^2 - 1)^3}$.

- (a) The slope of the line tangent to the graph of f at the point $\left(2, \frac{2}{27}\right)$ is given by $f'(2)$. Now,

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)^3 \cdot 1 - x \cdot 3(x^2 - 1)^{3-1} \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^6} = \frac{(x^2 - 1)^3 - 6x^2(x^2 - 1)^2}{(x^2 - 1)^6} \\ &= \frac{(x^2 - 1)^2(x^2 - 1 - 6x^2)}{(x^2 - 1)^6} = -\frac{5x^2 + 1}{(x^2 - 1)^4}, \end{aligned}$$

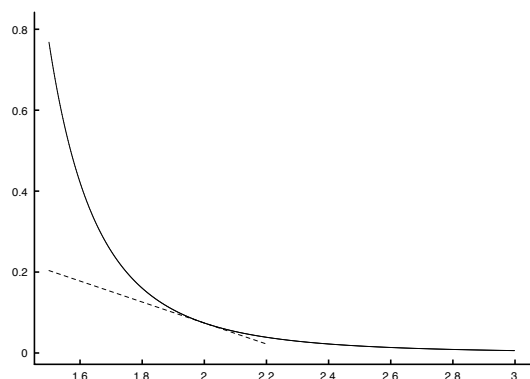
so

$$m_{\text{tan}} = f'(2) = -\frac{5(2)^2 + 1}{(2^2 - 1)^4} = -\frac{7}{27}.$$

Therefore, the equation of the tangent line is

$$y - \frac{2}{27} = -\frac{7}{27}(x - 2) \quad \text{or} \quad \boxed{y = -\frac{7}{27}x + \frac{16}{27}}.$$

- (b) The figure below displays the graph of f (the solid curve) and its tangent line (the dashed curve) at the point $\left(2, \frac{2}{27}\right)$.



76. Let $f(x) = \frac{x^2}{(x^2 - 1)^2}$.

- (a) The slope of the line tangent to the graph of f at the point $\left(2, \frac{4}{9}\right)$ is given by $f'(2)$. Now,

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)^2 \cdot 2x - x^2 \cdot 2(x^2 - 1)^{2-1} \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^4} = \frac{2x(x^2 - 1)^2 - 4x^3(x^2 - 1)}{(x^2 - 1)^4} \\ &= \frac{2x(x^2 - 1)(x^2 - 1 - 2x^2)}{(x^2 - 1)^4} = -\frac{2x(x^2 + 1)}{(x^2 - 1)^3}, \end{aligned}$$

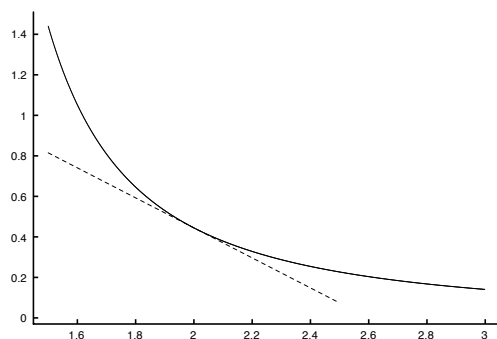
so

$$m_{\text{tan}} = f'(2) = -\frac{2(2)(2^2 + 1)}{(2^2 - 1)^3} = -\frac{20}{27}.$$

Therefore, the equation of the tangent line is

$$y - \frac{4}{9} = -\frac{20}{27}(x - 2) \quad \text{or} \quad \boxed{y = -\frac{20}{27}x + \frac{52}{27}}.$$

- (b) The figure below displays the graph of f (the solid curve) and its tangent line (the dashed curve) at the point $\left(2, \frac{4}{9}\right)$.



77. Let $f(x) = \sin(2x) + \cos \frac{x}{2}$.

- (a) The slope of the line tangent to the graph of f at the point $(0, 1)$ is given by $f'(0)$. Now,

$$f'(x) = \cos(2x) \frac{d}{dx}(2x) - \sin \frac{x}{2} \frac{d}{dx} \left(\frac{x}{2} \right) = 2 \cos(2x) - \frac{1}{2} \sin \frac{x}{2},$$

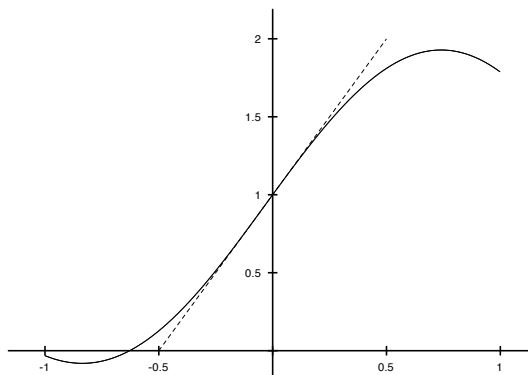
so

$$m_{\text{tan}} = f'(0) = 2 \cos 0 - \frac{1}{2} \sin 0 = 2.$$

Therefore, the equation of the tangent line is

$$y - 1 = 2(x - 0) \quad \text{or} \quad \boxed{y = 2x + 1}.$$

- (b) The figure below displays the graph of f (the solid curve) and its tangent line (the dashed curve) at the point $(0, 1)$.



78. Let $f(x) = \sin^2 x + \cos^3 x$.

- (a) The slope of the line tangent to the graph of f at the point $\left(\frac{\pi}{2}, 1\right)$ is given by $f'\left(\frac{\pi}{2}\right)$. Now,

$$f'(x) = 2 \sin x \frac{d}{dx} \sin x + 3 \cos^2 x \frac{d}{dx} \cos x = 2 \sin x \cos x - 3 \cos^2 x \sin x,$$

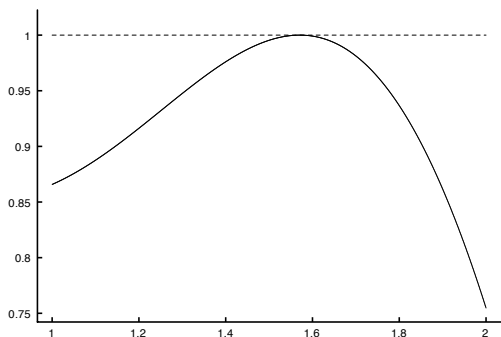
so

$$m_{\text{tan}} = f'\left(\frac{\pi}{2}\right) = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} - 3 \cos^2 \frac{\pi}{2} \sin \frac{\pi}{2} = 0.$$

Therefore, the equation of the tangent line is

$$y - 1 = 0 \left(x - \frac{\pi}{2} \right) \quad \text{or} \quad \boxed{y = 1}.$$

- (b) The figure below displays the graph of f (the solid curve) and its tangent line (the dashed curve) at the point $\left(\frac{\pi}{2}, 1\right)$.



79. $\frac{d}{dx} \cos(x^5) = -\sin(x^5) \frac{d}{dx} x^5 = -5x^4 \sin(x^5)$. Therefore,

$$\begin{aligned} \frac{d^2}{dx^2} \cos(x^5) &= \frac{d}{dx} [-5x^4 \sin(x^5)] = -5x^4 \frac{d}{dx} \sin(x^5) + \sin(x^5) \frac{d}{dx} (-5x^4) \\ &= -5x^4 \cos(x^5) \frac{d}{dx} x^5 - 20x^3 \sin(x^5) = \boxed{-25x^8 \cos(x^5) - 20x^3 \sin(x^5)}. \end{aligned}$$

80. $\frac{d}{dx} \sin^3 x = 3 \sin^2 x \frac{d}{dx} \sin x = 3 \sin^2 x \cos x$. Therefore,

$$\begin{aligned} \frac{d^2}{dx^2} \sin^3 x &= \frac{d}{dx} [3 \sin^2 x \cos x] = 3 \sin^2 x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} [3 \sin^2 x] \\ &= 3 \sin^2 x (-\sin x) + \cos x (6 \sin x) \frac{d}{dx} \sin x \\ &= -3 \sin^3 x + 6 \sin x \cos^2 x = -3 \sin^3 x + 6 \sin x (1 - \sin^2 x) = -9 \sin^3 x + 6 \sin x, \end{aligned}$$

and

$$\frac{d^3}{dx^3} \sin^3 x = \frac{d}{dx} (-9 \sin^3 x + 6 \sin x) = -27 \sin^2 x \frac{d}{dx} \sin x + 6 \cos x = \boxed{-27 \sin^2 x \cos x + 6 \cos x}.$$

81. Let $h = f \circ g$. Then

$$h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot (-2) = 6(-2) = \boxed{-12}.$$

82. Let $h = f \circ g$. Then

$$h'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot 3 = 4(3) = \boxed{12}.$$

83. Let $h = g \circ f$. Then

$$h'(0) = g'(f(0)) \cdot f'(0) = g'(3) \cdot (-1) = 0(-1) = \boxed{0}.$$

84. Let $h = g \circ f$. Then

$$h'(2) = g'(f(2)) \cdot f'(2) = g'(-3) \cdot 4 = 3(4) = \boxed{12}.$$

85. Let $y = u^5 + u$ and $u = 4x^3 + x - 4$. Then

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du} (u^5 + u) = 5u^4 + 1 = 5(4x^3 + x - 4)^4 + 1, \\ \frac{du}{dx} &= \frac{d}{dx} (4x^3 + x - 4) = 12x^2 + 1, \end{aligned}$$

and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = [5(4x^3 + x - 4)^4 + 1] (12x^2 + 1).$$

Therefore, at $x = 1$, $\frac{dy}{dx}$ is equal to

$$[5(4 + 1 - 4)^4 + 1] (12 + 1) = 6(13) = \boxed{78}.$$

86. Let $y = e^u + 3u$ and $u = \cos x$. Then

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du} (e^u + 3u) = e^u + 3 = e^{\cos x} + 3, \\ \frac{du}{dx} &= \frac{d}{dx} \cos x = -\sin x, \end{aligned}$$

and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -(e^{\cos x} + 3) \sin x.$$

Therefore, at $x = 0$, $\frac{dy}{dx}$ is equal to

$$-(e^{\cos 0} + 3) \sin 0 = -(e + 3)(0) = \boxed{0}.$$

Applications and Extensions

$$87. \frac{d}{dx} f(x^2 + 1) = f'(x^2 + 1) \cdot \frac{d}{dx} (x^2 + 1) = \boxed{f'(x^2 + 1) \cdot (2x)}.$$

$$88. \frac{d}{dx} f(1 - x^2) = f'(1 - x^2) \cdot \frac{d}{dx} (1 - x^2) = \boxed{f'(1 - x^2) \cdot (-2x)}.$$

89.

$$\begin{aligned} \frac{d}{dx} f\left(\frac{x+1}{x-1}\right) &= f'\left(\frac{x+1}{x-1}\right) \cdot \frac{d}{dx} \left(\frac{x+1}{x-1}\right) \\ &= f'\left(\frac{x+1}{x-1}\right) \cdot \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \boxed{f'\left(\frac{x+1}{x-1}\right) \cdot \left(-\frac{2}{(x-1)^2}\right)}. \end{aligned}$$

90.

$$\begin{aligned} \frac{d}{dx} f\left(\frac{1-x}{1+x}\right) &= f'\left(\frac{1-x}{1+x}\right) \cdot \frac{d}{dx} \left(\frac{1-x}{1+x}\right) \\ &= f'\left(\frac{1-x}{1+x}\right) \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = \boxed{f'\left(\frac{1-x}{1+x}\right) \cdot \left(-\frac{2}{(1+x)^2}\right)}. \end{aligned}$$

$$91. \frac{d}{dx} f(\sin x) = f'(\sin x) \cdot \frac{d}{dx} \sin x = \boxed{f'(\sin x) \cos x}.$$

$$92. \frac{d}{dx} f(\tan x) = f'(\tan x) \cdot \frac{d}{dx} \tan x = \boxed{f'(\tan x) \sec^2 x}.$$

$$93. \frac{d}{dx} f(\cos x) = f'(\cos x) \cdot \frac{d}{dx} \cos x = -f'(\cos x) \sin x. \text{ Therefore,}$$

$$\begin{aligned} \frac{d^2}{dx^2} f(\cos x) &= \frac{d}{dx} [-f'(\cos x) \sin x] = -f'(\cos x) \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} [-f'(\cos x)] \\ &= -f'(\cos x) \cos x - \sin x \cdot f''(\cos x) \frac{d}{dx} \cos x = -f'(\cos x) \cos x - \sin x \cdot f''(\cos x) \cdot (-\sin x) \\ &= \boxed{f''(\cos x) \sin^2 x - f'(\cos x) \cos x}. \end{aligned}$$

$$94. \frac{d}{dx} f(e^x) = f'(e^x) \cdot \frac{d}{dx} e^x = f'(e^x) e^x. \text{ Therefore,}$$

$$\begin{aligned} \frac{d^2}{dx^2} f(e^x) &= \frac{d}{dx} [f'(e^x) e^x] = f'(e^x) \frac{d}{dx} e^x + e^x \frac{d}{dx} f'(e^x) \\ &= f'(e^x) e^x + e^x \cdot f''(e^x) \frac{d}{dx} e^x = f'(e^x) e^x + e^x \cdot f''(e^x) e^x \\ &= \boxed{f'(e^x) e^x + f''(e^x) e^{2x}}. \end{aligned}$$

95. Let $s(t) = A \cos(\omega t + \phi)$, where A , ω , and ϕ are constants.

(a) Velocity is the derivative of position, so

$$v(t) = \frac{d}{dt}s(t) = -A \sin(\omega t + \phi) \frac{d}{dt}(\omega t + \phi) = \boxed{-A\omega \sin(\omega t + \phi)}.$$

(b) The velocity of the object will be zero when $\sin(\omega t + \phi) = 0$, which is when

$$\omega t + \phi = k\pi \quad \text{or} \quad \boxed{t = \frac{k\pi - \phi}{\omega}},$$

where k is any integer such that $k\pi - \phi \geq 0$.

(c) Acceleration is the derivative of velocity, so

$$a(t) = \frac{d}{dt}v(t) = -A\omega \cos(\omega t + \phi) \frac{d}{dt}(\omega t + \phi) = \boxed{-A\omega^2 \cos(\omega t + \phi)}.$$

(d) The acceleration of the object will be zero when $\cos(\omega t + \phi) = 0$, which is when

$$\omega t + \phi = \left(k - \frac{1}{2}\right)\pi \quad \text{or} \quad \boxed{t = \frac{\left(k - \frac{1}{2}\right)\pi - \phi}{\omega}},$$

where k is any integer such that $\left(k - \frac{1}{2}\right)\pi - \phi \geq 0$.

96. Let $s(t) = 8 - (2 - t)^3$.

(a) Velocity is the derivative of position, so

$$v(t) = \frac{d}{dt}s(t) = -3(2 - t)^{3-1} \frac{d}{dt}(2 - t) = \boxed{3(2 - t)^2}.$$

(b) Using the result from part (a),

$$\boxed{v(1) = 3(2 - 1)^2 = 3 \text{ meters/second}} \quad \text{and} \quad \boxed{v(2) = 3(2 - 2)^2 = 0 \text{ meters/second}}.$$

(c) Acceleration is the derivative of velocity, so

$$a(t) = \frac{d}{dt}v(t) = 6(2 - t)^{2-1} \frac{d}{dt}(2 - t) = \boxed{-6(2 - t)}.$$

(d) Using the result from part (c),

$$\boxed{a(1) = -6(2 - 1) = -6 \text{ meters/second}^2} \quad \text{and} \quad \boxed{a(2) = -6(2 - 2) = 0 \text{ meters/second}^2}.$$

(e) The bullet travels

$$s(2) = 8 - (2 - 2)^3 = 8 - 0 = \boxed{8 \text{ meters}}$$

into the bale of paper.

97. Let

$$s(t) = \frac{80}{3} \left[t + \frac{3}{\pi} \sin\left(\frac{\pi}{6}t\right) \right] = \frac{80}{3}t + \frac{80}{\pi} \sin\left(\frac{\pi}{6}t\right).$$

Then

$$v(t) = \frac{d}{dt}s(t) = \frac{80}{3} + \frac{80}{\pi} \cos\left(\frac{\pi}{6}t\right) \frac{d}{dt}\left(\frac{\pi}{6}t\right) = \frac{80}{3} + \frac{40}{3} \cos\left(\frac{\pi}{6}t\right)$$

and

$$a(t) = \frac{d}{dt}v(t) = -\frac{40}{3} \sin\left(\frac{\pi}{6}t\right) \frac{d}{dt}\left(\frac{\pi}{6}t\right) = \boxed{-\frac{20\pi}{9} \sin\left(\frac{\pi}{6}t\right)}.$$

98. Let $s(t) = \sin e^t$.

(a) The velocity of the object is

$$v(t) = \frac{d}{dt}s(t) = \cos e^t \frac{d}{dt}e^t = \boxed{e^t \cos e^t},$$

and the acceleration of the object is

$$a(t) = \frac{d}{dt}v(t) = e^t \frac{d}{dt} \cos e^t + \cos e^t \frac{d}{dt}e^t = e^t(-\sin e^t) \frac{d}{dt}e^t + e^t \cos e^t = \boxed{-e^{2t} \sin e^t + e^t \cos e^t}.$$

(b) For $t > 0$, the velocity of the object is first equal to zero when

$$e^t = \frac{\pi}{2} \quad \text{or} \quad \boxed{t = \ln \frac{\pi}{2} \text{ seconds}}.$$

(c) At the time found in part (b), the acceleration of the object is

$$-\left(\frac{\pi}{2}\right)^2 \sin \frac{\pi}{2} + \frac{\pi}{2} \cos \frac{\pi}{2} = \boxed{-\frac{\pi^2}{4} \text{ feet/second}^2}.$$

99. First,

$$\frac{dR}{dx} = \frac{d}{dx} \left(\frac{0.0048}{x^2} \right) = -\frac{0.0096}{x^3} = -\frac{0.0096}{(0.1991 + 0.000003T)^3}$$

and

$$\frac{dx}{dT} = \frac{d}{dT}(0.1991 + 0.000003T) = 0.000003,$$

so

$$\frac{dR}{dT} = \frac{dR}{dx} \cdot \frac{dx}{dT} = -\frac{2.88 \times 10^{-8}}{(0.1991 + 0.000003T)^3}.$$

Therefore, when $T = 320$ K,

$$\frac{dR}{dT} = -\frac{2.88 \times 10^{-8}}{(0.1991 + 0.000003(320))^3} = -\frac{2.88 \times 10^{-8}}{(0.20006)^3} \approx \boxed{-3.597 \times 10^{-6} \text{ ohms/K}}.$$

100. Let the position of the pendulum be given by $s(t) = 0.05 \sin(2t) + 3t$ for $0 \leq t \leq \pi$.

(a) Velocity is the derivative of position, so

$$v(t) = \frac{ds}{dt} = 0.05 \cos(2t) \frac{d}{dt}(2t) + 3 = 0.1 \cos(2t) + 3.$$

Therefore,

$$\begin{aligned} v(0) &= 0.1 \cos 0 + 3 = \boxed{3.1 \text{ meters/second}}; \\ v\left(\frac{\pi}{8}\right) &= 0.1 \cos \frac{\pi}{4} + 3 = \frac{\sqrt{2}}{20} + 3 \approx \boxed{3.07 \text{ meters/second}}; \\ v\left(\frac{\pi}{4}\right) &= 0.1 \cos \frac{\pi}{2} + 3 = \boxed{3 \text{ meters/second}}; \\ v\left(\frac{\pi}{2}\right) &= 0.1 \cos \pi + 3 = \boxed{2.9 \text{ meters/second}}; \\ v(\pi) &= 0.1 \cos(2\pi) + 3 = \boxed{3.1 \text{ meters/second}}. \end{aligned}$$

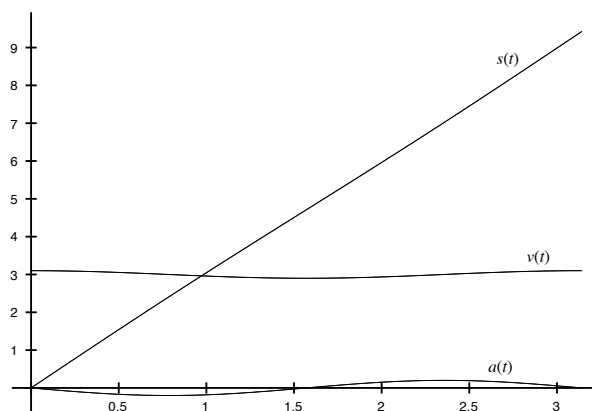
- (b) Acceleration is the derivative of velocity, so

$$a(t) = \frac{dv}{dt} = -0.1 \sin(2t) \frac{d}{dt}(2t) = -0.2 \sin(2t).$$

Therefore,

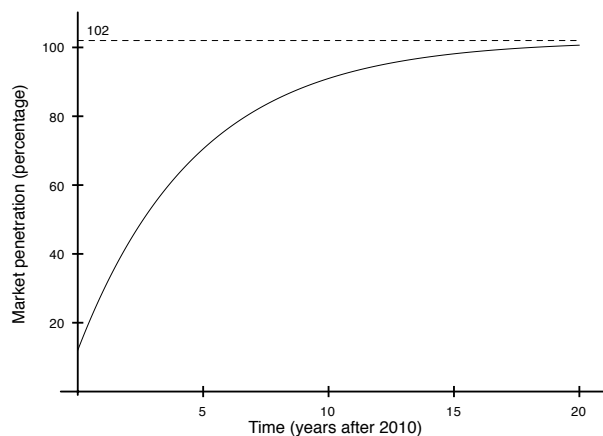
$$\begin{aligned} a(0) &= -0.2 \sin 0 = \boxed{0 \text{ meters/second}^2}; \\ a\left(\frac{\pi}{8}\right) &= -0.2 \sin \frac{\pi}{4} = -\frac{\sqrt{2}}{10} \approx \boxed{-0.14 \text{ meters/second}^2}; \\ a\left(\frac{\pi}{4}\right) &= -0.2 \sin \frac{\pi}{2} = \boxed{-0.2 \text{ meters/second}^2}; \\ a\left(\frac{\pi}{2}\right) &= -0.2 \sin \pi = \boxed{0 \text{ meters/second}^2}; \\ a(\pi) &= -0.2 \sin(2\pi) = \boxed{0 \text{ meters/second}^2}. \end{aligned}$$

- (c) The figure below displays the graphs of the position $s(t)$, the velocity $v(t)$, and the acceleration $a(t)$ of the pendulum.



101. Let $A(t) = 102 - 90e^{-0.21t}$.

- (a) $\lim_{t \rightarrow \infty} A(t) = 102 - 90 \lim_{t \rightarrow \infty} e^{-0.21t} = 102 - 90(0) = \boxed{102}$. In the long run, second-generation smart phones will penetrate 102% of the market.
- (b) The figure below displays the graph of A . The graph of A has a horizontal asymptote at $A = 102$, supporting the conclusion from part (a).



- (c) The rate of change of A with respect to time is

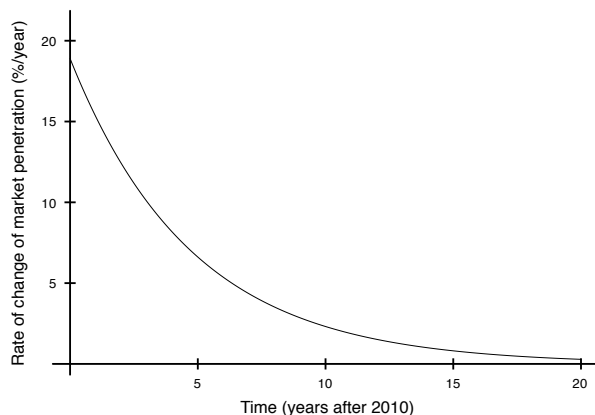
$$\frac{dA}{dt} = \frac{d}{dt}(102 - 90e^{-0.21t}) = -90e^{-0.21t} \frac{d}{dt}(-0.21t) = -90e^{-0.21t} \cdot (-0.21) = \boxed{18.9e^{-0.21t}}.$$

- (d) Using the result from part (c),

$$\boxed{A'(5) = 18.9e^{-0.21(5)} \approx 6.614\%/year} \quad \text{and} \quad \boxed{A'(10) = 18.9e^{-0.21(10)} \approx 2.314\%/year}.$$

Thus, in 2015, the market penetration of second-generation smart phones will be increasing at a rate of 6.614% per year, while in 2020, the market penetration of second-generation smart phones will be increasing at the slower rate of 2.314% per year.

- (e) The figure below displays the graph of A' . The graph of A' is decreasing from left to right, supporting the calculations at $t = 5$ and $t = 10$ in part (d).



102. Let $P(x) = 10^4 e^{-0.00012x}$. The rate of change of atmospheric pressure with respect to height is

$$P'(x) = 10^4 e^{-0.00012x} \frac{d}{dx}(-0.00012x) = 10^4 e^{-0.00012x} \cdot (-0.00012) = -1.2e^{-0.00012x}.$$

At $x = 500$ meters,

$$\boxed{P'(500) = -1.2e^{-0.00012(500)} \approx -1.13 \frac{\text{kilograms/square meter}}{\text{meter}}};$$

at $x = 750$ meters,

$$\boxed{P'(750) = -1.2e^{-0.00012(750)} \approx -1.10 \frac{\text{kilograms/square meter}}{\text{meter}}}.$$

103. Let $v(t) = \frac{mg}{k}(1 - e^{-kt/m})$.

- (a) Acceleration is the derivative of velocity, so

$$a(t) = \frac{d}{dt} \left[\frac{mg}{k} (1 - e^{-kt/m}) \right] = -\frac{mg}{k} e^{-kt/m} \frac{d}{dt} \left(-\frac{kt}{m} \right) = -\frac{mg}{k} e^{-kt/m} \cdot \left(-\frac{k}{m} \right) = \boxed{ge^{-kt/m}}.$$

- (b) $\lim_{t \rightarrow \infty} v(t) = \frac{mg}{k} - \frac{mg}{k} \lim_{t \rightarrow \infty} e^{-kt/m} = \frac{mg}{k} - \frac{mg}{k}(0) = \boxed{\frac{mg}{k}}$. Thus, in the long run, the speed of the hailstone stops increasing and approaches the constant value $\frac{mg}{k}$.

- (c) $\lim_{t \rightarrow \infty} a(t) = g \lim_{t \rightarrow \infty} e^{-kt/m} = g(0) = \boxed{0}$. Thus, in the long run, the acceleration of the hailstone approaches zero.

104. (a) Using the **ExpReg** function on a TI-84 Plus calculator, it is found that

$$E(t) = 9296(1.05)^t,$$

where t is the number of years since 1974.

- (b) The rate of change of E with respect to t is

$$\boxed{E'(t) = 9296(1.05)^t \ln 1.05}.$$

- (c) The rate of change at $t = 26$ is

$$E'(26) = 9296(1.05)^{26} \ln 1.05 \approx \boxed{1613 \text{ dollars per year}}.$$

- (d) The rate of change at $t = 31$ is

$$E'(31) = 9296(1.05)^{31} \ln 1.05 \approx \boxed{2058 \text{ dollars per year}}.$$

- (e) The rate of change at $t = 36$ is

$$E'(36) = 9296(1.05)^{36} \ln 1.05 \approx \boxed{2627 \text{ dollars per year}}.$$

- (f) From parts (c), (d), and (e), we can see that the rate of increase in mean earnings is growing over time, from roughly \$1600/year in 2000 to more than \$2000/year in 2005 and more than \$2600/year in 2010. An exponential function with a base larger than 1, as in this example, not only has a positive first derivative, it also has a positive second derivative, so the rate of increase of the function increases as the independent variable increases.

105. Suppose the velocity of the object $v = v(s)$ is expressed as a function of the position s . Then

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v = v \frac{dv}{ds}.$$

106. Let

$$Q(t) = 21 + \frac{10 \sin\left(\frac{2\pi t}{7}\right)}{\sqrt{t} - \sqrt{20}},$$

where Q is Professor Miller's approval rating and $0 \leq t \leq 16$ is the number of weeks since the beginning of the semester.

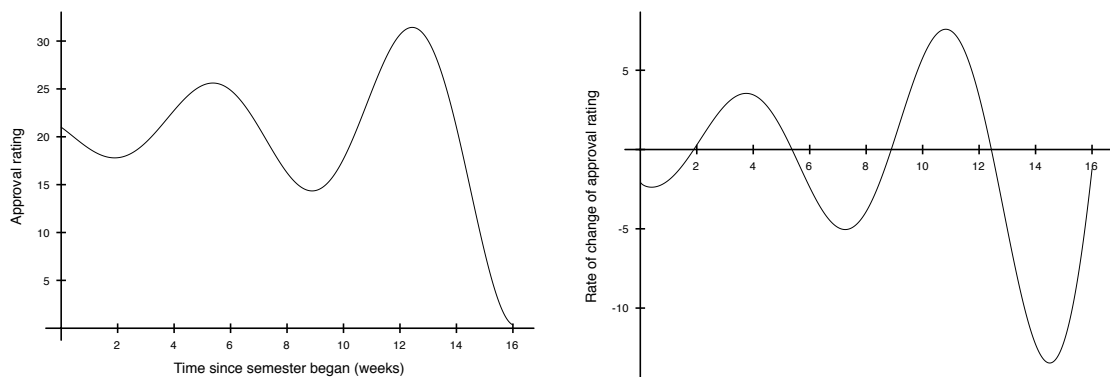
- (a)

$$\begin{aligned} Q'(t) &= \frac{(\sqrt{t} - \sqrt{20}) \cdot 10 \cos\left(\frac{2\pi t}{7}\right) \cdot \frac{d}{dt}\left(\frac{2\pi t}{7}\right) - 10 \sin\left(\frac{2\pi t}{7}\right) \cdot \frac{1}{2\sqrt{t}}}{(\sqrt{t} - \sqrt{20})^2} \\ &= \boxed{\frac{\frac{20\pi}{7}(\sqrt{t} - \sqrt{20}) \cos\left(\frac{2\pi t}{7}\right) - \frac{5}{\sqrt{t}} \sin\left(\frac{2\pi t}{7}\right)}{(\sqrt{t} - \sqrt{20})^2}}. \end{aligned}$$

- (b) Using the result from part (a),

$$\begin{aligned} Q'(1) &= \frac{\frac{20\pi}{7}(1 - \sqrt{20}) \cos\left(\frac{2\pi}{7}\right) - 5 \sin\left(\frac{2\pi}{7}\right)}{(1 - \sqrt{20})^2} \approx \boxed{-1.94} \\ Q'(5) &= \frac{\frac{20\pi}{7}(\sqrt{5} - \sqrt{20}) \cos\left(\frac{10\pi}{7}\right) - \sqrt{5} \sin\left(\frac{10\pi}{7}\right)}{(\sqrt{5} - \sqrt{20})^2} \approx \boxed{1.33} \\ Q'(10) &= \frac{\frac{20\pi}{7}(\sqrt{10} - \sqrt{20}) \cos\left(\frac{20\pi}{7}\right) - \frac{\sqrt{10}}{2} \sin\left(\frac{20\pi}{7}\right)}{(\sqrt{10} - \sqrt{20})^2} \approx \boxed{5.77}. \end{aligned}$$

- (c) The results in part (b) indicate: at the end of the first week of the semester, Professor Miller's approval rating was dropping at nearly 2 approval points per week; by the end of the fifth week of the semester, Professor Miller's approval rating was increasing at roughly one-and-one-third point per week; at the end of the tenth week of the semester, Professor Miller's approval rating was increasing at nearly 6 points per week.
- (d) The graphs of $Q(t)$ and $Q'(t)$ are shown below left and below right, respectively.



- (e) Professor Miller's approval rating oscillates with increasing amplitude throughout the semester. Peaks in approval rating occur at the start of the semester, during the sixth week of the semester and during the thirteenth week of the semester, with the highest approval occurring during the thirteenth week. Lows in student approval occur at the end of the second week, at the end of the ninth week and at the end of the semester, with the lowest approval occurring at the end of the semester.

The rate of change of Professor Miller's approval rating also oscillates with increasing amplitude throughout the semester. The approval rating is increasing most rapidly near the end of the fourth week and near the end of the eleventh week of the semester; the approval rating is decreasing most rapidly during the first week, the eighth week and the fifteenth week of the semester.

107. Let $\theta(t) = \frac{\pi}{3} \cos\left(\frac{1}{2}\sqrt{\frac{2k}{5}}t\right)$.

- (a) The angular velocity is given by

$$\begin{aligned}\omega(t) &= \frac{d\theta}{dt} = -\frac{\pi}{3} \sin\left(\frac{1}{2}\sqrt{\frac{2k}{5}}t\right) \cdot \frac{d}{dt}\left(\frac{1}{2}\sqrt{\frac{2k}{5}}t\right) \\ &= -\frac{\pi}{3} \left(\frac{1}{2}\sqrt{\frac{2k}{5}}\right) \sin\left(\frac{1}{2}\sqrt{\frac{2k}{5}}t\right) = \boxed{-\frac{\pi}{6}\sqrt{\frac{2k}{5}} \sin\left(\frac{1}{2}\sqrt{\frac{2k}{5}}t\right)}.\end{aligned}$$

- (b) The angular velocity at $t = 3$ is

$$\boxed{\omega(3) = -\frac{\pi}{6}\sqrt{\frac{2k}{5}} \sin\left(\frac{3}{2}\sqrt{\frac{2k}{5}}\right)}.$$

108. (a) Let $l(t) = 2 + \cos(2\pi t)$. Then

$$\begin{aligned} l(0) &= 2 + \cos 0 = 2 + 1 = \boxed{3 \text{ meters}}; \\ l\left(\frac{1}{2}\right) &= 2 + \cos \pi = 2 - 1 = \boxed{1 \text{ meter}}; \\ l(1) &= 2 + \cos(2\pi) = 2 + 1 = \boxed{3 \text{ meters}}; \\ l\left(\frac{3}{2}\right) &= 2 + \cos(3\pi) = 2 - 1 = \boxed{1 \text{ meter}}; \\ l\left(\frac{5}{8}\right) &= 2 + \cos\left(\frac{5\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2} \approx \boxed{1.3 \text{ meters}}. \end{aligned}$$

- (b) Velocity is the derivative of length, so

$$v(t) = l'(t) = -\sin(2\pi t) \frac{d}{dt}(2\pi t) = -2\pi \sin(2\pi t).$$

Then

$$v\left(\frac{1}{4}\right) = -2\pi \sin\left(\frac{\pi}{2}\right) = -2\pi \approx \boxed{-6.3 \text{ meters/second}}.$$

- (c) Acceleration is the derivative of velocity, so

$$a(t) = v'(t) = -2\pi \cos(2\pi t) \frac{d}{dt}(2\pi t) = -4\pi^2 \cos(2\pi t).$$

Then

$$a\left(\frac{1}{4}\right) = -4\pi^2 \cos\left(\frac{\pi}{2}\right) = \boxed{0 \text{ meters/second}^2}.$$

109. Let $F(t) = f(t^2 - 1)$ and $f(x) = \sin x$. Then

$$F'(t) = f'(t^2 - 1) \frac{d}{dt}(t^2 - 1) = 2tf'(t^2 - 1), \quad \text{so} \quad F'(1) = 2f'(0).$$

Now,

$$f'(x) = \cos x, \quad \text{so} \quad f'(0) = \cos 0 = 1.$$

Finally, $F'(1) = 2(1) = \boxed{2}$.

110. Let $f(x) = e^{-x}$, and suppose the point in question is (t, e^{-t}) . The slope of the line tangent to the graph of f at $x = t$ is $f'(t) = -e^{-t}$, so the slope of the normal line is

$$-\frac{1}{-e^{-t}} = e^t.$$

The equation of the normal line is then

$$y - e^{-t} = e^t(x - t).$$

In order for this line to pass through the origin, t must satisfy the equation

$$0 - e^{-t} = e^t(0 - t) \quad \text{or} \quad 1 = te^{2t}.$$

Using the computer algebra system *Mathematica*, the approximate solution to this equation is found to be $t \approx 0.426303$. The point on the graph of $y = e^{-x}$, where the normal line to the graph passes through the origin, is then

$$\boxed{(0.426303, e^{-0.426303}) \approx (0.426303, 0.652918)}.$$

111. $\frac{d}{dx} \cos x = \frac{d}{dx} \sin \left(\frac{\pi}{2} - x \right) = \cos \left(\frac{\pi}{2} - x \right) \frac{d}{dx} \left(\frac{\pi}{2} - x \right) = \sin x \cdot (-1) = -\sin x.$

112. Let $y = e^{2x}$. Then $y' = e^{2x} \cdot \frac{d}{dx}(2x) = 2e^{2x}$ and $y'' = 2e^{2x} \cdot \frac{d}{dx}(2x) = 4e^{2x}$. Therefore,

$$y'' - 4y = 4e^{2x} - 4e^{2x} = 0.$$

113. Let $y = e^{-2x}$. Then $y' = e^{-2x} \cdot \frac{d}{dx}(-2x) = -2e^{-2x}$ and $y'' = -2e^{-2x} \cdot \frac{d}{dx}(-2x) = 4e^{-2x}$. Therefore,

$$y'' - 4y = 4e^{-2x} - 4e^{-2x} = 0.$$

114. Let $y = Ae^{2x} + Be^{-2x}$. Then

$$\begin{aligned} y' &= Ae^{2x} \cdot \frac{d}{dx}(2x) + Be^{-2x} \cdot \frac{d}{dx}(-2x) = 2Ae^{2x} - 2Be^{-2x}, \\ y'' &= 2Ae^{2x} \cdot \frac{d}{dx}(2x) - 2Be^{-2x} \cdot \frac{d}{dx}(-2x) = 4Ae^{2x} + 4Be^{-2x}, \end{aligned}$$

and

$$y'' - 4y = 4Ae^{2x} + 4Be^{-2x} - 4(Ae^{2x} + Be^{-2x}) = 0.$$

115. Let $y = Ae^{ax} + Be^{-ax}$. Then

$$\begin{aligned} y' &= Ae^{ax} \cdot \frac{d}{dx}(ax) + Be^{-ax} \cdot \frac{d}{dx}(-ax) = aAe^{ax} - aBe^{-ax}, \\ y'' &= aAe^{ax} \cdot \frac{d}{dx}(ax) - aBe^{-ax} \cdot \frac{d}{dx}(-ax) = a^2Ae^{ax} + a^2Be^{-ax}, \end{aligned}$$

and

$$y'' - a^2y = a^2Ae^{ax} + a^2Be^{-ax} - a^2(Ae^{ax} + Be^{-ax}) = 0.$$

116. Let $y = Ae^{2x} + Be^{3x}$. Then

$$\begin{aligned} y' &= Ae^{2x} \cdot \frac{d}{dx}(2x) + Be^{3x} \cdot \frac{d}{dx}(3x) = 2Ae^{2x} + 3Be^{3x}, \\ y'' &= 2Ae^{2x} \cdot \frac{d}{dx}(2x) + 3Be^{3x} \cdot \frac{d}{dx}(3x) = 4Ae^{2x} + 9Be^{3x}, \end{aligned}$$

and

$$\begin{aligned} y'' - 5y' + 6y &= 4Ae^{2x} + 9Be^{3x} - 5(2Ae^{2x} + 3Be^{3x}) + 6(Ae^{2x} + Be^{3x}) \\ &= (4A - 10A + 6A)e^{2x} + (9B - 15B + 6B)e^{3x} = 0. \end{aligned}$$

117. Let $y = Ae^{-2x} + Be^{-x}$. Then

$$\begin{aligned} y' &= Ae^{-2x} \cdot \frac{d}{dx}(-2x) + Be^{-x} \cdot \frac{d}{dx}(-x) = -2Ae^{-2x} - Be^{-x}, \\ y'' &= -2Ae^{-2x} \cdot \frac{d}{dx}(-2x) - Be^{-x} \cdot \frac{d}{dx}(-x) = 4Ae^{-2x} + Be^{-x}, \end{aligned}$$

and

$$\begin{aligned} y'' + 3y' + 2y &= 4Ae^{-2x} + Be^{-x} + 3(-2Ae^{-2x} - Be^{-x}) + 2(Ae^{-2x} + Be^{-x}) \\ &= (4A - 6A + 2A)e^{-2x} + (B - 3B + 2B)e^{-x} = 0. \end{aligned}$$

118. Let $y = A \sin(\omega t) + B \cos(\omega t)$. Then

$$\begin{aligned} y' &= A \cos(\omega t) \cdot \frac{d}{dt}(\omega t) - B \sin(\omega t) \frac{d}{dt}(\omega t) = A\omega \cos(\omega t) - B\omega \sin(\omega t), \\ y'' &= -A\omega \sin(\omega t) \frac{d}{dt}(\omega t) - B\omega \cos(\omega t) \frac{d}{dt}(\omega t) = -A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t), \end{aligned}$$

and

$$y'' + \omega^2 y = -A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) + \omega^2(A \sin(\omega t) + B \cos(\omega t)) = 0.$$

119. $\frac{d}{dx}f(h(x)) = f'(h(x)) \cdot h'(x) = g(h(x)) \cdot 2x = 2xg(x^2)$.

120. Let $f(x) = (2x + 3)^n$. Then

$$\begin{aligned} \frac{df}{dx} &= n(2x + 3)^{n-1} \frac{d}{dx}(2x + 3) = 2n(2x + 3)^{n-1} \\ \frac{d^2f}{dx^2} &= 2n(n-1)(2x + 3)^{n-2} \frac{d}{dx}(2x + 3) = 2^2n(n-1)(2x + 3)^{n-2} \\ \frac{d^3f}{dx^3} &= 2^2n(n-1)(n-2)(2x + 3)^{n-3} \frac{d}{dx}(2x + 3) = 2^3n(n-1)(n-2)(2x + 3)^{n-3}, \end{aligned}$$

and so on. Based on this pattern, the n th derivative would be

$$\frac{d^n f}{dx^n} = 2^n n(n-1)(n-2) \cdots (2)(1)(2x + 3)^{n-n} = \boxed{2^n n!}.$$

121. (a) Let $y = e^{ax}$. Then

$$\begin{aligned} \frac{dy}{dx} &= e^{ax} \frac{d}{dx}(ax) = ae^{ax} \\ \frac{d^2y}{dx^2} &= ae^{ax} \frac{d}{dx}(ax) = a^2e^{ax} \\ \frac{d^3y}{dx^3} &= a^2e^{ax} \frac{d}{dx}(ax) = a^3e^{ax}, \end{aligned}$$

and so on. Each derivative results in multiplication by an additional factor of a , so the n th derivative of y would be $\boxed{a^n e^{ax}}$.

(b) Let $y = e^{-ax}$. Then

$$\begin{aligned} \frac{dy}{dx} &= e^{-ax} \frac{d}{dx}(-ax) = -ae^{-ax} \\ \frac{d^2y}{dx^2} &= -ae^{-ax} \frac{d}{dx}(-ax) = a^2e^{-ax} \\ \frac{d^3y}{dx^3} &= a^2e^{-ax} \frac{d}{dx}(-ax) = -a^3e^{-ax}, \end{aligned}$$

and so on. Each derivative results in multiplication by an additional factor of $-a$, so the n th derivative of y would be $\boxed{(-a)^n e^{-ax} = (-1)^n a^n e^{-ax}}$.

122. Let $f(x) = \sin(ax)$. Then

$$\begin{aligned} \frac{df}{dx} &= \cos(ax) \cdot \frac{d}{dx}(ax) = a \cos(ax); \\ \frac{d^2f}{dx^2} &= \frac{d}{dx}[a \cos(ax)] = -a \sin(ax) \cdot \frac{d}{dx}(ax) = -a^2 \sin(ax) \\ \frac{d^3f}{dx^3} &= \frac{d}{dx}[-a^2 \sin(ax)] = -a^2 \cos(ax) \cdot \frac{d}{dx}(ax) = -a^3 \cos(ax) \\ \frac{d^4f}{dx^4} &= \frac{d}{dx}[-a^3 \cos(ax)] = a^3 \sin(ax) \frac{d}{dx}(ax) = a^4 \sin(ax), \end{aligned}$$

and so on. At this point, the two elements of the pattern in these derivatives can be identified. First, each derivative introduces multiplication by a factor of a . Second, the trigonometric function in the derivatives cycles through $\sin(ax)$, $\cos(ax)$, $-\sin(ax)$, and $-\cos(ax)$, in that order, with each set of four derivatives.

- (a) Because the trigonometric function in the derivatives goes in a cycle of four, the tenth derivative will be like the second derivative in that it will contain the function $-\sin(ax)$. The tenth derivative will also include ten factors of a , so

$$\frac{d^{10}}{dx^{10}} \sin(ax) = \boxed{-a^{10} \sin(ax)}.$$

- (b) Because the trigonometric function in the derivatives goes in a cycle of four, the twenty-fifth derivative will be like the first derivative in that it will contain the function $\cos(ax)$. The twenty-fifth derivative will also contain 25 factors of a , so

$$\frac{d^{25}}{dx^{25}} \sin(ax) = \boxed{a^{25} \cos(ax)}.$$

- (c) The n th derivative of $f(x) = \sin(ax)$ will be

$$\begin{cases} a^n \sin(ax), & \text{if } n = 4k, k = 0, 1, 2, 3, \dots \\ a^n \cos(ax), & \text{if } n = 4k + 1, k = 0, 1, 2, 3, \dots \\ -a^n \sin(ax), & \text{if } n = 4k + 2, k = 0, 1, 2, 3, \dots \\ -a^n \cos(ax), & \text{if } n = 4k + 3, k = 0, 1, 2, 3, \dots \end{cases}$$

123. Let $f(x) = \cos(ax)$. Then

$$\begin{aligned} \frac{df}{dx} &= -\sin(ax) \cdot \frac{d}{dx}(ax) = -a \sin(ax); \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}[-a \sin(ax)] = -a \cos(ax) \cdot \frac{d}{dx}(ax) = -a^2 \cos(ax) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}[-a^2 \cos(ax)] = a^2 \sin(ax) \cdot \frac{d}{dx}(ax) = a^3 \sin(ax) \\ \frac{d^4 f}{dx^4} &= \frac{d}{dx}[a^3 \sin(ax)] = a^3 \cos(ax) \cdot \frac{d}{dx}(ax) = a^4 \cos(ax), \end{aligned}$$

and so on. At this point, the two elements of the pattern in these derivatives can be identified. First, each derivative introduces multiplication by a factor of a . Second, the trigonometric function in the derivatives cycles through $\cos(ax)$, $-\sin(ax)$, $-\cos(ax)$, and $\sin(ax)$, in that order, with each set of four derivatives.

- (a) Because the trigonometric function in the derivatives goes in a cycle of four, the eleventh derivative will be like the third derivative in that it will contain the function $\sin(ax)$. The eleventh derivative will also include eleven factors of a , so

$$\frac{d^{11}}{dx^{11}} \cos(ax) = \boxed{a^{11} \sin(ax)}.$$

- (b) Because the trigonometric function in the derivatives goes in a cycle of four, the twelfth derivative will be like the original function in that it will contain the function $\cos(ax)$. The twelfth derivative will also contain 12 factors of a , so

$$\frac{d^{12}}{dx^{12}} \cos(ax) = \boxed{a^{12} \cos(ax)}.$$

(c) The n th derivative of $f(x) = \cos(ax)$ will be

$$\begin{cases} a^n \cos(ax), & \text{if } n = 4k, k = 0, 1, 2, 3, \dots \\ -a^n \sin(ax), & \text{if } n = 4k + 1, k = 0, 1, 2, 3, \dots \\ -a^n \cos(ax), & \text{if } n = 4k + 2, k = 0, 1, 2, 3, \dots \\ a^n \sin(ax), & \text{if } n = 4k + 3, k = 0, 1, 2, 3, \dots \end{cases}$$

124. Let $y = e^{-at}[A \sin(\omega t) + B \cos(\omega t)]$, where A , B , a , and ω are constants. Then

$$\begin{aligned} y' &= e^{-at} \frac{d}{dt}[A \sin(\omega t) + B \cos(\omega t)] + [A \sin(\omega t) + B \cos(\omega t)] \frac{d}{dt}e^{-at} \\ &= e^{-at}[A\omega \cos(\omega t) - B\omega \sin(\omega t)] + [A \sin(\omega t) + B \cos(\omega t)] \cdot (-ae^{-at}) \\ &= \boxed{e^{-at}[(A\omega - Ba) \cos(\omega t) - (B\omega + Aa) \sin(\omega t)]}, \end{aligned}$$

and

$$\begin{aligned} y'' &= e^{-at} \frac{d}{dt}[(A\omega - Ba) \cos(\omega t) - (B\omega + Aa) \sin(\omega t)] + \\ &\quad [(A\omega - Ba) \cos(\omega t) - (B\omega + Aa) \sin(\omega t)] \frac{d}{dt}e^{-at} \\ &= e^{-at}[-(A\omega - Ba)\omega \sin(\omega t) - (B\omega + Aa)\omega \cos(\omega t)] + \\ &\quad [(A\omega - Ba) \cos(\omega t) - (B\omega + Aa) \sin(\omega t)] \cdot (-ae^{-at}) \\ &= \boxed{e^{-at}[(-A\omega^2 + 2Baw + Aa^2) \sin(\omega t) + (-B\omega^2 - 2Aa\omega + Ba^2) \cos(\omega t)]}. \end{aligned}$$

125. Consider the limit that appears in the definition of the derivative; that is,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Using the property that $f(u+v) = f(u)f(v)$ for all choices of u and v , it follows that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x)[f(h) - 1]}{h}.$$

Next, given that $f(x) = 1 + xg(x)$,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x)[(1 + hg(h)) - 1]}{h} = \lim_{h \rightarrow 0} \frac{hf(x)g(h)}{h} = f(x) \lim_{h \rightarrow 0} g(h).$$

We were able to pull the factor $f(x)$ outside of the limit because $f(x)$ is a constant with respect to h , the variable associated with the limit. Finally, because $\lim_{x \rightarrow 0} g(x) = 1$,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f(x) \cdot 1 = f(x).$$

Because

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists and is equal to $f(x)$, it follows that f is differentiable and $f'(x) = f(x)$.

Challenge Problems

126. Let $f(x) = \frac{1}{3x-4} = (3x-4)^{-1}$. Then

$$\begin{aligned}\frac{df}{dx} &= -1(3x-4)^{-1-1} \frac{d}{dx}(3x-4) = -3(3x-4)^{-2} = (-3)^1 \cdot 1! \cdot (3x-4)^{-2} \\ \frac{d^2f}{dx^2} &= (-3)(-2)(1)(3x-4)^{-2-1} \frac{d}{dx}(3x-4) = -(3)^2 \cdot 2! \cdot (3x-4)^{-3} \\ \frac{d^3f}{dx^3} &= (-3)^2(2!)(-3)(3x-4)^{-3-1} \frac{d}{dx}(3x-4) = (-3)^3 \cdot 3! \cdot (3x-4)^{-4},\end{aligned}$$

and so on. Following this pattern, the n th derivative is

$$\boxed{\frac{d^n f}{dx^n} = (-3)^n \cdot n! \cdot (3x-4)^{-n-1}}.$$

127. Repeatedly applying the Chain Rule yields

$$\begin{aligned}\frac{d}{dx} f_1(f_2(f_3(\cdots(f_n(x))\cdots))) \\ &= f'_1(f_2(f_3(\cdots(f_n(x))\cdots))) \frac{d}{dx} f_2(f_3(\cdots(f_n(x))\cdots)) \\ &= f'_1(f_2(f_3(\cdots(f_n(x))\cdots))) f'_2(f_3(\cdots(f_n(x))\cdots)) \frac{d}{dx} f_3(f_4(\cdots(f_n(x))\cdots)) \\ &= f'_1(f_2(f_3(\cdots(f_n(x))\cdots))) f'_2(f_3(\cdots(f_n(x))\cdots)) f'_3(f_4(\cdots(f_n(x))\cdots)) \frac{d}{dx} f_4(f_5(\cdots(f_n(x))\cdots)) \\ &\vdots \\ &= \boxed{f'_1(f_2(f_3(\cdots(f_n(x))\cdots))) f'_2(f_3(\cdots(f_n(x))\cdots)) f'_3(f_4(\cdots(f_n(x))\cdots)) \cdots f'_n(x)}.\end{aligned}$$

128. Consider the limit

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} \left[h \sin \frac{1}{h} \right].$$

Now,

$$\left| \sin \frac{1}{h} \right| \leq 1, \quad \text{so} \quad |h| \left| \sin \frac{1}{h} \right| = \left| h \sin \frac{1}{h} \right| \leq |h| \quad \text{and} \quad -|h| \leq h \sin \frac{1}{h} \leq |h|$$

for all $h \neq 0$. Because

$$\lim_{h \rightarrow 0} (-|h|) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} |h| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{h \rightarrow 0} \left[h \sin \frac{1}{h} \right] = 0.$$

Therefore,

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Because

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

exists and is equal to 0, f is differentiable at $x = 0$ with $f'(0) = 0$. For $x \neq 0$,

$$f'(x) = x^2 \frac{d}{dx} \sin \frac{1}{x} + \sin \frac{1}{x} \frac{d}{dx} x^2 = x^2 \cos \frac{1}{x} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + 2x \sin \frac{1}{x} = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}.$$

As x approaches 0, the second term in this derivative tends to 0 but the first term just oscillates between -1 and 1 . Therefore, $\lim_{x \rightarrow 0} f'(x)$ does not exist, so that $f'(x)$ is not continuous at 0.

129. For $x \neq 0$, f is the composition of the exponential function e^x , which is differentiable for all x , and the rational function $-1/x^2$, which is differentiable for $x \neq 0$. Therefore, f is differentiable for $x \neq 0$, and

$$f'(x) = \frac{d}{dx} e^{-1/x^2} = e^{-1/x^2} \frac{d}{dx} \left(-\frac{1}{x^2} \right) = \frac{2}{x^3} e^{-1/x^2}.$$

At $x = 0$, consider the limit

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \frac{e^{-1/h^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h}.$$

To proceed further, consider the function $g(x) = e^x - x$. Note that $g(0) = e^0 - 0 = 1 > 0$. Moreover, $g'(x) = e^x - 1 > 0$ for $x > 0$. Because the rate of change of g is positive for $x > 0$, it follows that $g(x) > g(0)$ for $x > 0$. Thus, for $x > 0$, $e^x - x > 0$, or $e^x > x$. Now, for $x \neq 0$, $1/x^2 > 0$, so

$$e^{1/x^2} > \frac{1}{x^2} \quad \text{or} \quad e^{-1/x^2} < x^2$$

for $x \neq 0$. As $e^{-1/x^2} > 0$ for $x \neq 0$, it follows that

$$\left| e^{-1/x^2} \right| < |x|^2, \quad \text{so that} \quad \left| \frac{e^{-1/x^2}}{x} \right| < |x|,$$

or

$$-|x| < \frac{e^{-1/x^2}}{x} < |x|.$$

Now,

$$\lim_{x \rightarrow 0} (-|x|) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} |x| = 0,$$

so, by the Squeeze Theorem,

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = 0.$$

It follows that

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0,$$

which implies that $f'(0)$ exists and is equal to 0. Therefore, f is differentiable on $(-\infty, \infty)$.

130. (a) Yes, the function $f(x) = x^2$ is differentiable at all real numbers x .
- (b) No, the function $g(x) = |x-1|$ is not differentiable at all real numbers x . Because of the presence of the absolute value function, g' does not exist at $x = 1$.
- (c) Because the Chain Rule calculates the derivative of the composition $(f \circ g)(x)$ as $f'(g(x))g'(x)$ and g' does not exist at $x = 1$, the Chain Rule cannot be used to differentiate this composite function for all x .
- (d) Yes, the composite function $(f \circ g)(x) = f(g(x)) = f(|x-1|) = |x-1|^2 = (x-1)^2$ is differentiable for all x . Its derivative is $2(x-1)$.
131. (a) Yes, the function $f(x) = x^4$ is differentiable at all real numbers x .
- (b) No, the function $g(x) = x^{1/3}$ is not differentiable at all real numbers x . The derivative does not exist at $x = 0$.
- (c) Because the Chain Rule calculates the derivative of the composition $(f \circ g)(x)$ as $f'(g(x))g'(x)$ and g' does not exist at $x = 0$, the Chain Rule cannot be used to differentiate this composite function for all x .

- (d) Yes, the composite function $(f \circ g)(x) = f(g(x)) = f(x^{1/3}) = (x^{1/3})^4 = x^{4/3}$ is differentiable for all x . Its derivative is $\frac{4}{3}x^{1/3}$.

132. Let $g(x) = 2e^x$. Then $g(x)$ is defined for all real x , $g'(x) = 2e^x = g(x)$ and $g(x) \neq f(x)$, as desired. In fact, any function of the form ce^x , where c is a real number, can be used as $g(x)$.

133. (a) Let $s(t) = 5 \sin(120\pi t)$. Then

$$v(t) = s'(t) = 5 \cos(120\pi t) \frac{d}{dt}(120\pi t) = 600\pi \cos(120\pi t) \text{ cm/sec}$$

and

$$a(t) = v'(t) = -600\pi \sin(120\pi t) \frac{d}{dt}(120\pi t) = -72000\pi^2 \sin(120\pi t) \text{ cm/sec}^2.$$

At $t = \frac{1}{240}$ seconds, the piston is at the end of its stroke, and the acceleration of the piston is

$$a\left(\frac{1}{240}\right) = -72,000\pi^2 \sin \frac{\pi}{2} \text{ cm/sec}^2 = \boxed{-72,000\pi^2 \text{ cm/sec}^2}.$$

- (b) If the piston has a mass of 1 kg, the force on the piston is

$$F = ma = \boxed{-72,000\pi^2 \text{ kg} \cdot \text{cm/sec}^2}.$$

3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

Concepts and Vocabulary

1. True. If f is a one-to-one differentiable function and if $f'(x) > 0$, then f has an inverse function whose derivative is positive.
2. True. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$, where $-\infty < x < \infty$.
3. True. Implicit differentiation is a technique for finding the derivative of an implicitly defined function.
4. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$.
5. False. If $y^q = x^p$ for integers p and q , then $qy^{q-1} \frac{dy}{dx} = px^{p-1}$.
6. $\frac{d}{dx}(3x^{1/3}) = 3 \cdot \frac{1}{3}x^{-2/3} = \frac{1}{x^{2/3}}$.

Skill Building

7. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned} \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}4 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \boxed{-\frac{x}{y}}, \end{aligned}$$

provided $y \neq 0$.

8. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned} \frac{d}{dx}[y^4 - 4x^2] &= \frac{d}{dx}4 \\ \frac{d}{dx}y^4 - 4 \frac{d}{dx}x^2 &= 0 \\ 4y^3 \frac{dy}{dx} - 8x &= 0 \\ \frac{dy}{dx} &= \boxed{\frac{2x}{y^3}}, \end{aligned}$$

provided $y \neq 0$.

9. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}e^y &= \frac{d}{dx}\sin x \\ e^y \frac{dy}{dx} &= \cos x \\ \frac{dy}{dx} &= \boxed{\frac{\cos x}{e^y}}.\end{aligned}$$

10. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}e^y &= \frac{d}{dx}\tan x \\ e^y \frac{dy}{dx} &= \sec^2 x \\ \frac{dy}{dx} &= \boxed{\frac{\sec^2 x}{e^y}}.\end{aligned}$$

11. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}e^{x+y} &= \frac{d}{dx}y \\ e^{x+y} \frac{d}{dx}(x+y) &= \frac{dy}{dx} \\ e^{x+y} \left(1 + \frac{dy}{dx}\right) &= \frac{dy}{dx} \\ e^{x+y} + e^{x+y} \frac{dy}{dx} &= \frac{dy}{dx} \\ (e^{x+y} - 1) \frac{dy}{dx} &= -e^{x+y} \\ \frac{dy}{dx} &= \boxed{-\frac{e^{x+y}}{e^{x+y} - 1}},\end{aligned}$$

provided $e^{x+y} - 1 \neq 0$, which is equivalent to $x + y \neq 0$.

12. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}e^{x+y} &= \frac{d}{dx}x^2 \\ e^{x+y} \frac{d}{dx}(x+y) &= 2x \\ e^{x+y} \left(1 + \frac{dy}{dx}\right) &= 2x \\ e^{x+y} + e^{x+y} \frac{dy}{dx} &= 2x \\ \frac{dy}{dx} &= \boxed{\frac{2x - e^{x+y}}{e^{x+y}}}.\end{aligned}$$

13. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}(x^2y) &= \frac{d}{dx}5 \\ x^2\frac{dy}{dx} + y\frac{d}{dx}x^2 &= 0 \\ x^2\frac{dy}{dx} + 2xy &= 0 \\ \frac{dy}{dx} &= -\frac{2xy}{x^2} = \boxed{-\frac{2y}{x}}.\end{aligned}$$

14. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}(x^3y) &= \frac{d}{dx}8 \\ x^3\frac{dy}{dx} + y\frac{d}{dx}x^3 &= 0 \\ x^3\frac{dy}{dx} + 3x^2y &= 0 \\ \frac{dy}{dx} &= -\frac{3x^2y}{x^3} = \boxed{-\frac{3y}{x}}.\end{aligned}$$

15. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}\frac{d}{dx}[x^2 - y^2 - xy] &= \frac{d}{dx}2 \\ \frac{d}{dx}x^2 - \frac{d}{dx}y^2 - \frac{d}{dx}(xy) &= 0 \\ 2x - 2y\frac{dy}{dx} - x\frac{dy}{dx} - y\frac{d}{dx}x &= 0 \\ 2x - 2y\frac{dy}{dx} - x\frac{dy}{dx} - y &= 0 \\ -(2y + x)\frac{dy}{dx} &= y - 2x \\ \frac{dy}{dx} &= \boxed{\frac{2x - y}{2y + x}},\end{aligned}$$

provided $2y + x \neq 0$.

16. Assume that y is a differentiable function of x and differentiate both sides of the given equation with

respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[x^2 - 4xy + y^2] &= \frac{d}{dx}y \\
 \frac{d}{dx}x^2 - 4\frac{d}{dx}(xy) + \frac{d}{dx}y^2 &= \frac{dy}{dx} \\
 2x - 4\left(x\frac{dy}{dx} + y\frac{d}{dx}x\right) + 2y\frac{dy}{dx} &= \frac{dy}{dx} \\
 2x - 4x\frac{dy}{dx} - 4y + 2y\frac{dy}{dx} &= \frac{dy}{dx} \\
 (2y - 4x - 1)\frac{dy}{dx} &= 4y - 2x \\
 \frac{dy}{dx} &= \boxed{\frac{4y - 2x}{2y - 4x - 1}},
 \end{aligned}$$

provided $2y - 4x - 1 \neq 0$.

17. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) &= \frac{d}{dx}1 \\
 \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}\left(\frac{1}{y}\right) &= 0 \\
 -\frac{1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= \boxed{-\frac{y^2}{x^2}}.
 \end{aligned}$$

18. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}\left(\frac{1}{x} - \frac{1}{y}\right) &= \frac{d}{dx}4 \\
 \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{d}{dx}\left(\frac{1}{y}\right) &= 0 \\
 -\frac{1}{x^2} + \frac{1}{y^2}\frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= \boxed{\frac{y^2}{x^2}}.
 \end{aligned}$$

19. Assume that y is a differentiable function of x and differentiate both sides of the given equation with

respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx} \left(\frac{2y}{x} \right) \\
 \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{x \cdot 2\frac{dy}{dx} - 2y}{x^2} \\
 2x + 2y\frac{dy}{dx} &= \frac{2}{x}\frac{dy}{dx} - \frac{2y}{x^2} \\
 \left(2y - \frac{2}{x}\right)\frac{dy}{dx} &= -2x - \frac{2y}{x^2} \\
 \frac{2xy - 2}{x}\frac{dy}{dx} &= -\frac{2x^3 + 2y}{x^2} \\
 \frac{dy}{dx} &= -\frac{x^3 + y}{x(xy - 1)} = \boxed{\frac{x^3 + y}{x(1 - xy)}},
 \end{aligned}$$

provided $1 - xy \neq 0$.

20. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx} \left(\frac{2y^2}{x^2} \right) \\
 \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{x^2 \cdot 4y\frac{dy}{dx} - 4xy^2}{x^4} \\
 2x + 2y\frac{dy}{dx} &= \frac{4y}{x^2}\frac{dy}{dx} - \frac{4y^2}{x^3} \\
 \left(2y - \frac{4y}{x^2}\right)\frac{dy}{dx} &= -2x - \frac{4y^2}{x^3} \\
 \frac{2x^2y - 4y}{x^2}\frac{dy}{dx} &= -\frac{2x^4 + 4y^2}{x^3} \\
 \frac{dy}{dx} &= \boxed{-\frac{x^4 + 2y^2}{xy(x^2 - 2)}},
 \end{aligned}$$

provided $y \neq 0$.

21. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[e^x \sin y + e^y \cos x] &= \frac{d}{dx}4 \\
 \frac{d}{dx}[e^x \sin y] + \frac{d}{dx}[e^y \cos x] &= 0 \\
 e^x \frac{d}{dx} \sin y + \sin y \frac{d}{dx}e^x + e^y \frac{d}{dx} \cos x + \cos x \frac{d}{dx}e^y &= 0 \\
 e^x \cos y \frac{dy}{dx} + e^x \sin y - e^y \sin x + e^y \cos x \frac{dy}{dx} &= 0 \\
 (e^x \cos y + e^y \cos x) \frac{dy}{dx} &= e^y \sin x - e^x \sin y \\
 \frac{dy}{dx} &= \boxed{\frac{e^y \sin x - e^x \sin y}{e^x \cos y + e^y \cos x}},
 \end{aligned}$$

provided $e^x \cos y + e^y \cos x \neq 0$.

22. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[e^y \cos x + e^{-x} \sin y] &= \frac{d}{dx} 10 \\
 \frac{d}{dx}[e^y \cos x] + \frac{d}{dx}[e^{-x} \sin y] &= 0 \\
 e^y \frac{d}{dx} \cos x + \cos x \frac{d}{dx} e^y + e^{-x} \frac{d}{dx} \sin y + \sin y \frac{d}{dx} e^{-x} &= 0 \\
 -e^y \sin x + e^y \cos x \frac{dy}{dx} + e^{-x} \cos y \frac{dy}{dx} - e^{-x} \sin y &= 0 \\
 (e^y \cos x + e^{-x} \cos y) \frac{dy}{dx} &= e^y \sin x + e^{-x} \sin y \\
 \frac{dy}{dx} &= \boxed{\frac{e^y \sin x + e^{-x} \sin y}{e^y \cos x + e^{-x} \cos y}},
 \end{aligned}$$

provided $e^y \cos x + e^{-x} \cos y \neq 0$.

23. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[(x^2 + y)^3] &= \frac{d}{dx} y \\
 3(x^2 + y)^2 \frac{d}{dx}(x^2 + y) &= \frac{dy}{dx} \\
 3(x^2 + y)^2 \left(\frac{d}{dx} x^2 + \frac{dy}{dx} \right) &= \frac{dy}{dx} \\
 3(x^2 + y)^2 \left(2x + \frac{dy}{dx} \right) &= \frac{dy}{dx} \\
 [3(x^2 + y)^2 - 1] \frac{dy}{dx} &= -6x(x^2 + y)^2 \\
 \frac{dy}{dx} &= \boxed{-\frac{6x(x^2 + y)^2}{3(x^2 + y)^2 - 1}},
 \end{aligned}$$

provided $3(x^2 + y)^2 - 1 \neq 0$.

24. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}[(x + y^2)^3] &= \frac{d}{dx}(3x) \\
 3(x + y^2)^2 \frac{d}{dx}(x + y^2) &= 3 \\
 3(x + y^2)^2 \left(\frac{d}{dx} x + \frac{d}{dx} y^2 \right) &= 3 \\
 3(x + y^2)^2 \left(1 + 2y \frac{dy}{dx} \right) &= 3 \\
 6y(x + y^2)^2 \frac{dy}{dx} &= 3 - 3(x + y^2)^2 \\
 \frac{dy}{dx} &= \boxed{\frac{1 - (x + y^2)^2}{2y(x + y^2)^2}},
 \end{aligned}$$

provided $y(x + y^2) \neq 0$.

25. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}y &= \frac{d}{dx}\tan(x-y) \\
 \frac{dy}{dx} &= \sec^2(x-y) \cdot \frac{d}{dx}(x-y) \\
 \frac{dy}{dx} &= \sec^2(x-y) \cdot \left(\frac{d}{dx}x - \frac{dy}{dx}\right) \\
 \frac{dy}{dx} &= \sec^2(x-y) \cdot \left(1 - \frac{dy}{dx}\right) \\
 (1 + \sec^2(x-y))\frac{dy}{dx} &= \sec^2(x-y) \\
 \frac{dy}{dx} &= \boxed{\frac{\sec^2(x-y)}{1 + \sec^2(x-y)}}.
 \end{aligned}$$

26. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}y &= \frac{d}{dx}\cos(x+y) \\
 \frac{dy}{dx} &= -\sin(x+y) \cdot \frac{d}{dx}(x+y) \\
 \frac{dy}{dx} &= -\sin(x+y) \cdot \left(\frac{d}{dx}x + \frac{dy}{dx}\right) \\
 \frac{dy}{dx} &= -\sin(x+y) \cdot \left(1 + \frac{dy}{dx}\right) \\
 (1 + \sin(x+y))\frac{dy}{dx} &= -\sin(x+y) \\
 \frac{dy}{dx} &= \boxed{-\frac{\sin(x+y)}{1 + \sin(x+y)}},
 \end{aligned}$$

provided $1 + \sin(x+y) \neq 0$.

27. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}y &= \frac{d}{dx}[x \sin y] \\
 \frac{dy}{dx} &= x \frac{d}{dx}\sin y + \sin y \frac{d}{dx}x \\
 \frac{dy}{dx} &= x \cos y \frac{dy}{dx} + \sin y \\
 (1 - x \cos y)\frac{dy}{dx} &= \sin y \\
 \frac{dy}{dx} &= \boxed{\frac{\sin y}{1 - x \cos y}},
 \end{aligned}$$

provided $1 - x \cos y \neq 0$.

28. Assume that y is a differentiable function of x and differentiate both sides of the given equation with

respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}y &= \frac{d}{dx}[x \cos y] \\
 \frac{dy}{dx} &= x \frac{d}{dx} \cos y + \cos y \frac{d}{dx}x \\
 \frac{dy}{dx} &= -x \sin y \frac{dy}{dx} + \cos y \\
 (1 + x \sin y) \frac{dy}{dx} &= \cos y \\
 \frac{dy}{dx} &= \boxed{\frac{\cos y}{1 + x \sin y}},
 \end{aligned}$$

provided $1 + x \sin y \neq 0$.

29. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}(x^2 y) &= \frac{d}{dx}e^{xy} \\
 x^2 \frac{dy}{dx} + y \frac{d}{dx}x^2 &= e^{xy} \frac{d}{dx}(xy) \\
 x^2 \frac{dy}{dx} + 2xy &= e^{xy} \left(x \frac{dy}{dx} + y \frac{d}{dx}x \right) \\
 x^2 \frac{dy}{dx} + 2xy &= xe^{xy} \frac{dy}{dx} + ye^{xy} \\
 (x^2 - xe^{xy}) \frac{dy}{dx} &= ye^{xy} - 2xy \\
 \frac{dy}{dx} &= \boxed{\frac{ye^{xy} - 2xy}{x^2 - xe^{xy}}},
 \end{aligned}$$

provided $x^2 - xe^{xy} \neq 0$.

30. Assume that y is a differentiable function of x and differentiate both sides of the given equation with respect to x . This yields

$$\begin{aligned}
 \frac{d}{dx}(ye^x) &= \frac{d}{dx}(y - x) \\
 y \frac{d}{dx}e^x + e^x \frac{dy}{dx} &= \frac{dy}{dx} - \frac{d}{dx}x \\
 ye^x + e^x \frac{dy}{dx} &= \frac{dy}{dx} - 1 \\
 (e^x - 1) \frac{dy}{dx} &= -(1 + ye^x) \\
 \frac{dy}{dx} &= \boxed{-\frac{1 + ye^x}{e^x - 1}}.
 \end{aligned}$$

31. Let $y = x^{2/3} + 4$. Then $y' = \frac{2}{3}x^{-1/3} = \boxed{\frac{2}{3x^{1/3}}}$.

32. Let $y = x^{1/3} - 1$. Then $y' = \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{3x^{2/3}}}$.

33. Let $y = \sqrt[3]{x^2} = x^{2/3}$. Then $y' = \frac{2}{3}x^{-1/3} = \boxed{\frac{2}{3x^{1/3}}}$.

34. Let $y = \sqrt[4]{x^5} = x^{5/4}$. Then $y' = \boxed{\frac{5}{4}x^{1/4}}$.

35. Let $y = \sqrt[3]{x} - \frac{1}{\sqrt[3]{x}} = x^{1/3} - x^{-1/3}$. Then $y' = \frac{1}{3}x^{-2/3} + \frac{1}{3}x^{-4/3} = \boxed{\frac{1}{3x^{2/3}} + \frac{1}{3x^{4/3}}}$.

36. Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$. Then $y' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \boxed{\frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}}$.

37. Let $y = (x^3 - 1)^{1/2}$. Then

$$y' = \frac{1}{2}(x^3 - 1)^{-1/2} \frac{d}{dx}(x^3 - 1) = \frac{1}{2}(x^3 - 1)^{-1/2}(3x^2) = \boxed{\frac{3x^2}{2(x^3 - 1)^{1/2}}}.$$

38. Let $y = (x^2 - 1)^{1/3}$. Then

$$y' = \frac{1}{3}(x^2 - 1)^{-2/3} \frac{d}{dx}(x^2 - 1) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x) = \boxed{\frac{2x}{3(x^2 - 1)^{2/3}}}.$$

39. Let $y = x\sqrt{x^2 - 1}$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \sqrt{x^2 - 1} + \sqrt{x^2 - 1} \frac{d}{dx} x = x \cdot \frac{1}{2}(x^2 - 1)^{-1/2} \frac{d}{dx}(x^2 - 1) + \sqrt{x^2 - 1} \\ &= \frac{x(2x)}{2(x^2 - 1)^{1/2}} + (x^2 - 1)^{1/2} = \boxed{\frac{x^2}{(x^2 - 1)^{1/2}} + (x^2 - 1)^{1/2}}. \end{aligned}$$

40. Let $y = x\sqrt{x^3 + 1}$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \sqrt{x^3 + 1} + \sqrt{x^3 + 1} \frac{d}{dx} x = x \cdot \frac{1}{2}(x^3 + 1)^{-1/2} \frac{d}{dx}(x^3 + 1) + \sqrt{x^3 + 1} \\ &= \frac{x(3x^2)}{2(x^3 + 1)^{1/2}} + (x^3 + 1)^{1/2} = \boxed{\frac{3x^3}{2(x^3 + 1)^{1/2}} + (x^3 + 1)^{1/2}}. \end{aligned}$$

41. Let $y = e^{\sqrt{x^2 - 9}}$. Then

$$y' = e^{\sqrt{x^2 - 9}} \frac{d}{dx} \sqrt{x^2 - 9} = e^{\sqrt{x^2 - 9}} \cdot \frac{1}{2}(x^2 - 9)^{-1/2} \frac{d}{dx}(x^2 - 9) = e^{(x^2 - 9)^{1/2}} \frac{2x}{2(x^2 - 9)^{1/2}} = \boxed{\frac{xe^{(x^2 - 9)^{1/2}}}{(x^2 - 9)^{1/2}}}.$$

42. Note that $\sqrt{e^x} = (e^x)^{1/2} = e^{x/2}$. Therefore, $y' = \boxed{\frac{1}{2}e^{x/2}}$.

43. Let $y = (x^2 \cos x)^{3/2}$. Then

$$\begin{aligned} y' &= \frac{3}{2}(x^2 \cos x)^{1/2} \frac{d}{dx}(x^2 \cos x) = \frac{3}{2}(x^2 \cos x)^{1/2} \left(x^2 \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x^2 \right) \\ &= \boxed{\frac{3}{2}(x^2 \cos x)^{1/2} (-x^2 \sin x + 2x \cos x)}. \end{aligned}$$

44. Let $y = (x^2 \sin x)^{3/2}$. Then

$$\begin{aligned} y' &= \frac{3}{2}(x^2 \sin x)^{1/2} \frac{d}{dx}(x^2 \sin x) = \frac{3}{2}(x^2 \sin x)^{1/2} \left(x^2 \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^2 \right) \\ &= \boxed{\frac{3}{2}(x^2 \sin x)^{1/2} (x^2 \cos x + 2x \sin x)}. \end{aligned}$$

45. Let $y = (x^2 - 3)^{3/2}(6x + 1)^{5/3}$. Then

$$\begin{aligned}
 y' &= (x^2 - 3)^{3/2} \frac{d}{dx} (6x + 1)^{5/3} + (6x + 1)^{5/3} \frac{d}{dx} (x^2 - 3)^{3/2} \\
 &= (x^2 - 3)^{3/2} \cdot \frac{5}{3} (6x + 1)^{2/3} \frac{d}{dx} (6x + 1) + (6x + 1)^{5/3} \cdot \frac{3}{2} (x^2 - 3)^{1/2} \frac{d}{dx} (x^2 - 3) \\
 &= 10(x^2 - 3)^{3/2}(6x + 1)^{2/3} + 3x(6x + 1)^{5/3}(x^2 - 3)^{1/2} \\
 &= (x^2 - 3)^{1/2}(6x + 1)^{2/3}(10(x^2 - 3) + 3x(6x + 1)) \\
 &= \boxed{(x^2 - 3)^{1/2}(6x + 1)^{2/3}(28x^2 + 3x - 30)}.
 \end{aligned}$$

46. Let $y = \frac{(2x^3 - 1)^{4/3}}{(3x + 4)^{5/2}}$. Then

$$\begin{aligned}
 y' &= \frac{(3x + 4)^{5/2} \frac{d}{dx} (2x^3 - 1)^{4/3} - (2x^3 - 1)^{4/3} \frac{d}{dx} (3x + 4)^{5/2}}{[(3x + 4)^{5/2}]^2} \\
 &= \frac{(3x + 4)^{5/2} \cdot \frac{4}{3} (2x^3 - 1)^{1/3} \frac{d}{dx} (2x^3 - 1) - (2x^3 - 1)^{4/3} \cdot \frac{5}{2} (3x + 4)^{3/2} \frac{d}{dx} (3x + 4)}{(3x + 4)^5} \\
 &= \frac{8x^2(2x^3 - 1)^{1/3}(3x + 4)^{5/2} - \frac{15}{2}(2x^3 - 1)^{4/3}(3x + 4)^{3/2}}{(3x + 4)^5} \\
 &= \frac{(2x^3 - 1)^{1/3}(3x + 4)^{3/2} (8x^2(3x + 4) - \frac{15}{2}(2x^3 - 1))}{(3x + 4)^5} \\
 &= \boxed{\frac{(2x^3 - 1)^{1/3}(9x^3 + 32x^2 + \frac{15}{2})}{(3x + 4)^{7/2}}}.
 \end{aligned}$$

47. Let $y = \sin^{-1}(4x)$. Then $y' = \frac{1}{\sqrt{1 - (4x)^2}} \frac{d}{dx} (4x) = \boxed{\frac{4}{\sqrt{1 - 16x^2}}}.$

48. Let $y = \cos^{-1} x^2$. Then $y' = -\frac{1}{\sqrt{1 - (x^2)^2}} \frac{d}{dx} x^2 = \boxed{-\frac{2x}{\sqrt{1 - x^4}}}.$

49. Let $y = \sec^{-1}(3x)$. Then $y' = \frac{1}{3x\sqrt{(3x)^2 - 1}} \frac{d}{dx} (3x) = \boxed{\frac{1}{x\sqrt{9x^2 - 1}}}.$

50. Let $y = \tan^{-1} \left(\frac{1}{x} \right)$. Then $y' = \frac{1}{1 + (\frac{1}{x})^2} \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{1}{1 + \frac{1}{x^2}} \left(-\frac{1}{x^2} \right) = \boxed{-\frac{1}{x^2 + 1}}.$

51. Let $y = \sin^{-1} e^x$. Then $y' = \frac{1}{\sqrt{1 - (e^x)^2}} \frac{d}{dx} e^x = \boxed{\frac{e^x}{\sqrt{1 - e^{2x}}}}.$

52. Let $y = \sin^{-1}(1 - x^2)$. Then

$$y' = \frac{1}{\sqrt{1 - (1 - x^2)^2}} \frac{d}{dx} (1 - x^2) = \frac{-2x}{\sqrt{1 - 1 + 2x^2 - x^4}} = -\frac{2x}{\sqrt{2x^2 - x^4}} = \boxed{-\frac{2x}{|x|\sqrt{2 - x^2}}}.$$

53. Let $y = x(\sin^{-1} x)$. Then $y' = x \frac{d}{dx} \sin^{-1} x + \sin^{-1} x \frac{d}{dx} x = \boxed{\frac{x}{\sqrt{1 - x^2}} + \sin^{-1} x}.$

54. Let $y = x \tan^{-1}(x+1)$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \tan^{-1}(x+1) + \tan^{-1}(x+1) \frac{d}{dx} x = \frac{x}{1+(x+1)^2} \frac{d}{dx} (x+1) + \tan^{-1}(x+1) \\ &= \boxed{\frac{x}{x^2+2x+2} + \tan^{-1}(x+1)}. \end{aligned}$$

55. Let $y = \tan^{-1}(\sin x)$. Then $y' = \frac{1}{1+\sin^2 x} \frac{d}{dx} \sin x = \boxed{\frac{\cos x}{1+\sin^2 x}}$.

56. Let $y = \sin(\tan^{-1} x)$. Then $y' = \cos(\tan^{-1} x) \frac{d}{dx} \tan^{-1} x = \boxed{\frac{\cos(\tan^{-1} x)}{1+x^2}}$.

57. Let $x^2 + y^2 = 4$. Then

$$\begin{aligned} \frac{d}{dx} [x^2 + y^2] &= \frac{d}{dx} 4 \\ \frac{d}{dx} x^2 + \frac{d}{dx} y^2 &= 0 \\ 2x + 2y \frac{dy}{dx} &= 0 \\ y' = \frac{dy}{dx} &= \boxed{-\frac{x}{y}}, \end{aligned}$$

provided $y \neq 0$. Next,

$$y'' = \frac{d}{dx} \left(-\frac{x}{y} \right) = -\frac{y \frac{d}{dx} x - x \frac{dy}{dx}}{y^2} = -\frac{y - x \left(-\frac{x}{y} \right)}{y^2} = \boxed{-\frac{y^2 + x^2}{y^3}},$$

provided $y \neq 0$. Using the original equation, $x^2 + y^2 = 4$, the second derivative can be simplified to

$$y'' = -\frac{4}{y^3}.$$

58. Let $x^2 - y^2 = 1$. Then

$$\begin{aligned} \frac{d}{dx} [x^2 - y^2] &= \frac{d}{dx} 1 \\ \frac{d}{dx} x^2 - \frac{d}{dx} y^2 &= 0 \\ 2x - 2y \frac{dy}{dx} &= 0 \\ y' = \frac{dy}{dx} &= \boxed{\frac{x}{y}}, \end{aligned}$$

provided $y \neq 0$. Next,

$$y'' = \frac{d}{dx} \left(\frac{x}{y} \right) = \frac{y \frac{d}{dx} x - x \frac{dy}{dx}}{y^2} = \frac{y - x \left(\frac{x}{y} \right)}{y^2} = \boxed{\frac{y^2 - x^2}{y^3}},$$

provided $y \neq 0$. Using the original equation, $x^2 - y^2 = 1$, the second derivative can be simplified to

$$y'' = -\frac{1}{y^3}.$$

59. Let $xy^2 + yx^2 = 2$. Then

$$\begin{aligned}
 \frac{d}{dx}[xy^2 + yx^2] &= \frac{d}{dx}2 \\
 \frac{d}{dx}(xy^2) + \frac{d}{dx}yx^2 &= 0 \\
 x\frac{d}{dx}y^2 + y^2\frac{d}{dx}x + y\frac{d}{dx}x^2 + x^2\frac{dy}{dx} &= 0 \\
 2xy\frac{dy}{dx} + y^2 + 2xy + x^2\frac{dy}{dx} &= 0 \\
 (2xy + x^2)\frac{dy}{dx} &= -(2xy + y^2) \\
 y' = \frac{dy}{dx} &= \boxed{-\frac{2xy + y^2}{2xy + x^2}},
 \end{aligned}$$

provided $2xy + x^2 \neq 0$. Next,

$$\begin{aligned}
 y'' &= -\frac{d}{dx}\left(\frac{2xy + y^2}{2xy + x^2}\right) = -\frac{(2xy + x^2)\frac{d}{dx}(2xy + y^2) - (2xy + y^2)\frac{d}{dx}(2xy + x^2)}{(2xy + x^2)^2} \\
 &= -\frac{(2xy + x^2)\left(2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx}\right) - (2xy + y^2)\left(2x\frac{dy}{dx} + 2y + 2x\right)}{(2xy + x^2)^2} \\
 &= -\frac{4xy^2 + 2x^2y - 4xy^2 - 4x^2y - 2y^3 - 2xy^2 + (4x^2y + 4xy^2 + 2x^3 + 2x^2y - 4x^2y - 2xy^2)\frac{dy}{dx}}{(2xy + x^2)^2} \\
 &= -\frac{-2(x^2y + xy^2 + y^3) + 2(x^2y + xy^2 + x^3)\frac{dy}{dx}}{(2xy + x^2)^2} = \frac{2(x^2 + xy + y^2)\left[y - x\left(-\frac{2xy + y^2}{2xy + x^2}\right)\right]}{(2xy + x^2)^2} \\
 &= -\frac{2(x^2 + xy + y^2)(2xy^2 + x^2y + 2x^2y + xy^2)}{(2xy + x^2)^3} = \frac{6(x^2 + xy + y^2)(xy^2 + x^2y)}{(2xy + x^2)^3} \\
 &= \boxed{\frac{6xy(x + y)(x^2 + xy + y^2)}{(2xy + x^2)^3}}.
 \end{aligned}$$

provided $2xy + x^2 \neq 0$. Using the original equation, $xy^2 + yx^2 = 2$, the second derivative can be simplified to

$$y'' = \frac{6(2)(x^2 + xy + y^2)}{(2xy + x^2)^3} = \frac{12(x^2 + xy + y^2)}{(2xy + x^2)^3}.$$

60. Let $4xy = x^2 + y^2$. Then

$$\begin{aligned}
 \frac{d}{dx}(4xy) &= \frac{d}{dx}[x^2 + y^2] \\
 4x\frac{dy}{dx} + 4y\frac{d}{dx}x &= \frac{d}{dx}x^2 + \frac{d}{dx}y^2 \\
 4x\frac{dy}{dx} + 4y &= 2x + 2y\frac{dy}{dx} \\
 (4x - 2y)\frac{dy}{dx} &= 2x - 4y \\
 y' = \frac{dy}{dx} &= \boxed{\frac{x - 2y}{2x - y}}.
 \end{aligned}$$

provided $2x - y \neq 0$. Next,

$$\begin{aligned}
 y'' &= \frac{d}{dx} \left(\frac{x - 2y}{2x - y} \right) = \frac{(2x - y) \frac{d}{dx}(x - 2y) - (x - 2y) \frac{d}{dx}(2x - y)}{(2x - y)^2} \\
 &= \frac{(2x - y) \left(1 - 2 \frac{dy}{dx} \right) - (x - 2y) \left(2 - \frac{dy}{dx} \right)}{(2x - y)^2} \\
 &= \frac{2x - y - 2x + 4y + (2y - 4x + x - 2y) \frac{dy}{dx}}{(2x - y)^2} \\
 &= \frac{3y - 3x \frac{x - 2y}{2x - y}}{(2x - y)^2} = \frac{3y(2x - y) - 3x(x - 2y)}{(2x - y)^3} \\
 &= \frac{6xy - 3y^2 - 3x^2 + 6xy}{(2x - y)^3} = \boxed{\frac{3(4xy - x^2 - y^2)}{(2x - y)^3}},
 \end{aligned}$$

provided $2x - y \neq 0$. Using the original equation, $4xy = x^2 + y^2$, the second derivative can be simplified to

$$y'' = \frac{3(0)}{(2x - y)^3} = 0.$$

61. Let $y = \sqrt{x^2 + 1}$. Then

$$y' = \frac{1}{2}(x^2 + 1)^{-1/2} \frac{d}{dx}(x^2 + 1) = \boxed{\frac{x}{(x^2 + 1)^{1/2}}},$$

and

$$\begin{aligned}
 y'' &= \frac{\sqrt{x^2 + 1} \frac{d}{dx}x - x \frac{d}{dx}\sqrt{x^2 + 1}}{(\sqrt{x^2 + 1})^2} = \frac{\sqrt{x^2 + 1} - x \cdot \frac{1}{2}(x^2 + 1)^{-1/2} \frac{d}{dx}(x^2 + 1)}{x^2 + 1} \\
 &= \frac{\sqrt{x^2 + 1} - \frac{x^2}{\sqrt{x^2 + 1}}}{x^2 + 1} = \frac{(x^2 + 1) - x^2}{(x^2 + 1)^{3/2}} = \boxed{\frac{1}{(x^2 + 1)^{3/2}}}.
 \end{aligned}$$

62. Let $y = \sqrt{4 - x^2}$. Then

$$y' = \frac{1}{2}(4 - x^2)^{-1/2} \frac{d}{dx}(4 - x^2) = \boxed{-\frac{x}{(4 - x^2)^{1/2}}},$$

and

$$\begin{aligned}
 y'' &= -\frac{\sqrt{4 - x^2} \frac{d}{dx}x - x \frac{d}{dx}\sqrt{4 - x^2}}{(\sqrt{4 - x^2})^2} = -\frac{\sqrt{4 - x^2} - x \cdot \frac{1}{2}(4 - x^2)^{-1/2} \frac{d}{dx}(4 - x^2)}{4 - x^2} \\
 &= -\frac{\sqrt{4 - x^2} + \frac{x^2}{\sqrt{4 - x^2}}}{4 - x^2} = -\frac{(4 - x^2) + x^2}{(4 - x^2)^{3/2}} = \boxed{-\frac{4}{(4 - x^2)^{3/2}}}.
 \end{aligned}$$

63. (a) Let $x^2 + y^2 = 5$. Then

$$\begin{aligned}
 \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}5 \\
 \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\
 2x + 2y \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= -\frac{x}{y}.
 \end{aligned}$$

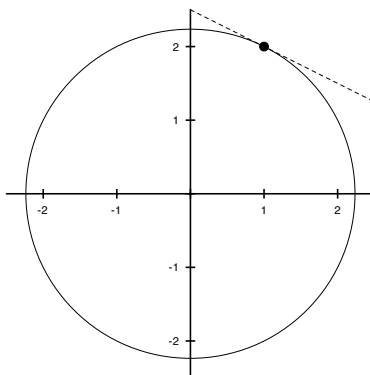
Therefore, the slope of the tangent line to the graph of the given equation at the point $(1, 2)$ is

$$m_{\text{tan}} = \boxed{-\frac{1}{2}}.$$

(b) The equation of the tangent line is

$$y - 2 = -\frac{1}{2}(x - 1) \quad \text{or} \quad \boxed{y = -\frac{x}{2} + \frac{5}{2}}.$$

(c) The figure below displays the graph of the equation $x^2 + y^2 = 5$ as the solid curve and the tangent line at $(1, 2)$ as the dashed curve.



64. (a) Let $(x - 3)^2 + (y + 4)^2 = 25$. Then

$$\begin{aligned} \frac{d}{dx}[(x - 3)^2 + (y + 4)^2] &= \frac{d}{dx}25 \\ \frac{d}{dx}(x - 3)^2 + \frac{d}{dx}(y + 4)^2 &= 0 \\ 2(x - 3)\frac{d}{dx}(x - 3) + 2(y + 4)\frac{d}{dx}(y + 4) &= 0 \\ 2(x - 3) + 2(y + 4)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x - 3}{y + 4}. \end{aligned}$$

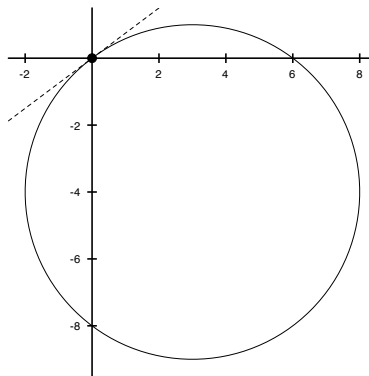
Therefore, the slope of the tangent line to the graph of the given equation at the point $(0, 0)$ is

$$m_{\text{tan}} = -\frac{0 - 3}{0 + 4} = \boxed{\frac{3}{4}}.$$

(b) The equation of the tangent line is

$$y - 0 = \frac{3}{4}(x - 0) \quad \text{or} \quad \boxed{y = \frac{3}{4}x}.$$

(c) The figure below displays the graph of the equation $(x - 3)^2 + (y + 4)^2 = 25$ as the solid curve and the tangent line at $(0, 0)$ as the dashed curve.



65. (a) Let $x^2 - y^2 = 8$. Then

$$\begin{aligned}\frac{d}{dx}[x^2 - y^2] &= \frac{d}{dx}8 \\ \frac{d}{dx}x^2 - \frac{d}{dx}y^2 &= 0 \\ 2x - 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y}.\end{aligned}$$

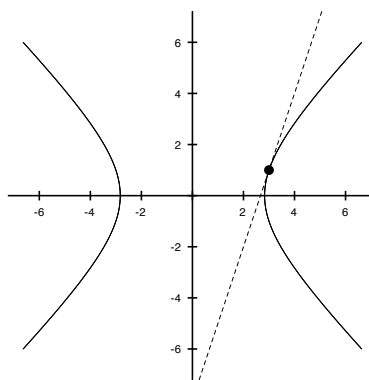
Therefore, the slope of the tangent line to the graph of the given equation at the point $(3, 1)$ is

$$m_{\text{tan}} = \frac{3}{1} = \boxed{3}.$$

- (b) The equation of the tangent line is

$$y - 1 = 3(x - 3) \quad \text{or} \quad \boxed{y = 3x - 8}.$$

- (c) The figure below displays the graph of the equation $x^2 - y^2 = 8$ as the solid curve and the tangent line at $(3, 1)$ as the dashed curve.



66. (a) Let $y^2 - 3x^2 = 6$. Then

$$\begin{aligned}\frac{d}{dx}[y^2 - 3x^2] &= \frac{d}{dx}6 \\ \frac{d}{dx}y^2 - 3\frac{d}{dx}x^2 &= 0 \\ 2y\frac{dy}{dx} - 6x &= 0 \\ \frac{dy}{dx} &= \frac{3x}{y}.\end{aligned}$$

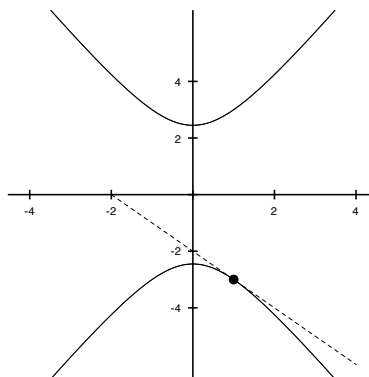
Therefore, the slope of the tangent line to the graph of the given equation at the point $(1, -3)$ is

$$m_{\text{tan}} = \frac{3(1)}{-3} = \boxed{-1}.$$

(b) The equation of the tangent line is

$$y - (-3) = -1(x - 1) \quad \text{or} \quad \boxed{y = -x - 2}.$$

(c) The figure below displays the graph of the equation $y^2 - 3x^2 = 6$ as the solid curve and the tangent line at $(1, -3)$ as the dashed curve.



67. (a) Let $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Then

$$\begin{aligned} \frac{d}{dx} \left[\frac{x^2}{4} + \frac{y^2}{3} \right] &= \frac{d}{dx} 1 \\ \frac{d}{dx} \left(\frac{x^2}{4} \right) + \frac{d}{dx} \left(\frac{y^2}{3} \right) &= 0 \\ \frac{1}{2}x + \frac{2}{3}y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{3x}{4y}. \end{aligned}$$

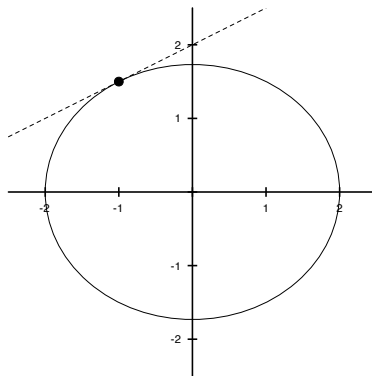
Therefore, the slope of the tangent line to the graph of the given equation at the point $\left(-1, \frac{3}{2}\right)$ is

$$m_{\text{tan}} = -\frac{3(-1)}{4(3/2)} = \boxed{\frac{1}{2}}.$$

(b) The equation of the tangent line is

$$y - \frac{3}{2} = \frac{1}{2}(x + 1) \quad \text{or} \quad \boxed{y = \frac{x}{2} + 2}.$$

(c) The figure below displays the graph of the equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$ as the solid curve and the tangent line at $\left(-1, \frac{3}{2}\right)$ as the dashed curve.



68. (a) Let $x^2 + \frac{y^2}{4} = 1$. Then

$$\begin{aligned}\frac{d}{dx} \left[x^2 + \frac{y^2}{4} \right] &= \frac{d}{dx} 1 \\ \frac{d}{dx} x^2 + \frac{d}{dx} \left(\frac{y^2}{4} \right) &= 0 \\ 2x + \frac{1}{2} y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{4x}{y}.\end{aligned}$$

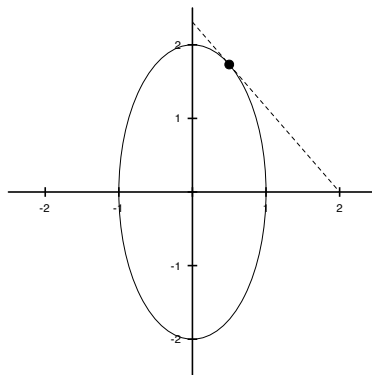
Therefore, the slope of the tangent line to the graph of the given equation at the point $\left(\frac{1}{2}, \sqrt{3}\right)$ is

$$m_{\text{tan}} = -\frac{4(1/2)}{\sqrt{3}} = \boxed{-\frac{2}{\sqrt{3}}}.$$

- (b) The equation of the tangent line is

$$y - \sqrt{3} = -\frac{2}{\sqrt{3}} \left(x - \frac{1}{2} \right) \quad \text{or} \quad \boxed{y = -\frac{2}{\sqrt{3}}x + \sqrt{3} + \frac{1}{\sqrt{3}}}.$$

- (c) The figure below displays the graph of the equation $x^2 + \frac{y^2}{4} = 1$ as the solid curve and the tangent line at $\left(\frac{1}{2}, \sqrt{3}\right)$ as the dashed curve.



69. Let $3x^2y + 2y^3 = 5x^2$. Then

$$\begin{aligned}\frac{d}{dx}[3x^2y + 2y^3] &= \frac{d}{dx}(5x^2) \\ \frac{d}{dx}(3x^2y) + \frac{d}{dx}(2y^3) &= 10x \\ 3x^2\frac{dy}{dx} + y\frac{d}{dx}(3x^2) + 6y^2\frac{dy}{dx} &= 10x \\ (3x^2 + 6y^2)\frac{dy}{dx} + 6xy &= 10x \\ \frac{dy}{dx} &= \frac{10x - 6xy}{3x^2 + 6y^2}\end{aligned}$$

and, at the point $(-1, 1)$,

$$\frac{dy}{dx} = \frac{10(-1) - 6(-1)(1)}{3(-1)^2 + 6(1)^2} = \boxed{-\frac{4}{9}}.$$

Next,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{10x - 6xy}{3x^2 + 6y^2}\right) = \frac{(3x^2 + 6y^2)\frac{d}{dx}(10x - 6xy) - (10x - 6xy)\frac{d}{dx}(3x^2 + 6y^2)}{(3x^2 + 6y^2)^2} \\ &= \frac{(3x^2 + 6y^2)\left(10 - 6x\frac{dy}{dx} - 6y\right) - (10x - 6xy)\left(6x + 12y\frac{dy}{dx}\right)}{(3x^2 + 6y^2)^2},\end{aligned}$$

and, at the point $(-1, 1)$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(3 + 6)\left(10 + 6\left(-\frac{4}{9}\right) - 6\right) - (-10 + 6)\left(-6 + 12\left(-\frac{4}{9}\right)\right)}{(3 + 6)^2} \\ &= \frac{9 \cdot \frac{4}{3} - (-4) \cdot \left(-\frac{34}{3}\right)}{9^2} = \boxed{-\frac{100}{243}}.\end{aligned}$$

70. Let $4x^3 + 2y^3 = x + y$. Then

$$\begin{aligned}\frac{d}{dx}[4x^3 + 2y^3] &= \frac{d}{dx}[x + y] \\ 12x^2 + 6y^2\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\ (6y^2 - 1)\frac{dy}{dx} &= 1 - 12x^2 \\ \frac{dy}{dx} &= \frac{1 - 12x^2}{6y^2 - 1}\end{aligned}$$

and, at the point $(0, 0)$,

$$\frac{dy}{dx} = \frac{1 - 12(0)^2}{6(0)^2 - 1} = \frac{1}{-1} = \boxed{-1}.$$

Next,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1 - 12x^2}{6y^2 - 1}\right) = \frac{(6y^2 - 1)\frac{d}{dx}(1 - 12x^2) - (1 - 12x^2)\frac{d}{dx}(6y^2 - 1)}{(6y^2 - 1)^2} \\ &= \frac{-24x(6y^2 - 1) - 12y(1 - 12x^2)\frac{dy}{dx}}{(6y^2 - 1)^2},\end{aligned}$$

and, at the point $(0, 0)$,

$$\frac{d^2y}{dx^2} = \frac{-24(0)(6(0)^2 - 1) - 12(0)(1 - 12(0)^2)(-1)}{(6(0)^2 - 1)^2} = \frac{0}{1} = \boxed{0}.$$

71. Because f and g are inverse functions,

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(0)} = \frac{1}{-2} = \boxed{-\frac{1}{2}}.$$

72. Because f and g are inverse functions,

$$g'(-2) = \frac{1}{f'(g(-2))} = \frac{1}{f'(1)} = \boxed{\frac{1}{4}}.$$

73. Because f and g are inverse functions,

$$f'(-2) = \frac{1}{g'(f(-2))} = \frac{1}{g'(3)} = \frac{1}{1/2} = \boxed{2}.$$

74. Because f and g are inverse functions,

$$f'(0) = \frac{1}{g'(f(0))} = \frac{1}{g'(-1)} = \frac{1}{-1/3} = \boxed{-3}.$$

75. Let $f(x) = x^3 + 2x$. First note that $f(0) = 0$, $f(1) = 3$, and $f'(x) = 3x^2 + 2$. Because g is the inverse function of f , it follows that $g(0) = 0$, $g(3) = 1$,

$$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(0)} = \frac{1}{3(0)^2 + 2} = \boxed{\frac{1}{2}},$$

and

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \boxed{\frac{1}{5}}.$$

76. Let $f(x) = 2x^3 + x - 3$. First note that $f(0) = -3$, $f(1) = 0$, and $f'(x) = 6x^2 + 1$. Because g is the inverse function of f , it follows that $g(-3) = 0$, $g(0) = 1$,

$$g'(-3) = \frac{1}{f'(g(-3))} = \frac{1}{f'(0)} = \frac{1}{6(0)^2 + 1} = \frac{1}{1} = \boxed{1},$$

and

$$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{f'(1)} = \frac{1}{6(1)^2 + 1} = \boxed{\frac{1}{7}}.$$

Applications and Extensions

77. Let $y = |3x| = 3|x| = 3\sqrt{x^2}$. Then

$$y' = 3 \cdot \frac{1}{2}(x^2)^{-1/2} \frac{d}{dx} x^2 = \frac{3x}{\sqrt{x^2}} = \boxed{\frac{3x}{|x|}}.$$

78. Let $y = |x^5| = \sqrt{(x^5)^2} = \sqrt{x^{10}}$. Then

$$y' = \frac{1}{2}(x^{10})^{-1/2} \frac{d}{dx} x^{10} = \frac{5x^9}{\sqrt{x^{10}}} = \boxed{\frac{5x^9}{|x^5|}}.$$

79. Let $y = |2x - 1| = \sqrt{(2x - 1)^2}$. Then

$$y' = \frac{1}{2}[(2x - 1)^2]^{-1/2} \frac{d}{dx}(2x - 1)^2 = \frac{2x - 1}{\sqrt{(2x - 1)^2}} \frac{d}{dx}(2x - 1) = \boxed{\frac{2(2x - 1)}{|2x - 1|}}.$$

80. Let $y = |5 - x^2| = \sqrt{(5 - x^2)^2}$. Then

$$y' = \frac{1}{2}[(5 - x^2)^2]^{-1/2} \frac{d}{dx}(5 - x^2)^2 = \frac{5 - x^2}{\sqrt{(5 - x^2)^2}} \frac{d}{dx}(5 - x^2) = \boxed{-\frac{2x(5 - x^2)}{|5 - x^2|}}.$$

81. Let $y = |\cos x| = \sqrt{\cos^2 x}$. Then

$$y' = \frac{1}{2}(\cos^2 x)^{-1/2} \frac{d}{dx} \cos^2 x = \frac{\cos x}{\sqrt{\cos^2 x}} \frac{d}{dx} \cos x = \boxed{-\frac{\cos x \sin x}{|\cos x|}}.$$

82. Let $y = |\sin x| = \sqrt{\sin^2 x}$. Then

$$y' = \frac{1}{2}(\sin^2 x)^{-1/2} \frac{d}{dx} \sin^2 x = \frac{\sin x}{\sqrt{\sin^2 x}} \frac{d}{dx} \sin x = \boxed{\frac{\sin x \cos x}{|\sin x|}}.$$

83. Let $y = \sin |x| = \sin \sqrt{x^2}$. Then

$$y' = \cos \sqrt{x^2} \frac{d}{dx} \sqrt{x^2} = \cos |x| \cdot \frac{1}{2}(x^2)^{-1/2} \frac{d}{dx} x^2 = \cos |x| \cdot \frac{x}{\sqrt{x^2}} = \boxed{\cos |x| \cdot \frac{x}{|x|}}.$$

84. Let $|x| + |y| = 1$ or $\sqrt{x^2} + \sqrt{y^2} = 1$. Then

$$\begin{aligned} \frac{d}{dx}[\sqrt{x^2} + \sqrt{y^2}] &= \frac{d}{dx} 1 \\ \frac{d}{dx} \sqrt{x^2} + \frac{d}{dx} \sqrt{y^2} &= 0 \\ \frac{1}{2}(x^2)^{-1/2} \frac{d}{dx} x^2 + \frac{1}{2}(y^2)^{-1/2} \frac{d}{dx} y^2 &= 0 \\ \frac{x}{\sqrt{x^2}} + \frac{y}{\sqrt{y^2}} \frac{dy}{dx} &= 0 \\ \frac{x}{|x|} + \frac{y}{|y|} \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \boxed{-\frac{x/|x|}{y/|y|}}, \end{aligned}$$

provided $y \neq 0$.

85. Let $x^{2/3} + y^{2/3} = 5$. Then

$$\begin{aligned} \frac{d}{dx}[x^{2/3} + y^{2/3}] &= \frac{d}{dx} 5 \\ \frac{d}{dx} x^{2/3} + \frac{d}{dx} y^{2/3} &= 0 \\ \frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}. \end{aligned}$$

The slope of the tangent line to the hypocycloid at the point $(1, 8)$ is therefore

$$m_{\text{tan}} = -\frac{8^{1/3}}{1^{1/3}} = -2,$$

so the equation of the tangent line is

$$y - 8 = -2(x - 1) \quad \text{or} \quad \boxed{y = -2x + 10}.$$

86. The line $x + 16y = 5$ has a slope of $-\frac{1}{16}$, while the slope of the tangent line to the graph of $y = \frac{1}{\sqrt{x}} = x^{-1/2}$ at the point (x, y) is

$$y' = -\frac{1}{2}x^{-3/2}.$$

The tangent line to the graph of $y = \frac{1}{\sqrt{x}}$ will therefore be parallel to the line $x + 16y = 5$ when

$$-\frac{1}{2}x^{-3/2} = -\frac{1}{16} \quad \text{or} \quad x^{3/2} = 8.$$

The solution of this last equation is $x = 4$, so the tangent line to the graph of $y = \frac{1}{\sqrt{x}}$ at the point

$\boxed{\left(4, \frac{1}{2}\right)}$ is parallel to the line $x + 16y = 5$.

87. Let $(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$. Then

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2 + 2x)^2 &= \frac{d}{dx}4(x^2 + y^2) \\ 2(x^2 + y^2 + 2x)\frac{d}{dx}(x^2 + y^2 + 2x) &= \frac{d}{dx}(4x^2) + \frac{d}{dx}(4y^2) \\ 2(x^2 + y^2 + 2x)\left(\frac{d}{dx}x^2 + \frac{d}{dx}y^2 + \frac{d}{dx}(2x)\right) &= 8x + 8y\frac{dy}{dx} \\ 2(x^2 + y^2 + 2x)\left(2x + 2y\frac{dy}{dx} + 2\right) &= 8x + 8y\frac{dy}{dx} \\ (x+1)(x^2 + y^2 + 2x) + y(x^2 + y^2 + 2x)\frac{dy}{dx} &= 2x + 2y\frac{dy}{dx} \\ y(x^2 + y^2 + 2x - 2)\frac{dy}{dx} &= 2x - (x+1)(x^2 + y^2 + 2x) \\ \frac{dy}{dx} &= \frac{2x - (x+1)(x^2 + y^2 + 2x)}{y(x^2 + y^2 + 2x - 2)}. \end{aligned}$$

- (a) The graph of the cardioid will have a horizontal tangent line when the numerator of the derivative is equal to zero but the denominator is not equal to zero. Now, $2x - (x+1)(x^2 + y^2 + 2x) = 0$ when $x^2 + y^2 + 2x = \frac{2x}{x+1}$. Substituting $\frac{2x}{x+1}$ for $x^2 + y^2 + 2x$ in the left-hand side of the cardioid equation yields $x^2 + y^2 = \frac{x^2}{(x+1)^2}$. Then,

$$2x = \frac{2x}{x+1} - \frac{x^2}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}.$$

Clearing fractions and collecting like terms leads to $2x^3 + 3x^2 = x^2(2x + 3) = 0$, so either $x = 0$ or $x = -\frac{3}{2}$.

- If $x = 0$, the numerator of the derivative reduces to $-y^2$, so y must be 0 for the numerator of the derivative to be 0. However, at the point $(0, 0)$, both the numerator and denominator of the derivative are 0. The tools needed to determine whether or not the derivative exists at the origin have not yet been covered, so we will ignore the origin for now.
- If $x = -\frac{3}{2}$, then

$$\left(-\frac{3}{2}\right)^2 + y^2 = \frac{(-3/2)^2}{(-3/2 + 1)^2} \quad \text{or} \quad y^2 = \frac{27}{4}.$$

Thus, $y = \pm \frac{3\sqrt{3}}{2}$. The denominator of the derivative is not 0 at the points $\left(-\frac{3}{2}, \pm \frac{3\sqrt{3}}{2}\right)$, so the graph of the cardioid has a horizontal tangent line at each.

In summary, the graph of the cardioid has a horizontal tangent line at the two points

$$\boxed{\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right) \quad \text{and} \quad \left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)}.$$

- (b) The graph of the cardioid will have a vertical tangent line when the denominator of the derivative is equal to zero but the numerator is not equal to zero. Now, $y(x^2 + y^2 + 2x - 2) = 0$ when $y = 0$ or when $x^2 + y^2 + 2x - 2 = 0$.

- If $y = 0$, then the cardioid equation reduces to $(x^2 + 2x)^2 = 4x^2$ or $x^4 + 4x^3 = 0$. Thus, $x = 0$ or $x = -4$. At the point $(0, 0)$, both the numerator and denominator of the derivative are 0. As noted above, the tools needed to determine whether or not the derivative exists at the origin have not yet been covered, so we will ignore the origin for now. At the point $(-4, 0)$, the numerator of the derivative is not zero, so the graph has a vertical tangent line at that point.
- If $x^2 + y^2 + 2x - 2 = 0$, then $x^2 + y^2 + 2x = 2$ and the cardioid equation yields $x^2 + y^2 = 1$. Subtracting $x^2 + y^2 = 1$ from $x^2 + y^2 + 2x = 2$, it follows that $2x = 1$, or $x = \frac{1}{2}$. Substituting this value into $x^2 + y^2 = 1$ then gives $y^2 = \frac{3}{4}$, or $y = \pm \frac{\sqrt{3}}{2}$. When $x^2 + y^2 + 2x = 2$, the numerator of the derivative is equal to -2 , so the graph has a vertical tangent line at both $\left(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)$.

In summary, the graph of the cardioid has a vertical tangent line at the three points

$$\boxed{(-4, 0), \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \text{and} \quad \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)}.$$

88. Let $(x^2 + y^2 + y)^2 = x^2 + y^2$. Then

$$\begin{aligned}
 \frac{d}{dx}(x^2 + y^2 + y)^2 &= \frac{d}{dx}(x^2 + y^2) \\
 2(x^2 + y^2 + y) \frac{d}{dx}(x^2 + y^2 + y) &= \frac{d}{dx}x^2 + \frac{d}{dx}y^2 \\
 2(x^2 + y^2 + y) \left(\frac{d}{dx}x^2 + \frac{d}{dx}y^2 + \frac{dy}{dx} \right) &= 2x + 2y \frac{dy}{dx} \\
 2(x^2 + y^2 + y) \left(2x + (2y + 1) \frac{dy}{dx} \right) &= 2x + 2y \frac{dy}{dx} \\
 2x(x^2 + y^2 + y) + (2y + 1)(x^2 + y^2 + y) \frac{dy}{dx} &= x + y \frac{dy}{dx} \\
 [(2y + 1)(x^2 + y^2 + y) - y] \frac{dy}{dx} &= x[1 - 2(x^2 + y^2 + y)] \\
 \frac{dy}{dx} &= \frac{x[1 - 2(x^2 + y^2 + y)]}{(2y + 1)(x^2 + y^2 + y) - y}.
 \end{aligned}$$

(a) The graph of the cardioid will have a horizontal tangent line when the numerator of the derivative is equal to zero but the denominator is not equal to zero. Now, $x[1 - 2(x^2 + y^2 + y)] = 0$ when $x = 0$ or when $x^2 + y^2 + y = \frac{1}{2}$.

- If $x = 0$, then the cardioid equation reduces to $(y^2 + y)^2 = y^2$, or $y^4 + 2y^2 = 0$. Thus $y = 0$ or $y = -2$. At the point $(0, 0)$, both the numerator and denominator of the derivative are 0. The tools needed to determine whether or not the derivative exists at the origin have not yet been covered, so we will ignore the origin for now. At the point $(0, -2)$, the denominator of the derivative is not zero, so the graph has a horizontal tangent line.
- If $x^2 + y^2 + y = \frac{1}{2}$, then the cardioid equation yields $x^2 + y^2 = \frac{1}{4}$. Subtracting the latter equation from the former, it follows that $y = \frac{1}{4}$. Substituting this value into either of the earlier equations yields $x^2 = \frac{3}{16}$, so that $x = \pm \frac{\sqrt{3}}{4}$. When $x^2 + y^2 + y = \frac{1}{2}$, the denominator of the derivative is equal to $\frac{1}{2}$, so the graph has a horizontal tangent line at both $\left(\pm \frac{\sqrt{3}}{4}, \frac{1}{4}\right)$.

In summary, the graph of the cardioid has a horizontal tangent line at the three points

$$\boxed{(0, -2), \quad \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right), \quad \text{and} \quad \left(-\frac{\sqrt{3}}{4}, \frac{1}{4}\right)}.$$

(b) The graph of the cardioid will have a vertical tangent line when the denominator of the derivative is equal to zero but the numerator is not equal to zero. Now, $(2y + 1)(x^2 + y^2 + y) - y = 0$ when $x^2 + y^2 + y = \frac{y}{2y + 1}$. Substituting $\frac{y}{2y + 1}$ for $x^2 + y^2 + y$ in the left-hand side of the cardioid equation yields $x^2 + y^2 = \frac{y^2}{(2y + 1)^2}$. Then,

$$y = \frac{y}{2y + 1} - \frac{y^2}{(2y + 1)^2} = \frac{y(2y + 1) - y^2}{(2y + 1)^2} = \frac{y^2 + y}{(2y + 1)^2}.$$

Clearing fractions and collecting like terms leads to $4y^3 + 3y^2 = y^2(4y + 3) = 0$, so either $y = 0$ or $y = -\frac{3}{4}$.

- If $y = 0$, the denominator of the derivative reduces to x^2 , so x must be 0 for the denominator of the derivative to be 0. As noted earlier, both the numerator and denominator of the

derivative are 0 at the point $(0,0)$, and the tools needed to determine whether or not the derivative exists at the origin have not yet been covered, so we will ignore the origin for now.

- If $y = -\frac{3}{4}$, then

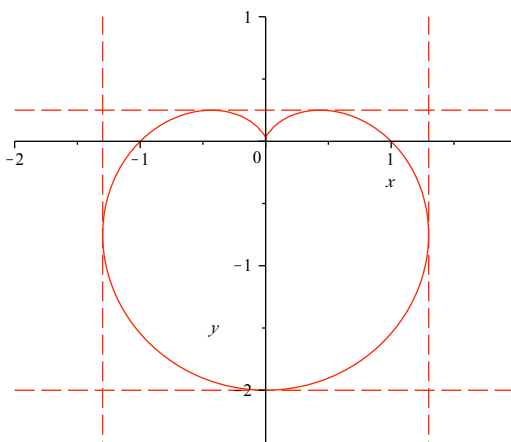
$$x^2 + \left(-\frac{3}{4}\right)^2 = \frac{(-3/4)^2}{(-3/2 + 1)^2} \quad \text{or} \quad x^2 = \frac{27}{16}.$$

Thus, $x = \pm \frac{3\sqrt{3}}{4}$. The numerator of the derivative is not 0 at the points $\left(\pm \frac{3\sqrt{3}}{4}, -\frac{3}{4}\right)$, so the graph of the cardioid has a vertical tangent line at each.

In summary, the graph of the cardioid has a vertical tangent line at the two points

$$\left(\frac{3\sqrt{3}}{4}, -\frac{3}{4}\right) \quad \text{and} \quad \left(-\frac{3\sqrt{3}}{4}, -\frac{3}{4}\right).$$

- (c) The figure below displays the graph of the cardioid as the solid curve and the horizontal and vertical tangent lines as dashed curves.



89. (a) Let $x + xy + 2y^2 = 6$. Then

$$\begin{aligned} \frac{d}{dx}[x + xy + 2y^2] &= \frac{d}{dx}6 \\ \frac{d}{dx}x + \frac{d}{dx}(xy) + \frac{d}{dx}(2y^2) &= 0 \\ 1 + x\frac{dy}{dx} + y\frac{d}{dx}x + 4y\frac{dy}{dx} &= 0 \\ 1 + y + (x + 4y)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-y-1}{x+4y}. \end{aligned}$$

- (b) The slope of the tangent line to the graph at the point $(2,1)$ is

$$m_{\text{tan}} = -\frac{1+1}{2+4(1)} = -\frac{1}{3},$$

so the equation of the tangent line is

$$y - 1 = -\frac{1}{3}(x - 2) \quad \text{or} \quad y = -\frac{1}{3}x + \frac{5}{3}.$$

(c) Solving

$$-\frac{1+y}{x+4y} = -\frac{1}{3}$$

yields $3+3y = x+4y$ or $x = 3-y$. Substituting this relationship into the equation $x+xy+2y^2 = 6$ then yields

$$3-y+(3-y)y+2y^2 = 6$$

$$3-y+3y-y^2+2y^2 = 6$$

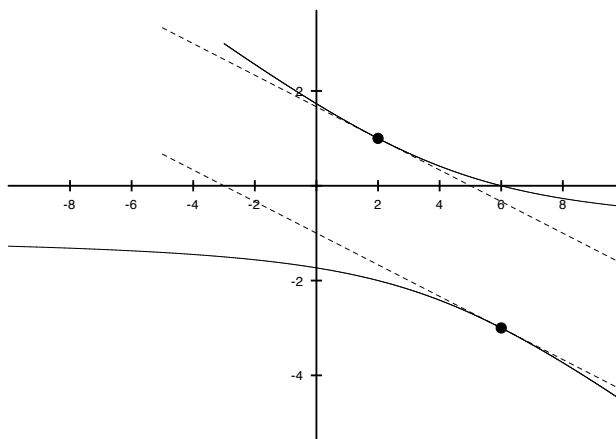
$$y^2+2y-3 = 0$$

$$(y+3)(y-1) = 0.$$

Then, $y = 1$, corresponding to the previously known point $(2, 1)$, or $y = -3$, corresponding to the point $(6, -3)$. The equation of the tangent line at the point $(6, -3)$ is

$$y - (-3) = -\frac{1}{3}(x - 6) \quad \text{or} \quad y = -\frac{1}{3}x - 1.$$

(d) The figure below displays the graph of $x+xy+2y^2 = 6$ as the solid curve and the tangent lines found in parts (b) and (c) as the dashed curves.



90. Let $(x^2 + y^2)^2 = x^2 - y^2$. Then

$$\frac{d}{dx}(x^2 + y^2)^2 = \frac{d}{dx}(x^2 - y^2)$$

$$2(x^2 + y^2) \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}x^2 - \frac{d}{dx}y^2$$

$$2(x^2 + y^2) \left(\frac{d}{dx}x^2 + \frac{d}{dx}y^2 \right) = 2x - 2y \frac{dy}{dx}$$

$$(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) = x - y \frac{dy}{dx}$$

$$2x(x^2 + y^2) + 2y(x^2 + y^2) \frac{dy}{dx} = x - y \frac{dy}{dx}$$

$$y[2(x^2 + y^2) + 1] \frac{dy}{dx} = x[1 - 2(x^2 + y^2)]$$

$$\frac{dy}{dx} = \frac{x[1 - 2(x^2 + y^2)]}{y[2(x^2 + y^2) + 1]}.$$

The graph of the lemniscate will have a horizontal tangent line when the numerator of the derivative is equal to zero but the denominator is not equal to zero. Now, $x[1 - 2(x^2 + y^2)] = 0$ when $x = 0$ or when $x^2 + y^2 = \frac{1}{2}$.

- If $x = 0$, then the lemniscate equation reduces to $y^4 = -y^2$. The only solution to this equation is $y = 0$. At the point $(0, 0)$, both the numerator and the denominator of the derivative are equal to 0, suggesting that the graph of the lemniscate might not have a horizontal tangent line at the origin. This is confirmed in the graph in the text.
- If $x^2 + y^2 = \frac{1}{2}$, then the lemniscate equation yields $x^2 - y^2 = \frac{1}{4}$. Adding the equations $x^2 + y^2 = \frac{1}{2}$ and $x^2 - y^2 = \frac{1}{4}$, it follows that $2x^2 = \frac{3}{4}$ so that $x = \pm \frac{\sqrt{6}}{4}$. Subtracting the two equations yields $2y^2 = \frac{1}{4}$ so that $y = \pm \frac{\sqrt{2}}{4}$. Also, note that when $x^2 + y^2 = \frac{1}{2}$, the denominator of the derivative is equal to $2y$, which is not 0 for the y -values just calculated.

Thus, the graph of the lemniscate has a horizontal tangent line at the four points

$$\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4} \right), \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4} \right), \left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4} \right), \text{ and } \left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4} \right).$$

91. (a) Let $y = \sin^{-1} \frac{x}{2}$. Then

$$y' = \frac{1}{\sqrt{1 - (x/2)^2}} \frac{d}{dx} \left(\frac{x}{2} \right) = \frac{2}{\sqrt{4 - x^2}} \cdot \frac{1}{2} = \frac{1}{\sqrt{4 - x^2}}.$$

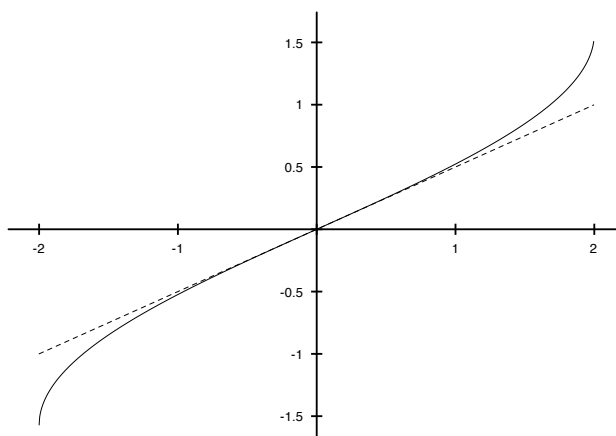
The slope of the tangent line to the graph at the origin is then

$$m_{\text{tan}} = \frac{1}{\sqrt{4 - 0^2}} = \frac{1}{2},$$

so the equation of the tangent line is

$$y - 0 = \frac{1}{2}(x - 0) \quad \text{or} \quad \boxed{y = \frac{1}{2}x}.$$

- (b) The figure below displays the graph of $y = \sin^{-1} \frac{x}{2}$ as the solid curve, and the graph of the tangent line at the origin as the dashed curve.



92. Let $P + \frac{a}{V^2} = \frac{C}{V-b}$, where C , a , and b are constants. Then

$$\begin{aligned}\frac{d}{dP} \left(P + \frac{a}{V^2} \right) &= \frac{d}{dP} \left(\frac{C}{V-b} \right) \\ \frac{d}{dP} P + \frac{d}{dP} \left(\frac{a}{V^2} \right) &= -\frac{C}{(V-b)^2} \frac{d}{dP} (V-b) \\ 1 - \frac{2a}{V^3} \frac{dV}{dP} &= -\frac{C}{(V-b)^2} \frac{dV}{dP} \\ 1 &= \left(\frac{2a}{V^3} - \frac{C}{(V-b)^2} \right) \frac{dV}{dP} \\ \boxed{\left(\frac{2a}{V^3} - \frac{C}{(V-b)^2} \right)^{-1}} &= \frac{dV}{dP}.\end{aligned}$$

93. Let $m(v^2 - v_0^2) = k(s_0^2 - s^2)$. Then

$$\begin{aligned}\frac{d}{dt}[m(v^2 - v_0^2)] &= \frac{d}{dt}[k(s_0^2 - s^2)] \\ \frac{d}{dt}(mv^2) - \frac{d}{dt}(mv_0^2) &= \frac{d}{dt}ks_0^2 - \frac{d}{dt}(ks^2) \\ 2mv \frac{dv}{dt} - 0 &= 0 - 2ks \frac{ds}{dt} \\ mva &= -ksv \\ ma &= -ks,\end{aligned}$$

where in moving from the next to the last line to the last line, the fact that $v > 0$ was used.

94. (a) Let

$$P(t) = \frac{300}{1 + \frac{1}{6}\sqrt{t}} + 100 = 300 \left(1 + \frac{1}{6}\sqrt{t} \right)^{-1} + 100.$$

Then

$$\begin{aligned}P'(t) &= -300 \left(1 + \frac{1}{6}\sqrt{t} \right)^{-2} \frac{d}{dt} \left(1 + \frac{1}{6}\sqrt{t} \right) = -300 \left(1 + \frac{1}{6}\sqrt{t} \right)^{-2} \frac{1}{12} t^{-1/2} \\ &= \boxed{-\frac{25}{\sqrt{t}} \left(1 + \frac{1}{6}\sqrt{t} \right)^{-2}}.\end{aligned}$$

- (b) Using the result from part (a), $P'(0)$ is undefined,

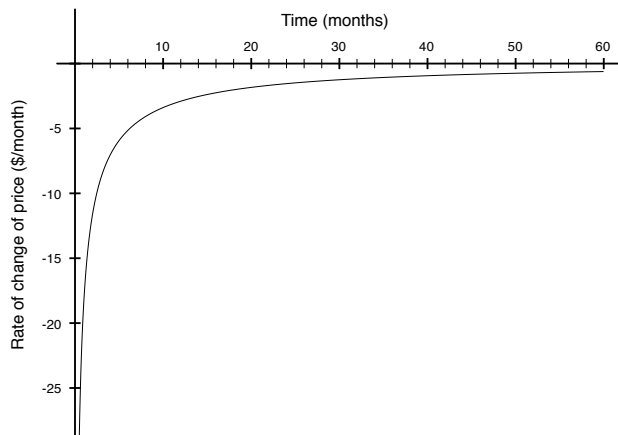
$$P'(16) = -\frac{25}{\sqrt{16}} \left(1 + \frac{1}{6}\sqrt{16} \right)^{-2} = -\frac{25}{4} \left(\frac{5}{3} \right)^{-2} = -\frac{25}{4} \cdot \frac{9}{25} = -\frac{9}{4} = \boxed{-2.25 \text{ dollars/month}},$$

and

$$P'(49) = -\frac{25}{\sqrt{49}} \left(1 + \frac{1}{6}\sqrt{49} \right)^{-2} = -\frac{25}{7} \left(\frac{13}{6} \right)^{-2} = -\frac{25}{7} \cdot \frac{36}{169} = -\frac{900}{1183} \approx \boxed{-0.76 \text{ dollars/month}}.$$

Thus, 16 months from now, the price of a tablet will be decreasing at the rate of \$2.25 per month, while 49 months from now, the price of a tablet will be decreasing at the rate of \$0.76 per month.

- (c) The figure below displays the graph of $P'(t)$. The graph appears to have a vertical asymptote at $t = 0$ and is always negative but increasing. This supports the answers in part (b), where it was found that $P'(49)$ was greater than $P'(16)$.



95. Let $z = x^{0.5}y^{0.4}$ and suppose z is held constant. Then

$$\begin{aligned}
 \frac{d}{dx}z &= \frac{d}{dx}[x^{0.5}y^{0.4}] \\
 0 &= x^{0.5}\frac{d}{dx}y^{0.4} + y^{0.4}\frac{d}{dx}x^{0.5} \\
 0 &= 0.4x^{0.5}y^{-0.6}\frac{dy}{dx} + 0.5y^{0.4}x^{-0.5} \\
 -0.4x^{0.5}y^{-0.6}\frac{dy}{dx} &= 0.5y^{0.4}x^{-0.5} \\
 \frac{dy}{dx} &= -\frac{0.5y^{0.4}x^{-0.5}}{0.4x^{0.5}y^{-0.6}} = -\frac{5y}{4x}.
 \end{aligned}$$

96. (a) Let $T = f(n) = Cn\sqrt{n-b}$, where C and b are constants. Then the rate of change in time with respect to the number of words to be memorized is

$$\begin{aligned}
 f'(n) &= \frac{d}{dn}[Cn\sqrt{n-b}] = Cn\frac{d}{dn}\sqrt{n-b} + \sqrt{n-b}\frac{d}{dn}(Cn) \\
 &= \frac{1}{2}Cn(n-b)^{-1/2}\frac{d}{dn}(n-b) + C\sqrt{n-b} \\
 &= \boxed{\frac{Cn}{2\sqrt{n-b}} + C\sqrt{n-b}}.
 \end{aligned}$$

(b) Let $C = 2$ and $b = 2$. Then

$$f'(10) = \frac{2(10)}{2\sqrt{10-2}} + 2\sqrt{10-2} = \frac{10}{\sqrt{8}} + 2\sqrt{8} = \frac{13\sqrt{2}}{2} \approx \boxed{9.2 \text{ minutes/word}}$$

and

$$f'(30) = \frac{2(30)}{2\sqrt{30-2}} + 2\sqrt{30-2} = \frac{30}{\sqrt{28}} + 2\sqrt{28} = \frac{43\sqrt{7}}{7} \approx \boxed{16.3 \text{ minutes/word}}.$$

(c) Answers will vary. One possible response is that $f'(30) > f'(10)$ implies it takes more time to memorize an additional word added to a longer list of words than it does to memorize an additional word added to a shorter list of words.

97. (a) Let $x^3 + y^3 = 2xy$. Then

$$\begin{aligned}\frac{d}{dx}[x^3 + y^3] &= \frac{d}{dx}(2xy) \\ \frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= 2x\frac{dy}{dx} + y\frac{d}{dx}(2x) \\ 3x^2 + 3y^2\frac{dy}{dx} &= 2x\frac{dy}{dx} + 2y \\ (3y^2 - 2x)\frac{dy}{dx} &= 2y - 3x^2 \\ y' = \frac{dy}{dx} &= \frac{2y - 3x^2}{3y^2 - 2x} = \boxed{\frac{3x^2 - 2y}{2x - 3y^2}}.\end{aligned}$$

- (b) The slope of the line tangent to the graph at the point $(1, 1)$ is

$$m_{\tan} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = \frac{-1}{1} = -1,$$

so the equation of the tangent line is

$$y - 1 = -1(x - 1) \quad \text{or} \quad \boxed{y = -x + 2}.$$

- (c) The tangent line to the graph of the Folium of Descartes will be horizontal when the numerator of the derivative is equal to zero but the denominator is not equal to zero. Now, $2y - 3x^2 = 0$ when $y = \frac{3}{2}x^2$. Substituting this expression into the equation for the Folium of Descartes yields

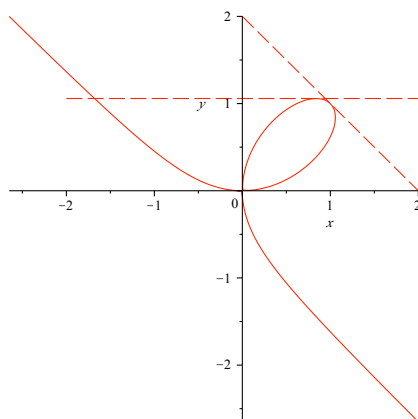
$$x^3 + \frac{27}{8}x^6 = 3x^3 \quad \text{or} \quad x^3 \left(\frac{27}{8}x^3 - 2 \right) = 0.$$

Thus, $x^3 = 0$ or $x^3 = \frac{16}{27}$, and finally, $x = 0$ or $x = \left(\frac{16}{27}\right)^{1/3} = \frac{2^{4/3}}{3}$.

- If $x = 0$, then $y = \frac{3}{2}x^2 = 0$. At the point $(0,0)$, both the numerator and the denominator of the derivative are equal to 0. The tools needed to determine whether or not the derivative exists at the origin have not yet been covered, so we will ignore the origin for now.
- If $x = \frac{2^{4/3}}{3}$, then $y = \frac{3}{2}x^2 = \frac{3}{2}\left(\frac{2^{4/3}}{3}\right)^2 = \frac{2^{5/3}}{3}$. The denominator of the derivative is not

zero at the point $\left(\frac{2^{4/3}}{3}, \frac{2^{5/3}}{3}\right)$, so the Folium of Descartes has a horizontal tangent line at this point.

- (d) The figure below displays the graph of the Folium of Descartes as the solid curve and the tangent lines from parts (b) and (c) as dashed curves.



98. Let n be an even positive integer. Then the slope of the tangent line to the graph of $y = \sqrt[n]{x}$ at the point $(1, 1)$ is

$$y'(1) = \frac{1}{n}(1)^{1/n-1} = \frac{1}{n},$$

while the slope of the tangent line to the graph of $y = x^n$ at the point $(-1, 1)$ is

$$y'(-1) = n(-1)^{n-1} = -n.$$

Because these slopes are negative reciprocals of one another, the two lines are perpendicular.

99. Let $y = \sqrt{25 - x^2}$. Then

$$y' = \frac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2) = \frac{1}{2\sqrt{25 - x^2}} \cdot (-2x) = -\frac{x}{\sqrt{25 - x^2}}.$$

The tangent line to the graph of $y = \sqrt{25 - x^2}$ at the point (x_0, y_0) has slope

$$m_{\text{tan}} = -\frac{x_0}{\sqrt{25 - x_0^2}}.$$

For the tangent line to be parallel to the line $y = x - 1$, the slope of the tangent line must equal 1. Solving the equation

$$-\frac{x_0}{\sqrt{25 - x_0^2}} = 1$$

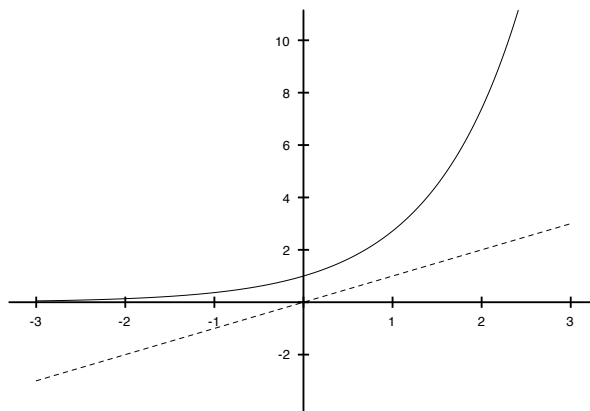
yields

$$-x_0 = \sqrt{25 - x_0^2} \quad \text{so that} \quad x_0^2 = 25 - x_0^2 \quad \text{and} \quad x_0 = \pm \frac{5\sqrt{2}}{2}.$$

The positive value for x_0 is extraneous because the slope of the tangent line at this location is -1 , so the only point on the graph of $y = \sqrt{25 - x^2}$ at which the tangent line is parallel to the line $y = x - 1$ is

$$\left(-\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \right).$$

100. Consider the equation $x + y = e^{x+y}$. The implicit differentiation calculations, and the conclusions drawn from them, that are presented in the problem statement hold for all points (x, y) whose coordinates satisfy the original implicit equation. So, let's suppose the coordinates of the point (x, y) satisfy the equation $x + y = e^{x+y}$. Further, suppose that $x + y = c$ for some real number c . Then, $c = e^c$. Now, consider the figure below, which displays the graphs of the function $f(c) = e^c$ (the solid curve) and $g(c) = c$ (the dashed curve). The two graphs do not intersect, which means that there is no real number c such that $f(c) = g(c)$, or $e^c = c$. Thus, the equation $x + y = e^{x+y}$ is not satisfied by any ordered pair (x, y) .



101. Let $x^n y^m + x^m y^n = k$, where k , m , and n are constants. Then

$$\begin{aligned}\frac{d}{dx}(x^n y^m + x^m y^n) &= \frac{d}{dx}k \\ \frac{d}{dx}(x^n y^m) + \frac{d}{dx}(x^m y^n) &= 0 \\ x^n \frac{d}{dx}y^m + y^m \frac{d}{dx}x^n + x^m \frac{d}{dx}y^n + y^n \frac{d}{dx}x^m &= 0 \\ mx^n y^{m-1} \frac{dy}{dx} + nx^{n-1} y^m + nx^m y^{n-1} \frac{dy}{dx} + mx^{m-1} y^n &= 0,\end{aligned}$$

so that

$$(mx^n y^{m-1} + nx^m y^{n-1}) \frac{dy}{dx} = -(nx^{n-1} y^m + mx^{m-1} y^n)$$

and

$$\begin{aligned}\frac{dy}{dx} &= -\frac{nx^{n-1} y^m + mx^{m-1} y^n}{mx^n y^{m-1} + nx^m y^{n-1}} \cdot \frac{xy}{xy} \\ &= -\frac{y(nx^n y^m + mx^m y^n)}{x(mx^n y^m + nx^m y^n)} = -\frac{x^m y^m \cdot y(nx^{n-m} + my^{n-m})}{x^m y^m \cdot x(mx^{n-m} + ny^{n-m})} \\ &= -\frac{y(nx^{n-m} + my^{n-m})}{x(mx^{n-m} + ny^{n-m})} = -\frac{y(nx^r + my^r)}{x(mx^r + ny^r)},\end{aligned}$$

where $r = n - m$.

102. Let $g(x) = \cos^{-1}(\cos x)$. Then

$$g'(x) = -\frac{1}{\sqrt{1 - \cos^2 x}} \frac{d}{dx} \cos x = -\frac{1}{\sqrt{\sin^2 x}} \cdot (-\sin x) = \frac{\sin x}{|\sin x|}.$$

103. $\frac{d}{dx} \tan^{-1}(\cot x) = \frac{1}{1 + \cot^2 x} \frac{d}{dx} \cot x = \frac{1}{\csc^2 x} \cdot (-\csc^2 x) = -1$.

104. For all real numbers x ,

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1 + x^2},$$

while for $x \neq 0$,

$$\frac{d}{dx} \tan^{-1} \frac{1}{x} = \frac{1}{1 + (1/x)^2} \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{x^2}{1 + x^2} \cdot \left(-\frac{1}{x^2} \right) = -\frac{1}{1 + x^2}.$$

Thus, for all $x \neq 0$,

$$\frac{d}{dx} \cot^{-1} x = \frac{d}{dx} \tan^{-1} \frac{1}{x}.$$

105. Let $y = \sin^{-1} x + \cos^{-1} x$. Then

$$y' = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0.$$

Thus, $y(x)$ is a constant function. Because

$$y(0) = \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2},$$

it follows that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

106. Let $y = \tan^{-1} x + \cot^{-1} x$. Then

$$y' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$$

Thus, $y(x)$ is a constant function. Because

$$y(1) = \tan^{-1} 1 + \cot^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$

it follows that

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

107. Consider the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) &= \frac{d}{dx} 1 \\ \frac{d}{dx} \left(\frac{x^2}{a^2} \right) + \frac{d}{dx} \left(\frac{y^2}{b^2} \right) &= 0 \\ \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{b^2 x}{a^2 y}. \end{aligned}$$

The slope of the tangent line to the graph of the ellipse at (x_0, y_0) is then

$$m_{\tan} = -\frac{b^2 x_0}{a^2 y_0},$$

so the equation of the tangent line is

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0) \quad \text{or} \quad b^2 x x_0 + a^2 y y_0 = b^2 x_0^2 + a^2 y_0^2.$$

Dividing by $a^2 b^2$ then yields

$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1,$$

because (x_0, y_0) is a point on the graph of the ellipse.

108. Let $x^{2/3} + y^{2/3} = a^{2/3}$, where $a > 0$ is a constant. Then

$$\begin{aligned} \frac{d}{dx}(x^{2/3} + y^{2/3}) &= \frac{d}{dx} a^{2/3} \\ \frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}. \end{aligned}$$

Thus, the slope of the tangent line to the graph of a hypocycloid is $-\frac{y^{1/3}}{x^{1/3}}$ at any point for which $x \neq 0$.

109. Let $P = (x_0, y_0)$ be a point on the circle $x^2 + y^2 = R^2$ with $x_0 \neq 0$ and $y_0 \neq 0$. The slope of the segment $\overline{OP}$, where O is the center of the circle, is y_0/x_0 . Applying implicit differentiation to the

equation of the circle yields

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}R^2 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}.\end{aligned}$$

The slope of the tangent line to the graph of the circle at the point $P = (x_0, y_0)$ is then

$$m_{\text{tan}} = -\frac{x_0}{y_0}.$$

The slope of the segment $\overline{OP}$ and the slope of the tangent line are negative reciprocals of one another, which means that the segment $\overline{OP}$ and the tangent line are perpendicular.

Now, what if $x_0 = 0$ or $y_0 = 0$. If $x_0 = 0$, then $y_0 = \pm R$. The segment $\overline{OP}$ is vertical while the tangent line is horizontal, so the segment $\overline{OP}$ and the tangent line are perpendicular. On the other hand, if $y_0 = 0$, then $x_0 = \pm R$. The segment $\overline{OP}$ is now horizontal while the tangent line is vertical, so the segment $\overline{OP}$ and the tangent line are again perpendicular.

Challenge Problems

110. As the line parallel to the secant line containing points A and B is moved upward, it will last touch the graph of f when it coincides with a tangent line. Because the secant line through points A and B has slope

$$m_{\text{sec}} = \frac{2-1}{5-2} = \frac{1}{3},$$

the tangent line must also have this slope. Therefore the x -coordinate of the last point on the graph of f before the secant loses contact with the graph must satisfy

$$f'(x) = \frac{1}{2}(x-1)^{-1/2} \frac{d}{dx}(x-1) = \frac{1}{2\sqrt{x-1}} = \frac{1}{3}.$$

Solving this equation for x yields

$$\sqrt{x-1} = \frac{3}{2} \quad \text{so that} \quad x = \frac{9}{4} + 1 = \frac{13}{4}.$$

The y -coordinate of the required point is then

$$f\left(\frac{13}{4}\right) = \sqrt{\frac{13}{4} - 1} = \sqrt{\frac{9}{4}} = \frac{3}{2}.$$

Thus, the last point on the graph of f before the line parallel to the secant line AB loses contact with the graph of f is

$$\boxed{\left(\frac{13}{4}, \frac{3}{2}\right)}.$$

111. (a) For the curve $xy = c_1$, implicit differentiation yields

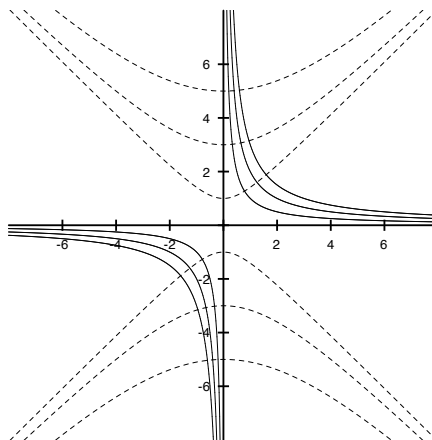
$$x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x};$$

for the curve $-x^2 + y^2 = c_2$, implicit differentiation yields

$$-2x + 2y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{y}.$$

Suppose the point (x_0, y_0) lies on both curves. At this point, the slope of the tangent line to the curve $xy = c_1$ is $-y_0/x_0$, while the slope of the tangent line to the curve $-x^2 + y^2 = c_2$ is x_0/y_0 . Because these slopes are negative reciprocals of one another, the two tangent lines are perpendicular. Therefore, the graphs of $xy = c_1$ and $-x^2 + y^2 = c_2$ are orthogonal.

- (b) The figure below displays the graphs of $xy = c_1$ with $c_1 = 1, 2$, and 3 as the solid curves and the graphs of $-x^2 + y^2 = c_2$ with $c_2 = 1, 9$, and 25 as the dashed curves. The value of both c_1 and c_2 increase as the graph moves further from the origin.



112. The parabolas $y^2 = 2ax + a^2$ and $y^2 = a^2 - 2ax$ intersect when

$$2ax + a^2 = a^2 - 2ax \quad \text{or when} \quad x = 0.$$

Substituting $x = 0$ into either parabola equation yields $y^2 = a^2$, so that $y = \pm a$. Therefore, the parabolas $y^2 = 2ax + a^2$ and $y^2 = a^2 - 2ax$ intersect at the points $(0, \pm a)$. For the parabola $y^2 = 2ax + a^2$, implicit differentiation yields

$$2y \frac{dy}{dx} = 2a \quad \text{or} \quad \frac{dy}{dx} = \frac{a}{y};$$

for the parabola $y^2 = a^2 - 2ax$, implicit differentiation yields

$$2y \frac{dy}{dx} = -2a \quad \text{or} \quad \frac{dy}{dx} = -\frac{a}{y}.$$

At the point $(0, a)$, the slopes of the tangent lines to the two parabolas are 1 and -1 , respectively, while at the point $(0, -a)$, the slopes of the tangent lines to the two parabolas are -1 and 1, respectively. These slopes are negative reciprocals of one another for any $a > 0$, so the graphs of the two parabolas $y^2 = 2ax + a^2$ and $y^2 = a^2 - 2ax$ are orthogonal for any $\boxed{a > 0}$.

113. Let p and $q > 0$ be integers. The functions $f(x) = x^p$ and $g(x) = x^{1/q} = \sqrt[q]{x}$ are differentiable on their domains, so the composite function $h(x) = f(g(x)) = (x^{1/q})^p = x^{p/q}$ is also differentiable on its domain.

114. Let y be an algebraic function of x and suppose

$$P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_{n-1}(x)y + P_n(x) = 0.$$

By implicit differentiation,

$$P_0(x) \cdot ny^{n-1}y' + y^n P_0'(x) + P_1(x) \cdot (n-1)y^{n-2}y' + y^{n-1}P_1'(x) + \cdots \\ + P_{n-1}(x) \cdot y' + yP_{n-1}'(x) + P_n'(x) = 0,$$

or

$$y' = -\frac{y^n P_0'(x) + y^{n-1}P_1'(x) + \cdots + yP_{n-1}'(x) + P_n'(x)}{nP_0(x)y^{n-1} + (n-1)P_1(x)y^{n-2} + \cdots + P_{n-1}(x)}.$$

115. Note that $x^{p/q} = (x^{1/q})^p$. Therefore,

$$\begin{aligned} \frac{d}{dx}x^{p/q} &= p(x^{1/q})^{p-1} \cdot \frac{d}{dx}x^{1/q} = p(x^{1/q})^{p-1} \cdot \frac{1}{q}x^{(1/q)-1} \\ &= \frac{p}{q}x^{(p/q)-(1/q)+(1/q)-1} = \frac{p}{q}x^{(p/q)-1}. \end{aligned}$$

116. (a) Let $y = \cot^{-1}x - \tan^{-1}\frac{1}{x}$. By Problem 104,

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}x - \frac{d}{dx}\tan^{-1}\frac{1}{x} = 0$$

for all $x \neq 0$. Thus, on the intervals $(-\infty, 0)$ and $(0, \infty)$, $y = \cot^{-1}x - \tan^{-1}\frac{1}{x}$ must be a constant function, but the constants can be different on the two intervals. For $x = -1$,

$$y = \cot^{-1}(-1) - \tan^{-1}(-1) = \frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \pi,$$

while for $x = 1$,

$$y = \cot^{-1}1 - \tan^{-1}1 = \frac{\pi}{4} - \frac{\pi}{4} = 0.$$

Thus,

$$\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x}, & x > 0 \\ \tan^{-1}\frac{1}{x} + \pi, & x < 0. \end{cases}$$

(b) The function $y = \cot^{-1}x - \tan^{-1}\frac{1}{x}$ is discontinuous at $x = 0$.

3.3: Derivatives of Logarithmic Functions

Concepts and Vocabulary

1. $\frac{d}{dx} \ln x = \boxed{\frac{1}{x}}$ for $x > 0$.
2. $\boxed{\text{True}}$. $\frac{d}{dx} x^e = ex^{e-1}$ because e is a constant so x^e is a power function.
3. $\boxed{\text{False}}$. $\frac{d}{dx} \ln[x \sin^2 x] = \frac{d}{dx} \ln x + \frac{d}{dx} \ln \sin^2 x$.
4. $\boxed{\text{False}}$. $\frac{d}{dx} \ln \pi = 0$ because $\ln \pi$ is a constant.
5. $\frac{d}{dx} \ln |x| = \boxed{\frac{1}{x}}$ for all $x \neq 0$.
6. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \boxed{e}$.

Skill Building

7. Let $y = 5 \ln x$. Then $y' = \boxed{\frac{5}{x}}$.
8. Let $y = -3 \ln x$. Then $y' = \boxed{-\frac{3}{x}}$.
9. Let $y = \log_2 u$. Then $y' = \boxed{\frac{1}{u \ln 2}}$.
10. Let $y = \log_3 u$. Then $y' = \boxed{\frac{1}{u \ln 3}}$.
11. Let $y = (\cos x)(\ln x)$. Then

$$y' = \cos x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \cos x = \boxed{\frac{1}{x} \cos x - (\sin x)(\ln x)}.$$
12. Let $y = (\sin x)(\ln x)$. Then

$$y' = \sin x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \sin x = \boxed{\frac{1}{x} \sin x + (\cos x)(\ln x)}.$$
13. Let $y = \ln(3x)$. Then $y' = \frac{1}{3x} \frac{d}{dx}(3x) = \frac{3}{3x} = \boxed{\frac{1}{x}}$.
14. Let $y = \ln \frac{x}{2}$. Then $y' = \frac{2}{x} \frac{d}{dx} \left(\frac{x}{2}\right) = \frac{2}{2x} = \boxed{\frac{1}{x}}$.
15. Let $y = \ln(e^t - e^{-t})$. Then

$$\begin{aligned} y' &= \frac{1}{e^t - e^{-t}} \frac{d}{dt}(e^t - e^{-t}) = \frac{1}{e^t - e^{-t}} \left(\frac{d}{dt} e^t - \frac{d}{dt} e^{-t} \right) \\ &= \frac{1}{e^t - e^{-t}} \left(e^t - e^{-t} \frac{d}{dt}(-t) \right) = \boxed{\frac{e^t + e^{-t}}{e^t - e^{-t}}}. \end{aligned}$$

16. Let $y = \ln(e^{at} + e^{-at})$. Then

$$\begin{aligned} y' &= \frac{1}{e^{at} + e^{-at}} \frac{d}{dt}(e^{at} + e^{-at}) = \frac{1}{e^{at} + e^{-at}} \left(\frac{d}{dt}e^{at} + \frac{d}{dt}e^{-at} \right) \\ &= \frac{1}{e^{at} + e^{-at}} \left(e^{at} \frac{d}{dt}(at) + e^{-at} \frac{d}{dt}(-at) \right) = \boxed{\frac{ae^{at} - ae^{-at}}{e^{at} + e^{-at}}}. \end{aligned}$$

17. Let $y = x \ln(x^2 + 4)$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \ln(x^2 + 4) + \ln(x^2 + 4) \frac{d}{dx} x = x \cdot \frac{1}{x^2 + 4} \frac{d}{dx}(x^2 + 4) + \ln(x^2 + 4) \\ &= \frac{x}{x^2 + 4} \cdot 2x + \ln(x^2 + 4) = \boxed{\frac{2x^2}{x^2 + 4} + \ln(x^2 + 4)}. \end{aligned}$$

18. Let $y = x \ln(x^2 + 5x + 1)$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \ln(x^2 + 5x + 1) + \ln(x^2 + 5x + 1) \frac{d}{dx} x \\ &= x \cdot \frac{1}{x^2 + 5x + 1} \frac{d}{dx}(x^2 + 5x + 1) + \ln(x^2 + 5x + 1) \\ &= \frac{x}{x^2 + 5x + 1} \cdot (2x + 5) + \ln(x^2 + 5x + 1) = \boxed{\frac{2x^2 + 5x}{x^2 + 5x + 1} + \ln(x^2 + 5x + 1)}. \end{aligned}$$

19. Let $y = v \ln \sqrt{v^2 + 1} = \frac{1}{2} v \ln(v^2 + 1)$. Then

$$\begin{aligned} y' &= \frac{1}{2} v \frac{d}{dv} \ln(v^2 + 1) + \ln(v^2 + 1) \frac{d}{dv} \left(\frac{1}{2} v \right) = \frac{1}{2} v \cdot \frac{1}{v^2 + 1} \frac{d}{dv}(v^2 + 1) + \frac{1}{2} \ln(v^2 + 1) \\ &= \frac{v}{2(v^2 + 1)} \cdot 2v + \ln \sqrt{v^2 + 1} = \boxed{\frac{v^2}{v^2 + 1} + \ln \sqrt{v^2 + 1}}. \end{aligned}$$

20. Let $y = v \ln \sqrt[3]{3v + 1} = \frac{1}{3} v \ln(3v + 1)$. Then

$$\begin{aligned} y' &= \frac{1}{3} v \frac{d}{dv} \ln(3v + 1) + \ln(3v + 1) \frac{d}{dv} \left(\frac{1}{3} v \right) = \frac{1}{3} v \cdot \frac{1}{3v + 1} \frac{d}{dv}(3v + 1) + \frac{1}{3} \ln(3v + 1) \\ &= \frac{v}{3(3v + 1)} \cdot 3 + \ln \sqrt[3]{3v + 1} = \boxed{\frac{v}{3v + 1} + \ln \sqrt[3]{3v + 1}}. \end{aligned}$$

21. Let $y = \frac{1}{2} \ln \frac{1+x}{1-x} = \frac{1}{2} (\ln(1+x) - \ln(1-x))$. Then

$$\begin{aligned} y' &= \frac{1}{2} \left(\frac{1}{1+x} \frac{d}{dx}(1+x) - \frac{1}{1-x} \frac{d}{dx}(1-x) \right) = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) \\ &= \frac{1}{2} \cdot \frac{(1-x) + (1+x)}{1-x^2} = \frac{1}{2} \cdot \frac{2}{1-x^2} = \boxed{\frac{1}{1-x^2}}. \end{aligned}$$

22. Let $y = \frac{1}{2} \ln \frac{1+x^2}{1-x^2} = \frac{1}{2} (\ln(1+x^2) - \ln(1-x^2))$. Then

$$\begin{aligned} y' &= \frac{1}{2} \left(\frac{1}{1+x^2} \frac{d}{dx}(1+x^2) - \frac{1}{1-x^2} \frac{d}{dx}(1-x^2) \right) = \frac{1}{2} \left(\frac{2x}{1+x^2} + \frac{2x}{1-x^2} \right) \\ &= x \cdot \frac{(1-x^2) + (1+x^2)}{1-x^4} = \boxed{\frac{2x}{1-x^4}}. \end{aligned}$$

23. Let $y = \ln(\ln x)$. Then $y' = \frac{1}{\ln x} \frac{d}{dx} \ln x = \boxed{\frac{1}{x \ln x}}$.

24. Let $y = \ln\left(\ln \frac{1}{x}\right) = \ln(-\ln x)$. Then

$$y' = \frac{1}{-\ln x} \frac{d}{dx} (-\ln x) = \boxed{\frac{1}{x \ln x}}.$$

25. Let $y = \ln \frac{x}{\sqrt{x^2+1}} = \ln x - \ln \sqrt{x^2+1} = \ln x - \frac{1}{2} \ln(x^2+1)$. Then

$$y' = \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x^2+1} \frac{d}{dx} (x^2+1) = \frac{1}{x} - \frac{1}{2(x^2+1)} \cdot 2x = \frac{1}{x} - \frac{x}{x^2+1} = \boxed{\frac{1}{x(x^2+1)}}.$$

26. Let $y = \ln \frac{4x^3}{\sqrt{x^2+4}} = \ln 4 + \ln x^3 - \ln \sqrt{x^2+4} = \ln 4 + 3 \ln x - \frac{1}{2} \ln(x^2+4)$. Then

$$y' = 0 + \frac{3}{x} - \frac{1}{2} \cdot \frac{1}{x^2+4} \frac{d}{dx} (x^2+4) = \frac{3}{x} - \frac{1}{2(x^2+4)} \cdot 2x = \boxed{\frac{3}{x} - \frac{x}{x^2+4}}.$$

27. Let

$$y = \ln \frac{(x^2+1)^2}{x\sqrt{x^2-1}} = \ln(x^2+1)^2 - \ln x - \ln \sqrt{x^2-1} = 2 \ln(x^2+1) - \ln x - \frac{1}{2} \ln(x^2-1).$$

Then

$$y' = \frac{2}{x^2+1} \frac{d}{dx} (x^2+1) - \frac{1}{x} - \frac{1}{2(x^2-1)} \frac{d}{dx} (x^2-1) = \boxed{\frac{4x}{x^2+1} - \frac{1}{x} - \frac{x}{x^2-1}}.$$

28. Let

$$y = \ln \frac{x\sqrt{3x-1}}{(x^2+1)^3} = \ln x + \ln \sqrt{3x-1} - \ln(x^2+1)^3 = \ln x + \frac{1}{2} \ln(3x-1) - 3 \ln(x^2+1).$$

Then

$$y' = \frac{1}{x} + \frac{1}{2(3x-1)} \frac{d}{dx} (3x-1) - \frac{3}{x^2+1} \frac{d}{dx} (x^2+1) = \boxed{\frac{1}{x} + \frac{3}{2(3x-1)} - \frac{6x}{x^2+1}}.$$

29. Let $y = \ln(\sin \theta)$. Then $y' = \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta = \frac{\cos \theta}{\sin \theta} = \boxed{\cot \theta}$.

30. Let $y = \ln(\cos \theta)$. Then $y' = \frac{1}{\cos \theta} \frac{d}{d\theta} \cos \theta = -\frac{\sin \theta}{\cos \theta} = \boxed{-\tan \theta}$.

31. Let $y = \ln(x + \sqrt{x^2+4})$. Then

$$\begin{aligned} y' &= \frac{1}{x + \sqrt{x^2+4}} \frac{d}{dx} (x + \sqrt{x^2+4}) = \frac{1}{x + \sqrt{x^2+4}} \left(1 + \frac{1}{2}(x^2+4)^{-1/2} \frac{d}{dx} (x^2+4) \right) \\ &= \frac{1}{x + \sqrt{x^2+4}} \left(1 + \frac{1}{2\sqrt{x^2+4}} \cdot 2x \right) = \frac{1}{x + \sqrt{x^2+4}} \left(1 + \frac{x}{\sqrt{x^2+4}} \right) \\ &= \frac{1}{x + \sqrt{x^2+4}} \cdot \frac{x + \sqrt{x^2+4}}{\sqrt{x^2+4}} = \boxed{\frac{1}{\sqrt{x^2+4}}}. \end{aligned}$$

32. Let $y = \ln(\sqrt{x+1} + \sqrt{x})$. Then

$$\begin{aligned} y' &= \frac{1}{\sqrt{x+1} + \sqrt{x}} \frac{d}{dx}(\sqrt{x+1} + \sqrt{x}) = \frac{1}{\sqrt{x+1} + \sqrt{x}} \left(\frac{1}{2}(x+1)^{-1/2} \frac{d}{dx}(x+1) + \frac{1}{2}x^{-1/2} \right) \\ &= \frac{1}{\sqrt{x+1} + \sqrt{x}} \left(\frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x}} \right) = \frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{2\sqrt{x+1}\sqrt{x}} \\ &= \boxed{\frac{1}{2\sqrt{x+1}\sqrt{x}}}. \end{aligned}$$

33. Let $y = \log_2(1+x^2)$. Then $y' = \frac{1}{(1+x^2)\ln 2} \frac{d}{dx}(1+x^2) = \boxed{\frac{2x}{(1+x^2)\ln 2}}$.

34. Let $y = \log_2(x^2-1)$. Then $y' = \frac{1}{(x^2-1)\ln 2} \frac{d}{dx}(x^2-1) = \boxed{\frac{2x}{(x^2-1)\ln 2}}$.

35. Let $y = \tan^{-1}(\ln x)$. Then $y' = \frac{1}{1+(\ln x)^2} \frac{d}{dx} \ln x = \boxed{\frac{1}{x(1+(\ln x)^2)}}$.

36. Let $y = \sin^{-1}(\ln x)$. Then $y' = \frac{1}{\sqrt{1-(\ln x)^2}} \frac{d}{dx} \ln x = \boxed{\frac{1}{x\sqrt{1-(\ln x)^2}}}$.

37. Let $y = \ln(\tan^{-1} t)$. Then $y' = \frac{1}{\tan^{-1} t} \frac{d}{dt} \tan^{-1} t = \boxed{\frac{1}{\tan^{-1} t} \cdot \frac{1}{1+t^2}}$.

38. Let $y = \ln(\sin^{-1} t)$. Then $y' = \frac{1}{\sin^{-1} t} \frac{d}{dt} \sin^{-1} t = \boxed{\frac{1}{\sin^{-1} t} \cdot \frac{1}{\sqrt{1-t^2}}}$.

39. Let $y = (\ln x)^{1/2}$. Then $y' = \frac{1}{2}(\ln x)^{-1/2} \frac{d}{dx} \ln x = \boxed{\frac{1}{2x(\ln x)^{1/2}}}$.

40. Let $y = (\ln x)^{-1/2}$. Then $y' = -\frac{1}{2}(\ln x)^{-3/2} \frac{d}{dx} \ln x = \boxed{-\frac{1}{2x(\ln x)^{3/2}}}$.

41. Let $y = \sin(\ln \theta)$. Then $y' = \cos(\ln \theta) \frac{d}{d\theta} \ln \theta = \boxed{\cos(\ln \theta) \cdot \frac{1}{\theta}}$.

42. Let $y = \cos(\ln \theta)$. Then $y' = -\sin(\ln \theta) \frac{d}{d\theta} \ln \theta = \boxed{-\sin(\ln \theta) \cdot \frac{1}{\theta}}$.

43. Let $y = x \ln \sqrt{\cos(2x)} = \frac{1}{2}x \ln(\cos(2x))$. Then

$$\begin{aligned} y' &= \frac{1}{2}x \frac{d}{dx} \ln(\cos(2x)) + \ln(\cos(2x)) \frac{d}{dx} \left(\frac{1}{2}x \right) = \frac{1}{2}x \cdot \frac{1}{\cos(2x)} \frac{d}{dx} \cos(2x) + \frac{1}{2} \ln(\cos(2x)) \\ &= \frac{x}{2\cos(2x)} \cdot (-\sin(2x)) \frac{d}{dx}(2x) + \ln \sqrt{\cos(2x)} \\ &= -\frac{x \sin(2x)}{\cos(2x)} + \ln \sqrt{\cos(2x)} = \boxed{-x \tan(2x) + \ln \sqrt{\cos(2x)}}. \end{aligned}$$

44. Let $y = x^2 \ln \sqrt{\sin(2x)} = \frac{1}{2}x^2 \ln(\sin(2x))$. Then

$$\begin{aligned}
 y' &= \frac{1}{2}x^2 \frac{d}{dx} \ln(\sin(2x)) + \ln(\sin(2x)) \frac{d}{dx} \left(\frac{1}{2}x^2 \right) = \frac{1}{2}x^2 \cdot \frac{1}{\sin(2x)} \frac{d}{dx} \sin(2x) + x \ln(\sin(2x)) \\
 &= \frac{x^2}{2 \sin(2x)} \cdot \cos(2x) \frac{d}{dx} (2x) + x \ln(\sin(2x)) \\
 &= \frac{x^2 \cos(2x)}{\sin(2x)} + x \ln(\sin(2x)) = \boxed{x^2 \cot(2x) + x \ln(\sin(2x))}.
 \end{aligned}$$

45. Let $x \ln y + y \ln x = 2$. Then

$$\begin{aligned}
 \frac{d}{dx} (x \ln y + y \ln x) &= \frac{d}{dx} 2 \\
 \frac{d}{dx} (x \ln y) + \frac{d}{dx} (y \ln x) &= 0 \\
 \frac{x}{y} \frac{dy}{dx} + \ln y + \frac{y}{x} + \ln x \frac{dy}{dx} &= 0 \\
 \left(\frac{x}{y} + \ln x \right) \frac{dy}{dx} &= - \left(\frac{y}{x} + \ln y \right) \\
 \frac{dy}{dx} &= - \frac{\frac{y}{x} + \ln y}{\frac{x}{y} + \ln x} = \boxed{- \frac{y^2 + xy \ln y}{x^2 + xy \ln x}}.
 \end{aligned}$$

46. Let $\frac{\ln y}{x} + \frac{\ln x}{y} = 2$. Then

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{\ln y}{x} + \frac{\ln x}{y} \right) &= \frac{d}{dx} 2 \\
 \frac{d}{dx} \left(\frac{\ln y}{x} \right) + \frac{d}{dx} \left(\frac{\ln x}{y} \right) &= 0 \\
 \frac{\frac{x}{y} \frac{dy}{dx} - \ln y}{x^2} + \frac{\frac{y}{x} - \ln x \frac{dy}{dx}}{y^2} &= 0 \\
 xy \frac{dy}{dx} - y^2 \ln y + xy - x^2 \ln x \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= \boxed{\frac{y^2 \ln y - xy}{xy - x^2 \ln x}}.
 \end{aligned}$$

47. Let $\ln(x^2 + y^2) = x + y$. Then

$$\begin{aligned}
 \frac{d}{dx} \ln(x^2 + y^2) &= \frac{d}{dx} (x + y) \\
 \frac{1}{x^2 + y^2} \frac{d}{dx} (x^2 + y^2) &= 1 + \frac{dy}{dx} \\
 \frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx} \right) &= 1 + \frac{dy}{dx} \\
 2x + 2y \frac{dy}{dx} &= x^2 + y^2 + (x^2 + y^2) \frac{dy}{dx} \\
 \frac{dy}{dx} &= \boxed{\frac{x^2 + y^2 - 2x}{2y - x^2 - y^2}}.
 \end{aligned}$$

48. Let $\ln(x^2 - y^2) = x - y$. Then

$$\begin{aligned}\frac{d}{dx} \ln(x^2 - y^2) &= \frac{d}{dx}(x - y) \\ \frac{1}{x^2 - y^2} \frac{d}{dx}(x^2 - y^2) &= 1 - \frac{dy}{dx} \\ \frac{1}{x^2 - y^2} \left(2x - 2y \frac{dy}{dx} \right) &= 1 - \frac{dy}{dx} \\ 2x - 2y \frac{dy}{dx} &= x^2 - y^2 - (x^2 - y^2) \frac{dy}{dx} \\ \frac{dy}{dx} &= \boxed{\frac{x^2 - y^2 - 2x}{x^2 - y^2 - 2y}}.\end{aligned}$$

49. Let $\ln \frac{y}{x} = y$ or $\ln y - \ln x = y$. Then

$$\begin{aligned}\frac{d}{dx}(\ln y - \ln x) &= \frac{dy}{dx} \\ \frac{1}{y} \frac{dy}{dx} - \frac{1}{x} &= \frac{dy}{dx} \\ x \frac{dy}{dx} - y &= xy \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{y}{x - xy} = \boxed{\frac{y}{x(1 - y)}}.\end{aligned}$$

50. Let $\ln \frac{y}{x} - \ln \frac{x}{y} = 1$ or $2 \ln y - 2 \ln x = 1$. Then

$$\begin{aligned}\frac{d}{dx}(2 \ln y - 2 \ln x) &= \frac{d}{dx} 1 \\ \frac{2}{y} \frac{dy}{dx} - \frac{2}{x} &= 0 \\ \frac{dy}{dx} &= \boxed{\frac{y}{x}}.\end{aligned}$$

51. Let $y = (x^2 + 1)^2(2x^3 - 1)^4$. Then

$$\ln y = \ln[(x^2 + 1)^2(2x^3 - 1)^4] = \ln[(x^2 + 1)^2] + \ln[(2x^3 - 1)^4] = 2 \ln(x^2 + 1) + 4 \ln(2x^3 - 1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [2 \ln(x^2 + 1) + 4 \ln(2x^3 - 1)].$$

This yields

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{2}{x^2 + 1} \frac{d}{dx}(x^2 + 1) + \frac{4}{2x^3 - 1} \frac{d}{dx}(2x^3 - 1) \\ &= \frac{2}{x^2 + 1} \cdot 2x + \frac{4}{2x^3 - 1} \cdot 6x^2 = \frac{4x}{x^2 + 1} + \frac{24x^2}{2x^3 - 1},\end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{4x}{x^2 + 1} + \frac{24x^2}{2x^3 - 1} \right) = \boxed{(x^2 + 1)^2(2x^3 - 1)^4 \left(\frac{4x}{x^2 + 1} + \frac{24x^2}{2x^3 - 1} \right)}.$$

52. Let $y = (3x^2 + 4)^3(x^2 + 1)^4$. Then

$$\ln y = \ln[(3x^2 + 4)^3(x^2 + 1)^4] = \ln[(3x^2 + 4)^3] + \ln[(x^2 + 1)^4] = 3 \ln(3x^2 + 4) + 4 \ln(x^2 + 1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [3 \ln(3x^2 + 4) + 4 \ln(x^2 + 1)].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{3}{3x^2 + 4} \frac{d}{dx} (3x^2 + 4) + \frac{4}{x^2 + 1} \frac{d}{dx} (x^2 + 1) \\ &= \frac{3}{3x^2 + 4} \cdot 6x + \frac{4}{x^2 + 1} \cdot 2x = \frac{18x}{3x^2 + 4} + \frac{8x}{x^2 + 1}, \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{18x}{3x^2 + 4} + \frac{8x}{x^2 + 1} \right) = \boxed{(3x^2 + 4)^3(x^2 + 1)^4 \left(\frac{18x}{3x^2 + 4} + \frac{8x}{x^2 + 1} \right)}.$$

53. Let $y = \frac{x^2(x^3 + 1)}{\sqrt{x^2 + 1}}$. Then

$$\ln y = \ln \left[\frac{x^2(x^3 + 1)}{\sqrt{x^2 + 1}} \right] = \ln x^2 + \ln(x^3 + 1) - \ln \sqrt{x^2 + 1} = 2 \ln x + \ln(x^3 + 1) - \frac{1}{2} \ln(x^2 + 1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[2 \ln x + \ln(x^3 + 1) - \frac{1}{2} \ln(x^2 + 1) \right].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{2}{x} + \frac{1}{x^3 + 1} \frac{d}{dx} (x^3 + 1) - \frac{1}{2(x^2 + 1)} \frac{d}{dx} (x^2 + 1) \\ &= \frac{2}{x} + \frac{1}{x^3 + 1} \cdot 3x^2 - \frac{1}{2(x^2 + 1)} \cdot 2x = \frac{2}{x} + \frac{3x^2}{x^3 + 1} - \frac{x}{x^2 + 1}, \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{2}{x} + \frac{3x^2}{x^3 + 1} - \frac{x}{x^2 + 1} \right) = \boxed{\frac{x^2(x^3 + 1)}{\sqrt{x^2 + 1}} \left(\frac{2}{x} + \frac{3x^2}{x^3 + 1} - \frac{x}{x^2 + 1} \right)}.$$

54. Let $y = \frac{\sqrt{x}(x^3 + 2)^2}{\sqrt[3]{3x + 4}}$. Then

$$\ln y = \ln \left[\frac{\sqrt{x}(x^3 + 2)^2}{\sqrt[3]{3x + 4}} \right] = \ln \sqrt{x} + \ln(x^3 + 2)^2 - \ln \sqrt[3]{3x + 4} = \frac{1}{2} \ln x + 2 \ln(x^3 + 2) - \frac{1}{3} \ln(3x + 4),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[\frac{1}{2} \ln x + 2 \ln(x^3 + 2) - \frac{1}{3} \ln(3x + 4) \right].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2x} + \frac{2}{x^3 + 2} \frac{d}{dx} (x^3 + 2) - \frac{1}{3(3x + 4)} \frac{d}{dx} (3x + 4) \\ &= \frac{1}{2x} + \frac{2}{x^3 + 2} \cdot 3x^2 - \frac{1}{3(3x + 4)} \cdot 3 = \frac{1}{2x} + \frac{6x^2}{x^3 + 2} - \frac{1}{3x + 4}, \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{1}{2x} + \frac{6x^2}{x^3 + 2} - \frac{1}{3x + 4} \right) = \boxed{\frac{\sqrt{x}(x^3 + 2)^2}{\sqrt[3]{3x + 4}} \left(\frac{1}{2x} + \frac{6x^2}{x^3 + 2} - \frac{1}{3x + 4} \right)}.$$

55. Let $y = \frac{x \cos x}{(x^2 + 1)^3 \sin x}$. Then

$$\begin{aligned}\ln y &= \ln \left[\frac{x \cos x}{(x^2 + 1)^3 \sin x} \right] = \ln x + \ln \cos x - \ln(x^2 + 1)^3 - \ln \sin x \\ &= \ln x + \ln \cos x - 3 \ln(x^2 + 1) - \ln \sin x,\end{aligned}$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\ln x + \ln \cos x - 3 \ln(x^2 + 1) - \ln \sin x).$$

This yields

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{\cos x} \frac{d}{dx} \cos x - \frac{3}{x^2 + 1} \frac{d}{dx} (x^2 + 1) - \frac{1}{\sin x} \frac{d}{dx} \sin x \\ &= \frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x) - \frac{3}{x^2 + 1} \cdot 2x - \frac{1}{\sin x} \cdot \cos x = \frac{1}{x} - \tan x - \frac{6x}{x^2 + 1} - \cot x,\end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{1}{x} - \tan x - \frac{6x}{x^2 + 1} - \cot x \right) = \boxed{\frac{x \cos x}{(x^2 + 1)^3 \sin x} \left(\frac{1}{x} - \tan x - \frac{6x}{x^2 + 1} - \cot x \right)}.$$

56. Let $y = \frac{x \sin x}{(1 + e^x)^3 \cos x}$. Then

$$\begin{aligned}\ln y &= \ln \left[\frac{x \sin x}{(1 + e^x)^3 \cos x} \right] = \ln x + \ln \sin x - \ln(1 + e^x)^3 - \ln \cos x \\ &= \ln x + \ln \sin x - 3 \ln(1 + e^x) - \ln \cos x,\end{aligned}$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\ln x + \ln \sin x - 3 \ln(1 + e^x) - \ln \cos x).$$

This yields

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{\sin x} \frac{d}{dx} \sin x - \frac{3}{1 + e^x} \frac{d}{dx} (1 + e^x) - \frac{1}{\cos x} \frac{d}{dx} \cos x \\ &= \frac{1}{x} + \frac{1}{\sin x} \cdot \cos x - \frac{3}{1 + e^x} \cdot e^x - \frac{1}{\cos x} \cdot (-\sin x) = \frac{1}{x} + \cot x - \frac{3e^x}{1 + e^x} + \tan x,\end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{1}{x} + \cot x - \frac{3e^x}{1 + e^x} + \tan x \right) = \boxed{\frac{x \sin x}{(1 + e^x)^3 \cos x} \left(\frac{1}{x} + \cot x - \frac{3e^x}{1 + e^x} + \tan x \right)}.$$

57. Let $y = (3x)^x$. Then

$$\ln y = \ln[(3x)^x] = x \ln(3x) = x (\ln 3 + \ln x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [x (\ln 3 + \ln x)].$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} (\ln 3 + \ln x) + (\ln 3 + \ln x) \frac{d}{dx} x = x \cdot \frac{1}{x} + \ln 3 + \ln x = 1 + \ln(3x),$$

so that

$$\frac{dy}{dx} = y(1 + \ln(3x)) = \boxed{(3x)^x (1 + \ln(3x))}.$$

58. Let $y = (x - 1)^x$. Then

$$\ln y = \ln[(x - 1)^x] = x \ln(x - 1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [x \ln(x - 1)].$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} \ln(x - 1) + \ln(x - 1) \frac{d}{dx} x = x \cdot \frac{1}{x - 1} + \ln(x - 1) = \frac{x}{x - 1} + \ln(x - 1),$$

so that

$$\frac{dy}{dx} = y \left(\frac{x}{x - 1} + \ln(x - 1) \right) = \boxed{(x - 1)^x \left(\frac{x}{x - 1} + \ln(x - 1) \right)}.$$

59. Let $y = x^{\ln x}$. Then

$$\ln y = \ln(x^{\ln x}) = \ln x \cdot \ln x = (\ln x)^2,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\ln x)^2.$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = 2 \ln x \frac{d}{dx} \ln x = \frac{2 \ln x}{x},$$

so that

$$\frac{dy}{dx} = y \frac{2 \ln x}{x} = x^{\ln x} \frac{2 \ln x}{x} = \boxed{2x^{\ln x} \frac{\ln x}{x}}.$$

60. Let $y = (2x)^{\ln x}$. Then

$$\ln y = \ln[(2x)^{\ln x}] = \ln x \cdot \ln(2x) = \ln x \cdot (\ln x + \ln 2) = (\ln x)^2 + \ln 2 \ln x,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [(\ln x)^2 + \ln 2 \ln x].$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = 2 \ln x \frac{d}{dx} \ln x + \frac{\ln 2}{x} = \frac{2 \ln x}{x} + \frac{\ln 2}{x} = \frac{2 \ln x + \ln 2}{x},$$

so that

$$\frac{dy}{dx} = y \frac{2 \ln x + \ln 2}{x} = \boxed{(2x)^{\ln x} \frac{2 \ln x + \ln 2}{x}}.$$

61. Let $y = x^{x^2}$. Then

$$\ln y = \ln x^{x^2} = x^2 \ln x,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (x^2 \ln x).$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = x^2 \frac{d}{dx} \ln x + \ln x \frac{d}{dx} x^2 = x^2 \cdot \frac{1}{x} + 2x \ln x = x + 2x \ln x,$$

so that

$$\frac{dy}{dx} = y(x + 2x \ln x) = \boxed{x^{x^2} (x + 2x \ln x)}.$$

62. Let $y = (3x)^{\sqrt{x}}$. Then

$$\ln y = \ln[(3x)^{\sqrt{x}}] = \sqrt{x} \ln(3x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [\sqrt{x} \ln(3x)].$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = \sqrt{x} \frac{d}{dx} \ln(3x) + \ln(3x) \frac{d}{dx} \sqrt{x} = \frac{\sqrt{x}}{3x} \frac{d}{dx} (3x) + \frac{1}{2} x^{-1/2} \ln(3x) = \frac{\sqrt{x}}{x} + \frac{\ln(3x)}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\ln(3x)}{2\sqrt{x}},$$

so that

$$\frac{dy}{dx} = y \left(\frac{1}{\sqrt{x}} + \frac{\ln(3x)}{2\sqrt{x}} \right) = \boxed{(3x)^{\sqrt{x}} \frac{2 + \ln(3x)}{2\sqrt{x}}}.$$

63. Let $y = x^{e^x}$. Then

$$\ln y = \ln(x^{e^x}) = e^x \ln x,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (e^x \ln x).$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = e^x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} e^x = \frac{e^x}{x} + e^x \ln x,$$

so that

$$\frac{dy}{dx} = y \left(\frac{e^x}{x} + e^x \ln x \right) = \boxed{x^{e^x} \left(\frac{e^x}{x} + e^x \ln x \right)}.$$

64. Let $y = (x^2 + 1)^{e^x}$. Then

$$\ln y = \ln[(x^2 + 1)^{e^x}] = e^x \ln(x^2 + 1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [e^x \ln(x^2 + 1)].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= e^x \frac{d}{dx} \ln(x^2 + 1) + \ln(x^2 + 1) \frac{d}{dx} e^x \\ &= \frac{e^x}{x^2 + 1} \frac{d}{dx} (x^2 + 1) + e^x \ln(x^2 + 1) = \frac{2xe^x}{x^2 + 1} + e^x \ln(x^2 + 1), \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{2xe^x}{x^2 + 1} + e^x \ln(x^2 + 1) \right) = \boxed{(x^2 + 1)^{e^x} \left(\frac{2xe^x}{x^2 + 1} + e^x \ln(x^2 + 1) \right)}.$$

65. Let $y = x^{\sin x}$. Then

$$\ln y = \ln[x^{\sin x}] = \sin x \ln x,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\sin x \ln x).$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = \sin x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \sin x = \frac{\sin x}{x} + \cos x \ln x,$$

so that

$$\frac{dy}{dx} = y \left(\frac{\sin x}{x} + \cos x \ln x \right) = \boxed{x^{\sin x} \left(\frac{\sin x}{x} + \cos x \ln x \right)}.$$

66. Let $y = x^{\cos x}$. Then

$$\ln y = \ln[x^{\cos x}] = \cos x \ln x,$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\cos x \ln x).$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = \cos x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \cos x = \frac{\cos x}{x} - \sin x \ln x,$$

so that

$$\frac{dy}{dx} = y \left(\frac{\cos x}{x} - \sin x \ln x \right) = \boxed{x^{\cos x} \left(\frac{\cos x}{x} - \sin x \ln x \right)}.$$

67. Let $y = (\sin x)^x$. Then

$$\ln y = \ln[(\sin x)^x] = x \ln(\sin x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [x \ln(\sin x)].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= x \frac{d}{dx} \ln(\sin x) + \ln(\sin x) \frac{d}{dx} x = \frac{x}{\sin x} \frac{d}{dx} \sin x + \ln(\sin x) \\ &= \frac{x}{\sin x} \cdot \cos x + \ln(\sin x) = x \cot x + \ln(\sin x), \end{aligned}$$

so that

$$\frac{dy}{dx} = y(x \cot x + \ln(\sin x)) = \boxed{(\sin x)^x (x \cot x + \ln(\sin x))}.$$

68. Let $y = (\cos x)^x$. Then

$$\ln y = \ln[(\cos x)^x] = x \ln(\cos x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [x \ln(\cos x)].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= x \frac{d}{dx} \ln(\cos x) + \ln(\cos x) \frac{d}{dx} x = \frac{x}{\cos x} \frac{d}{dx} \cos x + \ln(\cos x) \\ &= \frac{x}{\cos x} \cdot (-\sin x) + \ln(\cos x) = -x \tan x + \ln(\cos x), \end{aligned}$$

so that

$$\frac{dy}{dx} = y(-x \tan x + \ln(\cos x)) = \boxed{(\cos x)^x (-x \tan x + \ln(\cos x))}.$$

69. Let $y = (\sin x)^{\cos x}$. Then

$$\ln y = \ln[(\sin x)^{\cos x}] = \cos x \ln(\sin x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [\cos x \ln(\sin x)].$$

This yields

$$\frac{1}{y} \frac{dy}{dx} = \cos x \frac{d}{dx} \ln(\sin x) + \ln(\sin x) \frac{d}{dx} \cos x = \frac{\cos x}{\sin x} \frac{d}{dx} \sin x - \sin x \ln(\sin x) = \cos x \cot x - \sin x \ln(\sin x),$$

so that

$$\frac{dy}{dx} = y(\cos x \cot x - \sin x \ln(\sin x)) = \boxed{(\sin x)^{\cos x} (\cos x \cot x - \sin x \ln(\sin x))}.$$

70. Let $y = (\sin x)^{\tan x}$. Then

$$\ln y = \ln[(\sin x)^{\tan x}] = \tan x \ln(\sin x),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} [\tan x \ln(\sin x)].$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \tan x \frac{d}{dx} \ln(\sin x) + \ln(\sin x) \frac{d}{dx} \tan x = \frac{\tan x}{\sin x} \frac{d}{dx} \sin x + \sec^2 x \ln(\sin x) \\ &= \frac{1}{\cos x} \cdot \cos x + \sec^2 x \ln(\sin x) = 1 + \sec^2 x \ln(\sin x), \end{aligned}$$

so that

$$\frac{dy}{dx} = y(1 + \sec^2 x \ln(\sin x)) = \boxed{(\sin x)^{\tan x} (1 + \sec^2 x \ln(\sin x))}.$$

71. Let $x^y = 4$. Then

$$\ln(x^y) = y \ln x = \ln 4,$$

and

$$\frac{d}{dx} (y \ln x) = \frac{d}{dx} \ln 4.$$

This yields

$$\begin{aligned} y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} &= 0 \\ \ln x \cdot \frac{dy}{dx} &= -\frac{y}{x} \\ \frac{dy}{dx} &= \boxed{-\frac{y}{x \ln x}}. \end{aligned}$$

Here, we can solve for y explicitly as

$$y = \frac{\ln 4}{\ln x},$$

so the derivative can also be expressed explicitly in terms of x :

$$\frac{dy}{dx} = -\frac{\frac{\ln 4}{\ln x}}{x \ln x} = -\frac{\ln 4}{x(\ln x)^2}.$$

72. Let $y^x = 10$. Then

$$\ln(y^x) = \ln 10 \quad \text{or} \quad x \ln y = \ln 10.$$

By implicit differentiation it follows that

$$\begin{aligned} \frac{d}{dx} (x \ln y) &= \frac{d}{dx} \ln 10 \\ x \frac{d}{dx} \ln y + \ln y \frac{d}{dx} x &= 0 \\ \frac{x}{y} \frac{dy}{dx} + \ln y &= 0 \\ \frac{dy}{dx} &= \boxed{-\frac{y \ln y}{x}}. \end{aligned}$$

73. Let $f(x) = \ln(5x)$. Then

$$f'(x) = \frac{1}{5x} \frac{d}{dx}(5x) = \frac{1}{5x} \cdot 5 = \frac{1}{x},$$

and the slope of the tangent line at the point $\left(\frac{1}{5}, 0\right)$ is

$$m_{\text{sec}} = f'\left(\frac{1}{5}\right) = \frac{1}{1/5} = 5.$$

The equation of the tangent line is then

$$y - 0 = 5\left(x - \frac{1}{5}\right) \quad \text{or} \quad \boxed{y = 5x - 1}.$$

74. Let $f(x) = x \ln x$. Then

$$f'(x) = x \frac{d}{dx} \ln x + \ln x \frac{d}{dx} x = x \cdot \frac{1}{x} + \ln x = 1 + \ln x,$$

and the slope of the tangent line at the point $(1, 0)$ is

$$m_{\text{tan}} = f'(1) = 1 + \ln 1 = 1 + 0 = 1.$$

The equation of the tangent line is then

$$y - 0 = 1(x - 1) \quad \text{or} \quad \boxed{y = x - 1}.$$

75. Let $y = \frac{x^2 \sqrt{3x-2}}{(x-1)^2}$. Then

$$\ln y = \ln \left[\frac{x^2 \sqrt{3x-2}}{(x-1)^2} \right] = \ln x^2 + \ln \sqrt{3x-2} - \ln(x-1)^2 = 2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1) \right).$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{2}{x} + \frac{1}{2(3x-2)} \frac{d}{dx}(3x-2) - \frac{2}{x-1} \frac{d}{dx}(x-1) \\ &= \frac{2}{x} + \frac{1}{2(3x-2)} \cdot 3 - \frac{2}{x-1} \cdot 1 = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}, \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1} \right) = \frac{x^2 \sqrt{3x-2}}{(x-1)^2} \left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1} \right).$$

The slope of the tangent line at the point $(2, 8)$ is

$$m_{\text{tan}} = \frac{4\sqrt{4}}{1} \left(\frac{2}{2} + \frac{3}{2(4)} - \frac{2}{1} \right) = 8 \left(-\frac{5}{8} \right) = -5,$$

and the equation of the tangent line is

$$y - 8 = -5(x - 2) \quad \text{or} \quad \boxed{y = -5x + 18}.$$

76. Let $y = \frac{x(\sqrt[3]{x} + 1)^2}{\sqrt{x+1}}$. Then

$$\ln y = \ln \left[\frac{x(\sqrt[3]{x} + 1)^2}{\sqrt{x+1}} \right] = \ln x + \ln(\sqrt[3]{x} + 1)^2 - \ln \sqrt{x+1} = \ln x + 2 \ln(\sqrt[3]{x} + 1) - \frac{1}{2} \ln(x+1),$$

and

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(\ln x + 2 \ln(\sqrt[3]{x} + 1) - \frac{1}{2} \ln(x+1) \right).$$

This yields

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{2}{\sqrt[3]{x} + 1} \frac{d}{dx}(\sqrt[3]{x} + 1) - \frac{1}{2(x+1)} \frac{d}{dx}(x+1) \\ &= \frac{1}{x} + \frac{2}{\sqrt[3]{x} + 1} \cdot \frac{1}{3} x^{-2/3} - \frac{1}{2(x+1)} \cdot 1 = \frac{1}{x} + \frac{2}{3\sqrt[3]{x^2}(\sqrt[3]{x} + 1)} - \frac{1}{2(x+1)}, \end{aligned}$$

so that

$$\frac{dy}{dx} = y \left(\frac{1}{x} + \frac{2}{3\sqrt[3]{x^2}(\sqrt[3]{x} + 1)} - \frac{1}{2(x+1)} \right) = \frac{x(\sqrt[3]{x} + 1)^2}{\sqrt{x+1}} \left(\frac{1}{x} + \frac{2}{3\sqrt[3]{x^2}(\sqrt[3]{x} + 1)} - \frac{1}{2(x+1)} \right).$$

The slope of the tangent line at the point (8, 24) is

$$m_{\tan} = \frac{8(3)^2}{3} \left(\frac{1}{8} + \frac{2}{3(4)(3)} - \frac{1}{2(9)} \right) = 24 \cdot \frac{1}{8} = 3,$$

and the equation of the tangent line is

$$y - 24 = 3(x - 8) \quad \text{or} \quad \boxed{y = 3x}.$$

77. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{2n} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \right]^2 = \boxed{e^2}.$

78. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{n/2} = \sqrt{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n} = \boxed{\sqrt{e}}.$

79. First write

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n} \right)^{3n/3} = \sqrt[3]{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n} \right)^{3n}}.$$

Now, let $k = 3n$. As $n \rightarrow \infty$, $k \rightarrow \infty$ as well, so

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n} \right)^n = \sqrt[3]{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{3n} \right)^{3n}} = \sqrt[3]{\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k} = \sqrt[3]{e} = \boxed{e^{1/3}}.$$

80. First write

$$\lim_{n \rightarrow \infty} \left(1 + \frac{4}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/4} \right)^{4n/4} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/4} \right)^{n/4} \right]^4.$$

Now, let $k = n/4$. As $n \rightarrow \infty$, $k \rightarrow \infty$ as well, so

$$\lim_{n \rightarrow \infty} \left(1 + \frac{4}{n} \right)^n = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/4} \right)^{n/4} \right]^4 = \left[\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k \right]^4 = \boxed{e^4}.$$

Applications and Extensions

81. Let $f(x) = x^9 \ln x$. Then

$$\begin{aligned}\frac{df}{dx} &= x^9 \cdot \frac{1}{x} + \ln x \cdot 9x^8 = x^8 + 9x^8 \ln x \\ \frac{d^2 f}{dx^2} &= 8x^7 + 9x^8 \cdot \frac{1}{x} + \ln x \cdot 72x^7 = 17x^7 + 72x^7 \ln x \\ \frac{d^3 f}{dx^3} &= 119x^6 + 72x^7 \cdot \frac{1}{x} + \ln x \cdot 504x^6 = 191x^6 + 504x^6 \ln x,\end{aligned}$$

and so on. Note that with each derivative, the exponent on x decreases by one. By the ninth derivative, the exponent on x will be $9 - 9 = 0$; in other words, the ninth derivative will be of the form $c + d \ln x$, for some constants c and d . Then, the tenth derivative will be d/x . To identify the constant d , note that the multiplier on the natural logarithm term is 9 for the first derivative, $72 = 9(8)$ for the second derivative, $504 = 9(8)(7)$ for the third derivative, and so on. By the ninth derivative, the multiplier on the natural logarithm term will then be $9(8)(7) \cdots (2)(1) = 9!$. Finally,

$$\frac{d^{10}}{dx^{10}}(x^9 \ln x) = \boxed{\frac{9!}{x} = \frac{362880}{x}}.$$

82. Let $f(x) = \ln(x - 1)$. Then

$$\begin{aligned}f'(x) &= \frac{1}{x-1} \frac{d}{dx}(x-1) = \frac{1}{x-1} = (x-1)^{-1}; \\ f''(x) &= -(x-1)^{-2} \frac{d}{dx}(x-1) = -(x-1)^{-2}; \\ f'''(x) &= 2(x-1)^{-3} \frac{d}{dx}(x-1) = 2(x-1)^{-3}; \\ f^{(4)}(x) &= -6(x-1)^{-4} \frac{d}{dx}(x-1) = -6(x-1)^{-4};\end{aligned}$$

and so on. Note the elements in the pattern: the power on $x - 1$ in each derivative is the negative of the order of the derivative; the sign on the derivative alternates starting with positive on the first derivative; and the coefficients, 1, 1, 2, 6, are respectively $0!$, $1!$, $2!$, and $3!$. Thus

$$f^{(n)}(x) = \boxed{(-1)^{n-1}(n-1)!(x-1)^{-n}}.$$

83. Let $y = \ln(x^2 + y^2)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \ln(x^2 + y^2) \\ \frac{dy}{dx} &= \frac{1}{x^2 + y^2} \frac{d}{dx}(x^2 + y^2) \\ \frac{dy}{dx} &= \frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx} \right) \\ \left(1 - \frac{2y}{x^2 + y^2} \right) \frac{dy}{dx} &= \frac{2x}{x^2 + y^2} \\ \frac{dy}{dx} &= \frac{2x}{x^2 + y^2 - 2y}.\end{aligned}$$

Therefore, at the point $(1, 0)$,

$$\frac{dy}{dx} = \frac{2}{1 + 0 - 0} = \boxed{2}.$$

84. Let $f(x) = \tan\left(\ln x - \frac{1}{\ln x}\right)$. Then

$$\begin{aligned} f'(x) &= \sec^2\left(\ln x - \frac{1}{\ln x}\right) \frac{d}{dx}\left(\ln x - \frac{1}{\ln x}\right) \\ &= \sec^2\left(\ln x - \frac{1}{\ln x}\right) \cdot \left(\frac{1}{x} + \frac{1}{(\ln x)^2} \frac{d}{dx} \ln x\right) = \left(\frac{1}{x} + \frac{1}{x(\ln x)^2}\right) \sec^2\left(\ln x - \frac{1}{\ln x}\right), \end{aligned}$$

so

$$f'(e) = \left(\frac{1}{e} + \frac{1}{e(\ln e)^2}\right) \sec^2\left(\ln e - \frac{1}{\ln e}\right) = \left(\frac{1}{e} + \frac{1}{e}\right) \sec^2\left(1 - \frac{1}{1}\right) = \frac{2}{e} \sec^2 0 = \boxed{\frac{2}{e}}.$$

85. Let $x > 0$, and consider the function $y = x^x = e^{\ln x^x} = e^{x \ln x}$. Then

$$y' = e^{x \ln x} \frac{d}{dx}(x \ln x) = x^x \left(x \cdot \frac{1}{x} + \ln x \cdot 1\right) = \boxed{x^x(1 + \ln x)}.$$

86. Let $x > 0$ and $k > 0$ be a constant. Consider the function $y = \ln(kx) = \ln k + \ln x$. Then $y' = 0 + \frac{1}{x} = \frac{1}{x}$.

87. Let $y = x \tan^{-1} \frac{x}{a} - \frac{1}{2} a \ln(x^2 + a^2)$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \tan^{-1} \frac{x}{a} + \tan^{-1} \frac{x}{a} - \frac{a}{2(x^2 + a^2)} \frac{d}{dx}(x^2 + a^2) \\ &= x \frac{1}{1 + (x/a)^2} \frac{d}{dx} \left(\frac{x}{a}\right) + \tan^{-1} \frac{x}{a} - \frac{ax}{x^2 + a^2} \\ &= \frac{ax}{a^2 + x^2} + \tan^{-1} \frac{x}{a} - \frac{ax}{x^2 + a^2} = \boxed{\tan^{-1} \frac{x}{a}}. \end{aligned}$$

88. Let $y = x \sin^{-1} \frac{x}{a} + a \ln \sqrt{a^2 - x^2} = x \sin^{-1} \frac{x}{a} + \frac{1}{2} a \ln(a^2 - x^2)$. Then

$$\begin{aligned} y' &= x \frac{d}{dx} \sin^{-1} \frac{x}{a} + \sin^{-1} \frac{x}{a} + \frac{a}{2(a^2 - x^2)} \frac{d}{dx}(a^2 - x^2) \\ &= x \frac{1}{\sqrt{1 - (x/a)^2}} \frac{d}{dx} \left(\frac{x}{a}\right) + \sin^{-1} \frac{x}{a} - \frac{ax}{a^2 - x^2} \\ &= \boxed{\frac{x}{\sqrt{a^2 - x^2}} + \sin^{-1} \frac{x}{a} - \frac{ax}{a^2 - x^2}}. \end{aligned}$$

89. (a) First write

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/r}\right)^{rnt/r} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right]^{rt}.$$

Now, let $k = n/r$. As $n \rightarrow \infty$, $k \rightarrow \infty$ as well, so

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right]^{rt} = \left[\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k\right]^{rt} = e^{rt}.$$

Thus

$$A = \lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}.$$

- (b) With $P = \$5000$, $r = 0.02$, and $t = 10$ years,

$$A = 5000e^{0.02(10)} = 5000e^{0.2} \approx 6107.01,$$

so the investment is worth roughly $\boxed{\$6107.01}$ after 10 years.

- (c) With $P = \$10,000$ and $r = 0.024$, the investment will double to $\$20,000$ when t satisfies

$$20000 = 10000e^{0.024t} \quad \text{or} \quad e^{0.024t} = \frac{20000}{10000} = 2.$$

Thus, $0.024t = \ln 2$, or

$$t = \frac{1}{0.024} \ln 2 \approx \boxed{28.881 \text{ years}}.$$

- (d) With $A = Pe^{rt}$, then

$$\frac{dA}{dt} = Pe^{rt} \frac{d}{dt}(rt) = r(Pe^{rt}) = rA.$$

90. (a) With $P = \$2000$ and $r = 0.03$, $\boxed{A = 2000e^{0.03t}}$.

- (b) At maturity, $t = 10$ years, so

$$A = 2000e^{0.03(10)} = 2000e^{0.3} \approx 2699.72,$$

and the CD is worth roughly $\boxed{\$2699.72}$.

- (c) With $A = 2000e^{0.03t}$,

$$\frac{dA}{dt} = 2000e^{0.03t} \frac{d}{dt}(0.03t) = 60e^{0.03t}.$$

At $t = 3$, the rate of change of the amount A is

$$\frac{dA}{dt} = 60e^{0.03(3)} = 60e^{0.09} \approx 65.65,$$

or $\boxed{\text{roughly } \$65.65 \text{ per year}}$. At $t = 5$, the rate of change of the amount A is

$$\frac{dA}{dt} = 60e^{0.03(5)} = 60e^{0.15} \approx 69.71,$$

or $\boxed{\text{roughly } \$69.71 \text{ per year}}$. Finally, at $t = 8$, the rate of change of the amount A is

$$\frac{dA}{dt} = 60e^{0.03(8)} = 60e^{0.24} \approx 76.27,$$

or $\boxed{\text{roughly } \$76.27 \text{ per year}}$.

- (d) The rate at which the value of the CD increases is increasing in time. This is how compound interest works: interest is paid not only on the original deposit but also on any previously earned interest. Thus, as the value of the CD increases, the rate at which it increases also increases.

91. (a) With $I_0 = 10^{-12}$ and $I(x) = \frac{P}{4\pi x^2}$,

$$\begin{aligned} L(x) &= 10 \log \frac{I(x)}{I_0} = 10(\log I(x) - \log I_0) = 10 \left(\log \frac{P}{4\pi x^2} - \log I_0 \right) \\ &= 10 (\log P - \log(4\pi x^2) - \log I_0) \\ &= 10(\log P - \log(4\pi) - \log I_0) - 20 \log x. \end{aligned}$$

Thus

$$\frac{dL}{dx} = -\frac{20}{x \ln 10} \quad \text{and} \quad \frac{dL}{dx} = -\frac{20}{2 \ln 10} = \boxed{-\frac{10}{\ln 10} \frac{\text{dB}}{\text{m}}}.$$

- (b) The negative sign in the rate of change in part (a) indicates that the loudness of the sound decreases as the distance from the source of the sound increases.

92. Using the properties of the natural logarithm function, $\ln x + \ln y = \ln(xy)$. Thus,

$$\ln x + \ln y = 2x \quad \text{is equivalent to} \quad \ln(xy) = 2x,$$

and then exponentiating both sides of this last equation yields $xy = e^{2x}$, or

$$y = \frac{e^{2x}}{x}.$$

Thus,

$$y' = \frac{x \frac{d}{dx} e^{2x} - e^{2x} \frac{d}{dx} x}{x^2} = \frac{2xe^{2x} - e^{2x}}{x^2} = \frac{e^{2x}}{x} \cdot \frac{2x - 1}{x} = \boxed{\frac{y(2x - 1)}{x}},$$

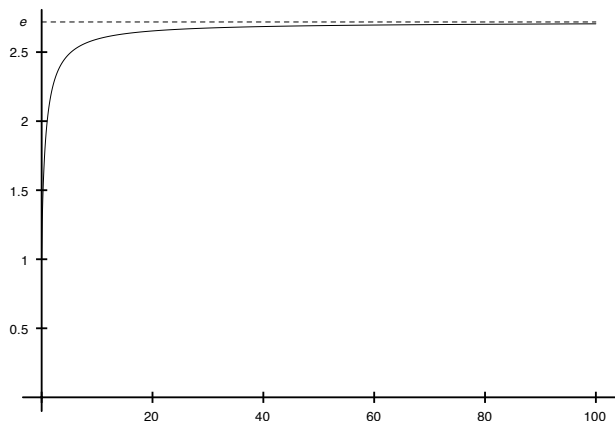
which matches the derivative calculated in Example 1(c).

93. Let $\ln T = kt$, where k is a constant. Then

$$\begin{aligned} \frac{d}{dt} \ln T &= \frac{d}{dt}(kt) \\ \frac{1}{T} \frac{dT}{dt} &= k \\ \frac{dT}{dt} &= kT. \end{aligned}$$

94. The figure below displays a graph of $y = \left(1 + \frac{1}{x}\right)^x$ and a graph of $y = e$. The graph of $y = \left(1 + \frac{1}{x}\right)^x$ appears to have a horizontal asymptote of $y = e$, supporting the definition that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$



95. Let $y = |[u(x)]^a| = |u(x)|^a$. Then

$$\ln y = \ln |u(x)|^a = a \ln |u(x)|,$$

so that

$$\frac{d}{dx} \ln y = \frac{d}{dx} [a \ln |u(x)|] \quad \text{and} \quad \frac{1}{y} \frac{dy}{dx} = \frac{a}{u(x)} u'(x).$$

Thus,

$$\frac{dy}{dx} = \frac{a}{u(x)} u'(x) \cdot y = \frac{a}{u(x)} u'(x) |u(x)|^a,$$

or

$$\frac{d}{dx} |u(x)|^a = a \frac{|u(x)|^a}{u(x)} u'(x).$$

When $u(x) > 0$, $|u(x)|^a = [u(x)]^a$, and

$$\frac{d}{dx} [u(x)]^a = a \frac{[u(x)]^a}{u(x)} u'(x) = a [u(x)]^{a-1} u'(x).$$

On the other hand, when $u(x) < 0$, $|u(x)|^a = [-u(x)]^a = (-1)^a [u(x)]^a$, so

$$\frac{d}{dx} \{(-1)^a [u(x)]^a\} = a \frac{(-1)^a [u(x)]^a}{u(x)} u'(x) = (-1)^a a [u(x)]^{a-1} u'(x),$$

or

$$\frac{d}{dx} [u(x)]^a = a [u(x)]^{a-1} u'(x).$$

96. Let $f(x) = -\frac{1}{2}x^2 + k$ and $g(x) = \ln(bx) + c$, where $b > 0$ and k and c are constants. Then

$$f'(x) = -x \quad \text{and} \quad g'(x) = \frac{1}{bx} \cdot b = \frac{1}{x}.$$

Thus, for any fixed $x \neq 0$, the slope of the tangent line to the graph of f is the negative reciprocal of the slope of the tangent line to the graph of g ; in other words, the tangent lines to the graphs of the family of parabolas given by $f(x)$ are always perpendicular to the tangent lines to the graphs of the family of natural logarithms given by $g(x)$.

Challenge Problems

97. Let

$$f(x) = 2x - \ln(3 + 6e^x + 3e^{2x}) = 2x - \ln[3(1 + e^x)^2] = 2x - \ln 3 - 2\ln(1 + e^x)$$

and $g(x) = -2\ln(1 + e^{-x})$. First, note that both functions have a domain of all real numbers. Next,

$$f'(x) = 2 - \frac{2}{1 + e^x} \cdot e^x = \frac{2(1 + e^x) - 2e^x}{1 + e^x} = \frac{2}{1 + e^x}$$

and

$$g'(x) = -\frac{2}{1 + e^{-x}} \cdot (-e^{-x}) = \frac{2e^{-x}}{1 + e^{-x}} = \frac{2}{1 + e^x}.$$

Thus, $f'(x) = g'(x)$ for all real numbers x , so there exists a constant C such that $f(x) = g(x) + C$ for all real numbers x .

98. Let $f(x) > 0$ and $y = f(x)^{g(x)}$. Then

$$\ln y = g(x) \ln f(x),$$

so

$$\begin{aligned} \frac{d}{dx} \ln y &= \frac{d}{dx} [g(x) \ln f(x)] \\ \frac{1}{y} \frac{dy}{dx} &= g(x) \frac{1}{f(x)} \cdot f'(x) + g'(x) \ln f(x) \\ &= \frac{g(x)}{f(x)} f'(x) + g'(x) \ln f(x), \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}f(x)^{g(x)} &= f(x)^{g(x)} \left(\frac{g(x)}{f(x)} f'(x) + g'(x) \ln f(x) \right) \\ &= g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}g'(x)\ln f(x).\end{aligned}$$

3.4: Differentials; Linear Approximations; Newton's Method

Concepts and Vocabulary

1. If $y = f(x)$ is a differentiable function, the differential $dy = \boxed{(d) f'(x)dx}$.
2. A linear approximation of a differentiable function f near x_0 is given by the function

$$L(x) = \boxed{f(x_0) + f'(x_0) \cdot (x - x_0)}.$$

3. $\boxed{\text{True}}$. The difference $|\Delta y - dy|$ measures the departure of the graph of $y = f(x)$ from the graph of the tangent line to f .
4. If Q is a quantity to be measured and ΔQ is the error made in measuring Q , then the relative error in the measurement is given by the ratio

$$\boxed{\frac{|\Delta Q|}{Q(x_0)}}.$$

5. $\boxed{\text{True}}$. Newton's Method uses tangent lines to the graph of f to approximate the zeros of f .
6. $\boxed{\text{True}}$. Before using Newton's Method, we need a first approximation for the zero.

Skill Building

7. Let $y = f(x) = x^3 - 2x + 1$. Then

$$dy = f'(x) dx = \boxed{(3x^2 - 2) dx}.$$

8. Let $y = f(x) = e^x + 2x - 1$. Then

$$dy = f'(x) dx = \boxed{(e^x + 2) dx}.$$

9. Let $y = f(x) = 4(x^2 + 1)^{3/2}$. Then

$$f'(x) = 4 \cdot \frac{3}{2}(x^2 + 1)^{1/2} \frac{d}{dx}(x^2 + 1) = 6(x^2 + 1)^{1/2} \cdot 2x = 12x(x^2 + 1)^{1/2},$$

and

$$dy = f'(x) dx = \boxed{12x(x^2 + 1)^{1/2} dx}.$$

10. Let $y = f(x) = \sqrt{x^2 - 1}$. Then

$$f'(x) = \frac{1}{2}(x^2 - 1)^{-1/2} \frac{d}{dx}(x^2 - 1) = \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x = \frac{x}{\sqrt{x^2 - 1}},$$

and

$$dy = f'(x) dx = \boxed{\frac{x}{\sqrt{x^2 - 1}} dx}.$$

11. Let $y = f(x) = 3 \sin(2x) + x$. Then

$$f'(x) = 3 \cos(2x) \frac{d}{dx}(2x) + 1 = 3 \cos(2x) \cdot 2 + 1 = 6 \cos(2x) + 1,$$

and

$$dy = f'(x) dx = \boxed{(6 \cos(2x) + 1) dx}.$$

12. Let $y = f(x) = \cos^2(3x) - x$. Then

$$\begin{aligned} f'(x) &= 2 \cos(3x) \frac{d}{dx} \cos(3x) - 1 = -2 \cos(3x) \sin(3x) \frac{d}{dx}(3x) - 1 \\ &= -2 \cos(3x) \sin(3x) \cdot 3 - 1 = -6 \cos(3x) \sin(3x) - 1, \end{aligned}$$

and

$$dy = f'(x) dx = \boxed{-(6 \cos(3x) \sin(3x) + 1) dx}.$$

13. Let $y = f(x) = e^{-x}$. Then

$$f'(x) = e^{-x} \frac{d}{dx}(-x) = -e^{-x},$$

and

$$dy = f'(x) dx = \boxed{-e^{-x} dx}.$$

14. Let $y = f(x) = e^{\sin x}$. Then

$$f'(x) = e^{\sin x} \frac{d}{dx} \sin x = \cos x e^{\sin x},$$

and

$$dy = f'(x) dx = \boxed{\cos x e^{\sin x} dx}.$$

15. Let $y = f(x) = xe^x$. Then

$$f'(x) = x \frac{d}{dx} e^x + e^x \frac{d}{dx} x = xe^x + e^x,$$

and

$$dy = f'(x) dx = \boxed{(xe^x + e^x) dx}.$$

16. Let $y = f(x) = \frac{e^{-x}}{x}$. Then

$$f'(x) = \frac{x \frac{d}{dx} e^{-x} - e^{-x} \frac{d}{dx} x}{x^2} = \frac{xe^{-x} \frac{d}{dx}(-x) - e^{-x}}{x^2} = -\frac{e^{-x}(x+1)}{x^2},$$

and

$$dy = f'(x) dx = \boxed{-\frac{e^{-x}(x+1)}{x^2} dx}.$$

17. Let $f(x) = e^x$.

(a) $dy = f'(x) dx = \boxed{e^x dx}.$

(b) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$dy = e^1(0.5) = \boxed{\frac{1}{2}e \approx 1.359141},$$

and

$$\Delta y = f(1 + 0.5) - f(1) = \boxed{e^{1.5} - e \approx 1.763407}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$dy = e^1(0.1) = \boxed{\frac{1}{10}e \approx 0.271828},$$

and

$$\Delta y = f(1 + 0.1) - f(1) = \boxed{e^{1.1} - e \approx 0.285884}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$dy = e^1(0.01) = \boxed{\frac{1}{100}e \approx 0.027183},$$

and

$$\Delta y = f(1 + 0.01) - f(1) = \boxed{e^{1.01} - e \approx 0.027319}.$$

(c) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|1.763407 - 1.359141| = 0.404266}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|0.285884 - 0.271828| = 0.014056}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|0.027319 - 0.027183| = 0.000136}.$$

18. Let $f(x) = e^{-x}$.

(a) $dy = f'(x) dx = \boxed{-e^{-x} dx}.$

(b) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$dy = -e^{-1}(0.5) = \boxed{-\frac{1}{2e} \approx -0.183940},$$

and

$$\Delta y = f(1 + 0.5) - f(1) = \boxed{e^{-1.5} - e^{-1} \approx -0.144749}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$dy = -e^{-1}(0.1) = \boxed{-\frac{1}{10e} \approx -0.036788},$$

and

$$\Delta y = f(1 + 0.1) - f(1) = \boxed{e^{-1.1} - e^{-1} \approx -0.035008}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$dy = -e^{-1}(0.01) = \boxed{-\frac{1}{100e} \approx -0.003679},$$

and

$$\Delta y = f(1 + 0.01) - f(1) = \boxed{e^{-1.01} - e^{-1} \approx -0.003660}.$$

(c) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|-0.144749 - (-0.183940)| = 0.039191}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|-0.035008 - (-0.036788)| = 0.001780}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|-0.003660 - (-0.003679)| = 0.000019}.$$

19. Let $f(x) = x^{2/3}$.

$$(a) \quad dy = f'(x) dx = \frac{2}{3} x^{-1/3} dx = \boxed{\frac{2}{3\sqrt[3]{x}} dx}.$$

(b) i. With $x = 2$ and $\Delta x = dx = 0.5$,

$$dy = \frac{2}{3\sqrt[3]{2}}(0.5) \approx 0.264567,$$

and

$$\Delta y = f(2 + 0.5) - f(2) = \boxed{(2.5)^{2/3} - 2^{2/3} \approx 0.254615}.$$

ii. With $x = 2$ and $\Delta x = dx = 0.1$,

$$dy = \frac{2}{3\sqrt[3]{2}}(0.1) \approx 0.052913,$$

and

$$\Delta y = f(2 + 0.1) - f(2) = \boxed{(2.1)^{2/3} - 2^{2/3} \approx 0.052482}.$$

iii. With $x = 2$ and $\Delta x = dx = 0.01$,

$$dy = \frac{2}{3\sqrt[3]{2}}(0.01) \approx 0.005291,$$

and

$$\Delta y = f(2 + 0.01) - f(2) = \boxed{(2.01)^{2/3} - 2^{2/3} \approx 0.005287}.$$

(c) i. With $x = 2$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|0.254615 - 0.264567| = 0.009952}.$$

ii. With $x = 2$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|0.052482 - 0.052913| = 0.000431}.$$

iii. With $x = 2$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|0.005287 - 0.005291| = 0.000004 = 4 \times 10^{-6}}.$$

20. Let $f(x) = x^{-1/2}$.

$$(a) \quad dy = f'(x) dx = \boxed{-\frac{1}{2} x^{-3/2} dx}.$$

(b) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$dy = -\frac{1}{2} 1^{-3/2}(0.5) = \boxed{-\frac{1}{4} = -0.25},$$

and

$$\Delta y = f(1 + 0.5) - f(1) = \boxed{(1.5)^{-1/2} - 1 \approx -0.183503}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$dy = -\frac{1}{2}1^{-3/2}(0.1) = \boxed{-\frac{1}{20} = -0.05},$$

and

$$\Delta y = f(1 + 0.1) - f(1) = \boxed{(1.1)^{-1/2} - 1 \approx -0.046537}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$dy = -\frac{1}{2}1^{-3/2}(0.01) = \boxed{-\frac{1}{200} = -0.005},$$

and

$$\Delta y = f(1 + 0.01) - f(1) = \boxed{(1.01)^{-1/2} - 1 \approx -0.004963}.$$

(c) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|-0.183503 - (-0.25)| = 0.066497}.$$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|-0.046537 - (-0.05)| = 0.003463}.$$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|-0.004963 - (-0.005)| = 0.000037}.$$

21. Let $f(x) = \cos x$.

(a) $dy = f'(x) dx = \boxed{-\sin x dx}.$

(b) i. With $x = \pi$ and $\Delta x = dx = 0.5$,

$$dy = -\sin \pi(0.5) = \boxed{0},$$

and

$$\Delta y = f(\pi + 0.5) - f(\pi) = \boxed{\cos(\pi + 0.5) - \cos(\pi) \approx 0.122417}.$$

ii. With $x = \pi$ and $\Delta x = dx = 0.1$,

$$dy = -\sin \pi(0.1) = \boxed{0},$$

and

$$\Delta y = f(\pi + 0.1) - f(\pi) = \boxed{\cos(\pi + 0.1) - \cos \pi \approx 0.004996}.$$

iii. With $x = \pi$ and $\Delta x = dx = 0.01$,

$$dy = -\sin \pi(0.01) = \boxed{0},$$

and

$$\Delta y = f(\pi + 0.01) - f(\pi) = \boxed{\cos(\pi + 0.01) - \cos \pi \approx 0.000050}.$$

(c) i. With $x = \pi$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|0.122417 - 0| = 0.122417}.$$

- ii. With $x = \pi$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|0.004996 - 0| = 0.004996}.$$

- iii. With $x = \pi$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|0.000050 - 0| = 0.000050}.$$

22. Let $f(x) = \tan x$.

(a) $dy = f'(x) dx = \boxed{\sec^2 x dx}$.

- (b) i. With $x = 0$ and $\Delta x = dx = 0.5$,

$$dy = \sec^2 0(0.5) = \boxed{0.5},$$

and

$$\Delta y = f(0 + 0.5) - f(0) = \boxed{\tan 0.5 - \tan 0 \approx 0.546302}.$$

- ii. With $x = 0$ and $\Delta x = dx = 0.1$,

$$dy = \sec^2 0(0.1) = \boxed{0.1},$$

and

$$\Delta y = f(0 + 0.1) - f(0) = \boxed{\tan 0.1 - \tan 0 \approx 0.100335}.$$

- iii. With $x = 0$ and $\Delta x = dx = 0.01$,

$$dy = \sec^2 0(0.01) = \boxed{0.01},$$

and

$$\Delta y = f(0 + 0.01) - f(0) = \boxed{\tan 0.01 - \tan 0 \approx 0.010000333}.$$

- (c) i. With $x = 0$ and $\Delta x = dx = 0.5$,

$$|\Delta y - dy| \approx \boxed{|0.546302 - 0.5| = 0.046302}.$$

- ii. With $x = 0$ and $\Delta x = dx = 0.1$,

$$|\Delta y - dy| \approx \boxed{|0.100335 - 0.1| = 0.000335}.$$

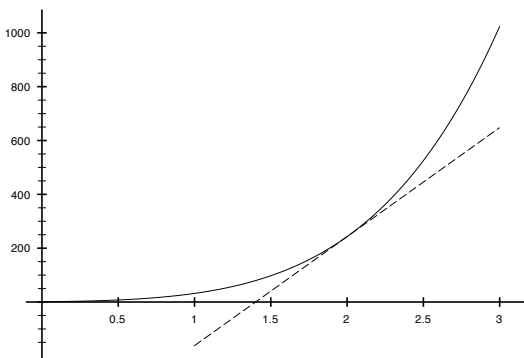
- iii. With $x = 0$ and $\Delta x = dx = 0.01$,

$$|\Delta y - dy| \approx \boxed{|0.010000333 - 0.01| = 3.33 \times 10^{-7}}.$$

23. (a) Let $f(x) = (x + 1)^5$ and $x_0 = 2$. Then $f(x_0) = f(2) = 3^5 = 243$, $f'(x) = 5(x + 1)^4$, $f'(x_0) = f'(2) = 5(3)^4 = 405$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 243 + 405(x - 2) = \boxed{405x - 567}.$$

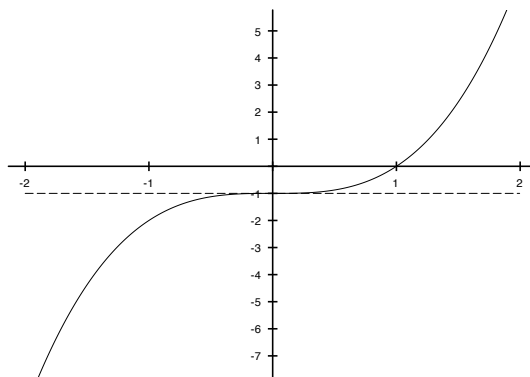
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



24. (a) Let $f(x) = x^3 - 1$ and $x_0 = 0$. Then $f(x_0) = f(0) = -1$, $f'(x) = 3x^2$, $f'(x_0) = f'(0) = 0$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = -1 + 0(x - 0) = \boxed{-1}.$$

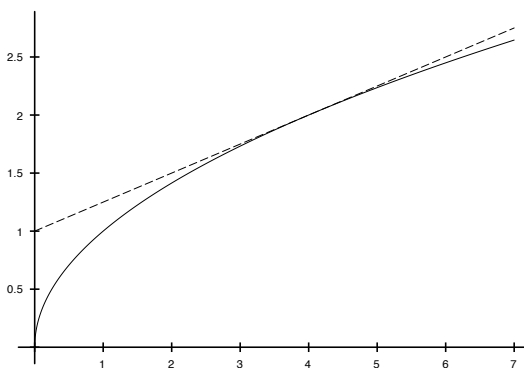
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



25. (a) Let $f(x) = \sqrt{x}$ and $x_0 = 4$. Then $f(x_0) = f(4) = 2$, $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(x_0) = f'(4) = \frac{1}{4}$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 2 + \frac{1}{4}(x - 4) = \boxed{\frac{1}{4}x + 1}.$$

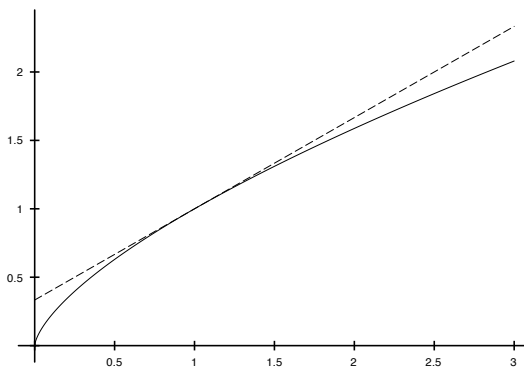
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



26. (a) Let $f(x) = x^{2/3}$ and $x_0 = 1$. Then $f(x_0) = f(1) = 1$, $f'(x) = \frac{2}{3}x^{-1/3}$, $f'(x_0) = f'(1) = \frac{2}{3}$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 1 + \frac{2}{3}(x - 1) = \boxed{\frac{2}{3}x + \frac{1}{3}}.$$

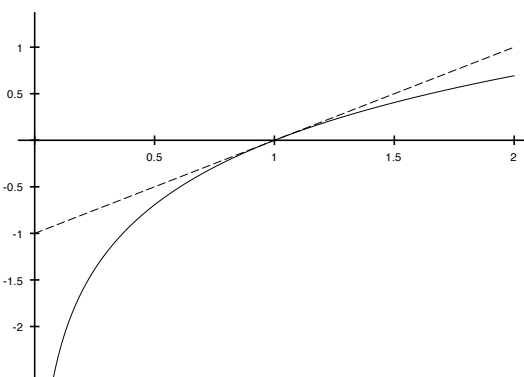
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



27. (a) Let $f(x) = \ln x$ and $x_0 = 1$. Then $f(x_0) = f(1) = 0$, $f'(x) = \frac{1}{x}$, $f'(x_0) = f'(1) = 1$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 1) = \boxed{x - 1}.$$

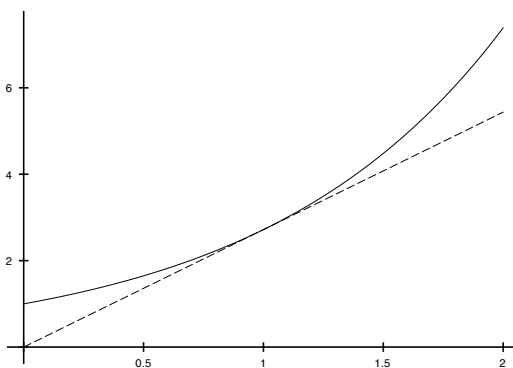
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



28. (a) Let $f(x) = e^x$ and $x_0 = 1$. Then $f(x_0) = f(1) = e$, $f'(x) = e^x$, $f'(x_0) = f'(1) = e$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = e + e(x - 1) = \boxed{ex}.$$

- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.

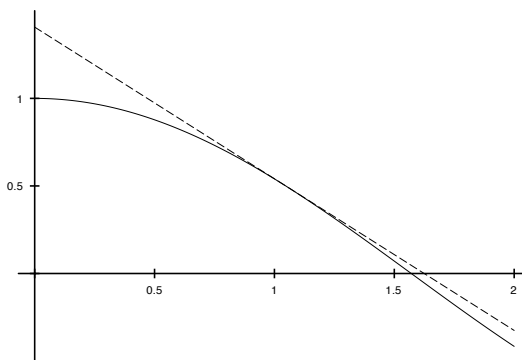


29. (a) Let $f(x) = \cos x$ and $x_0 = \frac{\pi}{3}$. Then $f(x_0) = f\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $f'(x) = -\sin x$,

$$f'(x_0) = f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \text{ and}$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = \frac{1}{2} - \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right) = \boxed{-\frac{\sqrt{3}}{2}x + \frac{1}{2} + \frac{\pi\sqrt{3}}{6}}.$$

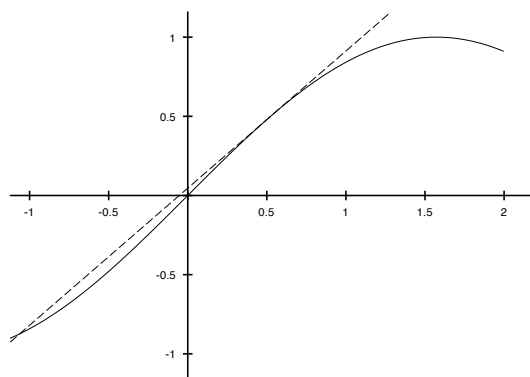
- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



30. (a) Let $f(x) = \sin x$ and $x_0 = \frac{\pi}{6}$. Then $f(x_0) = f\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $f'(x) = \cos x$, $f'(x_0) = f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, and

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}x + \frac{1}{2} - \frac{\pi\sqrt{3}}{12}}.$$

- (b) The figure below displays the graph of f as the solid curve and the graph of L as the dashed curve.



31. (a) Let $f(x) = x^2$ and $\Delta x = dx = 3.001 - 3 = 0.001$. Then $f'(x) = 2x$ and

$$\Delta y \approx dy = f'(3) dx = 6(0.001) = \boxed{0.006}.$$

- (b) Let $f(x) = \frac{1}{x+2}$ and $\Delta x = dx = 1.98 - 2 = -0.02$. Then $f'(x) = -\frac{1}{(x+2)^2}$ and

$$\Delta y \approx dy = f'(2) dx = -\frac{1}{16}(-0.02) = \boxed{0.00125}.$$

32. (a) Let $f(x) = x^3$ and $\Delta x = dx = 3.01 - 3 = 0.01$. Then $f'(x) = 3x^2$ and

$$\Delta y \approx dy = f'(3) dx = 27(0.01) = \boxed{0.27}.$$

- (b) Let $f(x) = \frac{1}{x-1}$ and $\Delta x = dx = 1.98 - 2 = -0.02$. Then $f'(x) = -\frac{1}{(x-1)^2}$ and

$$\Delta y \approx dy = f'(2) dx = -\frac{1}{1}(-0.02) = \boxed{0.02}.$$

33. (a) Let $f(x) = x^3 + 3x - 5$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 + 3 - 5 = -1 < 0$ and $f(2) = 2^3 + 3(2) - 5 = 9 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(1, 2)$.

- (b) With $f(x) = x^3 + 3x - 5$, $f'(x) = 3x^2 + 3$. Let $c_1 = 1.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{2.875}{9.75} \approx 1.205128.$$

Next, $f(c_2) \approx 0.365632$ and $f'(c_2) \approx 7.357000$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.205128 - \frac{0.365632}{7.357000} \approx \boxed{1.155429}.$$

34. (a) Let $f(x) = x^3 - 4x + 2$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 4 + 2 = -1 < 0$ and $f(2) = 2^3 - 4(2) + 2 = 2 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(1, 2)$.

- (b) With $f(x) = x^3 - 4x + 2$, $f'(x) = 3x^2 - 4$. Let $c_1 = 1.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-0.625}{2.75} \approx 1.727273.$$

Next, $f(c_2) \approx 0.244179$ and $f'(c_2) \approx 4.950416$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.727273 - \frac{0.244179}{4.950416} \approx \boxed{1.677948}.$$

35. (a) Let $f(x) = 2x^3 + 3x^2 + 4x - 1$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -1 < 0$ and $f(1) = 2 + 3 + 4 - 1 = 8 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.

- (b) With $f(x) = 2x^3 + 3x^2 + 4x - 1$, $f'(x) = 6x^2 + 6x + 4$. Let $c_1 = 0.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{2}{8.5} \approx 0.264706.$$

Next, $f(c_2) \approx 0.306127$ and $f'(c_2) \approx 6.008652$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.264706 - \frac{0.306127}{6.008652} \approx \boxed{0.213758}.$$

36. (a) Let $f(x) = x^3 - x^2 - 2x + 1$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = 1 > 0$ and $f(1) = 1 - 1 - 2 + 1 = -1 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.

- (b) With $f(x) = x^3 - x^2 - 2x + 1$, $f'(x) = 3x^2 - 2x - 2$. Let $c_1 = 0.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{-0.125}{-2.25} \approx 0.444444.$$

Next, $f(c_2) \approx 0.001373$ and $f'(c_2) \approx -2.296297$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.444444 - \frac{0.001373}{-2.296297} \approx \boxed{0.445042}.$$

37. (a) Let $f(x) = x^3 - 6x - 12$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[3, 4]$. Because $f(3) = 3^3 - 6(3) - 12 = -3 < 0$ and $f(4) = 4^3 - 6(4) - 12 = 28 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(3, 4)$.

- (b) With $f(x) = x^3 - 6x - 12$, $f'(x) = 3x^2 - 6$. Let $c_1 = 3.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.5 - \frac{9.875}{30.75} \approx 3.178862.$$

Next, $f(c_2) \approx 1.049749$ and $f'(c_2) \approx 24.315491$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 3.178862 - \frac{1.049749}{24.315491} \approx \boxed{3.135690}.$$

38. (a) Let $f(x) = 3x^3 + 5x - 40$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[2, 3]$. Because $f(2) = 3(2)^3 + 5(2) - 40 = -6 < 0$ and $f(3) = 3(3)^3 + 5(3) - 40 = 56 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(2, 3)$.

- (b) With $f(x) = 3x^3 + 5x - 40$, $f'(x) = 9x^2 + 5$. Let $c_1 = 2.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{19.375}{61.25} \approx 2.183673.$$

Next, $f(c_2) \approx 2.156426$ and $f'(c_2) \approx 47.915850$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 2.183673 - \frac{2.156426}{47.915850} \approx \boxed{2.138669}.$$

39. (a) Let $f(x) = x^4 - 2x^3 + 21x - 23$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 2 + 21 - 23 = -3 < 0$ and $f(2) = 2^4 - 2(2)^3 + 21(2) - 23 = 19 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(1, 2)$.

- (b) Answers will vary. Using the midpoint of the interval $(1, 2)$ as the first approximation c_1 yields the following results. With $f(x) = x^4 - 2x^3 + 21x - 23$, $f'(x) = 4x^3 - 6x^2 + 21$. Let $c_1 = 1.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{6.8125}{21} \approx 1.175595.$$

Next, $f(c_2) \approx 0.348084$ and $f'(c_2) \approx 19.206659$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.175595 - \frac{0.348084}{19.206659} \approx \boxed{1.157472}.$$

40. (a) Let $f(x) = x^4 - x^3 + x - 2$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 1 + 1 - 2 = -1 < 0$ and $f(2) = 2^4 - 2^3 + 2 - 2 = 8 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(1, 2)$.

- (b) Answers will vary. Using the midpoint of the interval $(1, 2)$ as the first approximation c_1 yields the following results. With $f(x) = x^4 - x^3 + x - 2$, $f'(x) = 4x^3 - 3x^2 + 1$. Let $c_1 = 1.5$. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{1.1875}{7.75} \approx 1.346774.$$

Next, $f(c_2) \approx 0.193866$ and $f'(c_2) \approx 5.329715$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.346774 - \frac{0.193866}{5.329715} \approx \boxed{1.310399}.$$

41. (a) Let $f(x) = x + e^x$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[-1, 0]$. Because $f(-1) = -1 + e^{-1} \approx -0.632 < 0$ and $f(0) = 1 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(-1, 0)$.
- (b) Begin by calculating $f'(x) = 1 + e^x$. Then, using a TI-84 Plus graphing calculator with $c_1 = -0.5$ yields

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx -0.5663110032; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx -0.5671431650; \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx -0.5671432904; \text{ and} \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{-0.5671432904}. \end{aligned}$$

42. (a) Let $f(x) = x - e^{-x}$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -1 < 0$ and $f(1) = 1 - e^{-1} \approx 0.632 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.
- (b) Begin by calculating $f'(x) = 1 + e^{-x}$. Then, using a TI-84 Plus graphing calculator with $c_1 = 0.5$ yields

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 0.5663110032; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.5671431650; \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 0.5671432904; \text{ and} \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{0.5671432904}. \end{aligned}$$

43. (a) Let $f(x) = x^3 + \cos^2 x$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[-1, 0]$. Because $f(-1) = -1 + \cos^2(-1) \approx -0.708 < 0$ and $f(0) = 1 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(-1, 0)$.
- (b) Begin by calculating

$$f'(x) = 3x^2 + 2 \cos x \frac{d}{dx} \cos x = 3x^2 - 2 \sin x \cos x.$$

Then, using a TI-84 Plus graphing calculator with $c_1 = -0.5$ yields

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx -0.9053804053; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx -0.8001514047; \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx -0.7908914208; \text{ and} \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{-0.7908208349}. \end{aligned}$$

44. (a) Let $f(x) = x^2 + 2 \sin x - 0.5$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -0.5 < 0$ and $f(1) = 1 + 2 \sin 1 - 0.5 \approx 2.183 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.

- (b) Begin by calculating $f'(x) = 2x + 2\cos x$. Then, using a TI-84 Plus graphing calculator with $c_1 = 0.5$ yields

$$\begin{aligned}c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 0.2427192037; \\c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.2264020798; \\c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 0.2263172819; \text{ and} \\c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{0.2263172795}.\end{aligned}$$

45. (a) Let $f(x) = 5 - \sqrt{x^2 + 2}$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[4, 5]$. Because $f(4) = 5 - \sqrt{18} \approx 0.757 > 0$ and $f(5) = 5 - \sqrt{27} \approx -0.196 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(4, 5)$.

- (b) Begin by calculating

$$f'(x) = -\frac{1}{2}(x^2 + 2)^{-1/2} \frac{d}{dx}(x^2 + 2) = -\frac{1}{2}(x^2 + 2)^{-1/2} \cdot 2x = -\frac{x}{\sqrt{x^2 + 2}}.$$

Then, using a TI-84 Plus graphing calculator with $c_1 = 4.5$ yields

$$\begin{aligned}c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 4.796656184; \\c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 4.795831529; \\c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 4.795831524; \text{ and} \\c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{4.795831523}.\end{aligned}$$

46. (a) Let $f(x) = 2x^2 + x^{2/3} - 4$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 2 + 1 - 4 = -1 < 0$ and $f(2) = 8 - 2^{2/3} - 4 \approx 2.413 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(1, 2)$.

- (b) Begin by calculating $f'(x) = 4x + \frac{2}{3}x^{-1/3}$. Then, using a TI-84 Plus graphing calculator with $c_1 = 1.5$ yields

$$\begin{aligned}c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 1.224967447; \\c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.198542475; \\c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 1.198295916; \text{ and} \\c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{1.198295895}.\end{aligned}$$

47. The formula for the area of a circle is $A = \pi R^2$, where R is the radius of the circle. With $\Delta R = dR = 10.1 - 10 = 0.1$ cm, the approximate increase in the area of the top surface is

$$\Delta A \approx dA = 2\pi R dR = 2\pi(10)(0.1) = \boxed{2\pi \text{ cm}^2}.$$

48. The volume V of a circular cylinder with a height of 3 cm and a radius r is $V = \pi r^2 h = 3\pi r^2$. With $\Delta r = dr = 2.2 - 2 = 0.2$ cm, the approximate volume of wood removed is

$$\Delta V \approx dV = 6\pi r dr = 6\pi(2)(0.2) = \boxed{2.4\pi \text{ cm}^3}.$$

49. The volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. With $\Delta r = dr = 3.1 - 3 = 0.1$ m, the approximate change in the volume of the balloon is

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi(3)^2(0.1) = \boxed{3.6\pi \text{ m}^3}.$$

50. The volume V of a right circular cone with a height h equal to four times its radius r is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(4r) = \frac{4}{3}\pi r^3$. With $\Delta r = dr = 1.9 - 2 = -0.1$ cm, the approximate change in volume of the cup is

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi(2)^2(-0.1) = -1.6\pi \text{ cm}^3 \approx -5.0 \text{ cm}^3.$$

The approximate loss in capacity of the cup is $\boxed{5.0 \text{ cm}^3}$.

51. (a) The volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. With $\Delta r = dr = 2.1 - 2 = 0.1$ m, the approximate volume of material needed for the specified hollow sphere is

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi(2)^2(0.1) = \boxed{1.6\pi \text{ m}^3} \approx 5.0 \text{ m}^3.$$

- (b) The actual amount of material needed for the specified hollow sphere is

$$\Delta V = V(2.1) - V(2) = \frac{4}{3}\pi(2.1)^3 - \frac{4}{3}\pi(2)^3 \approx 5.3 \text{ m}^3.$$

The approximation using differentials $\boxed{\text{underestimates}}$ the amount of material needed.

- (c) Answers will vary. One possible response is that knowing the approximation using differentials underestimates the actual material needed, the manufacturer will know to adjust upward the amount of material needed to complete the order for 10,000 spheres so as not to run out before the order has been filled.
52. (a) The difference in the distances traveled by the bugs is the difference between the circumference of a circle with a radius of 7 cm (the ant) and the circumference of a circle with a radius of 9 cm (the bee). The circumference C of a circle of radius r is $C = 2\pi r$. With $\Delta r = dr = 9 - 7 = 2$ cm, the approximate difference in the distances traveled by the two bugs is

$$\Delta C \approx dC = 2\pi dr = 2\pi(2) = \boxed{4\pi \text{ cm}}.$$

- (b) The actual difference in the distances traveled by the two bugs is

$$\Delta C = C(9) - C(7) = 2\pi(9) - 2\pi(7) = 4\pi \text{ cm}.$$

The linear approximation $\boxed{\text{neither underestimates nor overestimates}}$ the difference in the distances traveled. The linear approximation provides the exact value for the difference because the circumference of a circle is a linear function of the radius of the circle.

53. Let h denote the height of the building and x denote the length of the shadow of the 3-meter pole. Applying similar triangles to the figure in the text yields

$$\frac{h}{9+x} = \frac{3}{x} \quad \text{or} \quad h = \frac{27+3x}{x} = \frac{27}{x} + 3.$$

When $x = 1$ meter, the height of the building is calculated to be

$$h = \frac{27}{1} + 3 = 30 \text{ meters.}$$

Because the percentage error in the measurement of the length of the shadow of the 3-meter pole is 1%,

$$\frac{|\Delta x|}{x} = \frac{|dx|}{x} = 0.01,$$

so $|dx| = 0.01(1) = 0.01$, or $dx = \pm 0.01$. Using differentials, it then follows that

$$\Delta h \approx dh = -\frac{27}{x^2} dx = -\frac{27}{1^2} (\pm 0.01) = \pm 0.27 \text{ meters.}$$

Therefore, the height of the building is $\boxed{30 \pm 0.27 \text{ meters}}$. The percentage error in the estimate for the height of the building is

$$\frac{|\Delta h|}{h} \times 100\% \approx \frac{|\pm 0.27|}{30} \times 100\% = \boxed{0.9\%}.$$

54. Let $T = 2\pi\sqrt{\frac{l}{g}}$. The change in the period of the pendulum as a result of a change in the length of the pendulum is then

$$\Delta T \approx dT = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2\sqrt{l}} dl = \frac{\pi}{\sqrt{gl}} dl,$$

so

$$\frac{|\Delta T|}{T} \approx \frac{|dT|}{T} = \frac{\pi/\sqrt{gl}}{2\pi\sqrt{l/g}} |dl| = \frac{1}{2} \cdot \frac{|dl|}{l}.$$

Because the length of the pendulum has increased by 1%, $|dl|/l = 0.01$, so the percentage change in the period of the pendulum is

$$\frac{|\Delta T|}{T} \times 100\% \approx \frac{1}{2}(0.01) \times 100\% = \boxed{0.5\%}.$$

The length of the pendulum has increased, so the period of the pendulum has also increased, meaning that it now takes longer for the pendulum to swing through a full cycle than was designed. Accordingly, the clock loses time each day; specifically, it loses approximately 0.5% of the day, or

$$0.005(24 \text{ hours}) = 0.12 \text{ hours} = \boxed{7.2 \text{ minutes}}$$

each day.

55. If the length of the pendulum is normally 1 meter and the length has been increased by 10 cm, then

$$\frac{|\Delta l|}{l} \times 100\% = \frac{0.1}{1} \times 100\% = 10\%;$$

that is, the length of the pendulum has been increased by 10%. Based on Problem 54, the period of the pendulum will increase by 5%. Accordingly, the clock loses approximately 5% of the day, or

$$0.05(24 \text{ hours}) = 1.2 \text{ hours} = \boxed{72 \text{ minutes}}$$

each day.

56. (a) Let $L(t) = \sigma A[T(t)]^4$. Then

$$\frac{dL}{dt} = 4\sigma AT^3 \frac{dT}{dt},$$

so that

$$\frac{dT}{dt} = \frac{1}{4\sigma T^3} \frac{dL}{dt} = \frac{T}{4\sigma AT^4} \frac{dL}{dt} = \boxed{\frac{T}{4L} \frac{dL}{dt}}.$$

- (b) Let L_c denote the current luminosity and T_c the current temperature of the sun. By part (a),

$$\begin{aligned} \frac{dT}{dt} &= \frac{T_c}{4L_c} \frac{dL}{dt} \approx \frac{T_c}{4L_c} \cdot \frac{\Delta L}{\Delta t} = \frac{T_c}{4L_c} \cdot \frac{0.3L_c}{4.5} \\ &\approx 1.67 \times 10^{-2} T_c \frac{\text{K}}{\text{billion years}} \cdot \frac{1 \text{ billion years}}{10^7 \text{ centuries}} \\ &= \boxed{1.67 \times 10^{-9} T_c \frac{\text{K}}{\text{century}}}. \end{aligned}$$

57. With $\Delta r = dr = 8.8$ km and

$$G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \cdot \left(\frac{1 \text{ km}}{1000 \text{ m}} \right)^3 = 6.67 \times 10^{-20} \frac{\text{km}^3}{\text{kg} \cdot \text{s}^2},$$

it follows that

$$\begin{aligned} \Delta W &\approx dW = -\frac{2GmM}{r^3} dr \\ &= -\frac{2(6.67 \times 10^{-20})(70)(5.974 \times 10^{24})}{(6370)^3} (8.8) \approx -0.0019 \text{ kg}. \end{aligned}$$

Thus, a person who weighs 70 kg at sea level will weigh approximately $\boxed{70 - 0.0019 = 69.9981 \text{ kg}}$ at the top of Mount Everest.

58. (a) The man is 66 inches tall and weighs 142 lb in the morning. Thus, his BMI in the morning is

$$\text{BMI} = 703 \frac{142}{66^2} \approx \boxed{22.917}.$$

With $\Delta m = dm = 148 - 142 = 6$ lb, the change in this man's BMI in the afternoon is approximately

$$\Delta \text{BMI} \approx d\text{BMI} = \frac{703}{h^2} dm = \frac{703}{66^2} (6) \approx \boxed{0.968}.$$

- (b) The actual change in this man's BMI in the afternoon is

$$\Delta \text{BMI} = 703 \frac{148}{66^2} - 703 \frac{142}{66^2} = \frac{703(6)}{66^2} \approx 0.968.$$

Thus, the linear approximation $\boxed{\text{neither underestimates nor overestimates}}$ the change in BMI. The linear approximation provides the exact value for the change because BMI is a linear function of the person's weight.

- (c) A woman who weighs 165 pounds and is 68 inches tall has a BMI of

$$\text{BMI} = 703 \frac{165}{68^2} \approx \boxed{25.085}.$$

If the error in the height measurement is ± 1.5 inches, then $\Delta h = dh = \pm 1.5$ and the possible error in the calculation of BMI is

$$\Delta \text{BMI} \approx d\text{BMI} = -1406 \frac{m}{h^3} dh = -1406 \frac{165}{68^3} (\pm 1.5) \approx \boxed{\pm 1.107}.$$

- (d) Answers will vary. One possible response is that the man's BMI, both in the morning and in the afternoon, would lead to a classification of the man's weight as normal. Using the indicated height of 68 inches, the woman's BMI would lead to a classification as overweight. However, taking into account the possible error in the woman's height, her BMI could be as low as roughly $25.085 - 1.107 = 23.978$, which is in the normal weight range.

59. The volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$, so

$$r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3}{4\pi}} \cdot V^{1/3}.$$

If the volume is measured with an accuracy of $\pm 1\%$, then

$$\frac{|\Delta V|}{V} = \frac{|dV|}{V} \leq 0.01$$

and

$$\frac{|\Delta r|}{r} \approx \frac{|dr|}{r} = \frac{\sqrt[3]{\frac{3}{4\pi}} \cdot \frac{1}{3} V^{-2/3} |dV|}{\sqrt[3]{\frac{3}{4\pi}} \cdot V^{1/3}} = \frac{1}{3} \frac{|dV|}{V} \leq \frac{1}{300}.$$

The percentage error in the radius of the sphere caused by the error in measuring the volume of the sphere is approximately $\boxed{1/3 \%}$.

60. Let $V = \frac{1}{3}\pi h^2(3R - h) = \pi h^2 R - \frac{1}{3}\pi h^3$. Then

$$\Delta V \approx dV = (2\pi h R - \pi h^2) dh = \pi h(2R - h) dh$$

and

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\pi h(2R - h) dh}{\frac{1}{3}\pi h^2(3R - h)} = \frac{2R - h}{R - \frac{1}{3}h} \cdot \frac{dh}{h}.$$

If the margin of error in the measurement of h is 10%, then

$$\frac{|\Delta h|}{h} = \frac{|dh|}{h} = 0.1;$$

hence, with $h = 3$ cm and $R = 8$ cm,

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{16 - 3}{8 - 1}(0.1) \approx 0.186.$$

The approximate error in the calculated volume is $\boxed{18.6\%}$.

61. The volume V of a cube with edge length s is $V = s^3$. Suppose the percentage error in measuring the edge length is 2%; that is,

$$\frac{|\Delta s|}{s} = \frac{|ds|}{s} = 0.02.$$

Then

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{3s^2 |ds|}{s^3} = 3 \frac{|ds|}{s} = 0.06;$$

that is, the percentage error in the calculated volume is approximately $\boxed{6\%}$.

62. Solving the thin-lens equation for q yields

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p - f}{pf}, \quad \text{so that} \quad q = \frac{pf}{p - f}.$$

It then follows that

$$\Delta q \approx dq = q'(p) dp = \frac{(p-f)f - pf}{(p-f)^2} dp = -\frac{f^2}{(p-f)^2} dp.$$

With $f = 50 \text{ mm} = 0.05 \text{ m}$, $p = 15 \text{ m}$, and $\Delta p = dp = -0.33 \text{ m}$ (note the dog has moved closer to the lens so that p has decreased),

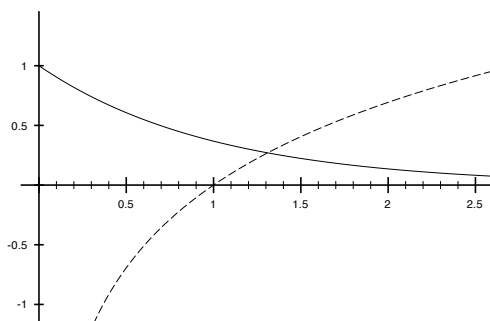
$$\Delta q \approx dq = -\frac{0.05^2}{(15 - 0.05)^2}(-0.33) \approx 3.69 \times 10^{-6} \text{ m}.$$

Thus, the image of the dog moves approximately $\boxed{3.69 \times 10^{-6} \text{ meters}}$ away from the lens.

63. The figure below displays the graph of $y = e^{-x}$ as the solid curve and the graph of $y = \ln x$ as the dashed curve. The graphs intersect near $x = 1.3$, so let $f(x) = e^{-x} - \ln x$ and take $c_1 = 1.3$. Newton's Method then generates the sequence of approximations:

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 1.309759929; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.309799585; \text{ and} \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 1.309799586. \end{aligned}$$

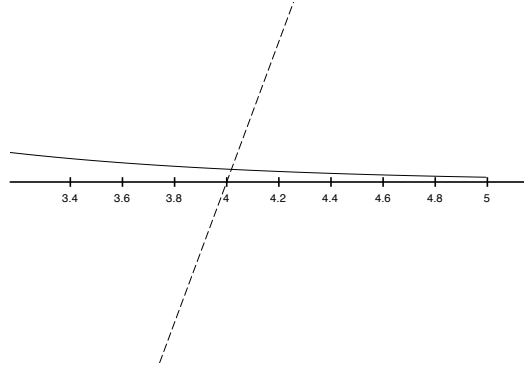
Correct to three decimal places, the solution of the equation $e^{-x} = \ln x$ is $\boxed{x = 1.309}$.



64. The figure below displays the graph of $y = e^{-x}$ as the solid curve and the graph of $y = x - 4$ as the dashed curve. The graphs intersect near $x = 4.0$, so let $f(x) = e^{-x} - x + 4$ and take $c_1 = 4.0$. Newton's Method then generates the sequence of approximations:

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 4.017986210; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 4.017989103; \text{ and} \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 4.017989103. \end{aligned}$$

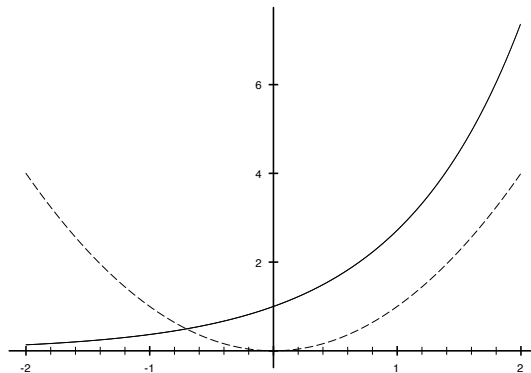
Correct to three decimal places, the solution of the equation $e^{-x} = x - 4$ is $\boxed{x = 4.017}$.



65. The figure below displays the graph of $y = e^x$ as the solid curve and the graph of $y = x^2$ as the dashed curve. The graphs intersect near $x = -0.7$, so let $f(x) = e^x - x^2$ and take $c_1 = -0.7$. Newton's Method then generates the sequence of approximations:

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx -0.7034721896; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx -0.7034674225; \text{ and} \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx -0.7034674225. \end{aligned}$$

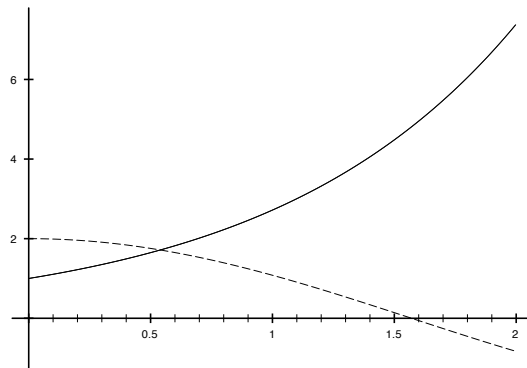
Correct to three decimal places, the solution of the equation $e^x = x^2$ is $\boxed{x = -0.703}$.



66. The equation $e^x = 2 \cos x$ has an infinite number of solutions for $x < 0$. For $x < 0$, the graph of $y = e^x$ asymptotically approaches the x -axis, so there will be a solution “near” each x -intercept of the cosine graph. However, the equation $e^x = 2 \cos x$ has only one solution for $x > 0$. The figure below displays the graph of $y = e^x$ as the solid curve and the graph of $y = 2 \cos x$ as the dashed curve. The graphs intersect near $x = 0.5$, so let $f(x) = e^x - 2 \cos x$ and take $c_1 = 0.5$. Newton's Method then generates the sequence of approximations:

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 0.5408210546; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.5397858311; \text{ and} \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 0.5397851608. \end{aligned}$$

Correct to three decimal places, the unique solution of the equation $e^x = 2 \cos x$ for $x > 0$ is $\boxed{x = 0.539}$.



67. Let $f(x) = \ln x - 1$ and take $c_1 = 3.0$. Newton's Method then generates the sequence of approximations:

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 2.704163134; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 2.718245099; \text{ and} \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx \boxed{2.718281828}. \end{aligned}$$

A TI-84 Plus calculator gives $e \approx 2.718281828$, so the fourth approximation obtained from Newton's Method matches the calculator to 9 decimal places.

68. Let $f(x) = (1+x)^k$, where x is near 0 and k is any real number. Then $f(0) = 1^k = 1$, $f'(x) = k(1+x)^{k-1}$, $f'(0) = k(1)^{k-1} = k$, and

$$L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx.$$

69. Yes, it seems reasonable that if a first-degree polynomial approximates a differentiable function in an interval near x_0 , then a higher-degree polynomial should approximate the function over a wider interval, provided enough derivatives exist near x_0 . The first-degree polynomial is able to match both the value of the original function and the value of the first derivative at x_0 . A higher-degree polynomial then ought to be able to match not only the function value and the value of the first derivative, but also the value of the second derivative, the value of the third derivative, and so on. As more information about the function near x_0 is used, the interval over which the function is approximated should increase.
70. A function needs to be differentiable at x_0 for a linear approximation to be used because the slope of the linear approximation is obtained from the derivative of the original function at x_0 .
71. The Newton's Method formula for calculating approximations is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $x_{n+1} = x_n$, it means that $f(x_n) = 0$, so $\boxed{x_n \text{ is a zero of the function } f}$.

72. Let $f(x) = -x^3 + 6x^2 - 9x + 6$. Being a polynomial function, f is continuous on the set of all real numbers, so it is continuous on the closed interval $[2, 5]$. Because $f(2) = -8 + 24 - 18 + 6 = 4 > 0$ and $f(5) = -125 + 150 - 45 + 6 = -14 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(2, 5)$.

Now, $f'(x) = -3x^2 + 12x - 9$, so the n th approximation to the zero obtained by Newton's method is given by

$$c_n = c_{n-1} - \frac{f(c_{n-1})}{f'(c_{n-1})} = c_{n-1} - \frac{-c_{n-1}^3 + 6c_{n-1}^2 - 9c_{n-1} + 6}{-3c_{n-1}^2 + 12c_{n-1} - 9}.$$

Starting with $c_1 = 2.9$, the list below shows the first 18 approximations produced by Newton's method. All calculations were performed using the computer algebra system *Mathematica*.

$c_1 = 2.9$	$c_7 = 0.595639$	$c_{13} = 5.28332$
$c_2 = -7.57544$	$c_8 = 1.47219$	$c_{14} = 4.54902$
$c_3 = -4.43931$	$c_9 = 0.287672$	$c_{15} = 4.25099$
$c_4 = -2.36601$	$c_{10} = 0.957711$	$c_{16} = 4.19749$
$c_5 = -0.998005$	$c_{11} = 8.69768$	$c_{17} = 4.19582$
$c_6 = -0.0819619$	$c_{12} = 6.59733$	$c_{18} = 4.19582$

Thus, Newton's Method does not fail to converge with an initial approximation of $c_1 = 2.9$. With an initial approximation of $c_1 = 3.0$, however, Newton's Method does fail because

$$\boxed{f'(3) = -3(3)^2 + 12(3) - 9 = -27 + 36 - 9 = 0}.$$

73. Let $f(x) = x^3 - 2x + 2$. and $c_1 = 0$. Then $f'(x) = 3x^2 - 2$, and Newton's Method generates the sequence

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{2}{-2} = 1; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = 1 - \frac{1}{1} = 0; \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} = 0 - \frac{2}{-2} = 1; \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} = 1 - \frac{1}{1} = 0; \end{aligned}$$

and so on. The approximations oscillate between 1 and 0 and never approach a limit; thus, Newton's Method fails.

74. Let $f(x) = x^8 - 1$. Then $f'(x) = 8x^7$. With $c_1 = 0.1$, Newton's Method produces the second approximation

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0.1 - \frac{(0.1)^8 - 1}{8(0.1)^7} = 1250000.088 \approx 1.250 \times 10^6.$$

Though this is a very poor approximation to the root of 1, using the computer algebra system *Mathematica* to continue calculations, we find that $c_{110} \approx 1.00000$. Thus, Newton's Method has not failed but has required a large number of iterations to converge.

75. Let $f(x) = (x-1)^{1/3}$. Then $f'(x) = \frac{1}{3}(x-1)^{-2/3}$, and the n th approximation to the zero of f obtained by Newton's Method is given by

$$c_n = c_{n-1} - \frac{f(c_{n-1})}{f'(c_{n-1})} = c_{n-1} - \frac{(c_{n-1} - 1)^{1/3}}{\frac{1}{3}(c_{n-1} - 1)^{-2/3}} = c_{n-1} - 3(c_{n-1} - 1) = 3 - 2c_{n-1}.$$

The first ten approximations generated by Newton's Method with an initial approximation of $c_1 = 2$ are

$$\begin{aligned} c_2 &= 3 - 2(2) = -1 & c_3 &= 3 - 2(-1) = 5 \\ c_4 &= 3 - 2(5) = -7 & c_5 &= 3 - 2(-7) = 17 \\ c_6 &= 3 - 2(17) = -31 & c_7 &= 3 - 2(-31) = 65 \\ c_8 &= 3 - 2(65) = -127 & c_9 &= 3 - 2(-127) = 257 \\ c_{10} &= 3 - 2(257) = -511 & c_{11} &= 3 - 2(-511) = 1025. \end{aligned}$$

With further calculations, the even subscripted approximations become unbounded in the negative direction while the odd subscripted approximations become unbounded in the positive direction. Thus, Newton's Method fails to locate the zero of f .

76. (a) Let $f(x) = x^4 + 2x^3 - 2x - 2$. The polynomial function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[-2, -1]$. Because $f(-2) = (-2)^4 + 2(-2)^3 - 2(-2) - 2 = 2 > 0$ and $f(-1) = (-1)^4 + 2(-1)^3 - 2(-1) - 2 = -1 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(-2, -1)$.

(b) Let $c_1 = -1.5$. Using Newton's Method generates the approximations

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -1.84375; \text{ and}$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx \boxed{-1.737643}.$$

(c) Because

$$\boxed{f'(-1) = 4(-1)^3 + 6(-1)^2 - 2 = -4 + 6 - 2 = 0},$$

an initial approximation of $c_1 = -1$ cannot be used in Newton's Method.

Challenge Problems

77. Let $S = 0.786$ and $f(x) = 2x^3 - 3x^2 - 0.786$. The figure below displays the graph of f ; the function has a single real zero at approximately $x = 1.6$. Using Newton's Method with a first approximation of $c_1 = 1.6$, yields

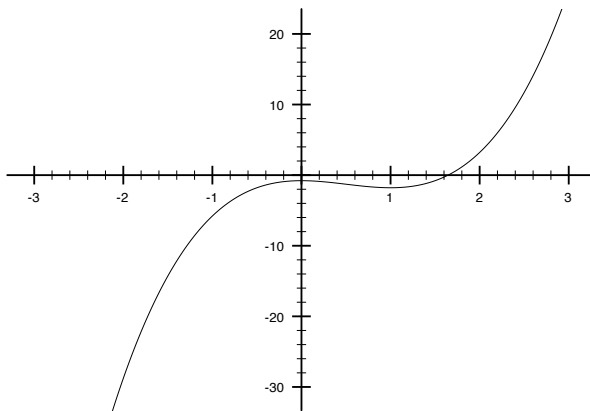
$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1.6 - \frac{2(1.6)^3 - 3(1.6)^2 - 0.786}{6(1.6)^2 - 6(1.6)} \approx 1.647569444.$$

and

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 1.645202792.$$

Therefore, the approximate depth of submersion is

$$h = dc_3 \approx 6(1.645202792) = 9.871216754 \approx \boxed{9.871 \text{ inches}}.$$



78. Let $p = 0.2$, $M = 0.85$, and $f(x) = x - 0.2 \sin x - 0.85$. Using Newton's Method with a first approximation of $c_1 = 1$ yields

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{0.15 - 0.2 \sin 1}{1 - 0.2 \cos 1} \approx \boxed{1.020511}.$$

3.5: Taylor Polynomials

Concepts and Vocabulary

1. False. If f is a function with both first and second derivatives defined on an interval containing x_0 , then f can be approximated by the Taylor Polynomial $P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$.
2. True. A Taylor Polynomial approximation P_n for a function f at x_0 has the following properties: $P_n(x_0) = f(x_0)$ and $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for derivatives of all order from $k = 1$ to $k = n$.

Skill Building

3. Let $f(x) = 3x^3 + 2x^2 - 6x + 5$ and $x_0 = 1$. Then

$$\begin{array}{ll}
 f(x) = 3x^3 + 2x^2 - 6x + 5, & f(1) = 4, \\
 f'(x) = 9x^2 + 4x - 6, & f'(1) = 7, \\
 f''(x) = 18x + 4, & f''(1) = 22, \\
 f'''(x) = 18, & f'''(1) = 18, \\
 f^{(4)}(x) = 0, & f^{(4)}(1) = 0, \\
 f^{(5)}(x) = 0, & f^{(5)}(1) = 0,
 \end{array}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 + \frac{f^{(5)}(1)}{5!}(x - 1)^5 \\
 &= 4 + 7(x - 1) + \frac{22}{2}(x - 1)^2 + \frac{18}{6}(x - 1)^3 + \frac{0}{24}(x - 1)^4 + \frac{0}{120}(x - 1)^5 \\
 &= \boxed{4 + 7(x - 1) + 11(x - 1)^2 + 3(x - 1)^3}.
 \end{aligned}$$

4. Let $f(x) = 4x^3 - 2x^2 - 4$ and $x_0 = 1$. Then

$$\begin{array}{ll}
 f(x) = 4x^3 - 2x^2 - 4, & f(1) = -2, \\
 f'(x) = 12x^2 - 4x, & f'(1) = 8, \\
 f''(x) = 24x - 4, & f''(1) = 20, \\
 f'''(x) = 24, & f'''(1) = 24, \\
 f^{(4)}(x) = 0, & f^{(4)}(1) = 0, \\
 f^{(5)}(x) = 0, & f^{(5)}(1) = 0,
 \end{array}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 + \frac{f^{(5)}(1)}{5!}(x - 1)^5 \\
 &= -2 + 8(x - 1) + \frac{20}{2}(x - 1)^2 + \frac{24}{6}(x - 1)^3 + \frac{0}{24}(x - 1)^4 + \frac{0}{120}(x - 1)^5 \\
 &= \boxed{-2 + 8(x - 1) + 10(x - 1)^2 + 4(x - 1)^3}.
 \end{aligned}$$

5. Let $f(x) = 2x^4 - 6x^3 + x$ and $x_0 = -1$. Then

$$\begin{aligned}
 f(x) &= 2x^4 - 6x^3 + x, & f(-1) &= 7, \\
 f'(x) &= 8x^3 - 18x^2 + 1, & f'(-1) &= -25, \\
 f''(x) &= 24x^2 - 36x, & f''(-1) &= 60, \\
 f'''(x) &= 48x - 36, & f'''(-1) &= -84, \\
 f^{(4)}(x) &= 48, & f^{(4)}(-1) &= 48, \\
 f^{(5)}(x) &= 0, & f^{(5)}(-1) &= 0,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \\
 &\quad \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5 \\
 &= 7 - 25(x-1) + \frac{60}{2}(x-1)^2 + \frac{-84}{6}(x-1)^3 + \frac{48}{24}(x-1)^4 + \frac{0}{120}(x-1)^5 \\
 &= \boxed{7 - 25(x+1) + 30(x+1)^2 - 14(x+1)^3 + 2(x+1)^4}.
 \end{aligned}$$

6. Let $f(x) = -3x^4 + 2x^2 - 5$ and $x_0 = -1$. Then

$$\begin{aligned}
 f(x) &= -3x^4 + 2x^2 - 5, & f(-1) &= -6, \\
 f'(x) &= -12x^3 + 4x, & f'(-1) &= 8, \\
 f''(x) &= -36x^2 + 4, & f''(-1) &= -32, \\
 f'''(x) &= -72x, & f'''(-1) &= 72, \\
 f^{(4)}(x) &= -72, & f^{(4)}(-1) &= -72, \\
 f^{(5)}(x) &= 0, & f^{(5)}(-1) &= 0,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \\
 &\quad \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5 \\
 &= -6 + 8(x-1) + \frac{-32}{2}(x-1)^2 + \frac{72}{6}(x-1)^3 + \frac{-72}{24}(x-1)^4 + \frac{0}{120}(x-1)^5 \\
 &= \boxed{-6 + 8(x+1) - 16(x+1)^2 + 12(x+1)^3 - 3(x+1)^4}.
 \end{aligned}$$

7. Let $f(x) = x^5$ and $x_0 = 2$. Then

$$\begin{aligned}
 f(x) &= x^5, & f(2) &= 32, \\
 f'(x) &= 5x^4, & f'(2) &= 80, \\
 f''(x) &= 20x^3, & f''(2) &= 160, \\
 f'''(x) &= 60x^2, & f'''(2) &= 240, \\
 f^{(4)}(x) &= 120x, & f^{(4)}(2) &= 240, \\
 f^{(5)}(x) &= 120, & f^{(5)}(2) &= 120,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \frac{f^{(5)}(2)}{5!}(x-2)^5 \\
 &= 32 + 80(x-2) + \frac{160}{2}(x-2)^2 + \frac{240}{6}(x-2)^3 + \frac{240}{24}(x-2)^4 + \frac{120}{120}(x-2)^5 \\
 &= \boxed{32 + 80(x-2) + 80(x-2)^2 + 40(x-2)^3 + 10(x-2)^4 + (x-2)^5}.
 \end{aligned}$$

8. Let $f(x) = x^6$ and $x_0 = 3$. Then

$$\begin{aligned}
 f(x) &= x^6, & f(3) &= 729, \\
 f'(x) &= 6x^5, & f'(3) &= 1458, \\
 f''(x) &= 30x^4, & f''(3) &= 2430, \\
 f'''(x) &= 120x^3, & f'''(3) &= 3240, \\
 f^{(4)}(x) &= 360x^2, & f^{(4)}(3) &= 3240, \\
 f^{(5)}(x) &= 720x, & f^{(5)}(3) &= 2160,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \frac{f^{(4)}(3)}{4!}(x-3)^4 + \frac{f^{(5)}(3)}{5!}(x-3)^5 \\
 &= 729 + 1458(x-3) + \frac{2430}{2}(x-3)^2 + \frac{3240}{6}(x-3)^3 + \frac{3240}{24}(x-3)^4 + \frac{2160}{120}(x-3)^5 \\
 &= \boxed{729 + 1458(x-3) + 1215(x-3)^2 + 540(x-3)^3 + 135(x-3)^4 + 18(x-3)^5}.
 \end{aligned}$$

9. Let $f(x) = \ln x$ and $x_0 = 1$. Then

$$\begin{aligned}
 f(x) &= \ln x, & f(1) &= 0, \\
 f'(x) &= \frac{1}{x} = x^{-1}, & f'(1) &= 1, \\
 f''(x) &= -x^{-2}, & f''(1) &= -1, \\
 f'''(x) &= 2x^{-3}, & f'''(1) &= 2, \\
 f^{(4)}(x) &= -6x^{-4}, & f^{(4)}(1) &= -6, \\
 f^{(5)}(x) &= 24x^{-5}, & f^{(5)}(1) &= 24,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= 0 + (x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 + \frac{-6}{24}(x-1)^4 + \frac{24}{120}(x-1)^5 \\
 &= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5}.
 \end{aligned}$$

10. Let $f(x) = \ln(1+x)$ and $x_0 = 0$. Then

$$\begin{aligned}
 f(x) &= \ln(1+x), & f(0) &= 0, \\
 f'(x) &= \frac{1}{1+x} = (1+x)^{-1}, & f'(0) &= 1, \\
 f''(x) &= -(1+x)^{-2}, & f''(0) &= -1, \\
 f'''(x) &= 2(1+x)^{-3}, & f'''(0) &= 2, \\
 f^{(4)}(x) &= -6(1+x)^{-4}, & f^{(4)}(0) &= -6, \\
 f^{(5)}(x) &= 24(1+x)^{-5}, & f^{(5)}(0) &= 24,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 0 + x + \frac{-1}{2}x^2 + \frac{2}{6}x^3 + \frac{-6}{24}x^4 + \frac{24}{120}x^5 = \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5}.
 \end{aligned}$$

11. Let $f(x) = \frac{1}{x} = x^{-1}$ and $x_0 = 1$. Then

$$\begin{array}{ll}
 f(x) = x^{-1}, & f(1) = 1, \\
 f'(x) = -x^{-2}, & f'(1) = -1, \\
 f''(x) = 2x^{-3}, & f''(1) = 2, \\
 f'''(x) = -6x^{-4}, & f'''(1) = -6, \\
 f^{(4)}(x) = 24x^{-5}, & f^{(4)}(1) = 24, \\
 f^{(5)}(x) = -120x^{-6}, & f^{(5)}(1) = -120,
 \end{array}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= 1 - (x-1) + \frac{2}{2}(x-1)^2 + \frac{-6}{6}(x-1)^3 + \frac{24}{24}(x-1)^4 + \frac{-120}{120}(x-1)^5 \\
 &= \boxed{1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5}.
 \end{aligned}$$

12. Let $f(x) = \frac{1}{x^2} = x^{-2}$ and $x_0 = 1$. Then

$$\begin{array}{ll}
 f(x) = x^{-2}, & f(1) = 1, \\
 f'(x) = -2x^{-3}, & f'(1) = -2, \\
 f''(x) = 6x^{-4}, & f''(1) = 6, \\
 f'''(x) = -24x^{-5}, & f'''(1) = -24, \\
 f^{(4)}(x) = 120x^{-6}, & f^{(4)}(1) = 120, \\
 f^{(5)}(x) = -720x^{-7}, & f^{(5)}(1) = -720,
 \end{array}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= 1 - 2(x-1) + \frac{6}{2}(x-1)^2 + \frac{-24}{6}(x-1)^3 + \frac{120}{24}(x-1)^4 + \frac{-720}{120}(x-1)^5 \\
 &= \boxed{1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - 6(x-1)^5}.
 \end{aligned}$$

13. Let $f(x) = \cos x$ and $x_0 = 0$. Then

$$\begin{array}{ll}
 f(x) = \cos x, & f(0) = 1, \\
 f'(x) = -\sin x, & f'(0) = 0, \\
 f''(x) = -\cos x, & f''(0) = -1, \\
 f'''(x) = \sin x, & f'''(0) = 0, \\
 f^{(4)}(x) = \cos x, & f^{(4)}(0) = 1, \\
 f^{(5)}(x) = -\sin x, & f^{(5)}(0) = 0,
 \end{array}$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 1 + 0 \cdot x + \frac{-1}{2}x^2 + \frac{0}{6}x^3 + \frac{1}{24}x^4 + \frac{0}{120}x^5 = \boxed{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4}.
 \end{aligned}$$

14. Let $f(x) = \sin x$ and $x_0 = \frac{\pi}{4}$. Then

$$\begin{aligned}
 f(x) &= \sin x, & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, \\
 f'(x) &= \cos x, & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, \\
 f''(x) &= -\sin x, & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}, \\
 f'''(x) &= -\cos x, & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}, \\
 f^{(4)}(x) &= \sin x, & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, \\
 f^{(5)}(x) &= \cos x, & f^{(5)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2},
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 + \\
 &\quad \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x - \frac{\pi}{4}\right)^4 + \frac{f^{(5)}\left(\frac{\pi}{4}\right)}{5!}\left(x - \frac{\pi}{4}\right)^5 \\
 &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{-\sqrt{2}/2}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{-\sqrt{2}/2}{6}\left(x - \frac{\pi}{4}\right)^3 + \\
 &\quad \frac{\sqrt{2}/2}{24}\left(x - \frac{\pi}{4}\right)^4 + \frac{\sqrt{2}/2}{120}\left(x - \frac{\pi}{4}\right)^5 \\
 &= \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{240}\left(x - \frac{\pi}{4}\right)^5}.
 \end{aligned}$$

15. Let $f(x) = e^{2x}$ and $x_0 = 0$. Then

$$\begin{aligned}
 f(x) &= e^{2x}, & f(0) &= 1, \\
 f'(x) &= 2e^{2x}, & f'(0) &= 2, \\
 f''(x) &= 4e^{2x}, & f''(0) &= 4, \\
 f'''(x) &= 8e^{2x}, & f'''(0) &= 8, \\
 f^{(4)}(x) &= 16e^{2x}, & f^{(4)}(0) &= 16, \\
 f^{(5)}(x) &= 32e^{2x}, & f^{(5)}(0) &= 32,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 + \frac{32}{120}x^5 = \boxed{1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5}.
 \end{aligned}$$

16. Let $f(x) = e^{-x}$ and $x_0 = 0$. Then

$$\begin{array}{ll} f(x) = e^{-x}, & f(0) = 1, \\ f'(x) = -e^{-x}, & f'(0) = -1, \\ f''(x) = e^{-x}, & f''(0) = 1, \\ f'''(x) = -e^{-x}, & f'''(0) = -1, \\ f^{(4)}(x) = e^{-x}, & f^{(4)}(0) = 1, \\ f^{(5)}(x) = -e^{-x}, & f^{(5)}(0) = -1, \end{array}$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 - x + \frac{1}{2}x^2 + \frac{-1}{6}x^3 + \frac{1}{24}x^4 + \frac{-1}{120}x^5 = \boxed{1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5}. \end{aligned}$$

17. Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ and $x_0 = 0$. Then

$$\begin{array}{ll} f(x) = (1-x)^{-1}, & f(0) = 1, \\ f'(x) = (1-x)^{-2}, & f'(0) = 1, \\ f''(x) = 2(1-x)^{-3}, & f''(0) = 2, \\ f'''(x) = 6(1-x)^{-4}, & f'''(0) = 6, \\ f^{(4)}(x) = 24(1-x)^{-5}, & f^{(4)}(0) = 24, \\ f^{(5)}(x) = 120(1-x)^{-6}, & f^{(5)}(0) = 120, \end{array}$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 + x + \frac{2}{2}x^2 + \frac{6}{6}x^3 + \frac{24}{24}x^4 + \frac{120}{120}x^5 = \boxed{1 + x + x^2 + x^3 + x^4 + x^5}. \end{aligned}$$

18. Let $f(x) = \frac{1}{1+x} = (1+x)^{-1}$ and $x_0 = 0$. Then

$$\begin{array}{ll} f(x) = (1+x)^{-1}, & f(0) = 1, \\ f'(x) = -(1+x)^{-2}, & f'(0) = -1, \\ f''(x) = 2(1+x)^{-3}, & f''(0) = 2, \\ f'''(x) = -6(1+x)^{-4}, & f'''(0) = -6, \\ f^{(4)}(x) = 24(1+x)^{-5}, & f^{(4)}(0) = 24, \\ f^{(5)}(x) = -120(1+x)^{-6}, & f^{(5)}(0) = -120, \end{array}$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 - x + \frac{2}{2}x^2 + \frac{-6}{6}x^3 + \frac{24}{24}x^4 + \frac{-120}{120}x^5 = \boxed{1 - x + x^2 - x^3 + x^4 - x^5}. \end{aligned}$$

19. Let $f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2}$ and $x_0 = 0$. Then

$$\begin{aligned} f(x) &= (1+x)^{-2}, & f(0) &= 1, \\ f'(x) &= -2(1+x)^{-3}, & f'(0) &= -2, \\ f''(x) &= 6(1+x)^{-4}, & f''(0) &= 6, \\ f'''(x) &= -24(1+x)^{-5}, & f'''(0) &= -24, \\ f^{(4)}(x) &= 120(1+x)^{-6}, & f^{(4)}(0) &= 120, \\ f^{(5)}(x) &= -720(1+x)^{-7}, & f^{(5)}(0) &= -720, \end{aligned}$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 - 2x + \frac{6}{2}x^2 + \frac{-24}{6}x^3 + \frac{120}{24}x^4 + \frac{-720}{120}x^5 = \boxed{1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5}. \end{aligned}$$

20. Let $f(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ and $x_0 = 0$. Then

$$\begin{aligned} f'(x) &= -(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2}, \\ f''(x) &= \frac{(1+x^2)^2(-2) - (-2x) \cdot 2(1+x^2)(2x)}{(1+x^2)^4} = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}, \\ f'''(x) &= \frac{(1+x^2)^2(12x) - (6x^2 - 2) \cdot 3(1+x^2)^2(2x)}{(1+x^2)^6} = \frac{12x + 12x^3 - 36x^3 + 12x}{(1+x^2)^4} = \frac{24x - 24x^3}{(1+x^2)^4}, \\ f^{(4)}(x) &= \frac{(1+x^2)^4(2x - 72x^2) - (24x - 24x^3) \cdot 4(1+x^2)^3(2x)}{(1+x^2)^8} \\ &= \frac{24 - 48x^2 - 72x^4 - 192x^2 + 192x^4}{(1+x^2)^5} = \frac{24 - 240x^2 + 120x^4}{(1+x^2)^5}, \text{ and} \\ f^{(5)}(x) &= \frac{(1+x^2)^5(-480x + 480x^3) - (24 - 240x^2 + 120x^4) \cdot 5(1+x^2)^4(2x)}{(1+x^2)^{10}} \\ &= \frac{-480x + 480x^5 - 240x + 2400x^2 - 1200x^5}{(1+x^2)^6} = \frac{-720x + 2400x^3 - 720x^5}{(1+x^2)^6}, \end{aligned}$$

so that

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -2, \quad f'''(0) = 0, \quad f^{(4)}(0) = 24, \quad f^{(5)}(0) = 0,$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 + 0 \cdot x + \frac{-2}{2}x^2 + \frac{0}{6}x^3 + \frac{24}{24}x^4 + \frac{0}{120}x^5 = \boxed{1 - x^2 + x^4}. \end{aligned}$$

21. Let $f(x) = x \ln x$ and $x_0 = 1$. Then

$$\begin{aligned} f(x) &= x \ln x, & f(1) &= 0, \\ f'(x) &= x \cdot \frac{1}{x} + \ln x = 1 + \ln x, & f'(1) &= 1, \\ f''(x) &= \frac{1}{x} = x^{-1}, & f''(1) &= 1, \\ f'''(x) &= -x^{-2}, & f'''(1) &= -1, \\ f^{(4)}(x) &= 2x^{-3}, & f^{(4)}(1) &= 2, \\ f^{(5)}(x) &= -6x^{-4}, & f^{(5)}(1) &= -6, \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= 0 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{-1}{6}(x-1)^3 + \frac{2}{24}(x-1)^4 + \frac{-6}{120}(x-1)^5 \\
 &= \boxed{(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5}.
 \end{aligned}$$

22. Let $f(x) = xe^x$ and $x_0 = 1$. Then

$$\begin{aligned}
 f(x) &= xe^x, & f(1) &= e, \\
 f'(x) &= xe^x + e^x = (x+1)e^x, & f'(1) &= 2e, \\
 f''(x) &= (x+1)e^x + e^x = (x+2)e^x, & f''(1) &= 3e, \\
 f'''(x) &= (x+2)e^x + e^x = (x+3)e^x, & f'''(1) &= 4e, \\
 f^{(4)}(x) &= (x+3)e^x + e^x = (x+4)e^x, & f^{(4)}(1) &= 5e, \\
 f^{(5)}(x) &= (x+4)e^x + e^x = (x+5)e^x, & f^{(5)}(1) &= 6e,
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{4e}{6}(x-1)^3 + \frac{5e}{24}(x-1)^4 + \frac{6e}{120}(x-1)^5 \\
 &= \boxed{e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{2e}{3}(x-1)^3 + \frac{5e}{24}(x-1)^4 + \frac{e}{20}(x-1)^5}.
 \end{aligned}$$

23. Let $f(x) = \sqrt{3+x^2}$ and $x_0 = 1$. Then

$$\begin{aligned}
 f'(x) &= \frac{1}{2}(3+x^2)^{-1/2}(2x) = \frac{x}{\sqrt{3+x^2}}, \\
 f''(x) &= \frac{\sqrt{3+x^2} \cdot 1 - x \cdot \frac{1}{2}(3+x^2)^{-1/2}(2x)}{3+x^2} = \frac{3+x^2-x^2}{(3+x^2)^{3/2}} = \frac{3}{(3+x^2)^{3/2}}, \\
 f'''(x) &= -\frac{9}{2}(3+x^2)^{-5/2}(2x) = -\frac{9x}{(3+x^2)^{5/2}}, \\
 f^{(4)}(x) &= \frac{(3+x^2)^{5/2}(-9) - (-9x) \cdot \frac{5}{2}(3+x^2)^{3/2}(2x)}{(3+x^2)^5} = \frac{-27-9x^2+45x^2}{(3+x^2)^{7/2}} = \frac{36x^2-27}{(3+x^2)^{7/2}}, \text{ and} \\
 f^{(5)}(x) &= \frac{(3+x^2)^{7/2}(72x) - (36x^2-27) \cdot \frac{7}{2}(3+x^2)^{5/2}(2x)}{(3+x^2)^7} \\
 &= \frac{216x+72x^3-252x^3+189x}{(3+x^2)^{9/2}} = \frac{405x-180x^3}{(3+x^2)^{9/2}},
 \end{aligned}$$

so that

$$f(1) = 2, \quad f'(1) = \frac{1}{2}, \quad f''(1) = \frac{3}{8}, \quad f'''(1) = -\frac{9}{32}, \quad f^{(4)}(1) = \frac{9}{128}, \quad f^{(5)}(1) = \frac{225}{512},$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= 2 + \frac{1}{2}(x-1) + \frac{3/8}{2}(x-1)^2 + \frac{-9/32}{6}(x-1)^3 + \frac{9/128}{24}(x-1)^4 + \frac{225/512}{120}(x-1)^5 \\
 &= \boxed{2 + \frac{1}{2}(x-1) + \frac{3}{16}(x-1)^2 - \frac{3}{64}(x-1)^3 + \frac{3}{1024}(x-1)^4 + \frac{15}{4096}(x-1)^5}.
 \end{aligned}$$

24. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ and $x_0 = 0$. Then

$$\begin{aligned} f(x) &= (1+x)^{1/2}, & f(0) &= 1, \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2}, & f'''(0) &= \frac{3}{8}, \\ f^{(4)}(x) &= -\frac{15}{16}(1+x)^{-7/2}, & f^{(4)}(0) &= -\frac{15}{16}, \\ f^{(5)}(x) &= \frac{105}{32}(1+x)^{-9/2}, & f^{(5)}(0) &= \frac{105}{32}, \end{aligned}$$

and

$$\begin{aligned} P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\ &= 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2 + \frac{3/8}{6}x^3 + \frac{-15/16}{24}x^4 + \frac{105/32}{120}x^5 \\ &= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5}. \end{aligned}$$

25. Let $f(x) = \tan x$ and $x_0 = \frac{\pi}{4}$. Then

$$\begin{aligned} f'(x) &= \sec^2 x, \\ f''(x) &= 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x, \\ f'''(x) &= 2 \sec^2 x \cdot \sec^2 x + \tan x \cdot 4 \sec x \cdot \sec x \tan x = 2 \sec^4 x + 4 \sec^2 x \tan^2 x \\ &= 2 \sec^4 x + 4 \sec^2 x (\sec^2 x - 1) = 6 \sec^4 x - 4 \sec^2 x, \\ f^{(4)}(x) &= 24 \sec^3 x \cdot \sec x \tan x - 8 \sec x \cdot \sec x \tan x = 8 \tan x (3 \sec^4 x - \sec^2 x), \text{ and} \\ f^{(5)}(x) &= 8 \tan x (12 \sec^3 x \cdot \sec x \tan x - 2 \sec x \cdot \sec x \tan x) + 8 \sec^2 x (3 \sec^4 x - \sec^2 x) \\ &= 8 \tan^2 x (12 \sec^4 x - 2 \sec^2 x) + 24 \sec^6 x - 8 \sec^4 x \\ &= 8 (\sec^2 x - 1) (12 \sec^4 x - 2 \sec^2 x) + 24 \sec^6 x - 8 \sec^4 x \\ &= 96 \sec^6 x - 16 \sec^4 x - 96 \sec^4 x + 16 \sec^2 x + 24 \sec^6 x - 8 \sec^4 x \\ &= 120 \sec^6 x - 120 \sec^4 x + 16 \sec^2 x, \end{aligned}$$

so that

$$f\left(\frac{\pi}{4}\right) = 1, \quad f'\left(\frac{\pi}{4}\right) = 2, \quad f''\left(\frac{\pi}{4}\right) = 4, \quad f'''\left(\frac{\pi}{4}\right) = 16, \quad f^{(4)}\left(\frac{\pi}{4}\right) = 80, \quad f^{(5)}\left(\frac{\pi}{4}\right) = 512,$$

and

$$\begin{aligned} P_5(x) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &\quad + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x - \frac{\pi}{4}\right)^4 + \frac{f^{(5)}\left(\frac{\pi}{4}\right)}{5!}\left(x - \frac{\pi}{4}\right)^5 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{6}\left(x - \frac{\pi}{4}\right)^3 + \frac{80}{24}\left(x - \frac{\pi}{4}\right)^4 + \frac{512}{120}\left(x - \frac{\pi}{4}\right)^5 \\ &= \boxed{1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5}. \end{aligned}$$

26. Let $f(x) = \sec x$ and $x_0 = 0$. Then

$$\begin{aligned}
 f'(x) &= \sec x \tan x, \\
 f''(x) &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x = \sec^3 x + \sec x \tan^2 x \\
 &= \sec^3 x + \sec x(\sec^2 x - 1) = 2\sec^3 x - \sec x, \\
 f'''(x) &= 6\sec^2 x \cdot \sec x \tan x - \sec x \tan x = \tan x(6\sec^3 x - \sec x), \\
 f^{(4)}(x) &= \tan x(18\sec^2 x \cdot \sec x \tan x - \sec x \tan x) + \sec^2 x(6\sec^3 x - \sec x) \\
 &= \tan^2 x(18\sec^3 x - \sec x) + 6\sec^5 x - \sec^3 x \\
 &= (\sec^2 x - 1)(18\sec^3 x - \sec x) + 6\sec^5 x - \sec^3 x \\
 &= 18\sec^5 x - 19\sec^3 x + \sec x + 6\sec^5 x - \sec^3 x \\
 &= 24\sec^5 x - 20\sec^3 x + \sec x, \text{ and} \\
 f^{(5)}(x) &= 120\sec^4 x \cdot \sec x \tan x - 60\sec^2 x \cdot \sec x \tan x + \sec x \tan x \\
 &= \tan x(120\sec^5 x - 60\sec^3 x + \sec x),
 \end{aligned}$$

so that

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 5, \quad f^{(5)}(0) = 0,$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 1 + 0 \cdot x + \frac{1}{2}x^2 + \frac{0}{6}x^3 + \frac{5}{24}x^4 + \frac{0}{120}x^5 = \boxed{1 + \frac{1}{2}x^2 + \frac{5}{24}x^4}.
 \end{aligned}$$

27. Let $f(x) = \tan^{-1} x$ and $x_0 = 0$. Then

$$\begin{aligned}
 f'(x) &= \frac{1}{1+x^2}, \\
 f''(x) &= -\frac{1}{(1+x^2)^2} \cdot 2x = -\frac{2x}{(1+x^2)^2}, \\
 f'''(x) &= \frac{(1+x^2)^2(-2) - (-2x) \cdot 2(1+x^2)(2x)}{(1+x^2)^4} \\
 &= \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}, \\
 f^{(4)}(x) &= \frac{(1+x^2)^2(12x) - (6x^2 - 2) \cdot 3(1+x^2)^2(2x)}{(1+x^2)^6} = \frac{12x + 12x^3 - 36x^3 + 12x}{(1+x^2)^4} = \frac{24x - 24x^3}{(1+x^2)^4},
 \end{aligned}$$

and

$$\begin{aligned}
 f^{(5)}(x) &= \frac{(1+x^2)^4(2x - 72x^2) - (24x - 24x^3) \cdot 4(1+x^2)^3(2x)}{(1+x^2)^8} \\
 &= \frac{24 - 48x^2 - 72x^4 - 192x^2 + 192x^4}{(1+x^2)^5} = \frac{24 - 240x^2 + 120x^4}{(1+x^2)^5},
 \end{aligned}$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -2, \quad f^{(4)}(0) = 0, \quad f^{(5)}(0) = 24,$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 0 + x + \frac{0}{2}x^2 + \frac{-2}{6}x^3 + \frac{0}{24}x^4 + \frac{24}{120}x^5 = \boxed{x - \frac{1}{3}x^3 + \frac{1}{5}x^5}.
 \end{aligned}$$

28. Let $f(x) = \sin^{-1} x$ and $x_0 = 0$. Then

$$\begin{aligned}
 f'(x) &= \frac{1}{\sqrt{1-x^2}}, \\
 f''(x) &= -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x) = \frac{x}{(1-x^2)^{3/2}}, \\
 f'''(x) &= \frac{(1-x^2)^{3/2} \cdot 1 - x \cdot \frac{3}{2}(1-x^2)^{1/2}(-2x)}{(1-x^2)^3} = \frac{1-x^2+3x^2}{(1-x^2)^{5/2}} = \frac{1+2x^2}{(1-x^2)^{5/2}}, \\
 f^{(4)}(x) &= \frac{(1-x^2)^{5/2}(4x) - (1+2x^2) \cdot \frac{5}{2}(1-x^2)^{3/2}(-2x)}{(1-x^2)^5} \\
 &= \frac{4x - 4x^3 + 5x + 10x^3}{(1-x^2)^{7/2}} = \frac{6x^3 + 9x}{(1-x^2)^{7/2}}, \text{ and} \\
 f^{(5)}(x) &= \frac{(1-x^2)^{7/2}(18x^2+9) - (6x^3+9x) \cdot \frac{7}{2}(1-x^2)^{5/2}(-2x)}{(1-x^2)^7} \\
 &= \frac{18x^2 - 18x^4 + 9 - 9x^2 + 42x^4 + 63x^2}{(1-x^2)^{9/2}} = \frac{24x^4 + 72x^2 + 9}{(1-x^2)^{9/2}},
 \end{aligned}$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 1, \quad f^{(4)}(0) = 0, \quad f^{(5)}(0) = 9,$$

and

$$\begin{aligned}
 P_5(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 \\
 &= 0 + x + \frac{0}{2}x^2 + \frac{1}{6}x^3 + \frac{0}{24}x^4 + \frac{9}{120}x^5 = \boxed{x + \frac{1}{6}x^3 + \frac{3}{40}x^5}.
 \end{aligned}$$

Applications and Extensions

29. Let $f(x) = 3x^2 - 6x + 4$, $x_0 = 1$, and $n = 2$ (because f is a second-degree polynomial). Then

$$\begin{aligned}
 f(x) &= 3x^2 - 6x + 4, & f(1) &= 1, \\
 f'(x) &= 6x - 6, & f'(1) &= 0, \\
 f''(x) &= 6, & f''(1) &= 6,
 \end{aligned}$$

and

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 1 + 3(x-1)^2.$$

Therefore, $\boxed{f(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1}.$

30. Let $f(x) = 4x^2 - x + 1$, $x_0 = 1$, and $n = 2$ (because f is a second-degree polynomial). Then

$$\begin{aligned}
 f(x) &= 4x^2 - x + 1, & f(1) &= 4, \\
 f'(x) &= 8x - 1, & f'(1) &= 7, \\
 f''(x) &= 8, & f''(1) &= 8,
 \end{aligned}$$

and

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 7(x-1) + 4(x-1)^2.$$

Therefore, $\boxed{f(x) = 4x^2 - x + 1 = 4(x-1)^2 + 7(x-1) + 4}.$

31. Let $f(x) = x^3 + x^2 - 8$, $x_0 = 1$, and $n = 3$ (because f is a third-degree polynomial). Then

$$\begin{aligned} f(x) &= x^3 + x^2 - 8, & f(1) &= -6, \\ f'(x) &= 3x^2 + 2x, & f'(1) &= 5, \\ f''(x) &= 6x + 2, & f''(1) &= 8, \\ f'''(x) &= 6, & f'''(1) &= 6, \end{aligned}$$

and

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= -6 + 5(x-1) + 4(x-1)^2 + (x-1)^3. \end{aligned}$$

Therefore, $\boxed{f(x) = x^3 + x^2 - 8 = (x-1)^3 + 4(x-1)^2 + 5(x-1) - 6}.$

32. Let $f(x) = x^4 + 1$, $x_0 = 1$, and $n = 4$ (because f is a fourth-degree polynomial). Then

$$\begin{aligned} f(x) &= x^4 + 1, & f(1) &= 2, \\ f'(x) &= 4x^3, & f'(1) &= 4, \\ f''(x) &= 12x^2, & f''(1) &= 12, \\ f'''(x) &= 24x, & f'''(1) &= 24, \\ f^{(4)}(x) &= 24, & f^{(4)}(1) &= 24, \end{aligned}$$

and

$$\begin{aligned} P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= 2 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4. \end{aligned}$$

Therefore, $\boxed{f(x) = x^4 + 1 = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 2}.$

33. (a) Let $f(x) = \sin^{-1} x$ and $x_0 = \frac{1}{2}$. Then

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-x^2}}, \\ f''(x) &= -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x) = \frac{x}{(1-x^2)^{3/2}}, \\ f'''(x) &= \frac{(1-x^2)^{3/2} \cdot 1 - x \cdot \frac{3}{2}(1-x^2)^{1/2}(-2x)}{(1-x^2)^3} = \frac{1-x^2+3x^2}{(1-x^2)^{5/2}} = \frac{1+2x^2}{(1-x^2)^{5/2}}, \text{ and} \\ f^{(4)}(x) &= \frac{(1-x^2)^{5/2}(4x) - (1+2x^2) \cdot \frac{5}{2}(1-x^2)^{3/2}(-2x)}{(1-x^2)^5} \\ &= \frac{4x-4x^3+5x+10x^3}{(1-x^2)^{7/2}} = \frac{6x^3+9x}{(1-x^2)^{7/2}}, \end{aligned}$$

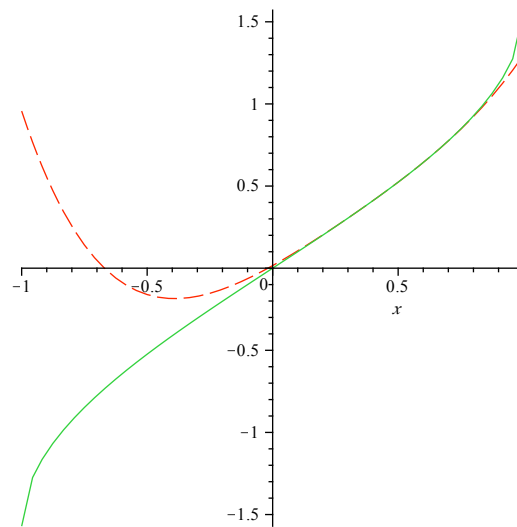
so that

$$f\left(\frac{1}{2}\right) = \frac{\pi}{6}, \quad f'\left(\frac{1}{2}\right) = \frac{2\sqrt{3}}{3}, \quad f''\left(\frac{1}{2}\right) = \frac{4\sqrt{3}}{9}, \quad f'''\left(\frac{1}{2}\right) = \frac{16\sqrt{3}}{9}, \quad f^{(4)}\left(\frac{1}{2}\right) = \frac{224\sqrt{3}}{27},$$

and

$$\begin{aligned}
 P_4(x) &= f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{f''\left(\frac{1}{2}\right)}{2!}\left(x - \frac{1}{2}\right)^2 + \\
 &\quad \frac{f'''\left(\frac{1}{2}\right)}{3!}\left(x - \frac{1}{2}\right)^3 + \frac{f^{(4)}\left(\frac{1}{2}\right)}{4!}\left(x - \frac{1}{2}\right)^4 \\
 &= \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{4\sqrt{3}/9}{2}\left(x - \frac{1}{2}\right)^2 + \frac{16\sqrt{3}/9}{6}\left(x - \frac{1}{2}\right)^3 + \frac{224\sqrt{3}/27}{24}x^4 \\
 &= \boxed{\frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2 + \frac{8\sqrt{3}}{27}\left(x - \frac{1}{2}\right)^3 + \frac{28\sqrt{3}}{81}\left(x - \frac{1}{2}\right)^4}.
 \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = \sin^{-1} x$ as the solid curve and the graph of $P_4(x)$ as the dashed curve.



34. (a) Let $f(x) = \tan^{-1} x$ and $x_0 = 1$. Then

$$\begin{aligned}
 f'(x) &= \frac{1}{1+x^2}, \\
 f''(x) &= -\frac{1}{(1+x^2)^2} \cdot 2x = -\frac{2x}{(1+x^2)^2}, \\
 f'''(x) &= \frac{(1+x^2)^2(-2) - (-2x) \cdot 2(1+x^2)(2x)}{(1+x^2)^4} \\
 &= \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}, \\
 f^{(4)}(x) &= \frac{(1+x^2)^2(12x) - (6x^2 - 2) \cdot 3(1+x^2)^2(2x)}{(1+x^2)^6} = \frac{12x + 12x^3 - 36x^3 + 12x}{(1+x^2)^4} = \frac{24x - 24x^3}{(1+x^2)^4},
 \end{aligned}$$

and

$$\begin{aligned}
 f^{(5)}(x) &= \frac{(1+x^2)^4(2x - 72x^2) - (24x - 24x^3) \cdot 4(1+x^2)^3(2x)}{(1+x^2)^8} \\
 &= \frac{24 - 48x^2 - 72x^4 - 192x^2 + 192x^4}{(1+x^2)^5} = \frac{24 - 240x^2 + 120x^4}{(1+x^2)^5},
 \end{aligned}$$

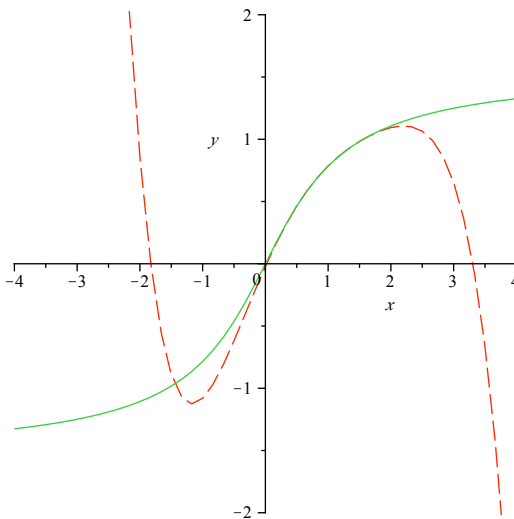
so that

$$f(1) = \frac{\pi}{4}, \quad f'(1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{2}, \quad f'''(1) = \frac{1}{2}, \quad f^{(4)}(1) = 0, \quad f^{(5)}(1) = -3,$$

and

$$\begin{aligned}
 P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \\
 &\quad \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
 &= \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{-1/2}{2}(x-1)^2 + \frac{1/2}{6}(x-1)^3 + \frac{0}{24}(x-1)^4 + \frac{-3}{120}(x-1)^5 \\
 &= \boxed{\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{40}(x-1)^5}.
 \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = \tan^{-1} x$ as the solid curve and the graph of $P_5(x)$ as the dashed curve.



35. (a) Let $f(x) = \frac{x}{\sqrt{x^2+3}}$ and $x_0 = 1$. Then

$$\begin{aligned}
 f'(x) &= \frac{\sqrt{x^2+3} \cdot 1 - x \cdot \frac{1}{2}(x^2+3)^{-1/2}(2x)}{x^2+3} = \frac{x^2+3-x^2}{(x^2+3)^{3/2}} = \frac{3}{(x^2+3)^{3/2}}, \\
 f''(x) &= -\frac{9x}{(x^2+3)^{5/2}}, \\
 f'''(x) &= \frac{(x^2+3)^{5/2} \cdot -9 + 9x \cdot \frac{5}{2}(x^2+3)^{3/2}(2x)}{(x^2+3)^5} = \frac{-9(x^2+3) + 45x^2}{(x^2+3)^{7/2}} = \frac{36x^2-27}{(x^2+3)^{7/2}}, \\
 f^{(4)}(x) &= \frac{(x^2+3)^{7/2}(72x) - (36x^2-27) \cdot \frac{7}{2}(x^2+3)^{5/2}}{(x^2+3)^7} = \frac{72x(x^2+3) - (36x^2-27)(7x)}{(x^2+3)^{9/2}} \\
 &= \frac{72x^3 + 216x - 252x^3 + 189x}{(x^2+3)^{9/2}} = \frac{-180x^3 + 405x}{(x^2+3)^{9/2}},
 \end{aligned}$$

and

$$\begin{aligned}
 f^{(5)}(x) &= \frac{(x^2+3)^{9/2}(-540x^2+405) - (-180x^3+405x) \cdot \frac{9}{2}(x^2+3)^{7/2}(2x)}{(x^2+3)^9} \\
 &= \frac{(x^2+3)(-540x^2+405) - (-180x^3+405x)(9x)}{(x^2+3)^{11/2}} \\
 &= \frac{-540x^4 - 1215x^2 + 1215 + 1620x^4 - 3645x^2}{(x^2+3)^{11/2}} = \frac{1080x^4 - 4860x^2 + 1215}{(x^2+3)^{11/2}},
 \end{aligned}$$

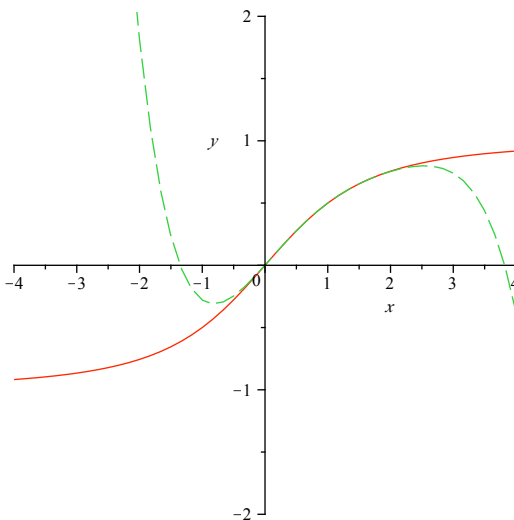
so that

$$f(1) = \frac{1}{2}, \quad f'(1) = \frac{3}{8}, \quad f''(1) = -\frac{9}{32}, \quad f'''(1) = \frac{9}{128}, \quad f^{(4)}(1) = \frac{225}{512}, \quad f^{(5)}(1) = -\frac{2565}{2048},$$

and

$$\begin{aligned} P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \\ &\quad \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\ &= \frac{1}{2} + \frac{3}{8}(x-1) + \frac{-9/32}{2}(x-1)^2 + \frac{9/128}{6}(x-1)^3 + \\ &\quad \frac{225/512}{24}(x-1)^4 + \frac{-2565/2048}{120}(x-1)^5 \\ &= \boxed{\frac{1}{2} + \frac{3}{8}(x-1) - \frac{9}{64}(x-1)^2 + \frac{3}{256}(x-1)^3 + \frac{75}{4096}(x-1)^4 - \frac{171}{16384}(x-1)^5}. \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = \frac{x}{\sqrt{x^2+3}}$ as the solid curve and the graph of $P_5(x)$ as the dashed curve.



36. (a) Let $f(x) = x\sqrt[3]{x^2+5}$ and $x_0 = 2$. Then

$$\begin{aligned}
 f'(x) &= x \cdot \frac{1}{3}(x^2+5)^{-2/3}(2x) + \sqrt[3]{x^2+5} = \frac{2x^2+3(x^2+5)}{3(x^2+5)^{2/3}} = \frac{5x^2+15}{3(x^2+5)^{2/3}}, \\
 f''(x) &= \frac{3(x^2+5)^{2/3}(10x) - (5x^2+15) \cdot 2(x^2+5)^{-1/3}(2x)}{9(x^2+5)^{4/3}} = \frac{30x(x^2+5) - 4x(5x^2+15)}{9(x^2+5)^{5/3}} \\
 &= \frac{10x^3+90x}{9(x^2+5)^{5/3}}, \\
 f'''(x) &= \frac{9(x^2+5)^{5/3}(30x^2+90) - (10x^3+90x) \cdot 15(x^2+5)^{2/3}(2x)}{81(x^2+5)^{10/3}} \\
 &= \frac{(x^2+5)(90x^2+270) - 10x(10x^3+90x)}{27(x^2+5)^{8/3}} = \frac{90x^4+720x^2+1350-100x^4-900x^2}{27(x^2+5)^{8/3}} \\
 &= \frac{-10x^4-180x^2+1350}{27(x^2+5)^{8/3}}, \\
 f^{(4)}(x) &= \frac{27(x^2+5)^{8/3}(-40x^3-360x) - (-10x^4-180x^2+1350) \cdot 72(x^2+5)^{5/3}(2x)}{729(x^2+5)^{16/3}} \\
 &= \frac{3(x^2+5)(-40x^3-360x) - 16x(-10x^4-180x^2+1350)}{81(x^2+5)^{11/3}} \\
 &= \frac{-120x^5-1680x^3-5400x+160x^5+2880x^3-21600x}{81(x^2+5)^{11/3}} \\
 &= \frac{40x^5+1200x^3-27000x}{81(x^2+5)^{11/3}},
 \end{aligned}$$

and

$$\begin{aligned}
 f^{(5)}(x) &= \frac{81(x^2+5)^{11/3}(200x^4+3600x^2-27000) - (40x^5+1200x^3-27000x) \cdot 297(x^2+5)^{8/3}(2x)}{6561(x^2+5)^{22/3}} \\
 &= \frac{3(x^2+5)(200x^4+3600x^2-27000) - 22x(40x^5+1200x^3-27000x)}{243(x^2+5)^{14/3}} \\
 &= \frac{600x^6+13800x^4-27000x^2-405000-880x^6-26400x^4+594000x^2}{243(x^2+5)^{14/3}} \\
 &= \frac{-280x^6-12600x^4+567000x^2-405000}{243(x^2+5)^{14/3}},
 \end{aligned}$$

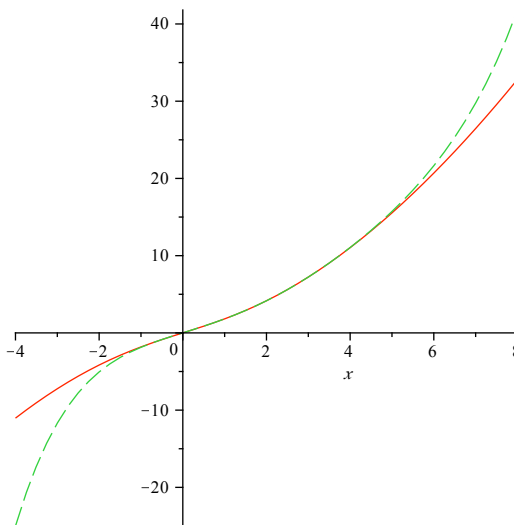
so that

$$\begin{aligned}
 f(2) &= 2\sqrt[3]{9}, \quad f'(2) = \frac{35\sqrt[3]{9}}{27}, \quad f''(2) = \frac{260\sqrt[3]{9}}{729}, \quad f'''(2) = \frac{470\sqrt[3]{9}}{19683}, \\
 f^{(4)}(2) &= -\frac{43120\sqrt[3]{9}}{531441}, \quad f^{(5)}(2) = \frac{1643480\sqrt[3]{9}}{14348907},
 \end{aligned}$$

and

$$\begin{aligned}
 P_5(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \\
 &\quad \frac{f^{(4)}(2)}{4!}(x-2)^4 + \frac{f^{(5)}(2)}{5!}(x-2)^5 \\
 &= 2\sqrt[3]{9} + \frac{35\sqrt[3]{9}}{27}(x-1) + \frac{260\sqrt[3]{9}/729}{2}(x-1)^2 + \frac{470\sqrt[3]{9}/19683}{6}(x-1)^3 + \\
 &\quad \frac{-43120\sqrt[3]{9}/531441}{24}(x-1)^4 + \frac{1643480\sqrt[3]{9}/14348907}{120}(x-1)^5 \\
 &= \boxed{2\sqrt[3]{9} + \frac{35\sqrt[3]{9}}{27}(x-2) + \frac{130\sqrt[3]{9}}{729}(x-2)^2 + \frac{235\sqrt[3]{9}}{59049}(x-2)^3 - \frac{5390\sqrt[3]{9}}{1594323}(x-2)^4 + \frac{41087\sqrt[3]{9}}{43046721}(x-2)^5}.
 \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = x\sqrt[3]{x^2 + 5}$ as the solid curve and the graph of $P_5(x)$ as the dashed curve.



37. (a) Let $f(x) = 0.34e^{-(\ln 2/5600)x}$ and $x_0 = 0$. Then

$$\begin{aligned} f'(x) &= -\frac{0.34 \ln 2}{5600} e^{-(\ln 2/5600)x}, \\ f''(x) &= 0.34 \left(\frac{\ln 2}{5600} \right)^2 e^{-(\ln 2/5600)x}, \\ f'''(x) &= -0.34 \left(\frac{\ln 2}{5600} \right)^3 e^{-(\ln 2/5600)x}, \text{ and} \\ f^{(4)}(x) &= 0.34 \left(\frac{\ln 2}{5600} \right)^4 e^{-(\ln 2/5600)x}, \end{aligned}$$

so that

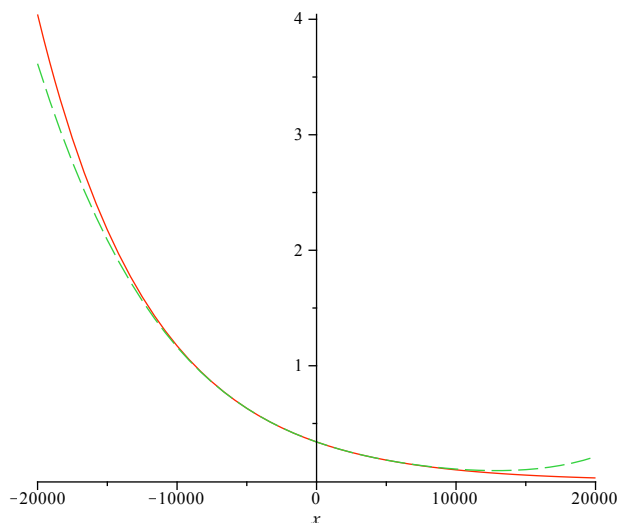
$$f(0) = 0.34, \quad f'(0) = -\frac{0.34 \ln 2}{5600}, \quad f''(0) = 0.34 \left(\frac{\ln 2}{5600} \right)^2,$$

$$f'''(0) = -0.34 \left(\frac{\ln 2}{5600} \right)^3, \quad f^{(4)}(0) = 0.34 \left(\frac{\ln 2}{5600} \right)^4,$$

and

$$\begin{aligned} P_4(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 \\ &= 0.34 - \frac{0.34 \ln 2}{5600}x + \frac{0.34(\ln 2/5600)^2}{2}x^2 + \frac{-0.34(\ln 2/5600)^3}{6}x^3 + \frac{0.34(\ln 2/5600)^4}{24}x^4 \\ &= \boxed{0.34 - \frac{0.34 \ln 2}{5600}x + 0.17 \left(\frac{\ln 2}{5600} \right)^2 x^2 - \frac{0.17}{3} \left(\frac{\ln 2}{5600} \right)^3 x^3 + \frac{0.17}{12} \left(\frac{\ln 2}{5600} \right)^4 x^4}. \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = 0.34e^{-(\ln 2/5600)x}$ as the solid curve and the graph of $P_4(x)$ as the dashed curve.



38. (a) Let $f(x) = 5000e^{0.04x}$ and $x_0 = 0$. Then

$$f'(x) = 200e^{0.04x}, \quad f''(x) = 8e^{0.04x}, \quad f'''(x) = \frac{8}{25}e^{0.04x}, \quad \text{and} \quad f^{(4)}(x) = \frac{8}{625}e^{0.04x},$$

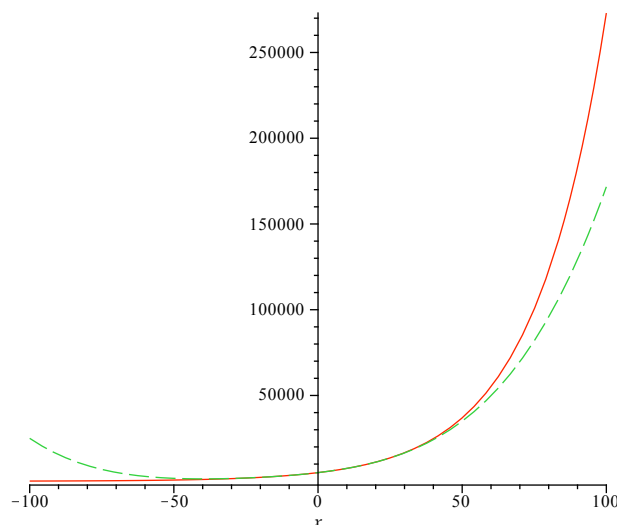
so that

$$f(0) = 5000, \quad f'(0) = 200, \quad f''(0) = 8, \quad f'''(0) = \frac{8}{25}, \quad f^{(4)}(0) = \frac{8}{625},$$

and

$$\begin{aligned} P_4(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 \\ &= 5000 + 200x + \frac{8}{2}x^2 + \frac{8/25}{6}x^3 + \frac{8/625}{24}x^4 = \boxed{5000 + 200x + 4x^2 + \frac{4}{75}x^3 + \frac{1}{1875}x^4}. \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = 5000e^{0.04x}$ as the solid curve and the graph of $P_4(x)$ as the dashed curve.



39. (a) Let $f(x) = \frac{100}{1 + 30.2e^{-0.2x}}$ and $x_0 = 0$. Then

$$\begin{aligned}
 f'(x) &= -\frac{100}{(1 + 30.2e^{-0.2x})^2}(-6.04e^{-0.2x}) = \frac{604e^{-0.2x}}{(1 + 30.2e^{-0.2x})^2} = 20 \frac{(1 + 30e^{-0.2x}) - 1}{(1 + 30.2e^{-0.2x})^2} \\
 &= 20 \left[\frac{1}{1 + 30.2e^{-0.2x}} - \frac{1}{(1 + 30.2e^{-0.2x})^2} \right], \\
 f''(x) &= 20 \left[-\frac{1}{(1 + 30.2e^{-0.2x})^2} + \frac{2}{(1 + 30.2e^{-0.2x})^3} \right] (-6.04e^{-0.2x}) \\
 &= 4 \left[\frac{(1 + 30.2e^{-0.2x}) - 1}{(1 + 30.2e^{-0.2x})^2} - 2 \frac{(1 + 30.2e^{-0.2x}) - 1}{(1 + 30.2e^{-0.2x})^3} \right] \\
 &= 4 \left[\frac{1}{1 + 30.2e^{-0.2x}} - \frac{3}{(1 + 30.2e^{-0.2x})^2} + \frac{2}{(1 + 30.2e^{-0.2x})^3} \right], \\
 f'''(x) &= 4 \left[-\frac{1}{(1 + 30.2e^{-0.2x})^2} + \frac{6}{(1 + 30.2e^{-0.2x})^3} - \frac{6}{(1 + 30.2e^{-0.2x})^4} \right] (-6.04e^{-0.2x}),
 \end{aligned}$$

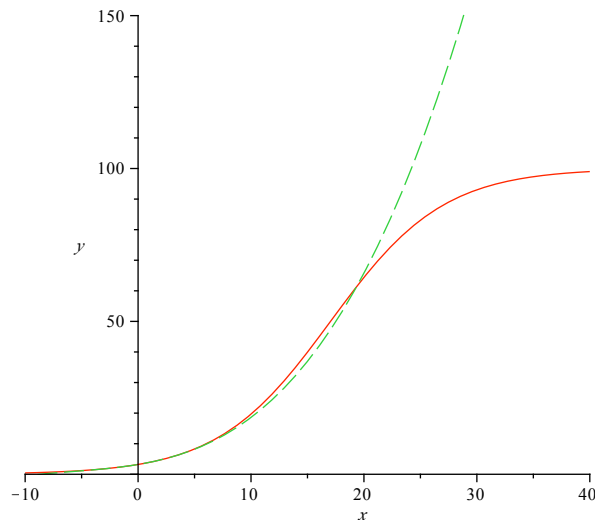
so that

$$f(0) = \frac{100}{31.2}, \quad f'(0) = \frac{604}{(31.2)^2}, \quad f''(0) = \frac{3527.36}{(31.2)^3}, \quad f'''(0) = \frac{19140.5184}{(31.2)^4},$$

and

$$\begin{aligned}
 P_3(x) &= f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 \\
 &= \frac{100}{31.2} + \frac{604}{(31.2)^2}x + \frac{3527.36/(31.2)^3}{2}x^2 + \frac{19140.5184/(31.2)^4}{6}x^3 \\
 &= \frac{125}{39} + \frac{3775}{6084}x + \frac{55115}{949104}x^2 + \frac{498451}{148060224}x^3 \\
 &\approx \boxed{3.205 + 0.620x + 0.058x^2 + 0.0034x^3}.
 \end{aligned}$$

(b) The figure below displays the graph of $f(x) = \frac{100}{1 + 30.2e^{-0.2x}}$ as the solid curve and the graph of $P_3(x)$ as the dashed curve.



40. (a) Let $f(x) = 100e^{-3e^{-0.2x}}$ and $x_0 = 0$. Then

$$\begin{aligned} f'(x) &= 100e^{-3e^{-0.2x}}(0.6e^{-0.2x}) = 60e^{-3e^{-0.2x}-0.2x}, \\ f''(x) &= 60e^{-3e^{-0.2x}-0.2x}(0.6e^{-0.2x} - 0.2) = 36e^{-3e^{-0.2x}-0.4x} - 12e^{-3e^{-0.2x}-0.2x}, \text{ and} \\ f'''(x) &= 36e^{-3e^{-0.2x}-0.4x}(0.6e^{-0.2x} - 0.4) - 12e^{-3e^{-0.2x}-0.2x}(0.6e^{-0.2x} - 0.2) \\ &= 21.6e^{-3e^{-0.2x}-0.6x} - 21.6e^{-3e^{-0.2x}-0.4x} + 2.4e^{-3e^{-0.2x}-0.2x}, \end{aligned}$$

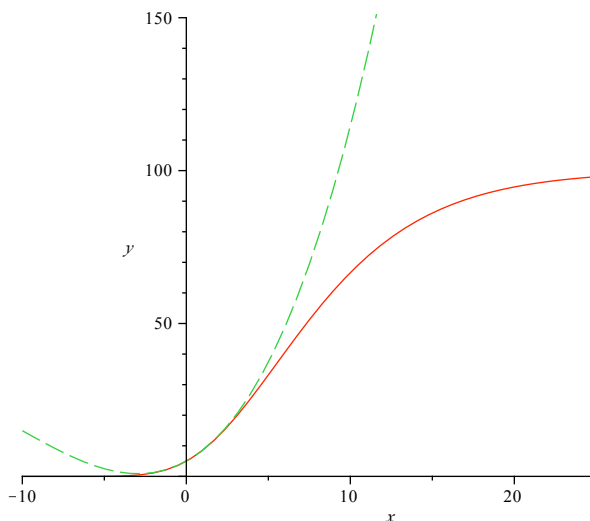
so that

$$f(0) = \frac{100}{e^3}, \quad f'(0) = \frac{60}{e^3}, \quad f''(0) = \frac{24}{e^3}, \quad f^{(4)}(0) = \frac{2.4}{e^3},$$

and

$$\begin{aligned} P_3(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= \frac{100}{e^3} + \frac{60}{e^3}x + \frac{24/e^3}{2}x^2 + \frac{2.4/e^3}{6}x^3 = \boxed{\frac{100}{e^3} + \frac{60}{e^3}x + \frac{12}{e^3}x^2 + \frac{2}{5e^3}x^3}. \end{aligned}$$

- (b) The figure below displays the graph of $f(x) = 100e^{-3e^{-0.2x}}$ as the solid curve and the graph of $P_3(x)$ as the dashed curve.



41. (a) Let $f(x) = \sin x$ and $x_0 = 0$. All even-order derivatives of f are equal to $\pm \sin x$, so that when these derivatives are evaluated at $x_0 = 0$, the result is always 0. As even-order derivatives are associated with even powers of x in a Taylor polynomial, it follows that Taylor polynomials for $\sin x$ at $x_0 = 0$ will always contain only terms with odd powers.
- (b) Let $f(x) = \sin x$ and $x_0 = \frac{\pi}{2}$. All even-order derivatives of f are equal to $\pm \sin x$, so that when these derivatives are evaluated at $x_0 = \frac{\pi}{2}$, the result is ± 1 ; on the other hand, all odd-order derivatives of f are equal to $\pm \cos x$, so that when these derivatives are evaluated at $x_0 = \frac{\pi}{2}$, the result is always 0. Thus, when $x_0 = \frac{\pi}{2}$, $P_7(x)$, and in fact all Taylor polynomials for f , will contain only terms with even powers.
42. (a) Let $f(x) = \cos x$ and $x_0 = 0$. All odd-order derivatives of f are equal to $\pm \sin x$, so that when these derivatives are evaluated at $x_0 = 0$, the result is always 0. As odd-order derivatives are associated with odd powers of x in a Taylor polynomial, it follows that Taylor polynomials for $\cos x$ at $x_0 = 0$ will always contain only terms with even powers.

- (b) Let $f(x) = \cos x$ and $x_0 = \frac{\pi}{2}$. All odd-order derivatives of f are equal to $\pm \sin x$, so that when these derivatives are evaluated at $x_0 = \frac{\pi}{2}$, the result is ± 1 ; on the other hand, all even-order derivatives of f are equal to $\pm \cos x$, so that when these derivatives are evaluated at $x_0 = \frac{\pi}{2}$, the result is always 0. Thus, when $x_0 = \frac{\pi}{2}$, $P_6(x)$, and in fact all Taylor polynomials for f , will contain only terms with odd powers.

Challenge Problems

43. Let $f(x) = \sin x - \lambda x$, $x_0 = \pi$, and $n = 2$. Then

$$\begin{aligned} f(x) &= \sin x - \lambda x, & f(\pi) &= -\lambda\pi, \\ f'(x) &= \cos x - \lambda, & f'(\pi) &= -1 - \lambda, \\ f''(x) &= -\sin x, & f''(\pi) &= 0, \end{aligned}$$

and

$$P_2(x) = f(\pi) + f'(\pi)(x - \pi) + \frac{f''(\pi)}{2!}(x - \pi)^2 = \boxed{-\lambda\pi - (1 + \lambda)(x - \pi)}.$$

Setting $P_2(x)$ equal to zero and solving for x yields

$$x = -\frac{\lambda\pi}{1 + \lambda} + \pi = \frac{-\lambda\pi + \pi + \lambda\pi}{1 + \lambda} = \frac{\pi}{1 + \lambda}.$$

44. Let $f(x) = \cot x - \lambda x$, $x_0 = \frac{\pi}{2}$, and $n = 2$. Then

$$\begin{aligned} f(x) &= \cot x - \lambda x, & f\left(\frac{\pi}{2}\right) &= -\frac{\lambda\pi}{2}, \\ f'(x) &= -\csc^2 x - \lambda, & f'\left(\frac{\pi}{2}\right) &= -1 - \lambda, \\ f''(x) &= -2\csc x \cdot (-\csc x \cot x) = 2\csc^2 x \cot x, & f''\left(\frac{\pi}{2}\right) &= 0, \end{aligned}$$

and

$$P_2(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x - \frac{\pi}{2}\right)^2 = \boxed{-\frac{\lambda\pi}{2} - (1 + \lambda)\left(x - \frac{\pi}{2}\right)}.$$

Setting $P_2(x)$ equal to zero and solving for x yields

$$x = -\frac{\lambda\pi}{2(1 + \lambda)} + \frac{\pi}{2} = \frac{-\lambda\pi + \pi + \lambda\pi}{2(1 + \lambda)} = \frac{\pi}{2(1 + \lambda)}.$$

3.6: Hyperbolic Functions

Concepts and Vocabulary

1. False. $\operatorname{csch} x = \frac{1}{\sinh x}$, and $\frac{1}{\cosh x} = \operatorname{sech} x$.

2. In terms of $\sinh x$ and $\cosh x$, $\tanh x = \frac{\sinh x}{\cosh x}$.

3. False. The domain of $y = \cosh x$ is all real numbers. The range of $y = \cosh x$ is $[1, \infty)$.

4. Because

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x,$$

$y = \cosh x$ is an (a) even function.

5. False. The correct identity is $\cosh^2 x - \sinh^2 x = 1$.

6. False. $\frac{d}{dx} \sinh x = \cosh x$.

7. True. Because the range of $y = \sinh x$ is all real numbers, the function $\sinh^{-1} x$ is defined for all real numbers.

8. False. $\tanh^{-1} x$ involves the natural logarithm of $\frac{1+x}{1-x}$, whereas $\coth^{-1} x$ involves the natural logarithm of $\frac{1+x}{x-1}$.

Skill Building

9. $\operatorname{csch}(\ln 3) = \frac{2}{e^{\ln 3} - e^{-\ln 3}} = \frac{2}{3 - \frac{1}{3}} = \frac{3}{4}$.

10. $\operatorname{sech}(\ln 2) = \frac{2}{e^{\ln 2} + e^{-\ln 2}} = \frac{2}{2 + \frac{1}{2}} = \frac{4}{5}$.

11. $\cosh^2 x - \sinh^2 x = 1$ for all real numbers x ; therefore,

$$\cosh^2(5) - \sinh^2(5) = 1.$$

12. $\cosh(-\ln 2) = \frac{e^{-\ln 2} + e^{-(-\ln 2)}}{2} = \frac{\frac{1}{2} + 2}{2} = \frac{5}{4}$.

13. $\tanh(0) = \frac{e^0 - e^0}{e^0 + e^0} = \frac{1 - 1}{1 + 1} = 0$.

14. Because $\ln \frac{1}{2} = -\ln 2$, it follows that

$$\sinh\left(\ln \frac{1}{2}\right) = \sinh(-\ln 2) = \frac{e^{-\ln 2} - e^{-(-\ln 2)}}{2} = \frac{\frac{1}{2} - 2}{2} = -\frac{3}{4}.$$

15. Working from the definitions,

$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 + \left(\frac{2}{e^x + e^{-x}} \right)^2 \\ &= \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} + \frac{4}{e^{2x} + 2 + e^{-2x}} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1.\end{aligned}$$

Alternately, divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\cosh^2 x$ to obtain $1 - \tanh^2 x = \operatorname{sech}^2 x$. Then add $\tanh^2 x$ to both sides to produce the required identity: $\tanh^2 x + \operatorname{sech}^2 x = 1$.

16. Working from the definitions,

$$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 - \left(\frac{2}{e^x - e^{-x}} \right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} - 2 + e^{-2x}} - \frac{4}{e^{2x} - 2 + e^{-2x}} = \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} - 2 + e^{-2x}} = 1.\end{aligned}$$

Alternately, divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$ to obtain $\coth^2 x - 1 = \operatorname{csch}^2 x$. Then add 1 to and subtract $\operatorname{csch}^2 x$ from both sides to produce the required identity: $\coth^2 x - \operatorname{csch}^2 x = 1$.

$$17. \sinh(-A) = \frac{e^{-A} - e^{-(-A)}}{2} = -\frac{e^A - e^{-A}}{2} = -\sinh(A).$$

$$18. \cosh(-A) = \frac{e^{-A} + e^{-(-A)}}{2} = \frac{e^A + e^{-A}}{2} = \cosh(A).$$

19. Let's start with the expression on the right-hand side and work toward the one on the left-hand side:

$$\begin{aligned}\sinh A \cosh B + \cosh A \sinh B &= \frac{e^A - e^{-A}}{2} \frac{e^B + e^{-B}}{2} + \frac{e^A + e^{-A}}{2} \frac{e^B - e^{-B}}{2} \\ &= \frac{e^{A+B} + e^{A-B} - e^{B-A} - e^{-A-B}}{4} + \frac{e^{A+B} - e^{A-B} + e^{B-A} - e^{-A-B}}{4} \\ &= \frac{2e^{A+B} - 2e^{-A-B}}{4} = \frac{e^{A+B} - e^{-(A+B)}}{2} = \sinh(A+B).\end{aligned}$$

20. Let's start with the expression on the right-hand side and work toward the one on the left-hand side:

$$\begin{aligned}\cosh A \cosh B + \sinh A \sinh B &= \frac{e^A + e^{-A}}{2} \frac{e^B + e^{-B}}{2} + \frac{e^A - e^{-A}}{2} \frac{e^B - e^{-B}}{2} \\ &= \frac{e^{A+B} + e^{A-B} + e^{B-A} + e^{-A-B}}{4} + \frac{e^{A+B} - e^{A-B} - e^{B-A} + e^{-A-B}}{4} \\ &= \frac{2e^{A+B} + 2e^{-A-B}}{4} = \frac{e^{A+B} + e^{-(A+B)}}{2} = \cosh(A+B).\end{aligned}$$

21. Working from the definition,

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2} = \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = 2 \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} = 2 \sinh x \cosh x.$$

Alternately, by Problem 19 with $A = B = x$,

$$\sinh(2x) = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$$

22. Working from the right-hand side to the left-hand side,

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x).\end{aligned}$$

Alternately, by Problem 20 with $A = B = x$,

$$\cosh(2x) = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

23. By Problems 20, 21, and 22,

$$\begin{aligned}\cosh(3x) &= \cosh(2x + x) = \cosh(2x) \cosh x + \sinh(2x) \sinh x \\ &= (\cosh^2 x + \sinh^2 x) \cosh x + (2 \sinh x \cosh x) \sinh x \\ &= \cosh^3 x + 3 \sinh^2 x \cosh x = \cosh^3 x + 3(\cosh^2 x - 1) \cosh x \\ &= 4 \cosh^3 x - 3 \cosh x.\end{aligned}$$

24. By Problems 21 and 22,

$$\tanh(2x) = \frac{\sinh(2x)}{\cosh(2x)} = \frac{2 \sinh x \cosh x}{\cosh^2 x + \sinh^2 x}.$$

Dividing the numerator and denominator of this last expression by $\cosh^2 x$ yields

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

25. By the Chain Rule,

$$\frac{d}{dx} \sinh(3x) = \cosh(3x) \frac{d}{dx} (3x) = \boxed{3 \cosh(3x)}.$$

26. By the Chain Rule,

$$\frac{d}{dx} \cosh\left(\frac{x}{2}\right) = \sinh\left(\frac{x}{2}\right) \frac{d}{dx} \left(\frac{x}{2}\right) = \boxed{\frac{1}{2} \sinh\left(\frac{x}{2}\right)}.$$

27. By the Chain Rule,

$$\frac{d}{dx} \cosh(x^2 + 1) = \sinh(x^2 + 1) \frac{d}{dx} (x^2 + 1) = \boxed{2x \sinh(x^2 + 1)}.$$

28. By the Chain Rule,

$$\frac{d}{dx} \cosh(2x^3 - 1) = \sinh(2x^3 - 1) \frac{d}{dx} (2x^3 - 1) = \boxed{6x^2 \sinh(2x^3 - 1)}.$$

29. By the Chain Rule,

$$\frac{d}{dx} \coth\left(\frac{1}{x}\right) = -\operatorname{csch}^2\left(\frac{1}{x}\right) \frac{d}{dx} \left(\frac{1}{x}\right) = \boxed{\frac{1}{x^2} \operatorname{csch}^2\left(\frac{1}{x}\right)}.$$

30. By the Chain Rule,

$$\frac{d}{dx} \tanh(x^2) = \operatorname{sech}^2(x^2) \frac{d}{dx} (x^2) = \boxed{2x \operatorname{sech}^2(x^2)}.$$

31. By the Product Rule and the Chain Rule,

$$\begin{aligned}
 \frac{d}{dx} \sinh x \cosh(4x) &= \sinh x \frac{d}{dx} \cosh(4x) + \cosh(4x) \frac{d}{dx} \sinh x \\
 &= \sinh x \sinh(4x) \frac{d}{dx} (4x) + \cosh(4x) \cosh x \\
 &= \boxed{4 \sinh x \sinh(4x) + \cosh(4x) \cosh x}.
 \end{aligned}$$

32. First note, by Problem 18, $\cosh(-x) = \cosh x$. Then, by the Product Rule and the Chain Rule

$$\begin{aligned}
 \frac{d}{dx} \sinh(2x) \cosh(-x) &= \frac{d}{dx} \sinh(2x) \cosh x = \sinh(2x) \frac{d}{dx} \cosh x + \cosh x \frac{d}{dx} \sinh(2x) \\
 &= \sinh(2x) \sinh x + \cosh x \cosh(2x) \frac{d}{dx} (2x) \\
 &= \boxed{\sinh(2x) \sinh x + 2 \cosh x \cosh(2x)}.
 \end{aligned}$$

33. By the Chain Rule,

$$\frac{d}{dx} \cosh^2 x = 2 \cosh x \frac{d}{dx} \cosh x = \boxed{2 \cosh x \sinh x}.$$

34. By the Chain Rule,

$$\frac{d}{dx} \tanh^2 x = 2 \tanh x \frac{d}{dx} \tanh x = \boxed{2 \tanh x \operatorname{sech}^2 x}.$$

35. By the Product Rule,

$$\frac{d}{dx} e^x \cosh x = e^x \frac{d}{dx} \cosh x + \cosh x \frac{d}{dx} e^x = \boxed{e^x \sinh x + e^x \cosh x}.$$

36. By the Product Rule,

$$\begin{aligned}
 \frac{d}{dx} e^x (\cosh x + \sinh x) &= e^x \frac{d}{dx} (\cosh x + \sinh x) + (\cosh x + \sinh x) \frac{d}{dx} e^x \\
 &= e^x (\sinh x + \cosh x) + e^x (\cosh x + \sinh x) \\
 &= \boxed{2e^x (\cosh x + \sinh x)}.
 \end{aligned}$$

37. By the Product Rule,

$$\frac{d}{dx} (x^2 \operatorname{sech} x) = x^2 \frac{d}{dx} \operatorname{sech} x + \operatorname{sech} x \frac{d}{dx} x^2 = \boxed{-x^2 \operatorname{sech} x \tanh x + 2x \operatorname{sech} x}.$$

38. By the Product Rule,

$$\frac{d}{dx} (x^3 \tanh x) = x^3 \frac{d}{dx} \tanh x + \tanh x \frac{d}{dx} x^3 = \boxed{x^3 \operatorname{sech}^2 x + 3x^2 \tanh x}.$$

39. By the Chain Rule,

$$\frac{d}{dx} \cosh^{-1}(4x) = \frac{1}{\sqrt{(4x)^2 - 1}} \frac{d}{dx} (4x) = \boxed{\frac{4}{\sqrt{16x^2 - 1}}}.$$

40. By the Chain Rule,

$$\frac{d}{dx} \sinh^{-1}(3x) = \frac{1}{\sqrt{(3x)^2 + 1}} \frac{d}{dx} (3x) = \boxed{\frac{3}{\sqrt{9x^2 + 1}}}.$$

41. By the Chain Rule,

$$\frac{d}{dx} \tanh^{-1}(x^2 - 1) = \frac{1}{1 - (x^2 - 1)^2} \frac{d}{dx}(x^2 - 1) = \boxed{\frac{2x}{1 - (x^2 - 1)^2}}.$$

42. By the Chain Rule,

$$\frac{d}{dx} \cosh^{-1}(2x + 1) = \frac{1}{\sqrt{(2x + 1)^2 - 1}} \frac{d}{dx}(2x + 1) = \frac{2}{\sqrt{4x^2 + 4x + 1 - 1}} = \boxed{\frac{1}{\sqrt{x^2 + x}}}.$$

43. By the Product Rule,

$$\frac{d}{dx} x \sinh^{-1} x = x \frac{d}{dx} \sinh^{-1} x + \sinh^{-1} x \frac{d}{dx} x = \boxed{\frac{x}{\sqrt{x^2 + 1}} + \sinh^{-1} x}.$$

44. By the Product Rule,

$$\frac{d}{dx} x^2 \cosh^{-1} x = x^2 \frac{d}{dx} \cosh^{-1} x + \cosh^{-1} x \frac{d}{dx} x^2 = \boxed{\frac{x^2}{\sqrt{x^2 - 1}} + 2x \cosh^{-1} x}.$$

45. By the Chain Rule,

$$\frac{d}{dx} \tanh^{-1}(\tan x) = \frac{1}{1 - \tan^2 x} \frac{d}{dx} \tan x = \boxed{\frac{\sec^2 x}{1 - \tan^2 x}}.$$

46. By the Chain Rule,

$$\frac{d}{dx} \sinh^{-1}(\sin x) = \frac{1}{\sqrt{\sin^2 x + 1}} \frac{d}{dx} \sin x = \boxed{\frac{\cos x}{\sqrt{\sin^2 x + 1}}}.$$

47. Using the Chain Rule twice,

$$\begin{aligned} \frac{d}{dx} \cosh^{-1}(\sqrt{x^2 - 1}) &= \frac{1}{\sqrt{(\sqrt{x^2 - 1})^2 - 1}} \frac{d}{dx} \sqrt{x^2 - 1} \\ &= \frac{1}{\sqrt{x^2 - 1 - 1}} \cdot \frac{1}{2}(x^2 - 1)^{-1/2} \frac{d}{dx}(x^2 - 1) = \boxed{\frac{x}{\sqrt{x^2 - 2\sqrt{x^2 - 1}}}}. \end{aligned}$$

48. Using the Chain Rule twice,

$$\begin{aligned} \frac{d}{dx} \sinh^{-1}(\sqrt{x^2 + 1}) &= \frac{1}{\sqrt{(\sqrt{x^2 + 1})^2 + 1}} \frac{d}{dx} \sqrt{x^2 + 1} \\ &= \frac{1}{\sqrt{x^2 + 1 + 1}} \cdot \frac{1}{2}(x^2 + 1)^{-1/2} \frac{d}{dx}(x^2 + 1) = \boxed{\frac{x}{\sqrt{x^2 + 2\sqrt{x^2 + 1}}}}. \end{aligned}$$

Applications and Extensions

49. Let $g(x) = \cosh x$, $x_0 = 0$, and $n = 6$. Then

$$\begin{aligned}
 g(x) &= \cosh x, & g(0) &= 1, \\
 g'(x) &= \sinh x, & g'(0) &= 0, \\
 g''(x) &= \cosh x, & g''(0) &= 1, \\
 g'''(x) &= \sinh x, & g'''(0) &= 0, \\
 g^{(4)}(x) &= \cosh x, & g^{(4)}(0) &= 1, \\
 g^{(5)}(x) &= \sinh x, & g^{(5)}(0) &= 0, \\
 g^{(6)}(x) &= \cosh x, & g^{(6)}(0) &= 1,
 \end{aligned}$$

and

$$\begin{aligned}
 P_6(x) &= g(0) + g'(0)(x-0) + \frac{g''(0)}{2!}(x-0)^2 + \frac{g'''(0)}{3!}(x-0)^3 + \frac{g^{(4)}(0)}{4!}(x-0)^4 + \\
 &\quad \frac{g^{(5)}(0)}{5!}(x-0)^5 + \frac{g^{(6)}(0)}{6!}(x-0)^6 \\
 &= \boxed{1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6}.
 \end{aligned}$$

50. Let $f(x) = \tanh x$, $x_0 = 0$, and $n = 7$. Then

$$\begin{aligned}
 f'(x) &= \operatorname{sech}^2 x, \\
 f''(x) &= 2 \operatorname{sech} x (-\operatorname{sech} x \tanh x) = -2 \operatorname{sech}^2 x \tanh x \\
 f'''(x) &= -2 \operatorname{sech}^2 x \operatorname{sech}^2 x + \tanh x [-4 \operatorname{sech} x (-\operatorname{sech} x \tanh x)] = -2 \operatorname{sech}^4 x + 4 \tanh^2 x \operatorname{sech}^2 x \\
 &= -2 \operatorname{sech}^4 x + 4(1 - \operatorname{sech}^2 x) \operatorname{sech}^2 x = 4 \operatorname{sech}^2 x - 6 \operatorname{sech}^4 x, \\
 f^{(4)}(x) &= 8 \operatorname{sech} x (-\operatorname{sech} x \tanh x) - 24 \operatorname{sech}^3 x (-\operatorname{sech} x \tanh x) = 8 \tanh x (3 \operatorname{sech}^4 x - \operatorname{sech}^2 x), \\
 f^{(5)}(x) &= 8 \tanh x (-12 \operatorname{sech}^4 x \tanh x + 2 \operatorname{sech}^2 x \tanh x) + 8 \operatorname{sech}^2 x (3 \operatorname{sech}^4 x - \operatorname{sech}^2 x) \\
 &= 16 \tanh^2 x (\operatorname{sech}^2 x - 6 \operatorname{sech}^4 x) + 24 \operatorname{sech}^6 x - 8 \operatorname{sech}^4 x \\
 &= 16(1 - \operatorname{sech}^2 x)(\operatorname{sech}^2 x - 6 \operatorname{sech}^4 x) + 24 \operatorname{sech}^6 x - 8 \operatorname{sech}^4 x \\
 &= 16 \operatorname{sech}^2 x - 112 \operatorname{sech}^4 x + 96 \operatorname{sech}^6 x + 24 \operatorname{sech}^6 x - 8 \operatorname{sech}^4 x \\
 &= 16 \operatorname{sech}^2 x - 120 \operatorname{sech}^4 x + 120 \operatorname{sech}^6 x, \\
 f^{(6)}(x) &= -32 \operatorname{sech}^2 x \tanh x + 480 \operatorname{sech}^4 x \tanh x - 720 \operatorname{sech}^6 x \tanh x \\
 &= -16 \tanh x (2 \operatorname{sech}^2 x - 30 \operatorname{sech}^4 x + 45 \operatorname{sech}^6 x), \text{ and} \\
 f^{(7)}(x) &= -16 \tanh x (-4 \operatorname{sech}^2 x \tanh x + 120 \operatorname{sech}^4 x \tanh x - 270 \operatorname{sech}^6 x \tanh x) - \\
 &\quad 16 \operatorname{sech}^2 x (2 \operatorname{sech}^2 x - 30 \operatorname{sech}^4 x + 45 \operatorname{sech}^6 x) \\
 &= 32 \tanh^2 x (2 \operatorname{sech}^2 x - 60 \operatorname{sech}^4 x + 135 \operatorname{sech}^6 x) - 32 \operatorname{sech}^4 x + 480 \operatorname{sech}^6 x - 720 \operatorname{sech}^8 x \\
 &= 32(1 - \operatorname{sech}^2 x)(2 \operatorname{sech}^2 x - 60 \operatorname{sech}^4 x + 135 \operatorname{sech}^6 x) - \\
 &\quad 32 \operatorname{sech}^4 x + 480 \operatorname{sech}^6 x - 720 \operatorname{sech}^8 x, \\
 &= 64 \operatorname{sech}^2 x - 1984 \operatorname{sech}^4 x + 6240 \operatorname{sech}^6 x - 4320 \operatorname{sech}^8 x - 32 \operatorname{sech}^4 x + 480 \operatorname{sech}^6 x - 720 \operatorname{sech}^8 x, \\
 &= 64 \operatorname{sech}^2 x - 2016 \operatorname{sech}^4 x + 6720 \operatorname{sech}^6 x - 5040 \operatorname{sech}^8 x,
 \end{aligned}$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -2, \quad f^{(4)}(0) = 0,$$

$$f^{(5)}(0) = 16, \quad f^{(6)}(0) = 0, \quad f^{(7)}(0) = -272,$$

and

$$\begin{aligned}
 P_7(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 + \\
 &\quad \frac{f^{(6)}(0)}{6!}(x-0)^6 + \frac{f^{(7)}(0)}{7!}(x-0)^7 \\
 &= \boxed{x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7}.
 \end{aligned}$$

51. Let $y = 12 \cosh \frac{x}{12} + 20$ model the height of the cable for $-50 \leq x \leq 50$. The slope of the tangent line to the cable is given by

$$y' = \frac{d}{dx} \left(12 \cosh \frac{x}{12} + 20 \right) = 12 \cdot \frac{1}{12} \sinh \frac{x}{12} = \sinh \frac{x}{12}.$$

At $x = 50$, the slope $m_{\tan}$ of the tangent line is

$$m_{\tan} = y'(50) = \sinh \frac{50}{12} = \sinh \frac{25}{6}.$$

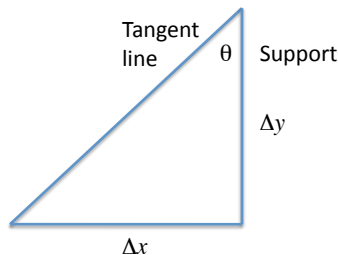
Based on the diagram below, we see that

$$\tan \theta = \frac{\Delta x}{\Delta y} \quad \text{and} \quad m_{\tan} = \frac{\Delta y}{\Delta x},$$

where θ is the angle at which the cable meets the support. Thus,

$$\theta = \tan^{-1} \left(\frac{1}{m_{\tan}} \right) = \tan^{-1} \left(\frac{1}{\sinh \frac{25}{6}} \right) \approx \boxed{0.031 \text{ radians}},$$

or approximately $\boxed{1.776^\circ}$.



52. (a) Let $y = 15 \cosh \frac{x}{15} - 10$ model the height of the string of holiday lights assuming that the utility poles are located at $x = \pm 6$. With

$$y' = \frac{d}{dx} \left(15 \cosh \frac{x}{15} - 10 \right) = 15 \cdot \frac{1}{15} \sinh \frac{x}{15} = \sinh \frac{x}{15},$$

the slope of the tangent line at $x = -6$ is

$$y'(-6) = \sinh \left(-\frac{6}{15} \right) = -\sinh \frac{2}{5},$$

and the slope of the tangent line at $x = 6$ is

$$y'(6) = \sinh \left(\frac{6}{15} \right) = \boxed{\sinh \frac{2}{5}},$$

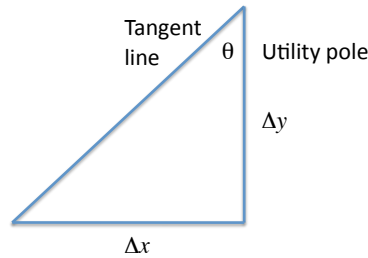
- (b) Because the graph of $y = 15 \cosh \frac{x}{15} - 10$ is symmetric with respect to the y -axis, the angle at which the string of lights meets the pole on the left is the same as the angle at which the string of lights meets the pole on the right. Based on the diagram below, we see that

$$\tan \theta = \frac{\Delta x}{\Delta y} \quad \text{and} \quad m_{\tan} = \frac{\Delta y}{\Delta x},$$

where θ is the angle at which the cable meets the support and $m_{\tan}$ is the slope of the tangent line where the string of lights meets the utility pole. Using the result from part (a), it follows that

$$\theta = \tan^{-1} \left(\frac{1}{m_{\tan}} \right) = \tan^{-1} \left(\frac{1}{\sinh \frac{2}{5}} \right) \approx \boxed{1.181 \text{ radians}},$$

or approximately $\boxed{67.666^\circ}$.



53. Let $y = -68.767 \cosh \left(\frac{0.711x}{68.767} \right) + 693.859$.

- (a) The height of the arch is the value of y when $x = 0$; that is, the height of the arch is

$$\begin{aligned} -68.767 \cosh \left(\frac{0.711(0)}{68.767} \right) + 693.859 &= -68.767 \cosh 0 + 693.859 \\ &= -68.767 \cdot 1 + 693.859 = \boxed{625.092 \text{ feet}}. \end{aligned}$$

- (b) The width of the arch is the distance between the x -intercepts of the graph of y . Solving

$$-68.767 \cosh \left(\frac{0.711x}{68.767} \right) + 693.859 = 0$$

for x yields

$$x = \pm \frac{68.767}{0.711} \cosh^{-1} \frac{693.859}{68.767} \approx \pm 290.372.$$

The width of the arch is therefore approximately $\boxed{580.744 \text{ feet}}$.

- (c) The slope of the arch 50 feet to the left of its center is

$$y'(-50) = -0.711 \sinh \left(\frac{0.711(-50)}{68.767} \right) = -0.711 \sinh \left(-\frac{35.55}{68.767} \right) \approx \boxed{0.384},$$

and the slope of the arch 50 feet to the right of its center is

$$y'(50) = -0.711 \sinh \left(\frac{0.711(50)}{68.767} \right) = -0.711 \sinh \frac{35.55}{68.767} \approx \boxed{-0.384}.$$

- (d) The slope of the arch 200 feet to the left of its center is

$$y'(-200) = -0.711 \sinh \left(\frac{0.711(-200)}{68.767} \right) = -0.711 \sinh \left(-\frac{142.2}{68.767} \right) \approx \boxed{2.766},$$

and the slope of the arch 200 feet to the right of its center is

$$y'(200) = -0.711 \sinh \left(\frac{0.711(200)}{68.767} \right) = -0.711 \sinh \frac{142.2}{68.767} \approx \boxed{-2.766}.$$

- (e) Because the graph of $y = -68.767 \cosh\left(\frac{0.711x}{68.767}\right) + 693.859$ is symmetric with respect to the y -axis, the angle at which the arch meets the ground on the left is the same as the angle at which the arch meets the ground on the right. Moreover, the angle at which the arch meets the ground on the left is

$$\tan^{-1}(m_{\tan}),$$

where $m_{\tan}$ is the slope of the tangent line to the graph of $y = -68.767 \cosh\left(\frac{0.711x}{68.767}\right) + 693.859$ at $x = -290.372$. Now

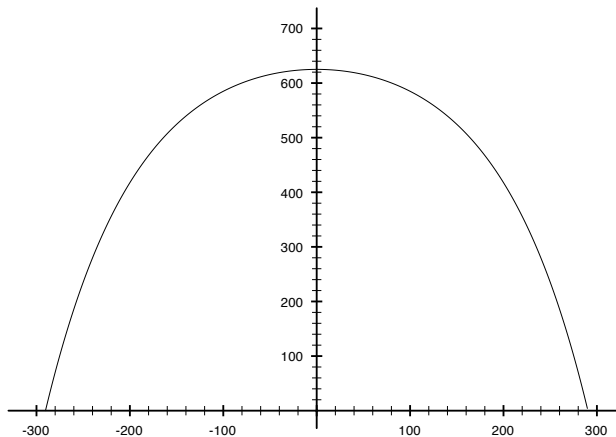
$$\begin{aligned} m_{\tan} &= y'(-290.372) = -0.711 \sinh\left(\frac{0.711(-290.372)}{68.767}\right) \\ &= -0.711 \sinh\left(-\frac{206.454492}{68.767}\right) = 0.711 \sinh\frac{206.454492}{68.767}, \end{aligned}$$

so the angle at which the arch meets the ground is approximately

$$\tan^{-1}\left(0.711 \sinh\frac{206.454492}{68.767}\right) \approx \boxed{1.432 \text{ radians}},$$

or approximately $\boxed{82.048^\circ}$.

- (f) The figure below displays the graph of the height of the Gateway Arch. Observe that the y -intercept of the graph is slightly larger than 620, and the x -intercepts are roughly $x \pm 290$. These observations support the results found in parts (a) and (b). Moreover, the magnitude of the rate of change of the height of the Gateway Arch is larger at $x = \pm 200$ than it is at $x = \pm 50$, supporting the results found in parts (c) and (d).



54. Let n be any real number. Now

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x,$$

so

$$\begin{aligned} (\cosh x + \sinh x)^n &= (e^x)^n = e^{nx} = \frac{2e^{nx}}{2} = \frac{e^{nx} + e^{nx}}{2} \\ &= \frac{e^{nx} + e^{-nx} + e^{nx} - e^{-nx}}{2} = \frac{e^{nx} + e^{-nx}}{2} + \frac{e^{nx} - e^{-nx}}{2} = \cosh(nx) + \sinh(nx). \end{aligned}$$

55. (a)

$$\begin{aligned} \frac{d}{dx} \cosh^{-1} x &= \frac{d}{dx} \ln(x + \sqrt{x^2 - 1}) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{1}{2}(x^2 - 1)^{-1/2}(2x)\right) \\ &= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}. \end{aligned}$$

- (b) The domain of $y = \cosh^{-1} x$ is the set $\{x|x \geq 1\}$, so the domain of y' is at most this same set. However, from part (a), y' is not defined for $x = 1$; therefore, the domain of y' is the set $\boxed{\{x|x > 1\}}$.

56. (a)

$$\begin{aligned}\frac{d}{dx} \tanh^{-1} x &= \frac{d}{dx} \left[\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \right] = \frac{1}{2} \frac{d}{dx} (\ln(1+x) - \ln(1-x)) \\ &= \frac{1}{2} \left(\frac{1}{1+x} - \frac{1}{1-x} \cdot (-1) \right) = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) \\ &= \frac{1}{2} \cdot \frac{(1-x) + (1+x)}{1-x^2} = \frac{1}{1-x^2}.\end{aligned}$$

- (b) The domain of $y = \tanh^{-1} x$ is the set $\{x|-1 < x < 1\}$, so the domain of y' is at most this same set. There are no additional restrictions on the domain of y' based on the result from part (a), so the domain of y' is the set $\boxed{\{x|-1 < x < 1\}}$.

57. Using the definition of $\tanh x$ in terms of $\sinh x$ and $\cosh x$ and the Quotient Rule,

$$\begin{aligned}\frac{d}{dx} \tanh x &= \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) = \frac{\cosh x \frac{d}{dx} \sinh x - \sinh x \frac{d}{dx} \cosh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.\end{aligned}$$

58. Using the definition of $\coth x$ in terms of $\sinh x$ and $\cosh x$ and the Quotient Rule,

$$\begin{aligned}\frac{d}{dx} \coth x &= \frac{d}{dx} \left(\frac{\cosh x}{\sinh x} \right) = \frac{\sinh x \frac{d}{dx} \cosh x - \cosh x \frac{d}{dx} \sinh x}{\sinh^2 x} \\ &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x.\end{aligned}$$

59. Using the definition of $\operatorname{sech} x$ and the Chain Rule,

$$\begin{aligned}\frac{d}{dx} \operatorname{sech} x &= \frac{d}{dx} (\cosh x)^{-1} = -(\cosh x)^{-2} \frac{d}{dx} \cosh x \\ &= -\frac{1}{\cosh^2 x} \cdot \sinh x = -\frac{1}{\cosh x} \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x.\end{aligned}$$

60. Using the definition of $\operatorname{csch} x$ and the Chain Rule,

$$\begin{aligned}\frac{d}{dx} \operatorname{csch} x &= \frac{d}{dx} (\sinh x)^{-1} = -(\sinh x)^{-2} \frac{d}{dx} \sinh x \\ &= -\frac{1}{\sinh^2 x} \cdot \cosh x = -\frac{1}{\sinh x} \frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x.\end{aligned}$$

61. Let $-1 < x < 1$ and $y = \tanh^{-1} x$. Then

$$\begin{aligned}x &= \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \\ x(e^y + e^{-y}) &= e^y - e^{-y} \\ e^y(x-1) &= -(1+x)e^{-y} \\ e^{2y} &= -\frac{1+x}{x-1} = \frac{1+x}{1-x}.\end{aligned}$$

With the restriction $-1 < x < 1$, it follows that $0 < 1 + x < 2$ and $0 < 1 - x < 2$. The expression $\frac{1+x}{1-x}$ is therefore defined and positive for $-1 < x < 1$. Taking the natural logarithm of both sides of the last equation and then dividing by 2 yields

$$y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

Thus, $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$.

62. Let $x \geq 1$ and $y = \cosh^{-1} x$. Then

$$x = \cosh y = \frac{e^y + e^{-y}}{2} \quad \text{and} \quad e^{2y} - 2xe^y + 1 = 0.$$

By the quadratic formula,

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Note that the term $\sqrt{x^2 - 1}$ is defined because of the restriction $x \geq 1$. To determine which sign on the radical is needed, note that $y = \cosh^{-1} x \geq 0$, so $e^y \geq 1$. Pick an x greater than 1, say $x = 3$. Because

$$3 + \sqrt{9 - 1} = 3 + 2\sqrt{2} > 1 \quad \text{and} \quad 3 - \sqrt{9 - 1} = 3 - 2\sqrt{2} \approx 0.172 < 1,$$

it follows that the plus sign is needed. Therefore,

$$e^y = x + \sqrt{x^2 - 1}, \quad \text{so that} \quad y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

63. Let $|x| > 1$ and $y = \coth^{-1} x$. Then

$$\begin{aligned} x &= \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}} \\ x(e^y - e^{-y}) &= e^y + e^{-y} \\ e^y(x - 1) &= (1 + x)e^{-y} \\ e^{2y} &= \frac{1 + x}{x - 1}. \end{aligned}$$

When $x > 1$, it follows that $1 + x > 2$ and $x - 1 > 0$; when $x < -1$, it follows that $1 + x < 0$ and $x - 1 < 0$. The expression $\frac{1+x}{x-1}$ is therefore defined and positive for $|x| > 1$. Taking the natural logarithm of both sides of the last equation and then dividing by 2 yields

$$y = \frac{1}{2} \ln \left(\frac{1+x}{x-1} \right).$$

Thus, $\coth^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{x-1} \right) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$.

Challenge Problems

64. Recognize that

$$\lim_{x \rightarrow 0} \frac{\sinh x}{x} = \lim_{x \rightarrow 0} \frac{\sinh x - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{\sinh x - \sinh 0}{x - 0} = f'(0),$$

where $f(x) = \sinh x$. Thus,

$$\lim_{x \rightarrow 0} \frac{\sinh x}{x} = \cosh 0 = \boxed{1}.$$

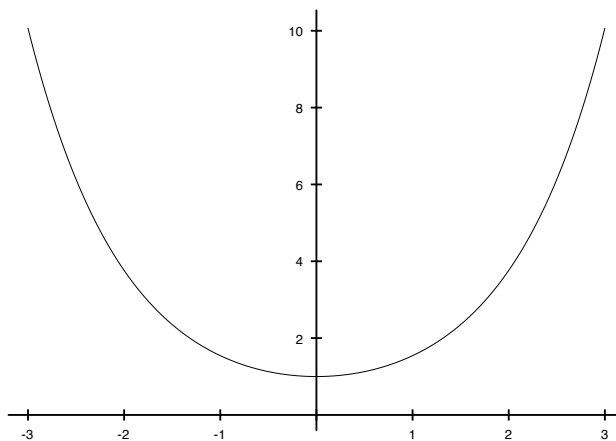
65. Recognize that

$$\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x} = \lim_{x \rightarrow 0} \frac{\cosh x - \cosh 0}{x - 0} = f'(0),$$

where $f(x) = \cosh x$. Thus,

$$\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x} = \sinh 0 = \boxed{0}.$$

66. (a) The figure below displays the graph of $y = f(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.



(b) The slope of the tangent line to the graph of f at the point R is

$$m_{\text{tan}} = f'(r) = \sinh r,$$

so the equation of the tangent line is

$$y - \cosh r = \sinh r(x - r) \quad \text{or} \quad y = x \sinh r + \cosh r - r \sinh r.$$

For $r \neq 0$, $\sinh r \neq 0$, and the x -intercept of this line is $x = r - \coth r$. Therefore, the coordinates of the point Q are $\boxed{(0, r - \coth r)}$.

(c) The length of the line segment $\overline{PQ}$ is

$$|r - (r - \coth r)| = \boxed{|\coth r|}.$$

In the limit as r increases without bound, the length of the segment $\overline{PQ}$ approaches

$$\lim_{r \rightarrow \pm\infty} |\coth r| = \boxed{1}.$$

67. By the Chain Rule,

$$\frac{d}{dx} \sin^{-1}(\cosh x) = \frac{1}{\sqrt{1 - \cosh^2 x}} \cdot \sinh x = \frac{\sinh x}{\sqrt{-\sinh^2 x}},$$

where the identity $\cosh^2 x - \sinh^2 x = 1$ has been used in the last step. Note, however, that the function

$$\frac{\sinh x}{\sqrt{-\sinh^2 x}}$$

is undefined for all real numbers x . The problem here is that the domain of the inverse sine function is the set $\{x | -1 \leq x \leq 1\}$ but the range of the hyperbolic cosine function is the set $\{y | y \geq 1\}$. Thus, the original function $f(x) = \sin^{-1}(\cosh x)$ is defined only for $x = 0$, and so it is not possible to find the derivative of f at any real value of x .

68. Let $f(x) = x \sinh^{-1} x$.

(a) Note that

$$\begin{aligned} \sinh^{-1}(-x) &= \ln(-x + \sqrt{x^2 + 1}) = \ln \left[(-x + \sqrt{x^2 + 1}) \cdot \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}} \right] \\ &= \ln \frac{-x^2 + x^2 + 1}{x + \sqrt{x^2 + 1}} = \ln \frac{1}{x + \sqrt{x^2 + 1}} = -\ln(x + \sqrt{x^2 + 1}) = -\sinh^{-1} x. \end{aligned}$$

Therefore,

$$f(-x) = (-x) \sinh^{-1}(-x) = -(-x) \sinh^{-1} x = x \sinh^{-1} x = f(x),$$

and f is an even function.

(b) By the Product Rule,

$$f'(x) = x \cdot \frac{1}{\sqrt{x^2 + 1}} + \sinh^{-1} x = \boxed{\frac{x}{\sqrt{x^2 + 1}} + \sinh^{-1} x},$$

and by the Quotient Rule and the Chain Rule,

$$\begin{aligned} f''(x) &= \frac{\sqrt{x^2 + 1} \cdot 1 - x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x)}{x^2 + 1} + \frac{1}{\sqrt{x^2 + 1}} \\ &= \frac{(x^2 + 1) - x^2}{(x^2 + 1)^{3/2}} + \frac{1}{\sqrt{x^2 + 1}} = \boxed{\frac{x^2 + 2}{(x^2 + 1)^{3/2}}}. \end{aligned}$$

Chapter 3 Review Exercises

$$1. \frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \frac{d}{dx}(ax+b) = \boxed{an(ax+b)^{n-1}}.$$

$$2. \frac{d}{dx}\sqrt{2ax} = \sqrt{2a} \frac{d}{dx}x^{1/2} = \sqrt{2a} \cdot \frac{1}{2}x^{-1/2} = \frac{\sqrt{2a}}{2\sqrt{x}} = \boxed{\sqrt{\frac{a}{2x}}}.$$

3.

$$\begin{aligned} \frac{d}{dx}(x\sqrt{1-x}) &= x \frac{d}{dx}(1-x)^{1/2} + \sqrt{1-x} \frac{d}{dx}x \\ &= x \cdot \frac{1}{2}(1-x)^{-1/2} \frac{d}{dx}(1-x) + \sqrt{1-x} = \boxed{-\frac{x}{2(1-x)^{1/2}} + (1-x)^{1/2}}. \end{aligned}$$

$$4. \frac{d}{dx} \left(\frac{1}{\sqrt{x^2+1}} \right) = \frac{d}{dx}(x^2+1)^{-1/2} = -\frac{1}{2}(x^2+1)^{-3/2} \frac{d}{dx}(x^2+1) = \boxed{-\frac{x}{(x^2+1)^{3/2}}}.$$

$$5. \frac{d}{dx}(x^2+4)^{3/2} = \frac{3}{2}(x^2+4)^{1/2} \frac{d}{dx}(x^2+4) = \boxed{3x(x^2+4)^{1/2}}.$$

6.

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2}{\sqrt{x^2-1}} \right) &= \frac{\sqrt{x^2-1} \frac{d}{dx}x^2 - x^2 \frac{d}{dx}(x^2-1)^{1/2}}{x^2-1} = \frac{\sqrt{x^2-1} \cdot 2x - x^2 \cdot \frac{1}{2}(x^2-1)^{-1/2} \frac{d}{dx}(x^2-1)}{x^2-1} \\ &= \frac{2x\sqrt{x^2-1} - x^3(x^2-1)^{-1/2}}{x^2-1} \cdot \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} = \frac{2x(x^2-1) - x^3}{(x^2-1)^{3/2}} = \boxed{\frac{x^3-2x}{(x^2-1)^{3/2}}}. \end{aligned}$$

7.

$$\begin{aligned} \frac{d}{dx} \left(\frac{\sqrt{2ax-x^2}}{x} \right) &= \frac{x \frac{d}{dx}(2ax-x^2)^{1/2} - \sqrt{2ax-x^2} \frac{d}{dx}x}{x^2} \\ &= \frac{x \cdot \frac{1}{2}(2ax-x^2)^{-1/2} \frac{d}{dx}(2ax-x^2) - \sqrt{2ax-x^2} \cdot 1}{x^2} \\ &= \frac{x(a-x)(2ax-x^2)^{-1/2} - \sqrt{2ax-x^2}}{x^2} \cdot \frac{\sqrt{2ax-x^2}}{\sqrt{2ax-x^2}} \\ &= \frac{x(a-x) - (2ax-x^2)}{x^2\sqrt{2ax-x^2}} = -\frac{ax}{x^2\sqrt{2ax-x^2}} = \boxed{-\frac{a}{x\sqrt{2ax-x^2}}}. \end{aligned}$$

$$8. \frac{d}{dx}(\sqrt{x} + \sqrt[3]{x}) = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}}.$$

9. Let $y = (e^x - x)^{5x}$. Then

$$\ln y = \ln[(e^x - x)^{5x}] = 5x \ln(e^x - x),$$

so

$$\begin{aligned} \frac{d}{dx} \ln y &= \frac{d}{dx}[5x \ln(e^x - x)] \\ \frac{1}{y} \frac{dy}{dx} &= 5x \frac{d}{dx} \ln(e^x - x) + \ln(e^x - x) \frac{d}{dx}(5x) \\ &= 5x \cdot \frac{1}{e^x - x} \frac{d}{dx}(e^x - x) + \ln(e^x - x) \cdot 5 \\ &= \frac{5x(e^x - 1)}{e^x - x} + 5 \ln(e^x - x). \end{aligned}$$

Thus,

$$\frac{d}{dx}(e^x - x)^{5x} = \boxed{(e^x - x)^{5x} \left(\frac{5x(e^x - 1)}{e^x - x} + 5 \ln(e^x - x) \right)}.$$

10. Let $\phi(x) = \frac{(x^2 - a^2)^{3/2}}{\sqrt{x + a}}$. Then

$$\ln \phi = \ln \frac{(x^2 - a^2)^{3/2}}{\sqrt{x + a}} = \frac{3}{2} \ln(x^2 - a^2) - \frac{1}{2} \ln(x + a),$$

so

$$\begin{aligned} \frac{d}{dx} \ln \phi &= \frac{d}{dx} \left[\frac{3}{2} \ln(x^2 - a^2) - \frac{1}{2} \ln(x + a) \right] \\ \frac{1}{\phi} \frac{d\phi}{dx} &= \frac{3}{2} \cdot \frac{1}{x^2 - a^2} \frac{d}{dx}(x^2 - a^2) - \frac{1}{2} \cdot \frac{1}{x + a} \frac{d}{dx}(x + a) = \frac{3x}{x^2 - a^2} - \frac{1}{2(x + a)}. \end{aligned}$$

Thus,

$$\frac{d}{dx} \frac{(x^2 - a^2)^{3/2}}{\sqrt{x + a}} = \boxed{\frac{(x^2 - a^2)^{3/2}}{\sqrt{x + a}} \left[\frac{3x}{x^2 - a^2} - \frac{1}{2(x + a)} \right]}.$$

11.

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2}{(x-1)^2} \right) &= \frac{(x-1)^2 \frac{d}{dx} x^2 - x^2 \frac{d}{dx} (x-1)^2}{(x-1)^4} \\ &= \frac{2x(x-1)^2 - x^2 \cdot 2(x-1) \frac{d}{dx} (x-1)}{(x-1)^4} = \frac{2x(x-1) - 2x^2}{(x-1)^3} = \boxed{-\frac{2x}{(x-1)^3}}. \end{aligned}$$

$$12. \frac{d}{dx} (b^{1/2} - x^{1/2})^2 = 2(b^{1/2} - x^{1/2}) \frac{d}{dx} (b^{1/2} - x^{1/2}) = 2(b^{1/2} - x^{1/2}) \left(-\frac{1}{2} x^{-1/2} \right) = \boxed{-\frac{b^{1/2} - x^{1/2}}{x^{1/2}}}.$$

13.

$$\begin{aligned} \frac{d}{dx} [x \sec(2x)] &= x \frac{d}{dx} \sec(2x) + \sec(2x) \frac{d}{dx} x \\ &= x \sec(2x) \tan(2x) \frac{d}{dx} (2x) + \sec(2x) = \boxed{2x \sec(2x) \tan(2x) + \sec(2x)}. \end{aligned}$$

$$14. \frac{d}{dx} \cos^3 x = 3 \cos^2 x \frac{d}{dx} \cos x = \boxed{-3 \cos^2 x \sin x}.$$

15.

$$\begin{aligned} \frac{d}{dx} \sqrt{a^2 \sin \left(\frac{x}{a} \right)} &= \frac{1}{2} \left[a^2 \sin \left(\frac{x}{a} \right) \right]^{-1/2} \frac{d}{dx} \left[a^2 \sin \left(\frac{x}{a} \right) \right] \\ &= \frac{1}{2} \left[a^2 \sin \left(\frac{x}{a} \right) \right]^{-1/2} \cdot a^2 \cos \left(\frac{x}{a} \right) \frac{d}{dx} \left(\frac{x}{a} \right) = \boxed{\frac{a \cos \left(\frac{x}{a} \right)}{2 \left(a^2 \sin \left(\frac{x}{a} \right) \right)^{1/2}}}. \end{aligned}$$

$$16. \frac{d}{dz} \sqrt{1 + \sin z} = \frac{1}{2} (1 + \sin z)^{-1/2} \frac{d}{dz} (1 + \sin z) = \boxed{\frac{\cos z}{2\sqrt{1 + \sin z}}}.$$

$$17. \frac{d}{dv} \left(\sin v - \frac{1}{3} \sin^3 v \right) = \cos v - \sin^2 v \frac{d}{dv} \sin v = \cos v - \sin^2 v \cos v = \cos v (1 - \sin^2 v) = \boxed{\cos^3 v}.$$

$$18. \frac{d}{dx} \tan \sqrt{\frac{\pi}{x}} = \sec^2 \sqrt{\frac{\pi}{x}} \frac{d}{dx} \sqrt{\pi x^{-1/2}} = \sec^2 \sqrt{\frac{\pi}{x}} \cdot \left(-\frac{1}{2} \sqrt{\pi} x^{-3/2} \right) = \boxed{-\frac{1}{2} \sqrt{\frac{\pi}{x^3}} \sec^2 \sqrt{\frac{\pi}{x}}}.$$

$$19. \frac{d}{dx} (1.05)^x = \boxed{(1.05)^x \ln 1.05}.$$

$$20. \frac{d}{dy} \ln(y^2 + 1) = \frac{1}{y^2 + 1} \frac{d}{dy} (y^2 + 1) = \boxed{\frac{2y}{y^2 + 1}}.$$

21.

$$\begin{aligned} \frac{d}{du} \ln(\sqrt{u^2 + 25} - u) &= \frac{1}{\sqrt{u^2 + 25} - u} \frac{d}{du} [(u^2 + 25)^{1/2} - u] \\ &= \frac{1}{\sqrt{u^2 + 25} - u} \cdot \left(\frac{1}{2} (u^2 + 25)^{-1/2} \frac{d}{du} (u^2 + 25) - 1 \right) \\ &= \frac{1}{\sqrt{u^2 + 25} - u} \cdot \left(\frac{u}{\sqrt{u^2 + 25}} - 1 \right) = \frac{1}{\sqrt{u^2 + 25} - u} \cdot \left(\frac{u - \sqrt{u^2 + 25}}{\sqrt{u^2 + 25}} \right) \\ &= \boxed{-\frac{1}{\sqrt{u^2 + 25}}}. \end{aligned}$$

$$22. \frac{d}{dx} (x^2 + 2^x) = \boxed{2x + 2^x \ln 2}.$$

$$23. \frac{d}{dx} \ln[\sin(2x)] = \frac{1}{\sin(2x)} \frac{d}{dx} \sin(2x) = \frac{\cos(2x)}{\sin(2x)} \frac{d}{dx} (2x) = \boxed{2 \cot(2x)}.$$

24.

$$\begin{aligned} \frac{d}{dx} [e^{-x} \sin(2x + \pi)] &= e^{-x} \frac{d}{dx} \sin(2x + \pi) + \sin(2x + \pi) \frac{d}{dx} e^{-x} \\ &= e^{-x} \cos(2x + \pi) \frac{d}{dx} (2x + \pi) + e^{-x} \sin(2x + \pi) \frac{d}{dx} (-x) \\ &= \boxed{e^{-x} [2 \cos(2x + \pi) - \sin(2x + \pi)]}. \end{aligned}$$

$$25. \frac{d}{dx} \ln(x^2 - 2x) = \frac{1}{x^2 - 2x} \frac{d}{dx} (x^2 - 2x) = \boxed{\frac{2x - 2}{x^2 - 2x}}.$$

26.

$$\begin{aligned} \frac{d}{dx} \ln \frac{x^2 + 1}{x^2 - 1} &= \frac{d}{dx} \ln(x^2 + 1) - \frac{d}{dx} \ln(x^2 - 1) = \frac{1}{x^2 + 1} \frac{d}{dx} (x^2 + 1) - \frac{1}{x^2 - 1} \frac{d}{dx} (x^2 - 1) \\ &= 2x \left(\frac{1}{x^2 + 1} - \frac{1}{x^2 - 1} \right) = 2x \frac{(x^2 - 1) - (x^2 + 1)}{x^4 - 1} = \boxed{-\frac{4x}{x^4 - 1}}. \end{aligned}$$

$$27. \frac{d}{dx} e^{-x} \ln x = e^{-x} \frac{d}{dx} \ln x + \ln x \frac{d}{dx} e^{-x} = \frac{e^{-x}}{x} + e^{-x} \ln x \frac{d}{dx} (-x) = \boxed{e^{-x} \left(\frac{1}{x} - \ln x \right)}.$$

28.

$$\begin{aligned}
\frac{d}{dx} \ln(\sqrt{x+7} - \sqrt{x}) &= \frac{1}{\sqrt{x+7} - \sqrt{x}} \frac{d}{dx} [(x+7)^{1/2} - x^{1/2}] \\
&= \frac{1}{\sqrt{x+7} - \sqrt{x}} \cdot \left(\frac{1}{2} (x+7)^{-1/2} \frac{d}{dx} (x+7) - \frac{1}{2} x^{-1/2} \right) \\
&= \frac{1}{\sqrt{x+7} - \sqrt{x}} \cdot \left(\frac{1}{2\sqrt{x+7}} - \frac{1}{2\sqrt{x}} \right) \\
&= \frac{1}{\sqrt{x+7} - \sqrt{x}} \cdot \frac{\sqrt{x} - \sqrt{x+7}}{2\sqrt{x(x+7)}} = \boxed{-\frac{1}{2\sqrt{x(x+7)}}}.
\end{aligned}$$

29.

$$\begin{aligned}
\frac{d}{dx} \left[\frac{1}{12} \ln \left(\frac{x}{\sqrt{144-x^2}} \right) \right] &= \frac{1}{12} \frac{d}{dx} \ln x - \frac{1}{24} \frac{d}{dx} \ln(144-x^2) = \frac{1}{12x} - \frac{1}{24(144-x^2)} \frac{d}{dx} (144-x^2) \\
&= \frac{1}{12x} + \frac{x}{12(144-x^2)} = \frac{144-x^2+x^2}{12x(144-x^2)} = \boxed{\frac{12}{x(144-x^2)}}.
\end{aligned}$$

$$30. \quad \frac{d}{dx} \ln(\tan^2 x) = 2 \frac{d}{dx} \ln(\tan x) = \frac{2}{\tan x} \frac{d}{dx} \tan x = \boxed{\frac{2 \sec^2 x}{\tan x}}.$$

$$31. \quad \text{Let } f(x) = \frac{e^x(x^2+4)}{x-2}. \text{ Then}$$

$$\ln f(x) = \ln \left(\frac{e^x(x^2+4)}{x-2} \right) = \ln e^x + \ln(x^2+4) - \ln(x-2) = x + \ln(x^2+4) - \ln(x-2),$$

so that

$$\begin{aligned}
\frac{d}{dx} \ln f(x) &= \frac{d}{dx} [x + \ln(x^2+4) - \ln(x-2)] \\
\frac{1}{f(x)} \frac{df}{dx} &= 1 + \frac{1}{x^2+4} \frac{d}{dx} (x^2+4) - \frac{1}{x-2} \frac{d}{dx} (x-2) = 1 + \frac{2x}{x^2+4} - \frac{1}{x-2}.
\end{aligned}$$

Therefore,

$$\frac{d}{dx} \left(\frac{e^x(x^2+4)}{x-2} \right) = \boxed{\frac{e^x(x^2+4)}{x-2} \left(1 + \frac{2x}{x^2+4} - \frac{1}{x-2} \right)}.$$

32.

$$\begin{aligned}
\frac{d}{dx} [\sin^{-1}(x-1) + \sqrt{2x-x^2}] &= \frac{1}{\sqrt{1-(x-1)^2}} \frac{d}{dx} (x-1) + \frac{1}{2} (2x-x^2)^{-1/2} \frac{d}{dx} (2x-x^2) \\
&= \frac{1}{\sqrt{1-x^2+2x-1}} + \frac{1-x}{\sqrt{2x-x^2}} = \boxed{\frac{2-x}{\sqrt{2x-x^2}}}.
\end{aligned}$$

33.

$$\begin{aligned}
\frac{d}{dx} [2\sqrt{x} - 2 \tan^{-1} \sqrt{x}] &= 2 \cdot \frac{1}{2} x^{-1/2} - 2 \cdot \frac{1}{1+(\sqrt{x})^2} \frac{d}{dx} x^{1/2} = \frac{1}{\sqrt{x}} - \frac{2}{1+x} \cdot \frac{1}{2} x^{-1/2} \\
&= \frac{1}{\sqrt{x}} - \frac{1}{(1+x)\sqrt{x}} = \frac{(1+x)-1}{(1+x)\sqrt{x}} = \frac{x}{(1+x)\sqrt{x}} = \boxed{\frac{\sqrt{x}}{1+x}}.
\end{aligned}$$

$$34. \frac{d}{dx} \left(4 \tan^{-1} \frac{x}{2} + x \right) = 4 \cdot \frac{1}{1 + (x/2)^2} \frac{d}{dx} \left(\frac{x}{2} \right) + 1 = \frac{16}{4 + x^2} \cdot \frac{1}{2} + 1 = \frac{8}{4 + x^2} + 1 = \boxed{\frac{12 + x^2}{4 + x^2}}.$$

$$35. \frac{d}{dx} \sin^{-1}(2x - 1) = \frac{1}{\sqrt{1 - (2x - 1)^2}} \frac{d}{dx}(2x - 1) = \frac{2}{\sqrt{1 - 4x^2 + 4x - 1}} = \boxed{\frac{1}{\sqrt{x - x^2}}}.$$

36.

$$\begin{aligned} \frac{d}{dx} \left(x^2 \tan^{-1} \frac{1}{x} \right) &= x^2 \frac{d}{dx} \tan^{-1} \frac{1}{x} + \tan^{-1} \frac{1}{x} \frac{d}{dx} x^2 = x^2 \cdot \frac{1}{1 + (1/x)^2} \frac{d}{dx} \left(\frac{1}{x} \right) + 2x \tan^{-1} \frac{1}{x} \\ &= \frac{x^4}{1 + x^2} \cdot \left(-\frac{1}{x^2} \right) + 2x \tan^{-1} \frac{1}{x} = \boxed{-\frac{x^2}{1 + x^2} + 2x \tan^{-1} \frac{1}{x}}. \end{aligned}$$

37.

$$\begin{aligned} \frac{d}{dx} \left[x \tan^{-1} x - \ln \sqrt{1 + x^2} \right] &= x \frac{d}{dx} \tan^{-1} x + \tan^{-1} x \frac{d}{dx} x - \frac{1}{2} \frac{d}{dx} \ln(1 + x^2) \\ &= \frac{x}{1 + x^2} + \tan^{-1} x - \frac{1}{2(1 + x^2)} \frac{d}{dx} (1 + x^2) \\ &= \frac{x}{1 + x^2} + \tan^{-1} x - \frac{x}{1 + x^2} = \boxed{\tan^{-1} x}. \end{aligned}$$

38.

$$\begin{aligned} \frac{d}{dx} \left[\sqrt{1 - x^2} (\sin^{-1} x) \right] &= \sqrt{1 - x^2} \frac{d}{dx} \sin^{-1} x + \sin^{-1} x \frac{d}{dx} (\sqrt{1 - x^2})^{1/2} \\ &= \sqrt{1 - x^2} \cdot \frac{1}{\sqrt{1 - x^2}} + \sin^{-1} x \cdot \frac{1}{2} (1 - x^2)^{-1/2} \frac{d}{dx} (1 - x^2) \\ &= \boxed{1 - \frac{x \sin^{-1} x}{\sqrt{1 - x^2}}}. \end{aligned}$$

39.

$$\begin{aligned} \frac{d}{dx} \left(\tanh \frac{x}{2} + \frac{2x}{4 + x^2} \right) &= \operatorname{sech}^2 \frac{x}{2} \frac{d}{dx} \left(\frac{x}{2} \right) + \frac{(4 + x^2) \frac{d}{dx} (2x) - 2x \frac{d}{dx} (4 + x^2)}{(4 + x^2)^2} \\ &= \frac{1}{2} \operatorname{sech}^2 \frac{x}{2} + \frac{2(4 + x^2) - 2x(2x)}{(4 + x^2)^2} = \boxed{\frac{1}{2} \operatorname{sech}^2 \frac{x}{2} + \frac{8 - 2x^2}{(4 + x^2)^2}}. \end{aligned}$$

$$40. \frac{d}{dx} (x \sinh x) = x \frac{d}{dx} \sinh x + \sinh x \frac{d}{dx} x = \boxed{x \cosh x + \sinh x}.$$

$$41. \frac{d}{dx} \sqrt{\sinh x} = \frac{1}{2} (\sinh x)^{-1/2} \frac{d}{dx} \sinh x = \boxed{\frac{\cosh x}{2(\sinh x)^{1/2}}}.$$

$$42. \frac{d}{dx} \sinh^{-1} e^x = \frac{1}{\sqrt{1 + (e^x)^2}} \frac{d}{dx} e^x = \boxed{\frac{e^x}{\sqrt{1 + e^{2x}}}}.$$

43. Let $x = y^5 + y$. Then

$$\begin{aligned} \frac{d}{dx} x &= \frac{d}{dx} (y^5 + y) \\ 1 &= 5y^4 \frac{dy}{dx} + \frac{dy}{dx} \\ \frac{dy}{dx} &= \boxed{\frac{1}{5y^4 + 1}}. \end{aligned}$$

44. Let $x = \cos^5 y + \cos y$. Then

$$\begin{aligned}\frac{d}{dx}x &= \frac{d}{dx}(\cos^5 y + \cos y) \\ 1 &= 5\cos^4 y \frac{d}{dx} \cos y - \sin y \frac{dy}{dx} \\ 1 &= -5\cos^4 y \sin y \frac{dy}{dx} - \sin y \frac{dy}{dx} \\ \frac{dy}{dx} &= \boxed{-\frac{1}{\sin y(5\cos^4 y + 1)}}.\end{aligned}$$

45. Let $\ln x + \ln y = x \cos y$. Then

$$\begin{aligned}\frac{d}{dx}(\ln x + \ln y) &= \frac{d}{dx}(x \cos y) \\ \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} &= x \frac{d}{dx} \cos y + \cos y \\ \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} &= -x \sin y \frac{dy}{dx} + \cos y \\ \left(\frac{1}{y} + x \sin y\right) \frac{dy}{dx} &= \cos y - \frac{1}{x} \\ \frac{dy}{dx} &= \frac{\cos y - \frac{1}{x}}{\frac{1}{y} + x \sin y} = \boxed{\frac{y(x \cos y - 1)}{x(1 + xy \sin y)}}.\end{aligned}$$

46. Let $\tan(xy) = x$. Then

$$\begin{aligned}\frac{d}{dx} \tan(xy) &= \frac{d}{dx} x \\ \sec^2(xy) \cdot \left(x \frac{dy}{dx} + y\right) &= 1 \\ x \sec^2(xy) \frac{dy}{dx} &= 1 - y \sec^2(xy) \\ \frac{dy}{dx} &= \boxed{\frac{1 - y \sec^2(xy)}{x \sec^2(xy)}}.\end{aligned}$$

47. Let $y = x + \sin(xy)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x + \sin(xy)) \\ \frac{dy}{dx} &= 1 + \cos(xy) \cdot \left(x \frac{dy}{dx} + y\right) \\ (1 - x \cos(xy)) \frac{dy}{dx} &= 1 + y \cos(xy) \\ \frac{dy}{dx} &= \boxed{\frac{1 + y \cos(xy)}{1 - x \cos(xy)}}.\end{aligned}$$

48. Let $x = \ln(\csc y + \cot y)$. Then

$$\begin{aligned}\frac{d}{dx}x &= \frac{d}{dx}\ln(\csc y + \cot y) \\ 1 &= \frac{1}{\csc y + \cot y} \frac{d}{dx}(\csc y + \cot y) \\ 1 &= \frac{1}{\csc y + \cot y} \cdot \left(-\csc y \cot y \frac{dy}{dx} - \csc^2 y \frac{dy}{dx} \right) \\ \frac{dy}{dx} &= -\frac{\csc y + \cot y}{\csc y \cot y + \csc^2 y} = -\frac{\csc y + \cot y}{\csc y(\cot y + \csc y)} = -\frac{1}{\csc y} = \boxed{-\sin y}.\end{aligned}$$

49. Let $xy + 3y^2 = 10x$. Then

$$\begin{aligned}\frac{d}{dx}(xy + 3y^2) &= \frac{d}{dx}(10x) \\ x \frac{dy}{dx} + y + 6y \frac{dy}{dx} &= 10 \\ \frac{dy}{dx} &= \boxed{\frac{10 - y}{x + 6y}},\end{aligned}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(x + 6y) \cdot -\frac{dy}{dx} - (10 - y) \cdot \left(1 + 6\frac{dy}{dx}\right)}{(x + 6y)^2} \\ &= \frac{-(10 - y) - (10 - y) - 6(10 - y)\frac{10 - y}{x + 6y}}{(x + 6y)^2} \\ &= \frac{-2(10 - y)(x + 6y) - 6(10 - y)^2}{(x + 6y)^3} \\ &= -\frac{2(10 - y)(x + 6y + 3(10 - y))}{(x + 6y)^3} = \boxed{\frac{2(y - 10)(3y + x + 30)}{(x + 6y)^3}}.\end{aligned}$$

50. Let $y^3 + y = x^2$. Then

$$\begin{aligned}\frac{d}{dx}(y^3 + y) &= \frac{d}{dx}x^2 \\ 3y^2 \frac{dy}{dx} + \frac{dy}{dx} &= 2x \\ \frac{dy}{dx} &= \boxed{\frac{2x}{3y^2 + 1}},\end{aligned}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(3y^2 + 1) \cdot 2 - 2x \cdot 6y \frac{dy}{dx}}{(3y^2 + 1)^2} = \frac{2(3y^2 + 1) - \frac{24x^2y}{3y^2 + 1}}{(3y^2 + 1)^2} \\ &= \frac{2(3y^2 + 1)^2 - 24(y^3 + y)y}{(3y^2 + 1)^3} = \frac{18y^4 + 12y^2 + 2 - 24y^4 - 24y^2}{(3y^2 + 1)^3} \\ &= \boxed{\frac{2(1 - 6y^2 - 3y^4)}{(3y^2 + 1)^3}},\end{aligned}$$

where, in going from the end of the first line to the beginning of the second line, we have substituted $y^3 + y$ for x^2 based on the original equation.

51. Let $xe^y = 4x^2$. Then

$$\begin{aligned}\frac{d}{dx}(xe^y) &= \frac{d}{dx}(4x^2) \\ xe^y \frac{dy}{dx} + e^y &= 8x \\ \frac{dy}{dx} &= \boxed{\frac{8x - e^y}{xe^y}},\end{aligned}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{xe^y \left(8 - e^y \frac{dy}{dx}\right) - (8x - e^y) \left(xe^y \frac{dy}{dx} + e^y\right)}{x^2 e^{2y}} \\ &= \frac{8xe^y - xe^{2y} \frac{dy}{dx} - 8x^2 e^y \frac{dy}{dx} - 8xe^y + xe^{2y} \frac{dy}{dx} + e^{2y}}{x^2 e^{2y}} \\ &= \frac{-8x^2 e^y \frac{dy}{dx} + e^{2y}}{x^2 e^{2y}} = \frac{-8x^2 e^y \left(\frac{8x - e^y}{xe^y}\right) + e^{2y}}{x^2 e^{2y}} \\ &= \boxed{\frac{-64x^2 + 8xe^y + e^{2y}}{x^2 e^{2y}}}.\end{aligned}$$

For $x \neq 0$, the equation $xe^y = 4x^2$ reduces to $e^y = 4x$. Thus, for $x \neq 0$, the derivatives can be simplified to

$$\frac{dy}{dx} = \frac{8x - 4x}{x(4x)} = \frac{1}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{-64x^2 + 8x(4x) + (4x)^2}{x^2(4x)^2} = -\frac{1}{x^2}.$$

52. Let $\ln(x+y) = 8x$. Then

$$\begin{aligned}\frac{d}{dx} \ln(x+y) &= \frac{d}{dx}(8x) \\ \frac{1}{x+y} \left(1 + \frac{dy}{dx}\right) &= 8 \\ \frac{dy}{dx} &= \boxed{8(x+y) - 1},\end{aligned}$$

and

$$\frac{d^2y}{dx^2} = 8 + 8 \frac{dy}{dx} = 8 + 8[8(x+y) - 1] = \boxed{64(x+y)}.$$

From the equation $\ln(x+y) = 8x$, it follows that $x+y = e^{8x}$, so the derivatives can be simplified to

$$\frac{dy}{dx} = 8e^{8x} - 1 \quad \text{and} \quad \frac{d^2y}{dx^2} = 64e^{8x}.$$

53. Let $g(x)$ be the inverse function of the function $f(x) = e^{2x}$. Because $f(0) = 1$ and $f'(x) = 2e^{2x}$, it follows that $g(1) = 0$ and

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \boxed{\frac{1}{2}}.$$

54. Let $g(x)$ be the inverse function of the function $f(x) = \sin x$ restricted to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{and} \quad f'(x) = \cos x,$$

it follows that $g\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and

$$g'\left(\frac{1}{2}\right) = \frac{1}{f'\left(g\left(\frac{1}{2}\right)\right)} = \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \boxed{\frac{2\sqrt{3}}{3}}.$$

55. First write

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{5n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n/2}\right)^{(2/5)(5n/2)} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n/2}\right)^{5n/2} \right]^{2/5}.$$

Now, let $k = 5n/2$. As $n \rightarrow \infty$, $k \rightarrow \infty$ as well, so

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{5n}\right)^n = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n/2}\right)^{5n/2} \right]^{2/5} = \left[\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k \right]^{2/5} = \boxed{e^{2/5}}.$$

56. Let $k = 3h$. Then $\frac{2}{h} = \frac{6}{k}$ and $k \rightarrow 0$ as $h \rightarrow 0$. Therefore,

$$\lim_{h \rightarrow 0} (1 + 3h)^{2/h} = \lim_{k \rightarrow 0} (1 + k)^{6/k} = \left[\lim_{k \rightarrow 0} (1 + k)^{1/k} \right]^6 = \boxed{e^6}.$$

57. $\sinh 0 = \frac{e^0 - e^0}{2} = \boxed{0}.$

58. $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \boxed{\frac{5}{3}}.$

59. $\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x.$

60. $\tanh(x + y) = \frac{\sinh(x + y)}{\cosh(x + y)} = \frac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y + \sinh x \sinh y} \cdot \frac{\frac{1}{\cosh x \cosh y}}{\frac{1}{\cosh x \cosh y}} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$

61. Let $f(x) = \sqrt{1 - \sin^2 x}$, and note that the domain of f is the set of all real numbers because $\sin^2 x \leq 1$ for all real numbers x . Now,

$$f'(x) = \frac{1}{2}(1 - \sin^2 x)^{-1/2} \frac{d}{dx}(1 - \sin^2 x) = \frac{1}{2\sqrt{1 - \sin^2 x}} \cdot -2 \sin x \frac{d}{dx} \sin x = -\frac{\sin x \cos x}{\sqrt{1 - \sin^2 x}}.$$

This function is not defined whenever $\sin^2 x = 1$, or $\sin x = \pm 1$. The domain of f' is therefore the set

$$\boxed{\{x | x \neq \frac{\pi}{2} + n\pi\}, \text{ where } n \text{ is any integer}}.$$

62. Let $f(x) = x^{1/2}(x - 2)^{3/2}$ for all $x \geq 2$. Then

$$f'(x) = x^{1/2} \frac{d}{dx}(x - 2)^{3/2} + (x - 2)^{3/2} \frac{d}{dx} x^{1/2} = \frac{3}{2} x^{1/2} (x - 2)^{1/2} + \frac{1}{2} (x - 2)^{3/2} x^{-1/2}.$$

Thus, the domain of f' is the set $\boxed{\{x | x \geq 2\}}.$

63. Let $f(x) = \sqrt{1 + 6x}$.

(a) $1 + 6x \geq 0$ for $x \geq -\frac{1}{6}$, so the domain of f is the set $\boxed{\left\{x | x \geq -\frac{1}{6}\right\}}$. The range of f is the set

$$\boxed{\{y | y \geq 0\}}.$$

(b) With $f(x) = \sqrt{1 + 6x}$,

$$f'(x) = \frac{1}{2}(1 + 6x)^{-1/2} \frac{d}{dx}(1 + 6x) = \frac{3}{\sqrt{1 + 6x}},$$

so the slope of the line tangent to the graph of f at $x = 4$ is

$$m_{\text{tan}} = f'(4) = \frac{3}{\sqrt{1 + 6(4)}} = \boxed{\frac{3}{5}}.$$

- (c) Using the result of part (b) and the fact that $f(4) = \sqrt{1+6(4)} = 5$, the equation of the line tangent to the graph of f at $x = 4$ is

$$y - 5 = \frac{3}{5}(x - 4) \quad \text{or} \quad y = \frac{3}{5}x + \frac{13}{5}.$$

The y -intercept of this line is $\boxed{y = \frac{13}{5}}$.

- (d) The line $y = x + 12$ has a slope of 1. From part (b),

$$f'(x) = \frac{3}{\sqrt{1+6x}},$$

so the tangent line to the graph of f will have slope 1 when

$$\sqrt{1+6x} = 3 \quad \text{or when} \quad x = \frac{4}{3}.$$

Because $f\left(\frac{4}{3}\right) = \sqrt{1+6\left(\frac{4}{3}\right)} = 3$, the coordinates of the point on the graph of f where the tangent line is parallel to the line $y = x + 12$ are $\boxed{\left(\frac{4}{3}, 3\right)}$.

64. Let $y = x\sqrt{x + (x-1)^2} = x\sqrt{x^2 - x + 1}$. Then

$$\begin{aligned} y' &= x \frac{d}{dx}(x^2 - x + 1)^{1/2} + \sqrt{x^2 - x + 1} \frac{d}{dx}x \\ &= x \cdot \frac{1}{2}(x^2 - x + 1)^{-1/2} \frac{d}{dx}(x^2 - x + 1) + \sqrt{x^2 - x + 1} = \frac{x(2x-1)}{2\sqrt{x^2 - x + 1}} + \sqrt{x^2 - x + 1}. \end{aligned}$$

Thus, the slope of the tangent line to the graph of $y = x\sqrt{x + (x-1)^2}$ at $x = 2$ is

$$m_{\text{tan}} = y'(2) = \frac{2(3)}{2\sqrt{4-2+1}} + \sqrt{4-2+1} = \frac{3}{\sqrt{3}} + \sqrt{3} = 2\sqrt{3},$$

so that the equation of the tangent line is

$$y - 2\sqrt{3} = 2\sqrt{3}(x - 2) \quad \text{or} \quad \boxed{y = 2\sqrt{3}x - 2\sqrt{3}}.$$

Moreover, the slope of the normal line to the graph of $y = x\sqrt{x + (x-1)^2}$ at $x = 2$ is $-\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{6}$, so the equation of the normal line is

$$y - 2\sqrt{3} = -\frac{\sqrt{3}}{6}(x - 2) \quad \text{or} \quad \boxed{y = -\frac{\sqrt{3}}{6}x + \frac{7\sqrt{3}}{3}}.$$

65. Let $x^3 + 2y^2 = x^2y$. Then

$$\begin{aligned} \frac{d}{dx}(x^3 + 2y^2) &= \frac{d}{dx}(x^2y) \\ 3x^2 + 4y \frac{dy}{dx} &= x^2 \frac{dy}{dx} + 2xy \\ (4y - x^2) \frac{dy}{dx} &= 2xy - 3x^2 \\ \frac{dy}{dx} &= \frac{2xy - 3x^2}{4y - x^2}. \end{aligned}$$

Therefore,

$$dy = \boxed{\frac{2xy - 3x^2}{4y - x^2} dx}.$$

66. If there can be up to a 2% error in the measurement of p , then

$$\frac{|\Delta p|}{p} = \frac{|dp|}{p} = 0.02.$$

With $g = (4\pi^2 L)p^{-2}$,

$$\Delta g \approx dg = -2(4\pi^2 L)p^{-3} dp,$$

so

$$\frac{|\Delta g|}{g} \approx \frac{|dg|}{g} = \frac{2(4\pi^2 L)p^{-3} |dp|}{(4\pi^2 L)p^{-2}} = 2 \frac{|dp|}{p} = 0.04.$$

Thus, the approximate percentage error in the computation of g is $\boxed{4\%}$.

67. Let $y = f(x) = x + \ln x$ and $x_0 = 1$. Then $f(1) = 1 + \ln 1 = 1$, $f'(x) = 1 + \frac{1}{x}$, $f'(1) = 2$, and

$$L(x) = f(1) + f'(1)(x - 1) = 1 + 2(x - 1) = \boxed{2x - 1}.$$

68. The volume V of a cube with edge length s is $V = s^3$. Suppose the percentage error in measuring the edge length is 5%; that is,

$$\frac{|\Delta s|}{s} = \frac{|ds|}{s} = 0.05.$$

Then

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{3s^2 |ds|}{s^3} = 3 \frac{|ds|}{s} = 0.15;$$

that is, the percentage error in the calculated volume is approximately $\boxed{15\%}$.

69. Let $f(x) = \tan x$.

(a) When $x = 0$,

$$dy = f'(0) dx = \sec^2 0 dx = \boxed{dx}$$

and

$$\Delta y = f(0 + \Delta x) - f(0) = \tan \Delta x - \tan 0 = \boxed{\tan(\Delta x)}.$$

(b) i. With $x = 0$ and $\Delta x = dx = 0.5$,

$$dy = dx = 0.5$$

and

$$\Delta y = \tan 0.5 \approx 0.546302.$$

Thus, $\boxed{|\Delta y - dy| \approx 0.046302}$.

ii. With $x = 0$ and $\Delta x = dx = 0.1$,

$$dy = dx = 0.1$$

and

$$\Delta y = \tan 0.1 \approx 0.100335.$$

Thus, $\boxed{|\Delta y - dy| \approx 0.000335}$.

iii. With $x = 0$ and $\Delta x = dx = 0.01$,

$$dy = dx = 0.01$$

and

$$\Delta y = \tan 0.01 \approx 0.010000333.$$

Thus, $\boxed{|\Delta y - dy| \approx 3.33 \times 10^{-7}}.$

70. Let $f(x) = \ln x$.

(a) When $x = 1$,

$$dy = f'(1) dx = \frac{1}{1} dx = \boxed{dx}$$

and

$$\Delta y = f(1 + \Delta x) - f(1) = \ln(1 + \Delta x) - \ln 1 = \boxed{\ln(1 + \Delta x)}.$$

(b) i. With $x = 1$ and $\Delta x = dx = 0.5$,

$$dy = dx = 0.5$$

and

$$\Delta y = \ln(1 + 0.5) = \ln 1.5 \approx 0.405465.$$

Thus, $\boxed{|\Delta y - dy| \approx 0.094535}.$

ii. With $x = 1$ and $\Delta x = dx = 0.1$,

$$dy = dx = 0.1$$

and

$$\Delta y = \ln(1 + 0.1) = \ln 1.1 \approx 0.095310.$$

Thus, $\boxed{|\Delta y - dy| \approx 0.004690}.$

iii. With $x = 1$ and $\Delta x = dx = 0.01$,

$$dy = dx = 0.01$$

and

$$\Delta y = \ln(1 + 0.01) = \ln 1.01 \approx 0.009950.$$

Thus, $\boxed{|\Delta y - dy| \approx 0.000050}.$

71. Let $f(x) = (x^2 + 1)^{(2-3x)}$. Then

$$\ln f(x) = \ln(x^2 + 1)^{(2-3x)} = (2 - 3x) \ln(x^2 + 1)$$

and

$$\begin{aligned} \frac{d}{dx} \ln f(x) &= \frac{d}{dx} [(2 - 3x) \ln(x^2 + 1)] \\ \frac{1}{f(x)} \frac{df}{dx} &= (2 - 3x) \frac{d}{dx} \ln(x^2 + 1) + \ln(x^2 + 1) \frac{d}{dx} (2 - 3x) \\ &= (2 - 3x) \cdot \frac{1}{x^2 + 1} \frac{d}{dx} (x^2 + 1) - 3 \ln(x^2 + 1) = \frac{2x(2 - 3x)}{x^2 + 1} - 3 \ln(x^2 + 1). \end{aligned}$$

Thus,

$$f'(x) = \frac{df}{dx} = (x^2 + 1)^{(2-3x)} \left(\frac{2x(2 - 3x)}{x^2 + 1} - 3 \ln(x^2 + 1) \right),$$

and

$$f'(1) = 2^{-1} \left(\frac{2(-1)}{2} - 3 \ln 2 \right) = \boxed{-\frac{1}{2}(1 + 3 \ln 2) = -\frac{1}{2}(1 + \ln 8)}.$$

72. Note that

$$\lim_{x \rightarrow 2} \frac{\ln x - \ln 2}{x - 2} = f'(2),$$

where $f(x) = \ln x$. Now,

$$f'(x) = \frac{1}{x},$$

so

$$\lim_{x \rightarrow 2} \frac{\ln x - \ln 2}{x - 2} = f'(2) = \boxed{\frac{1}{2}}.$$

73. Let $x \sin y + y \cos x = 0$. Then

$$\begin{aligned} \frac{d}{dx}(x \sin y + y \cos x) &= \frac{d}{dx} 0 \\ x \frac{d}{dx} \sin y + \sin y + y \frac{d}{dx} \cos x + \cos x \frac{dy}{dx} &= 0 \\ x \cos y \frac{dy}{dx} + \sin y - y \sin x + \cos x \frac{dy}{dx} &= 0 \\ (x \cos y + \cos x) \frac{dy}{dx} &= y \sin x - \sin y \\ y' = \frac{dy}{dx} &= \frac{y \sin x - \sin y}{x \cos y + \cos x}. \end{aligned}$$

Thus, when $x = \frac{\pi}{2}$ and $y = \pi$,

$$y' = \frac{\pi \sin \frac{\pi}{2} - \sin \pi}{\frac{\pi}{2} \cos \pi + \cos \frac{\pi}{2}} = \frac{\pi - 0}{-\frac{\pi}{2} - 0} = \boxed{-2}.$$

74. Let $f(x) = e^{2x}$, $x_0 = 3$, and $n = 4$. Then

$$\begin{aligned} f(x) &= e^{2x}, & f(3) &= e^6, \\ f'(x) &= 2e^{2x}, & f'(3) &= 2e^6, \\ f''(x) &= 4e^{2x}, & f''(3) &= 4e^6, \\ f'''(x) &= 8e^{2x}, & f'''(3) &= 8e^6, \\ f^{(4)}(x) &= 16e^{2x}, & f^{(4)}(3) &= 16e^6, \end{aligned}$$

and

$$\begin{aligned} P_4(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \frac{f^{(4)}(3)}{4!}(x-3)^4 \\ &= \boxed{e^6 + 2e^6(x-3) + 2e^6(x-3)^2 + \frac{4e^6}{3}(x-3)^3 + \frac{2e^6}{3}(x-3)^4}. \end{aligned}$$

75. Let $f(x) = \tan x$, $x_0 = 0$, and $n = 4$. Then

$$\begin{aligned} f'(x) &= \sec^2 x, \\ f''(x) &= 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x, \\ f'''(x) &= 2 \sec^2 x \cdot \sec^2 x + \tan x \cdot 4 \sec x \cdot \sec x \tan x = 2 \sec^4 x + 4 \sec^2 x \tan^2 x \\ &= 2 \sec^4 x + 4 \sec^2 x (\sec^2 x - 1) = 6 \sec^4 x - 4 \sec^2 x, \text{ and} \\ f^{(4)}(x) &= 24 \sec^3 x \cdot \sec x \tan x - 8 \sec x \cdot \sec x \tan x = 24 \sec^4 x \tan x - 8 \sec^2 x \tan x, \end{aligned}$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 2, \quad f^{(4)}(0) = 0,$$

and

$$\begin{aligned} P_4(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 \\ &= \boxed{x + \frac{1}{3}x^3}. \end{aligned}$$

76. Let $f(x) = \frac{1}{1+x} = (1+x)^{-1}$, $x_0 = 1$, and $n = 4$. Then

$$\begin{aligned} f(x) &= (1+x)^{-1}, & f(1) &= \frac{1}{2}, \\ f'(x) &= -(1+x)^{-2}, & f'(1) &= -\frac{1}{4}, \\ f''(x) &= 2(1+x)^{-3}, & f''(1) &= \frac{1}{4}, \\ f'''(x) &= -6(1+x)^{-4}, & f'''(1) &= -\frac{3}{8}, \\ f^{(4)}(x) &= 24(1+x)^{-5}, & f^{(4)}(1) &= \frac{3}{4}, \end{aligned}$$

and

$$\begin{aligned} P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= \boxed{\frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4}. \end{aligned}$$

77. Let $f(x) = \ln x$, $x_0 = 2$, and $n = 6$. Then

$$\begin{aligned} f(x) &= \ln x, & f(2) &= \ln 2, \\ f'(x) &= \frac{1}{x} = x^{-1}, & f'(2) &= \frac{1}{2}, \\ f''(x) &= -x^{-2}, & f''(2) &= -\frac{1}{4}, \\ f'''(x) &= 2x^{-3}, & f'''(2) &= \frac{1}{4}, \\ f^{(4)}(x) &= -6x^{-4}, & f^{(4)}(2) &= -\frac{3}{8}, \\ f^{(5)}(x) &= 24x^{-5}, & f^{(5)}(2) &= \frac{3}{4}, \\ f^{(6)}(x) &= -120x^{-6}, & f^{(6)}(2) &= -\frac{15}{8}, \end{aligned}$$

and

$$\begin{aligned} P_6(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \\ &\quad \frac{f^{(5)}(2)}{5!}(x-2)^5 + \frac{f^{(6)}(2)}{6!}(x-2)^6 \\ &= \boxed{\ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \frac{1}{160}(x-2)^5 - \frac{1}{384}(x-2)^6}. \end{aligned}$$

78. (a) Let $f(x) = 8x^4 - 2x^2 + 5x - 1$. The polynomial function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -1 < 0$ and $f(1) = 8 - 2 + 5 - 1 = 10 > 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.

(b) With $f(x) = 8x^4 - 2x^2 + 5x - 1$, $f'(x) = 32x^3 - 4x + 5$. Let $c_1 = 0.5$ be the first approximation for Newton's Method. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0.5 - \frac{1.5}{7} \approx 0.285714.$$

Next, $f(c_2) \approx 0.318616$ and $f'(c_2) \approx 4.603497$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.285714 - \frac{0.318616}{4.603497} \approx \boxed{0.216502}.$$

79. (a) Let $f(x) = 2 - x + \sin x$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $\left[\frac{\pi}{2}, \pi\right]$. Because $f\left(\frac{\pi}{2}\right) = 2 - \frac{\pi}{2} + 1 \approx 1.429 > 0$ and $f(\pi) = 2 - \pi + 0 \approx -1.142 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $\left(\frac{\pi}{2}, \pi\right)$.

- (b) With $f(x) = 2 - x + \sin x$, $f'(x) = -1 + \cos x$. Let $c_1 = \frac{\pi}{2}$ be the first approximation for Newton's Method. Then

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = \frac{\pi}{2} - \frac{3 - \frac{\pi}{2}}{-1} = 3.$$

Next, $f(c_2) \approx -0.858880$ and $f'(c_2) \approx -1.989992$, so

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} \approx 3 - \frac{-0.858880}{-1.989992} \approx \boxed{2.568400}.$$

80. (a) Let $f(x) = 2 \cos x - e^x$. The function f is continuous on the set of all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = 2 - 1 = 1 > 0$ and $f(1) = 2 \cos 1 - e \approx -1.638 < 0$, the Intermediate Value Theorem guarantees that f has a zero on the interval $(0, 1)$.

- (b) Begin by calculating $f'(x) = -2 \sin x - e^x$. Then, using a TI-84 Plus graphing calculator with $c_1 = 0.5$ yields

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \approx 0.5408210546; \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \approx 0.5397858311; \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \approx 0.5397851608; \text{ and} \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} \approx \boxed{0.5397851608}. \end{aligned}$$

81. Let $4xy - y^2 = 3$. Then

$$\begin{aligned} \frac{d}{dx}(4xy - y^2) &= \frac{d}{dx}3 \\ 4x \frac{dy}{dx} + 4y - 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2y}{2x - y}. \end{aligned}$$

The slope of the tangent line to the graph of $4xy - y^2 = 3$ at the point $(1, 3)$ is

$$m_{\text{tan}} = -\frac{2(3)}{2 - 3} = 6,$$

so the equation of the tangent line is

$$y - 3 = 6(x - 1) \quad \text{or} \quad \boxed{y = 6x - 3}.$$

Chapter 4: Applications of the Derivative

4.1: Related Rates

Concepts and Vocabulary

1. If a spherical balloon of volume V is inflated at a rate of $10 \text{ m}^3/\text{min}$, where t is the time (in minutes), then the rate of change of V with respect to t is

$$\frac{dV}{dt} = \boxed{10 \text{ m}^3/\text{min}}.$$

2. For the balloon in Problem 1, if the radius r is increasing at the rate of $0.5 \text{ m}/\text{min}$, then the rate of change of r with respect to t is

$$\frac{dr}{dt} = \boxed{0.5 \text{ m}/\text{min}}.$$

3. Let $x^2 + y^2 = 25$. Then, differentiating with respect to t yields

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}.$$

Given that $x = 3$, $y = 4$ and $\frac{dy}{dt} = 2$, it follows that

$$\frac{dx}{dt} = -\frac{4}{3} \cdot 2 = \boxed{-\frac{8}{3}}.$$

4. Let $x^3 y^2 = 432$. Then, differentiating with respect to t yields

$$2x^3 y \frac{dy}{dt} + 3x^2 y^2 \frac{dx}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = -\frac{2x}{3y} \frac{dy}{dt}.$$

Given that $x = 3$, $y = 4$ and $\frac{dy}{dt} = 2$, it follows that

$$\frac{dx}{dt} = -\frac{2(3)}{3(4)} \cdot 2 = \boxed{-1}.$$

Skill Building

5. Let $V = \frac{1}{12}\pi h^3$. Then, differentiating with respect to t yields

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}.$$

Given that $\frac{dh}{dt} = \frac{5\pi}{16}$ when $h = 8$, it follows that when $h = 8$,

$$\frac{dV}{dt} = \frac{1}{4}\pi(8)^2 \cdot \frac{5\pi}{16} = \boxed{5\pi^2}.$$

6. Let $s^2 = x^2 + y^2$. Then, differentiating with respect to t yields

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \quad \text{or} \quad \frac{ds}{dt} = \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right).$$

Given that $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 4$, $x = 150$, and $y = 80$ when $t = 20$, it follows that when $t = 20$

$$s^2 = 150^2 + 80^2 = 28,900 \quad \text{so that} \quad s = \pm\sqrt{28900} = \pm 170,$$

and

$$\frac{ds}{dt} = \pm \frac{1}{170} (150 \cdot 5 + 80 \cdot 4) = \boxed{\pm \frac{1070}{170} \approx \pm 6.294}.$$

7. Let $V = 80h^2$. Then, differentiating with respect to t yields

$$\frac{dV}{dt} = 160h \frac{dh}{dt}.$$

Given that $\frac{dh}{dt} = \frac{1}{12}$ when $h = 3$, it follows that when $h = 3$,

$$\frac{dV}{dt} = 160(3) \cdot \frac{1}{12} = \boxed{40}.$$

8. Let $y^2 = 625 - x^2$. Then, differentiating with respect to t yields

$$2y \frac{dy}{dt} = -2x \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 3$ when $x = 15$ and that $y \geq 0$, it follows that when $x = 15$

$$y^2 = 625 - 15^2 = 400 \quad \text{so that} \quad y = 20,$$

and

$$\frac{dy}{dt} = \frac{15}{20} \cdot 3 = \boxed{\frac{9}{4}}.$$

9. The volume V of a cube with edge length x is $V = x^3$. Differentiating with respect to time t yields

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 3$ cm/s, it follows that when $x = 10$ cm,

$$\frac{dV}{dt} = 3(10)^2 \cdot 3 = 900.$$

When the length of a side of the cube is 10 cm, the volume is increasing at a rate of $\boxed{900 \text{ cm}^3/\text{s}}$.

10. The volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Given that $\frac{dr}{dt} = 1$ cm/s, it follows that when $r = 6$ cm,

$$\frac{dV}{dt} = 4\pi(6)^2 \cdot 1 = 144\pi.$$

When the radius of the sphere is 6 cm, the volume is increasing at a rate of $\boxed{144\pi \text{ cm}^3/\text{s} \approx 452.389 \text{ cm}^3/\text{s}}$.

11. The surface area S of a sphere of radius r is $S = 4\pi r^2$. Differentiating with respect to time t yields

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad \text{so that} \quad \frac{dr}{dt} = \frac{1}{8\pi r} \frac{dS}{dt}.$$

Given that $\frac{dS}{dt} = -0.1 \text{ cm}^2/\text{h}$, it follows that when $r = \frac{20}{\pi} \text{ cm}$,

$$\frac{dr}{dt} = \frac{1}{8\pi(20/\pi)} \cdot -0.1 = -\frac{1}{1600} = -0.000625.$$

When the radius of the sphere is $\frac{20}{\pi} \text{ cm}$, the radius is decreasing at a rate of $\boxed{0.000625 \text{ cm/h}}$.

12. The surface area S of a sphere of radius r is $S = 4\pi r^2$. Differentiating with respect to time t yields

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Given that $\frac{dr}{dt} = 2 \text{ cm/min}$, it follows that when $r = 100 \text{ cm}$

$$\frac{dS}{dt} = 8\pi(100) \cdot 2 = 1600\pi.$$

When the radius of the sphere is 100 cm , the surface area is increasing at a rate of $\boxed{1600\pi \text{ cm}^2/\text{min} \approx 5026.548 \text{ cm}^2/\text{min}}$.

13. Let x and y denote the lengths of the legs of a right triangle with hypotenuse of length 45 cm . By the Pythagorean theorem, $x^2 + y^2 = 45^2$. Differentiating with respect to time t yields

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 2 \text{ cm/min}$, it follows that when $x = 4 \text{ cm}$,

$$y = \sqrt{45^2 - 4^2} = \sqrt{2009} \text{ cm},$$

and

$$\frac{dy}{dt} = -\frac{4}{\sqrt{2009}} \cdot 2 = -\frac{8}{\sqrt{2009}}.$$

When $x = 4 \text{ cm}$, y is decreasing at a rate of $\boxed{\frac{8}{\sqrt{2009}} \text{ cm/min} \approx 0.178 \text{ cm/min}}$.

14. Let x and θ be defined as indicated in the figure below. These quantities are related by the formula $x = \sin \theta$, and the area A of the triangle is given by

$$A = \frac{1}{2}(1)^2 \sin(2\theta) = \frac{1}{2} \sin(2\theta).$$

Differentiating both formulas with respect to time t yields

$$\frac{dx}{dt} = \cos \theta \frac{d\theta}{dt} \quad \text{and} \quad \frac{dA}{dt} = \cos(2\theta) \frac{d\theta}{dt}.$$

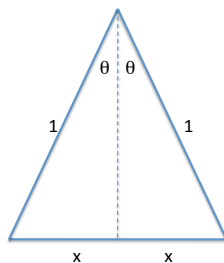
Solving the former equation for $\frac{d\theta}{dt}$ and substituting into the latter equation leads to

$$\frac{dA}{dt} = \cos(2\theta) \cdot \frac{1}{\cos \theta} \frac{dx}{dt} = \frac{\cos(2\theta)}{\cos \theta} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 1$ cm/min, it follows that when $\theta = 30^\circ$,

$$\frac{dA}{dt} = \frac{\cos 60^\circ}{\cos 30^\circ}(1) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

When $\theta = 30^\circ$, the area of the triangle is increasing at a rate of $\boxed{\frac{\sqrt{3}}{3} \text{ cm}^2/\text{min} \approx 0.577 \text{ cm}^2/\text{min}}.$



15. The area of an isosceles triangle with equal sides of length 4 cm and an included angle θ is $A = \frac{1}{2}4^2 \sin \theta = 8 \sin \theta$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = 8 \cos \theta \frac{d\theta}{dt}.$$

Given that $\frac{d\theta}{dt} = 2^\circ/\text{min} = \frac{\pi}{90}$ radians/min, it follows that when $\theta = 30^\circ$

$$\frac{dA}{dt} = 8 \cos 30^\circ \frac{\pi}{90} = \frac{2\sqrt{3}\pi}{45}.$$

When $\theta = 30^\circ$, the area of the triangle is increasing at a rate of $\boxed{\frac{2\sqrt{3}\pi}{45} \text{ cm}^2/\text{min} \approx 0.242 \text{ cm}^2/\text{min}}.$

16. Let s_1 and s_2 denote the lengths of the sides of the rectangle and suppose that s_1 is increasing at the rate of $2\sqrt{5}$ cm/s. Then

$$s_2 = \sqrt{15^2 - s_1^2} = \sqrt{225 - s_1^2},$$

and the area A of the rectangle is

$$A = s_1 s_2 = s_1 \sqrt{225 - s_1^2}.$$

Differentiating with respect to time t yields

$$\frac{dA}{dt} = s_1 \cdot \frac{1}{2}(225 - s_1^2)^{-1/2}(-2s_1) \frac{ds_1}{dt} + \sqrt{225 - s_1^2} \frac{ds_1}{dt} = \left(\sqrt{225 - s_1^2} - \frac{s_1^2}{\sqrt{225 - s_1^2}} \right) \frac{ds_1}{dt}.$$

Given that $\frac{ds_1}{dt} = 2\sqrt{5}$ cm/s, it follows that when $s_1 = 10$ cm,

$$\frac{dA}{dt} = \left(\sqrt{225 - 100} - \frac{100}{\sqrt{225 - 100}} \right) 2\sqrt{5} = \left(5\sqrt{5} - \frac{20}{\sqrt{5}} \right) 2\sqrt{5} = 50 - 40 = 10.$$

When $s_1 = 10$ cm, the area of the rectangle is increasing at a rate of $\boxed{10 \text{ cm}^2/\text{s}}.$

17. The volume V and surface area S of a sphere of radius r are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Differentiating both formulas with respect to time t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Solving the former equation for $\frac{dr}{dt}$ and substituting that expression into the latter formula gives

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt} \quad \text{so that} \quad \frac{dS}{dt} = \frac{8\pi r}{4\pi r^2} \frac{dV}{dt} = \frac{2}{r} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = -1.5 \text{ m}^3/\text{min}$, it follows that when $r = 4 \text{ m}$,

$$\frac{dS}{dt} = \frac{2}{4}(-1.5) = -0.75.$$

When the radius of the balloon is 4 m, the surface area is shrinking at a rate of $\boxed{0.75 \text{ m}^2/\text{min}}$.

18. The volume V and surface area S of a sphere of radius r are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Differentiating the volume formula with respect to time t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Given that the volume of ice decreases at a rate proportional to the surface area, it also follows that

$$\frac{dV}{dt} = kS = k(4\pi r^2),$$

where $k < 0$ is a constant of proportionality. Equating the two expressions for dV/dt yields

$$4\pi r^2 \frac{dr}{dt} = k(4\pi r^2) \quad \text{or} \quad \frac{dr}{dt} = k.$$

Therefore, the radius of the sphere decreases at a constant rate.

Applications and Extensions

19. Let x denote the distance from the lower end of the ladder to the wall against which the ladder is leaning. Then

$$\cos \theta = \frac{x}{5} \quad \text{and} \quad -\sin \theta \frac{d\theta}{dt} = \frac{1}{5} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 0.5 \text{ m/s}$, it follows that when $x = 4 \text{ m}$,

$$\cos \theta = \frac{4}{5} \quad \text{so that} \quad \sin \theta = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

and

$$-\frac{3}{5} \frac{d\theta}{dt} = \frac{1}{5}(0.5).$$

Therefore,

$$\frac{d\theta}{dt} = -\frac{1}{6}.$$

When the lower end of the ladder is 4 m from the wall, the inclination θ of the ladder is decreasing at a rate of $\boxed{\frac{1}{6} \text{ rad/s}}$.

20. Let x denote the distance from the man to the tower. Then

$$\tan \theta = \frac{23}{x} \quad \text{and} \quad \sec^2 \theta \frac{d\theta}{dt} = -\frac{23}{x^2} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = -1$ m/s, it follows that when $x = 10$ m,

$$\tan \theta = \frac{23}{10} = 2.3 \quad \text{so that} \quad \sec^2 \theta = 1 + \tan^2 \theta = 1 + (2.3)^2 = 6.29$$

and

$$6.29 \frac{d\theta}{dt} = -\frac{23}{100}(-1) = 0.23.$$

Therefore,

$$\frac{d\theta}{dt} = \frac{0.23}{6.29} \approx 0.037.$$

When the man is 10 m from the tower, the angle of elevation is increasing at a rate of approximately 0.037 rad/s.

21. Let h denote the depth of the water in the pool at the deep end, and let L denote the horizontal length of the surface of the water at depth h (see the diagram below). By similar triangles

$$\frac{L}{h} = \frac{30}{2} = 15 \quad \text{so that} \quad L = 15h$$

and the volume of water in the pool is

$$V = \left(\frac{1}{2} Lh \right) (5) = \frac{5}{2} (15h)h = \frac{75}{2} h^2.$$

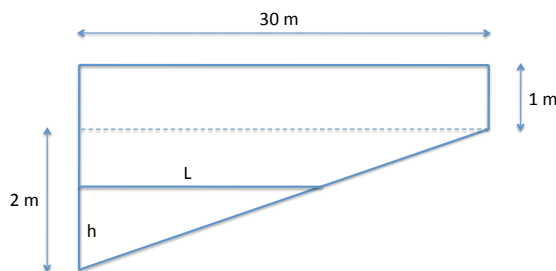
Differentiating with respect to time t yields

$$\frac{dV}{dt} = 75h \frac{dh}{dt} \quad \text{so} \quad \frac{dh}{dt} = \frac{1}{75h} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 15$ m³/min, it follows that when $h = 1$ m,

$$\frac{dh}{dt} = \frac{1}{75(1)} 15 = \frac{1}{5} = 0.2.$$

When the water level is 1 m deep at the deep end of the pool, the water level is rising at a rate of 0.2 m/min.



22. The volume V of water in a cylindrical tank with diameter 6 m (that is, the radius is 3 m) and depth h is $V = \pi(3)^2 h = 9\pi h$. Differentiating with respect to time t yields

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{1}{9\pi} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 5 \text{ m}^3/\text{min}$, it follows that

$$\frac{dh}{dt} = \frac{1}{9\pi}(5) = \frac{5}{9\pi}.$$

The depth of the water is rising at a rate of $\boxed{\frac{5}{9\pi} \text{ m/min} \approx 0.177 \text{ m/min}}.$

23. The volume V of water in the shape of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Consider the diagram on the right in the text. By similar triangles

$$\frac{r}{h} = \frac{4}{16} = \frac{1}{4} \quad \text{so that} \quad r = \frac{1}{4}h$$

and

$$V = \frac{1}{3}\pi \left(\frac{1}{4}h\right)^2 h = \frac{1}{48}\pi h^3.$$

Differentiating with respect to time t yields

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{16}{\pi h^2} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 16 \text{ m}^3/\text{min}$, it follows that when $h = 8 \text{ m}$,

$$\frac{dh}{dt} = \frac{16}{\pi(8)^2}(16) = \frac{4}{\pi}.$$

When the water is 8 m deep, the water level is rising at a rate of $\boxed{\frac{4}{\pi} \text{ m/min} \approx 1.273 \text{ m/min}}.$

24. The volume V of sand in a conical pile of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. If the height of the pile is always one-fourth the diameter, then the height is one-half the radius; in other words, the radius of the conical pile is always twice the height. Therefore,

$$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3.$$

Differentiating with respect to time t yields

$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{1}{4\pi h^2} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 20 \text{ cm}^3/\text{s}$, it follows that when $h = 3 \text{ cm}$,

$$\frac{dh}{dt} = \frac{1}{4\pi(3)^2}(20) = \frac{5}{9\pi}.$$

When the height of the sand pile is 3 cm, the height of the sand pile is increasing at a rate of

$$\boxed{\frac{5}{9\pi} \text{ cm/s} \approx 0.177 \text{ cm/s}}.$$

25. (a) The volume V of water in the shape of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. By similar triangles (see the diagram below),

$$\frac{r}{h} = \frac{1}{4} \quad \text{so that} \quad r = \frac{1}{4}h$$

and

$$V = \frac{1}{3}\pi \left(\frac{1}{4}h\right)^2 h = \frac{1}{48}\pi h^3.$$

Differentiating with respect to time t yields

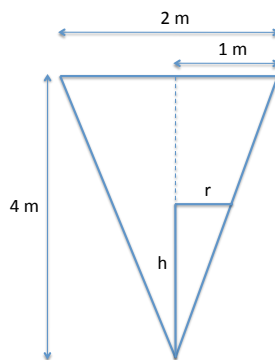
$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{16}{\pi h^2} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$, it follows that when $h = 3 \text{ m}$,

$$\frac{dh}{dt} = \frac{16}{\pi(3)^2}(3) = \frac{16}{3\pi}.$$

When the water in the tank is 3 m deep, the water level is rising at a rate of

$$\boxed{\frac{16}{3\pi} \text{ m/min} \approx 1.698 \text{ m/min}}.$$



- (b) Based on the observation that the depth of the water is increasing at only 0.5 m/min when the depth is 3 m, the rate at which the volume of water in the tank is increasing is

$$\frac{dV}{dt} = \frac{1}{16}\pi(3)^2(0.5) = \frac{9\pi}{32} \text{ m}^3/\text{min} \approx 0.884 \text{ m}^3/\text{min}.$$

Because water is entering the tank at a rate of 3 m³/min, the rate at which water is leaking from the tank is

$$3 - \frac{9\pi}{32} = \boxed{\frac{96 - 9\pi}{32} \text{ m}^3/\text{min} \approx 2.116 \text{ m}^3/\text{min}}.$$

26. As a function of x , the area A of the rectangle is $A = xe^x$. Differentiating with respect to time t then yields

$$\frac{dA}{dt} = xe^x \frac{dx}{dt} + e^x \frac{dx}{dt} = e^x(x+1) \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 1$ unit per second, it follows that when $x = 10$ units,

$$\frac{dA}{dt} = e^{10}(11)(1) = 11e^{10}.$$

When $x = 10$ units, the area of the rectangle is increasing at a rate of

$$\boxed{11e^{10} \text{ units}^2/\text{s} \approx 242,291.124 \text{ units}^2/\text{s}}.$$

27. Let (x, y) be a point on the graph of the parabola $y^2 = 4(3 - x) = 12 - 4x$. The distance D from the point (x, y) to the origin is

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 - 4x + 12}.$$

Differentiating both the equation of the parabola and the distance formula with respect to time t yields

$$2y \frac{dy}{dt} = -4 \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{2} \frac{dy}{dt}$$

and

$$\frac{dD}{dt} = \frac{1}{2}(x^2 - 4x + 12)^{-1/2}(2x - 4) \frac{dx}{dt} = \frac{x - 2}{\sqrt{x^2 - 4x + 12}} \frac{dx}{dt}.$$

Given that $\frac{dy}{dt} = 3$ units per second when the object is at the point $(-1, 4)$, it follows that when the object is at the point $(-1, 4)$,

$$\frac{dx}{dt} = -\frac{4}{2}(3) = -6 \text{ units per second,}$$

and

$$\frac{dD}{dt} = \frac{-1 - 2}{\sqrt{1 + 4 + 12}}(-6) = \frac{18}{\sqrt{17}}.$$

When the object is at the point $(-1, 4)$, the distance between the object and the origin is increasing at a rate of $\frac{18}{\sqrt{17}}$ units per second ≈ 4.366 units per second.

28. The volume V of liquid in the shape of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. By similar triangles (see the diagram below),

$$\frac{r}{h} = \frac{7.5}{15} = \frac{1}{2} \quad \text{so that} \quad r = \frac{1}{2}h$$

and

$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3.$$

Differentiating with respect to time t yields

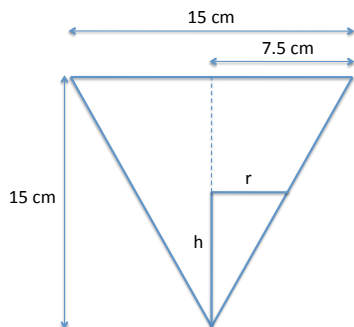
$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = -5 \text{ cm}^3/\text{min}$, it follows that when $h = 8 \text{ cm}$,

$$\frac{dh}{dt} = \frac{4}{\pi(8)^2}(-5) = -\frac{5}{16\pi}.$$

When the depth of the liquid is 8 cm, the depth of the liquid is decreasing at a rate of

$\frac{5}{16\pi} \text{ cm/min} \approx 0.099 \text{ cm/min}$.



29. The distance x between the ball and first base is

$$x = \sqrt{s^2 + 90^2} = \sqrt{s^2 + 8100},$$

where s is the distance between the ball and home plate along the third-base line. Differentiating with respect to time t yields

$$\frac{dx}{dt} = \frac{1}{2}(s^2 + 8100)^{-1/2}(2s)\frac{ds}{dt} = \frac{s}{\sqrt{s^2 + 8100}} \frac{ds}{dt}.$$

Given that $\frac{ds}{dt} = 100$ ft/s, it follows that when the ball crosses third base ($s = 90$ ft),

$$\frac{dx}{dt} = \frac{90}{\sqrt{90^2 + 8100}}(100) = \frac{100}{\sqrt{2}} = 50\sqrt{2}.$$

When the ball crosses third base, the distance from the ball to first base is increasing at a rate of $50\sqrt{2}$ ft/s ≈ 70.711 ft/s.

30. Let x denote the horizontal distance between the falcon and the falconer, and let y denote the height of the falcon. With a speed of 88 ft/s, the falcon flies the first 200 ft in

$$\frac{200}{88} = \frac{25}{11} \text{ seconds.}$$

During these first $25/11$ seconds of flight, the falcon is traveling at an angle of 60° , so

$$x = 88(\cos 60^\circ)t = 44t \quad \text{and} \quad y = 88(\sin 60^\circ)t = 44t\sqrt{3}.$$

When the falcon levels off, it has traveled

$$44 \cdot \frac{25}{11} = 100 \text{ feet}$$

horizontally, and

$$44 \cdot \frac{25}{11} \cdot \sqrt{3} = 100\sqrt{3} \text{ feet}$$

vertically. Thereafter, the falcon maintains a constant height, so that

$$x = \begin{cases} 44t, & 0 \leq t \leq \frac{25}{11} \\ 100 + 88t, & t > \frac{25}{11} \end{cases} \quad \text{and} \quad y = \begin{cases} 44t\sqrt{3}, & 0 \leq t \leq \frac{25}{11} \\ 100\sqrt{3}, & t > \frac{25}{11}. \end{cases}$$

Now, let $D = \sqrt{x^2 + y^2}$ be the distance between the falcon and the falconer. Then

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

At $t = 6$ seconds, $x = 100 + 88(6) = 628$ ft, $y = 100\sqrt{3}$ ft, $\frac{dx}{dt} = 88$ ft/s, and $\frac{dy}{dt} = 0$ ft/s, so

$$\frac{dD}{dt} = \frac{628(88) + 0}{\sqrt{628^2 + (100\sqrt{3})^2}} \approx 84.833.$$

After the falcon has been in flight for 6 seconds, the falcon is moving away from the falconer at a rate of approximately 84.833 ft/s.

31. Differentiating the equation $PV^{1.4} = k$ with respect to time t yields

$$P \cdot 1.4V^{0.4} \frac{dV}{dt} + V^{1.4} \frac{dP}{dt} = 0$$

so that

$$\frac{dP}{dt} = -\frac{1.4PV^{0.4}}{V^{1.4}} \frac{dV}{dt} = -\frac{1.4P}{V} \frac{dV}{dt}.$$

Given that $\frac{dV}{dt} = -2 \text{ cm}^3/\text{min}$ at the instant when $P = 20 \text{ kg/cm}^2$ and $V = 32 \text{ cm}^3$, it follows that

$$\frac{dP}{dt} = -\frac{1.4(20)}{32}(-2) = \frac{28}{16} = 1.75.$$

At the instant when $P = 20 \text{ kg/cm}^2$ and $V = 32 \text{ cm}^3$, the pressure is increasing at a rate of

$$\boxed{1.75 \frac{\text{kg/cm}^2}{\text{min}}}.$$

32. The area A of a circle of radius r is $A = \pi r^2$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Given that $\frac{dr}{dt} = 0.02 \text{ cm/s}$, it follows that when $r = 3 \text{ cm}$,

$$\frac{dA}{dt} = 2\pi(3)(0.02) = 0.12\pi.$$

When the radius of the plate is 3 cm, the area of the top surface of the plate is increasing at a rate of

$$\boxed{0.12\pi \text{ cm}^2/\text{s} \approx 0.377 \text{ cm}^2/\text{s}}.$$

33. The area A of a circle of radius r is $A = \pi r^2$. Differentiating with respect to time t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Given that $\frac{dr}{dt} = 0.42 \text{ ft/min}$, it follows that when $r = 120 \text{ ft}$,

$$\frac{dA}{dt} = 2\pi(120)(0.42) = 100.8\pi.$$

When the radius of the oil spill is 120 ft, the area of the spill is increasing at a rate of

$$\boxed{100.8\pi \text{ ft}^2/\text{min} \approx 316.673 \text{ ft}^2/\text{min}}.$$

34. Let L denote the length of the string released and x denote the horizontal distance from the kite to the girl. Then $L^2 = 30^2 + x^2 = 900 + x^2$. Differentiating with respect to time t yields

$$2L \frac{dL}{dt} = 2x \frac{dx}{dt} \quad \text{so that} \quad \frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt}.$$

Given that $\frac{dx}{dt} = 2 \text{ m/s}$, it follows that when $L = 70 \text{ m}$,

$$x = \sqrt{70^2 - 900} = \sqrt{4000} \text{ m},$$

and

$$\frac{dL}{dt} = \frac{\sqrt{4000}}{70}(2) = \frac{4\sqrt{10}}{7}.$$

When the length of the string released is 70 ft, the string is being let out at a rate of

$$\boxed{\frac{4\sqrt{10}}{7} \text{ m/s} \approx 1.807 \text{ m/s}}.$$

35. Let x denote the distance from the base of the ladder to the wall and y denote the distance up the wall from the top of the ladder to the ground. Then $x^2 + y^2 = 8^2 = 64$. Differentiating with respect to time t yields

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

- (a) Given that $\frac{dx}{dt} = 0.5$ m/s, it follows that when $x = 3$ m,

$$y = \sqrt{64 - 3^2} = \sqrt{55} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{3}{\sqrt{55}}(0.5) = -\frac{1.5}{\sqrt{55}}.$$

The top of the ladder is moving down the wall at a rate of $\frac{1.5}{\sqrt{55}} \text{ m/s} \approx 0.202 \text{ m/s}$.

- (b) Given that $\frac{dx}{dt} = 0.5$ m/s, it follows that when $x = 4$ m,

$$y = \sqrt{64 - 4^2} = \sqrt{48} = 4\sqrt{3} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{4}{4\sqrt{3}}(0.5) = -\frac{0.5}{\sqrt{3}}.$$

The top of the ladder is moving down the wall at a rate of $\frac{0.5}{\sqrt{3}} \text{ m/s} \approx 0.289 \text{ m/s}$.

- (c) Given that $\frac{dx}{dt} = 0.5$ m/s, it follows that when $x = 6$ m,

$$y = \sqrt{64 - 6^2} = \sqrt{28} = 2\sqrt{7} \text{ m},$$

and

$$\frac{dy}{dt} = -\frac{6}{2\sqrt{7}}(0.5) = -\frac{1.5}{\sqrt{7}}.$$

The top of the ladder is moving down the wall at a rate of $\frac{1.5}{\sqrt{7}} \text{ m/s} \approx 0.567 \text{ m/s}$.

36. Let x denote the distance from the point on the shore opposite the lighthouse to the point where the beam of light from the lighthouse touches the shoreline, and let θ denote the angle made by the beam and the perpendicular from the lighthouse to the shoreline (see the diagram below). Then

$$\frac{x}{2000} = \tan \theta \quad \text{so that} \quad \frac{1}{2000} \frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt}.$$

Given that

$$\frac{d\theta}{dt} = 2 \frac{\text{revolutions}}{\text{minute}} \cdot \frac{2\pi \text{ rad}}{\text{revolution}} = 4\pi \frac{\text{rad}}{\text{minute}},$$

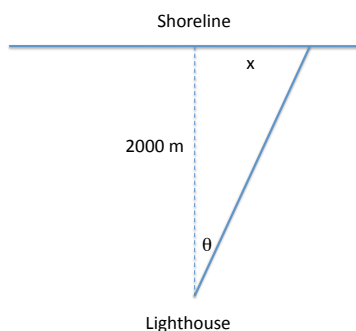
and $x = 500$ m, it follows that

$$\begin{aligned} \tan \theta &= \frac{500}{2000} = \frac{1}{4}, \\ \sec^2 \theta &= 1 + \tan^2 \theta = 1 + \left(\frac{1}{4}\right)^2 = \frac{17}{16}, \end{aligned}$$

and

$$\frac{dx}{dt} = 2000 \left(\frac{17}{16} \right) (4\pi) = 8500\pi.$$

When the beam passes a point 500 m from the point on the shore opposite the lighthouse, the beam is moving along the shore at a rate of $8500\pi \text{ m/min} \approx 26,703.538 \text{ m/min}$.



37. Let x denote the distance from the point on the shore opposite the ship to the point where the radar beam intersects the shoreline, and let θ denote the angle made by the beam and the perpendicular from the ship to the shoreline (see the diagram below). Then

$$\frac{x}{6} = \tan \theta \quad \text{so that} \quad \frac{1}{6} \frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt}.$$

Given that

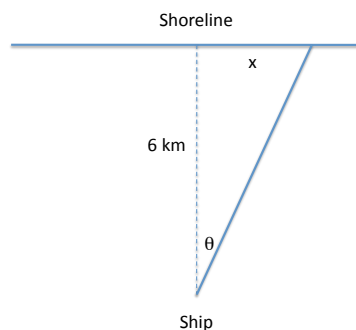
$$\frac{d\theta}{dt} = \frac{1 \text{ revolution}}{5 \text{ s}} \cdot \frac{2\pi \text{ rad}}{\text{revolution}} = \frac{2\pi}{5} \frac{\text{rad}}{\text{s}},$$

and $\theta = 45^\circ$, it follows that

$$\frac{dx}{dt} = 6 \sec^2 45^\circ \cdot \frac{2\pi}{5} = \frac{24\pi}{5}.$$

When the beam makes an angle of 45° with the shore, the beam is moving along the shore at a rate of

$$\frac{24\pi}{5} \text{ km/s} \approx 15.080 \text{ km/s}.$$



38. Let y denote the height of the rocket above the ground. Then

$$\tan \theta = \frac{y}{150} \quad \text{and} \quad \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{150} \frac{dy}{dt}.$$

- (a) Just after launch, $y = 0$, so $\tan \theta = 0$ and $\sec^2 \theta = 1 + \tan^2 \theta = 1$. With $\frac{dy}{dt} = 2.0 \text{ m/s}$, it follows that

$$\frac{d\theta}{dt} = \frac{1}{150 \sec^2 \theta} \frac{dy}{dt} = \frac{1}{150} (2.0) = \frac{1}{75}.$$

Just after launch, the angle of elevation of the tracking dish is increasing at a rate of

$$\boxed{\frac{1}{75} \text{ rad/s} \approx 0.0133 \text{ rad/s}}.$$

- (b) When the rocket is 100 m above the ground, $y = 100$, so

$$\tan \theta = \frac{100}{150} = \frac{2}{3} \quad \text{and} \quad \sec^2 \theta = 1 + \tan^2 \theta = \frac{13}{9}.$$

With $\frac{dy}{dt} = 2.0$ m/s, it follows that

$$\frac{d\theta}{dt} = \frac{1}{150 \sec^2 \theta} \frac{dy}{dt} = \frac{1}{150(13/9)} (2.0) = \frac{18}{1950}.$$

When the rocket is 100 m above the ground, the angle of elevation of the tracking dish is increasing

at a rate of $\boxed{\frac{18}{1950} \text{ rad/s} \approx 0.00923 \text{ rad/s}}.$

- (c) When the rocket is 1.0 km above the ground, $y = 1000$, so

$$\tan \theta = \frac{1000}{150} = \frac{20}{3} \quad \text{and} \quad \sec^2 \theta = 1 + \tan^2 \theta = \frac{409}{9}.$$

With $\frac{dy}{dt} = 2.0$ m/s, it follows that

$$\frac{d\theta}{dt} = \frac{1}{150 \sec^2 \theta} \frac{dy}{dt} = \frac{1}{150(409/9)} (2.0) = \frac{18}{61350}.$$

When the rocket is 1 km above the ground, the angle of elevation of the tracking dish is increasing

at a rate of $\boxed{\frac{18}{61350} \text{ rad/s} \approx 0.000293 \text{ rad/s}}.$

- (d) Based on the results from parts (a), (b), and (c), $\boxed{\text{the tracking rate decreases with increasing height of the rocket. In the limit as the rocket gets extremely high (that is, as } \theta \rightarrow \pi/2), \tan \theta \rightarrow \infty \text{ so } \sec^2 \theta \rightarrow \infty \text{ and } \frac{d\theta}{dt} \rightarrow 0}.$

39. Let x denote the horizontal distance from the boy to the street lamp, and let s denote the length of the boy's shadow (see the diagram below). By similar triangles

$$\frac{s}{1} = \frac{x+s}{6} \quad \text{so that} \quad 6s = x+s \quad \text{or} \quad s = \frac{1}{5}x.$$

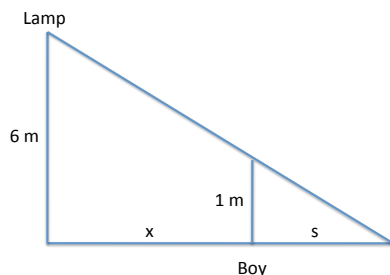
Therefore,

$$\frac{ds}{dt} = \frac{1}{5} \frac{dx}{dt},$$

and with $\frac{dx}{dt} = 20$ m/min,

$$\frac{ds}{dt} = \frac{1}{5}(20) = 4.$$

The child's shadow is lengthening at a rate of $\boxed{4 \text{ m/min}}.$



40. Let x denote the distance from the boy's feet to the base of the pole. Then, the distance D from the boy's feet to the top of the pole is given by

$$D = \sqrt{x^2 + 20^2} = \sqrt{x^2 + 400}.$$

Differentiating with respect to time t yields

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + 400)^{-1/2}(2x)\frac{dx}{dt} = \frac{x}{\sqrt{x^2 + 400}} \frac{dx}{dt}.$$

Given that

$$\frac{dx}{dt} = -4 \frac{\text{km}}{\text{h}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = -\frac{10}{9} \frac{\text{m}}{\text{s}},$$

where the derivative is negative because the boy is walking toward the pole, it follows that when $x = 5 \text{ m}$,

$$\frac{dD}{dt} = \frac{5}{\sqrt{5^2 + 400}} \left(-\frac{10}{9} \right) \approx -0.269.$$

When the boy is 5 m from the pole, the distance from his feet to the top of the pole is decreasing at a rate of approximately $\boxed{0.269 \text{ m/s}}$.

41. Place the center of the ferris wheel at the point $(0, 30)$, and suppose that the ferris wheel rotates counterclockwise. Then the x - and y -coordinates of a passenger on the ferris wheel can be modeled by

$$x = 25 \cos \theta \quad \text{and} \quad y = 30 + 25 \sin \theta.$$

When a passenger is 42.5 ft above the ground,

$$42.5 = 30 + 25 \sin \theta \quad \text{so that} \quad \sin \theta = \frac{1}{2}.$$

Given that the passenger is rising, $\theta = 30^\circ$. Moreover, with

$$\frac{d\theta}{dt} = \frac{1 \text{ revolution}}{2 \text{ min}} \cdot \frac{2\pi \text{ rad}}{\text{revolution}} = \pi \frac{\text{rad}}{\text{min}},$$

it follows that

$$\frac{dy}{dt} = 25 \cos \theta \frac{d\theta}{dt} = 25\pi \cos 30^\circ = \frac{25\pi\sqrt{3}}{2} \text{ ft/min} \approx 68.017 \text{ ft/min}.$$

Additionally,

$$\frac{dx}{dt} = -25 \sin \theta \frac{d\theta}{dt} = -25\pi \sin 30^\circ = -\frac{25\pi}{2} \text{ ft/min} \approx -39.270 \text{ ft/min}.$$

The passenger is rising at a rate of $\boxed{\frac{25\pi\sqrt{3}}{2} \text{ ft/min} \approx 68.017 \text{ ft/min}}$ and is moving horizontally backward at a rate of $\boxed{\frac{25\pi}{2} \text{ ft/min} \approx 39.270 \text{ ft/min}}$.

42. Let x denote the distance from the intersection of the car traveling east, and y denote the distance from the intersection of the car traveling south. The distance D between the cars is then given by $D = \sqrt{x^2 + y^2}$, and

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

Given that $\frac{dx}{dt} = -30 \text{ km/h}$, $\frac{dy}{dt} = -40 \text{ km/h}$, $x = 100 \text{ m} = 0.1 \text{ km}$, and $y = 75 \text{ m} = 0.075 \text{ km}$, it follows that

$$\frac{dD}{dt} = \frac{0.1 \cdot -30 + 0.075 \cdot -40}{\sqrt{0.1^2 + 0.075^2}} = -48 \text{ km/h}.$$

The cars are approaching one another at a rate of $\boxed{48 \text{ km/h}}$.

43. Let x denote the horizontal distance between the delivery truck and the elevator, and let h denote the elevation of the elevator. The distance D between the delivery truck and the elevator is then given by $D = \sqrt{x^2 + h^2}$, and

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + h^2)^{-1/2} \left(2x \frac{dx}{dt} + 2h \frac{dh}{dt} \right) = \frac{x \frac{dx}{dt} + h \frac{dh}{dt}}{\sqrt{x^2 + h^2}}.$$

One second after the elevator and delivery truck begin moving, $x = 8$ m and $h = 20$ m. Given that $\frac{dx}{dt} = 8$ m/s and $\frac{dh}{dt} = -5$ m/s, it follows that

$$\frac{dD}{dt} = \frac{8 \cdot 8 + 20 \cdot -5}{\sqrt{8^2 + 20^2}} = -\frac{9}{\sqrt{29}} = -\frac{9\sqrt{29}}{\sqrt{29}} \text{ m/s} \approx -1.671 \text{ m/s}.$$

The elevator and the delivery truck are separating at a rate of $-\frac{9\sqrt{29}}{\sqrt{29}}$ m/s ≈ -1.671 m/s; that is,

they are approaching one another at a rate of $\frac{9\sqrt{29}}{\sqrt{29}}$ m/s ≈ 1.671 m/s.

44. Let x denote the horizontal distance between the person and the pulley, and let h denote the distance through which the container has been lifted. Given that the person holds the end of the rope and walks away from an initial position beneath the pulley, the rope has length 20 m. The right triangle formed by the pulley, the person's hand and the initial location of the top of the container therefore has hypotenuse of length $10 + h$ and legs of length x and 10. Therefore,

$$(10 + h)^2 = x^2 + 10^2,$$

and

$$2(10 + h) \frac{dh}{dt} = 2x \frac{dx}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{x}{10 + h} \frac{dx}{dt}.$$

When the person is 5 m away from the container, $x = 5$ and

$$h = \sqrt{x^2 + 10^2} - 10 = \sqrt{125} - 10 = 5\sqrt{5} - 10.$$

Given that $\frac{dx}{dt} = 2$ m/s, it follows that

$$\frac{dh}{dt} = \frac{5}{10 + 5\sqrt{5} - 10} (2) = \frac{2}{\sqrt{5}}.$$

When the person is 5 m away, the container is rising at a rate of $\frac{2}{\sqrt{5}}$ m/s ≈ 0.894 m/s.

45. (a) Differentiating $C = 10,000 + 3x$ with respect to time t yields

$$\frac{dC}{dt} = 3 \frac{dx}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dC}{dt} = 3(50) = 150.$$

When production is 1000 switches per day, cost increases at a rate of \$150/day.

(b) Differentiating

$$R = 5x - \frac{x^2}{2000}$$

with respect to time t yields

$$\frac{dR}{dt} = \left(5 - \frac{x}{1000}\right) \frac{dx}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dR}{dt} = \left(5 - \frac{1000}{1000}\right) \cdot 50 = 200.$$

When production is 1000 switches per day, revenue increases at a rate of $\boxed{\$200/\text{day}}$.

(c) Let P denote profit, so that $P = R - C$ and

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}.$$

When $x = 1000$ switches and $\frac{dx}{dt} = 50$ switches per day,

$$\frac{dP}{dt} = 200 - 150 = 50,$$

based on the answers from parts (a) and (b). When production is 1000 switches per day, profit increases at a rate of $\boxed{\$50/\text{day}}$.

46. (a) With $R = \frac{2Gm}{c^2}$ and G and c constant,

$$\frac{dR}{dt} = \frac{2G}{c^2} \frac{dm}{dt}.$$

Given that $\frac{dm}{dt} = 1.99 \times 10^{30}$ kg/yr, it follows that

$$\frac{dR}{dt} = \frac{2(6.67 \times 10^{-11})}{(3.00 \times 10^8)^2} \cdot 1.99 \times 10^{30} \approx 2949.622.$$

If one of these huge black holes swallows one Sun-like star per year, the event horizon grows at a rate of approximately $\boxed{2949.622 \text{ m/yr} \approx 2.95 \text{ km/yr}}$.

(b) Using differentials,

$$\frac{|\Delta R|}{R} \approx \frac{|dR|}{R} = \frac{\frac{2G}{c^2} |dm|}{\frac{2Gm}{c^2}} = \frac{|dm|}{m}.$$

Assuming m is the mass of 10 billion Suns and $dm = 1$ Sun, the percentage increase in the event horizon each year is

$$\frac{|dm|}{m} \times 100\% = \frac{1}{10^{10}} \times 100\% = \boxed{10^{-8}\%}.$$

47. Because the object weighs 1000 lb on Earth's surface, $K = 1000$, and

$$W = 1000 \left(\frac{3960}{3960 + R} \right)^2 = 1000(3960)^2(3960 + R)^{-2}.$$

Differentiating with respect to time t then yields

$$\frac{dW}{dt} = -2000(3960)^2(3960 + R)^{-3} \frac{dR}{dt}.$$

Given that $\frac{dR}{dt} = 10$ mi/s, it follows that when $R = 50$ mi,

$$\frac{dW}{dt} = -2000 \frac{3960^2}{(3960 + 50)^3} (10) \approx -4.864.$$

When the object is 50 mi above Earth's surface, the weight of the object is decreasing at a rate of approximately $\boxed{-4.864 \text{ lb/s}}$.

48. (a) Let $x = \cos \theta + \sqrt{16 - \sin^2 \theta}$. Differentiating with respect to time t ,

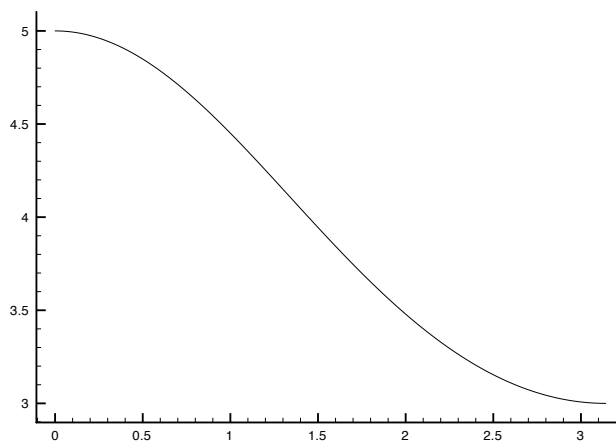
$$\frac{dx}{dt} = -\sin \theta \frac{d\theta}{dt} + \frac{1}{2}(16 - \sin^2 \theta)^{-1/2}(-2 \sin \theta \cos \theta) \frac{d\theta}{dt} = -\left(1 + \frac{\cos \theta}{\sqrt{16 - \sin^2 \theta}}\right) \sin \theta \frac{d\theta}{dt}.$$

Given that $\frac{d\theta}{dt} = 45$ rad/s, it follows that when $\theta = \pi/6$ rad,

$$\frac{dx}{dt} = -\left(1 + \frac{\frac{\sqrt{3}}{2}}{\sqrt{16 - \frac{1}{4}}}\right) \cdot \frac{1}{2}(45) \approx 27.410.$$

When $\theta = \frac{\pi}{6}$, the speed of the piston head is approximately $\boxed{27.410 \text{ m/s}}$.

- (b) The figure below displays the graph of x as a function of θ . Based on the graph, the maximum distance of the piston head from the center of the crank shaft is $\boxed{5 \text{ m}}$, and the minimum distance is $\boxed{3 \text{ m}}$.



49. Using the diagram in the text,

$$\cot \theta = \frac{x}{4500},$$

so that

$$-\csc^2 \theta \frac{d\theta}{dt} = \frac{1}{4500} \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = -4500 \csc^2 \theta \frac{d\theta}{dt}.$$

Given that $\frac{d\theta}{dt} = 1^\circ/\text{s} = \frac{\pi}{180}$ rad/s, it follows that when $\theta = 30^\circ$,

$$\frac{dx}{dt} = -4500 \cdot 4 \cdot \frac{\pi}{180} = -100\pi.$$

The plane is approaching the battery at a rate of $\boxed{100\pi \text{ ft/s} \approx 314.159 \text{ ft/s}}$.

50. Let h denote the elevation of the balloon, and let θ denote the angle of elevation of the observer's line of sight. Then

$$\tan \theta = \frac{h}{200} \quad \text{so that} \quad \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{200} \frac{dh}{dt}.$$

When the balloon is 600 m high, $h = 600$ and $\tan \theta = 3$ so

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + 3^2 = 10.$$

Given that $\frac{dh}{dt} = 100$ m/min, it follows that

$$\frac{d\theta}{dt} = \frac{1}{200 \sec^2 \theta} \frac{dh}{dt} = \frac{1}{200(10)} \cdot 100 = \frac{1}{20}.$$

When the balloon is 600 m high, the angle of elevation of the observer's line of sight increases at a rate of $\boxed{\frac{1}{20} \text{ rad/min} = 0.05 \text{ rad/min}}.$

51. Let x denote the horizontal distance between the searchlight and the plane. Then

$$\cot \theta = \frac{x}{3000 \text{ ft}} \cdot \frac{5280 \text{ ft}}{\text{mi}} = 1.76x,$$

so that

$$-\csc^2 \theta \frac{d\theta}{dt} = 1.76 \frac{dx}{dt} \quad \text{or} \quad \frac{d\theta}{dt} = -\frac{1.76}{\csc^2 \theta} \frac{dx}{dt}.$$

When the distance between the plane and the searchlight is 5000 ft, then $x = 4000$ ft and $\cot \theta = 4/3$ so

$$\csc^2 \theta = 1 + \cot^2 \theta = 1 + \left(\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}.$$

Given that $\frac{dx}{dt} = 500$ mi/h, it follows that

$$\frac{d\theta}{dt} = -\frac{1.76}{25/9}(500) = -316.8.$$

When the distance between the searchlight and the plane is 5000 ft, the searchlight is turning at a rate of $\boxed{-316.8 \text{ rad/h} = -0.088 \text{ rad/s}}.$

52. Let x denote the distance from the police car to the intersection, and let y denote the distance from the suspect car to the intersection. Then

$$\tan \theta = \frac{y}{x}.$$

Differentiating with respect to time t yields

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2},$$

or

$$\frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2 \sec^2 \theta}.$$

Two seconds after the suspect car has crossed the intersection, $x = 210 - 2(80) = 50$ ft, $y = 2(60) = 120$ ft, $\tan \theta = 120/50 = 2.4$, and

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + (2.4)^2 = 6.76.$$

Given that $\frac{dx}{dt} = -80$ ft/s and $\frac{dy}{dt} = 60$ ft/s, it follows that

$$\frac{d\theta}{dt} = \frac{50 \cdot 60 - 120 \cdot (-80)}{50^2(6.76)} \approx 0.746.$$

In order to follow the suspicious vehicle, the light beam must be turning at a rate of approximately $\boxed{0.746 \text{ rad/s}}$.

53. Let $V = hx^2$.

(a) If h decreases with time but x remains constant, then

$$\frac{dV}{dt} = \boxed{x^2 \frac{dh}{dt}},$$

so that the volume of the box decreases at a rate x^2 time the rate of decrease in the height.

(b) If both h and x change with time, then

$$\frac{dV}{dt} = h \cdot 2x \frac{dx}{dt} + x^2 \frac{dh}{dt} = \boxed{2hx \frac{dx}{dt} + x^2 \frac{dh}{dt}}.$$

Answers will now vary. For example, note that the rate of change of the height is multiplied by the area of the square base while the rate of change of the side length of the base is multiplied by twice the area of the vertical sides.

54. Let $y = 2e^{\cos x}$. If both x and y vary with time, then

$$\frac{dy}{dt} = 2e^{\cos x}(-\sin x) \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = -\frac{1}{2 \sin x e^{\cos x}} \frac{dy}{dt}.$$

Given that $\frac{dy}{dt} = 5$ units per second, it follows that when $x = \frac{\pi}{2}$,

$$\frac{dx}{dt} = -\frac{1}{2 \cdot 1 \cdot e^0}(5) = -\frac{5}{2}.$$

When $x = \frac{\pi}{2}$, x is decreasing at a rate of $\boxed{\frac{5}{2} \text{ units per second}}$.

Challenge Problems

55. Place the origin at the center of the dome, so the surface of the dome over which the shadow travels can be modeled by the equation $x^2 + y^2 = 30^2 = 900$. As the shadow of the ball moves along the surface of the dome,

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt},$$

and the speed with which the shadow is moving is

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{1 + \frac{y^2}{x^2}} \left| \frac{dy}{dt} \right| \\ &= \sqrt{\frac{x^2 + y^2}{x^2}} \left| \frac{dy}{dt} \right| = \frac{30}{\sqrt{900 - y^2}} \left| \frac{dy}{dt} \right|. \end{aligned}$$

Because it is sunset, the rays from the sun are traveling parallel to the ground, and the y -coordinate of the shadow is the same as the y -coordinate of the ball itself. Let $t = 0$ denote the time when the ball begins falling. Then the y -coordinate of the ball (and of the shadow) is given by $y = 30 - 16t^2$, and

$$\frac{dy}{dt} = -32t.$$

At $t = 1/2$ s, $y = 30 - 16(1/4) = 26$ ft, $\frac{dy}{dt} = -16$ ft/s, and the speed of the shadow is

$$\frac{30}{\sqrt{900 - 26^2}} |-16| \approx \boxed{32.071 \text{ ft/s}}.$$

56. Because the sun is rising in the east and the wall of the station runs north-south, the speed with which the shadow of the train moves along the wall of the station is equal to the rate of change in the y -coordinate of the train as it moves along the track. Given that $y^2 = 600x$ along the track, it follows that

$$2y \frac{dy}{dt} = 600 \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{y}{300} \frac{dy}{dt}.$$

The speed with which the train moves along the track is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1 + \frac{y^2}{300^2}} \left| \frac{dy}{dt} \right|.$$

At the instant the shadow of the train reaches the end of the station wall, $y = 400$ ft and the speed of the train along the track is 15 mi/h, so

$$\sqrt{1 + \frac{400^2}{300^2}} \left| \frac{dy}{dt} \right| = 15$$

and

$$\left| \frac{dy}{dt} \right| = \frac{15}{\sqrt{1 + \frac{16}{9}}} = 15 \cdot \frac{3}{5} = 9 \text{ mi/h}.$$

Because the train is traveling in a northeasterly direction, the shadow is moving north at 9 mi/h as it reaches the end of the station wall.

57. The area A of the triangle OAB is

$$A = \frac{1}{2}(2)(3) \sin \theta = 3 \sin \theta,$$

where θ is the angle between the two hands of the clock. Now the hour hand moves one-twelfth of the circle in 60 minutes, so it moves at a rate of

$$\frac{1}{12} \cdot 2\pi \cdot \frac{1}{60} = \frac{\pi}{360} \text{ rad/min},$$

while the minute hand makes one revolution of the circle in 60 minutes, so it moves at a rate of

$$\frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/min}.$$

At noon, the hour hand and the minute hand overlay, so ten minutes later the angle between the hands is

$$\theta = \frac{\pi}{30} \cdot 10 - \frac{\pi}{360} \cdot 10 = \frac{\pi}{3} - \frac{\pi}{36} = \frac{11\pi}{36},$$

and the rate at which the angle θ is changing is

$$\frac{d\theta}{dt} = \frac{\pi}{30} - \frac{\pi}{360} = \frac{11\pi}{360} \text{ rad/min.}$$

At 12:10 p.m.,

$$\frac{dA}{dt} = 3 \cos \theta \frac{d\theta}{dt} = 3 \cos \frac{11\pi}{36} \cdot \frac{11\pi}{360} \approx 0.165;$$

that is, the area of the triangle is increasing at a rate of approximately $\boxed{0.165 \text{ in}^2/\text{min}}$.

58. Place the origin at the “intersection” of the track and the street, with the positive x -axis pointing along the track in the direction the train is traveling. Let $t = 0$ correspond to the time the car passes under the track, and note that at this time, the train has already traveled 125 feet (5 seconds at 25 ft/s) along the track. Therefore, the x -, y - and z -coordinates of the train at any time t are given by

$$x_T = 125 + 25t, \quad y_T = 0, \quad \text{and} \quad z_T = 20,$$

while the coordinates of the car are given by

$$x_C = 40 \cos(-30^\circ)t = 20\sqrt{3}t, \quad y_C = 40 \sin(-30^\circ)t = -20t, \quad \text{and} \quad z_C = 0.$$

The distance D between the train and the car is

$$\begin{aligned} D &= \sqrt{(x_T - x_C)^2 + (y_T - y_C)^2 + (z_T - z_C)^2} \\ &= \sqrt{(125 + 25t - 20\sqrt{3}t)^2 + (20t)^2 + 20^2} \\ &= \sqrt{125^2 + 20^2 + 250(25 - 20\sqrt{3})t + [(25 - 20\sqrt{3})^2 + 20^2]t^2} \\ &= \sqrt{16025 + (6250 - 5000\sqrt{3})t + (2225 - 1000\sqrt{3})t^2} \end{aligned}$$

and

$$\frac{dD}{dt} = \frac{6250 - 5000\sqrt{3} + 2(2225 - 1000\sqrt{3})t}{2\sqrt{16025 + (6250 - 5000\sqrt{3})t + (2225 - 1000\sqrt{3})t^2}}.$$

Therefore, at $t = 3$,

$$\begin{aligned} \frac{dD}{dt} &= \frac{6250 - 5000\sqrt{3} + 2(2225 - 1000\sqrt{3})(3)}{2\sqrt{16025 + (6250 - 5000\sqrt{3})(3) + (2225 - 1000\sqrt{3})(3)^2}} \\ &= \frac{19600 - 11000\sqrt{3}}{2\sqrt{54800 - 24000\sqrt{3}}} \approx 2.380 \text{ ft/s.} \end{aligned}$$

The train and the car are separating at a rate of approximately $\boxed{2.380 \text{ ft/s}}$.

4.2: Maximum and Minimum Values; Critical Numbers

Concepts and Vocabulary

1. False. Any function f that is defined **and continuous** on a closed interval $[a, b]$ will have both an absolute maximum value and an absolute minimum value.
2. A number c in the domain of a function f is called a (b) critical number of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
3. False. At a critical number, there **may be** a local extreme value.
4. False. If a function f is continuous on a closed interval $[a, b]$, then its absolute maximum value is found **either** at a critical number **or at an endpoint of the interval**.
5. False. The Extreme Value Theorem tells us when the absolute maximum and absolute minimum **exist**.
6. False. If f is differentiable on the interval $(0, 4)$ and $f'(2) = 0$, then f **may have** a local maximum or a local minimum at 2.

Skill Building

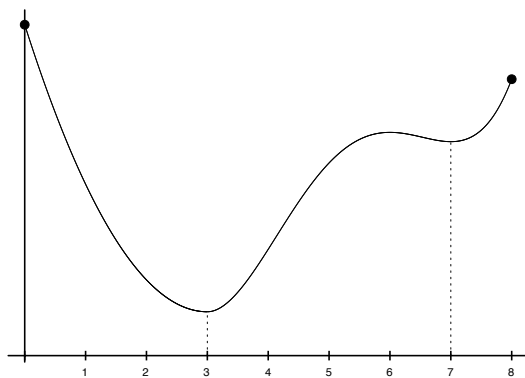
7. For the function given in the graph,

x_1 :	neither
x_2 :	local maximum
x_3 :	local minimum and absolute minimum
x_4 :	neither
x_5 :	local maximum
x_6 :	neither
x_7 :	local minimum
x_8 :	absolute maximum

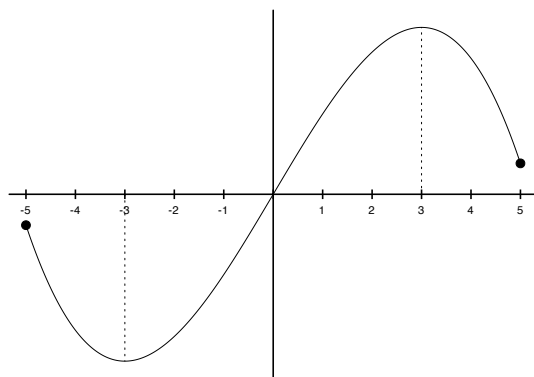
8. For the function given in the graph,

x_1 :	neither
x_2 :	local maximum
x_3 :	local maximum
x_4 :	local minimum
x_5 :	neither
x_6 :	local maximum
x_7 :	local minimum and absolute minimum
x_8 :	absolute maximum

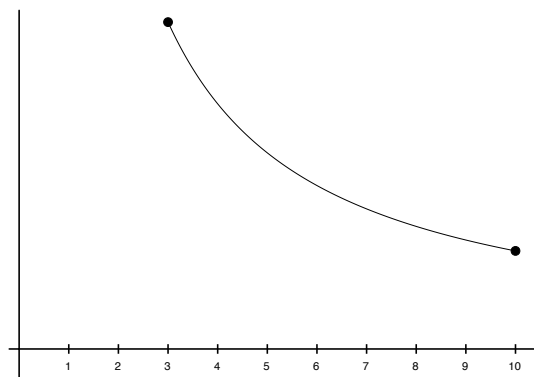
9. The figure below displays the graph of a function with the following properties: domain $[0, 8]$, absolute maximum at 0, absolute minimum at 3, local minimum at 7.



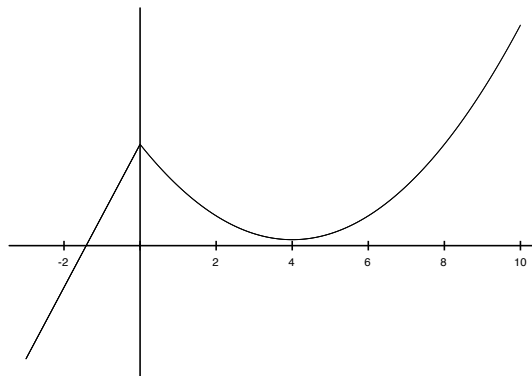
10. The figure below displays the graph of a function with the following properties: domain $[-5, 5]$, absolute maximum at 3, absolute minimum at -3 .



11. The figure below displays the graph of a function with the following properties: domain $[3, 10]$ and no local extreme values.



12. The figure below displays the graph of a function with the following properties: no absolute extreme values, differentiable at 4 and a local minimum at 4, not differentiable at 0 but a local maximum at 0.



13. Let $f(x) = x^2 - 8x$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 2x - 8 = 2(x - 4) = 0$$

when $x = 4$. Therefore, $\boxed{4}$ is the only critical number of f .

14. Let $f(x) = 1 - 6x + x^2$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = -6 + 2x = 2(x - 3) = 0$$

when $x = 3$. Therefore, $\boxed{3}$ is the only critical number of f .

15. Let $f(x) = x^3 - 3x^2$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$

when $x = 0$ and when $x = 2$. Therefore, $\boxed{0 \text{ and } 2}$ are the critical numbers of f .

16. Let $f(x) = x^3 - 6x$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 3x^2 - 6 = 3(x^2 - 2) = 0$$

when $x = \pm\sqrt{2}$. Therefore, $\boxed{-\sqrt{2} \text{ and } \sqrt{2}}$ are the critical numbers of f .

17. Let $f(x) = x^4 - 2x^2 + 1$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$$

when $x = 0$ and when $x = \pm 1$. Therefore, $\boxed{-1, 0, \text{ and } 1}$ are the critical numbers of f .

18. Let $f(x) = 3x^4 - 4x^3$. Because f is a polynomial function, it is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1) = 0$$

when $x = 0$ and when $x = 1$. Therefore, $\boxed{0 \text{ and } 1}$ are the critical numbers of f .

19. Let $f(x) = x^{2/3}$. The domain of f is the set of all real numbers, and $f'(x) = \frac{2}{3}x^{-1/3}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Therefore, $\boxed{0}$ is the only critical number of f .

20. Let $f(x) = x^{1/3}$. The domain of f is the set of all real numbers, and $f'(x) = \frac{1}{3}x^{-2/3}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Therefore, $\boxed{0}$ is the only critical number of f .

21. Let $f(x) = 2\sqrt{x}$. The domain of f is the set $\{x|x \geq 0\}$, and $f'(x) = \frac{1}{\sqrt{x}}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Therefore, $\boxed{0}$ is the only critical number of f .

22. Let $f(x) = 4 - \sqrt{x}$. The domain of f is the set $\{x|x \geq 0\}$, and $f'(x) = -\frac{1}{2\sqrt{x}}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Therefore, $\boxed{0}$ is the only critical number of f .

23. Let $f(x) = x + \sin x$, and consider the interval $0 \leq x \leq \pi$. Because f is differentiable on $0 \leq x \leq \pi$, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 1 + \cos x = 0$$

when $\cos x = -1$. On the interval $0 \leq x \leq \pi$, $\cos x = -1$ when $x = \pi$. Therefore, $\boxed{\pi}$ is the only critical number of f on the interval $0 \leq x \leq \pi$.

24. Let $f(x) = x - \cos x$, and consider the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Because f is differentiable on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 1 + \sin x = 0$$

when $\sin x = -1$. On the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $\sin x = -1$ when $x = -\frac{\pi}{2}$. Therefore, $\boxed{-\frac{\pi}{2}}$ is the only critical number of f on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

25. Let $f(x) = x\sqrt{1-x^2}$. The domain of f is the set $\{x|-1 \leq x \leq 1\}$, and

$$\begin{aligned} f'(x) &= x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + \sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-x^2 + 1 - x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $1 - 2x^2 = 0$, or when $x = \pm\frac{\sqrt{2}}{2}$, and $f'(x)$ does not exist when $1 - x^2 = 0$, or when $x = \pm 1$. All four

of these numbers are in the domain of f , so all four are critical numbers. Therefore, $\boxed{\pm 1 \text{ and } \pm\frac{\sqrt{2}}{2}}$ are critical numbers of f .

26. Let $f(x) = x^2\sqrt{2-x}$. The domain of f is the set $\{x|x \leq 2\}$, and

$$\begin{aligned} f'(x) &= x^2 \cdot \frac{1}{2}(2-x)^{-1/2}(-1) + 2x\sqrt{2-x} \\ &= -\frac{x^2}{2\sqrt{2-x}} + 2x\sqrt{2-x} = \frac{-x^2 + 4x(2-x)}{2\sqrt{2-x}} = \frac{8x - 5x^2}{2\sqrt{2-x}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $8x - 5x^2 = x(8 - 5x) = 0$, or when $x = 0$ or $x = \frac{8}{5}$. On the other hand, $f'(x)$ does not exist when $2 - x = 0$, or when $x = 2$. All three of these numbers are in the domain of f , so all three are critical numbers. Therefore, $\boxed{0, \frac{8}{5}, \text{ and } 2}$ are critical numbers of f .

27. Let $f(x) = \frac{x^2}{x-1}$. The domain of f is the set $\{x|x \neq 1\}$, and

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$, or when $x = 0$ or $x = 2$. On the other hand, $f'(x)$ does not exist when $x-1 = 0$, or when $x = 1$. As $x = 1$ is not in the domain of f , 1 is not a critical number. Therefore, $\boxed{0 \text{ and } 2}$ are the critical numbers of f .

28. Let $f(x) = \frac{x}{x^2-1}$. The domain of f is the set $\{x|x \neq \pm 1\}$, and

$$f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} = -\frac{1+x^2}{(x^2-1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is never equal to zero, but $f'(x)$ does not exist at $x = \pm 1$. As neither $x = 1$ nor $x = -1$ are in the domain of f , neither number is a critical number. Therefore, f has $\boxed{\text{no critical numbers}}$.

29. Let $f(x) = (x+3)^2(x-1)^{2/3}$. The domain of f is the set of all real numbers, and

$$\begin{aligned} f'(x) &= (x+3)^2 \cdot \frac{2}{3}(x-1)^{-1/3} + (x-1)^{2/3} \cdot 2(x+3) \\ &= \frac{2(x+3)^2}{3(x-1)^{1/3}} + 2(x+3)(x-1)^{2/3} = \frac{2(x+3)^2 + 6(x+3)(x-1)}{3(x-1)^{1/3}} \\ &= \frac{2(x+3)[x+3+3(x-1)]}{3(x-1)^{1/3}} = \frac{8x(x+3)}{3(x-1)^{1/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $8x(x+3) = 0$, or when $x = 0$ or $x = -3$. On the other hand, $f'(x)$ does not exist when $x-1 = 0$, or when $x = 1$. All three of these numbers are in the domain of f , so all three are critical numbers. Therefore, $\boxed{-3, 0, \text{ and } 1}$ are critical numbers of f .

30. Let $f(x) = (x-1)^2(x+1)^{1/3}$. The domain of f is the set of all real numbers, and

$$\begin{aligned} f'(x) &= (x-1)^2 \cdot \frac{1}{3}(x+1)^{-2/3} + (x+1)^{1/3} \cdot 2(x-1) \\ &= \frac{(x-1)^2}{3(x+1)^{2/3}} + 2(x-1)(x+1)^{1/3} = \frac{(x-1)^2 + 6(x-1)(x+1)}{3(x+1)^{2/3}} \\ &= \frac{(x-1)[x-1+6(x+1)]}{3(x+1)^{2/3}} = \frac{(x-1)(7x+5)}{3(x+1)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $(x-1)(7x+5) = 0$, or when $x = 1$ or $x = -\frac{5}{7}$. On the other hand, $f'(x)$ does not exist when $x+1 = 0$, or when $x = -1$. All three of these numbers are in the domain of f , so all three are critical numbers.

Therefore, $\boxed{-1, -\frac{5}{7}, \text{ and } 1}$ are critical numbers of f .

31. Let $f(x) = \frac{(x-3)^{1/3}}{x-1}$. The domain of f is the set $\{x|x \neq 1\}$, and

$$\begin{aligned} f'(x) &= \frac{(x-1) \cdot \frac{1}{3}(x-3)^{-2/3} - (x-3)^{1/3}}{(x-1)^2} \cdot \frac{3(x-3)^{2/3}}{3(x-3)^{2/3}} \\ &= \frac{(x-1) - 3(x-3)}{3(x-1)^2(x-3)^{2/3}} = \frac{8-2x}{3(x-1)^2(x-3)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $8 - 2x = 0$, or when $x = 4$. On the other hand, $f'(x)$ does not exist when $(x-1)^2(x-3)^{2/3} = 0$, or when $x = 1$ or $x = 3$. As $x = 1$ is not in the domain of f , 1 is not a critical number. Therefore, 3 and 4 are critical numbers of f .

32. Let $f(x) = \frac{(x+3)^{2/3}}{x+1}$. The domain of f is the set $\{x|x \neq -1\}$, and

$$\begin{aligned} f'(x) &= \frac{(x+1) \cdot \frac{2}{3}(x+3)^{-1/3} - (x+3)^{2/3}}{(x+1)^2} \cdot \frac{3(x+3)^{1/3}}{3(x+3)^{1/3}} \\ &= \frac{2(x+1) - 3(x+3)}{3(x+1)^2(x+3)^{1/3}} = -\frac{x+7}{3(x+1)^2(x+3)^{1/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x + 7 = 0$, or when $x = -7$. On the other hand, $f'(x)$ does not exist when $(x+1)^2(x+3)^{1/3} = 0$, or when $x = -1$ or $x = -3$. As $x = -1$ is not in the domain of f , -1 is not a critical number. Therefore, -7 and -3 are critical numbers of f .

33. Let $f(x) = \frac{\sqrt[3]{x^2-9}}{x}$. The domain of f is the set $\{x|x \neq 0\}$, and

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(x^2-9)^{-2/3}(2x) - \sqrt[3]{x^2-9}}{x^2} \cdot \frac{3(x^2-9)^{2/3}}{3(x^2-9)^{2/3}} \\ &= \frac{2x^2 - 3(x^2-9)}{3x^2(x^2-9)^{2/3}} = \frac{27-x^2}{3x^2(x^2-9)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $27 - x^2 = 0$, or when $x = \pm\sqrt{27} = \pm 3\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(x^2-9)^{2/3} = 0$, or when $x = 0$ or $x = \pm 3$. As $x = 0$ is not in the domain of f , 0 is not a critical number. Therefore, ± 3 and $\pm 3\sqrt{3}$ are critical numbers of f .

34. Let $f(x) = \frac{\sqrt[3]{4-x^2}}{x}$. The domain of f is the set $\{x|x \neq 0\}$, and

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(4-x^2)^{-2/3}(-2x) - \sqrt[3]{4-x^2}}{x^2} \cdot \frac{3(4-x^2)^{2/3}}{3(4-x^2)^{2/3}} \\ &= \frac{-2x^2 - 3(4-x^2)}{3x^2(4-x^2)^{2/3}} = \frac{x^2-12}{3x^2(4-x^2)^{2/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x^2 - 12 = 0$, or when $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(4-x^2)^{2/3} = 0$, or when $x = 0$ or $x = \pm 2$. As $x = 0$ is not in the domain of f , 0 is not a critical number. Therefore, ± 2 and $\pm 2\sqrt{3}$ are critical numbers of f .

35. On the interval $[0, 1)$, $f(x) = 3x$ so $f'(x) = 3$. Because $f'(x)$ exists everywhere on the interval $[0, 1)$ and is never equal to 0, f has no critical numbers in $[0, 1)$. On the interval $(1, 2]$, $f(x) = 4 - x$, so

$f'(x) = -1$. Because $f'(x)$ exists everywhere on the interval $(1, 2]$ and is never equal to 0, f has no critical numbers in $(1, 2]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{3x - 3}{x - 1} = \lim_{x \rightarrow 1^-} \frac{3(x - 1)}{x - 1} = \lim_{x \rightarrow 1^-} 3 = 3,$$

and

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(4 - x) - 3}{x - 1} = \lim_{x \rightarrow 1^+} \frac{-(x - 1)}{x - 1} = \lim_{x \rightarrow 1^+} -1 = -1.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, $\boxed{1}$ is the only critical number of f .

36. On the interval $[0, 1)$, $f(x) = x^2$ so $f'(x) = 2x$. Because $f'(x)$ exists everywhere on the interval $[0, 1)$ but is equal to 0 at $x = 0$, 0 is a critical number in $[0, 1)$. On the interval $(1, 2]$, $f(x) = 1 - x^2$, so $f'(x) = -2x$. Because $f'(x)$ exists everywhere and is never equal to 0 on the interval $(1, 2]$, f has no critical numbers in $(1, 2]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1 - x^2) = 0.$$

Because these two one-sided limits are not equal, f is not continuous at $x = 1$ and $f'(x)$ does not exist at $x = 1$. Therefore, $\boxed{0 \text{ and } 1}$ are the critical numbers of f .

37. Let $f(x) = x^2 - 8x$, and consider the closed interval $[-1, 10]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 10]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 2x - 8 = 2(x - 4) = 0$$

when $x = 4$. Therefore, 4 is the only critical number of f . Evaluating f at the endpoints of the interval $[-1, 10]$ and at the critical number 4 yields

$$\begin{aligned} f(-1) &= (-1)^2 - 8(-1) = 9 \\ f(4) &= 4^2 - 8(4) = -16 \\ f(10) &= 10^2 - 8(10) = 20. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 10]$ is $\boxed{20}$ (and this occurs at the endpoint $x = 10$), while the absolute minimum value of f is $\boxed{-16}$ (and this occurs at the critical number $x = 4$).

38. Let $f(x) = 1 - 6x + x^2$, and consider the closed interval $[0, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = -6 + 2x = 2(x - 3) = 0$$

when $x = 3$. Therefore, 3 is the only critical number of f . Evaluating f at the endpoints of the interval $[0, 4]$ and at the critical number 3 yields

$$\begin{aligned} f(0) &= 1 - 6(0) + 0^2 = 1 \\ f(3) &= 1 - 6(3) + 3^2 = -8 \\ f(4) &= 1 - 6(4) + 4^2 = -7. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 4]$ is $\boxed{1}$ (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is $\boxed{-8}$ (and this occurs at critical number $x = 3$).

39. Let $f(x) = x^3 - 3x^2$, and consider the closed interval $[1, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[1, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$

when $x = 0$ or $x = 2$. Therefore, 0 and 2 are the critical numbers of f ; however, $x = 0$ is not inside the interval $(1, 4)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $[1, 4]$ and at the critical number 2 yields

$$\begin{aligned} f(1) &= 1^3 - 3(1)^2 = -2 \\ f(2) &= 2^3 - 3(2)^2 = -4 \\ f(4) &= 4^3 - 3(4)^2 = 16. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[1, 4]$ is $\boxed{16}$ (and this occurs at the endpoint $x = 4$), while the absolute minimum value of f is $\boxed{-4}$ (and this occurs at critical number $x = 2$).

40. Let $f(x) = x^3 - 6x$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 3x^2 - 6 = 3(x^2 - 2) = 0$$

when $x = \pm\sqrt{2}$. Therefore, $-\sqrt{2}$ and $\sqrt{2}$ are the critical numbers of f ; however, neither of these numbers is inside the interval $(-1, 1)$, so both can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[-1, 1]$ yields

$$\begin{aligned} f(-1) &= (-1)^3 - 6(-1) = 5 \\ f(1) &= 1^3 - 6(1) = -5. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{5}$ (and this occurs at the endpoint $x = -1$), while the absolute minimum value of f is $\boxed{-5}$ (and this occurs at the endpoint $x = 1$).

41. Let $f(x) = x^4 - 2x^2 + 1$, and consider the closed interval $[0, 2]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 2]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0$$

when $x = 0$ or $x = \pm 1$. Therefore, $-1, 0$ and 1 are the critical numbers of f ; however, of these numbers, only $x = 1$ is inside the open interval $(0, 2)$. Evaluating f at the endpoints of the interval $[0, 2]$ and at the critical number 1 yields

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1^4 - 2(1)^2 + 1 = 0 \\ f(2) &= 2^4 - 2(2)^2 + 1 = 9. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 2]$ is $\boxed{9}$ (and this occurs at the endpoint $x = 2$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 1$).

42. Let $f(x) = 3x^4 - 4x^3$, and consider the closed interval $[-2, 0]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-2, 0]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1) = 0$$

when $x = 0$ or $x = 1$. Therefore, 0 and 1 are the critical numbers of f ; however, neither of these numbers is inside the interval $(-2, 0)$, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, 2]$ yields

$$\begin{aligned} f(-2) &= 3(-2)^4 - 4(-2)^3 = 80 \\ f(0) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-2, 0]$ is $\boxed{80}$ (and this occurs at the endpoint $x = -2$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoint $x = 0$).

43. Let $f(x) = x^{2/3}$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{2}{3}x^{-1/3},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= (-1)^{2/3} = 1 \\ f(0) &= 0 \\ f(1) &= 1^{2/3} = 1. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{1}$ (and this occurs at both endpoints $x = \pm 1$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 0$).

44. Let $f(x) = x^{1/3}$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{1}{3}x^{-2/3},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f . Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= (-1)^{1/3} = -1 \\ f(0) &= 0 \\ f(1) &= 1^{1/3} = 1. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{1}$ (and this occurs at the endpoint $x = 1$), while the absolute minimum value of f is $\boxed{-1}$ (and this occurs at the endpoint $x = -1$).

45. Let $f(x) = 2\sqrt{x}$, and consider the closed interval $[1, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[1, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{1}{\sqrt{x}},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f ; however, $x = 0$ is not inside the interval $(1, 4)$, so this critical number can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[1, 4]$ yields

$$\begin{aligned} f(1) &= 2\sqrt{1} = 2 \\ f(4) &= 2\sqrt{4} = 4. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[1, 4]$ is $\boxed{4}$ (and this occurs at the endpoint $x = 4$), while the absolute minimum value of f is $\boxed{2}$ (and this occurs at the endpoint $x = 1$).

46. Let $f(x) = 4 - \sqrt{x}$, and consider the closed interval $[0, 4]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 4]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = -\frac{1}{2\sqrt{x}},$$

so $f'(x)$ is never equal to 0, and $f'(x)$ does not exist at $x = 0$. Because $x = 0$ is in the domain of f , 0 is a critical number of f ; however, $x = 0$ is not inside the open interval $(0, 4)$, so this critical number can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, 4]$ yields

$$\begin{aligned} f(0) &= 4 \\ f(4) &= 4 - \sqrt{4} = 2. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 4]$ is $\boxed{4}$ (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is $\boxed{2}$ (and this occurs at the endpoint $x = 4$).

47. Let $f(x) = x + \sin x$, and consider the closed interval $[0, \pi]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, \pi]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 1 + \cos x = 0$$

when $\cos x = -1$. On the interval $0 \leq x \leq \pi$, $\cos x = -1$ when $x = \pi$; however, $x = \pi$ is not inside the interval $(0, \pi)$, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, \pi]$ yields

$$\begin{aligned} f(0) &= \sin 0 = 0 \\ f(\pi) &= \pi + \sin \pi = \pi. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, \pi]$ is $\boxed{\pi}$ (and this occurs at the endpoint $x = \pi$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoint $x = 0$).

48. Let $f(x) = x - \cos x$, and consider the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 1 + \sin x = 0$$

when $\sin x = -1$. On the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $\sin x = -1$ when $x = -\frac{\pi}{2}$; however, $x = -\frac{\pi}{2}$ is not inside the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ yields

$$\begin{aligned} f\left(-\frac{\pi}{2}\right) &= -\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \\ f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is $\boxed{\frac{\pi}{2}}$ (and this occurs at the endpoint $x = \frac{\pi}{2}$), while the absolute minimum value of f is $\boxed{-\frac{\pi}{2}}$ (and this occurs at the endpoint $x = -\frac{\pi}{2}$).

49. Let $f(x) = x\sqrt{1-x^2}$, and consider the closed interval $[-1, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-1, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + \sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-x^2 + 1 - x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}, \end{aligned}$$

so $f'(x) = 0$ when $1 - 2x^2 = 0$, or when $x = \pm \frac{\sqrt{2}}{2}$, and $f'(x)$ does not exist when $1 - x^2 = 0$, or when $x = \pm 1$. All four of these numbers are in the domain of f , so all four are critical numbers; however, of these numbers, only $x = \pm \frac{\sqrt{2}}{2}$ are inside the open interval $(-1, 1)$. Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical numbers $\pm \frac{\sqrt{2}}{2}$ yields

$$\begin{aligned} f(-1) &= 0 \\ f\left(-\frac{\sqrt{2}}{2}\right) &= -\frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}} = -\frac{1}{2} \\ f\left(\frac{\sqrt{2}}{2}\right) &= \frac{\sqrt{2}}{2}\sqrt{1-\frac{1}{2}} = \frac{1}{2} \\ f(1) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{\frac{1}{2}}$ (and this occurs at the critical number $x = \frac{\sqrt{2}}{2}$), while the absolute minimum value of f is $\boxed{-\frac{1}{2}}$ (and this occurs at the critical number $x = -\frac{\sqrt{2}}{2}$).

50. Let $f(x) = x^2\sqrt{2-x}$, and consider the closed interval $[0, 2]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 2]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= x^2 \cdot \frac{1}{2}(2-x)^{-1/2}(-1) + 2x\sqrt{2-x} \\ &= -\frac{x^2}{2\sqrt{2-x}} + 2x\sqrt{2-x} = \frac{-x^2 + 4x(2-x)}{2\sqrt{2-x}} = \frac{8x - 5x^2}{2\sqrt{2-x}}, \end{aligned}$$

so $f'(x) = 0$ when $8x - 5x^2 = x(8 - 5x) = 0$, or when $x = 0$ or $x = \frac{8}{5}$. On the other hand, $f'(x)$ does not exist when $2 - x = 0$, or when $x = 2$. All three of these numbers are in the domain of f , so all three are critical numbers; however, of these numbers, only $\frac{8}{5}$ is inside the open interval $(0, 2)$. Evaluating f at the endpoints of the interval $[0, 2]$ and the critical number $\frac{8}{5}$ yields

$$\begin{aligned} f(0) &= 0 \\ f\left(\frac{8}{5}\right) &= \frac{64}{25}\sqrt{2 - \frac{8}{5}} = \frac{64\sqrt{10}}{125} \\ f(2) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 2]$ is $\boxed{\frac{64\sqrt{10}}{125}}$ (and this occurs at the critical number $x = \frac{8}{5}$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoints $x = 0$ and $x = 2$).

51. Let $f(x) = \frac{x^2}{x-1}$, and consider the closed interval $\left[-1, \frac{1}{2}\right]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $\left[-1, \frac{1}{2}\right]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2},$$

so $f'(x) = 0$ when $x^2 - 2x = x(x - 2) = 0$, or when $x = 0$ or $x = 2$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. As $x = 1$ is not in the domain of f , 1 is not a critical number. Therefore, 0 and 2 are the critical numbers of f ; however, $x = 2$ is not inside the open interval $\left(-1, \frac{1}{2}\right)$, so this critical number can be excluded. Evaluating f at the endpoints of the

interval $\left[-1, \frac{1}{2}\right]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= \frac{(-1)^2}{-1-1} = -\frac{1}{2} \\ f(0) &= 0 \\ f\left(\frac{1}{2}\right) &= \frac{1/4}{-1/2} = -\frac{1}{2}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $\left[-1, \frac{1}{2}\right]$ is $\boxed{0}$ (and this occurs at the critical number $x = 0$), while the absolute minimum value of f is $\boxed{-\frac{1}{2}}$ (and this occurs at both endpoints $x = -1$ and $x = \frac{1}{2}$).

52. Let $f(x) = \frac{x}{x^2-1}$, and consider the closed interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} = -\frac{1+x^2}{(x^2-1)^2},$$

so $f'(x)$ is never equal to zero, and $f'(x)$ does not exist at $x = \pm 1$. As neither $x = 1$ nor $x = -1$ are in the domain of f , neither number is a critical number. Therefore, f has no critical numbers. Evaluating f at the endpoints of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ yields

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= \frac{-1/2}{-3/4} = \frac{2}{3} \\ f\left(\frac{1}{2}\right) &= \frac{1/2}{-3/4} = -\frac{2}{3}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is $\boxed{\frac{2}{3}}$ (and this occurs at the endpoint $x = -\frac{1}{2}$), while the absolute minimum value of f is $\boxed{-\frac{2}{3}}$ (and this occurs at the endpoint $x = \frac{1}{2}$).

53. Let $f(x) = (x+3)^2(x-1)^{2/3}$, and consider the closed interval $[-4, 5]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-4, 5]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= (x+3)^2 \cdot \frac{2}{3}(x-1)^{-1/3} + (x-1)^{2/3} \cdot 2(x+3) \\ &= \frac{2(x+3)^2}{3(x-1)^{1/3}} + 2(x+3)(x-1)^{2/3} = \frac{2(x+3)^2 + 6(x+3)(x-1)}{3(x-1)^{1/3}} \\ &= \frac{2(x+3)[x+3+3(x-1)]}{3(x-1)^{1/3}} = \frac{8x(x+3)}{3(x-1)^{1/3}}, \end{aligned}$$

so $f'(x) = 0$ when $8x(x+3) = 0$, or when $x = 0$ or $x = -3$. On the other hand, $f'(x)$ does not exist when $x - 1 = 0$, or when $x = 1$. All three of these numbers are in the domain of f , so all three are critical numbers. Evaluating f at the endpoints of the interval $[-4, 5]$ and the critical numbers -3 , 0 , and 1 yields

$$\begin{aligned} f(-4) &= (-1)^2(-5)^{2/3} = 25^{1/3} = \sqrt[3]{25} \approx 2.924 \\ f(-3) &= 0 \\ f(0) &= 3^2(-1)^{2/3} = 9 \\ f(1) &= 0 \\ f(5) &= 8^2(4)^{2/3} = 64(4)^{2/3} = 64\sqrt[3]{16} \approx 161.270. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-4, 5]$ is $\boxed{64\sqrt[3]{16}}$ (and this occurs at the endpoint $x = 5$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical numbers $x = -3$ and $x = 1$).

54. Let $f(x) = (x-1)^2(x+1)^{1/3}$, and consider the closed interval $[-2, 7]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-2, 7]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= (x-1)^2 \cdot \frac{1}{3}(x+1)^{-2/3} + (x+1)^{1/3} \cdot 2(x-1) \\ &= \frac{(x-1)^2}{3(x+1)^{2/3}} + 2(x-1)(x+1)^{1/3} = \frac{(x-1)^2 + 6(x-1)(x+1)}{3(x+1)^{2/3}} \\ &= \frac{(x-1)[x-1+6(x+1)]}{3(x+1)^{2/3}} = \frac{(x-1)(7x+5)}{3(x+1)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $(x-1)(7x+5) = 0$, or when $x = 1$ or $x = -\frac{5}{7}$. On the other hand, $f'(x)$ does not exist when $x+1 = 0$, or when $x = -1$. All three of these numbers are in the domain of f , so all three are critical numbers. Evaluating f at the endpoints of the interval $[-2, 7]$ and the critical numbers -1 , $-\frac{5}{7}$, and 1 yields

$$\begin{aligned} f(-2) &= (-3)^2(-1)^{1/3} = -9 \\ f(-1) &= 0 \\ f\left(-\frac{5}{7}\right) &= \left(-\frac{12}{7}\right)^2 \left(\frac{2}{7}\right)^{1/3} \approx 1.936 \\ f(1) &= 0 \\ f(7) &= 6^2(8)^{1/3} = 72. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-2, 7]$ is $\boxed{72}$ (and this occurs at the endpoint $x = 7$), while the absolute minimum value of f is $\boxed{-9}$ (and this occurs at the endpoint $x = -2$).

55. Let $f(x) = \frac{(x-3)^{1/3}}{x-1}$, and consider the closed interval $[2, 11]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[2, 11]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur

where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{(x-1) \cdot \frac{1}{3}(x-3)^{-2/3} - (x-3)^{1/3}}{(x-1)^2} \cdot \frac{3(x-3)^{2/3}}{3(x-3)^{2/3}} \\ &= \frac{(x-1) - 3(x-3)}{3(x-1)^2(x-3)^{2/3}} = \frac{8-2x}{3(x-1)^2(x-3)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $8 - 2x = 0$, or when $x = 4$. On the other hand, $f'(x)$ does not exist when $(x-1)^2(x-3)^{2/3} = 0$, or when $x = 1$ or $x = 3$. As $x = 1$ is not in the domain of f , 1 is not a critical number. Therefore, 3 and 4 are critical numbers of f . Evaluating f at the endpoints of the interval $[2, 11]$ and the critical numbers 3 and 4 yields

$$\begin{aligned} f(2) &= \frac{(-1)^{1/3}}{1} = -1 \\ f(3) &= 0 \\ f(4) &= \frac{1^{1/3}}{3} = \frac{1}{3} \\ f(11) &= \frac{8^{1/3}}{10} = \frac{1}{5}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[2, 11]$ is $\boxed{\frac{1}{3}}$ (and this occurs at the critical number $x = 4$), while the absolute minimum value of f is $\boxed{-1}$ (and this occurs at the endpoint $x = 2$).

56. Let $f(x) = \frac{(x+3)^{2/3}}{x+1}$, and consider the closed interval $[-4, -2]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-4, -2]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{(x+1) \cdot \frac{2}{3}(x+3)^{-1/3} - (x+3)^{2/3}}{(x+1)^2} \cdot \frac{3(x+3)^{1/3}}{3(x+3)^{1/3}} \\ &= \frac{2(x+1) - 3(x+3)}{3(x+1)^2(x+3)^{1/3}} = -\frac{x+7}{3(x+1)^2(x+3)^{1/3}}, \end{aligned}$$

so $f'(x) = 0$ when $x + 7 = 0$, or when $x = -7$. On the other hand, $f'(x)$ does not exist when $(x+1)^2(x+3)^{1/3} = 0$, or when $x = -1$ or $x = -3$. As $x = -1$ is not in the domain of f , -1 is not a critical number. Therefore, -7 and -3 are critical numbers of f ; however, $x = -7$ is not inside the interval $(-4, -2)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $[-4, -2]$ and the critical number -3 yields

$$\begin{aligned} f(-4) &= \frac{(-1)^{2/3}}{-3} = -\frac{1}{3} \\ f(-3) &= 0 \\ f(-2) &= \frac{1^{2/3}}{-1} = -1. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-4, -2]$ is $\boxed{0}$ (and this occurs at the critical number $x = -3$), while the absolute minimum value of f is $\boxed{-1}$ (and this occurs at the endpoint $x = -2$).

57. Let $f(x) = \frac{\sqrt[3]{x^2-9}}{x}$, and consider the closed interval $[3, 6]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute

minimum on $[3, 6]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(x^2 - 9)^{-2/3}(2x) - \sqrt[3]{x^2 - 9}}{x^2} \cdot \frac{3(x^2 - 9)^{2/3}}{3(x^2 - 9)^{2/3}} \\ &= \frac{2x^2 - 3(x^2 - 9)}{3x^2(x^2 - 9)^{2/3}} = \frac{27 - x^2}{3x^2(x^2 - 9)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $27 - x^2 = 0$, or when $x = \pm\sqrt{27} = \pm 3\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(x^2 - 9)^{2/3} = 0$, or when $x = 0$ or $x = \pm 3$. As $x = 0$ is not in the domain of f , 0 is not a critical number. Therefore, ± 3 and $\pm 3\sqrt{3}$ are critical numbers of f ; however, of these numbers, only $3\sqrt{3}$ is inside the open interval $(3, 6)$. Evaluating f at the endpoints of the interval $[3, 6]$ and the critical number $3\sqrt{3}$ yields

$$\begin{aligned} f(3) &= 0 \\ f(3\sqrt{3}) &= \frac{\sqrt[3]{18}}{3\sqrt{3}} \approx 0.504 \\ f(6) &= \frac{\sqrt[3]{27}}{6} = \frac{1}{2}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[3, 6]$ is $\boxed{\frac{\sqrt[3]{18}}{3\sqrt{3}}}$ (and this occurs at the critical number $x = 3\sqrt{3}$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the endpoint $x = 3$).

58. Let $f(x) = \frac{\sqrt[3]{4 - x^2}}{x}$, and consider the closed interval $[-4, -1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-4, -1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{x \cdot \frac{1}{3}(4 - x^2)^{-2/3}(-2x) - \sqrt[3]{4 - x^2}}{x^2} \cdot \frac{3(4 - x^2)^{2/3}}{3(4 - x^2)^{2/3}} \\ &= \frac{-2x^2 - 3(4 - x^2)}{3x^2(4 - x^2)^{2/3}} = \frac{x^2 - 12}{3x^2(4 - x^2)^{2/3}}, \end{aligned}$$

so $f'(x) = 0$ when $x^2 - 12 = 0$, or when $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. On the other hand, $f'(x)$ does not exist when $x^2(4 - x^2)^{2/3} = 0$, or when $x = 0$ or $x = \pm 2$. As $x = 0$ is not in the domain of f , 0 is not a critical number. Therefore, ± 2 and $\pm 2\sqrt{3}$ are critical numbers of f ; however, $x = 2$ and $x = 2\sqrt{3}$ are not inside the interval $(-4, -1)$, so these critical numbers can be excluded. Evaluating f at the endpoints of the interval $[-4, -1]$ and the critical numbers $-2\sqrt{3}$ and -2 yields

$$\begin{aligned} f(-4) &= \frac{\sqrt[3]{-12}}{-4} = \frac{\sqrt[3]{12}}{4} \approx 0.572 \\ f(-2\sqrt{3}) &= \frac{\sqrt[3]{-8}}{-2\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577 \\ f(-2) &= 0 \\ f(-1) &= \frac{\sqrt[3]{3}}{-1} = -\sqrt[3]{3}. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-4, -1]$ is $\boxed{\frac{\sqrt{3}}{3}}$ (and this occurs at the critical number $x = -2\sqrt{3}$), while the absolute minimum value of f is $\boxed{-\frac{\sqrt{3}}{3}}$ (and this occurs at the endpoint $x = -1$).

59. Let $f(x) = e^x - 3x$, and consider the closed interval $[0, 1]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 1]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = e^x - 3 = 0$$

when $x = \ln 3$. Therefore, $\ln 3$ is the only critical number of f ; however, $x = \ln 3 \approx 1.099$ is not inside the interval $(0, 1)$, so this critical number can be excluded, leaving no critical numbers inside the open interval. Evaluating f at the endpoints of the interval $[0, 1]$ yields

$$\begin{aligned} f(0) &= e^0 - 3(0) = 1 \\ f(1) &= e^1 - 3(1) = e - 3 \approx -0.282. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 1]$ is $\boxed{1}$ (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is $\boxed{e - 3}$ (and this occurs at the endpoint $x = 1$).

60. Let $f(x) = e^{\cos x}$, and consider the closed interval $[-\pi, 2\pi]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-\pi, 2\pi]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = -\sin x e^{\cos x} = 0$$

when $\sin x = 0$. On the interval $-\pi \leq x \leq 2\pi$, $\sin x = 0$ when $x = -\pi$, $x = 0$, $x = \pi$, and $x = 2\pi$; however, $-\pi$ and 2π are not inside the interval $(-\pi, 2\pi)$, so these critical numbers can be excluded, leaving only 0 and π inside the open interval. Evaluating f at the endpoints of the interval $[-\pi, 2\pi]$ and the critical numbers 0 and π yields

$$\begin{aligned} f(-\pi) &= e^{\cos(-\pi)} = e^{-1} \\ f(0) &= e^{\cos 0} = e \\ f(\pi) &= e^{\cos \pi} = e^{-1} \\ f(2\pi) &= e^{\cos(2\pi)} = e. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-\pi, 2\pi]$ is $\boxed{e}$ (and this occurs at the critical number $x = 0$ and at the endpoint $x = 2\pi$), while the absolute minimum value of f is $\boxed{e^{-1}}$ (and this occurs at the endpoint $x = -\pi$ and at the critical number $x = \pi$).

61. The function f is continuous on the intervals $[0, 1)$ and $(1, 3]$ because the component functions $2x + 1$ and $3x$ are continuous on these intervals, respectively. At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x + 1) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3x) = 3,$$

so that $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 3. As $f(1) = 3(1) = 3$, it follows that f is continuous at $x = 1$. Hence, f is continuous on the closed interval $[0, 3]$.

Now, on the interval $[0, 1)$, $f(x) = 2x + 1$ so $f'(x) = 2$. Because $f'(x)$ exists everywhere on the interval $[0, 1)$ and is never equal to 0, f has no critical numbers in $[0, 1)$. On the interval $(1, 3]$, $f(x) = 3x$,

so $f'(x) = 3$. Because $f'(x)$ exists everywhere on the interval $(1, 3]$ and is never equal to 0, f has no critical numbers in $(1, 3]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(2x + 1) - 3}{x - 1} = \lim_{x \rightarrow 1^-} \frac{2(x - 1)}{x - 1} = \lim_{x \rightarrow 1^-} 2 = 2,$$

and

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{3x - 3}{x - 1} = \lim_{x \rightarrow 1^+} \frac{3(x - 1)}{x - 1} = \lim_{x \rightarrow 1^+} 3 = 3.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, 1 is the only critical number of f . Evaluating f at the endpoints of the interval $[0, 3]$ and the critical number 1 yields

$$\begin{aligned} f(0) &= 2(0) + 1 = 1 \\ f(1) &= 3(1) = 3 \\ f(3) &= 3(3) = 9. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 3]$ is $\boxed{9}$ (and this occurs at the endpoint $x = 3$), while the absolute minimum value of f is $\boxed{1}$ (and this occurs at the endpoint $x = 0$).

62. The function f is continuous on the intervals $[-1, 2)$ and $(2, 4]$ because the component functions $x + 3$ and $2x + 1$ are continuous on these intervals, respectively. At $x = 2$,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 3) = 5 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x + 1) = 5,$$

so that $\lim_{x \rightarrow 2} f(x)$ exists and is equal to 5. As $f(2) = 2 + 3 = 5$, it follows that f is continuous at $x = 2$. Hence, f is continuous on the closed interval $[-1, 4]$.

Now, on the interval $[-1, 2)$, $f(x) = x + 3$ so $f'(x) = 1$. Because $f'(x)$ exists everywhere on the interval $[-1, 2)$ and is never equal to 0, f has no critical numbers in $[-1, 2)$. On the interval $(2, 4]$, $f(x) = 2x + 1$, so $f'(x) = 2$. Because $f'(x)$ exists everywhere on the interval $(2, 4]$ and is never equal to 0, f has no critical numbers in $(2, 4]$. At $x = 2$, the rule for f changes, so it is necessary to investigate the existence of $f'(2)$. Now,

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x + 3) - 5}{x - 2} = \lim_{x \rightarrow 2^-} \frac{x - 2}{x - 2} = \lim_{x \rightarrow 2^-} 1 = 1,$$

and

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(2x + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^+} \frac{2(x - 2)}{x - 2} = \lim_{x \rightarrow 2^+} 2 = 2.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 2$. Therefore, 2 is the only critical number of f . Evaluating f at the endpoints of the interval $[-1, 4]$ and the critical number 2 yields

$$\begin{aligned} f(-1) &= -1 + 3 = 2 \\ f(2) &= 2 + 3 = 5 \\ f(4) &= 2(4) + 1 = 9. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 4]$ is $\boxed{9}$ (and this occurs at the endpoint $x = 4$), while the absolute minimum value of f is $\boxed{2}$ (and this occurs at the endpoint $x = -1$).

63. The function f is continuous on the intervals $[-2, 1)$ and $(1, 2]$ because the component functions x^2 and x^3 are continuous on these intervals, respectively. At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^3 = 1,$$

so that $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 1. As $f(1) = 1^3 = 1$, it follows that f is continuous at $x = 1$. Hence, f is continuous on the closed interval $[-2, 2]$.

Now, on the interval $[-2, 1)$, $f(x) = x^2$ so $f'(x) = 2x$. Because $f'(x)$ exists everywhere on the interval $[-2, 1)$ but is equal to 0 at $x = 0$, 0 is a critical number in $[-2, 1)$. On the interval $(1, 2]$, $f(x) = x^3$, so $f'(x) = 3x^2$. Because $f'(x)$ exists everywhere and is never equal to 0 on the interval $(1, 2]$, f has no critical numbers in $(1, 2]$. At $x = 1$, the rule for f changes, so it is necessary to investigate the existence of $f'(1)$. Now,

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^-} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1^-} (x + 1) = 2,$$

and

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1^+} (x^2 + x + 1) = 3.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 1$. Therefore, 1 is a critical number of f . Evaluating f at the endpoints of the interval $[-2, 2]$ and the critical numbers 0 and 1 yields

$$\begin{aligned} f(-2) &= (-2)^2 = 4 \\ f(0) &= 0^2 = 0 \\ f(1) &= 1^3 = 1 \\ f(2) &= 2^3 = 8. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-2, 2]$ is $\boxed{8}$ (and this occurs at the endpoint $x = 2$), while the absolute minimum value of f is $\boxed{0}$ (and this occurs at the critical number $x = 0$).

64. The function f is continuous on the intervals $[-1, 0)$ and $(0, 1]$ because the component functions $x + 2$ and $2 - x$ are continuous on these intervals, respectively. At $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 2) = 2 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2 - x) = 2,$$

so that $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 2. As $f(0) = 2 - 0 = 2$, it follows that f is continuous at $x = 2$. Hence, f is continuous on the closed interval $[-1, 1]$.

Now, on the interval $[-1, 0)$, $f(x) = x + 2$ so $f'(x) = 1$. Because $f'(x)$ exists everywhere on the interval $[-1, 0)$ and is never equal to 0, f has no critical numbers in $[-1, 0)$. On the interval $(0, 1]$, $f(x) = 2 - x$, so $f'(x) = -1$. Because $f'(x)$ exists everywhere on the interval $(0, 1]$ and is never equal to 0, f has no critical numbers in $(0, 1]$. At $x = 0$, the rule for f changes, so it is necessary to investigate the existence of $f'(0)$. Now,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{(x + 2) - 2}{x} = \lim_{x \rightarrow 0^-} \frac{x}{x} = \lim_{x \rightarrow 0^-} 1 = 1,$$

and

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{(2 - x) - 2}{x} = \lim_{x \rightarrow 0^+} \frac{-x}{x} = \lim_{x \rightarrow 0^+} -1 = -1.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 0$. Therefore, 0 is the only critical number of f . Evaluating f at the endpoints of the interval $[-1, 1]$ and the critical number 0 yields

$$\begin{aligned} f(-1) &= -1 + 2 = 1 \\ f(0) &= 2 - 0 = 2 \\ f(1) &= 2 - 1 = 1. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is $\boxed{2}$ (and this occurs at the critical number $x = 0$), while the absolute minimum value of f is $\boxed{1}$ (and this occurs at both endpoints $x = \pm 1$).

Applications and Extensions

65. Let $f(x) = 3x^4 - 2x^3 - 21x^2 + 36x$.

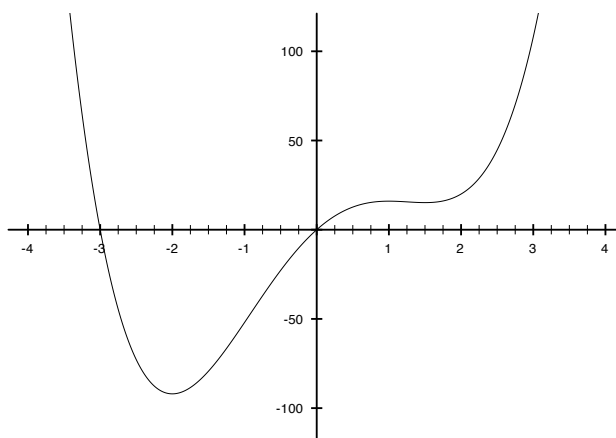
(a) Then $\boxed{f'(x) = 12x^3 - 6x^2 - 42x + 36}$.

(b) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Using the computer algebra system *Maple*,

$$12x^3 - 6x^2 - 42x + 36 = 0$$

when $x = -2$, $x = 1$, and $x = \frac{3}{2}$. Therefore, $\boxed{-2, 1, \text{ and } \frac{3}{2}}$ are the critical numbers of f .

(c) The figure below displays the graph of f . The graph suggests that f has an $\boxed{\text{absolute minimum at } -2}$, a $\boxed{\text{local maximum at } 1}$, and a $\boxed{\text{local minimum at } \frac{3}{2}}$.



66. Let $f(x) = x^2 + 2x - \frac{2}{x}$.

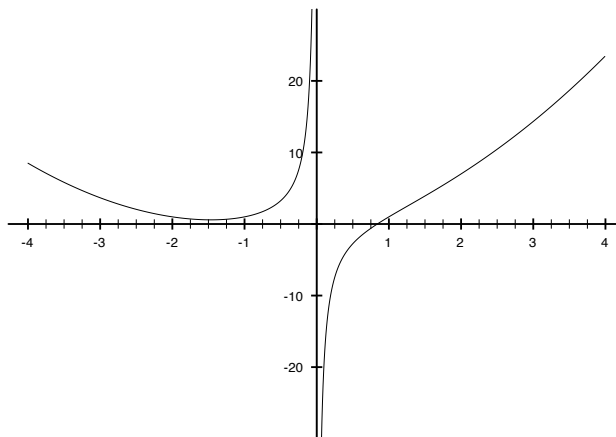
(a) Then $\boxed{f'(x) = 2x + 2 + \frac{2}{x^2}}$.

(b) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ does not exist at $x = 0$; however, $x = 0$ is not in the domain of f , so 0 is not a critical number. Using the computer algebra system *Maple*,

$$2x + 2 + \frac{2}{x^2} = 0$$

when $x \approx -1.466$. Therefore, $\boxed{-1.466}$ is the only critical number of f .

(c) The figure below displays the graph of f . The graph suggests that f has a $\boxed{\text{local minimum at } -1.466}$.



67. Let $f(x) = \frac{(x^2 - 5x + 2)\sqrt{x + 5}}{\sqrt{x^2 + 2}}$.

(a) Then

$$\ln f(x) = \ln \left(\frac{(x^2 - 5x + 2)\sqrt{x + 5}}{\sqrt{x^2 + 2}} \right) = \ln(x^2 - 5x + 2) + \frac{1}{2} \ln(x + 5) - \frac{1}{2} \ln(x^2 + 2)$$

and

$$\begin{aligned} \frac{d}{dx} \ln f(x) &= \frac{d}{dx} \left(\ln(x^2 - 5x + 2) + \frac{1}{2} \ln(x + 5) - \frac{1}{2} \ln(x^2 + 2) \right) \\ \frac{1}{f(x)} \frac{df}{dx} &= \frac{1}{x^2 - 5x + 2} \cdot (2x - 5) + \frac{1}{2} \cdot \frac{1}{x + 5} - \frac{1}{2} \cdot \frac{1}{x^2 + 2} \cdot 2x \\ &= \frac{2x - 5}{x^2 - 5x + 2} + \frac{1}{2x + 10} - \frac{x}{x^2 + 2} \\ &= \frac{(2x - 5)(2x + 10)(x^2 + 2) + (x^2 - 5x + 2)(x^2 + 2) - x(x^2 - 5x + 2)(2x + 10)}{2(x^2 - 5x + 2)(x + 5)(x^2 + 2)} \\ &= \frac{(4x^4 + 10x^3 - 42x^2 + 20x - 100) + (x^4 - 5x^3 + 4x^2 - 10x + 4) - (2x^4 - 46x^2 + 20x)}{2(x^2 - 5x + 2)(x + 5)(x^2 + 2)} \\ &= \frac{3x^4 + 5x^3 + 8x^2 - 10x - 96}{2(x^2 - 5x + 2)(x + 5)(x^2 + 2)}. \end{aligned}$$

Therefore,

$$f'(x) = \frac{(x^2 - 5x + 2)\sqrt{x + 5}}{\sqrt{x^2 + 2}} \cdot \frac{3x^4 + 5x^3 + 8x^2 - 10x - 96}{2(x^2 - 5x + 2)(x + 5)(x^2 + 2)} = \boxed{\frac{3x^4 + 5x^3 + 8x^2 - 10x - 96}{2\sqrt{x + 5}(x^2 + 2)^{3/2}}}.$$

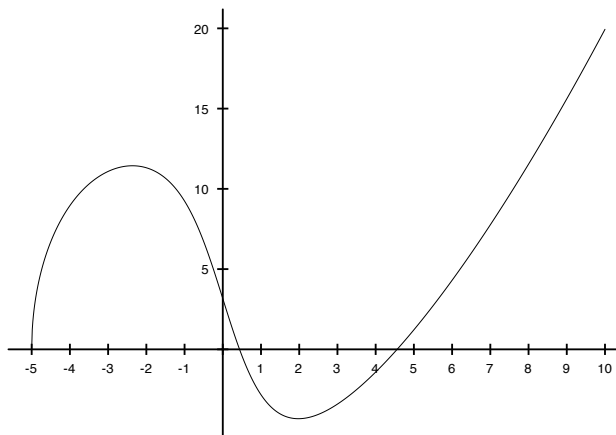
- (b) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ does not exist at $x = -5$. Using the computer algebra system *Maple*,

$$\frac{3x^4 + 5x^3 + 8x^2 - 10x - 96}{2\sqrt{x + 5}(x^2 + 2)^{3/2}} = 0$$

when $x \approx -2.364$ and $x \approx 1.977$. Therefore, $\boxed{-5, -2.364, \text{ and } 1.977}$ are the critical numbers of f .

- (c) The figure below displays the graph of f . The graph suggests that f has an $\boxed{\text{absolute minimum at } 1.977}$, a $\boxed{\text{local maximum at } -2.364}$, and

$\boxed{\text{neither a local maximum nor a local minimum at } -5}$.



68. Let $f(x) = \frac{(x^2 - 9x + 16)\sqrt{x+3}}{\sqrt{x^2 - 4x + 6}}$.

(a) Then

$$\ln f(x) = \ln \left(\frac{(x^2 - 9x + 16)\sqrt{x+3}}{\sqrt{x^2 - 4x + 6}} \right) = \ln(x^2 - 9x + 16) + \frac{1}{2} \ln(x+3) - \frac{1}{2} \ln(x^2 - 4x + 6)$$

and

$$\begin{aligned} \frac{d}{dx} \ln f(x) &= \frac{d}{dx} \left(\ln(x^2 - 9x + 16) + \frac{1}{2} \ln(x+3) - \frac{1}{2} \ln(x^2 - 4x + 6) \right) \\ \frac{1}{f(x)} \frac{df}{dx} &= \frac{1}{x^2 - 9x + 16} \cdot (2x - 9) + \frac{1}{2} \cdot \frac{1}{x+3} - \frac{1}{2} \cdot \frac{1}{x^2 - 4x + 6} \cdot (2x - 4) \\ &= \frac{2x - 9}{x^2 - 9x + 16} + \frac{1}{2x + 6} - \frac{x - 2}{x^2 - 4x + 6} \\ &= \frac{(2x - 9)(2x + 6)(x^2 - 4x + 6) + (x^2 - 9x + 16)(x^2 - 4x + 6) - (x - 2)(x^2 - 9x + 16)(2x + 6)}{2(x^2 - 9x + 16)(x + 3)(x^2 - 4x + 6)} \\ &= \frac{3x^4 - 19x^3 + 50x^2 - 78x - 36}{2(x^2 - 9x + 16)(x + 3)(x^2 - 4x + 6)}. \end{aligned}$$

Therefore,

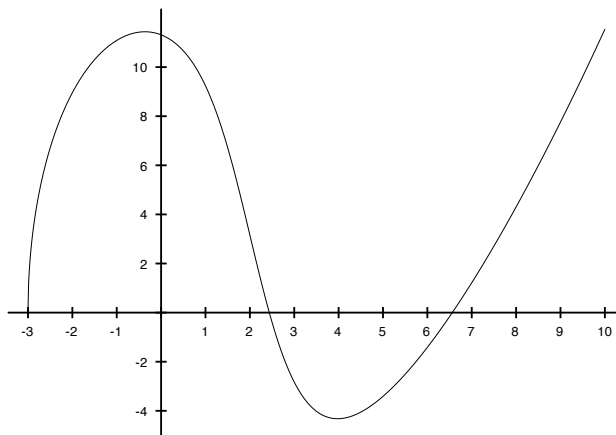
$$f'(x) = \frac{(x^2 - 9x + 16)\sqrt{x+3}}{\sqrt{x^2 - 4x + 6}} \cdot \frac{3x^4 - 19x^3 + 50x^2 - 78x - 36}{2(x^2 - 9x + 16)(x + 3)(x^2 - 4x + 6)} = \boxed{\frac{3x^4 - 19x^3 + 50x^2 - 78x - 36}{2\sqrt{x+3}(x^2 - 4x + 6)^{3/2}}}.$$

(b) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ does not exist at $x = -3$. Using the computer algebra system *Maple*,

$$\frac{3x^4 - 19x^3 + 50x^2 - 78x - 36}{2\sqrt{x+3}(x^2 - 4x + 6)^{3/2}} = 0$$

when $x \approx -0.364$ and $x \approx 3.977$. Therefore, $\boxed{-3, -0.364, \text{ and } 3.977}$ are the critical numbers of f .

(c) The figure below displays the graph of f . The graph suggests that f has a $\boxed{\text{local minimum at } 3.977}$, a $\boxed{\text{local maximum at } -0.364}$, and $\boxed{\text{neither a local maximum nor a local minimum at } -3}$.



69. Let $f(x) = x^4 - 12.4x^3 + 49.24x^2 - 68.64x$.

(a) Then $f'(x) = 4x^3 - 37.2x^2 + 98.48x - 68.64$.

(b) Consider the closed interval $[0, 5]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 5]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is a polynomial function, so it is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Using the computer algebra system *Maple*,

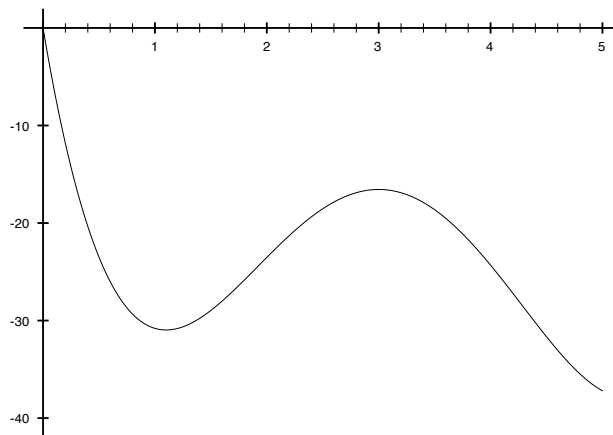
$$4x^3 - 37.2x^2 + 98.48x - 68.64 = 0$$

when $x = 1.1$, when $x = 3$, and when $x = 5.2$. Therefore, 1.1, 3, and 5.2 are the critical numbers of f ; however, $x = 5.2$ is not inside the open interval $(0, 5)$, so this critical number can be excluded. Evaluating f at the endpoints of the interval $[0, 5]$ and at the critical numbers 1.1 and 3 yields

$$\begin{aligned} f(0) &= 0 \\ f(1.1) &= -30.9639 \\ f(3) &= -16.56 \\ f(5) &= -37.2. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 5]$ is 0 (and this occurs at the endpoint $x = 0$), while the absolute minimum value of f is -37.2 (and this occurs at the endpoint $x = 5$).

(c) The figure below displays the graph of f , which does support the conclusions from part (b): the absolute maximum occurs at 0 and the absolute minimum occurs at 5.



70. Let $f(x) = e^{-x} \sin(2x) + e^{-x/2} \cos(2x)$.

(a) Then

$$\begin{aligned} f'(x) &= e^{-x} \cdot 2 \cos(2x) - e^{-x} \sin(2x) - e^{-x/2} \cdot 2 \sin(2x) - \frac{1}{2} e^{-x/2} \cos(2x) \\ &= \left(2e^{-x} - \frac{1}{2} e^{-x/2} \right) \cos(2x) - \left(e^{-x} + 2e^{-x/2} \right) \sin(2x). \end{aligned}$$

(b) Consider the closed interval $[0, 5]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 5]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Using the computer algebra system *Maple*,

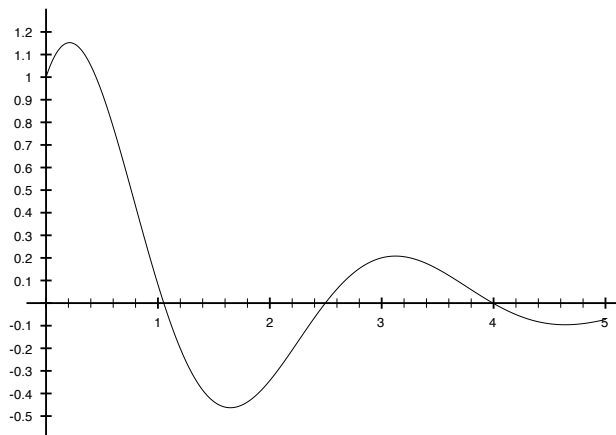
$$\left(2e^{-x} - \frac{1}{2} e^{-x/2} \right) \cos(2x) - \left(e^{-x} + 2e^{-x/2} \right) \sin(2x) = 0$$

on the open interval $(0, 5)$ when $x \approx 0.211$, when $x \approx 1.648$, when $x \approx 3.123$ and when $x \approx 4.641$. Therefore, 0.211, 1.648, 3.123, and 4.641 are the critical numbers of f . Evaluating f at the endpoints of the interval $[0, 5]$ and at the critical numbers 0.211, 1.648, 3.123, and 4.641 yields

$$\begin{aligned} f(0) &= 1 \\ f(0.211) &\approx 1.153 \\ f(1.648) &\approx -0.463 \\ f(3.123) &\approx 0.208 \\ f(4.641) &\approx -0.096 \\ f(5) &\approx -0.073. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 5]$ is approximately 1.153 (and this occurs at the critical number $x \approx 0.211$), while the absolute minimum value of f is approximately -0.463 (and this occurs at the critical number $x \approx 1.648$).

(c) The figure below displays the graph of f , which does support the conclusions from part (b): the absolute maximum occurs at approximately 0.211 and the absolute minimum occurs at approximately 1.648.



71. (a) Let $C(x) = 3.60 \left(\frac{2500}{x} + x \right)$ and consider the interval $[10, 75]$. Because C is continuous on this closed interval, the Extreme Value Theorem guarantees that C has an absolute maximum and an absolute minimum on $[10, 75]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

$$C'(x) = 3.60 \left(-\frac{2500}{x^2} + 1 \right),$$

so $C'(x) = 0$ when $x = \pm 50$ and $C'(x)$ does not exist when $x = 0$. Because $x = 0$ is not in the domain of C , 0 is not a critical number. Therefore, ± 50 are the critical numbers of C ; however, $x = -50$ is not inside the open interval $(10, 75)$, so this number can be excluded. Evaluating C at the endpoints of the interval $[10, 75]$ and at the critical number 50 yields

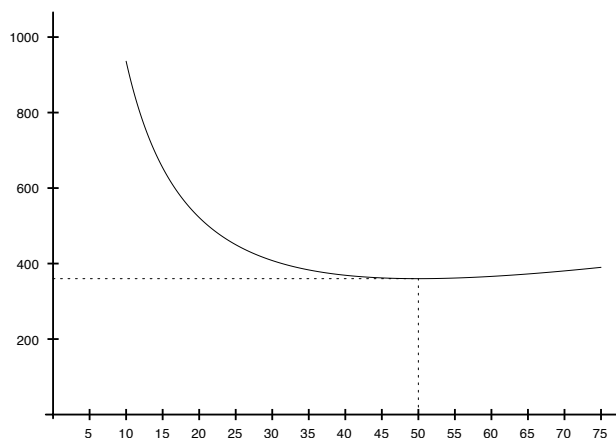
$$C(10) = 3.60 \left(\frac{2500}{10} + 10 \right) = 3.60(260) = 936$$

$$C(50) = 3.60 \left(\frac{2500}{50} + 50 \right) = 3.60(100) = 360$$

$$C(75) = 3.60 \left(\frac{2500}{75} + 75 \right) = 3.60 \cdot \frac{325}{3} = 390.$$

Therefore, the absolute minimum value of C on the interval $[10, 75]$ is \$360 and this occurs at $x = 50$. The most economical speed for the truck to travel is 50 miles per hour.

- (b) The figure below displays the graph of C .



72. Working from the set-up of the previous problem, the time it takes to drive 200 miles at the rate of x miles per hour is $\frac{200}{x}$ hours. If the driver is paid \$28.00 per hour, then the total cost of the trip is

$$C(x) = 3.60 \left(\frac{2500}{x} + x \right) + 28 \cdot \frac{200}{x} = \frac{14600}{x} + 3.60x.$$

The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

$$C'(x) = -\frac{14600}{x^2} + 3.6,$$

so $C'(x) = 0$ when $x \approx \pm 63.68$ and $C'(x)$ does not exist when $x = 0$. Because $x = 0$ is not in the domain of C , 0 is not a critical number. Therefore, ± 63.68 are the critical numbers of C ; however, $x = -63.68$ is not inside the interval $(10, 75)$, so this number can be excluded. Evaluating C at the endpoints of the interval $[10, 75]$ and at the critical number 63.68 yields

$$\begin{aligned} C(10) &= \frac{14600}{10} + 3.60(10) = 1496 \\ C(63.68) &= \frac{14600}{63.68} + 3.60(63.68) \approx 458.52 \\ C(75) &= \frac{14600}{75} + 3.60(75) \approx 464.67. \end{aligned}$$

Therefore, the absolute minimum value of C on the interval $[10, 75]$ is approximately \$458.52 and this occurs at $x \approx 63.68$. The most economical speed for the truck to travel is approximately 63.68 miles per hour.

73. Let $R(\theta) = \frac{v_0^2 \sqrt{2}}{16} \cos \theta (\sin \theta - \cos \theta)$.

- (a) Consider the closed interval $[45^\circ, 90^\circ]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[45^\circ, 90^\circ]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, R is differentiable everywhere, which means that the critical numbers of R occur where $R'(\theta) = 0$. Moreover,

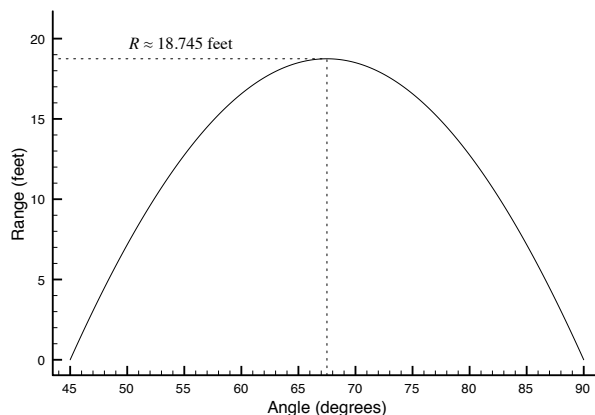
$$\begin{aligned} R'(\theta) &= \frac{v_0^2 \sqrt{2}}{16} [\cos \theta (\cos \theta + \sin \theta) - (\sin \theta - \cos \theta) \sin \theta] \\ &= \frac{v_0^2 \sqrt{2}}{16} (\cos^2 \theta - \sin^2 \theta + 2 \sin \theta \cos \theta) \\ &= \frac{v_0^2 \sqrt{2}}{16} (\cos 2\theta + \sin 2\theta). \end{aligned}$$

Therefore, $R'(\theta) = 0$ when $\cos 2\theta + \sin 2\theta = 0$. On the interval $45^\circ \leq \theta \leq 90^\circ$, $\cos 2\theta + \sin 2\theta = 0$ when $\theta = 67.5^\circ$. Evaluating f at the endpoints of the interval $[45^\circ, 90^\circ]$ and the critical number 67.5° yields

$$\begin{aligned} R(45^\circ) &= \frac{v_0^2 \sqrt{2}}{16} \cos 45^\circ (\sin 45^\circ - \cos 45^\circ) = 0 \\ R(67.5^\circ) &= \frac{v_0^2 \sqrt{2}}{16} \cos 67.5^\circ (\sin 67.5^\circ - \cos 67.5^\circ) \approx 0.0183v_0^2 \\ R(90^\circ) &= \frac{v_0^2 \sqrt{2}}{16} \cos 90^\circ (\sin 90^\circ - \cos 90^\circ) = 0. \end{aligned}$$

Therefore, the absolute maximum value of R on the interval $[45^\circ, 90^\circ]$ occurs when $\theta = 67.5^\circ$. The maximum value of R is approximately $0.0183v_0^2$.

- (b) The figure below displays the graph of R using $v_0 = 32$ ft/s. The dashed vertical line is drawn at $\theta = 67.5^\circ$.



- (c) Based on the graph in part (b), the angle that maximizes R is $\theta = 67.5^\circ$, and the maximum range is approximately

$$0.0183(32)^2 \approx 18.745 \text{ feet.}$$

74. Let $y = 10 \cosh \frac{x}{10} + 15$ and consider the interval $[-10, 10]$. Because y is continuous on this closed interval, the Extreme Value Theorem guarantees that y has an absolute maximum and an absolute minimum on $[-10, 10]$. The absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, y is differentiable everywhere, which means that the critical numbers of y occur where $y'(x) = 0$. Moreover,

$$y'(x) = \sinh \frac{x}{10} = 0$$

when $x = 0$. Evaluating y at the endpoints of the interval $[-10, 10]$ and the critical number 0 yields

$$y(-10) = 10 \cosh(-1) + 15 \approx 30.431$$

$$y(0) = 10 \cosh 0 + 15 = 25$$

$$y(10) = 10 \cosh 1 + 15 \approx 30.431.$$

Therefore, the absolute minimum value of y on the interval $[-10, 10]$ is 25; that is, the height of the cable at its lowest point is 25 m.

75. (a) Let $R = \frac{2v_0^2}{g} \sin \theta \cos \theta = \frac{v_0^2}{g} \sin 2\theta$, and consider the interval $[0^\circ, 90^\circ]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[0^\circ, 90^\circ]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, R is differentiable everywhere, which means that the critical numbers of R occur where $R'(\theta) = 0$. Moreover,

$$R'(\theta) = \frac{2v_0^2}{g} \cos 2\theta = 0$$

when $\cos 2\theta = 0$. On the interval $[0^\circ, 90^\circ]$, $\cos 2\theta = 0$ when $\theta = 45^\circ$. Evaluating f at the endpoints of the interval $[0^\circ, 90^\circ]$ and the critical number 45° yields

$$R(0^\circ) = \frac{v_0^2}{g} \sin 0^\circ = 0$$

$$R(45^\circ) = \frac{v_0^2}{g} \sin 90^\circ = \frac{v_0^2}{g}$$

$$R(90^\circ) = \frac{v_0^2}{g} \sin 180^\circ = 0 = 0.$$

Therefore, the absolute maximum value of R is achieved when $\theta = 45^\circ$; that is, the golf ball achieves its maximum range when the golfer hits the ball at an angle of 45° .

- (b) With $v_0 = 91.1$ m/s and $g = 9.8$ m/s², the maximum range is

$$\frac{v_0^2}{g} = \frac{91.1^2}{9.8} \approx \boxed{846.858 \text{ m}}.$$

76. (a) Recall that

$$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1,$$

so

$$\lim_{\alpha \rightarrow 0} I = I_0 \lim_{\alpha \rightarrow 0} \left(\frac{\sin \alpha}{\alpha} \right)^2 = I_0 \left(\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} \right)^2 = I_0(1)^2 = \boxed{I_0}.$$

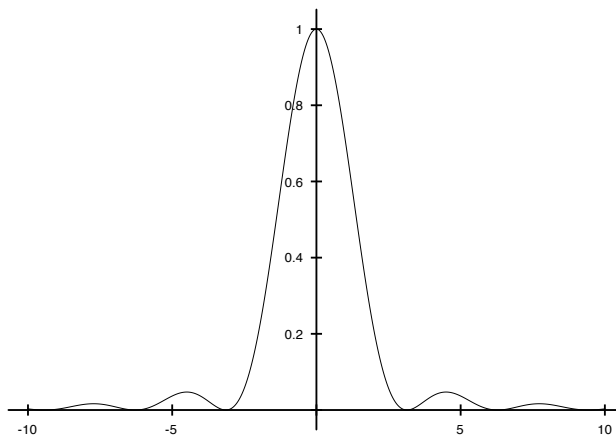
- (b) Differentiating the intensity formula with respect to α yields

$$I'(\alpha) = 2I_0 \frac{\sin \alpha}{\alpha} \cdot \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2} = 2I_0 \frac{\sin \alpha (\alpha \cos \alpha - \sin \alpha)}{\alpha^3}.$$

Though $I'(\alpha)$ does not exist when $\alpha = 0$, 0 is not in the domain of I , so 0 is not a critical number. Now, $I'(\alpha) = 0$ when either $\sin \alpha = 0$ (but $\alpha \neq 0$) or $\alpha \cos \alpha - \sin \alpha = 0$. When $\sin \alpha = 0$ (but $\alpha \neq 0$), it follows that I is also equal to 0 and because $I \geq 0$, these critical numbers correspond to absolute and local minima. On the other hand, when

$$\alpha \cos \alpha - \sin \alpha = 0 \quad \text{or equivalently} \quad \tan \alpha = \alpha,$$

I could have a local maximum or neither a maximum nor a minimum. As the graph of I shows (see below with $I_0 = 1$), every location where I has a horizontal tangent line corresponds to either a local maximum or a local minimum of I . The critical numbers corresponding to $\tan \alpha = \alpha$ must therefore be local maxima.



- (c) Based on the graph of the intensity in part (b), the intensity of light is not the same at each bright spot, but rather decreases as the angle α increases.

77. With $t = \sqrt{27 - 3q^2}$, the tax revenue R is given by

$$R = tq = q\sqrt{27 - 3q^2}.$$

The domain of this function is the set $\{q \mid -3 \leq q \leq 3\}$; however, from an economics standpoint, the quantity consumed must be greater than or equal to 0, so consider the function R over the interval $[0, 3]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[0, 3]$. The absolute maximum can only occur

at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of R occur where $R'(q) = 0$ or where $R'(q)$ does not exist. Moreover,

$$\begin{aligned} R'(q) &= q \cdot \frac{1}{2}(27 - 3q^2)^{-1/2}(-6q) + \sqrt{27 - 3q^2} \\ &= -\frac{3q^2}{\sqrt{27 - 3q^2}} + \sqrt{27 - 3q^2} = \frac{-3q^2 + 27 - 3q^2}{\sqrt{27 - 3q^2}} = \frac{27 - 6q^2}{\sqrt{27 - 3q^2}}, \end{aligned}$$

so $R'(q) = 0$ when $27 - 6q^2 = 0$, or when $q = \pm \frac{3\sqrt{2}}{2}$, and $R'(q)$ does not exist when $27 - 3q^2 = 0$, or when $q = \pm 3$. Of these numbers, only $\frac{3\sqrt{2}}{2}$ lies inside the open interval $(0, 3)$. Evaluating R at the endpoints of the interval $[0, 3]$ and the critical number $\frac{3\sqrt{2}}{2}$ yields

$$\begin{aligned} R(0) &= 0\sqrt{27} = 0 \\ R\left(\frac{3\sqrt{2}}{2}\right) &= \frac{3\sqrt{2}}{2} \sqrt{27 - 3\left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{9\sqrt{3}}{2} \approx 7.794 \\ R(3) &= 3\sqrt{27 - 3(3)^2} = 0. \end{aligned}$$

Therefore, the absolute maximum value of R on the interval $[0, 3]$ is $\frac{9\sqrt{3}}{2}$ (and this occurs at the critical number $q = \frac{3\sqrt{2}}{2}$). Substituting this value for q into the formula for t gives

$$t = \sqrt{27 - 3\left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{3\sqrt{6}}{2} \approx 3.674.$$

The maximum tax revenue is approximately 7.794, and this is produced with a tax rate of approximately 3.674.

78. With $t = 18 - 3q^2$, the tax revenue R is given by

$$R = tq = q(18 - 3q^2) = 18q - 3q^3.$$

The domain of this function is the set of all real numbers; however, from an economics standpoint, the quantity consumed must be greater than or equal to 0 and the tax rate must also be greater than or equal to 0, so consider the function R over the interval $[0, \sqrt{6}]$. Because R is continuous on this closed interval, the Extreme Value Theorem guarantees that R has an absolute maximum and an absolute minimum on $[0, \sqrt{6}]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, R is differentiable everywhere, which means that the critical numbers of R occur where $R'(\theta) = 0$. Moreover,

$$R'(q) = 18 - 9q^2 = 9(2 - q^2),$$

so $R'(q) = 0$ when $2 - q^2 = 0$, or when $q = \pm\sqrt{2}$. As $q = -\sqrt{2}$ is not inside the open interval $(0, \sqrt{6})$, this number can be excluded. This leaves $\sqrt{2}$ as the only critical number. Evaluating R at the endpoints of the interval $[0, \sqrt{6}]$ and the critical number $\sqrt{2}$ yields

$$\begin{aligned} R(0) &= 0 \\ R(\sqrt{2}) &= \sqrt{2}(12) = 12\sqrt{2} \approx 16.971 \\ R(\sqrt{6}) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of R on the interval $[0, \sqrt{6}]$ is $12\sqrt{2}$ (and this occurs at the critical number $q = \sqrt{2}$). Substituting this value for q into the formula for t gives

$$t = 18 - 3(\sqrt{2})^2 = 12.$$

The maximum tax revenue is approximately 16.971, and this is produced with a tax rate of 12.

79. Let $y = 15 \cosh \frac{x}{15} - 10$ and consider the interval $[-6, 6]$, thus assuming that the x -axis is placed along the ground with the two poles at $(-6, 0)$ and $(6, 0)$. Because y is continuous on this closed interval, the Extreme Value Theorem guarantees that y has an absolute maximum and an absolute minimum on $[-6, 6]$. The absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, y is differentiable everywhere, which means that the critical numbers of y occur where $y'(x) = 0$. Moreover,

$$y'(x) = \sinh \frac{x}{15} = 0$$

when $x = 0$. Evaluating y at the endpoints of the interval $[-6, 6]$ and the critical number 0 yields

$$\begin{aligned} y(-6) &= 15 \cosh \left(-\frac{6}{15} \right) - 10 \approx 6.216 \\ y(0) &= 15 \cosh 0 - 10 = 5 \\ y(6) &= 15 \cosh \left(\frac{6}{15} \right) - 10 \approx 6.216. \end{aligned}$$

Therefore, the absolute minimum value of y on the interval $[-6, 6]$ is 5; that is, the height of the cable at its lowest point is 5 m.

80. Let the amplitude of the harmonic motion be $A = 0.24$ m and the period be 4 s, so that

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2}.$$

- (a) The position s is then given by

$$s(t) = 0.24 \cos \left(\frac{\pi}{2} t \right).$$

When $t = 0.5$ s,

$$s(0.5) = 0.24 \cos \left(\frac{\pi}{2} \cdot 0.5 \right) = 0.24 \cos \left(\frac{\pi}{4} \right) = \boxed{0.12\sqrt{2} \text{ m} \approx 0.170 \text{ m}}.$$

- (b) The velocity of the object is

$$v(t) = s'(t) = \boxed{-0.12\pi \sin \left(\frac{\pi}{2} t \right)}.$$

- (c) When $t = 0.5$ s,

$$v(0.5) = -0.12\pi \sin \left(\frac{\pi}{2} \cdot 0.5 \right) = -0.12\pi \sin \left(\frac{\pi}{4} \right) = \boxed{-0.06\pi\sqrt{2} \text{ m/s} \approx -0.267 \text{ m/s}}.$$

- (d) The acceleration of the object is

$$a(t) = v'(t) = \boxed{-0.06\pi^2 \cos \left(\frac{\pi}{2} t \right)}.$$

(e) When $t = 0.5$ s,

$$a(0.5) = -0.06\pi^2 \cos\left(\frac{\pi}{2} \cdot 0.5\right) = -0.06\pi^2 \cos\left(\frac{\pi}{4}\right) = -0.03\pi^2\sqrt{2} \approx -0.419 \text{ m/s}^2.$$

With $m = 1$ kg, $F = ma \approx -0.419$ N. The magnitude of the force is approximately $\boxed{0.419 \text{ N}}$, and the direction of the force is $\boxed{\text{toward the equilibrium position}}$ of $s = 0$.

(f) The position of the object will be -0.12 m when

$$-0.12 = s(t) = 0.24 \cos\left(\frac{\pi}{2}t\right) \quad \text{or when} \quad \cos\left(\frac{\pi}{2}t\right) = -\frac{1}{2}.$$

Starting from $\theta = 0$, $\cos \theta$ first reaches the value $-\frac{1}{2}$ when $\theta = \frac{2\pi}{3}$. Therefore,

$$\cos\left(\frac{\pi}{2}t\right)$$

first reaches the value $-\frac{1}{2}$ when

$$\frac{\pi}{2}t = \frac{2\pi}{3} \quad \text{or when} \quad \boxed{t = \frac{4}{3} \text{ seconds}}.$$

(g) When $t = 4/3$ s,

$$v\left(\frac{4}{3}\right) = -0.12\pi \sin\left(\frac{\pi}{2} \cdot \frac{4}{3}\right) = -0.12\pi \sin\left(\frac{2\pi}{3}\right) = \boxed{-0.06\pi\sqrt{3} \text{ m/s} \approx -0.326 \text{ m/s}}.$$

81. (a) Let $f(x) = \frac{x^3}{3} - 0.055x^2 + 0.0028x - 4$. Because f is a polynomial function, it is differentiable everywhere. Therefore, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = x^2 - 0.11x + 0.0028 = 0$$

when

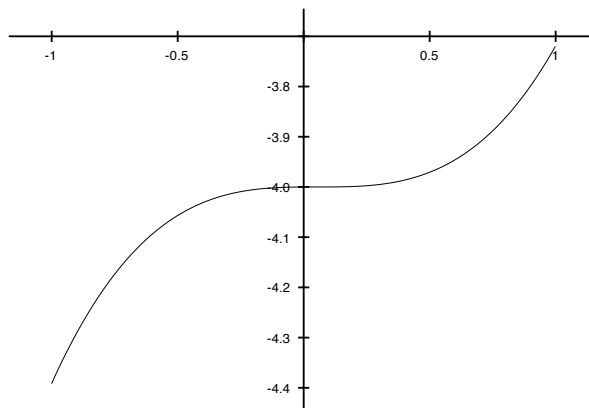
$$x = \frac{0.11 \pm \sqrt{0.11^2 - 4(0.0028)}}{2} = \frac{0.11 \pm \sqrt{0.0009}}{2} = \frac{0.11 \pm 0.03}{2} = \boxed{0.07, 0.04}.$$

- (b) Evaluating f at the endpoints of the closed interval $[-1, 1]$ and at the critical numbers 0.04 and 0.07 yields

$$\begin{aligned} f(-1) &\approx -4.391133333 \\ f(0.04) &\approx -3.999954667 \\ f(0.07) &\approx -3.999959167 \\ f(1) &\approx -3.718866667. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[-1, 1]$ is approximately $\boxed{-3.719}$ (which occurs at the endpoint $x = 1$), and the absolute minimum value is approximately $\boxed{-4.391}$ (which occurs at the endpoint $x = -1$).

- (c) The figure below displays the graph of f on the interval $[-1, 1]$.



82. (a) Let $y = x - \cosh^{-1} x$, and note the domain of y is the set $\{x|x \geq 1\}$. Now,

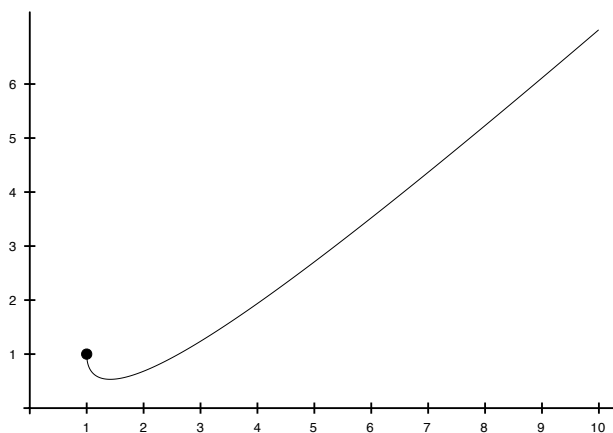
$$y' = 1 - \frac{1}{\sqrt{x^2 - 1}},$$

so, on the domain of y , $y'(x) = 0$ when $\sqrt{x^2 - 1} = 1$ or $x = \sqrt{2}$, and $y'(x)$ does not exist when $x = 1$. Because

$$\begin{aligned} y(1) &= 1 - \cosh^{-1} 1 = 1 - 0 = 1 \\ y(\sqrt{2}) &= \sqrt{2} - \cosh^{-1} \sqrt{2} \approx 0.533 \\ \lim_{x \rightarrow \infty} y(x) &= \infty, \end{aligned}$$

it follows that the minimum value of $y = x - \cosh^{-1} x$ is $\boxed{\sqrt{2} - \cosh^{-1} \sqrt{2}}$.

- (b) The figure below displays the graph of $y = x - \cosh^{-1} x$.



83. Let $f(x) = \sqrt{1+x^2} + |x-2|$, and consider the interval $[0, 3]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 3]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. On the interval $[0, 2)$, $x-2 < 0$, so $f(x) = \sqrt{1+x^2} - x + 2$ and

$$f'(x) = \frac{1}{2}(1+x^2)^{-1/2}(2x) - 1 = \frac{x}{\sqrt{1+x^2}} - 1.$$

This derivative exists and is never equal to zero for $0 \leq x < 2$, so there are no critical numbers on this interval. On the interval $(2, 3]$, $x - 2 > 0$, so $f(x) = \sqrt{1+x^2} + x - 2$ and

$$f'(x) = \frac{1}{2}(1+x^2)^{-1/2}(2x) + 1 = \frac{x}{\sqrt{1+x^2}} + 1.$$

This derivative exists and is never equal to zero for $2 < x \leq 3$, so there are no critical numbers on this interval. At $x = 2$, $f'(x)$ does not exist because of the $|x - 2|$ term, so 2 is a critical number of f . Evaluating f at the endpoints of the interval $[0, 3]$ and at the critical number 2 yields

$$\begin{aligned} f(0) &= \sqrt{1} + |0 - 2| = 1 + 2 = 3 \\ f(2) &= \sqrt{1+4} + 0 = \sqrt{5} \\ f(3) &= \sqrt{1+9} + |3 - 2| = \sqrt{10} + 1. \end{aligned}$$

Therefore, the absolute maximum value of the function f on the interval $[0, 3]$ is $\boxed{\sqrt{10} + 1}$ (and this occurs at the endpoint $\boxed{x = 3}$), while the absolute minimum value is $\boxed{\sqrt{5}}$ (and this occurs at the critical point $\boxed{x = 2}$).

84. Let $f(x) = Ax^2 + Bx + C$. Because the graph of f contains the points $(0, 2)$ and $(1, 8)$, it follows that $f(0) = 2$ and $f(1) = 8$. Now, $f(0) = C$, so $C = 2$, and $f(1) = A + B + C$, so $A + B + C = 8$ or $A + B = 6$. If f has a local minimum at 0, then 0 must be a critical number of f ; since the polynomial function f is differentiable everywhere, it follows that $f'(0) = 0$. Now, $f'(x) = 2Ax + B$, so $f'(0) = B$. Therefore, $B = 0$, and finally, $A = 6$. In summary, $\boxed{A = 6, B = 0, \text{ and } C = 2}$.

85. Let $f(x) = [(16 - x^2)(x^2 - 9)]^{1/2}$.

- (a) The domain of f is determined by the inequality $(16 - x^2)(x^2 - 9) \geq 0$. Now, $16 - x^2$ is positive for $-4 < x < 4$ and is negative for $|x| > 4$; on the other hand, $x^2 - 9$ is positive for $|x| > 3$ and is negative for $-3 < x < 3$. It follows that $16 - x^2$ and $x^2 - 9$ have the same sign for $-4 < x < -3$ and $3 < x < 4$. Additionally, $(16 - x^2)(x^2 - 9) = 0$ for $x = \pm 3$ and $x = \pm 4$. Therefore, the domain of f is the set

$$\boxed{\{x \mid -4 \leq x \leq -3 \text{ or } 3 \leq x \leq 4\}},$$

or in interval notation $\boxed{[-4, -3] \cup [3, 4]}$.

- (b) Note that

$$f(-x) = [(16 - (-x)^2)((-x)^2 - 9)]^{1/2} = [(16 - x^2)(x^2 - 9)]^{1/2} = f(x),$$

so f is an even function. To determine the absolute maximum value of f on its domain, it is therefore sufficient to consider the closed interval $[3, 4]$. The absolute maximum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$\begin{aligned} f'(x) &= \frac{1}{2}[(16 - x^2)(x^2 - 9)]^{-1/2}[(16 - x^2)(2x) + (x^2 - 9)(-2x)] \\ &= \frac{1}{[(16 - x^2)(x^2 - 9)]^{1/2}} \cdot (16x - x^3 - x^3 + 9x) \\ &= \frac{x(25 - 2x^2)}{[(16 - x^2)(x^2 - 9)]^{1/2}}, \end{aligned}$$

so $f'(x) = 0$ when $25 - 2x^2 = 0$ or $x = \pm \frac{5\sqrt{2}}{2}$ and $f'(x)$ does not exist when $x = \pm 3$ and when $x = \pm 4$. Of these numbers, only $\frac{5\sqrt{2}}{2}$ is inside the open interval $(3, 4)$. Evaluating f at the

endpoints of the closed interval $[3, 4]$ and the critical number $\frac{5\sqrt{2}}{2}$ yields

$$\begin{aligned} f(3) &= 0 \\ f\left(\frac{5\sqrt{2}}{2}\right) &= \sqrt{\left(16 - \frac{25}{2}\right)\left(\frac{25}{2} - 9\right)} = \frac{7}{2} \\ f(4) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[3, 4]$ is $\frac{7}{2}$. Because f is an even function, it follows that the absolute maximum value on the interval $[-4, -3]$ is also $\frac{7}{2}$, so the absolute maximum value of f on its domain is $\boxed{\frac{7}{2}}$.

86. Let $f(x) = \sqrt{x(2-x)}$. Because $x(2-x) \geq 0$ for $0 \leq x \leq 2$, the domain of f is the set $\{x | 0 \leq x \leq 2\}$. On this closed interval, f is continuous, so the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 2]$. Without calculus, note that $f(x) = \sqrt{x(2-x)} \geq 0$ on its domain and $f(0) = f(2) = 0$. It follows that 0 is the absolute minimum value of f . As for the absolute maximum value, note that the graph of $y = x(2-x) = 2x - x^2$ is a downward opening parabola with its vertex at $(1, 1)$; therefore, $2x - x^2 \leq 1$, so that $\sqrt{x(2-x)} \leq 1$, and 1 is the absolute maximum value of f . Next, using calculus, the absolute maximum and absolute minimum can only occur at the endpoints of the interval $[0, 2]$ or at the critical numbers inside the open interval. Now, the critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Moreover,

$$f'(x) = \frac{1}{2}[x(2-x)]^{-1/2}(2-2x) = \frac{1-x}{\sqrt{x(2-x)}},$$

so $f'(x) = 0$ when $x = 1$ and $f'(x)$ does not exist when $x = 0$ and $x = 2$. Of these numbers, only $x = 1$ is inside the open interval $(0, 2)$. Evaluating f at the endpoints of the closed interval $[0, 2]$ and the critical number 1 yields

$$\begin{aligned} f(0) &= 0 \\ f(1) &= \sqrt{1(1)} = 1 \\ f(2) &= 0. \end{aligned}$$

Therefore, the absolute maximum value of f on its domain is $\boxed{1}$ (and this occurs at the critical number $x = 1$), while the absolute minimum value is $\boxed{0}$ (and this occurs at both endpoints $x = 0$ and $x = 2$).

87. (a) $\boxed{\text{Not necessarily true}}$. Just because a function is continuous on a closed interval $[a, b]$ is no guarantee that the function is differentiable on (a, b) .
- (b) $\boxed{\text{Not necessarily true}}$. An absolute maximum can occur at an endpoint of the interval $[a, b]$; even if the absolute maximum occurs at a critical number, the derivative might not exist at that critical number.
- (c) $\boxed{\text{True}}$. This is the definition of continuity on the open interval (a, b) .
- (d) $\boxed{\text{Not necessarily true}}$. Just because a function is continuous on a closed interval $[a, b]$ is no guarantee that the function is differentiable on (a, b) ; even if the function is differentiable, the derivative does not need to equal 0 for some x , $a \leq x \leq b$.
- (e) $\boxed{\text{True}}$. This follows from the Extreme Value Theorem.
88. Let f be a function defined on an interval I . An extreme value of f , whether absolute or local, is a value of f that is larger than or equal to in the case of a maximum or smaller than or equal to in

the case of a minimum other values of f . To be an absolute extreme value, the value of f must be the largest or smallest value taken on by f throughout the entire interval I . In contrast, to be a local extreme value, the value of f only needs to be the largest or smallest value taken on by f in some open interval contained inside I .

89. The absolute extreme values of a continuous function defined on a closed interval $[a, b]$ can only occur at the endpoints of the interval or at the critical numbers inside the open interval (a, b) . Therefore, first determine the critical numbers of the function: determine where inside the open interval that the derivative is equal to zero and where the derivative does not exist. Next, evaluate the function at the endpoints of the interval and at the critical numbers inside the interval. Finally, identify the absolute maximum value (the largest of the function values obtained in the previous step) and the absolute minimum value (the smallest of the function values obtained in the previous step).
90. By definition, a local maximum for a function f occurs at c when $f(c) \geq f(x)$ for all x in some open interval containing c . Similarly, f has a local minimum at c when $f(c) \leq f(x)$ for all x in some open interval containing c . In each case, it is required that f be defined in an open interval containing c . Here, the function is defined on the closed interval $[a, b]$, so there is no open interval containing a on which f is defined. It is therefore impossible for $f(a)$ to be a local extreme value of f – though it is still possible for $f(a)$ to be an absolute extreme value of f .
91. Suppose f has a local minimum at c . Then, for all x in an open interval containing c , $f(c) \leq f(x)$. Multiplying this inequality by -1 yields $-f(c) \geq -f(x)$. Now, let $g(x) = -f(x)$. Then the last inequality becomes $g(c) \geq g(x)$ for all x in an open interval containing c . Therefore, g has a local maximum at c .
92. Suppose f has a local minimum at c . Then $f(c) \leq f(x)$ for all x in some open interval containing c . Equivalently, $f(x) - f(c) \geq 0$. The derivative of f at c may be written as

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists. If this limit does not exist, then $f'(c)$ does not exist and there is nothing further to prove. If this limit does exist, then the two one-sided limits

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

must exist and be equal to one another. For the limit from the left, $x < c$, so $x - c < 0$,

$$\frac{f(x) - f(c)}{x - c} \leq 0, \quad \text{and} \quad \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0.$$

For the limit from the right, $x > c$, so $x - c > 0$,

$$\frac{f(x) - f(c)}{x - c} \geq 0, \quad \text{and} \quad \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

For the two one-sided limits to be equal, they must equal zero; therefore

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0.$$

Challenge Problems

93. (a) Let a , b , c , and d be real numbers, $n \geq 1$ an integer, and $f(x) = \frac{ax^{2n} + b}{cx^n + d}$. If $a = 0$, then

$$f(x) = \frac{b}{cx^n + d} = b(cx^n + d)^{-1} \quad \text{and} \quad f'(x) = -bcn x^{n-1} (cx^n + d)^{-2}.$$

Now, $f'(x)$ does not exist when $cx^n + d = 0$, but any x that satisfies this equation is not in the domain of f , so it cannot be a critical number. Therefore, the only critical numbers of f occur where $f'(x) = 0$, which is at $x = 0$. So, if $a = 0$, then f has only one critical number.

Next, consider the case $c = 0$ but $d \neq 0$ (c and d cannot be simultaneously equal to zero, as that would cause division by zero for all x). Then

$$f(x) = \frac{a}{d}x^{2n} + \frac{b}{d} \quad \text{and} \quad f'(x) = \frac{2an}{d}x^{2n-1}.$$

Now, $f'(x)$ exists for all x and is equal to zero only at $x = 0$. When $c = 0$ but $d \neq 0$, f has only one critical number.

Finally, suppose $a \neq 0$ and $c \neq 0$. Then

$$\begin{aligned} f'(x) &= \frac{(cx^n + d)(2anx^{2n-1}) - (ax^{2n} + b)(cnx^{n-1})}{(cx^n + d)^2} \\ &= \frac{2acnx^{3n-1} + 2adnx^{2n-1} - acnx^{3n-1} - bcnx^{n-1}}{(cx^n + d)^2} \\ &= \frac{acnx^{n-1} \left(x^{2n} + 2\frac{d}{c}x^n - \frac{b}{a} \right)}{(cx^n + d)^2}. \end{aligned}$$

Now, $f'(x)$ does not exist when $cx^n + d = 0$, but any x that satisfies this equation is not in the domain of f , so it cannot be a critical number. Therefore, the only critical numbers of f occur where $f'(x) = 0$. Clearly, $f'(x) = 0$ when $x = 0$. Any other critical numbers are the zeros of the function

$$g(x) = x^{2n} + 2\frac{d}{c}x^n - \frac{b}{a} = \left(x^n + \frac{d}{c} \right)^2 - \left(\frac{b}{a} + \frac{d^2}{c^2} \right).$$

The number of zeros of g depends on the values of a , b , c , and d :

- i. If $\frac{b}{a} + \frac{d^2}{c^2} < 0$, g has no zeros.
- ii. If $\frac{b}{a} + \frac{d^2}{c^2} = 0$, then g has one zero if n is odd (which will not be 0 provided $d \neq 0$) but 2 zeroes if n is even and $d/c < 0$.
- iii. If $\frac{b}{a} + \frac{d^2}{c^2} > 0$, then g has two zeros if n is odd (neither of which will be 0 provided $b \neq 0$) but can have 0, 2, or 4 zeros if n is even depending on the signs of

$$-\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} \quad \text{and} \quad -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}.$$

It follows that f can have at most five critical numbers.

- (b) Based on part (a), in order for f to have five critical numbers, n must be even and

$$\frac{b}{a} + \frac{d^2}{c^2}, \quad -\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}, \quad \text{and} \quad -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}}$$

must all be positive. Answers will vary, but one choice of parameter values is $n = 2$, $a = 1$, $b = -1$, $c = 1$, and $d = -2$. Note that

$$\begin{aligned} \frac{b}{a} + \frac{d^2}{c^2} &= -1 + 4 = 3 > 0 \\ -\frac{d}{c} + \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} &= 2 + \sqrt{3} > 0 \\ -\frac{d}{c} - \sqrt{\frac{b}{a} + \frac{d^2}{c^2}} &= 2 - \sqrt{3} > 0. \end{aligned}$$

The function

$$f(x) = \frac{x^4 - 1}{x^2 - 2}$$

has exactly five critical numbers:

$$0, \quad \pm\sqrt{2 + \sqrt{3}}, \quad \text{and} \quad \pm\sqrt{2 - \sqrt{3}}.$$

4.3: The Mean Value Theorem

Concepts and Vocabulary

1. False. If a function f is defined and continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b) , and if $f(a) = f(b)$, then Rolle's Theorem guarantees that there is at least one number c in the interval (a, b) for which $f'(c) = 0$. The conclusion of Rolle's Theorem involves the derivative of the function, not the function itself.
2. Answers will vary. Here is one possible response. For a function f that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , the Mean Value Theorem guarantees that there is at least one tangent line to the graph of f on the open interval (a, b) which is parallel to the secant line between the points $(a, f(a))$ and $(b, f(b))$.
3. True. If two functions f and g are differentiable on an open interval (a, b) and if $f'(x) = g'(x)$ for all numbers x in (a, b) , then f and g differ by a constant on (a, b) .
4. False. When the derivative f' is positive on an open interval I , then f is **increasing** on I .

Skill Building

5. Let $f(x) = x^2 - 3x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 3]$ and differentiable on the open interval $(0, 3)$. Additionally,

$$f(0) = 0 \quad \text{and} \quad f(3) = 3^2 - 3(3) = 0,$$

so $f(0) = f(3)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 3]$. Now, $f'(x) = 2x - 3$, so $f'(c) = 0$ when $c = \frac{3}{2}$.

6. Let $f(x) = x^2 + 2x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 0]$ and differentiable on the open interval $(-2, 0)$. Additionally,

$$f(-2) = (-2)^2 + 2(-2) = 0 \quad \text{and} \quad f(0) = 0,$$

so $f(-2) = f(0)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[-2, 0]$. Now, $f'(x) = 2x + 2$, so $f'(c) = 0$ when $c = -1$.

7. Let $g(x) = x^2 - 2x - 2$. The polynomial function g is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$. Additionally,

$$g(0) = -2 \quad \text{and} \quad g(2) = 2^2 - 2(2) - 2 = -2,$$

so $g(0) = g(2)$. The function g therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 2]$. Now, $g'(x) = 2x - 2$, so $g'(c) = 0$ when $c = 1$.

8. Let $g(x) = x^2 + 1$. The polynomial function g is continuous and differentiable everywhere, so it is continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $(-1, 1)$. Additionally,

$$g(-1) = (-1)^2 + 1 = 2 \quad \text{and} \quad g(1) = 1^2 + 1 = 2,$$

so $g(-1) = g(1)$. The function g therefore satisfies all three conditions of Rolle's Theorem on the interval $[-1, 1]$. Now, $g'(x) = 2x$, so $g'(c) = 0$ when $c = 0$.

9. Let $f(x) = x^3 - x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-1, 0]$ and differentiable on the open interval $(-1, 0)$. Additionally,

$$f(-1) = (-1)^3 - (-1) = 0 \quad \text{and} \quad f(0) = 0,$$

so $f(-1) = f(0)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[-1, 0]$. Now, $f'(x) = 3x^2 - 1$, so $f'(c) = 0$ when $c = \pm \frac{\sqrt{3}}{3}$. Of these two numbers, only

$$\boxed{c = -\frac{\sqrt{3}}{3}} \text{ is in the interval } [-1, 0].$$

10. Let $f(x) = x^3 - 4x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. Additionally,

$$f(-2) = (-2)^3 - 4(-2) = 0 \quad \text{and} \quad f(2) = 2^3 - 4(2) = 0,$$

so $f(-2) = f(2)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the

interval $[-2, 2]$. Now, $f'(x) = 3x^2 - 4$, so $f'(c) = 0$ when $\boxed{c = \pm \frac{2\sqrt{3}}{3}}$.

11. Let $f(t) = t^3 - t + 2$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $(-1, 1)$. Additionally,

$$f(-1) = (-1)^3 - (-1) + 2 = 2 \quad \text{and} \quad f(1) = 1^3 - 1 + 2 = 2,$$

so $f(-1) = f(1)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the

interval $[-1, 1]$. Now, $f'(t) = 3t^2 - 1$, so $f'(c) = 0$ when $\boxed{c = \pm \frac{\sqrt{3}}{3}}$.

12. Let $f(t) = t^4 - 3$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. Additionally,

$$f(-2) = (-2)^4 - 3 = 13 \quad \text{and} \quad f(2) = 2^4 - 3 = 13,$$

so $f(-2) = f(2)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[-2, 2]$. Now, $f'(t) = 4t^3$, so $f'(c) = 0$ when $\boxed{c = 0}$.

13. Let $s(t) = t^4 - 2t^2 + 1$. The polynomial function s is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. Additionally,

$$s(-2) = (-2)^4 - 2(-2)^2 + 1 = 9 \quad \text{and} \quad s(2) = 2^4 - 2(2)^2 + 1 = 9,$$

so $s(-2) = s(2)$. The function s therefore satisfies all three conditions of Rolle's Theorem on the interval $[-2, 2]$. Now, $s'(t) = 4t^3 - 4t = 4t(t^2 - 1)$, so $s'(c) = 0$ when $\boxed{c = 0 \text{ or } c = \pm 1}$.

14. Let $s(t) = t^4 + t^2$. The polynomial function s is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. Additionally,

$$s(-2) = (-2)^4 + (-2)^2 = 20 \quad \text{and} \quad s(2) = 2^4 + (2)^2 = 20,$$

so $s(-2) = s(2)$. The function s therefore satisfies all three conditions of Rolle's Theorem on the interval $[-2, 2]$. Now, $s'(t) = 4t^3 + 2t = 2t(2t^2 + 1)$, so $s'(c) = 0$ when $\boxed{c = 0}$.

15. Let $f(x) = \sin(2x)$. The trigonometric function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, \pi]$ and differentiable on the open interval $(0, \pi)$. Additionally,

$$f(0) = \sin 0 = 0 \quad \text{and} \quad f(\pi) = \sin(2\pi) = 0,$$

so $f(0) = f(\pi)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval

$[0, \pi]$. Now, $f'(x) = 2\cos(2x)$, so $f'(c) = 0$ on the interval $(0, \pi)$ when $\boxed{c = \frac{\pi}{4} \text{ or } c = \frac{3\pi}{4}}$.

16. Let $f(x) = \sin x + \cos x$. The function f is the sum of the trigonometric functions $\sin x$ and $\cos x$, both of which are continuous and differentiable everywhere. Therefore, f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2\pi]$ and differentiable on the open interval $(0, 2\pi)$. Additionally,

$$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1 \quad \text{and} \quad f(2\pi) = \sin(2\pi) + \cos(2\pi) = 0 + 1 = 1,$$

so $f(0) = f(2\pi)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 2\pi]$. Now, $f'(x) = \cos x - \sin x$, so $f'(c) = 0$ on the interval $(0, 2\pi)$ when $c = \frac{\pi}{4}$ or $c = \frac{5\pi}{4}$.

17. Let $f(x) = x^2 - 2x + 1$. Though the polynomial function f is continuous on the closed interval $[-2, 1]$ and differentiable on the open interval $(-2, 1)$,

$$f(-2) = (-2)^2 - 2(-2) + 1 = 9 \quad \text{and} \quad f(1) = 1 - 2 + 1 = 0.$$

Therefore, $f(-2) \neq f(1)$, so Rolle's Theorem does not apply for the function f on the interval $[-2, 1]$.

18. Let $f(x) = x^3 - 3x$. Though the polynomial function f is continuous on the closed interval $[2, 4]$ and differentiable on the open interval $(2, 4)$,

$$f(2) = 2^3 - 3(2) = 2 \quad \text{and} \quad f(4) = 4^3 - 3(4) = 52.$$

Therefore, $f(2) \neq f(4)$, so Rolle's Theorem cannot be applied to the function f on the interval $[2, 4]$.

19. Let $f(x) = x^{1/3} - x$. Though f is continuous on the closed interval $[-1, 1]$,

$$f'(x) = \frac{1}{3}x^{-2/3} - 1,$$

so f is not differentiable at 0 and therefore is not differentiable on the open interval $(-1, 1)$. Consequently, Rolle's Theorem cannot be applied to the function f on the interval $[-1, 1]$.

20. Let $f(x) = x^{2/5}$. Though f is continuous on the closed interval $[-1, 1]$,

$$f'(x) = \frac{2}{5}x^{-3/5},$$

so f is not differentiable at 0 and therefore is not differentiable on the open interval $(-1, 1)$. Consequently, Rolle's Theorem cannot be applied to the function f on the interval $[-1, 1]$.

21. Let $f(x) = x^2 + 1$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[0, 2]$. Now,

$$f'(x) = 2x \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0} = \frac{5 - 1}{2} = 2,$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 2c = 2,$$

or when $c = 1$.

22. Let $f(x) = x + 2 + \frac{3}{x-1}$. The function f is continuous and differentiable on the set $\{x | x \neq 1\}$, so it is continuous on the closed interval $[2, 7]$ and differentiable on the open interval $(2, 7)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[2, 7]$. Now,

$$f'(x) = 1 - \frac{3}{(x-1)^2} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(7) - f(2)}{7 - 2} = \frac{\frac{19}{2} - 7}{5} = \frac{1}{2},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 1 - \frac{3}{(c-1)^2} = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \frac{3}{(c-1)^2} &= \frac{1}{2} \\ (c-1)^2 &= 6 \\ c &= 1 \pm \sqrt{6}. \end{aligned}$$

Of these numbers, only $\boxed{c = 1 + \sqrt{6}}$ is in the interval $(2, 7)$.

23. Let $f(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$. The function f is continuous and differentiable on the set $\{x|x > 0\}$, so it is continuous on the closed interval $[1, e]$ and differentiable on the open interval $(1, e)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, e]$. Now,

$$f'(x) = \frac{1}{2x} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(e) - f(1)}{e - 1} = \frac{\frac{1}{2} - 0}{e - 1} = \frac{1}{2(e - 1)},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{2c} = \frac{1}{2(e - 1)},$$

or when $\boxed{c = e - 1}$. Note that $e - 1 \approx 1.718 > 1$.

24. Let $f(x) = xe^x$. The function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[0, 1]$. Now,

$$f'(x) = xe^x + e^x \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = \frac{e - 0}{1} = e,$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad e^c(c + 1) = e,$$

or when $\boxed{c \approx 0.557}$. This value was obtained using the computer algebra system *Maple*.

25. Let $f(x) = x^3 - 5x^2 + 4x - 2$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[1, 3]$ and differentiable on the open interval $(1, 3)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 3]$. Now,

$$f'(x) = 3x^2 - 10x + 4 \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{-8 - (-2)}{2} = -3,$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 3c^2 - 10c + 4 = -3.$$

Therefore,

$$\begin{aligned} 3c^2 - 10c + 7 &= 0 \\ (3c - 7)(c - 1) &= 0 \\ c &= \frac{7}{3}, 1. \end{aligned}$$

Of these numbers, only $\boxed{c = \frac{7}{3}}$ is in the interval $(1, 3)$.

26. Let $f(x) = x^3 - 7x^2 + 5x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[-2, 2]$ and differentiable on the open interval $(-2, 2)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[-2, 2]$. Now,

$$f'(x) = 3x^2 - 14x + 5 \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{-10 - (-46)}{4} = 9,$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 3c^2 - 14c + 5 = 9.$$

Therefore, $3c^2 - 14c - 4 = 0$ and

$$c = \frac{14 \pm \sqrt{(-14)^2 - 4(3)(-4)}}{2(3)} = \frac{14 \pm \sqrt{244}}{6} = \frac{7 \pm \sqrt{61}}{3}.$$

Of these numbers, only $c = \frac{7 - \sqrt{61}}{3} \approx -0.270$ is in the interval $(-2, 2)$. Note that $c = \frac{7 + \sqrt{61}}{3} \approx 4.937$.

27. Let $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$. The function f is continuous and differentiable on the set $\{x|x \neq 0\}$, so it is continuous on the closed interval $[1, 3]$ and differentiable on the open interval $(1, 3)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 3]$. Now,

$$f'(x) = -\frac{1}{x^2} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{\frac{4}{3} - 2}{2} = -\frac{1}{3},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad -\frac{1}{c^2} = -\frac{1}{3}.$$

Therefore, $c^2 = 3$ and $c = \pm\sqrt{3}$. Of these numbers, only $c = \sqrt{3}$ is in the interval $(1, 3)$.

28. Let $f(x) = \frac{x^2}{x+1}$. The function f is continuous and differentiable on the set $\{x|x \neq -1\}$, so it is continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[0, 1]$. Now,

$$f'(x) = \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = \frac{\frac{1}{2} - 0}{1} = \frac{1}{2},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{c^2 + 2c}{(c+1)^2} = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} 2(c^2 + 2c) &= (c+1)^2 \\ 2c^2 + 4c &= c^2 + 2c + 1 \\ c^2 + 2c &= 1 \\ c^2 + 2c + 1 &= 2 \\ (c+1)^2 &= 2 \end{aligned}$$

so that $c = -1 \pm \sqrt{2}$. Of these numbers, only $c = -1 + \sqrt{2}$ is in the interval $(0, 1)$.

29. Let $f(x) = \sqrt[3]{x^2}$. The function f is continuous on the set of all real numbers and differentiable on the set $\{x|x \neq 0\}$, so it is continuous on the closed interval $[1, 8]$ and differentiable on the open interval $(1, 8)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 8]$. Now,

$$f'(x) = \frac{2}{3}x^{-1/3} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(1)}{8 - 1} = \frac{4 - 1}{7} = \frac{3}{7},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{2}{3}c^{-1/3} = \frac{3}{7}.$$

Therefore, $c^{1/3} = \frac{14}{9}$ and $\boxed{c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729}}$.

30. Let $f(x) = \sqrt{x-2}$. The function f is continuous on the set $\{x|x \leq 2\}$ and differentiable on the set $\{x|x > 2\}$, so it is continuous on the closed interval $[2, 4]$ and differentiable on the open interval $(2, 4)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[2, 4]$. Now,

$$f'(x) = \frac{1}{2\sqrt{x-2}} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(2)}{4 - 2} = \frac{\sqrt{2} - 0}{2} = \frac{\sqrt{2}}{2},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{2\sqrt{c-2}} = \frac{\sqrt{2}}{2}.$$

Therefore, $\sqrt{c-2} = \frac{1}{\sqrt{2}}$ and $\boxed{c = \frac{5}{2}}$.

31. Let $f(x) = x^3 + 6x^2 + 12x + 1$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 + 12x + 12 = 3(x^2 + 4x + 4) = 3(x+2)^2,$$

so -2 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) > 0$ for all $x \neq -2$, it follows that f is increasing on the intervals $(-\infty, -2)$ and $(-2, \infty)$. Since f is continuous on its domain, we can say that f is increasing on $(-\infty, \infty)$.

32. Let $f(x) = -x^3 + 3x^2 + 4$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = -3x^2 + 6x = 3x(2 - x),$$

so 0 and 2 are critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 2 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x$	Sign of $2 - x$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	f is decreasing
$(0, 2)$	+	+	+	f is increasing
$(2, \infty)$	+	—	—	f is decreasing

Therefore, f is increasing on the interval $(0, 2)$ and decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$. Since f is continuous on its domain, we can say that f is increasing on the interval $[0, 2]$ and decreasing on the intervals $(-\infty, 0]$ and $[2, \infty)$.

33. Let $f(x) = x^{2/3}(x^2 - 4)$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{2/3}(2x) + (x^2 - 4) \cdot \frac{2}{3}x^{-1/3} = \frac{3x(2x) + 2(x^2 - 4)}{3x^{1/3}} = \frac{8x^2 - 8}{3x^{1/3}} = \frac{8(x-1)(x+1)}{3x^{1/3}},$$

so -1 , 0 , and 1 are the critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers -1 , 0 , and 1 to form four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x-1)$	Sign of $x+1$	Sign of $3x^{1/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$-$	$-$	$-$	$-$	f is decreasing
$(-1, 0)$	$-$	$+$	$-$	$+$	f is increasing
$(0, 1)$	$-$	$+$	$+$	$-$	f is decreasing
$(1, \infty)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-1, 0)$ and $(1, \infty)$ and decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$. Since f is continuous on its domain, we can say that f is increasing on the intervals $[-1, 0]$ and $[1, \infty)$ and decreasing on the intervals $(-\infty, -1]$ and $[0, 1]$.

34. Let $f(x) = x^{1/3}(x^2 - 7)$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{1/3}(2x) + (x^2 - 7) \cdot \frac{1}{3}x^{-2/3} = \frac{3x(2x) + (x^2 - 7)}{3x^{2/3}} = \frac{7x^2 - 7}{3x^{2/3}} = \frac{7(x-1)(x+1)}{3x^{2/3}},$$

so -1 , 0 , and 1 are the critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers -1 , 0 , and 1 to form four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $7(x-1)$	Sign of $x+1$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$-$	$-$	$+$	$+$	f is increasing
$(-1, 0)$	$-$	$+$	$+$	$-$	f is decreasing
$(0, 1)$	$-$	$+$	$+$	$-$	f is decreasing
$(1, \infty)$	$+$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and decreasing on the intervals $(-1, 0)$ and $(0, 1)$. Since f is continuous on its domain, we can say that f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$ and decreasing on the interval $[-1, 1]$.

35. Let

$$f(x) = |x^3 + 3| = \begin{cases} -(x^3 + 3), & x < -\sqrt[3]{3} \\ x^3 + 3, & x \geq -\sqrt[3]{3}. \end{cases}$$

On the interval $(-\infty, -\sqrt[3]{3})$, $f'(x) = -3x^2$ exists and is never equal to 0; f has no critical numbers on this interval. On the interval $(-\sqrt[3]{3}, \infty)$, $f'(x) = 3x^2$ exists and is equal to 0 when $x = 0$. It follows that 0 is a critical number. At $x = -\sqrt[3]{3}$, the rule for f changes, so it is necessary to investigate the

existence of $f'(-\sqrt[3]{3})$. Now,

$$\begin{aligned}\lim_{x \rightarrow -\sqrt[3]{3}^-} \frac{f(x) - f(-\sqrt[3]{3})}{x - (-\sqrt[3]{3})} &= \lim_{x \rightarrow -\sqrt[3]{3}^-} \frac{-(x^3 + 3) - 0}{x + \sqrt[3]{3}} \\ &= \lim_{x \rightarrow -\sqrt[3]{3}^-} \frac{-(x + \sqrt[3]{3})(x^2 - \sqrt[3]{3}x + \sqrt[3]{9})}{x + \sqrt[3]{3}} \\ &= \lim_{x \rightarrow -\sqrt[3]{3}^-} -(x^2 - \sqrt[3]{3}x + \sqrt[3]{9}) = -3\sqrt[3]{9},\end{aligned}$$

and

$$\begin{aligned}\lim_{x \rightarrow -\sqrt[3]{3}^+} \frac{f(x) - f(-\sqrt[3]{3})}{x - (-\sqrt[3]{3})} &= \lim_{x \rightarrow -\sqrt[3]{3}^+} \frac{(x^3 + 3) - 0}{x + \sqrt[3]{3}} \\ &= \lim_{x \rightarrow -\sqrt[3]{3}^+} \frac{(x + \sqrt[3]{3})(x^2 - \sqrt[3]{3}x + \sqrt[3]{9})}{x + \sqrt[3]{3}} \\ &= \lim_{x \rightarrow -\sqrt[3]{3}^+} (x^2 - \sqrt[3]{3}x + \sqrt[3]{9}) = 3\sqrt[3]{9}.\end{aligned}$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = -\sqrt[3]{3}$. Therefore, $-\sqrt[3]{3}$ is also a critical number of f .

The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(-\infty, -\sqrt[3]{3})$, $f'(x) = -3x^2 < 0$, while on the intervals $(-\sqrt[3]{3}, 0)$ and $(0, \infty)$, $f'(x) = 3x^2 > 0$. Therefore, f is decreasing on the interval $(-\infty, -\sqrt[3]{3})$ and increasing on the intervals $(-\sqrt[3]{3}, 0)$ and $(0, \infty)$. Since f is continuous on its domain, we can say that f is decreasing on the interval $(-\infty, -\sqrt[3]{3}]$ and is increasing on the interval $[-\sqrt[3]{3}, \infty)$.

36. Let

$$f(x) = |x^2 - 4| = \begin{cases} x^2 - 4, & x \leq -2 \\ -(x^2 - 4), & -2 < x < 2 \\ x^2 - 4, & x \geq 2. \end{cases}$$

On the interval $(-\infty, -2)$, $f'(x) = 2x$ exists and is never equal to 0; f has no critical numbers on this interval. On the interval $(-2, 2)$, $f'(x) = -2x$ exists and is equal to 0 when $x = 0$. It follows that 0 is a critical number. On the interval $(2, \infty)$, $f'(x) = 2x$ exists and is never equal to 0; f has no critical numbers on this interval. At $x = \pm 2$, the rule for f changes, so it is necessary to investigate the existence of $f'(-2)$ and $f'(2)$. Now,

$$\lim_{x \rightarrow -2^-} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2^-} \frac{(x^2 - 4) - 0}{x + 2} = \lim_{x \rightarrow -2^-} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x \rightarrow -2^-} (x - 2) = -4,$$

and

$$\lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2^+} \frac{-(x^2 - 4) - 0}{x + 2} = \lim_{x \rightarrow -2^+} \frac{-(x - 2)(x + 2)}{x + 2} = \lim_{x \rightarrow -2^+} -(x - 2) = 4.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = -2$. Additionally,

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-(x^2 - 4) - 0}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2^-} -(x + 2) = -4,$$

and

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x^2 - 4) - 0}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2^+} (x + 2) = 4.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 2$. Therefore, ± 2 are also critical numbers of f .

The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(-\infty, -2)$, $f'(x) = 2x < 0$; on the interval $(-2, 0)$, $f'(x) = -2x > 0$; on the interval $(0, 2)$, $f'(x) = -2x < 0$; finally, on the interval $(2, \infty)$, $f'(x) = 2x > 0$. Therefore, f is decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$ and increasing on the intervals $(-2, 0)$ and $(2, \infty)$. Since f is continuous on its domain, we can say that f is decreasing on the intervals $(-\infty, -2]$ and $[0, 2]$ and increasing on the intervals $[-2, 0]$ and $[2, \infty)$.

37. Let $f(x) = 3 \sin x$. The function f is a constant multiple of the trigonometric function $\sin x$, which is differentiable everywhere; therefore, f is differentiable everywhere. The critical numbers of f therefore occur where $f'(x) = 0$. Now,

$$f'(x) = 3 \cos x,$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(0, \frac{\pi}{2})$, $f'(x) > 0$; on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, $f'(x) < 0$; finally, on the interval $(\frac{3\pi}{2}, 2\pi)$, $f'(x) > 0$. Therefore, f is increasing on the intervals $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$ and decreasing on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. Since f is continuous on its domain, we can say that f is

increasing on the intervals $[0, \frac{\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$ and decreasing on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

38. Let $f(x) = \cos(2x)$. The trigonometric function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = -2 \sin(2x),$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{\pi}{2}$, π , and $\frac{3\pi}{2}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(0, \frac{\pi}{2})$, $f'(x) < 0$; on the interval $(\frac{\pi}{2}, \pi)$, $f'(x) > 0$; on the interval $(\pi, \frac{3\pi}{2})$, $f'(x) < 0$; finally, on the interval $(\frac{3\pi}{2}, 2\pi)$, $f'(x) > 0$. Therefore, f is increasing on the intervals $(\frac{\pi}{2}, \pi)$ and $(\frac{3\pi}{2}, 2\pi)$ and decreasing on the intervals $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$. Since f is

continuous on its domain, we can say that f is increasing on the intervals $[\frac{\pi}{2}, \pi]$ and $[\frac{3\pi}{2}, 2\pi]$ and decreasing on the intervals $[0, \frac{\pi}{2}]$ and $[\pi, \frac{3\pi}{2}]$.

39. Let $f(x) = xe^x$. The function g is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = xe^x + e^x = (x + 1)e^x,$$

so -1 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number -1 to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x + 1$	Sign of e^x	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	$-$	$+$	$-$	f is decreasing
$(-1, \infty)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the interval $(-1, \infty)$ and decreasing on the interval $(-\infty, -1)$. Since f is continuous on its domain, we can say that f is increasing on the interval $[-1, \infty)$ and decreasing on the interval $(-\infty, -1]$.

40. Let $f(x) = x + e^x$. The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 1 + e^x,$$

which is never equal to 0 because $e^x > 0$ for all x , so there are no critical numbers. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Here, $f'(x) > 1 > 0$ for all x , so f is increasing for all x .

41. Let $f(x) = e^x \sin x$. The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = e^x \cos x + e^x \sin x = e^x(\sin x + \cos x),$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $\sin x + \cos x$	Sign of $f'(x)$	Conclusion
$(0, \frac{3\pi}{4})$	$+$	$+$	$+$	f is increasing
$(\frac{3\pi}{4}, \frac{7\pi}{4})$	$+$	$-$	$-$	f is decreasing
$(\frac{7\pi}{4}, 2\pi)$	$+$	$+$	$+$	f is increasing

Therefore, f is increasing on the intervals $(0, \frac{3\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$ and decreasing on the interval $(\frac{3\pi}{4}, \frac{7\pi}{4})$. Since f is continuous on its domain, we can say that f is

increasing on the intervals $[0, \frac{3\pi}{4}]$ and $[\frac{7\pi}{4}, 2\pi]$ and decreasing on the interval $[\frac{3\pi}{4}, \frac{7\pi}{4}]$.

42. Let $f(x) = e^x \cos x$. The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = -e^x \sin x + e^x \cos x = e^x(\cos x - \sin x),$$

so the critical numbers on the interval $(0, 2\pi)$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $\cos x - \sin x$	Sign of $f'(x)$	Conclusion
$(0, \frac{\pi}{4})$	+	+	+	f is increasing
$(\frac{\pi}{4}, \frac{5\pi}{4})$	+	-	-	f is decreasing
$(\frac{5\pi}{4}, 2\pi)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(0, \frac{\pi}{4})$ and $(\frac{5\pi}{4}, 2\pi)$ and decreasing on the interval $(\frac{\pi}{4}, \frac{5\pi}{4})$. Since f is continuous on its domain, we can say that f is

increasing on the intervals $[0, \frac{\pi}{4}]$ and $[\frac{5\pi}{4}, 2\pi]$ and decreasing on the interval $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

Applications and Extensions

43. Let $f(x) = 2x^3 - 6x^2 + 6x - 5$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x - 1)^2,$$

so 1 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) > 0$ for all $x \neq 1$, it follows that f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. Since f is continuous on its domain, we can say that f is increasing for all x .

44. Let $f(x) = x^3 - 3x^2 + 3x$. The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2,$$

so 1 is the only critical number. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) > 0$ for all $x \neq 1$, it follows that f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. Since f is continuous on its domain, we can say that f is increasing for all x .

45. Let $f(x) = \frac{x}{x+1}$. The rational function f is differentiable on its domain, the set $\{x|x \neq -1\}$, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2},$$

so f has no critical numbers. For $x \neq -1$, $f'(x) > 0$, so the Increasing/Decreasing Function Test indicates that f is increasing on the intervals $(-\infty, -1)$ and $(-1, \infty)$; that is, f is increasing on any interval that does not contain $x = -1$.

46. Let $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$. The rational function f is differentiable on its domain, the set $\{x|x \neq 0\}$, so the critical numbers of f occur where $f'(x) = 0$. Now,

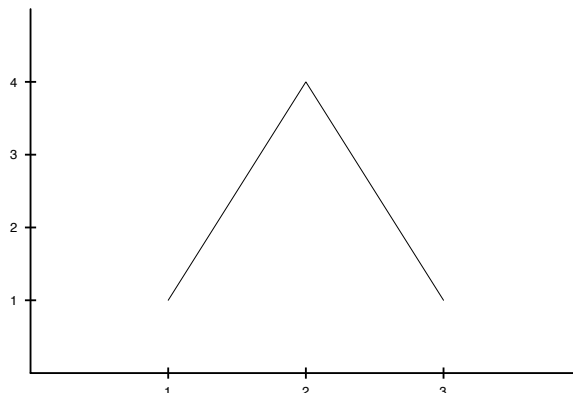
$$f'(x) = -\frac{1}{x^2},$$

so f has no critical numbers. For $x \neq 0$, $f'(x) < 0$, so the Increasing/Decreasing Function Test indicates that f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$; that is, f is decreasing on any interval that does not contain $x = 0$.

47. Answers will vary. The figure below displays the graph of a function that is continuous on the closed interval $[1, 3]$ but not differentiable on the open interval $(1, 3)$ – the function is not differentiable at $x = 2$ – and for which the conclusion of the Mean Value Theorem does not hold. Note that

$$\frac{f(3) - f(1)}{3 - 1} = \frac{1 - 1}{2} = 0,$$

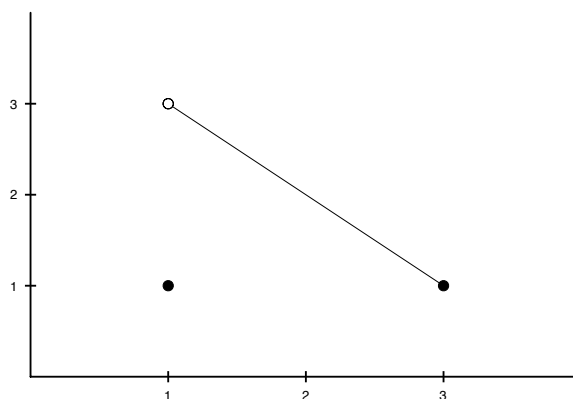
but the graph of the function does not have a horizontal tangent line anywhere on the open interval $(1, 3)$.



48. Answers will vary. The figure below displays the graph of a function that is differentiable on the open interval $(1, 3)$ but not continuous on the closed interval $[1, 3]$ – the function is not continuous at $x = 1$ – and for which the conclusion of the Mean Value Theorem does not hold. Note that

$$\frac{f(3) - f(1)}{3 - 1} = \frac{1 - 1}{2} = 0,$$

but the graph of the function does not have a horizontal tangent line anywhere on the open interval $(1, 3)$.



49. Let $s(t)$ denote the distance function of the automobile, and assume that this function is differentiable. If the automobile traveled 20 miles at an average speed of 40 mph, the trip took one half hour to complete. Let $t = 0$ denote the beginning of the trip and $t = 0.5$ denote the end of the trip. Because s is continuous on the closed interval $[0, 0.5]$ and differentiable on the open interval $(0, 0.5)$, the conditions of the Mean Value Theorem are satisfied. Therefore, there exists at least one c in $(0, 0.5)$ such that

$$s'(c) = \frac{s(0.5) - s(0)}{0.5 - 0} = \frac{20 - 0}{0.5} = 40 \text{ mph.}$$

Now, s' is the speed of the automobile, so the speed was exactly 40 mph at some time during the trip.

50. Let $v(t)$ denote the speed of the car as indicated by the car's speedometer, where t is measured in hours, and assume that this function is differentiable. Let $t = 0$ correspond to 4:00 p.m. and $t = 0.2$ correspond to 4:12 p.m. Because v is continuous on the closed interval $[0, 0.2]$ and differentiable on the open interval $(0, 0.2)$, the conditions of the Mean Value Theorem are satisfied. Therefore, there exists at least one c in $(0, 0.2)$ such that

$$v'(c) = \frac{v(0.2) - v(0)}{0.2 - 0} = \frac{60 - 40}{0.2} = 100 \text{ mi/h}^2.$$

Now, v' is the acceleration of the automobile, so the acceleration was exactly 100 mi/h^2 at some time between 4:00 p.m. and 4:12 p.m.

51. Let $f(t) = f_2(t) - f_1(t)$, and let $t = 0$ denote the start of the race and $t = T$ denote the end of the race for these two cars. Because the cars start the race together and finish in a tie,

$$f(0) = f_2(0) - f_1(0) = 0 \quad \text{and} \quad f(T) = f_2(T) - f_1(T) = 0.$$

Assuming that f_1 and f_2 are continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$, the function f will also be continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$. The Mean Value Theorem therefore applies to f over $[0, T]$, so there is at least one c in $(0, T)$ such that

$$f'(c) = \frac{f(T) - f(0)}{T - 0} = \frac{0 - 0}{T - 0} = 0.$$

Recalling the definition of f , this last statement is equivalent to $f'_2(c) - f'_1(c) = 0$, or $f'_2(c) = f'_1(c)$. Therefore, at some time during the race, the two cars are traveling at the same speed.

52. Assuming f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then the Mean Value Theorem applies and there is at least one c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

53. Let $d = -\frac{1}{192}x^4 + \frac{25}{384}x^3 - \frac{25}{128}x^2$ on the closed interval $[0, 5]$.

- (a) The polynomial function d is continuous on the closed interval $[0, 5]$ and differentiable on the open interval $(0, 5)$. Moreover, $d(0) = 0$ and

$$d(5) = -\frac{1}{192}5^4 + \frac{25}{384}5^3 - \frac{25}{128}5^2 = \frac{5^4}{384}(-2 + 5 - 3) = 0.$$

Therefore, d satisfies the conditions of Rolle's Theorem on the interval $[0, 5]$.

- (b) According to part (a), $\boxed{d(0) = d(5) = 0}$, so the deflection at the ends of the beam is always zero; therefore, the ends of the beam must be held fixed in place.
- (c) Differentiating d yields

$$d'(x) = -\frac{1}{48}x^3 + \frac{25}{128}x^2 - \frac{25}{64}x = -\frac{x}{384}(8x^2 - 75x + 150).$$

Therefore, $d'(c) = 0$ when $c = 0$ and when $8c^2 - 75c + 150 = 0$. Applying the quadratic formula to this last equation gives

$$c = \frac{75 \pm \sqrt{(-75)^2 - 4(8)(150)}}{16} = \frac{75 \pm \sqrt{825}}{16} = \frac{75 \pm 5\sqrt{33}}{16}.$$

Now,

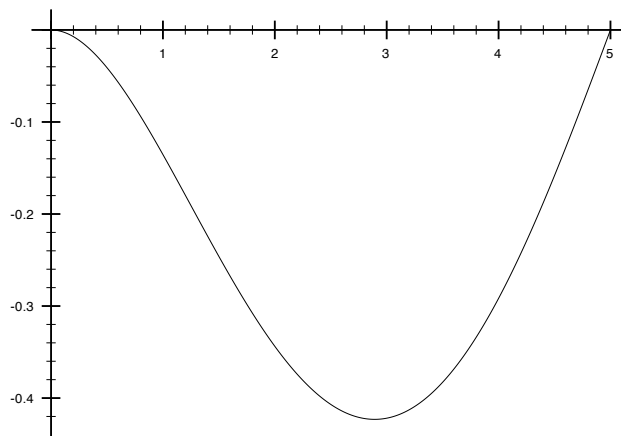
$$c = \frac{75 + 5\sqrt{33}}{16} \approx 6.483 \quad \text{and} \quad c = \frac{75 - 5\sqrt{33}}{16} \approx 2.892,$$

so the only c in $(0, 5)$ that satisfies the conclusion of Rolle's Theorem is $c = \frac{75 - 5\sqrt{33}}{16} \approx 2.892$ ft.

Using the computer algebra system *Maple*,

$$d\left(\frac{75 - 5\sqrt{33}}{16}\right) = -\frac{24375 + 34375\sqrt{33}}{524288} \text{ ft} \approx -0.423 \text{ ft}.$$

(d) The figure below displays the graph of d on the interval $[0, 5]$.



54. Let $f(x) = x^4 - 2x^3 - 4x^2 + 7x + 3$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now, $f'(x) = 4x^3 - 6x^2 - 8x + 7$. Using the computer algebra system *Maple*, the zeros of $f'(x)$ are found to be -1.243 , 0.684 , and 2.059 , rounded to three decimal places. It follows that the critical numbers of f are approximately -1.243 , 0.684 , and 2.059 .
- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. The critical numbers of f divide the number line into four intervals: $(-\infty, -1.243)$, $(-1.243, 0.684)$, $(0.684, 2.059)$, and $(2.059, \infty)$. Select a test number in each interval and evaluate f' at that number. If $f'(x) < 0$ at the test number, then $f'(x) < 0$ throughout the interval; if $f'(x) > 0$ at the test number, then $f'(x) > 0$ throughout the interval. The following table summarizes the results:

Interval	Test Number	Value of $f'(x)$ at Test Number	Sign of $f'(x)$	Conclusion
$(-\infty, -1.243)$	-2	-33	$-$	f is decreasing
$(-1.243, 0.684)$	0	7	$+$	f is increasing
$(0.684, 2.059)$	1	-3	$-$	f is decreasing
$(2.059, \infty)$	3	37	$+$	f is increasing

Therefore, f is increasing on the intervals $(-1.243, 0.684)$ and $(2.059, \infty)$ and decreasing on the intervals $(-\infty, -1.243)$ and $(0.684, 2.059)$. Since f is continuous on its domain, we can say that f is

increasing on the intervals $[-1.243, 0.684]$ and $[2.059, \infty)$

and

decreasing on the intervals $(-\infty, -1.243]$ and $[0.684, 2.059]$.

55. Let $f(x) = (x-1)\sin x$. The function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. Additionally, $f(0) = (-1)\sin 0 = 0$, and $f(1) = (0)\sin 1 = 0$, so $f(0) = f(1)$, and all three conditions of Rolle's Theorem are satisfied. Therefore, there exists at least one c in $(0, 1)$ such that $f'(c) = 0$. Now,

$$f'(x) = (x-1)\cos x + \sin x,$$

so

$$(c-1)\cos c + \sin c = 0.$$

Because $c \in (0, 1)$, $\cos c \neq 0$, so the previous equation can be divided by $\cos c$ to yield

$$c - 1 + \tan c = 0 \quad \text{or} \quad \tan c + c = 1.$$

In conclusion, the equation $\tan x + x = 1$ has a solution in the interval $(0, 1)$.

56. Let $f(x) = x^3 - 2$. Then $f(\sqrt[3]{2}) = (\sqrt[3]{2})^3 - 2 = 2 - 2 = 0$, so $\sqrt[3]{2}$ is a real zero of f . Now, if $x \leq 0$, then $x^3 \leq 0$ and $x^3 - 2 \leq -2 < 0$; consequently, if f does have another real zero, that other real zero must be a positive number. Suppose, for the sake of contradiction, that f does have another real zero at $z > 0$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval between z and $\sqrt[3]{2}$ and differentiable on the open interval between z and $\sqrt[3]{2}$. Additionally, $f(z) = f(\sqrt[3]{2}) = 0$, so Rolle's Theorem applies and there must exist a c between z and $\sqrt[3]{2}$ such that $f'(c) = 0$. However, $f'(x) = 3x^2 = 0$ only for $x = 0$ which is not between the positive numbers z and $\sqrt[3]{2}$. Therefore, f has exactly one real zero.

57. Let $f(x) = (x-8)^3$. Then $f(8) = (8-8)^3 = 0$, so 8 is a real zero of f . Suppose, for the sake of contradiction, that f does have another real zero at z . The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval between z and 8 and differentiable on the open interval between z and 8. Additionally, $f(z) = f(8) = 0$, so Rolle's Theorem applies and there must exist a c between z and 8 such that $f'(c) = 0$. However, $f'(x) = 3(x-8)^2 = 0$ only for $x = 8$ which is not between the positive numbers z and 8. Therefore, f has exactly one real zero.

58. Let $f(x) = (x^2 - 4x + 3)(x^2 + x + 1)$. The function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[1, 3]$ and differentiable on the open interval $(1, 3)$. Additionally,

$$f(1) = (1 - 4 + 3)(1 + 1 + 1) = 0(3) = 0$$

and

$$f(3) = (9 - 12 + 3)(9 + 3 + 1) = 0(13) = 0,$$

so $f(1) = f(3)$ and Rolle's Theorem applies. Therefore, there must exist at least one c between 1 and 3 such that $f'(c) = 0$. To verify this directly, note that

$$\begin{aligned} f'(x) &= (x^2 - 4x + 3)(2x + 1) + (x^2 + x + 1)(2x - 4) \\ &= 2x^3 - 8x^2 + 6x + x^2 - 4x + 3 + 2x^3 + 2x^2 + 2x - 4x^2 - 4x - 4 \\ &= 4x^3 - 9x^2 - 1. \end{aligned}$$

This polynomial function is continuous everywhere, so it is continuous on the closed interval $[1, 3]$, and $f'(1) = 4 - 9 - 1 = -6 < 0$ and $f'(3) = 108 - 81 - 1 = 26 > 0$. The Intermediate Value Theorem then guarantees there exists a number c between 1 and 3 such that $f'(c) = 0$.

59. Though the function $f(x) = |x|$ is continuous on the closed interval $[-1, 1]$, it is not differentiable on the open interval $(-1, 1)$ because $f'(x)$ does not exist at $x = 0$. Therefore, Rolle's Theorem does not apply to f on the interval $[-1, 1]$.

60. Let $f(x) = x^{2/3}$. Then

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - 1}{2} = 0;$$

however, $f'(x) = \frac{2}{3}x^{-1/3}$, which is never equal to zero. Therefore, there is no c in $[-1, 1]$ for which

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}.$$

Though f is continuous on the closed interval $[-1, 1]$, it is not differentiable on the open interval $(-1, 1)$ because $f'(x)$ does not exist at $x = 0$. Therefore, the Mean Value Theorem does not apply to f on the interval $[-1, 1]$.

61. Let $f(x) = \sin^{-1} x$. Then

$$f(1) - f(0) = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

and $f'(x) = \frac{1}{\sqrt{1-x^2}}$. Therefore,

$$f'(N) = \frac{1}{\sqrt{1-N^2}} = \frac{\pi}{2}$$

when

$$\begin{aligned} \sqrt{1-N^2} &= \frac{2}{\pi} \\ 1-N^2 &= \frac{4}{\pi^2} \\ N^2 &= 1 - \frac{4}{\pi^2} \\ N &= \pm \sqrt{1 - \frac{4}{\pi^2}}. \end{aligned}$$

Of these numbers, $N = \sqrt{1 - \frac{4}{\pi^2}}$ is contained in the interval $(0, 1)$.

62. Let $f(x) = Ax^2 + Bx + C$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) for any real numbers a and b with $a < b$. The Mean Value Theorem applies and guarantees there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Now,

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} = A(b + a) + B, \end{aligned}$$

and

$$f'(x) = 2Ax + B.$$

Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad 2Ac + B = A(b + a) + B,$$

or $c = \frac{b+a}{2}$, the midpoint of the interval $[a, b]$.

63. (a) Let $f(x) = \sqrt{x}$. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Now,

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} > 0$$

for $x > 0$, so $\boxed{f \text{ is increasing on the interval } (0, \infty)}$.

- (b) Because f is increasing on the interval $(0, \infty)$ and is continuous for $x \geq 0$, $\boxed{f \text{ is increasing for all } x \geq 0}$; that is, f is increasing on its domain.
64. Consider the function $g(x) = \frac{a}{5}x^5 + \frac{b}{4}x^4 + \frac{c}{3}x^3 + \frac{d}{2}x^2 + ex$ and suppose that $\frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} + e = 0$. The polynomial function g is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. Additionally,

$$g(0) = 0 \quad \text{and} \quad g(1) = \frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} + e = 0,$$

so $g(0) = g(1)$, and Rolle's Theorem applies. Therefore, there exists at least one c in $(0, 1)$ such that $g'(c) = 0$. But

$$g'(x) = ax^4 + bx^3 + cx^2 + dx + e = f(x).$$

In conclusion, the function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ must have a zero between 0 and 1 if $\frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} + e = 0$.

65. (a) $\boxed{\text{Not necessarily true}}$. Let $f(x) = e^x$ and $g(x) = -1$. Then $f'(x) = e^x$ and $g'(x) = 0$ exist and $f'(x) > g'(x)$ for all real x ; however, the graphs of $y = f(x)$ and $y = g(x)$ never intersect because $f(x) > 0$ for all real x .
- (b) $\boxed{\text{True}}$. Suppose for sake of contradiction that the graphs of $y = f(x)$ and $y = g(x)$ intersect more than once. In particular, suppose the graphs intersect at a and b with $a < b$. Consider the function $h(x) = f(x) - g(x)$. The functions f and g are continuous and differentiable everywhere, so the function h is also continuous and differentiable everywhere. Additionally,

$$h(a) = f(a) - g(a) = 0 \quad \text{and} \quad h(b) = f(b) - g(b) = 0,$$

so $h(a) = h(b)$, and Rolle's Theorem applies. It follows that there exists at least one c in (a, b) such that $h'(c) = f'(c) - g'(c) = 0$; equivalently, $f'(c) = g'(c)$, in violation of the condition that $f'(x) > g'(x)$ for all real x . Therefore, the graphs of $y = f(x)$ and $y = g(x)$ can intersect no more than once.

- (c) $\boxed{\text{Not necessarily true}}$. Let $f(x) = 2x$ and $g(x) = x$. Then $f'(x) = 2$ and $g'(x) = 1$ exist and $f'(x) > g'(x)$ for all real x ; however, the graphs of $y = f(x)$ and $y = g(x)$ intersect at the origin.
- (d) $\boxed{\text{False}}$. See the proof for part (b).
- (e) $\boxed{\text{False}}$. Even if the graphs intersect, the tangent lines will be different because the condition $f'(x) > g'(x)$ guarantees that the tangent lines will have different slopes.

66. Suppose, for sake of contradiction, that there exists a number k for which the function $f(x) = x^3 - 3x + k$ has two distinct zeros in the interval $[0, 1]$. Let x_1 and x_2 denote the two zeros, with $x_1 < x_2$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies. It follows that there exists a number c in $(x_1, x_2) \subset (0, 1)$ such that $f'(c) = 0$. However, $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ which is never equal to zero in $(0, 1)$. Therefore, there is no k for which the function $f(x) = x^3 - 3x + k$ has two distinct zeros in the interval $[0, 1]$.

67. Consider the function $f(x) = \frac{e^x}{x^2}$ for $x > 0$. Now,

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4} = \frac{e^x(x-2)}{x^3},$$

so the only critical number on $x > 0$ is 2. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number 2 to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $x-2$	Sign of x^3	Sign of $f'(x)$	Conclusion
$(0, 2)$	+	-	+	-	f is decreasing
$(2, \infty)$	+	+	+	+	f is increasing

Because f is decreasing on the interval $(0, 2)$ and increasing on the interval $(2, \infty)$, f must have an absolute minimum at 2. Therefore,

$$f(x) = \frac{e^x}{x^2} \geq f(2) = \frac{e^2}{4} \approx 1.847 > 1,$$

so that, for $x > 0$

$$\frac{e^x}{x^2} > 1 \quad \text{or} \quad e^x > x^2.$$

68. Consider the function $f(x) = e^x - 1 - x$ for $x > 0$. Now, $f'(x) = e^x - 1$. Because $e^x > 1$ for $x > 0$, it follows that $f'(x) > 0$ for $x > 0$. Therefore, by the Increasing/Decreasing Function Test, f is increasing for $x > 0$, so that $f(x) > f(0) = 1 - 1 - 0 = 0$. Therefore, for $x > 0$

$$f(x) = e^x - 1 - x > 0 \quad \text{or} \quad e^x > 1 + x.$$

69. For $x > 1$, $\ln x > 0$. Next, consider the function $f(x) = x - \ln x$ for $x > 1$. Now, $f'(x) = 1 - \frac{1}{x}$. For $x > 1$

$$\frac{1}{x} < 1 \quad \text{so that} \quad 1 - \frac{1}{x} > 0;$$

that is, $f'(x) > 0$. Therefore, by the Increasing/Decreasing Function Test, f is increasing for $x > 1$, so that $f(x) > f(1) = 1 - \ln 1 = 1 > 0$. Therefore, for $x > 1$

$$f(x) = x - \ln x > 0 \quad \text{or} \quad x > \ln x.$$

70. Consider the function $f(\theta) = \tan \theta - \theta$ on the interval $\left(0, \frac{\pi}{2}\right)$. Now, $f'(\theta) = \sec^2 \theta - 1$. On the interval $\left(0, \frac{\pi}{2}\right)$, $\sec^2 \theta > 1$, so $f'(\theta) > 0$. Therefore, by the Increasing/Decreasing Function Test, f is increasing on the interval $\left(0, \frac{\pi}{2}\right)$, so that $f(\theta) > f(0) = \tan 0 - 0 = 0$. Therefore, on the interval $\left(0, \frac{\pi}{2}\right)$,

$$\tan \theta - \theta > 0 \quad \text{or} \quad \tan \theta > \theta.$$

71. Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . Further, suppose that $f(x) = 0$ for three different numbers in (a, b) . Let x_1 , x_2 , and x_3 denote the three zeros with $x_1 < x_2 < x_3$. Consider the interval $[x_1, x_2]$. Because $[x_1, x_2]$ is contained in $[a, b]$, f is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Next, consider the interval $[x_2, x_3]$. Because $[x_2, x_3]$ is contained in $[a, b]$, f is continuous on the closed interval $[x_2, x_3]$ and differentiable on the open interval (x_2, x_3) . Additionally, $f(x_2) = f(x_3) = 0$, so Rolle's Theorem applies and there exists at least one number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. By construction, $c_1 < x_2$ and $c_2 > x_2$, so c_1 cannot be equal to c_2 . Therefore, there are at least two numbers in (a, b) at which $f'(x) = 0$.

72. Because f is differentiable on (a, b) , it is continuous on (a, b) . To show that f is decreasing on (a, b) , choose two numbers x_1 and x_2 in (a, b) with $x_1 < x_2$. With x_1 and x_2 in (a, b) , it follows that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , and the Mean Value Theorem applies. It follows that there exists a number c in (x_1, x_2) for which

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(x) < 0$ for all x in (a, b) , it follows that $f'(c) < 0$. With $x_1 < x_2$, $x_2 - x_1 > 0$, so

$$f(x_2) - f(x_1) < 0 \quad \text{or} \quad f(x_2) < f(x_1).$$

Therefore, f is decreasing on (a, b) .

73. Suppose, for sake of contradiction, that f has an absolute extreme value on (a, b) , say at c . Because c is in (a, b) – and, in particular, not at an endpoint of the interval – $f(c)$ must also be a local extreme value. Now, local extreme values can only occur at critical numbers. It follows that $f'(c)$ must be equal to zero or not exist. However, $f'(x)$ exists and is never equal to 0 for x in (a, b) . Therefore, f cannot have an extreme value on (a, b) .

Challenge Problems

74. Let f be a polynomial function with real zeros x_1 and x_2 , $x_1 < x_2$. Because f is continuous and differentiable everywhere, it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies. It follows that there exists at least one number c in (x_1, x_2) such that $f'(c) = 0$. Therefore, between any two real zeros of the polynomial function f , there is a real zero of its derivative f' .
75. Let a , b , c , and d be real numbers and consider the function $f(x) = ax^3 + bx^2 + cx + d$. This polynomial function is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3ax^2 + 2bx + c.$$

The discriminant of this quadratic function is $(2b)^2 - 4(3a)(c) = 4(b^2 - 3ac)$. Consider the following cases:

- Case I: $b^2 - 3ac < 0$: In this case, $f'(x)$ is never equal to zero and is always positive when $a > 0$ and always negative when $a < 0$. Therefore, f is increasing for all x when $a > 0$ and decreasing for all x when $a < 0$.
- Case II: $b^2 - 3ac = 0$: In this case, $f'(x) = 0$ when $x = -\frac{b}{3a}$. For all other x , $f'(x) > 0$ when $a > 0$ and $f'(x) < 0$ when $a < 0$. Therefore, f is increasing for all $x \neq -\frac{b}{3a}$ when $a > 0$ and decreasing for all $x \neq -\frac{b}{3a}$ when $a < 0$. Because f is continuous on its domain, we can say f is increasing for all x when $a > 0$ and decreasing for all x when $a < 0$.
- Case III: $b^2 - 3ac > 0$: In this case, $f'(x) = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

For notation, let

$$x_1 = \min \left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \right) \quad \text{and} \quad x_2 = \max \left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \right).$$

If $a > 0$, then $f'(x) > 0$ and f is increasing on the intervals $(-\infty, x_1)$ and (x_2, ∞) and $f'(x) < 0$ and f is decreasing on the interval (x_1, x_2) . If $a < 0$, then $f'(x) > 0$ and f is increasing on the interval (x_1, x_2) and $f'(x) < 0$ and f is decreasing on the intervals $(-\infty, x_1)$ and (x_2, ∞) .

76. Let $f(x) = x^n + ax + b$, where n is a positive even integer. Suppose, for sake of contradiction, that f has more than two distinct real zeros. Let x_1, x_2 , and x_3 , with $x_1 < x_2 < x_3$, denote three of the distinct real zeros of f . Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Next, consider the interval $[x_2, x_3]$. Following the same reasoning as above, f is continuous on the closed interval $[x_2, x_3]$ and differentiable on the open interval (x_2, x_3) . Additionally, $f(x_2) = f(x_3) = 0$, so Rolle's Theorem applies and there exists at least one number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. It follows that there are at least two distinct numbers in (a, b) at which $f'(x) = 0$. However, $f'(x) = nx^{n-1} + a$, which has only one real zero because n is an even integer, so $n - 1$ is an odd integer. Therefore, the function $f(x) = x^n + ax + b$, where n is a positive even integer, has at most two distinct real zeros.
77. Let $f(x) = x^n + ax + b$, where n is a positive odd integer. Suppose, for sake of contradiction, that f has more than three distinct real zeros. Let x_1, x_2, x_3 , and x_4 , with $x_1 < x_2 < x_3 < x_4$, denote four of the distinct real zeros of f . Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Next, consider the interval $[x_2, x_3]$. Following the same reasoning as above, f is continuous on the closed interval $[x_2, x_3]$ and differentiable on the open interval (x_2, x_3) . Additionally, $f(x_2) = f(x_3) = 0$, so Rolle's Theorem applies and there exists at least one number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. Finally, consider the interval $[x_3, x_4]$. Following the same reasoning as above, f is continuous on the closed interval $[x_3, x_4]$ and differentiable on the open interval (x_3, x_4) . Additionally, $f(x_3) = f(x_4) = 0$, so Rolle's Theorem applies and there exists at least one number c_3 in (x_3, x_4) such that $f'(c_3) = 0$. It follows that there are at least three distinct numbers in (a, b) at which $f'(x) = 0$. However, $f'(x) = nx^{n-1} + a$, which has only at most two distinct real zeros because n is an odd integer, so $n - 1$ is an even integer. Therefore, the function $f(x) = x^n + ax + b$, where n is a positive odd integer, has at most three distinct real zeros.
78. Let $f(x) = x^n + ax^2 + b$, where n is a positive odd integer. Suppose, for sake of contradiction, that f has more than three distinct real zeros. Let x_1, x_2, x_3 , and x_4 , with $x_1 < x_2 < x_3 < x_4$, denote four of the distinct real zeros of f . Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. By similar reasoning, there exists at least one number c_2 in (x_2, x_3) and at least one number c_3 in (x_3, x_4) such that $f'(c_2) = f'(c_3) = 0$. However, $f'(x) = nx^{n-1} + 2ax = x(nx^{n-2} + 2a)$, which has only two distinct real zeros because n is an odd integer, so $n - 2$ is also an odd integer. Therefore, the function $f(x) = x^n + ax^2 + b$, where n is a positive odd integer, has at most three distinct real zeros.
79. Let $f(x) = x^n + ax^2 + b$, where n is a positive even integer. Suppose, for sake of contradiction, that f has more than four distinct real zeros. Let x_1, x_2, x_3, x_4 , and x_5 , with $x_1 < x_2 < x_3 < x_4 < x_5$, denote five of the distinct real zeros of f . Consider the interval $[x_1, x_2]$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2) . Additionally, $f(x_1) = f(x_2) = 0$, so Rolle's Theorem applies and there exists at least one number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. By similar reasoning, there exists at least one number c_2 in (x_2, x_3) , at least one number c_3 in (x_3, x_4) , and at least one number c_4 in (x_4, x_5) such that $f'(c_2) = f'(c_3) = f'(c_4) = 0$. However, $f'(x) = nx^{n-1} + 2ax = x(nx^{n-2} + 2a)$, which has at most three distinct real zeros because n is an even integer, so $n - 2$ is also an even integer. Therefore, the function $f(x) = x^n + ax^2 + b$, where n is a positive even integer, has at most four distinct real zeros.
80. Consider the function $f(x) = \sqrt{x}$ on the interval $[64, 66]$. Because f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, it is continuous on the closed interval $[64, 66]$ and differentiable on the open interval $(64, 66)$. By the Mean Value Theorem, there exists a number c in $(64, 66)$ for which

$$f'(c) = \frac{f(66) - f(64)}{66 - 64} = \frac{\sqrt{66} - \sqrt{64}}{2} = \frac{\sqrt{66} - 8}{2}.$$

Now,

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{so} \quad f'(c) = \frac{1}{2\sqrt{c}}.$$

With $64 < c < 66$, it follows that $8 < \sqrt{c} < \sqrt{66}$ and $16 < 2\sqrt{c} < 2\sqrt{66}$ so that

$$\frac{1}{2\sqrt{66}} < \frac{1}{2\sqrt{c}} = f'(c) < \frac{1}{16}.$$

Substituting for $f'(c)$ from the Mean Value Theorem gives

$$\frac{1}{2\sqrt{66}} < \frac{\sqrt{66} - 8}{2} < \frac{1}{16}.$$

Multiplying this compound inequality by 2 and noting that

$$\frac{1}{9} = \frac{1}{\sqrt{81}} < \frac{1}{\sqrt{66}},$$

then yields

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}.$$

81. Let a , b , c , and d be real numbers with $ad - bc \neq 0$. Additionally, let $n \geq 2$ be an integer. Consider the function $f(x) = \frac{ax^n + b}{cx^n + d}$. The rational function f is differentiable on its domain, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$\begin{aligned} f'(x) &= \frac{(cx^n + d)(anx^{n-1}) - (ax^n + b)(cnx^{n-1})}{(cx^n + d)^2} \\ &= \frac{acnx^{2n-1} + adnx^{n-1} - acnx^{2n-1} - bcnx^{n-1}}{(cx^n + d)^2} = \frac{(ad - bc)nx^{n-1}}{(cx^n + d)^2}, \end{aligned}$$

so 0 is the only critical number. Suppose that n is odd, so that $n - 1$ is even. If $ad - bc > 0$, then $f'(x) > 0$ and f is increasing everywhere on the domain of f except at $x = 0$. Because f is continuous on its domain, we can say that f is increasing on its domain. If $ad - bc < 0$, then $f'(x) < 0$ and f is decreasing everywhere on the domain of f except at $x = 0$. Because f is continuous on its domain, we can say that f is decreasing on its domain. On the other hand, suppose that n is even, so that $n - 1$ is odd. If $ad - bc > 0$, then $f'(x) > 0$ and f is increasing on that portion of the domain of f where $x > 0$, while $f'(x) < 0$ and f is decreasing on that portion of the domain of f where $x < 0$. If $ad - bc < 0$, then $f'(x) < 0$ and f is decreasing on that portion of the domain of f where $x < 0$, while $f'(x) > 0$ and f is increasing on that portion of the domain of f where $x > 0$.

82. Let $f(x) = -x + \ln(1 + x)^{1+x} = -x + (1 + x)\ln(1 + x)$. The domain of this function is the set $\{x | x > -1\}$. Now,

$$f'(x) = -1 + (1 + x) \cdot \frac{1}{1 + x} + \ln(1 + x) = -1 + 1 + \ln(1 + x) = \ln(1 + x),$$

so the only critical number is 0. The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. For $-1 < x < 0$, $0 < 1 + x < 1$, so $f'(x) = \ln(1 + x) < 0$; on the other hand, for $x > 0$, $1 + x > 1$ and $f'(x) = \ln(1 + x) > 0$. Therefore, f is decreasing on the interval $(-1, 0)$ and increasing on the interval $(0, \infty)$, so that f must have an absolute minimum at 0. Therefore,

$$f(x) = -x + \ln(1 + x)^{1+x} \geq f(0) = -0 + \ln 1^1 = 0,$$

so that, for $x > -1$

$$-x + \ln(1 + x)^{1+x} \geq 0 \quad \text{or} \quad \ln(1 + x)^{1+x} \geq x.$$

83. Let a and b be real numbers with $0 < a < b$ and consider the function $f(x) = \ln x$ on the interval $[a, b]$. The natural logarithm function is continuous and differentiable on its domain, so it is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . By the Mean Value Theorem, there exists a number c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln b - \ln a}{b - a} = \frac{1}{b - a} \ln \frac{b}{a}.$$

Now,

$$f'(x) = \frac{1}{x} \quad \text{so} \quad f'(c) = \frac{1}{c}.$$

With $a < c < b$, it follows that

$$\frac{1}{b} < \frac{1}{c} = f'(c) < \frac{1}{a},$$

so, substituting for $f'(c)$ from the Mean Value Theorem,

$$\frac{1}{b} < \frac{1}{b - a} \ln \frac{b}{a} < \frac{1}{a}.$$

Taking the reciprocal of each component of this compound inequality and then multiplying by $\ln(b/a)$, which is positive because $b > a$ so that $b/a > 1$, yields

$$a \ln \frac{b}{a} < b - a < b \ln \frac{b}{a}.$$

84. Let n be a positive integer and consider the function $f(x) = \ln x$ on the interval $[n, n + 1]$. The natural logarithm function is continuous and differentiable on its domain, so it is continuous on the closed interval $[n, n + 1]$ and differentiable on the open interval $(n, n + 1)$. By the Mean Value Theorem, there exists a number c in $(n, n + 1)$ for which

$$f'(c) = \frac{f(n + 1) - f(n)}{(n + 1) - n} = \frac{\ln(n + 1) - \ln n}{1} = \ln \left(\frac{n + 1}{n} \right) = \ln \left(1 + \frac{1}{n} \right).$$

Now,

$$f'(x) = \frac{1}{x} \quad \text{so} \quad f'(c) = \frac{1}{c}.$$

With $n < c < n + 1$, it follows that

$$\frac{1}{n + 1} < \frac{1}{c} = f'(c) < \frac{1}{n},$$

so, substituting for $f'(c)$ from the Mean Value Theorem,

$$\frac{1}{n + 1} < \ln \left(1 + \frac{1}{n} \right) < \frac{1}{n}.$$

Multiplying this compound inequality by n then yields

$$\frac{n}{n + 1} < n \ln \left(1 + \frac{1}{n} \right) < 1 \quad \text{or} \quad \frac{n}{n + 1} < \ln \left(1 + \frac{1}{n} \right)^n < 1.$$

4.4: Local Extrema and Concavity

Concepts and Vocabulary

1. **False**. If a function f is continuous on the interval $[a, b]$, differentiable on the interval (a, b) , and changes from an increasing function to a decreasing function at the point $(c, f(c))$, then $f(c)$ is a **local maximum value of f** .
2. **False**. Suppose c is a critical number of f and (a, b) is an open interval containing c . If $f'(x)$ is positive on both sides of c , then $f(c)$ is **neither a local maximum value nor a local minimum value** of f .
3. Suppose a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) . If the graph of f lies above each of its tangent lines on the interval (a, b) , then f is **(a) concave up** on (a, b) .
4. If the acceleration of an object in rectilinear motion is negative, then the velocity of the object is **(b) decreasing**.
5. Suppose f is a function that is differentiable on an open interval containing c and the concavity of f changes at the point $(c, f(c))$. Then $(c, f(c))$ is an **(a) inflection point** of f .
6. Suppose a function f is continuous on a closed interval $[a, b]$ and both f' and f'' exist on the open interval (a, b) . If $f''(x) > 0$ on the interval (a, b) , then f is **(c) concave up** on (a, b) .
7. **True**. Suppose f is a function for which f' and f'' exist on an open interval (a, b) and suppose c , $a < c < b$, is a critical number of f . If $f''(c) = 0$, then the Second Derivative Test cannot be used to determine if there is a local extremum at c .
8. **False**. Suppose a function f is differentiable on the open interval (a, b) . If either $f''(c) = 0$ or f'' does not exist at the number c in (a, b) , then $(c, f(c))$ is a **candidate** for an inflection point of f . The point $(c, f(c))$ will be an inflection point if concavity changes at c .

Skill Building

9. (a) Based on the graph of f , f has
 - a **local maximum value at the point $(-1, 0)$** ,
 - a **local minimum value at the points $(-2.5, -4)$ and $(0.5, -4)$** , and
 - a **point of inflection at the points $(-1.8, -2)$ and $(-0.2, -2)$** .
 (b) Based on the graph of f , f is increasing on the intervals $(-2.5, -1)$ and $(0.5, \infty)$ and decreasing on the intervals $(-\infty, -2.5)$ and $(-1, 0.5)$. Since f is continuous on its domain, we can say that f is
 - **increasing on the intervals $[-2.5, -1]$ and $[0.5, \infty)$** and
 - **decreasing on the intervals $(-\infty, -2.5]$ and $[-1, 0.5]$** .
 Additionally, f is
 - **concave up on the intervals $(-\infty, -1.8)$ and $(-0.2, \infty)$** , and
 - **concave down on the interval $(-1.8, -0.2)$** .
10. (a) Based on the graph of f , f has

- a local maximum value at the points $(-1, 2)$ and $(7, 7)$,
 - a local minimum value at the point $(2, -6)$, and
 - a point of inflection at the points $(0, -2)$ and $(4, 0)$.
- (b) Based on the graph of f , f is increasing on the intervals $(-4, -1)$ and $(2, 7)$ and decreasing on the intervals $(-1, 2)$ and $(7, 10)$. Since f is continuous on its domain, we can say that f is
- increasing on the intervals $[-4, -1]$ and $[2, 7]$ and
 - decreasing on the intervals $[-1, 2]$ and $[7, 10]$.

Additionally, f is

- concave up on the interval $(0, 4)$, and
 - concave down on the intervals $(-4, 0)$ and $(4, 10)$.
11. (a) Based on the graph of f , f has
- a local maximum value at the points $(-2, 3)$ and $(12, 10)$,
 - a local minimum value at the point $(0, 0)$, and
 - no points of inflection.
- (b) Based on the graph of f , f is increasing on the intervals $(-\infty, -2)$ and $(0, 12)$ and decreasing on the intervals $(-2, 0)$ and $(12, \infty)$. Since f is continuous on its domain, we can say that f is
- increasing on the intervals $(-\infty, -2]$ and $[0, 12]$ and
 - decreasing on the intervals $[-2, 0]$ and $[12, \infty)$.

Additionally, f is concave down on the intervals $(-\infty, 0)$ and $(0, \infty)$. The function is never concave up.

12. (a) Based on the graph of f , f has
- a local maximum value at the point $(-2, 4)$,
 - a local minimum value at the point $(3, -1)$, and
 - a point of inflection at the point $(-2, 4)$.
- (b) Based on the graph of f , f is increasing on the intervals $(-7, -2)$ and $(3, \infty)$ and decreasing on the interval $(-2, 3)$. Since f is continuous on its domain, we can say that f is

increasing on the intervals $[-7, -2]$ and $[3, \infty)$ and decreasing on the interval $[-2, 3]$.

Additionally, f is concave up on the interval $(-2, \infty)$ and concave down on the interval $(-7, -2)$.

13. Let $f(x) = x^3 - 6x^2 + 2$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 - 12x = 3x(x - 4),$$

so 0 and 4 are critical numbers.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 4 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x$	Sign of $x - 4$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	−	−	+	f is increasing
$(0, 4)$	+	−	−	f is decreasing
$(4, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has a local maximum value at 0 and a local minimum value at 4. The local maximum value is $f(0) = 2$; the local minimum value is $f(4) = -30$.

14. Let $f(x) = x^3 + 6x^2 + 12x + 1$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 + 12x + 12 = 3(x^2 + 4x + 4) = 3(x + 2)^2,$$

so -2 is the only critical number.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. Because $f'(x) = 3(x + 2)^2$ is positive for $x \neq -2$, f is increasing on the intervals $(-\infty, -2)$ and $(-2, \infty)$. Therefore, by the First Derivative Test, f has neither a local maximum value nor a local minimum value at -2 .

15. Let $f(x) = 3x^4 - 4x^3$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1),$$

so 0 and 1 are critical numbers.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 1 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $12x^2$	Sign of $x - 1$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	+	−	−	f is decreasing
$(0, 1)$	+	−	−	f is decreasing
$(1, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has neither a local maximum value nor a local minimum value at 0 and a local minimum value at 1. The local minimum value is $f(1) = -1$.

16. Let $h(x) = x^4 + 2x^3 - 3$.

- (a) The polynomial function h is differentiable everywhere, so the critical numbers of h occur where $h'(x) = 0$. Now,

$$h'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3),$$

so $-\frac{3}{2}$ and 0 are critical numbers.

- (b) The Increasing/Decreasing Function Test states that h is increasing on intervals where $h'(x) > 0$ and that h is decreasing on intervals where $h'(x) < 0$. These inequalities are solved by using the critical numbers $-\frac{3}{2}$ and 0 to form three intervals. The sign of $h'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^2$	Sign of $2x + 3$	Sign of $h'(x)$	Conclusion
$(-\infty, -\frac{3}{2})$	+	-	-	h is decreasing
$(-\frac{3}{2}, 0)$	+	+	+	h is increasing
$(0, \infty)$	+	+	+	h is increasing

Therefore, by the First Derivative Test, h has a local minimum value at $-\frac{3}{2}$ and neither a local maximum value nor a local minimum value at 0. The local minimum value is $h\left(-\frac{3}{2}\right) = -\frac{75}{16}$.

17. Let $f(x) = (5 - 2x)e^x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = (5 - 2x)e^x - 2e^x = (3 - 2x)e^x,$$

so $\frac{3}{2}$ is the only critical number.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number $\frac{3}{2}$ to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3 - 2x$	Sign of e^x	Sign of $f'(x)$	Conclusion
$(-\infty, \frac{3}{2})$	+	+	+	f is increasing
$(\frac{3}{2}, \infty)$	-	+	-	f is decreasing

Therefore, by the First Derivative Test, f has a local maximum value at $\frac{3}{2}$. The

$$\text{local maximum value is } f\left(\frac{3}{2}\right) = 2e^{3/2}.$$

18. Let $f(x) = (x - 8)e^x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = (x - 8)e^x + e^x = (x - 7)e^x,$$

so 7 is the only critical number.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number 7 to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x - 7$	Sign of e^x	Sign of $f'(x)$	Conclusion
$(-\infty, 7)$	-	+	-	f is decreasing
$(7, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has a local minimum value at 7. The local minimum value is $f(7) = -e^7$.

19. Let $g(u) = u^{-1}e^u$.

(a) The critical numbers of g occur where $g'(u) = 0$ or where $g'(u)$ does not exist. Now,

$$g'(u) = u^{-1}e^u - u^{-2}e^u = \frac{(u-1)e^u}{u^2},$$

so $g'(u) = 0$ when $u = 1$ and $g'(u)$ does not exist when $u = 0$. As 0 is not in the domain of g , 0 is not a critical number. Therefore, $\boxed{1}$ is the only critical number of g .

(b) The Increasing/Decreasing Function Test states that g is increasing on intervals where $g'(u) > 0$ and that g is decreasing on intervals where $g'(u) < 0$. These inequalities are solved by using 0 and the critical number 1 to form three intervals. The sign of $g'(u)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $u-1$	Sign of e^u	Sign of u^2	Sign of $g'(u)$	Conclusion
$(-\infty, 0)$	—	+	+	—	g is decreasing
$(0, 1)$	—	+	+	—	g is decreasing
$(1, \infty)$	+	+	+	+	g is increasing

Therefore, by the First Derivative Test, g has a local minimum value at 1. The local minimum value is $g(1) = e^1 = e$.

20. Let $f(x) = x^{-2}e^x$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{-2}e^x - 2x^{-3}e^x = \frac{(x-2)e^x}{x^3},$$

so $f'(x) = 0$ when $x = 2$ and $f'(x)$ does not exist when $x = 0$. As 0 is not in the domain of f , 0 is not a critical number. Therefore, $\boxed{2}$ is the only critical number of f .

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using 0 and the critical number 2 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x-2$	Sign of e^x	Sign of x^3	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	+	f is increasing
$(0, 2)$	—	+	+	—	f is decreasing
$(2, \infty)$	+	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has a local minimum value at 2. The local minimum value is $f(2) = \frac{1}{4}e^2$. Note that f does not have a local maximum value at 0 because f is not defined at 0.

21. Let $f(x) = x^{2/3} + x^{1/3}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3} = \frac{2x^{1/3} + 1}{3x^{2/3}},$$

so $f'(x) = 0$ when $x = -\frac{1}{8}$ and $f'(x)$ does not exist when $x = 0$. Therefore, $\boxed{-\frac{1}{8}}$ and $\boxed{0}$ are the critical numbers of f .

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $-\frac{1}{8}$ and 0 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^{1/3} + 1$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{1}{8})$	—	+	—	f is decreasing
$(-\frac{1}{8}, 0)$	+	+	+	f is increasing
$(0, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has a local minimum value at $-\frac{1}{8}$ but has neither a local maximum value nor a local minimum value at 0. The local minimum value is $f\left(-\frac{1}{8}\right) = -\frac{1}{4}$.

22. Let $f(x) = \frac{1}{2}x^{2/3} - x^{1/3}$.

- (a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{1}{3}x^{-1/3} - \frac{1}{3}x^{-2/3} = \frac{x^{1/3} - 1}{3x^{2/3}},$$

so $f'(x) = 0$ when $x = 1$ and $f'(x)$ does not exist when $x = 0$. Therefore, $\boxed{0 \text{ and } 1}$ are the critical numbers of f .

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers 0 and 1 to form three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x^{1/3} - 1$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	f is decreasing
$(0, 1)$	—	+	—	f is decreasing
$(1, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has neither a local maximum value nor a local minimum value at 0, but has a local minimum value at 1. The local minimum value is $f(1) = -\frac{1}{2}$.

23. Let $g(x) = x^{2/3}(x^2 - 4)$.

- (a) The critical numbers of g occur where $g'(x) = 0$ or where $g'(x)$ does not exist. Now,

$$g'(x) = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{6x^2 + 2(x^2 - 4)}{3x^{1/3}} = \frac{8(x-1)(x+1)}{3x^{1/3}},$$

so $g'(x) = 0$ when $x = \pm 1$ and $g'(x)$ does not exist when $x = 0$. Therefore, $\boxed{-1, 0, \text{ and } 1}$ are the critical numbers of g .

- (b) The Increasing/Decreasing Function Test states that g is increasing on intervals where $g'(x) > 0$ and that g is decreasing on intervals where $g'(x) < 0$. These inequalities are solved by using the critical numbers -1 , 0, and 1 to form four intervals. The sign of $g'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x-1)$	Sign of $x+1$	Sign of $3x^{1/3}$	Sign of $g'(x)$	Conclusion
$(-\infty, -1)$	—	—	—	—	g is decreasing
$(-1, 0)$	—	+	—	+	g is increasing
$(0, 1)$	—	+	+	—	g is decreasing
$(1, \infty)$	+	+	+	+	g is increasing

Therefore, by the First Derivative Test, g has a local minimum value at -1 , a local maximum value at 0 , and a local minimum value at 1 . The local minimum values are $g(-1) = -3$ and $g(1) = -3$, and the local maximum value is $g(0) = 0$.

24. Let $f(x) = x^{1/3}(x^2 - 9)$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{1/3}(2x) + \frac{1}{3}x^{-2/3}(x^2 - 9) = \frac{6x^2 + (x^2 - 9)}{3x^{2/3}} = \frac{7x^2 - 9}{3x^{2/3}},$$

so $f'(x) = 0$ when $x = \pm \frac{3\sqrt{7}}{7}$ and $f'(x)$ does not exist when $x = 0$. Therefore, $-\frac{3\sqrt{7}}{7}$, 0 , and $\frac{3\sqrt{7}}{7}$ are the critical numbers of f .

(b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical numbers $-\frac{3\sqrt{7}}{7}$, 0 , $\frac{3\sqrt{7}}{7}$ to form four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $7x^2 - 9$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{3\sqrt{7}}{7})$	+	+	+	f is increasing
$(-\frac{3\sqrt{7}}{7}, 0)$	—	+	—	f is decreasing
$(0, \frac{3\sqrt{7}}{7})$	—	+	—	f is decreasing
$(\frac{3\sqrt{7}}{7}, \infty)$	+	+	+	f is increasing

Therefore, by the First Derivative Test, f has a local maximum value at $-\frac{3\sqrt{7}}{7}$, neither a local maximum value nor a local minimum value at 0 , and a local minimum value at $\frac{3\sqrt{7}}{7}$. The local maximum value is

$$f\left(-\frac{3\sqrt{7}}{7}\right) = \left(-\frac{3\sqrt{7}}{7}\right)^{1/3} \left(\frac{9}{7} - 9\right) \approx 8.044,$$

and the local minimum value is

$$f\left(\frac{3\sqrt{7}}{7}\right) = \left(\frac{3\sqrt{7}}{7}\right)^{1/3} \left(\frac{9}{7} - 9\right) \approx -8.044.$$

25. Let $f(x) = \frac{\ln x}{x^3}$.

- (a) The function f is differentiable on its domain, the set $\{x|x > 0\}$, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - 3x^2 \ln x}{x^6} = \frac{1 - 3 \ln x}{x^4},$$

so $f'(x) = 0$ when $x = \sqrt[3]{e}$. Therefore, $\sqrt[3]{e}$ is the only critical number of f .

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. These inequalities are solved by using the critical number $\sqrt[3]{e}$ to form two intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $1 - 3 \ln x$	Sign of x^4	Sign of $f'(x)$	Conclusion
$(0, \sqrt[3]{e})$	+	+	+	f is increasing
$(\sqrt[3]{e}, \infty)$	-	+	-	f is decreasing

Therefore, by the First Derivative Test, f has a local maximum value at $\sqrt[3]{e}$. The

local maximum value is $f(\sqrt[3]{e}) = \frac{1}{3e}$.

26. Let $h(x) = \frac{\ln x}{\sqrt{x^3}}$.

- (a) The function h is differentiable on its domain, the set $\{x|x > 0\}$, so the critical numbers of h occur where $h'(x) = 0$. Now,

$$h'(x) = \frac{x^{3/2} \cdot \frac{1}{x} - \frac{3}{2}x^{1/2} \ln x}{x^3} = \frac{2 - 3 \ln x}{2x^{5/2}},$$

so $h'(x) = 0$ when $x = \sqrt[3]{e^2}$. Therefore, $\sqrt[3]{e^2}$ is the only critical number of h .

- (b) The Increasing/Decreasing Function Test states that h is increasing on intervals where $h'(x) > 0$ and that h is decreasing on intervals where $h'(x) < 0$. These inequalities are solved by using the critical number $\sqrt[3]{e^2}$ to form two intervals. The sign of $h'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2 - 3 \ln x$	Sign of $2x^{5/2}$	Sign of $h'(x)$	Conclusion
$(0, \sqrt[3]{e^2})$	+	+	+	h is increasing
$(\sqrt[3]{e^2}, \infty)$	-	+	-	h is decreasing

Therefore, by the First Derivative Test, h has a local maximum value at $\sqrt[3]{e^2}$. The

local maximum value is $h(\sqrt[3]{e^2}) = \frac{2}{3e}$.

27. Let

$$g(x) = |x^2 - 1| = \begin{cases} x^2 - 1, & x \leq -1 \\ -(x^2 - 1), & -1 < x < 1 \\ x^2 - 1, & x \geq 1. \end{cases}$$

- (a) On the interval $(-\infty, -1)$, $g'(x) = 2x$ exists and is never equal to 0; g has no critical numbers on this interval. On the interval $(-1, 1)$, $g'(x) = -2x$ exists and is equal to 0 when $x = 0$. Therefore, 0 is a critical number. On the interval $(1, \infty)$, $g'(x) = 2x$ exists and is never equal to 0; g has

no critical numbers on this interval. At $x = \pm 1$, the rule for g changes, so it is necessary to investigate the existence of $g'(-1)$ and $g'(1)$. Now,

$$\lim_{x \rightarrow -1^-} \frac{g(x) - g(-1)}{x - (-1)} = \lim_{x \rightarrow -1^-} \frac{(x^2 - 1) - 0}{x + 1} = \lim_{x \rightarrow -1^-} \frac{(x - 1)(x + 1)}{x + 1} = \lim_{x \rightarrow -1^-} (x - 1) = -2,$$

and

$$\lim_{x \rightarrow -1^+} \frac{g(x) - g(-1)}{x - (-1)} = \lim_{x \rightarrow -1^+} \frac{-(x^2 - 1) - 0}{x + 1} = \lim_{x \rightarrow -1^+} \frac{-(x - 1)(x + 1)}{x + 1} = \lim_{x \rightarrow -1^+} -(x - 1) = 2.$$

Because these two one-sided limits are not equal, $g'(x)$ does not exist at $x = -1$. Additionally,

$$\lim_{x \rightarrow 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x^2 - 1) - 0}{x - 1} = \lim_{x \rightarrow 1^-} \frac{-(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1^-} -(x + 1) = -2,$$

and

$$\lim_{x \rightarrow 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x^2 - 1) - 0}{x - 1} = \lim_{x \rightarrow 1^+} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

Because these two one-sided limits are not equal, $g'(x)$ does not exist at $x = 1$. Therefore, $\boxed{\pm 1}$ are also critical numbers of g .

- (b) The Increasing/Decreasing Function Test states that g is increasing on intervals where $g'(x) > 0$ and that g is decreasing on intervals where $g'(x) < 0$. On the interval $(-\infty, -1)$, $g'(x) = 2x < 0$; on the interval $(-1, 0)$, $g'(x) = -2x > 0$; on the interval $(0, 1)$, $g'(x) = -2x < 0$; finally, on the interval $(1, \infty)$, $g'(x) = 2x > 0$. Therefore, g is decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$ and increasing on the intervals $(-1, 0)$ and $(1, \infty)$. By the First Derivative Test, it follows that g has a local minimum value at -1 , a local maximum value at 0 , and a local minimum value at 1 . The local minimum values are $g(-1) = 0$ and $g(1) = 0$, and the local maximum value is $g(0) = 1$.

28. Let

$$f(x) = |x^2 - 4| = \begin{cases} x^2 - 4, & x \leq -2 \\ -(x^2 - 4), & -2 < x < 2 \\ x^2 - 4, & x \geq 2. \end{cases}$$

- (a) On the interval $(-\infty, -2)$, $f'(x) = 2x$ exists and is never equal to 0; f has no critical numbers on this interval. On the interval $(-2, 2)$, $f'(x) = -2x$ exists and is equal to 0 when $x = 0$. Therefore, $\boxed{0}$ is a critical number. On the interval $(2, \infty)$, $f'(x) = 2x$ exists and is never equal to 0; f has no critical numbers on this interval. At $x = \pm 2$, the rule for f changes, so it is necessary to investigate the existence of $f'(-2)$ and $f'(2)$. Now,

$$\lim_{x \rightarrow -2^-} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2^-} \frac{(x^2 - 4) - 0}{x + 2} = \lim_{x \rightarrow -2^-} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x \rightarrow -2^-} (x - 2) = -4,$$

and

$$\lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2^+} \frac{-(x^2 - 4) - 0}{x + 2} = \lim_{x \rightarrow -2^+} \frac{-(x - 2)(x + 2)}{x + 2} = \lim_{x \rightarrow -2^+} -(x - 2) = 4.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = -2$. Additionally,

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-(x^2 - 4) - 0}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2^-} -(x + 2) = -4,$$

and

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x^2 - 4) - 0}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2^+} (x + 2) = 4.$$

Because these two one-sided limits are not equal, $f'(x)$ does not exist at $x = 2$. Therefore, $\boxed{\pm 2}$ are also critical numbers of f .

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. On the interval $(-\infty, -2)$, $f'(x) = 2x < 0$; on the interval $(-2, 0)$, $f'(x) = -2x > 0$; on the interval $(0, 2)$, $f'(x) = -2x < 0$; finally, on the interval $(2, \infty)$, $f'(x) = 2x > 0$. Therefore, f is decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$ and increasing on the intervals $(-2, 0)$ and $(2, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at -2 , a local maximum value at 0 , and a local minimum value at 2 . The local minimum values are $f(-2) = 0$ and $f(2) = 0$, and the local maximum value is $f(0) = 4$.

29. Let $f(\theta) = \sin \theta - 2 \cos \theta$.

- (a) The function f is the difference of the trigonometric function $\sin x$ and a constant multiple of the trigonometric function $\cos x$, so it is differentiable everywhere. The critical numbers of f therefore occur where $f'(\theta) = 0$. Now,

$$f'(\theta) = \cos \theta + 2 \sin \theta,$$

so $f'(\theta) = 0$ when $\cos \theta = -2 \sin \theta$ or $\tan \theta = -\frac{1}{2}$. It follows that the critical numbers of f are

$-\tan^{-1} \frac{1}{2} + k\pi$, where k is any integer. Remember that the tangent function is periodic with period π .

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(\theta) > 0$ and that f is decreasing on intervals where $f'(\theta) < 0$. The critical numbers of f divide the number line into intervals of the form

$$\left(-\tan^{-1} \frac{1}{2} + 2k\pi, -\tan^{-1} \frac{1}{2} + (2k+1)\pi \right)$$

and

$$\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi, -\tan^{-1} \frac{1}{2} + (2k+2)\pi \right)$$

for each integer k . Select a test number in each interval and evaluate f' at that number. If $f'(\theta) < 0$ at the test number, then $f'(\theta) < 0$ throughout the interval; if $f'(\theta) > 0$ at the test number, then $f'(\theta) > 0$ throughout the interval. The following table summarizes the results:

Interval	Test Number	Value of $f'(\theta)$ at Test Number	Sign of $f'(\theta)$	Conclusion
$\left(-\tan^{-1} \frac{1}{2} + 2k\pi, -\tan^{-1} \frac{1}{2} + (2k+1)\pi \right)$	$2k\pi$	1	+	f is increasing
$\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi, -\tan^{-1} \frac{1}{2} + (2k+2)\pi \right)$	$(2k+1)\pi$	-1	-	f is decreasing

Therefore, f is increasing on the intervals $\left(-\tan^{-1} \frac{1}{2} + 2k\pi, -\tan^{-1} \frac{1}{2} + (2k+1)\pi \right)$ and decreasing on the intervals $\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi, -\tan^{-1} \frac{1}{2} + (2k+2)\pi \right)$. By the First Derivative Test, it follows that f has a local maximum value at all points of the form $-\tan^{-1} \frac{1}{2} + (2k+1)\pi$ and a local minimum value at all points of the form $-\tan^{-1} \frac{1}{2} + 2k\pi$. The local maximum values are

$$\begin{aligned} f\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi\right) &= \sin\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi\right) - 2\cos\left(-\tan^{-1} \frac{1}{2} + (2k+1)\pi\right) \\ &= \sin\left(\tan^{-1} \frac{1}{2}\right) + 2\cos\left(\tan^{-1} \frac{1}{2}\right) = \frac{1}{\sqrt{5}} + 2\frac{2}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \boxed{\sqrt{5}}, \end{aligned}$$

and the local minimum values are

$$\begin{aligned} f\left(-\tan^{-1}\frac{1}{2} + 2k\pi\right) &= \sin\left(-\tan^{-1}\frac{1}{2} + 2k\pi\right) - 2\cos\left(-\tan^{-1}\frac{1}{2} + 2k\pi\right) \\ &= -\sin\left(\tan^{-1}\frac{1}{2}\right) - 2\cos\left(\tan^{-1}\frac{1}{2}\right) = -\frac{1}{\sqrt{5}} - 2\frac{2}{\sqrt{5}} = -\frac{5}{\sqrt{5}} = \boxed{-\sqrt{5}}. \end{aligned}$$

30. Let $f(x) = x + 2\sin x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 1 + 2\cos x,$$

so $f'(x) = 0$ when $\cos x = -\frac{1}{2}$. Therefore, the critical numbers of f are $\boxed{\frac{2\pi}{3} + 2k\pi}$ and $\boxed{\frac{4\pi}{3} + 2k\pi}$, where k is any integer.

- (b) The Increasing/Decreasing Function Test states that f is increasing on intervals where $f'(x) > 0$ and that f is decreasing on intervals where $f'(x) < 0$. The critical numbers of f divide the number line into intervals of the form

$$\left(\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi\right) \quad \text{and} \quad \left(\frac{4\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2(k+1)\pi\right),$$

for each integer k . Select a test number in each interval and evaluate f' at that number. If $f'(x) < 0$ at the test number, then $f'(x) < 0$ throughout the interval; if $f'(x) > 0$ at the test number, then $f'(x) > 0$ throughout the interval. The following table summarizes the results:

Interval	Test Number	Value of $f'(x)$ at Test Number	Sign of $f'(x)$	Conclusion
$\left(\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi\right)$	$\pi + 2k\pi$	-1	$-$	f is decreasing
$\left(\frac{4\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2(k+1)\pi\right)$	$2(k+1)\pi$	3	$+$	f is increasing

Therefore, f is decreasing on the intervals $\left(\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi\right)$ and increasing on the intervals $\left(\frac{4\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2(k+1)\pi\right)$. By the First Derivative Test, it follows that f has a local minimum value at all points of the form $\frac{4\pi}{3} + 2k\pi$ and a local maximum value at all points of the form $\frac{2\pi}{3} + 2k\pi$. The local minimum values are

$$\boxed{f\left(\frac{4\pi}{3} + 2k\pi\right) = \frac{4\pi}{3} + 2k\pi + 2\sin\left(\frac{4\pi}{3} + 2k\pi\right) = \frac{4\pi}{3} + 2k\pi - \sqrt{3}},$$

and the local maximum values are

$$\boxed{f\left(\frac{2\pi}{3} + 2k\pi\right) = \frac{2\pi}{3} + 2k\pi + 2\sin\left(\frac{2\pi}{3} + 2k\pi\right) = \frac{2\pi}{3} + 2k\pi + \sqrt{3}}.$$

31. Let $s = t^2 - 2t + 3$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 2t - 2 = 2(t - 1),$$

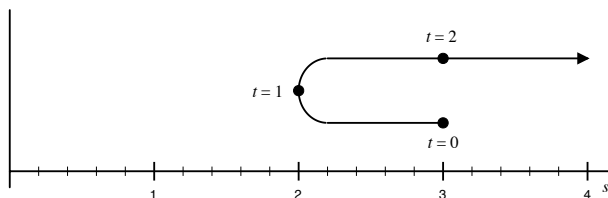
so 1 is the only critical number of s . When $t < 1$, $v(t) < 0$ and when $t > 1$, $v(t) > 0$. Therefore, the object moves to the right on the interval $(1, \infty)$ and moves to the left on the interval $(0, 1)$.

- (b) The object reverses direction at $t = 1$.
- (c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

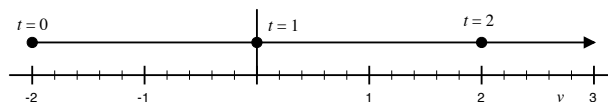
$$a(t) = v'(t) = 2 > 0$$

for all $t \geq 0$. Therefore, the velocity is increasing on the interval $(0, \infty)$.

- (d) The figure below illustrates the motion of the object.



- (e) The figure below illustrates the velocity of the object.



32. Let $s = 2t^2 + 8t - 7$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 4t + 8 = 4(t + 2).$$

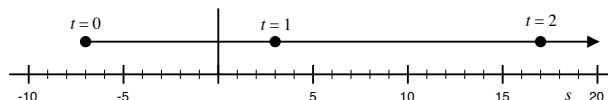
As the domain of s is the set $\{t | t \geq 0\}$ and -2 is not in this domain, -2 is not a critical number. Because $v(t) > 0$ for all $t \geq 0$, the object moves to the right on the interval $(0, \infty)$.

- (b) The object does not reverse direction.
- (c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

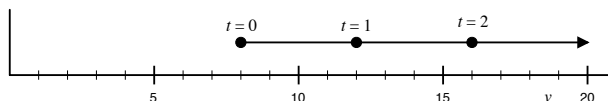
$$a(t) = v'(t) = 4 > 0$$

for all $t \geq 0$. Therefore, the velocity is increasing on the interval $(0, \infty)$.

- (d) The figure below illustrates the motion of the object.



- (e) The figure below illustrates the velocity of the object.



33. Let $s = 2t^3 + 6t^2 - 18t + 1$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 6t^2 + 12t - 18 = 6(t^2 + 2t - 3) = 6(t + 3)(t - 1).$$

As the domain of s is the set $\{t | t \geq 0\}$ and -3 is not in this domain, -3 is not a critical number. When $t < 1$, $v(t) < 0$ and when $t > 1$, $v(t) > 0$. Therefore, the object moves to the right on the interval $(1, \infty)$ and moves to the left on the interval $(0, 1)$.

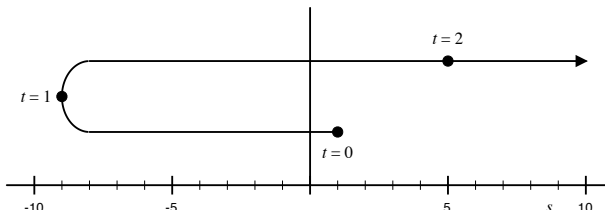
(b) The object reverses direction at $t = 1$.

(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

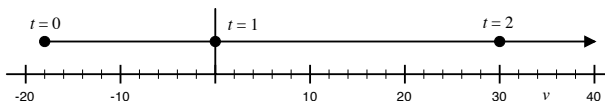
$$a(t) = v'(t) = 12t + 12 = 12(t + 1) > 0$$

for all $t \geq 0$. Therefore, the velocity is increasing on the interval $(0, \infty)$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



34. Let $s = 3t^4 - 16t^3 + 24t^2$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 12t^3 - 48t^2 + 48t = 12t(t^2 - 4t + 4) = 12t(t - 2)^2,$$

so $v(t) > 0$ on the intervals $(0, 2)$ and $(2, \infty)$. Therefore, the object is moving to the right on the intervals $(0, 2)$ and $(2, \infty)$.

(b) The object does not reverse direction.

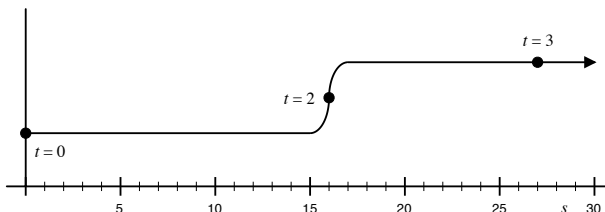
(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

$$a(t) = v'(t) = 36t^2 - 96t + 48 = 12(3t^2 - 8t + 4) = 12(3t - 2)(t - 2).$$

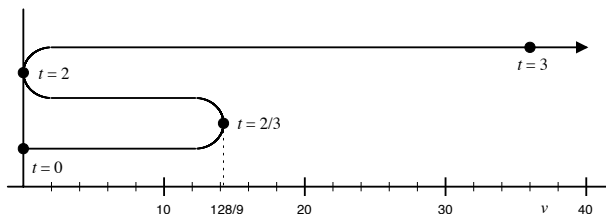
When $t < \frac{2}{3}$ and when $t > 2$, $a(t) > 0$; when $\frac{2}{3} < t < 2$, $a(t) < 0$. Therefore, the velocity of the ob-

ject is increasing on the intervals $\left(0, \frac{2}{3}\right)$ and $(2, \infty)$, and the velocity is decreasing on the interval $\left(\frac{2}{3}, 2\right)$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



35. Let $s = 2t - \frac{6}{t}$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 2 + \frac{6}{t^2} > 0$$

for all $t > 0$. Therefore, the object is moving to the right on the interval $(0, \infty)$.

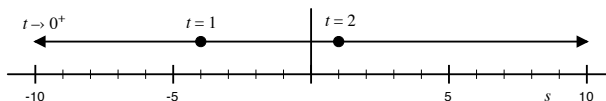
- (b) The object does not reverse direction.

- (c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

$$a(t) = v'(t) = -\frac{12}{t^3} < 0$$

for all $t > 0$. Therefore, the velocity of the object is decreasing on the interval $(0, \infty)$.

- (d) The figure below illustrates the motion of the object.



- (e) The figure below illustrates the velocity of the object.



36. Let $s = 3\sqrt{t} - \frac{1}{\sqrt{t}}$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = \frac{3}{2\sqrt{t}} + \frac{1}{2\sqrt{t^3}} = \frac{3t+1}{2\sqrt{t^3}} > 0$$

for all $t > 0$. Therefore, the object is moving to the right on the interval $(0, \infty)$.

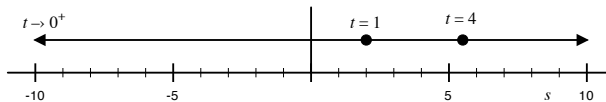
- (b) The object does not reverse direction.

- (c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

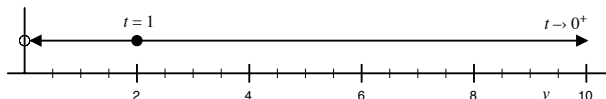
$$a(t) = v'(t) = -\frac{3}{4\sqrt{t^3}} - \frac{3}{4\sqrt{t^5}} = -\frac{3}{4} \frac{t+1}{\sqrt{t^5}} < 0$$

for all $t > 0$. Therefore, the velocity of the object is decreasing on the interval $(0, \infty)$.

- (d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



37. Let $s = 2 \sin(3t)$ for $0 \leq t \leq \frac{2\pi}{3}$.

(a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 6 \cos(3t).$$

When $0 < t < \frac{\pi}{6}$, $v(t) > 0$; when $\frac{\pi}{6} < t < \frac{\pi}{2}$, $v(t) < 0$; and when $\frac{\pi}{2} < t < \frac{2\pi}{3}$, $v(t) > 0$.

Therefore, the object is moving to the right on the intervals $\left(0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and is moving to the left on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

(b) The object reverses direction at $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$.

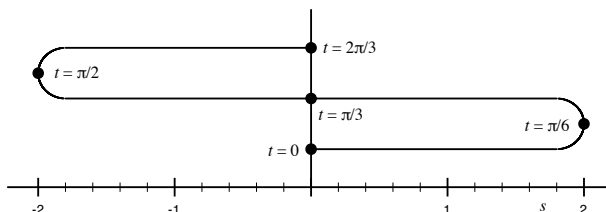
(c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now,

$$a(t) = v'(t) = -18 \sin(3t).$$

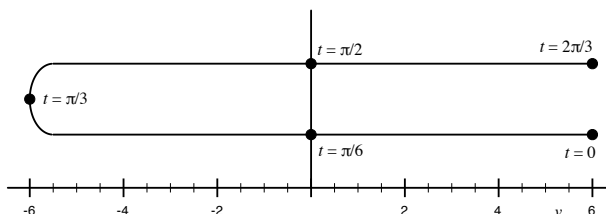
When $0 < t < \frac{\pi}{3}$, $a(t) < 0$, and when $\frac{\pi}{3} < t < \frac{2\pi}{3}$, $a(t) > 0$. Therefore, the velocity of the object

is decreasing on the interval $\left(0, \frac{\pi}{3}\right)$ and is increasing on the interval $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$.

(d) The figure below illustrates the motion of the object.



(e) The figure below illustrates the velocity of the object.



38. Let $s = 3 \cos(\pi t)$ for $0 \leq t \leq 2$.

- (a) The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = -3\pi \sin(\pi t).$$

When $0 < t < 1$, $v(t) < 0$, and when $1 < t < 2$, $v(t) > 0$. Therefore, the object is moving to the left on the interval $(0, 1)$ and is moving to the right on the interval $(1, 2)$.

- (b) The object reverses direction at $t = 1$.
 (c) The velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now,

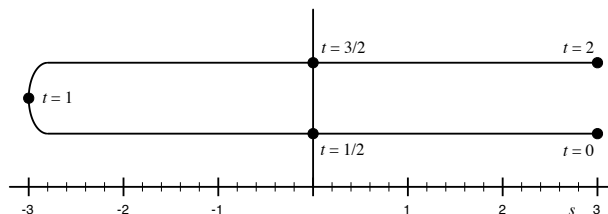
$$a(t) = v'(t) = -3\pi^2 \cos(\pi t).$$

When $0 < t < \frac{1}{2}$, $a(t) < 0$; when $\frac{1}{2} < t < \frac{3}{2}$, $a(t) > 0$; and when $\frac{3}{2} < t < 2$, $a(t) < 0$.

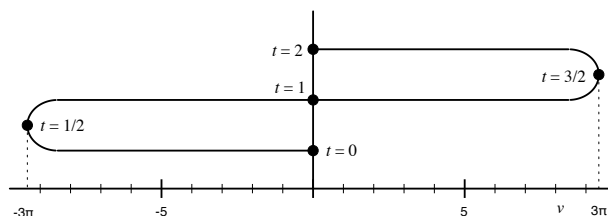
Therefore, the velocity of the object is decreasing on the intervals $(0, \frac{1}{2})$ and $(\frac{3}{2}, 2)$ and is

increasing on the interval $(\frac{1}{2}, \frac{3}{2})$.

- (d) The figure below illustrates the motion of the object.



- (e) The figure below illustrates the velocity of the object.



39. Let $f(x) = x^2 - 2x + 5$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 2x - 2 \quad \text{and} \quad f''(x) = 2 > 0$$

for all x . Therefore, f is concave up on the interval $(-\infty, \infty)$.

- (b) Because the concavity of f never changes, f has no points of inflection.

40. Let $f(x) = x^2 + 4x - 2$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 2x + 4 \quad \text{and} \quad f''(x) = 2 > 0$$

for all x . Therefore, f is concave up on the interval $(-\infty, \infty)$.

- (b) Because the concavity of f never changes, f has no points of inflection.

41. Let $f(x) = x^3 - 9x^2 + 2$.

(a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 3x^2 - 18x \quad \text{and} \quad f''(x) = 6x - 18 = 6(x - 3),$$

so $f''(x) > 0$ when $x > 3$ and $f''(x) < 0$ when $x < 3$. Therefore, f is
concave up on the interval $(3, \infty)$ and concave down on the interval $(-\infty, 3)$.

(b) Because the concavity of f changes at 3, the point $(3, f(3)) = (3, -52)$ is a point of inflection of f .

42. Let $f(x) = x^3 - 6x^2 + 9x + 1$.

(a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 3x^2 - 12x + 9 \quad \text{and} \quad f''(x) = 6x - 12 = 6(x - 2),$$

so $f''(x) > 0$ when $x > 2$ and $f''(x) < 0$ when $x < 2$. Therefore, f is
concave up on the interval $(2, \infty)$ and concave down on the interval $(-\infty, 2)$.

(b) Because the concavity of f changes at 2, the point $(2, f(2)) = (2, 3)$ is a point of inflection of f .

43. Let $f(x) = x^4 - 4x^3 + 10$.

(a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 4x^3 - 12x^2 \quad \text{and} \quad f''(x) = 12x^2 - 24x = 12x(x - 2),$$

so $f''(x) = 0$ when $x = 0$ and when $x = 2$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and 2 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $12x$	Sign of $x - 2$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	—	—	+	f is concave up
$(0, 2)$	+	—	—	f is concave down
$(2, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$ and is
concave down on the interval $(0, 2)$.

(b) Because the concavity of f changes at 0 and 2, the points $(0, f(0)) = (0, 10)$ and $(2, f(2)) = (2, -6)$ are points of inflection of f .

44. Let $f(x) = 3x^4 - 8x^3 + 6x + 1$.

(a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = 12x^3 - 24x^2 + 6 \quad \text{and} \quad f''(x) = 36x^2 - 48x = 12x(3x - 4),$$

so $f''(x) = 0$ when $x = 0$ and when $x = \frac{4}{3}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and $\frac{4}{3}$ to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $12x$	Sign of $3x - 4$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	-	-	+	f is concave up
$(0, \frac{4}{3})$	+	-	-	f is concave down
$(\frac{4}{3}, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(\frac{4}{3}, \infty)$ and is concave down on the interval $(0, \frac{4}{3})$.

- (b) Because the concavity of f changes at 0 and $\frac{4}{3}$, the points $(0, f(0)) = (0, 1)$ and $(\frac{4}{3}, f(\frac{4}{3})) = (\frac{4}{3}, -\frac{13}{27})$ are points of inflection of f .

45. Let $f(x) = x^{2/3}e^x$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = x^{2/3}e^x + \frac{2}{3}x^{-1/3}e^x = \frac{e^x(3x+2)}{3x^{1/3}},$$

and

$$\begin{aligned} f''(x) &= \frac{3x^{1/3}[e^x(3) + (3x+2)e^x] - e^x(3x+2)x^{-2/3}}{9x^{2/3}} \\ &= \frac{3x(3x+5)e^x - (3x+2)e^x}{9x^{4/3}} = \frac{e^x(9x^2 + 12x - 2)}{9x^{4/3}}, \end{aligned}$$

so $f''(x) = 0$ when

$$x = \frac{-12 \pm \sqrt{12^2 - 4(9)(-2)}}{18} = \frac{-12 \pm \sqrt{216}}{18} = \frac{-12 \pm 6\sqrt{6}}{18} = \frac{-2 \pm \sqrt{6}}{3}$$

and $f''(x)$ does not exist when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and $\frac{-2 \pm \sqrt{6}}{3}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $9x^2 + 12x - 2$	Sign of $e^x/(9x^{4/3})$	Sign of $f''(x)$	Conclusion
$(-\infty, \frac{-2-\sqrt{6}}{3})$	+	+	+	f is concave up
$(\frac{-2-\sqrt{6}}{3}, 0)$	-	+	-	f is concave down
$(0, \frac{-2+\sqrt{6}}{3})$	-	+	-	f is concave down
$(\frac{-2+\sqrt{6}}{3}, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals

$$\left(-\infty, \frac{-2-\sqrt{6}}{3}\right) \quad \text{and} \quad \left(\frac{-2+\sqrt{6}}{3}, \infty\right)$$

and is concave down on the intervals

$$\left(\frac{-2 - \sqrt{6}}{3}, 0 \right) \quad \text{and} \quad \left(0, \frac{-2 + \sqrt{6}}{3} \right).$$

- (b) Because the concavity of f changes at $\frac{-2 - \sqrt{6}}{3}$ and $\frac{-2 + \sqrt{6}}{3}$, the points

$$\left(\frac{-2 - \sqrt{6}}{3}, f\left(\frac{-2 - \sqrt{6}}{3}\right) \right) \approx (-1.483, 0.295)$$

and

$$\left(\frac{-2 + \sqrt{6}}{3}, f\left(\frac{-2 + \sqrt{6}}{3}\right) \right) \approx (0.150, 0.328)$$

are points of inflection of f . The concavity of f does not change at 0, so the point $(0, f(0)) = (0, 0)$ is not a point of inflection.

46. Let $f(x) = x^{2/3}e^{-x}$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = -x^{2/3}e^{-x} + \frac{2}{3}x^{-1/3}e^{-x} = \frac{e^{-x}(2 - 3x)}{3x^{1/3}},$$

and

$$\begin{aligned} f''(x) &= \frac{3x^{1/3}[e^{-x}(-3) - (2 - 3x)e^{-x}] - e^{-x}(2 - 3x)x^{-2/3}}{9x^{2/3}} \\ &= \frac{3x(3x - 5)e^{-x} - (2 - 3x)e^{-x}}{9x^{4/3}} = \frac{e^{-x}(9x^2 - 12x - 2)}{9x^{4/3}}, \end{aligned}$$

so $f''(x) = 0$ when

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(9)(-2)}}{18} = \frac{12 \pm \sqrt{216}}{18} = \frac{12 \pm 6\sqrt{6}}{18} = \frac{2 \pm \sqrt{6}}{3}$$

and $f''(x)$ does not exist when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and $\frac{2 \pm \sqrt{6}}{3}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $9x^2 - 12x - 2$	Sign of $e^{-x}/(9x^{4/3})$	Sign of $f''(x)$	Conclusion
$\left(-\infty, \frac{2 - \sqrt{6}}{3}\right)$	+	+	+	f is concave up
$\left(\frac{2 - \sqrt{6}}{3}, 0\right)$	-	+	-	f is concave down
$\left(0, \frac{2 + \sqrt{6}}{3}\right)$	-	+	-	f is concave down
$\left(\frac{2 + \sqrt{6}}{3}, \infty\right)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals

$$\left(-\infty, \frac{2 - \sqrt{6}}{3} \right) \quad \text{and} \quad \left(\frac{2 + \sqrt{6}}{3}, \infty \right)$$

and is concave down on the intervals

$$\left(\frac{2 - \sqrt{6}}{3}, 0 \right) \quad \text{and} \quad \left(0, \frac{2 + \sqrt{6}}{3} \right).$$

- (b) Because the concavity of f changes at $\frac{2 - \sqrt{6}}{3}$ and $\frac{2 + \sqrt{6}}{3}$, the points

$$\left(\frac{2 - \sqrt{6}}{3}, f\left(\frac{2 - \sqrt{6}}{3}\right) \right) \approx (-0.150, 0.328)$$

and

$$\left(\frac{2 + \sqrt{6}}{3}, f\left(\frac{2 + \sqrt{6}}{3}\right) \right) \approx (1.483, 0.295)$$

are points of inflection of f . The concavity of f does not change at 0, so the point $(0, f(0)) = (0, 0)$ is not a point of inflection.

47. Let $f(x) = \frac{\ln x}{x^3}$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - 3x^2 \ln x}{x^6} = \frac{1 - 3 \ln x}{x^4},$$

and

$$f''(x) = \frac{x^4 \cdot \left(-\frac{3}{x}\right) - 4x^3(1 - 3 \ln x)}{x^8} = \frac{-3x^3 - 4x^3 + 12x^3 \ln x}{x^8} = \frac{12 \ln x - 7}{x^5},$$

so $f''(x)$ exists on the domain of f , the set $\{x|x > 0\}$, and $f''(x) = 0$ when $x = e^{7/12}$. When $0 < x < e^{7/12}$, $f''(x) < 0$, and when $x > e^{7/12}$, $f''(x) > 0$. Therefore, f is concave down on the interval $(0, e^{7/12})$ and is concave up on the interval $(e^{7/12}, \infty)$.

- (b) Because the concavity of f changes at $e^{7/12}$, the point $(e^{7/12}, f(e^{7/12})) = \left(e^{7/12}, \frac{7}{12}e^{-7/4}\right)$ is a point of inflection of f .

48. Let $f(x) = \frac{\ln x}{\sqrt{x^3}}$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = \frac{\sqrt{x^3} \cdot \frac{1}{x} - \frac{3}{2}\sqrt{x} \ln x}{x^3} = \frac{2 - 3 \ln x}{2x^{5/2}},$$

and

$$f''(x) = \frac{2x^{5/2} \cdot \left(-\frac{3}{x}\right) - 5x^{3/2}(2 - 3 \ln x)}{4x^5} = \frac{-6x^{3/2} - 10x^{3/2} + 15x^{3/2} \ln x}{4x^5} = \frac{15 \ln x - 16}{4x^{7/2}},$$

so $f''(x)$ exists on the domain of f , the set $\{x|x > 0\}$, and $f''(x) = 0$ when $x = e^{16/15}$. When $0 < x < e^{16/15}$, $f''(x) < 0$, and when $x > e^{16/15}$, $f''(x) > 0$. Therefore, f is concave down on the interval $(0, e^{16/15})$ and is concave up on the interval $(e^{16/15}, \infty)$.

- (b) Because the concavity of f changes at $e^{16/15}$, the point $(e^{16/15}, f(e^{16/15})) = \left(e^{16/15}, \frac{16}{15e^{8/5}}\right)$ is a point of inflection of f .
49. Let $f(x) = x + \frac{1}{x}$.
- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,
- $$f'(x) = 1 - \frac{1}{x^2} \quad \text{and} \quad f''(x) = \frac{2}{x^3},$$
- so $f''(x)$ exists on the domain of f , the set $\{x|x \neq 0\}$, and is never equal to 0. When $x < 0$, $f''(x) < 0$ and when $x > 0$, $f''(x) > 0$. Therefore, f is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
- (b) Note that f does not have a point of inflection at 0 because f is not defined at 0. Therefore, f has no points of inflection.
50. Let $f(x) = 2x^2 - \frac{1}{x}$.
- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,
- $$f'(x) = 4x + \frac{1}{x^2} \quad \text{and} \quad f''(x) = 4 - \frac{2}{x^3},$$
- so $f''(x)$ exists on the domain of f , the set $\{x|x \neq 0\}$, and $f''(x) = 0$ when $x = \frac{1}{\sqrt[3]{2}}$. When $x < 0$, $f''(x) > 0$; when $0 < x < \frac{1}{\sqrt[3]{2}}$, $f''(x) < 0$; and when $x > \frac{1}{\sqrt[3]{2}}$, $f''(x) > 0$. Therefore, f is concave up on the intervals $(-\infty, 0)$ and $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$ and concave down on the interval $\left(0, \frac{1}{\sqrt[3]{2}}\right)$.
- (b) Because the concavity of f changes at $\frac{1}{\sqrt[3]{2}}$, the point $\left(\frac{1}{\sqrt[3]{2}}, f\left(\frac{1}{\sqrt[3]{2}}\right)\right) = \left(\frac{1}{\sqrt[3]{2}}, 0\right)$ is a point of inflection of f . Note that f does not have a point of inflection at 0 because f is not defined at 0. Therefore, f has no points of inflection.
51. Let $f(x) = 3x^{1/3} + 2x$.
- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,
- $$f'(x) = x^{-2/3} + 2 \quad \text{and} \quad f''(x) = -\frac{2}{3}x^{-5/3},$$
- so $f''(x)$ does not exist at $x = 0$ and is never equal to 0. When $x < 0$, $f''(x) > 0$ and when $x > 0$, $f''(x) < 0$. Therefore, f is concave up on the interval $(-\infty, 0)$ and concave down on the interval $(0, \infty)$.
- (b) Because the concavity of f changes at 0, the point $(0, f(0)) = (0, 0)$ is a point of inflection of f .
52. Let $f(x) = x^{4/3} - 8x^{1/3}$.
- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,
- $$f'(x) = \frac{4}{3}x^{1/3} - \frac{8}{3}x^{-2/3} \quad \text{and} \quad f''(x) = \frac{4}{9}x^{-2/3} + \frac{16}{9}x^{-5/3} = \frac{4x + 16}{9x^{5/3}},$$
- so $f''(x)$ does not exist at $x = 0$ and $f''(x) = 0$ when $x = -4$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers -4 and 0 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $4x + 32$	Sign of $9x^{5/3}$	Sign of $f''(x)$	Conclusion
$(-\infty, -4)$	—	—	+	f is concave up
$(-4, 0)$	+	—	—	f is concave down
$(0, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\infty, -4)$ and $(0, \infty)$ and concave down on the interval $(-4, 0)$.

- (b) Because the concavity of f changes at -4 and 0 , the points $(-4, f(-4)) = (-4, 12\sqrt[3]{4})$ and $(0, f(0)) = (0, 0)$ are points of inflection of f .

53. Let $f(x) = 3 - \frac{4}{x} + \frac{4}{x^2}$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = \frac{4}{x^2} - \frac{8}{x^3} \quad \text{and} \quad f''(x) = -\frac{8}{x^3} + \frac{24}{x^4} = \frac{24 - 8x}{x^4},$$

so $f''(x)$ exists on the domain of f , the set $\{x | x \neq 0\}$, and $f''(x) = 0$ when $x = 3$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and 3 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $24 - 8x$	Sign of x^4	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	+	+	+	f is concave up
$(0, 3)$	+	+	+	f is concave up
$(3, \infty)$	—	+	—	f is concave down

Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(0, 3)$ and concave down on the interval $(3, \infty)$.

- (b) Because the concavity of f changes at 3 , the point $(3, f(3)) = \left(3, \frac{19}{9}\right)$ is a point of inflection of f .

54. Let $f(x) = (x - 1)^{3/2}$.

- (a) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f'(x) = \frac{3}{2}(x - 1)^{1/2} \quad \text{and} \quad f''(x) = \frac{3}{4}(x - 1)^{-1/2},$$

so $f''(x)$ does not exist at $x = 1$ and is never equal to zero. When $x > 1$, $f''(x) > 0$, so f is concave up on the interval $(1, \infty)$.

- (b) Because the concavity of f does not change, f has no points of inflection.

55. Let $f(x) = 2x^3 - 6x^2 + 6x - 3$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x - 1)^2,$$

so 1 is a critical number of f . For $x \neq 1$, $f'(x) > 0$, so that f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. By the First Derivative Test, it follows that f has neither a local maximum value nor a local minimum value at 1. Hence, f has no local extrema.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x - 12 = 12(x - 1).$$

For $x < 1$, $f''(x) < 0$, and for $x > 1$, $f''(x) > 0$. Therefore, f is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

- (c) Because the concavity of f changes at 1, the point $(1, f(1)) = (1, -1)$ is a point of inflection of f .

56. Let $f(x) = 2x^3 + 9x^2 + 12x - 4$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2) = 6(x + 1)(x + 2),$$

so -2 and -1 are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -2 and -1 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $6(x + 1)$	Sign of $x + 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	—	—	+	f is increasing
$(-2, -1)$	—	+	—	f is decreasing
$(-1, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(-1, \infty)$ and decreasing on the interval $(-2, -1)$. By the First Derivative Test, it follows that f has a local maximum value at -2 and a local minimum value at -1 . The local maximum value is $f(-2) = -8$, and the local minimum value is $f(-1) = -9$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x + 18 = 6(2x + 3).$$

For $x < -\frac{3}{2}$, $f''(x) < 0$, and for $x > -\frac{3}{2}$, $f''(x) > 0$. Therefore, f is concave down on the interval $(-\infty, -\frac{3}{2})$ and concave up on the interval $(-\frac{3}{2}, \infty)$.

- (c) Because the concavity of f changes at $-\frac{3}{2}$, the point $(-\frac{3}{2}, f(-\frac{3}{2})) = (-\frac{3}{2}, -\frac{17}{2})$ is a point of inflection of f .

57. Let $f(x) = x^4 - 4x$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1),$$

so 1 is a critical number of f . For $x < 1$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 1)$; for $x > 1$, $f'(x) > 0$, so f is increasing on the interval $(1, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at 1. The local minimum value is $f(1) = -3$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x^2,$$

so $f''(x) > 0$ for $x \neq 0$. Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(0, \infty)$.

- (c) Because the concavity of f never changes, f has no points of inflection.

58. Let $f(x) = x^4 + 4x$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1),$$

so -1 is a critical number of f . For $x < -1$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, -1)$; for $x > -1$, $f'(x) > 0$, so f is increasing on the interval $(-1, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at -1 . The local minimum value is $f(-1) = -3$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 12x^2,$$

so $f''(x) > 0$ for $x \neq 0$. Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(0, \infty)$.

- (c) Because the concavity of f never changes, f has no points of inflection.

59. Let $f(x) = 5x^4 - x^5$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 20x^3 - 5x^4 = 5x^3(4 - x),$$

so 0 and 4 are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and 4 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $5x^3$	Sign of $4 - x$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	f is decreasing
$(0, 4)$	+	+	+	f is increasing
$(4, \infty)$	+	—	—	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$ and increasing on the interval $(0, 4)$. By the First Derivative Test, it follows that f has a local minimum value at 0 and a local maximum value at 4. The local minimum value is $f(0) = 0$, and the local maximum value is $f(4) = 256$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 60x^2 - 20x^3 = 20x^2(3 - x),$$

so $f''(x) = 0$ when $x = 0$ and when $x = 3$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and 3 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $20x^2$	Sign of $3 - x$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	+	+	+	f is concave up
$(0, 3)$	+	+	+	f is concave up
$(3, \infty)$	+	—	—	f is concave down

Therefore, f is concave up on the intervals $(-\infty, 0)$ and $(0, 3)$ and concave down on the interval $(3, \infty)$.

- (c) Because the concavity of f changes at 3, the point $(3, f(3)) = (3, 162)$ is a point of inflection of f .

60. Let $f(x) = 4x^6 + 6x^4$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 24x^5 + 24x^3 = 24x^3(x^2 + 1),$$

so 0 is a critical number of f . For $x < 0$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 0)$; for $x > 0$, $f'(x) > 0$, so f is increasing on the interval $(0, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at 0. The local minimum value is $f(0) = 0$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 120x^4 + 72x^2 = 24x^2(5x^2 + 3),$$

so $f''(x) = 0$ when $x = 0$. For $x \neq 0$, $f''(x) > 0$, so f is

concave up on the intervals $(-\infty, 0)$ and $(0, \infty)$.

- (c) Because the concavity of f never changes, f has no points of inflection.

61. Let $f(x) = 3x^5 - 20x^3$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x - 2)(x + 2),$$

so -2 , 0 , and 2 are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -2 , 0 , and 2 to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $15x^2$	Sign of $x - 2$	Sign of $x + 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	+	−	−	+	f is increasing
$(-2, 0)$	+	−	+	−	f is decreasing
$(0, 2)$	+	−	+	−	f is decreasing
$(2, \infty)$	+	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing on the intervals $(-2, 0)$ and $(0, 2)$. By the First Derivative Test, it follows that f has a local maximum value at -2 , neither a local maximum value nor a local minimum value at 0 , and a local minimum value at 2 . The local maximum value is $f(-2) = 64$, and the local minimum value is $f(2) = -64$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 60x^3 - 120x = 60x(x^2 - 2) = 60x(x - \sqrt{2})(x + \sqrt{2}),$$

so $f''(x) = 0$ when $x = \pm\sqrt{2}$ and when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-\sqrt{2}$, 0 , and $\sqrt{2}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $60x$	Sign of $x - \sqrt{2}$	Sign of $x + \sqrt{2}$	Sign of $f''(x)$	Conclusion
$(-\infty, -\sqrt{2})$	-	-	-	-	f is concave down
$(-\sqrt{2}, 0)$	-	-	+	+	f is concave up
$(0, \sqrt{2})$	+	-	+	-	f is concave down
$(\sqrt{2}, \infty)$	+	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$ and concave down on the intervals $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$.

- (c) Because the concavity of f changes at $-\sqrt{2}$, 0 , and $\sqrt{2}$, the points $(-\sqrt{2}, f(-\sqrt{2})) = (-\sqrt{2}, 28\sqrt{2})$, $(0, f(0)) = (0, 0)$, and $(\sqrt{2}, f(\sqrt{2})) = (\sqrt{2}, -28\sqrt{2})$ are points of inflection of f .

62. Let $f(x) = 3x^5 + 5x^3$.

- (a) The polynomial function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = 15x^4 + 15x^2 = 15x^2(x^2 + 1),$$

so 0 is a critical number of f . For $x \neq 0$, $f'(x) > 0$, so f is increasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. By the First Derivative Test, it follows that f has neither a local maximum value nor a local minimum value at 0 . Hence, f has no local extrema.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 60x^3 + 30x = 30x(2x^2 + 1),$$

so $f''(x) = 0$ when $x = 0$. For $x < 0$, $f''(x) < 0$, so f is concave down on the interval $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is concave up on the interval $(0, \infty)$.

- (c) Because the concavity of f changes at 0 , the point $(0, f(0)) = (0, 0)$ is a point of inflection of f .

63. Let $f(x) = x^2e^x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = x^2e^x + 2xe^x = xe^x(x + 2),$$

so -2 and 0 are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -2 and 0 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of xe^x	Sign of $x + 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -2)$	-	-	+	f is increasing
$(-2, 0)$	-	+	-	f is decreasing
$(0, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$ and decreasing on the interval $(-2, 0)$. By the First Derivative Test, it follows that f has a local maximum value at -2 and a local minimum value at 0 . The local maximum value is $f(-2) = 4e^{-2}$, and the local minimum value is $f(0) = 0$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = (x^2 + 2x)e^x + (2x + 2)e^x = e^x(x^2 + 4x + 2) = e^x[(x + 2)^2 - 2],$$

so $f''(x) = 0$ when $x = -2 \pm \sqrt{2}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-2 - \sqrt{2}$ and $-2 + \sqrt{2}$ to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of e^x	Sign of $x^2 + 4x + 2$	Sign of $f''(x)$	Conclusion
$(-\infty, -2 - \sqrt{2})$	+	+	+	f is concave up
$(-2 - \sqrt{2}, -2 + \sqrt{2})$	+	-	-	f is concave down
$(-2 + \sqrt{2}, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\infty, -2 - \sqrt{2})$ and $(-2 + \sqrt{2}, \infty)$ and concave down on the interval $(-2 - \sqrt{2}, -2 + \sqrt{2})$.

- (c) Because the concavity of f changes at $-2 - \sqrt{2}$ and $-2 + \sqrt{2}$, the points

$$(-2 - \sqrt{2}, f(-2 - \sqrt{2})) = (-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}})$$

and

$$(-2 + \sqrt{2}, f(-2 + \sqrt{2})) = (-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}})$$

are points of inflection of f .

64. Let $f(x) = x^3e^x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = x^3e^x + 3x^2e^x = x^2e^x(x + 3),$$

so -3 and 0 are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -3 and 0 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of x^2e^x	Sign of $x + 3$	Sign of $f'(x)$	Conclusion
$(-\infty, -3)$	+	-	-	f is decreasing
$(-3, 0)$	+	+	+	f is increasing
$(0, \infty)$	+	+	+	f is increasing

Therefore, f is decreasing on the interval $(-\infty, -3)$ and increasing on the intervals $(-3, 0)$ and $(0, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at -3 and neither a local maximum value nor a local minimum value at 0 . The local minimum value is $f(-3) = -27e^{-3}$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = (x^3 + 3x^2)e^x + (3x^2 + 6x)e^x = xe^x(x^2 + 6x + 6) = xe^x[(x + 3)^2 - 3],$$

so $f''(x) = 0$ when $x = -3 \pm \sqrt{3}$ and when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-3 - \sqrt{3}$, $-3 + \sqrt{3}$, and 0 to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of xe^x	Sign of $x^2 + 6x + 6$	Sign of $f''(x)$	Conclusion
$(-\infty, -3 - \sqrt{3})$	-	+	-	f is concave down
$(-3 - \sqrt{3}, -3 + \sqrt{3})$	-	-	+	f is concave up
$(-3 + \sqrt{3}, 0)$	-	+	-	f is concave down
$(0, \infty)$	+	+	+	f is concave up

Therefore, f is concave down on the intervals $(-\infty, -3 - \sqrt{3})$ and $(-3 + \sqrt{3}, 0)$ and concave up on the intervals $(-3 - \sqrt{3}, -3 + \sqrt{3})$ and $(0, \infty)$.

- (c) Because the concavity of f changes at $-3 - \sqrt{3}$, $-3 + \sqrt{3}$, and 0 , the points

$$\boxed{(-3 - \sqrt{3}, f(-3 - \sqrt{3})) = (-3 - \sqrt{3}, -(54 + 30\sqrt{3})e^{-3-\sqrt{3}})},$$

$$\boxed{(-3 + \sqrt{3}, f(-3 + \sqrt{3})) = (-3 + \sqrt{3}, (30\sqrt{3} - 54)e^{-3+\sqrt{3}})},$$

and $(0, f(0)) = (0, 0)$ are points of inflection of f .

65. Let $f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = \sinh x,$$

so 0 is a critical number of f . For $x < 0$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 0)$; for $x > 0$, $f'(x) > 0$, so f is increasing on the interval $(0, \infty)$. By the First Derivative Test, it follows that f has a local minimum value at 0 . The local minimum value is $f(0) = \cosh 0 = 1$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \cosh x \geq 1 > 0$$

for all x . Therefore, f is concave up on the interval $(-\infty, \infty)$.

- (c) Because the concavity of f does not change, f has no points of inflection.

66. Let $f(x) = \frac{e^x - e^{-x}}{2} = \sinh x$.

- (a) The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = \cosh x \geq 1 > 0$$

for all x , so f is increasing on the interval $(-\infty, \infty)$. Because f has no critical numbers, f has no local extrema.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \sinh x,$$

so $f''(x) = 0$ when $x = 0$. For $x < 0$, $f''(x) < 0$, so f is concave down on the interval $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is concave up on the interval $(0, \infty)$.

- (c) Because the concavity of f changes at 0 , the point $(0, f(0)) = (0, 0)$ is a point of inflection of f .

67. Let $f(x) = 6x^{4/3} - 3x^{1/3}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = 8x^{1/3} - x^{-2/3} = \frac{8x - 1}{x^{2/3}},$$

so 0 and $\frac{1}{8}$ are critical numbers. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\frac{1}{8}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8x - 1$	Sign of $x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	-	+	-	f is decreasing
$(0, \frac{1}{8})$	-	+	-	f is decreasing
$(\frac{1}{8}, \infty)$	+	+	+	f is increasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(0, \frac{1}{8})$ and increasing on the interval $(\frac{1}{8}, \infty)$. By the First Derivative Test, f has neither a local maximum value nor a local minimum

value at 0 and a local minimum value at $\frac{1}{8}$. The local minimum value is $f\left(\frac{1}{8}\right) = -\frac{9}{8}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{8}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{8x + 2}{3x^{5/3}},$$

so $f''(x) = 0$ when $x = -\frac{1}{4}$ and $f''(x)$ does not exist when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $-\frac{1}{4}$ and 0 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8x + 2$	Sign of $3x^{5/3}$	Sign of $f''(x)$	Conclusion
$(-\infty, -\frac{1}{4})$	-	-	+	f is concave up
$(-\frac{1}{4}, 0)$	+	-	-	f is concave down
$(0, \infty)$	+	+	+	f is concave up

Therefore, f is concave up on the intervals $(-\infty, -\frac{1}{4})$ and $(0, \infty)$ and

concave down on the interval $(-\frac{1}{4}, 0)$.

(c) Because the concavity of f changes at $-\frac{1}{4}$ and 0, the points $\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = \left(-\frac{1}{4}, \frac{9\sqrt[3]{2}}{4}\right)$

and $(0, f(0)) = (0, 0)$ are points of inflection of f .

68. Let $f(x) = x^{2/3} - x^{1/3}$.

- (a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-2/3} = \frac{2x^{1/3} - 1}{3x^{2/3}},$$

so 0 and $\frac{1}{8}$ are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\frac{1}{8}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^{1/3} - 1$	Sign of $x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	f is decreasing
$(0, \frac{1}{8})$	—	+	—	f is decreasing
$(\frac{1}{8}, \infty)$	+	+	+	f is increasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(0, \frac{1}{8})$ and increasing on the interval $(\frac{1}{8}, \infty)$. By the First Derivative Test, f has neither a local maximum value nor a local minimum

value at 0 and a local minimum value at $\frac{1}{8}$. The local minimum value is $f\left(\frac{1}{8}\right) = -\frac{1}{4}$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = -\frac{2}{9}x^{-4/3} + \frac{2}{9}x^{-5/3} = \frac{2}{9} \frac{1 - x^{1/3}}{x^{5/3}},$$

so $f''(x) = 0$ when $x = 1$ and $f''(x)$ does not exist when $x = 0$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and 1 to divide the number line into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $\frac{2}{9}(1 - x^{1/3})$	Sign of $x^{5/3}$	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	+	—	—	f is concave down
$(0, 1)$	+	+	+	f is concave up
$(1, \infty)$	—	+	—	f is concave down

Therefore, f is concave down on the intervals $(-\infty, 0)$ and $(1, \infty)$ and concave up on the interval $(0, 1)$.

- (c) Because the concavity of f changes at 0 and 1, the points $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, 0)$ are points of inflection of f .

69. Let $f(x) = x^{2/3}(x^2 - 8)$.

- (a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{2/3}(2x) + (x^2 - 8) \cdot \frac{2}{3}x^{-1/3} = \frac{6x^2 + 2(x^2 - 8)}{3x^{1/3}} = \frac{8(x - \sqrt{2})(x + \sqrt{2})}{3x^{1/3}},$$

so 0 and $\pm\sqrt{2}$ are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\pm\sqrt{2}$ to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $8(x - \sqrt{2})$	Sign of $x + \sqrt{2}$	Sign of $3x^{1/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\sqrt{2})$	—	—	—	—	f is decreasing
$(-\sqrt{2}, 0)$	—	+	—	+	f is increasing
$(0, \sqrt{2})$	—	+	+	—	f is decreasing
$(\sqrt{2}, \infty)$	+	+	+	+	f is increasing

Therefore, f is decreasing on the intervals $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$ and increasing on the intervals $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$. By the First Derivative Test, it follows that f has local minimum values at $-\sqrt{2}$ and $\sqrt{2}$ and a local maximum value at 0. The

$$\text{local minimum values are } f(-\sqrt{2}) = -6\sqrt[3]{2} \text{ and } f(\sqrt{2}) = -6\sqrt[3]{2},$$

and the local maximum value is $f(0) = 0$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{3x^{1/3}(16x) - 8(x^2 - 2) \cdot x^{-2/3}}{9x^{2/3}} = \frac{40x^2 + 16}{9x^{4/3}},$$

so $f''(x)$ does not exist when $x = 0$ and is never equal to zero. For $x \neq 0$, $f''(x) > 0$, so f is concave up on the intervals $(-\infty, 0)$ and $(0, \infty)$.

- (c) Because the concavity of f does not change, f has no points of inflection.

70. Let $f(x) = x^{1/3}(x^2 - 2)$.

- (a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^{1/3}(2x) + (x^2 - 2) \cdot \frac{1}{3}x^{-2/3} = \frac{6x^2 + x^2 - 2}{3x^{2/3}} = \frac{7x^2 - 2}{3x^{2/3}},$$

so 0 and $\pm \frac{2\sqrt{7}}{7}$ are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\pm \frac{2\sqrt{7}}{7}$ to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $7x^2 - 2$	Sign of $3x^{2/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{2\sqrt{7}}{7})$	+	+	+	f is increasing
$(-\frac{2\sqrt{7}}{7}, 0)$	—	+	—	f is decreasing
$(0, \frac{2\sqrt{7}}{7})$	—	+	—	f is decreasing
$(\frac{2\sqrt{7}}{7}, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -\frac{2\sqrt{7}}{7})$ and $(\frac{2\sqrt{7}}{7}, \infty)$ and decreasing on the intervals $(-\frac{2\sqrt{7}}{7}, 0)$ and $(0, \frac{2\sqrt{7}}{7})$. By the First Derivative Test, f has a local maximum value at $-\frac{2\sqrt{7}}{7}$, neither a local maximum value nor a local minimum value at 0, and a local minimum

value at $\frac{2\sqrt{7}}{7}$. The local maximum value is

$$f\left(-\frac{2\sqrt{7}}{7}\right) = \left(-\frac{2\sqrt{7}}{7}\right)^{1/3} \left(\frac{2}{7} - 2\right) \approx 1.562,$$

and the local minimum value is

$$f\left(\frac{2\sqrt{7}}{7}\right) = \left(\frac{2\sqrt{7}}{7}\right)^{1/3} \left(\frac{2}{7} - 2\right) \approx -1.562.$$

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{3x^{2/3}(14x) - 2(7x^2 - 2) \cdot x^{-1/3}}{9x^{4/3}} = \frac{28x^2 + 4}{9x^{5/3}},$$

so $f''(x)$ does not exist when $x = 0$ and is never equal to zero. For $x < 0$, $f''(x) < 0$, so f is concave down on the interval $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is concave up on the interval $(0, \infty)$.

- (c) Because the concavity of f changes at 0, the point $(0, f(0)) = (0, 0)$ is a point of inflection of f .

71. Let $f(x) = x^2 - \ln x$. Note that the domain of f is the set $\{x|x > 0\}$.

- (a) The function f is differentiable on its domain, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 2x - \frac{1}{x} = \frac{2x^2 - 1}{x},$$

so $\frac{\sqrt{2}}{2}$ is a critical number of f . For $0 < x < \frac{\sqrt{2}}{2}$, $f'(x) < 0$, so f is decreasing on the interval $\left(0, \frac{\sqrt{2}}{2}\right)$; for $x > \frac{\sqrt{2}}{2}$, $f'(x) > 0$, so f is increasing on the interval $\left(\frac{\sqrt{2}}{2}, \infty\right)$. By the First Derivative Test, it follows that f has a local minimum value at $\frac{\sqrt{2}}{2}$. The

$$\text{local minimum value is } f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} \ln 2.$$

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 2 + \frac{1}{x^2} > 2 > 0$$

for all $x > 0$. Therefore, f is concave up on the interval $(0, \infty)$.

- (c) Because the concavity of f does not change, f has no points of inflection.

72. Let $f(x) = \ln x - x$. Note that the domain of f is the set $\{x|x > 0\}$.

- (a) The function f is differentiable on its domain, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x},$$

so 1 is a critical number of f . For $0 < x < 1$, $f'(x) > 0$, so f is increasing on the interval $(0, 1)$; for $x > 1$, $f'(x) < 0$, so f is decreasing on the interval $(1, \infty)$. By the First Derivative Test, it follows that f has a local maximum value at 1. The local minimum value is $f(1) = -1$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = -\frac{1}{x^2} < 0$$

for all $x > 0$. Therefore, f is concave down on the interval $(0, \infty)$.

(c) Because the concavity of f does not change, f has no points of inflection.

73. Let $f(x) = \frac{x}{(1+x^2)^{5/2}}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{(1+x^2)^{5/2} - x \cdot \frac{5}{2}(1+x^2)^{3/2}(2x)}{(1+x^2)^5} = \frac{1-4x^2}{(1+x^2)^{7/2}} = \frac{(1-2x)(1+2x)}{(1+x^2)^{7/2}},$$

so $\pm \frac{1}{2}$ are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers $\pm \frac{1}{2}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $1-2x$	Sign of $1+2x$	Sign of $(1+x^2)^{7/2}$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{1}{2})$	+	-	+	-	f is decreasing
$(-\frac{1}{2}, \frac{1}{2})$	+	+	+	+	f is increasing
$(\frac{1}{2}, \infty)$	-	+	+	-	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, -\frac{1}{2})$ and $(\frac{1}{2}, \infty)$ and increasing on the interval $(-\frac{1}{2}, \frac{1}{2})$. By the First Derivative Test, it follows that f has a local minimum value at $-\frac{1}{2}$ and a

local maximum value at $\frac{1}{2}$. The local minimum value is $f\left(-\frac{1}{2}\right) = -\frac{16}{5^{5/2}} = -\frac{16\sqrt{5}}{125}$, and the

local maximum value is $f\left(\frac{1}{2}\right) = \frac{16}{5^{5/2}} = \frac{16\sqrt{5}}{125}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^{7/2}(-8x) - (1-4x^2) \cdot \frac{7}{2}(1+x^2)^{5/2}(2x)}{(1+x^2)^7} \\ &= \frac{-8x - 8x^3 - 7x + 28x^3}{(1+x^2)^{9/2}} = \frac{5x(4x^2-3)}{(1+x^2)^{9/2}}, \end{aligned}$$

so $f''(x)$ exists for all x and is equal to zero when $x = 0$ and when $x = \pm \frac{\sqrt{3}}{2}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers 0 and $\pm \frac{\sqrt{3}}{2}$ to divide the number line into four intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $5x$	Sign of $4x^2 - 3$	Sign of $(1 + x^2)^{9/2}$	Sign of $f''(x)$	Conclusion
$\left(-\infty, -\frac{\sqrt{3}}{2}\right)$	-	+	+	-	f is concave down
$\left(-\frac{\sqrt{3}}{2}, 0\right)$	-	-	+	+	f is concave up
$\left(0, \frac{\sqrt{3}}{2}\right)$	+	-	+	-	f is concave down
$\left(\frac{\sqrt{3}}{2}, \infty\right)$	+	+	+	+	f is concave up

Therefore, f is concave down on the intervals $\left(-\infty, -\frac{\sqrt{3}}{2}\right)$ and $\left(0, \frac{\sqrt{3}}{2}\right)$ and concave up on the intervals $\left(-\frac{\sqrt{3}}{2}, 0\right)$ and $\left(\frac{\sqrt{3}}{2}, \infty\right)$.

(c) Because the concavity of f changes at $\pm\frac{\sqrt{3}}{2}$ and 0, the points

$$\left(-\frac{\sqrt{3}}{2}, f\left(-\frac{\sqrt{3}}{2}\right)\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{16\sqrt{3}}{7^{5/2}}\right),$$

$$\left(\frac{\sqrt{3}}{2}, f\left(\frac{\sqrt{3}}{2}\right)\right) = \left(\frac{\sqrt{3}}{2}, \frac{16\sqrt{3}}{7^{5/2}}\right),$$

and $(0, f(0)) = (0, 0)$ are points of inflection of f .

74. Let $f(x) = \frac{\sqrt{x}}{1+x}$. Note that the domain of f is the set $\{x|x \geq 0\}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{(1+x) \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot 1}{(1+x)^2} = \frac{1+x-2x}{2\sqrt{x}(1+x)^2} = \frac{1-x}{2\sqrt{x}(1+x)^2},$$

so 0 and 1 are critical numbers of f . For $0 < x < 1$, $f'(x) > 0$, so f is increasing on the interval $(0, 1)$; for $x > 1$, $f'(x) < 0$, so f is decreasing on the interval $(1, \infty)$. By the First Derivative Test,

it follows that f has a local maximum value at 1. The local maximum value is $f(1) = \frac{1}{2}$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$\begin{aligned} f''(x) &= \frac{2\sqrt{x}(1+x)^2 \cdot (-1) - (1-x) \left[4\sqrt{x}(1+x) + \frac{1}{\sqrt{x}}(1+x)^2\right]}{4x(1+x)^4} \\ &= \frac{-2x - 2x^2 - 5x - 1 + 5x^2 + x}{4x^{3/2}(1+x)^3} = \frac{3x^2 - 6x - 1}{4x^{3/2}(1+x)^3}, \end{aligned}$$

so $f''(x)$ does not exist when $x = 0$ and $f''(x) = 0$ when

$$x = \frac{6 \pm \sqrt{36 - 4(3)(-1)}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = 1 \pm \frac{2\sqrt{3}}{3}.$$

Note that $1 - \frac{2\sqrt{3}}{3} \approx -1.15 < 0$ is not in the domain of f , so this number may be excluded.

For $0 < x < 1 + \frac{2\sqrt{3}}{3}$, $f''(x) < 0$, so f is concave down on the interval $\left(0, 1 + \frac{2\sqrt{3}}{3}\right)$; for

$x > 1 + \frac{2\sqrt{3}}{3}$, $f''(x) > 0$, so f is concave up on the interval $\left(1 + \frac{2\sqrt{3}}{3}, \infty\right)$.

(c) Because the concavity of f changes at $1 + \frac{2\sqrt{3}}{3}$, the point

$$\left(1 + \frac{2\sqrt{3}}{3}, f\left(1 + \frac{2\sqrt{3}}{3}\right)\right) = \left(1 + \frac{2\sqrt{3}}{3}, \frac{\sqrt{9+6\sqrt{3}}}{2(3+\sqrt{3})}\right)$$

is a point of inflection of f .

75. Let $f(x) = x^2\sqrt{1-x^2}$. Note that the domain of f is the set $\{x \mid -1 \leq x \leq 1\}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x^2 \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + 2x\sqrt{1-x^2} = \frac{-x^3 + 2x - 2x^3}{\sqrt{1-x^2}} = \frac{x(2-3x^2)}{\sqrt{1-x^2}},$$

so ± 1 , 0 , and $\pm \frac{\sqrt{6}}{3}$ are critical numbers of f . To determine where $f'(x) > 0$ and $f'(x) < 0$,

use the numbers ± 1 , 0 , and $\pm \frac{\sqrt{6}}{3}$ to divide $[-1, 1]$ into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of x	Sign of $2-3x^2$	Sign of $\sqrt{1-x^2}$	Sign of $f'(x)$	Conclusion
$\left(-1, -\frac{\sqrt{6}}{3}\right)$	-	-	+	+	f is increasing
$\left(-\frac{\sqrt{6}}{3}, 0\right)$	-	+	+	-	f is decreasing
$\left(0, \frac{\sqrt{6}}{3}\right)$	+	+	+	+	f is increasing
$\left(\frac{\sqrt{6}}{3}, 1\right)$	+	-	+	-	f is decreasing

Therefore, f is increasing on the intervals $\left(-1, -\frac{\sqrt{6}}{3}\right)$ and $\left(0, \frac{\sqrt{6}}{3}\right)$ and decreasing on the intervals $\left(-\frac{\sqrt{6}}{3}, 0\right)$ and $\left(\frac{\sqrt{6}}{3}, 1\right)$. By the First Derivative Test, it follows that f has local maximum values at $\pm \frac{\sqrt{6}}{3}$ and a local minimum value at 0 . The

local maximum values are $f\left(\pm \frac{\sqrt{6}}{3}\right) = \frac{2\sqrt{3}}{9}$, and the local minimum value is $f(0) = 0$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$\begin{aligned} f''(x) &= \frac{\sqrt{1-x^2}(2-9x^2) - (2x-3x^3) \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x)}{1-x^2} \\ &= \frac{(1-x^2)(2-9x^2) + x(2x-3x^3)}{(1-x^2)^{3/2}} = \frac{6x^4 - 9x^2 + 2}{(1-x^2)^{3/2}}, \end{aligned}$$

so $f''(x)$ does not exist at $x = \pm 1$ and is equal to zero when

$$x^2 = \frac{9 \pm \sqrt{81 - 4(6)(2)}}{12} = \frac{9 \pm \sqrt{33}}{12},$$

or when

$$x = \pm \sqrt{\frac{9 \pm \sqrt{33}}{12}} = \pm \frac{1}{6} \sqrt{27 \pm 3\sqrt{33}}.$$

Note that

$$\pm \frac{1}{6} \sqrt{27 + 3\sqrt{33}} \approx \pm 1.108$$

are not in the domain of f and can be excluded. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $\pm \frac{1}{6} \sqrt{27 - 3\sqrt{33}}$ to divide $[-1, 1]$ into three intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $6x^4 - 9x^2 + 2$	Sign of $(1 - x^2)^{3/2}$	Sign of $f''(x)$	Conclusion
$\left(-1, -\frac{1}{6}\sqrt{27 - 3\sqrt{33}}\right)$	−	+	−	f is concave down
$\left(-\frac{1}{6}\sqrt{27 - 3\sqrt{33}}, \frac{1}{6}\sqrt{27 - 3\sqrt{33}}\right)$	+	+	+	f is concave up
$\left(\frac{1}{6}\sqrt{27 - 3\sqrt{33}}, 1\right)$	−	+	−	f is concave down

Therefore, f is concave down on the intervals

$$\left(-1, -\frac{1}{6}\sqrt{27 - 3\sqrt{33}}\right) \quad \text{and} \quad \left(\frac{1}{6}\sqrt{27 - 3\sqrt{33}}, 1\right)$$

and concave up on the interval

$$\left(-\frac{1}{6}\sqrt{27 - 3\sqrt{33}}, \frac{1}{6}\sqrt{27 - 3\sqrt{33}}\right).$$

(c) Because the concavity of f changes at $\pm \frac{1}{6} \sqrt{27 - 3\sqrt{33}}$, the points

$$\left(\pm \frac{1}{6} \sqrt{27 - 3\sqrt{33}}, f\left(\pm \frac{1}{6} \sqrt{27 - 3\sqrt{33}}\right)\right) = \left(\pm \frac{1}{6} \sqrt{27 - 3\sqrt{33}}, \frac{9 - \sqrt{33}}{12} \sqrt{\frac{3 + \sqrt{33}}{12}}\right)$$

are points of inflection of f .

76. Let $f(x) = x\sqrt{1-x}$. Note that the domain of f is the set $\{x|x \leq 1\}$.

(a) The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = x \cdot \frac{1}{2}(1-x)^{-1/2}(-1) + \sqrt{1-x} = \frac{2-3x}{2\sqrt{1-x}},$$

so 1 and $\frac{2}{3}$ are critical numbers. For $x < \frac{2}{3}$, $f'(x) > 0$, so f is increasing on the interval $\left(-\infty, \frac{2}{3}\right)$; for $x > \frac{2}{3}$, $f'(x) < 0$, so f is decreasing on the interval $\left(\frac{2}{3}, 1\right)$. By the First Derivative Test, it

follows that f has a local maximum value at $\frac{2}{3}$. The local maximum value is $f\left(\frac{2}{3}\right) = \frac{2\sqrt{3}}{9}$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = \frac{2\sqrt{1-x}(-3) - (2-3x) \cdot \frac{-1}{\sqrt{1-x}}}{4(1-x)} = \frac{3x-4}{4(1-x)^{3/2}}.$$

For $x < 1$, $f''(x) < 0$, so f is concave down on the interval $(-\infty, 1)$.

- (c) Because the concavity of f does not change, f has no points of inflection.

77. Let $f(x) = \sin^2 x$. Using the half-angle formula $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, we can see that f is a periodic function with period π .

- (a) The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 2 \sin x \cos x = \sin(2x),$$

so $f'(x) = 0$ when $2x = k\pi$, where k is any integer. The critical numbers of f are therefore $\frac{k\pi}{2}$ for any integer k . To determine where $f'(x) > 0$ and $f'(x) < 0$, consider the intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. The increasing/decreasing pattern demonstrated on these two intervals will repeat indefinitely in either direction. The results are shown in the table below.

Interval	Sign of $2 \sin x$	Sign of $\cos x$	Sign of $f'(x)$	Conclusion
$\left(0, \frac{\pi}{2}\right)$	+	+	+	f is increasing
$\left(\frac{\pi}{2}, \pi\right)$	+	-	-	f is decreasing

Therefore, f is increasing on intervals of the form $\left(k\pi, k\pi + \frac{\pi}{2}\right)$ and decreasing on intervals of the form $\left(k\pi + \frac{\pi}{2}, (k+1)\pi\right)$, where k is any integer. By the First Derivative Test, it follows that f has a local maximum value at $k\pi + \frac{\pi}{2}$ and a local minimum value at $k\pi$, for any integer k . The

local maximum values are $f\left(k\pi + \frac{\pi}{2}\right) = 1$, and the local minimum values are $f(k\pi) = 0$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 2 \cos(2x),$$

so $f''(x) = 0$ when $2x = \frac{(2k+1)\pi}{2}$ or $x = \frac{(2k+1)\pi}{4}$, where k is any integer. For $-\frac{\pi}{4} < x < \frac{\pi}{4}$, $f''(x) > 0$ while for $\frac{\pi}{4} < x < \frac{3\pi}{4}$, $f''(x) < 0$. Taking into account the period of the function f , it follows that f is concave up on intervals of the form $\left(k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}\right)$ and

concave down on intervals of the form $\left(k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right)$ for any integer k .

- (c) Because the concavity of f changes at $\frac{(2k+1)\pi}{4}$ for any integer k , the points

$$\left(\frac{(2k+1)\pi}{4}, f\left(\frac{(2k+1)\pi}{4}\right)\right) = \left(\frac{(2k+1)\pi}{4}, \frac{1}{2}\right)$$

are points of inflection of f for any integer k .

78. Let $f(x) = \cos^2 x$. Using the half-angle formula $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, we can see that f is a periodic function with period π .

(a) The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -2 \sin x \cos x = -\sin(2x),$$

so $f'(x) = 0$ when $2x = k\pi$, where k is any integer. The critical numbers of f are therefore $\frac{k\pi}{2}$ for any integer k . To determine where $f'(x) > 0$ and $f'(x) < 0$, consider the intervals $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$. The increasing/decreasing pattern demonstrated on these two intervals will repeat indefinitely in either direction in increments of the period π . The results are shown in the table below.

Interval	Sign of $-2 \sin x$	Sign of $\cos x$	Sign of $f'(x)$	Conclusion
$(0, \frac{\pi}{2})$	—	+	—	f is decreasing
$(\frac{\pi}{2}, \pi)$	—	—	+	f is increasing

Therefore, f is decreasing on intervals of the form $(k\pi, k\pi + \frac{\pi}{2})$ and increasing on intervals of the form $(k\pi + \frac{\pi}{2}, (k+1)\pi)$, where k is any integer. By the First Derivative Test, it follows that f has a local minimum value at $k\pi + \frac{\pi}{2}$ and a local maximum value at $k\pi$, for any integer k . The

local minimum values are $f(k\pi + \frac{\pi}{2}) = 0$, and the local maximum values are $f(k\pi) = 1$.

(b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = -2 \cos(2x),$$

so $f''(x) = 0$ when $2x = \frac{(2k+1)\pi}{2}$ or $x = \frac{(2k+1)\pi}{4}$, where k is any integer. For $-\frac{\pi}{4} < x < \frac{\pi}{4}$, $f''(x) < 0$ while for $\frac{\pi}{4} < x < \frac{3\pi}{4}$, $f''(x) > 0$. Taking into account the period of the function f , it follows that f is

concave down on intervals of the form $(k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4})$ and concave up on intervals of the form $(k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4})$ for any integer k .

(c) Because the concavity of f changes at $\frac{(2k+1)\pi}{4}$ for any integer k , the points

$$\left(\frac{(2k+1)\pi}{4}, f\left(\frac{(2k+1)\pi}{4}\right) \right) = \left(\frac{(2k+1)\pi}{4}, \frac{1}{2} \right)$$

are points of inflection of f for any integer k .

79. Let $f(x) = x - 2 \sin x$, and consider the interval $[0, 2\pi]$.

(a) The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 1 - 2 \cos x,$$

so $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are critical numbers. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ to divide $[0, 2\pi]$ into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $f'(x)$	Conclusion
$(0, \frac{\pi}{3})$	−	f is decreasing
$(\frac{\pi}{3}, \frac{5\pi}{3})$	+	f is increasing
$(\frac{5\pi}{3}, 2\pi)$	−	f is decreasing

Therefore, f is decreasing on the intervals $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, 2\pi)$ and increasing on the interval $(\frac{\pi}{3}, \frac{5\pi}{3})$. By the First Derivative Test, it follows that f has a local minimum value at $\frac{\pi}{3}$ and a local maximum value at $\frac{5\pi}{3}$. The local minimum value is $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$, and the

local maximum value is $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = 2 \sin x,$$

so $f''(x) = 0$ when $x = 0, \pi$, and 2π . For $0 < x < \pi$, $f''(x) > 0$, so f is concave up on the interval $(0, \pi)$; for $\pi < x < 2\pi$, $f''(x) < 0$, so f is concave down on the interval $(\pi, 2\pi)$.

- (c) Because the concavity of f changes at π , the point $(\pi, f(\pi)) = (\pi, \pi)$ is a point of inflection of f .

80. Let $f(x) = 2 \cos^2 x - \sin^2 x$, and consider the interval $[0, 2\pi]$.

- (a) The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -4 \cos x \sin x - 2 \sin x \cos x = -6 \sin x \cos x = -3 \sin(2x),$$

so $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π are critical numbers. To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers $\frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$ to divide $[0, 2\pi]$ into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $-6 \sin x$	Sign of $\cos x$	Sign of $f'(x)$	Conclusion
$(0, \frac{\pi}{2})$	−	+	−	f is decreasing
$(\frac{\pi}{2}, \pi)$	−	−	+	f is increasing
$(\pi, \frac{3\pi}{2})$	+	−	−	f is decreasing
$(\frac{3\pi}{2}, 2\pi)$	+	+	+	f is increasing

Therefore, f is decreasing on the intervals $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$ and increasing on the intervals $(\frac{\pi}{2}, \pi)$ and $(\frac{3\pi}{2}, 2\pi)$. By the First Derivative Test, it follows that f has local minimum values at

$\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and a local maximum value at π . The local minimum values are $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = -1$, and the local maximum value is $f(\pi) = 2$.

- (b) The function f is concave up where $f''(x) > 0$ and concave down where $f''(x) < 0$. Now,

$$f''(x) = -6 \cos(2x),$$

so $f''(x) = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4},$ and $\frac{7\pi}{4}$. To determine where $f''(x) > 0$ and $f''(x) < 0$, use the numbers $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ to divide $[0, 2\pi]$ into five intervals. The sign of $f''(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $f''(x)$	Conclusion
$\left(0, \frac{\pi}{4}\right)$	—	f is concave down
$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	+	f is concave up
$\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$	—	f is concave down
$\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$	+	f is concave up
$\left(\frac{7\pi}{4}, 2\pi\right)$	—	f is concave down

Therefore, f is concave down on the intervals $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right),$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and concave up on the intervals $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$.

- (c) Because the concavity of f changes at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4},$ and $\frac{7\pi}{4}$, the points

$$\left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \frac{1}{2}\right), \quad \left(\frac{3\pi}{4}, f\left(\frac{3\pi}{4}\right)\right) = \left(\frac{3\pi}{4}, \frac{1}{2}\right),$$

$$\left(\frac{5\pi}{4}, f\left(\frac{5\pi}{4}\right)\right) = \left(\frac{5\pi}{4}, \frac{1}{2}\right), \quad \text{and} \quad \left(\frac{7\pi}{4}, f\left(\frac{7\pi}{4}\right)\right) = \left(\frac{7\pi}{4}, \frac{1}{2}\right)$$

are points of inflection of f .

81. Let $f(x) = -2x^3 + 15x^2 - 36x + 7$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -6x^2 + 30x - 36 = -6(x^2 - 5x + 6) = -6(x - 3)(x - 2),$$

so 2 and 3 are the critical numbers of f .

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 2 and 3 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $-6(x - 3)$	Sign of $x - 2$	Sign of $f'(x)$	Conclusion
$(-\infty, 2)$	+	—	—	f is decreasing
$(2, 3)$	+	+	+	f is increasing
$(3, \infty)$	—	+	—	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 2)$ and $(3, \infty)$ and increasing on the interval $(2, 3)$. By the First Derivative Test, it follows that f has a local minimum value at 2 and a local maximum value at 3. The

$$\text{local minimum value is } f(2) = -21, \text{ and the local maximum value is } f(3) = -20.$$

- (b) The second derivative of
- f
- is

$$f''(x) = -12x + 30.$$

Evaluating the second derivative at the critical numbers yields

$$f''(2) = 6 > 0 \quad \text{and} \quad f''(3) = -6 < 0.$$

By the Second Derivative Test, it follows that f has a local minimum value at 2 and a local maximum value at 3.

- (c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.

82. Let $f(x) = x^3 + 10x^2 + 25x - 25$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 + 20x + 25 = (3x + 5)(x + 5),$$

so -5 and $-\frac{5}{3}$ are the critical numbers of f .

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -5 and $-\frac{5}{3}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x + 5$	Sign of $x + 5$	Sign of $f'(x)$	Conclusion
$(-\infty, -5)$	—	—	+	f is increasing
$(-5, -\frac{5}{3})$	—	+	—	f is decreasing
$(-\frac{5}{3}, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -5)$ and $(-\frac{5}{3}, \infty)$ and decreasing on the interval $(-5, -\frac{5}{3})$. By the First Derivative Test, it follows that f has a local maximum value at -5 and a local minimum value at $-\frac{5}{3}$. The

local maximum value is $f(-5) = -25$, and the local minimum value is $f\left(-\frac{5}{3}\right) = -\frac{1175}{27}$.

- (b) The second derivative of f is

$$f''(x) = 6x + 20.$$

Evaluating the second derivative at the critical numbers yields

$$f''(-5) = -10 < 0 \quad \text{and} \quad f''\left(-\frac{5}{3}\right) = 10 > 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at -5 and a local minimum value at $-\frac{5}{3}$.

- (c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.

83. Let $f(x) = (x - 3)^2 e^x$. The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = (x - 3)^2 e^x + 2(x - 3)e^x = (x^2 - 4x + 3)e^x = (x - 3)(x - 1)e^x,$$

so 1 and 3 are the critical numbers of f

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 1 and 3 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x - 3$	Sign of $(x - 1)e^x$	Sign of $f'(x)$	Conclusion
$(-\infty, 1)$	—	—	+	f is increasing
$(1, 3)$	—	+	—	f is decreasing
$(3, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, 1)$ and $(3, \infty)$ and decreasing on the interval $(1, 3)$. By the First Derivative Test, it follows that f has a local maximum value at 1 and a local minimum value at 3. The

local maximum value is $f(1) = 4e$, and the local minimum value is $f(3) = 0$.

- (b) The second derivative of f is

$$f''(x) = (x^2 - 4x + 3)e^x + (2x - 4)e^x = (x^2 - 2x - 1)e^x.$$

Evaluating the second derivative at the critical numbers yields

$$f''(1) = -2e < 0 \quad \text{and} \quad f''(3) = 2e > 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at 1 and a local minimum value at 3.

- (c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.
84. Let $f(x) = (x + 1)^2 e^{-x}$. The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -(x + 1)^2 e^{-x} + 2(x + 1)e^{-x} = (-x^2 + 1)e^{-x} = (x + 1)(1 - x)e^{-x},$$

so -1 and 1 are the critical numbers of f .

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers -1 and 1 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $x + 1$	Sign of $(1 - x)e^{-x}$	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	—	+	—	f is decreasing
$(-1, 1)$	+	+	+	f is increasing
$(1, \infty)$	+	—	—	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and increasing on the interval $(-1, 1)$. By the First Derivative Test, it follows that f has a local minimum value at -1 and a local maximum value at 1 . The

$$\boxed{\text{local minimum value is } f(-1) = 0, \text{ and the local maximum value is } f(1) = 4e^{-1}}.$$

(b) The second derivative of f is

$$f''(x) = -(-x^2 + 1)e^{-x} - 2xe^{-x} = (x^2 - 2x - 1)e^{-x}.$$

Evaluating the second derivative at the critical numbers yields

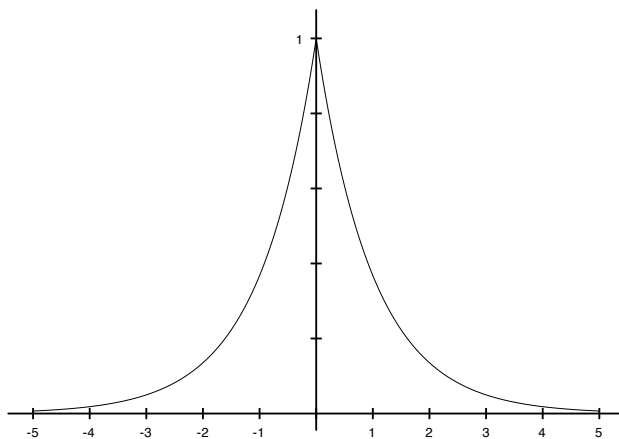
$$f''(-1) = 2e > 0 \quad \text{and} \quad f''(1) = -2e^{-1} < 0.$$

By the Second Derivative Test, it follows that f has a $\boxed{\text{local minimum value at } -1}$ and a $\boxed{\text{local maximum value at } 1}$.

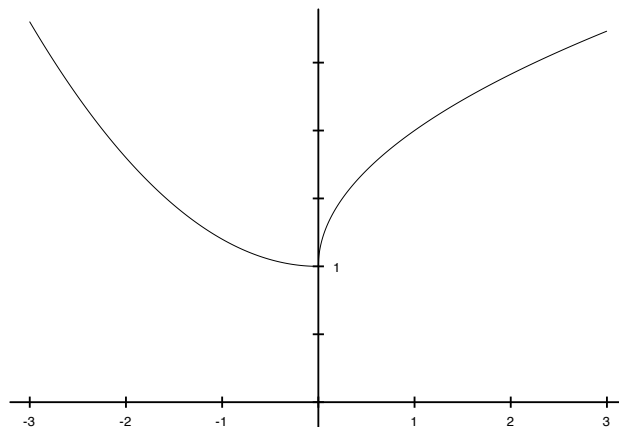
(c) Answers will vary, but here is one possible response. Neither test is particularly difficult to use. The First Derivative Test requires the calculation of only one derivative, but the Second Derivative Test requires fewer steps.

Applications and Extensions

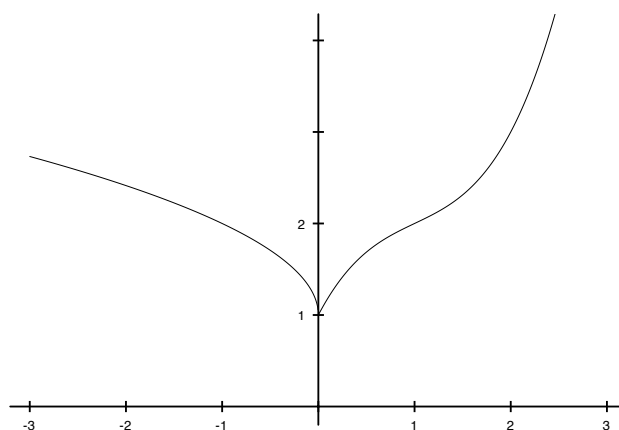
85. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave up on $(-\infty, \infty)$, increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and $f(0) = 1$.



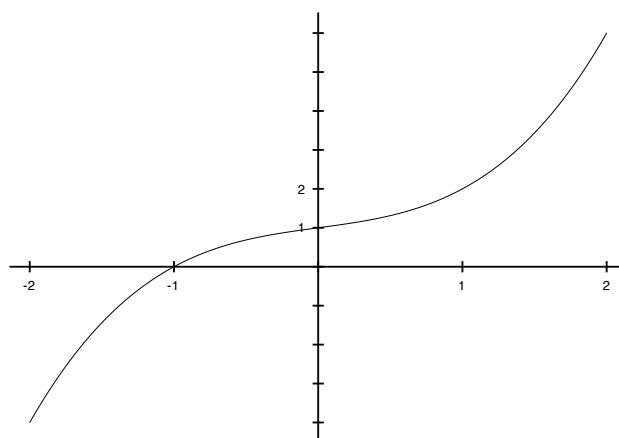
86. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, and $f(0) = 1$.



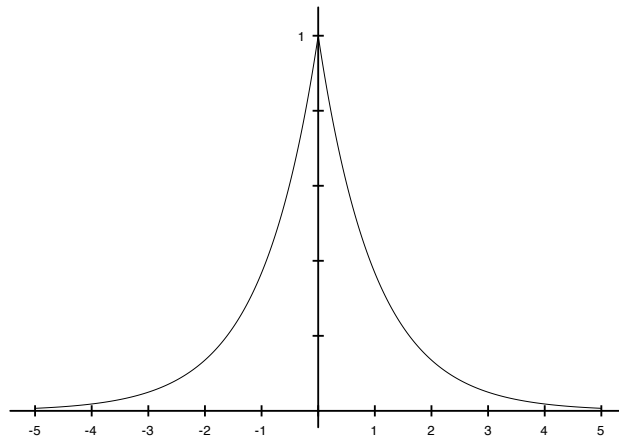
87. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave down on $(-\infty, 1)$, concave up on $(1, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, and $f(0) = 1$ and $f(1) = 2$.



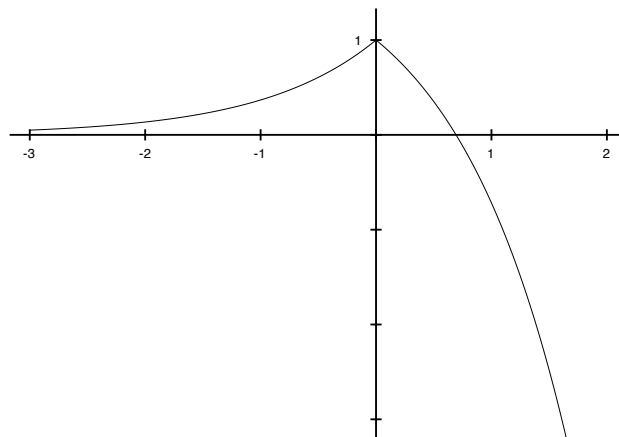
88. Answers will vary. The figure below displays the graph of a function f with the properties: f is concave down on $(-\infty, 0)$, concave up on $(0, \infty)$, increasing on $(-\infty, \infty)$, and $f(0) = 1$ and $f(1) = 2$.



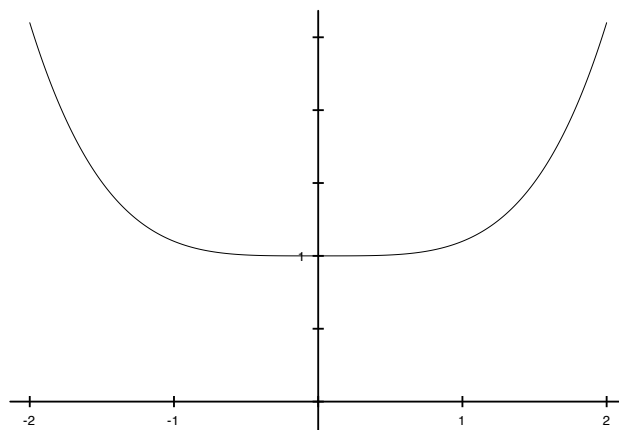
89. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(x) > 0$ if $x < 0$, $f'(x) < 0$ if $x > 0$, $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x > 0$, and $f(0) = 1$.



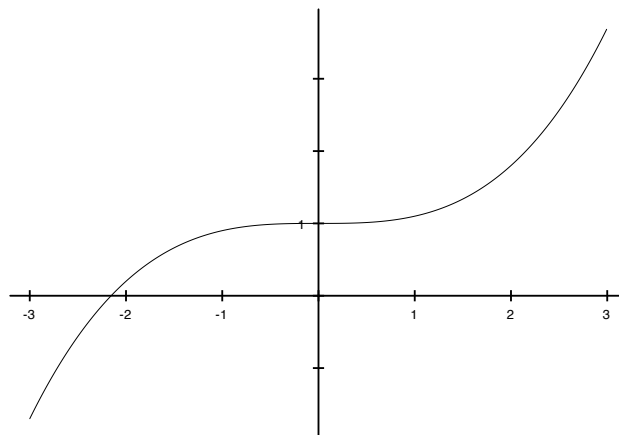
90. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(x) > 0$ if $x < 0$, $f'(x) < 0$ if $x > 0$, $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x > 0$, and $f(0) = 1$.



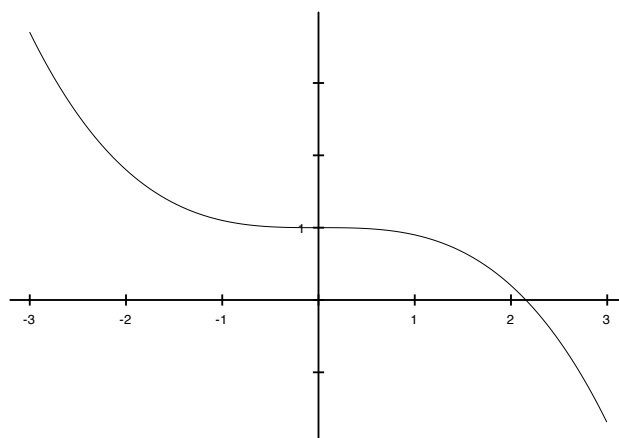
91. Answers will vary. The figure below displays the graph of a function f with the properties: $f''(0) = 0$, $f'(0) = 0$, $f''(x) > 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



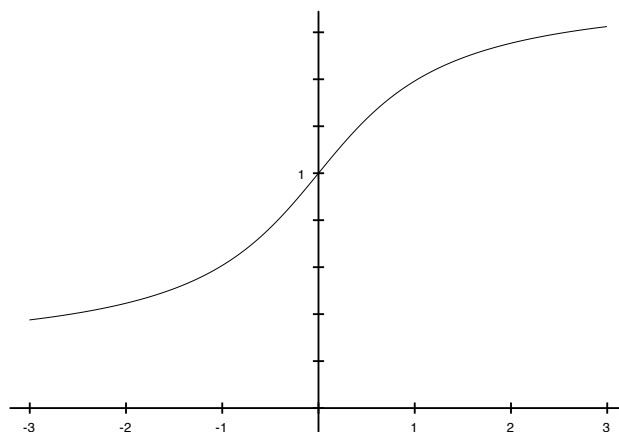
92. Answers will vary. The figure below displays the graph of a function f with the properties: $f''(0) = 0$, $f'(x) > 0$ if $x \neq 0$, $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



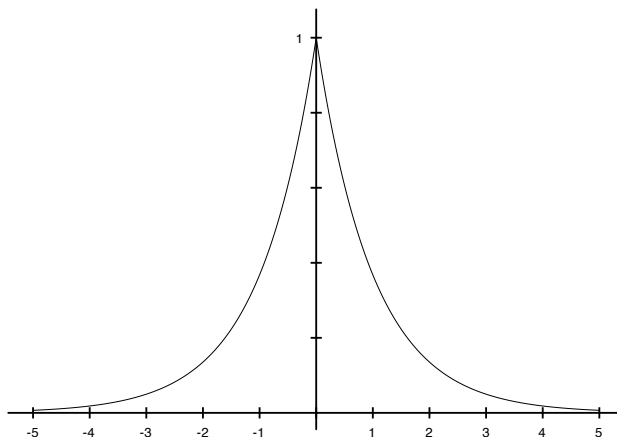
93. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(0) = 0$, $f'(x) < 0$ if $x \neq 0$, $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x > 0$, and $f(0) = 1$.



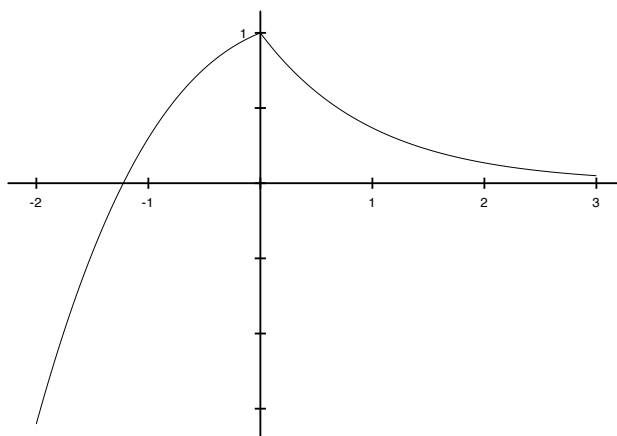
94. Answers will vary. The figure below displays the graph of a function f with the properties: $f''(0) = 0$, $f'(0) = \frac{1}{2}$, $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x > 0$, and $f(0) = 1$.



95. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(0)$ does not exist, $f''(x) > 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



96. Answers will vary. The figure below displays the graph of a function f with the properties: $f'(0)$ does not exist, $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$, and $f(0) = 1$.



97. Let $f(x) = e^{-(x-2)^2}$.

(a) Using the command

`Reduce [ D [ Exp[-(x-2)^2], {x,2} ] > 0, x ]`

in *Mathematica* yields the solution

$$x < 2 - \frac{\sqrt{2}}{2} \quad \text{or} \quad x > 2 + \frac{\sqrt{2}}{2}$$

to the inequality $f''(x) > 0$; the command

`Reduce [ D [ Exp[-(x-2)^2], {x,2} ] < 0, x ]`

yields the solution

$$2 - \frac{\sqrt{2}}{2} < x < 2 + \frac{\sqrt{2}}{2}$$

to the inequality $f''(x) < 0$. Therefore, f is

concave up on the intervals $\left(-\infty, 2 - \frac{\sqrt{2}}{2}\right)$ and $\left(2 + \frac{\sqrt{2}}{2}, \infty\right)$

and $\text{concave down on the interval } \left(2 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2}\right)$.

- (b) Because the concavity of f changes at $2 \pm \frac{\sqrt{2}}{2}$, the points

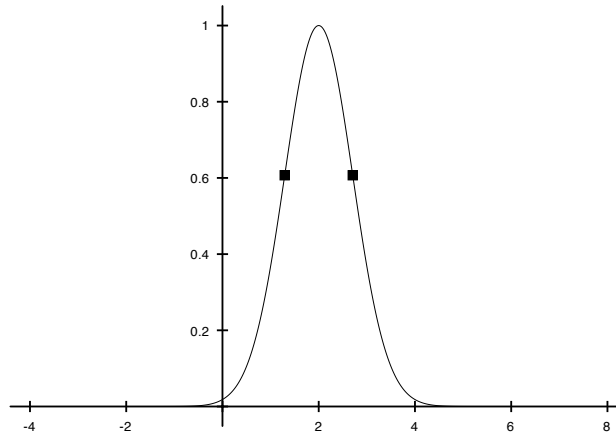
$$\left(2 - \frac{\sqrt{2}}{2}, f\left(2 - \frac{\sqrt{2}}{2}\right)\right) = \left(2 - \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}}\right)$$

and

$$\left(2 + \frac{\sqrt{2}}{2}, f\left(2 + \frac{\sqrt{2}}{2}\right)\right) = \left(2 + \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{e}}\right)$$

are points of inflection of f .

- (c) The figure below displays the graph of f with the points of inflection marked with black squares. At the point on the left, the concavity changes from up to down, while at the point on the right, the concavity changes from down to up.



98. Let $f(x) = x^2\sqrt{5-x}$.

- (a) Using the command

`Reduce [ D [  $x^2 \text{ Sqrt}[5-x]$ , {x,2} ] > 0, x ]`

in *Mathematica* yields the solution

$$x < 4 - \frac{2\sqrt{6}}{3}$$

to the inequality $f''(x) > 0$; the command

`Reduce [ D [  $x^2 \text{ Sqrt}[5-x]$ , {x,2} ] < 0, x ]`

yields the solution

$$4 - \frac{2\sqrt{6}}{3} < x < 5$$

to the inequality $f''(x) < 0$. Therefore, f is $\text{concave up on the interval } \left(-\infty, 4 - \frac{2\sqrt{6}}{3}\right)$ and

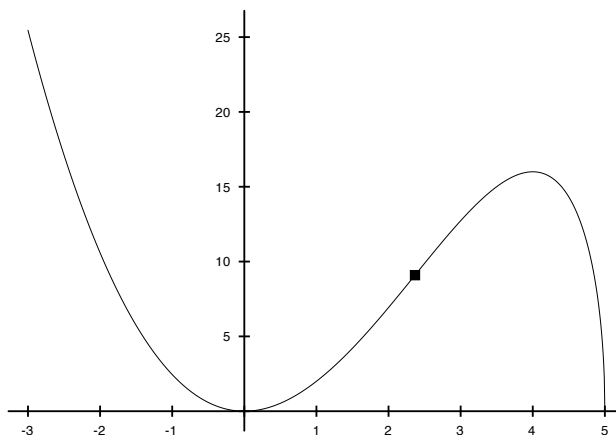
$\text{concave down on the interval } \left(4 - \frac{2\sqrt{6}}{3}, 5\right)$.

- (b) Because the concavity of f changes at $4 - \frac{2\sqrt{6}}{3}$, the point

$$\left(4 - \frac{2\sqrt{6}}{3}, f\left(4 - \frac{2\sqrt{6}}{3}\right)\right) = \left(4 - \frac{2\sqrt{6}}{3}, \frac{1}{27}(12 - 2\sqrt{6})^2 \sqrt{9 + 6\sqrt{6}}\right)$$

is a point of inflection of f .

- (c) The figure below displays the graph of f with the point of inflection marked with a black square. At the point inflection, the concavity changes from up to down.



99. Let $f(x) = \frac{2-x}{2x^2-2x+1}$.

- (a) Using the command

$$\text{N [Reduce [D [(2-x)/(2x^2 - 2x + 1), \{x,2\}] > 0, x]]}$$

in *Mathematica* yields the approximate solution

$$x < 0.134792 \quad \text{or} \quad 0.7211 < x < 5.14411$$

to the inequality $f''(x) > 0$; the command

$$\text{N [Reduce [D [(2-x)/(2x^2 - 2x + 1), \{x,2\}] < 0, x]]}$$

yields the approximate solution

$$0.134792 < x < 0.7211 \quad \text{or} \quad x > 5.14411$$

to the inequality $f''(x) < 0$. Therefore, f is

$$\text{concave up on the intervals } (-\infty, 0.134792) \text{ and } (0.7211, 5.14411)$$

and $\text{concave down on the intervals } (0.134792, 0.7211) \text{ and } (5.14411, \infty)$.

- (b) Because the concavity of f changes at approximately 0.134792, 0.7211, and 5.14411, the points

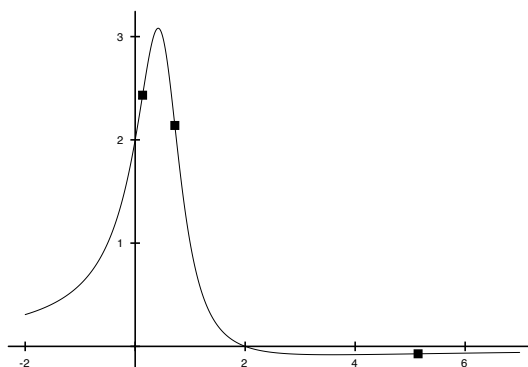
$$(0.134792, f(0.134792)) = (0.134792, 2.43260),$$

$$(0.7211, f(0.7211)) = (0.7211, 2.13945), \text{ and}$$

$$(5.14411, f(5.14411)) = (5.14411, -0.07205)$$

are approximately the points of inflection of f .

- (c) The figure below displays the graph of f with the points of inflection marked with black squares. From left to right, the concavity changes from up to down, down to up, and up to down at the inflection points.



100. Let $f(x) = \frac{3x}{x^2 + 3x + 5}$.

- (a) Using the command

$$\text{N [Reduce [D [(3x)/(x^2 + 3x + 5), \{x, 2\}] > 0, x]]}$$

in *Mathematica* yields the approximate solution

$$-3.21463 < x < -1.0852 \quad \text{or} \quad x > 4.29983$$

to the inequality $f''(x) > 0$; the command

$$\text{N [Reduce [D [(3x)/(x^2 + 3x + 5), \{x, 2\}] < 0, x]]}$$

yields the approximate solution

$$x < -3.21463 \quad \text{or} \quad -1.0852 < x < 4.29983$$

to the inequality $f''(x) < 0$. Therefore, f is

$$\boxed{\text{concave up on the intervals } (-3.21463, -1.0852) \text{ and } (4.29983, \infty)}$$

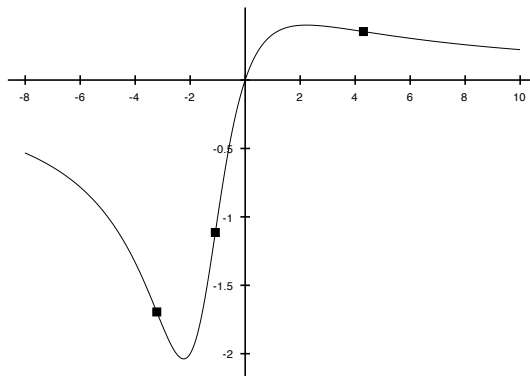
and $\boxed{\text{concave down on the intervals } (-\infty, -3.21463) \text{ and } (-1.0852, 4.29983)}$.

- (b) Because the concavity of f changes at approximately -3.21463 , -1.0852 , and 4.29983 , the points

$$\begin{aligned} (-3.21463, f(-3.21463)) &= \boxed{(-3.21463, -1.69490)}, \\ (-1.0852, f(-1.0852)) &= \boxed{(-1.0852, -1.11415)}, \text{ and} \\ (4.29983, f(4.29883)) &= \boxed{(4.29883, 0.35450)} \end{aligned}$$

are approximately the points of inflection of f .

- (c) The figure below displays the graph of f with the points of inflection marked with black squares. From left to right, the concavity changes from down to up, up to down, and down to up at the inflection points.



101. Let $f(x) = \sqrt[3]{x}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}},$$

so $f'(x)$ does not exist when $x = 0$ and is never equal to 0. Therefore, 0 is the only critical number of f . Additionally, $f'(x) > 0$ for $x \neq 0$, so f is increasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. By the First Derivative Test, it follows that f has neither a local maximum nor a local minimum at 0; therefore, f has no local extrema.

102. Let $f(x) = \sqrt[3]{x^2}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}},$$

so $f'(x)$ does not exist when $x = 0$ and is never equal to 0. Therefore, 0 is the only critical number of f . Additionally, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, so f is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$. By the First Derivative Test, it follows that f has a local minimum at 0.

103. Let $f(x) = ax^3 + bx^2$. For the point $(1, 6)$ to be on the graph of f , $f(1)$ must be equal to 6. As $f(1) = a + b$, the numbers a and b must satisfy the equation $a + b = 6$. Because the polynomial function f is twice differentiable everywhere, if f is to have an inflection point at the point $(1, 6)$, $f''(1)$ must be equal to 0. Now,

$$f'(x) = 3ax^2 + 2bx \quad \text{and} \quad f''(x) = 6ax + 2b,$$

so $f''(1) = 6a + 2b$ and another equation that a and b must satisfy is $6a + 2b = 0$. Solving $a + b = 6$ and $6a + 2b = 0$ yields $\boxed{a = -3 \text{ and } b = 9}$.

104. Let $f(x) = ax^3 + bx^2 + cx + d$. The condition $f(0) = 4$ requires that $d = 4$. For the point $(1, -2)$ to be on the graph of f , $f(1)$ must be equal to -2 . As $f(1) = a + b + c + d$, the numbers a , b , c , and d must satisfy the equation $a + b + c + d = -2$. Because the polynomial function f is twice differentiable everywhere, for 0 to be a critical number of f , $f'(0)$ must be equal to 0. Additionally, for f to have an inflection point at $(1, -2)$, $f''(1)$ must be equal to 0. Now,

$$f'(x) = 3ax^2 + 2bx + c \quad \text{and} \quad f''(x) = 6ax + 2b,$$

so $f'(0) = c$ and $f''(1) = 6a + 2b$. It follows that a , b , c , and d must also satisfy the equations $c = 0$ and $6a + 2b = 0$. Solving $d = 4$, $a + b + c + d = -2$, $c = 0$, and $6a + 2b = 0$ yields $\boxed{a = 3, b = -9, c = 0, \text{ and } d = 4}$.

105. Let $N(t) = \frac{10,000}{1 + 9999e^{-t}}$.

- (a) The rate of change of the infection is

$$N'(t) = \frac{(1 + 9999e^{-t}) \cdot 0 - 10,000 \cdot -9999e^{-t}}{(1 + 9999e^{-t})^2} = \frac{99,990,000e^{-t}}{(1 + 9999e^{-t})^2}.$$

- (b) $N'(t)$ is increasing when $N''(t) > 0$ and decreasing when $N''(t) < 0$. Now,

$$\begin{aligned} N''(t) &= \frac{(1 + 9999e^{-t})^2 \cdot -99,990,000e^{-t} - 99,990,000e^{-t} \cdot 2(1 + 9999e^{-t})(-9999e^{-t})}{(1 + 9999e^{-t})^4} \\ &= \frac{99,990,000e^{-t}(19998e^{-t} - 1 - 9999e^{-t})}{(1 + 9999e^{-t})^3} = \frac{99,990,000e^{-t}(9999e^{-t} - 1)}{(1 + 9999e^{-t})^3}, \end{aligned}$$

so $N''(t) = 0$ when

$$9999e^{-t} - 1 = 0 \quad \text{or} \quad t = \ln 9999.$$

For $0 < t < \ln 9999$, $N''(t) > 0$, so $N'(t)$ is increasing on the interval $(0, \ln 9999)$.

For $t > \ln 9999$, $N''(t) < 0$, so $N'(t)$ is decreasing on the interval $(\ln 9999, \infty)$.

- (c) Based on the results of part (b) and the First Derivative Test, the rate of change of the infection is a maximum when $t = \ln 9999$.
- (d) Based on the results of part (b), N is concave up on the interval $(0, \ln 9999)$ and concave down on the interval $(\ln 9999, \infty)$. Therefore, the concavity of N changes at $\ln 9999$, and the point

$$(\ln 9999, N(\ln 9999)) = (\ln 9999, 5000)$$

is an inflection point of N .

- (e) Comparing the results of parts (c) and (d), we see that the point of inflection of N is the point at which $N'(t)$ is maximum.

106. Let $P(t) = \frac{300}{1 + 50e^{-0.2t}}$.

- (a) The annual profit is increasing when $P'(t) > 0$ and decreasing when $P'(t) < 0$. Now,

$$P'(t) = \frac{(1 + 50e^{-0.2t}) \cdot 0 - 300 \cdot -10e^{-0.2t}}{(1 + 50e^{-0.2t})^2} = \frac{3000e^{-0.2t}}{(1 + 50e^{-0.2t})^2}.$$

Note that $P'(t) > 0$ for all t , so annual profit is increasing on the interval $(0, \infty)$ and is never decreasing.

- (b) From part (a), the rate of change in profit is

$$P'(t) = \frac{3000e^{-0.2t}}{(1 + 50e^{-0.2t})^2}.$$

- (c) The rate of change in profit $P'(t)$ is increasing when $P''(t) > 0$ and decreasing when $P''(t) < 0$. Now,

$$\begin{aligned} P''(t) &= \frac{(1 + 50e^{-0.2t})^2 \cdot -600e^{-0.2t} - 3000e^{-0.2t} \cdot 2(1 + 50e^{-0.2t})(-10e^{-0.2t})}{(1 + 50e^{-0.2t})^4} \\ &= \frac{600e^{-0.2t}(100e^{-0.2t} - 1 - 50e^{-0.2t})}{(1 + 50e^{-0.2t})^3} = \frac{600e^{-0.2t}(50e^{-0.2t} - 1)}{(1 + 50e^{-0.2t})^3}, \end{aligned}$$

so $P''(t) = 0$ when

$$50e^{-0.2t} - 1 = 0 \quad \text{or} \quad t = 5 \ln 50.$$

For $0 < t < 5 \ln 50$, $P''(t) > 0$, so $P'(t)$ is increasing on the interval $(0, 5 \ln 50)$. For $t > 5 \ln 50$, $P''(t) < 0$, so $P'(t)$ is decreasing on the interval $(5 \ln 50, \infty)$.

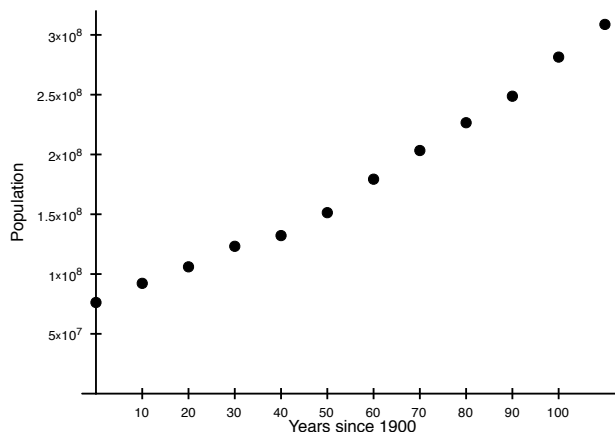
- (d) Based on the results of part (c) and the First Derivative Test, the rate of change in profit is a maximum when $t = 5 \ln 50$.
- (e) Based on the results of part (c), P is concave up on the interval $(0, 5 \ln 50)$ and concave down on the interval $(5 \ln 50, \infty)$. Therefore, the concavity of P changes at $5 \ln 50$, and the point

$$(5 \ln 50, P(5 \ln 50)) = (5 \ln 50, 150)$$

is an inflection point of P .

- (f) Comparing the results of parts (d) and (e), we see that the point of inflection of P is the point at which $P'(t)$ is maximum.

107. (a) The figure below displays a scatterplot of the population data.



- (b) Using the **Logistic** regression function on a TI-84 Plus calculator, the logistic function that best fits the given data is

$$P(t) = \frac{762,176,717.8}{1 + 8.743e^{-0.0162t}}.$$

- (c) The rate of change in population is

$$P'(t) = \frac{(1 + 8.743e^{-0.0162t}) \cdot 0 - 762,176,717.8 \cdot -0.1416366e^{-0.0162t}}{(1 + 8.743e^{-0.0162t})^2} = \frac{107,952,118.9e^{-0.0162t}}{(1 + 8.743e^{-0.0162t})^2}.$$

- (d) $P'(t)$ is increasing when $P''(t) > 0$ and decreasing when $P''(t) < 0$. Now,

$$\begin{aligned} P''(t) &= \frac{(1 + 8.743e^{-0.0162t})^2 \cdot -1,748,824.326e^{-0.0162t}}{(1 + 8.743e^{-0.0162t})^4} - \frac{107,952,118.9e^{-0.0162t} \cdot 2(1 + 8.743e^{-0.0162t})(-0.1416366e^{-0.0162t})}{(1 + 8.743e^{-0.0162t})^4} \\ &= \frac{1,748,824.326e^{-0.0162t}(17.486e^{-0.0162t} - 1 - 8.743e^{-0.0162t})}{(1 + 8.743e^{-0.0162t})^3} \\ &= \frac{1,748,824.326e^{-0.0162t}(8.743e^{-0.0162t} - 1)}{(1 + 8.743e^{-0.0162t})^3}, \end{aligned}$$

so $P''(t) = 0$ when

$$8.743e^{-0.0162t} - 1 = 0 \quad \text{or} \quad t = \frac{1}{0.0162} \ln 8.743 \approx 133.843.$$

For $0 < t < 133.843$, $P''(t) > 0$, so $P'(t)$ is increasing on the interval $(0, 133.843)$. For $t > 133.843$, $P''(t) < 0$, so $P'(t)$ is decreasing on the interval $(133.843, \infty)$.

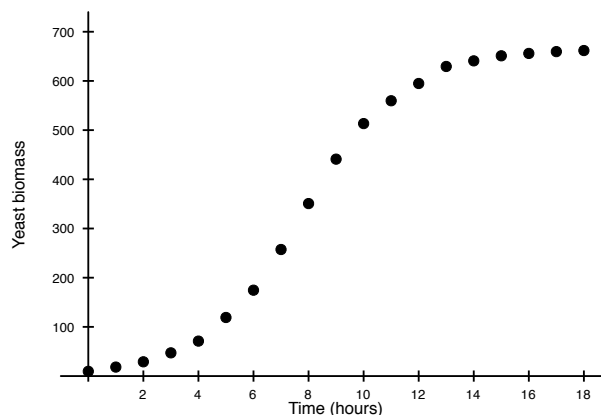
- (e) Based on the results of part (d) and the First Derivative Test, the rate of change in population is a maximum when $t \approx 133.843$.
- (f) Based on the results of part (d), P is concave up on the interval $(0, 133.843)$ and concave down on the interval $(133.843, \infty)$. Therefore, the concavity of P changes at 133.843, and the point

$$(133.843, P(133.843)) \approx (133.843, 381088358.9)$$

is an inflection point of P .

- (g) Comparing the results of parts (e) and (f), we see that the point of inflection of P is the point at which $P'(t)$ is maximum.

108. (a) The figure below displays a scatterplot of the yeast biomass data.



- (b) Using the **Logistic** regression function on a TI-84 Plus calculator, the logistic function that best fits the given data is

$$y = \frac{663.0}{1 + 71.6e^{-0.5470t}}.$$

- (c) The rate of change in biomass is

$$y'(t) = \frac{(1 + 71.6e^{-0.5470t}) \cdot 0 - 663.0 \cdot -39.1652e^{-0.5470t}}{(1 + 71.6e^{-0.5470t})^2} = \frac{25,966.5276e^{-0.5470t}}{(1 + 71.6e^{-0.5470t})^2}.$$

- (d) $y'(t)$ is increasing when $y''(t) > 0$ and decreasing when $y''(t) < 0$. Now,

$$\begin{aligned} y''(t) &= \frac{(1 + 71.6e^{-0.5470t})^2 \cdot -14,203.6906e^{-0.5470t}}{(1 + 71.6e^{-0.5470t})^4} - \frac{25,966.5276e^{-0.5470t} \cdot 2(1 + 71.6e^{-0.5470t})(-39.1652e^{-0.5470t})}{(1 + 71.6e^{-0.5470t})^4} \\ &= \frac{14,203.6906e^{-0.5470t}(143.2e^{-0.5470t} - 1 - 71.6e^{-0.5470t})}{(1 + 71.6e^{-0.5470t})^3} \\ &= \frac{14,203.6906e^{-0.5470t}(71.6e^{-0.5470t} - 1)}{(1 + 71.6e^{-0.5470t})^3}, \end{aligned}$$

so $y''(t) = 0$ when

$$71.6e^{-0.5470t} - 1 = 0 \quad \text{or} \quad t = \frac{1}{0.5470} \ln 71.6 \approx 7.808.$$

For $0 < t < 7.808$, $y''(t) > 0$, so $y'(t)$ is increasing on the interval $(0, 7.808)$. For $t > 7.808$, $y''(t) < 0$, so $y'(t)$ is decreasing on the interval $(7.808, \infty)$.

- (e) Based on the results of part (d) and the First Derivative Test, the rate of change in biomass is a maximum when $t \approx 7.808$.
- (f) Based on the results of part (d), y is concave up on the interval $(0, 7.808)$ and concave down on the interval $(7.808, \infty)$. Therefore, the concavity of y changes at 7.808, and the point

$$(7.808, y(7.808)) \approx (7.808, 331.5)$$

is an inflection point of y .

- (g) Comparing the results of parts (e) and (f), we see that the point of inflection of y is the point at which $y'(t)$ is maximum.

109. Let $B(t) = -12.8t^3 + 163.4t^2 - 614.0t + 390.6$.

- (a) The polynomial function B is differentiable everywhere, so the critical numbers of B occur where $B'(t) = 0$. Now,

$$B'(t) = -38.4t^2 + 326.8t - 614.0,$$

so the critical numbers of B are

$$t = \frac{-326.8 \pm \sqrt{326.8^2 - 4(-38.4)(-614.0)}}{-76.8} = \frac{-326.8 \pm \sqrt{12487.84}}{-76.8} \approx 2.80, 5.71.$$

Evaluating $B''(t) = -76.8t + 326.8$ at the critical numbers yields

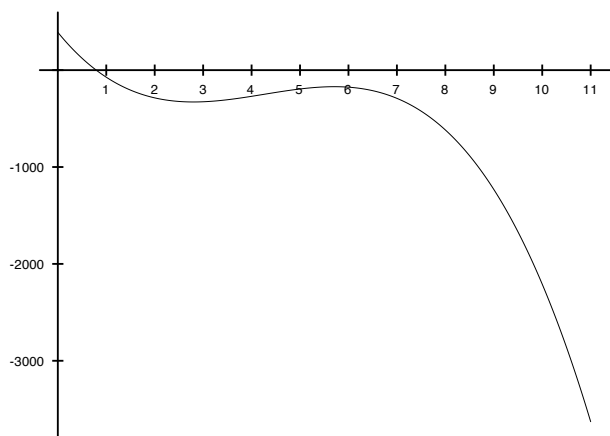
$$B''(2.80) = 111.76 > 0 \quad \text{and} \quad B''(5.71) = -111.728 < 0,$$

so B has a local minimum value at 2.80 and a local maximum value at 5.71. The

$$\text{local minimum value is } B(2.80) \approx -328.53 \text{ billion dollars},$$

and the $\text{local maximum value is } B(5.71) \approx -170.80 \text{ billion dollars}.$

- (b) Because both local extreme values are negative, both represent a $\text{budget deficit}.$
- (c) From part (a), $B''(t) = -76.8t + 326.8$, so $B''(t) = 0$ when $t \approx 4.26$. For $0 < t < 4.26$, $B''(t) > 0$, so B is $\text{concave up on the interval } (0, 4.26);$ for $4.26 < t < 9$, $B''(t) < 0$, so B is $\text{concave down on the interval } (4.26, 9).$ Because the concavity of B changes at 4.26, the point $(4.26, B(4.26)) \approx (4.26, -249.27)$ is a point of inflection of B .
- (d) Because $B''(t) > 0$ for $t < 4.26$, to the left of the point of inflection, the rate of change of the budget is increasing at an increasing rate; because $B''(t) < 0$ for $t > 4.26$, to the right of the point of inflection, the rate of change of the budget is increasing at a decreasing rate.
- (e) The figure below displays the graph of B . Given that B predicts a 3.6 trillion dollar budget deficit in 2011, which would indicate that the government took in nothing in 2011, B does not seem to be an accurate predictor for the budget for the years 2010 and beyond.



110. Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Because local extrema can only occur at critical numbers and the polynomial function f is differentiable everywhere, local extrema of f can only occur where $f'(x) = 0$. Now,

$$f'(x) = 3ax^2 + 2bx + c,$$

so $f'(x) = 0$ when

$$x = \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)(c)}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

Therefore, the number of critical numbers, and therefore the number of potential local extrema, depends on the sign of the discriminant $b^2 - 3ac$.

111. Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. In order for the graph of f to contain the points $(0, 5)$ and $(4, 33)$, $f(0)$ must be equal to 5 and $f(4)$ must be equal to 33. These conditions yield the equations $d = 5$ and $64a + 16b + 4c + d = 33$. The polynomial function f is differentiable everywhere, so for f to have a local minimum at 0 and a local maximum at 4, both $f'(0)$ and $f'(4)$ must be equal to 0. Now,

$$f'(x) = 3ax^2 + 2bx + c,$$

so the derivative conditions yield the equations $c = 0$ and $48a + 8b + c = 0$. The solution of the equations

$$d = 5, 64a + 16b + 4c + d = 33, c = 0, \text{ and } 48a + 8b + c = 0 \text{ is } \boxed{a = -\frac{7}{8}, b = \frac{21}{4}, c = 0, \text{ and } d = 5}.$$

112. Let $f(x) = \frac{2}{x} + \frac{8}{1-x}$. Then

$$f'(x) = -\frac{2}{x^2} + \frac{8}{(1-x)^2},$$

so $f'(x) = 0$ when

$$\begin{aligned} \frac{2}{x^2} &= \frac{8}{(1-x)^2} \\ (1-x)^2 &= 4x^2 \\ 1-2x+x^2 &= 4x^2 \\ 0 &= 3x^2 + 2x - 1. \end{aligned}$$

Using the quadratic formula,

$$x = \frac{-2 \pm \sqrt{(-2)^2 - 4(3)(-1)}}{6} = \frac{-2 \pm \sqrt{16}}{6} = -1, \frac{1}{3},$$

so the only critical number on the interval $(0, 1)$ is $1/3$. For $0 < x < 1/3$, $f'(x) < 0$, so f is decreasing on the interval $(0, 1/3)$; for $1/3 < x < 1$, $f'(x) > 0$, so f is increasing on the interval $(1/3, 1)$. By the First Derivative Test, it follows that f has a local minimum value at $1/3$. The local minimum value is $f(1/3) = 18$.

113. Let $y = \sqrt{3} \sin x + \cos x$, and consider the interval $0 \leq x \leq 2\pi$. Because the function y is differentiable everywhere, local extrema can only occur where $y'(x) = 0$. Now,

$$y'(x) = \sqrt{3} \cos x - \sin x,$$

so $y'(x) = 0$ when $\tan x = \sqrt{3}$, which is when $x = \frac{\pi}{3}$ and when $x = \frac{4\pi}{3}$. For $0 < x < \frac{\pi}{3}$, $y'(x) > 0$, so y is increasing on the interval $(0, \frac{\pi}{3})$; for $\frac{\pi}{3} < x < \frac{4\pi}{3}$, $y'(x) < 0$, so y is decreasing on the interval $(\frac{\pi}{3}, \frac{4\pi}{3})$; and for $\frac{4\pi}{3} < x < 2\pi$, $y'(x) > 0$, so y is increasing on the interval $(\frac{4\pi}{3}, 2\pi)$. By the First

Derivative Test, it follows that y has a local maximum value at $\frac{\pi}{3}$ and a local minimum value at $\frac{4\pi}{3}$.

The local maximum value is $f\left(\frac{\pi}{3}\right) = 2$, and the local minimum value is $f\left(\frac{4\pi}{3}\right) = -2$.

Next,

$$y''(x) = -\sqrt{3}\sin x - \cos x,$$

so $y''(x) = 0$ when $\tan x = -\frac{\sqrt{3}}{3}$, which is when $x = \frac{5\pi}{6}$ and when $x = \frac{11\pi}{6}$. For $0 < x < \frac{5\pi}{6}$, $y''(x) < 0$, so y is concave down on the interval $\left(0, \frac{5\pi}{6}\right)$; for $\frac{5\pi}{6} < x < \frac{11\pi}{6}$, $y''(x) > 0$, so y is concave up on the interval $\left(\frac{5\pi}{6}, \frac{11\pi}{6}\right)$; and for $\frac{11\pi}{6} < x < 2\pi$, $y''(x) < 0$, so y is concave down on the interval $\left(\frac{11\pi}{6}, 2\pi\right)$. Because the concavity of y changes at $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$, the points

$\left(\frac{5\pi}{6}, f\left(\frac{5\pi}{6}\right)\right) = \left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, f\left(\frac{11\pi}{6}\right)\right) = \left(\frac{11\pi}{6}, 0\right)$ are points of inflection of y .

114. Let $n > 1$ and consider the function $f(x) = x^n - n(x-1) - 1$ for $x > 0$. Now,

$$f'(x) = nx^{n-1} - n = n(x^{n-1} - 1),$$

so 1 is a critical number of f . For $0 < x < 1$, $f'(x) < 0$, so f is decreasing on the interval $(0, 1)$, and for $x > 1$, $f'(x) > 0$, so f is increasing on the interval $(1, \infty)$. It follows that $f(x)$ is always greater than or equal to $f(1) = 1 - n(0) - 1 = 0$. Therefore,

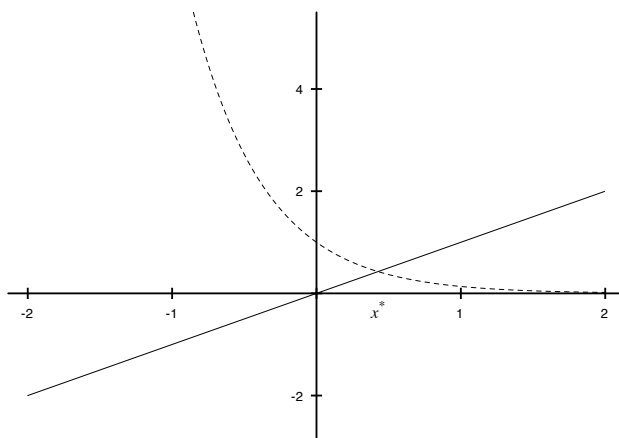
the expression $x^n - n(x-1) - 1$ can never be negative for $x > 0$ and $n > 1$.

115. Let $f(x) = x^{2/3}$. Then

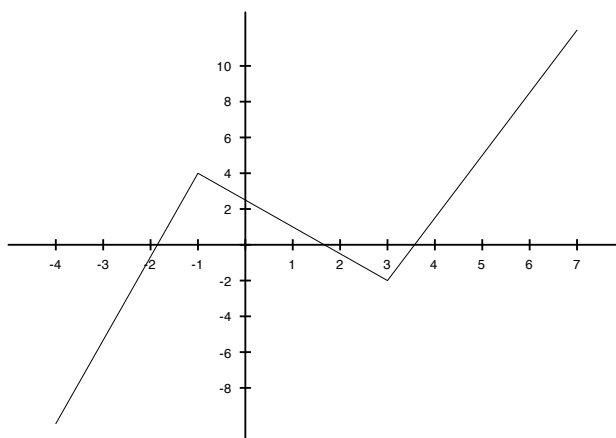
$$f'(x) = \frac{2}{3}x^{-1/3} \quad \text{and} \quad f''(x) = -\frac{2}{9}x^{-4/3}.$$

It follows that 0 is the only critical number of f , but $f''(x)$ does not exist at 0; therefore, the Second Derivative Test cannot be applied to identify the extreme value of $f(x) = x^{2/3}$.

116. Let $f(x) = x^2 + e^{-2x}$. Then $f'(x) = 2x - 2e^{-2x} = 2(x - e^{-2x})$ and $f''(x) = 2 + 4e^{-2x}$. The figure below displays the graphs of $y = x$ (the solid curve) and $y = e^{-2x}$ (the dashed curve). Let x^* denote the x -coordinate of the point of intersection between the two curves.



- (a) **False**. Based on the graph above, $x > e^{-2x}$ for $x > x^*$, so $f'(x) > 0$ and f is increasing on the interval (x^*, ∞) .
- (b) **False**. Based on the graph above, $x < e^{-2x}$ for $x < x^*$, so $f'(x) < 0$ and f is decreasing on the interval $(-\infty, x^*)$.
- (c) **False**. As f is the sum of two functions that are continuous everywhere, f is continuous everywhere.
- (d) **True**. Because $e^{-2x} > 0$ for all x , $f''(x) = 2 + 4e^{-2x} > 2 > 0$ for all x . Therefore, f is concave up for all x .
- (e) **False**. As noted in part (d), f is concave up for all x .
117. (a) **Not necessarily true**. The figure below displays a function f that is continuous for all x and has a local maximum at $(-1, 4)$ and a local minimum at $(3, -2)$, but does not have a point of inflection somewhere between $x = -1$ and $x = 3$.



- (b) **Not necessarily true**. See the graph in part (a) for which $f'(-1)$ does not exist.
- (c) **Not necessarily true**. See the graph in part (a) which does not have a horizontal asymptote.
- (d) **Not necessarily true**. See the graph in part (a) which does not have a tangent line at $x = 3$.
- (e) **True**. The function is continuous for all x , so $f(0)$ is defined, meaning the graph of f has a y -intercept, and the graph of f intersects the y -axis. Moreover, because f is continuous on the closed interval $[-1, 3]$ with $f(-1) = 4 > 0$ and $f(3) = -2 < 0$, the Intermediate Value Theorem guarantees there exists a c in $(-1, 3)$ such that $f(c) = 0$. Therefore, the graph of f intersects the x -axis as well.
118. Let $f(x) = ax^2 + bx + c$, where $a \neq 0$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 2ax + b = 0 \quad \text{when} \quad x = -\frac{b}{2a}.$$

Moreover, $f''(x) = 2a$. It follows that when $a < 0$, $f''(x) = 2a < 0$ for all x and f has a local maximum at $-\frac{b}{2a}$ by the Second Derivative Test. On the other hand, when $a > 0$, $f''(x) = 2a > 0$ for all x and f has a local minimum at $-\frac{b}{2a}$ by the Second Derivative Test.

119. Consider the function $f(x) = x - \sin x$. Now,

$$f'(x) = 1 + \cos x \geq 0$$

on the interval $0 \leq x \leq 2\pi$. Therefore, f is increasing on the interval $0 \leq x \leq 2\pi$, so $f(x) \geq f(0) = 0$ on the interval $0 \leq x \leq 2\pi$. That is, $x - \sin x \geq 0$, or $x \geq \sin x$ on the interval $0 \leq x \leq 2\pi$.

120. Consider the function $f(x) = \cos x - 1 + \frac{x^2}{2}$. Now,

$$f'(x) = -\sin x + x \geq 0$$

on the interval $0 \leq x \leq 2\pi$ by the result of Problem 119. Therefore, f is increasing on the interval $0 \leq x \leq 2\pi$, so $f(x) \geq f(0) = 0$ on the interval $0 \leq x \leq 2\pi$. That is, $\cos x - 1 + \frac{x^2}{2} \geq 0$, or $\cos x \geq 1 - \frac{x^2}{2}$ on the interval $0 \leq x \leq 2\pi$.

121. Let $f(x) = 2\sqrt{x} - 3 + \frac{1}{x}$. Now,

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}.$$

For $x > 1$,

$$x^2 > \sqrt{x} \quad \text{so that} \quad \frac{1}{x^2} < \frac{1}{\sqrt{x}}.$$

It follows that $f'(x) > 0$ and f is increasing for $x > 1$. Therefore, $f(x) > f(1) = 0$ for $x > 1$; that is,

$$2\sqrt{x} - 3 + \frac{1}{x} > 0 \quad \text{or} \quad 2\sqrt{x} > 3 - \frac{1}{x}$$

for $x > 1$.

122. Let $f(x) = x^2 - 8x + 21$. Then $f'(x) = 2x - 8 = 2(x - 4)$, and 4 is the only critical number of f . For $x < 4$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, 4)$; for $x > 4$, $f'(x) > 0$, so f is increasing on the interval $(4, \infty)$. It follows that $f(x) \geq f(4)$ for all x . Now, $f(4) = 5$, so $f(x) \geq 5 > 0$ for all x .

123. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 40$. Then

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x + 1)(x - 2),$$

so -1 , 0 , and 2 are critical numbers of f . Consider the closed interval $[-1, 2]$. Evaluating f at -1 , 0 , and 2 yields

$$f(-1) = 35, \quad f(0) = 40, \quad \text{and} \quad f(2) = 8,$$

so the absolute minimum value of f on the interval $[-1, 2]$ is 8. For $x < -1$, $f'(x) < 0$, so f is decreasing on the interval $(-\infty, -1)$. This means that $f(x) \geq f(-1) = 35$ for $x < -1$. Moreover, for $x > 2$, $f'(x) > 0$, so f is increasing on the interval $(2, \infty)$. This means that $f(x) > f(2) = 8$ for $x > 2$. Bringing all of this information together, it follows that f has an absolute minimum value of 8 at 2. Therefore, $f(x) \geq 8 > 0$ for all x .

124. Let $f(x) = ax^2 + bx + c$, where $a \neq 0$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. If $a < 0$, then $f''(x) = 2a < 0$ for all x , so f is concave down for all x ; if $a > 0$, then $f''(x) = 2a > 0$ for all x , so f is concave up for all x . Because the concavity of f never changes, the function f has no points of inflection.

125. Let $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$, so $f''(x) = 0$ when $x = -\frac{b}{3a}$. If $a < 0$, then

$$f''(x) > 0 \text{ when } x < -\frac{b}{3a} \quad \text{and} \quad f''(x) < 0 \text{ when } x > -\frac{b}{3a},$$

so the concavity of f changes and there is a point of inflection at $-\frac{b}{3a}$; if $a > 0$, then

$$f''(x) < 0 \text{ when } x < -\frac{b}{3a} \quad \text{and} \quad f''(x) > 0 \text{ when } x > -\frac{b}{3a},$$

so, again, the concavity of f changes and there is a point of inflection at $-\frac{b}{3a}$. Therefore, every polynomial of degree 3 has exactly one point of inflection.

126. Let f be a polynomial of degree $n \geq 3$. Because the polynomial function f is differentiable everywhere, critical numbers can only occur where $f'(x) = 0$. Now, $f'(x)$ is a polynomial of degree $n - 1 \geq 2$, so there are at most $n - 1$ critical numbers. Moreover, the only candidates for points of inflection are where $f''(x) = 0$; but, $f''(x)$ is a polynomial of degree $n - 2 \geq 1$, so there are at most $n - 2$ points of inflection.

127. Let $f(x) = (x - a)^n$, where a is a constant and $n \geq 3$ is an odd integer. Then,

$$f'(x) = n(x - a)^{n-1} \quad \text{and} \quad f''(x) = n(n - 1)(x - a)^{n-2},$$

so $f''(x)$ exists everywhere and is equal to zero only for $x = a$. Because n is an odd integer, $n - 2$ is also an odd integer, so $f''(x) < 0$ for $x < a$ and $f''(x) > 0$ for $x > a$. It follows that the concavity of f changes and f has a point of inflection at a . There are no other candidates for points of inflection, so the function f has exactly one point of inflection.

128. Let $f(x) = (x - a)^n$, where a is a constant and $n \geq 2$ is an even integer. Then,

$$f'(x) = n(x - a)^{n-1} \quad \text{and} \quad f''(x) = n(n - 1)(x - a)^{n-2},$$

so $f''(x)$ exists everywhere and is equal to zero only for $x = a$. Because n is an even integer, $n - 2$ is also an even integer, so $f''(x) > 0$ for all $x \neq a$. Therefore, the concavity of f never changes, and f has no points of inflection.

129. Consider the function $f(x) = \frac{ax + b}{ax + d}$. The rational function f is differentiable on its domain, so critical numbers occur only when $f'(x) = 0$ and points of inflection can only occur when $f''(x) = 0$. Now,

$$f'(x) = \frac{(ax + d)(a) - (ax + b)(a)}{(ax + d)^2} = \frac{a(d - b)}{(ax + d)^2},$$

and

$$f''(x) = -\frac{2a^2(d - b)}{(ax + d)^3}.$$

Provided $a \neq 0$ and $d \neq b$, $f'(x)$ and $f''(x)$ are never equal to zero, so the function f has no critical numbers and no points of inflection. If $a = 0$ or $d = b$, then f is a constant function so that all real numbers x are critical numbers but f still has no points of inflection.

130. Let f be a continuous function on some interval I . Suppose c is a critical number of f and (a, b) is some open interval in I containing c with $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$. Then the function f is decreasing on the interval (a, c) and increasing on the interval (c, b) ; in other words, for all x in (a, b) , $f(x) \geq f(c)$. Therefore, $f(c)$ is a local minimum value.
131. Let f be a continuous function on some interval I . Suppose c is a critical number of f and (a, b) is some open interval in I containing c with $f'(x)$ having the same sign on both sides of c . If $f'(x) < 0$ on the intervals (a, c) and (c, b) , then f is decreasing on both intervals. This implies that $f(x) \geq f(c)$ on the interval (a, c) but $f(x) \leq f(c)$ on the interval (c, b) . On the other hand, if $f'(x) > 0$ on the intervals (a, c) and (c, b) , then f is increasing on both intervals. This implies that $f(x) \leq f(c)$ on the interval (a, c) but $f(x) \geq f(c)$ on the interval (c, b) . In either case, $f(c)$ is neither a local maximum value nor a local minimum value.
132. Let f be a function that is continuous on the closed interval $[a, b]$. Moreover, suppose that f' and f'' exist on the open interval (a, b) with $f''(x) < 0$ on (a, b) . Let c be any fixed number in (a, b) . An equation of the tangent line to f at the point $(c, f(c))$ is

$$y = f(c) + f'(c)(x - c).$$

To show that f is concave down on (a, b) , it must be established that the graph of f lies below each of its tangent lines for all x in (a, b) ; that is, that

$$f(x) \leq f(c) + f'(c)(x - c)$$

for all x in (a, b) . If $x = c$, then $f(x) = f(c)$ and there is nothing more to prove. If $x \neq c$, then by applying the Mean Value Theorem to the function f , there is a number x_1 between c and x for which

$$f'(x_1) = \frac{f(x) - f(c)}{x - c}.$$

Solve this equation for $f(x)$ to obtain $f(x) = f(c) + f'(x_1)(x - c)$. There are two cases to consider:

- Case I: $c < x_1 < x$: Because $f''(x) < 0$ on the interval (a, b) , it follows that f' is decreasing on (a, b) . For $x_1 > c$, this means that $f'(x_1) < f'(c)$. Multiplying by $x - c$, which is positive, and adding $f(c)$ then yields

$$f(x) = f(c) + f'(x_1)(x - c) < f(c) + f'(c)(x - c);$$

that is, the graph of f lies below each of its tangent lines to the right of c in (a, b) .

- Case II: $x < x_1 < c$: Because $f''(x) < 0$ on the interval (a, b) , it follows that f' is decreasing on (a, b) . For $x_1 < c$, this means that $f'(x_1) > f'(c)$. Multiplying by $x - c$, which is negative, and adding $f(c)$ then yields

$$f(x) = f(c) + f'(x_1)(x - c) < f(c) + f'(c)(x - c);$$

that is, the graph of f lies below each of its tangent lines to the left of c in (a, b) .

Therefore, f is concave down.

133. Let f be a function for which f' and f'' exist on an open interval (a, b) , and let $c \in (a, b)$ be a critical number of f . Suppose $f''(c) > 0$. Then

$$f''(c) = \lim_{x \rightarrow c} \frac{f'(x) - f'(c)}{x - c} > 0,$$

so there is an open interval about c for which

$$\frac{f'(x) - f'(c)}{x - c} > 0$$

everywhere in the interval, except possibly at c itself. Because f' exists at c and c is a critical number of f , $f'(c) = 0$, so

$$\frac{f'(x)}{x - c} > 0.$$

For $x < c$ on this interval, $f'(x) < 0$, while for $x > c$ on this interval, $f'(x) > 0$. By the First Derivative Test, it follows that $f(c)$ is a local minimum value.

Challenge Problems

134. Let $f(x) = (x + 1)\tan^{-1} x$. Then

$$f'(x) = (x + 1) \cdot \frac{1}{1 + x^2} + \tan^{-1} x = \frac{x + 1}{1 + x^2} + \tan^{-1} x,$$

and

$$f''(x) = \frac{(1 + x^2) \cdot 1 - (x + 1)(2x)}{(1 + x^2)^2} + \frac{1}{1 + x^2} = \frac{1 + x^2 - 2x^2 - 2x}{(1 + x^2)^2} + \frac{1 + x^2}{(1 + x^2)^2} = \frac{2 - 2x}{(1 + x^2)^2},$$

so $f''(x)$ exists everywhere and is equal to zero when $x = 1$. For $x < 1$, $f''(x) > 0$, so f is concave up on the interval $(-\infty, 1)$; for $x > 1$, $f''(x) < 0$, so f is concave down on the interval $(1, \infty)$. Because

the concavity of f changes at 1, the point $\boxed{(1, f(1)) = \left(1, \frac{\pi}{2}\right)}$ is a point of inflection of f .

135. Let $f(x) = (1+x)^n - (1+nx)$, where $n > 1$ and $x \neq 0$. Then

$$f'(x) = n(1+x)^{n-1} - n = n((1+x)^{n-1} - 1).$$

If $-1 < x < 0$, then $0 < 1+x < 1$ and $(1+x)^{n-1} < 1$ so $f'(x) < 0$. This means that f is decreasing on the interval $(-1, 0)$, so that $f(x) > f(0)$ for $-1 < x < 0$. On the other hand, if $x > 0$, then $1+x > 1$ and $(1+x)^{n-1} > 1$, so $f'(x) > 0$. This means that f is increasing on the interval $(0, \infty)$, so that $f(x) > f(0)$ for $x > 0$. Therefore, for $x > -1$, $x \neq 0$, it follows that $f(x) > f(0)$; equivalently,

$$(1+x)^n - (1+nx) > 0 \quad \text{or} \quad (1+x)^n > 1+nx.$$

4.5: Indeterminate Forms and L'Hôpital's Rule

Concepts and Vocabulary

1. False. $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$ if $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$.
2. False. If $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$, then L'Hôpital's Rule states that

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}.$$

3. False. $\frac{1}{x}$ is not an indeterminate form at 0 because, in the numerator, $\lim_{x \rightarrow 0} 1 = 1 \neq 0$.
4. False. $x \ln x$ is an indeterminate form at 0^+ of the type $0 \cdot \infty$ because $\lim_{x \rightarrow 0^+} x = 0$ and $\lim_{x \rightarrow 0^+} \ln x = -\infty$.
5. Answers will vary, but here is one possible response. Suppose

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = \infty.$$

The following argument would also hold if both limits were $-\infty$. Consider the expression $f(x) - g(x)$ as x approaches c . If the value of f becomes unbounded faster than the value of g , the value of $f(x) - g(x)$ should approach ∞ , whereas if the value of g becomes unbounded faster than the value of f , the value of $f(x) - g(x)$ should approach $-\infty$. Moreover, if f and g become unbounded at similar rates, there could be cancellation leading the value of $f(x) - g(x)$ to approach any real number. In other words, in an expression of the form $\infty - \infty$, there is competition between the two terms in the expression, and an analysis of these competing forces is needed to determine the value of the limit. This is why $\infty - \infty$ is an indeterminate form.

Next, consider the expression $f(x) + g(x)$. In this expression, there is no competition between the two terms. The values of f and g are working together to make the value of $f(x) + g(x)$ approach ∞ . This is why $\infty + \infty$ is not an indeterminate form.

6. Answers will vary, but here is one possible response. Remember that the notation $0 \cdot \infty$ refers to a limit for which the value of one factor is *approaching* 0 and the value of the other factor is *approaching* ∞ and not to multiplication of the number 0 and a quantity that is becoming unbounded. If the value of the first factor approaches 0 faster than the value of the second factor becomes unbounded, the value of the product will approach 0; however, if the value of the second factor becomes unbounded faster than the value of the first factor approaches 0, the value of the product will become unbounded. Additionally, if the values of the two factors change at similar rates, the value of the product could approach any real number. Therefore, $0 \cdot \infty$ is an indeterminate form.

Skill Building

7. (a) Because

$$\lim_{x \rightarrow 0} (1 - e^x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $\frac{1 - e^x}{x}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\frac{1 - e^x}{x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

8. (a) Because

$$\lim_{x \rightarrow 0} (1 - e^x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x - 1) = -1 \neq 0,$$

the expression $\frac{1 - e^x}{x}$ is not an indeterminate form at 0.

- (b) The value of a limit of the form $\frac{0}{-1}$ approaches 0, so it is not an indeterminate form at 0.

9. (a) Because

$$\lim_{x \rightarrow 0} e^x = 1 \neq 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $\frac{e^x}{x}$ is not an indeterminate form at 0.

- (b) The value of a limit of the form $\frac{1}{0}$ approaches $\pm\infty$, so it is not an indeterminate form at 0.

10. (a) Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\frac{e^x}{x}$ is an indeterminate form at ∞ .

- (b) Based on the limits in part (a), the expression $\frac{e^x}{x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$.

11. (a) Because

$$\lim_{x \rightarrow \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x^2 = \infty,$$

the expression $\frac{\ln x}{x^2}$ is an indeterminate form at ∞ .

- (b) Based on the limits in part (a), the expression $\frac{\ln x}{x^2}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$.

12. (a) Because

$$\lim_{x \rightarrow 0} \ln(x + 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (e^x - 1) = 0,$$

the expression $\frac{\ln(x + 1)}{e^x - 1}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\frac{\ln(x + 1)}{e^x - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

13. (a) Because

$$\lim_{x \rightarrow 0} \sec x = 1 \neq 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $\frac{\sec x}{x}$ is not an indeterminate form at 0.

- (b) The value of a limit of the form $\frac{1}{0}$ approaches $\pm\infty$, so it is not an indeterminate form at 0.

14. (a) Because

$$\lim_{x \rightarrow 0} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (\sec x - 1) = 0,$$

the expression $\frac{x}{\sec x - 1}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\frac{x}{\sec x - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

15. (a) Because

$$\lim_{x \rightarrow 0^+} [\sin x(1 - \cos x)] = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 = 0,$$

the expression $\frac{\sin x(\cos x - 1)}{x^2}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\frac{\sin x(\cos x - 1)}{x^2}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

16. (a) Because

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow \frac{\pi}{2}} \cos x = 0,$$

the expression $\frac{(\sin x - 1)}{\cos x}$ is an indeterminate form at $\frac{\pi}{2}$.

- (b) Based on the limits in part (a), the expression $\frac{(\sin x - 1)}{\cos x}$ is an indeterminate form at $\frac{\pi}{2}$ of the type $\frac{0}{0}$.

17. (a) Because

$$\lim_{x \rightarrow \frac{\pi}{4}} (\tan x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow \frac{\pi}{4}} \sin(4x - \pi) = 0,$$

the expression $\frac{(\tan x - 1)}{\sin(4x - \pi)}$ is an indeterminate form at $\frac{\pi}{4}$.

- (b) Based on the limits in part (a), the expression $\frac{(\tan x - 1)}{\sin(4x - \pi)}$ is an indeterminate form at $\frac{\pi}{4}$ of the type $\frac{0}{0}$.

18. (a) Because

$$\lim_{x \rightarrow 0} (e^x - e^{-x}) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 - \cos x) = 0,$$

the expression $\frac{e^x - e^{-x}}{1 - \cos x}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\frac{e^x - e^{-x}}{1 - \cos x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$.

19. (a) Because

$$\lim_{x \rightarrow \infty} x^2 = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^{-x} = 0,$$

the expression $x^2 e^{-x}$ is an indeterminate form at ∞ .

- (b) Based on the limits in part (a), the expression $x^2 e^{-x}$ is an indeterminate form at ∞ of the type $0 \cdot \infty$.

20. (a) Because

$$\lim_{x \rightarrow 0^-} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \cot x = -\infty,$$

the expression $x \cot x$ is an indeterminate form at 0^- . Additionally,

$$\lim_{x \rightarrow 0^+} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \cot x = \infty,$$

so the expression $x \cot x$ is also an indeterminate form at 0^+ . Therefore, the expression $x \cot x$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $x \cot x$ is an indeterminate form at 0 of the type $0 \cdot \infty$.

21. (a) Because

$$\lim_{x \rightarrow 0^-} \csc \frac{x}{2} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \cot \frac{x}{2} = -\infty,$$

the expression $\csc \frac{x}{2} - \cot \frac{x}{2}$ is an indeterminate form at 0^- . Additionally,

$$\lim_{x \rightarrow 0^+} \csc \frac{x}{2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \cot \frac{x}{2} = \infty,$$

so the expression $\csc \frac{x}{2} - \cot \frac{x}{2}$ is also an indeterminate form at 0^+ . Therefore, the expression $\csc \frac{x}{2} - \cot \frac{x}{2}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\csc \frac{x}{2} - \cot \frac{x}{2}$ is an indeterminate form at 0 of the type $\infty - \infty$.

22. (a) Because

$$\lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{\ln x} = -\infty,$$

the expression $\frac{x}{x-1} + \frac{1}{\ln x}$ at 1^- is of the form $-\infty + (-\infty)$, which is not an indeterminate form. On the other hand,

$$\lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{\ln x} = \infty,$$

so the expression $\frac{x}{x-1} + \frac{1}{\ln x}$ at 1^+ is of the form $\infty + \infty$, which is again not an indeterminate form. Therefore, the expression $\frac{x}{x-1} + \frac{1}{\ln x}$ is not an indeterminate form at 1.

- (b) The value of a limit of the form $-\infty + (-\infty)$ becomes unbounded in the negative direction; the value of a limit of the form $\infty + \infty$ becomes unbounded in the positive direction. Therefore, neither is an indeterminate form.

23. (a) Because

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0} \sin x = 0,$$

the expression $\left(\frac{1}{x^2}\right)^{\sin x}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $\left(\frac{1}{x^2}\right)^{\sin x}$ is an indeterminate form at 0 of the type ∞^0 .

24. (a) Because

$$\lim_{x \rightarrow 0^-} (e^x + x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty,$$

the expression $(e^x + x)^{1/x}$ is an indeterminate form at 0^- . Additionally,

$$\lim_{x \rightarrow 0^+} (e^x + x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty,$$

so the expression $(e^x + x)^{1/x}$ is also an indeterminate form at 0^+ . Therefore, the expression $(e^x + x)^{1/x}$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $(e^x + x)^{1/x}$ is an indeterminate form at 0 of the type 1^∞ .

25. (a) Because

$$\lim_{x \rightarrow 0} (x^2 - 1) = -1 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $(x^2 - 1)^x$ is not an indeterminate form at 0.

- (b) The value of a limit of the form $(-1)^0$ approaches 1, so it is not an indeterminate form at 0.

26. (a) Because

$$\lim_{x \rightarrow 0} \sin x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $(\sin x)^x$ is an indeterminate form at 0.

- (b) Based on the limits in part (a), the expression $(\sin x)^x$ is an indeterminate form at 0 of the type 0^0 .

27. Because

$$\lim_{x \rightarrow 2} (x^2 + x - 6) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2} (x^2 - 3x + 2) = 0,$$

the expression $\frac{x^2 + x - 6}{x^2 - 3x + 2}$ is an indeterminate form at 2 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 3x + 2} = \lim_{x \rightarrow 2} \frac{\frac{d}{dx}(x^2 + x - 6)}{\frac{d}{dx}(x^2 - 3x + 2)} = \lim_{x \rightarrow 2} \frac{2x + 1}{2x - 3} = \frac{5}{1} = 5.$$

28. Because

$$\lim_{x \rightarrow 1} (2x^3 + 5x^2 - 4x - 3) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x^3 + x^2 - 10x + 8) = 0,$$

the expression $\frac{2x^3 + 5x^2 - 4x - 3}{x^3 + x^2 - 10x + 8}$ is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1} \frac{2x^3 + 5x^2 - 4x - 3}{x^3 + x^2 - 10x + 8} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(2x^3 + 5x^2 - 4x - 3)}{\frac{d}{dx}(x^3 + x^2 - 10x + 8)} = \lim_{x \rightarrow 1} \frac{6x^2 + 10x - 4}{3x^2 + 2x - 10} = \frac{12}{-5} = -\frac{12}{5}.$$

29. Because

$$\lim_{x \rightarrow 1} \ln x = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x^2 - 1) = 0,$$

the expression $\frac{\ln x}{x^2 - 1}$ is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(x^2 - 1)} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}.$$

30. Because

$$\lim_{x \rightarrow 0} \ln(1-x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (e^x - 1) = 0,$$

the expression $\frac{\ln(1-x)}{e^x - 1}$ is an indeterminate form at 0 of the type $\left[\frac{0}{0}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\ln(1-x)}{e^x - 1} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \ln(1-x)}{\frac{d}{dx} (e^x - 1)} = \lim_{x \rightarrow 0} \frac{\frac{-1}{1-x}}{e^x} = \frac{-1}{1} = \boxed{-1}.$$

31. Because

$$\lim_{x \rightarrow 0} (e^x - e^{-x}) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin x = 0,$$

the expression $\frac{e^x - e^{-x}}{\sin x}$ is an indeterminate form at 0 of the type $\left[\frac{0}{0}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (e^x - e^{-x})}{\frac{d}{dx} \sin x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{\cos x} = \frac{2}{1} = \boxed{2}.$$

32. Because

$$\lim_{x \rightarrow 0} \tan(2x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \ln(1+x) = 0,$$

the expression $\frac{\tan(2x)}{\ln(1+x)}$ is an indeterminate form at 0 of the type $\left[\frac{0}{0}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan(2x)}{\ln(1+x)} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tan(2x)}{\frac{d}{dx} \ln(1+x)} = \lim_{x \rightarrow 0} \frac{2 \sec^2(2x)}{\frac{1}{1+x}} = \frac{2}{1} = \boxed{2}.$$

33. Because

$$\lim_{x \rightarrow 1} \sin(\pi x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x-1) = 0,$$

the expression $\frac{\sin(\pi x)}{x-1}$ is an indeterminate form at 1 of the type $\left[\frac{0}{0}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1} \frac{\sin(\pi x)}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \sin(\pi x)}{\frac{d}{dx} (x-1)} = \lim_{x \rightarrow 1} \frac{\pi \cos(\pi x)}{1} = \boxed{-\pi}.$$

34. Because

$$\lim_{x \rightarrow \pi} (1 + \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi} \sin(2x) = 0,$$

the expression $\frac{1 + \cos x}{\sin(2x)}$ is an indeterminate form at π of the type $\left[\frac{0}{0}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \pi} \frac{1 + \cos x}{\sin(2x)} = \lim_{x \rightarrow \pi} \frac{\frac{d}{dx} (1 + \cos x)}{\frac{d}{dx} \sin(2x)} = \lim_{x \rightarrow \pi} \frac{-\sin x}{2 \cos(2x)} = \frac{0}{2} = \boxed{0}.$$

35. Because

$$\lim_{x \rightarrow \infty} x^2 = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

the expression $\frac{x^2}{e^x}$ is an indeterminate form at ∞ of the type $\left[\frac{\infty}{\infty}\right]$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x}.$$

Because

$$\lim_{x \rightarrow \infty} (2x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

the expression $\frac{2x}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = \boxed{0}.$$

36. Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x^4 = \infty,$$

the expression $\frac{e^x}{x^4}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}x^4} = \lim_{x \rightarrow \infty} \frac{e^x}{4x^3}.$$

Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (4x^3) = \infty,$$

the expression $\frac{e^x}{4x^3}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \lim_{x \rightarrow \infty} \frac{e^x}{4x^3} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}(4x^3)} = \lim_{x \rightarrow \infty} \frac{e^x}{12x^2}.$$

Now,

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (12x^2) = \infty,$$

so the expression $\frac{e^x}{12x^2}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule for a third time,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \lim_{x \rightarrow \infty} \frac{e^x}{12x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}(12x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{24x}.$$

Finally, because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (24x) = \infty,$$

the expression $\frac{e^x}{24x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule for a fourth time,

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \lim_{x \rightarrow \infty} \frac{e^x}{24x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}e^x}{\frac{d}{dx}(24x)} = \lim_{x \rightarrow \infty} \frac{e^x}{24} = \boxed{\infty}.$$

37. Because

$$\lim_{x \rightarrow \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

the expression $\frac{\ln x}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{e^x} = \boxed{0}.$$

38. Because

$$\lim_{x \rightarrow \infty} (x + \ln x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (x \ln x) = \infty,$$

the expression $\frac{x + \ln x}{x \ln x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{x + \ln x}{x \ln x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x + \ln x)}{\frac{d}{dx}(x \ln x)} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{x \cdot \frac{1}{x} + \ln x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{1 + \ln x} = \boxed{0}.$$

39. Because

$$\lim_{x \rightarrow 0} (e^x - 1 - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 - \cos x) = 0,$$

the expression $\frac{e^x - 1 - \sin x}{1 - \cos x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - \sin x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - 1 - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{e^x - \cos x}{\sin x}.$$

Because

$$\lim_{x \rightarrow 0} (e^x - \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin x = 0,$$

the expression $\frac{e^x - \cos x}{\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - \sin x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{e^x - \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - \cos x)}{\frac{d}{dx} \sin x} = \lim_{x \rightarrow 0} \frac{e^x + \sin x}{\cos x} = \frac{1}{1} = \boxed{1}.$$

40. Because

$$\lim_{x \rightarrow 0} (e^x - e^{-x} - 2 \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (3x^3) = 0,$$

the expression $\frac{e^x - e^{-x} - 2 \sin x}{3x^3}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \sin x}{3x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - e^{-x} - 2 \sin x)}{\frac{d}{dx}(3x^3)} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \cos x}{9x^2}.$$

Because

$$\lim_{x \rightarrow 0} (e^x + e^{-x} - 2 \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (9x^2) = 0,$$

the expression $\frac{e^x + e^{-x} - 2 \cos x}{9x^2}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \sin x}{3x^3} &= \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \cos x}{9x^2} \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x + e^{-x} - 2 \cos x)}{\frac{d}{dx}(9x^2)} = \lim_{x \rightarrow 0} \frac{e^x - e^{-x} + 2 \sin x}{18x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 0} (e^x - e^{-x} + 2 \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (18x) = 0,$$

so the expression $\frac{e^x - e^{-x} + 2 \sin x}{18x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's rule for a third time,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \sin x}{3x^3} &= \lim_{x \rightarrow 0} \frac{e^x - e^{-x} + 2 \sin x}{18x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - e^{-x} + 2 \sin x)}{\frac{d}{dx}(18x)} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x}{18} = \frac{4}{18} = \boxed{\frac{2}{9}}. \end{aligned}$$

41. Because

$$\lim_{x \rightarrow 0} (\sin x - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^3 = 0,$$

the expression $\frac{\sin x - x}{x^3}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sin x - x)}{\frac{d}{dx}x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2}.$$

Because

$$\lim_{x \rightarrow 0} (\cos x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (3x^2) = 0,$$

the expression $\frac{\cos x - 1}{3x^2}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(3x^2)} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x}.$$

Now,

$$\lim_{x \rightarrow 0} (-\sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (6x) = 0,$$

so the expression $\frac{-\sin x}{6x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's rule for a third time,

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(6x)} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}.$$

42. Because

$$\lim_{x \rightarrow 0} x^3 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (\cos x - 1) = 0,$$

the expression $\frac{x^3}{\cos x - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{x^3}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}x^3}{\frac{d}{dx}(\cos x - 1)} = \lim_{x \rightarrow 0} \frac{3x^2}{-\sin x}.$$

Because

$$\lim_{x \rightarrow 0} (3x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-\sin x) = 0,$$

the expression $\frac{3x^2}{-\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{x^3}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{3x^2}{-\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(3x^2)}{\frac{d}{dx}(-\sin x)} = \lim_{x \rightarrow 0} \frac{6x}{-\cos x} = \frac{0}{-1} = \boxed{0}.$$

43. Because

$$\lim_{x \rightarrow 0^+} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \ln x = -\infty,$$

the expression $x^2 \ln x$ is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$x^2 \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{x^2}} = \frac{\ln x}{x^{-2}},$$

which is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} (x^2 \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x \rightarrow 0^+} \frac{x^2}{-2} = \boxed{0}.$$

44. Because

$$\lim_{x \rightarrow \infty} x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^{-x} = 0,$$

the expression xe^{-x} is an indeterminate form at ∞ of the type $\boxed{0 \cdot \infty}$. Rewrite

$$xe^{-x} \quad \text{as} \quad \frac{x}{e^x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} (xe^{-x}) = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \boxed{0}.$$

45. Because

$$\lim_{x \rightarrow \infty} x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (e^{1/x} - 1) = 0,$$

the expression $x(e^{1/x} - 1)$ is an indeterminate form at ∞ of the type $\boxed{0 \cdot \infty}$. Rewrite

$$x(e^{1/x} - 1) \quad \text{as} \quad \frac{e^{1/x} - 1}{1/x},$$

which is an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} [x(e^{1/x} - 1)] = \lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^{1/x} - 1)}{\frac{d}{dx}(1/x)} = \lim_{x \rightarrow \infty} \frac{-x^{-2}e^{1/x}}{-x^{-2}} = \lim_{x \rightarrow \infty} e^{1/x} = e^0 = \boxed{1}.$$

46. Because

$$\lim_{x \rightarrow \pi/2^-} (1 - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \tan x = \infty,$$

while

$$\lim_{x \rightarrow \pi/2^+} (1 - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/2^+} \tan x = -\infty,$$

the expression $(1 - \sin x) \tan x$ is an indeterminate form at $\pi/2$ of the type $\boxed{0 \cdot \infty}$. Rewrite

$$(1 - \sin x) \tan x \quad \text{as} \quad \frac{1 - \sin x}{\cot x},$$

which is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \pi/2} [(1 - \sin x) \tan x] = \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cot x} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx}(1 - \sin x)}{\frac{d}{dx} \cot x} = \lim_{x \rightarrow \pi/2} \frac{-\cos x}{-\csc^2 x} = \frac{0}{-1} = \boxed{0}.$$

47. Because

$$\lim_{x \rightarrow \pi/2^-} \sec x = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \tan x = \infty,$$

while

$$\lim_{x \rightarrow \pi/2^+} \sec x = -\infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^+} \tan x = -\infty,$$

the expression $\sec x - \tan x$ is an indeterminate form at $\pi/2$ of the type $\boxed{\infty - \infty}$. Rewrite

$$\sec x - \tan x \quad \text{as} \quad \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x},$$

which is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \pi/2} (\sec x - \tan x) = \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx}(1 - \sin x)}{\frac{d}{dx} \cos x} = \lim_{x \rightarrow \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = \boxed{0}.$$

48. Because

$$\lim_{x \rightarrow 0^-} \cot x = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty,$$

while

$$\lim_{x \rightarrow 0^+} \cot x = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty,$$

the expression $\cot x - \frac{1}{x}$ is an indeterminate form at 0 of the type $\boxed{\infty - \infty}$. Rewrite

$$\cot x - \frac{1}{x} \quad \text{as} \quad \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x \cos x - \sin x}{x \sin x},$$

which is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x \cos x - \sin x)}{\frac{d}{dx}(x \sin x)} \\ &= \lim_{x \rightarrow 0} \frac{-x \sin x + \cos x - \cos x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{x \cos x + \sin x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 0} (-x \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x \cos x + \sin x) = 0,$$

so the expression $\frac{-x \sin x}{x \cos x + \sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{-x \sin x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(-x \sin x)}{\frac{d}{dx}(x \cos x + \sin x)} \\ &= \lim_{x \rightarrow 0} \frac{-x \cos x - \sin x}{-x \sin x + \cos x + \cos x} = \frac{0}{2} = \boxed{0}. \end{aligned}$$

49. Because

$$\lim_{x \rightarrow 1^-} \frac{1}{\ln x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{x}{\ln x} = -\infty,$$

while

$$\lim_{x \rightarrow 1^+} \frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{x}{\ln x} = \infty,$$

the expression $\frac{1}{\ln x} - \frac{x}{\ln x}$ is an indeterminate form at 1 of the type $\boxed{\infty - \infty}$. Rewrite

$$\frac{1}{\ln x} - \frac{x}{\ln x} \quad \text{as} \quad \frac{1-x}{\ln x},$$

which is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{x}{\ln x} \right) = \lim_{x \rightarrow 1} \frac{1-x}{\ln x} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(1-x)}{\frac{d}{dx} \ln x} = \lim_{x \rightarrow 1} \frac{-1}{\frac{1}{x}} = \frac{-1}{1} = \boxed{-1}.$$

50. Because

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{e^x - 1} = -\infty,$$

while

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{e^x - 1} = \infty,$$

the expression $\frac{1}{x} - \frac{1}{e^x - 1}$ is an indeterminate form at 0 of the type $\boxed{\infty - \infty}$. Rewrite

$$\frac{1}{x} - \frac{1}{e^x - 1} \quad \text{as} \quad \frac{e^x - 1 - x}{x(e^x - 1)},$$

which is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}[x(e^x - 1)]} = \lim_{x \rightarrow 0} \frac{e^x - 1}{xe^x + e^x - 1}.$$

Now,

$$\lim_{x \rightarrow 0} (e^x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (xe^x + e^x - 1) = 0,$$

so the expression $\frac{e^x - 1}{xe^x + e^x - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \rightarrow 0} \frac{e^x - 1}{xe^x + e^x - 1} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(xe^x + e^x - 1)} = \lim_{x \rightarrow 0} \frac{e^x}{xe^x + e^x + e^x} = \boxed{\frac{1}{2}}.$$

51. Because

$$\lim_{x \rightarrow 0^+} (2x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} (3x) = 0,$$

the expression $(2x)^{3x}$ is an indeterminate form at 0^+ of the type $\boxed{0^0}$. Let $y = (2x)^{3x}$. Then

$$\ln y = \ln(2x)^{3x} = (3x) \ln(2x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$(3x) \ln(2x) \quad \text{as} \quad \frac{\ln(2x)}{\frac{1}{3x}},$$

which is now an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} [(3x) \ln(2x)] = \lim_{x \rightarrow 0^+} \frac{\ln(2x)}{\frac{1}{3x}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(2x)}{\frac{d}{dx} \frac{1}{3x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{2x} \cdot 2}{-\frac{1}{(3x)^2} \cdot 3} = \lim_{x \rightarrow 0^+} (-3x) = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (2x)^{3x} = e^0 = \boxed{1}.$$

52. Because

$$\lim_{x \rightarrow 0^+} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x^2 = 0,$$

the expression x^{x^2} is an indeterminate form at 0 of the type $\boxed{0^0}$. Let $y = x^{x^2}$. Then

$$\ln y = \ln(x^{x^2}) = x^2 \ln x,$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$x^2 \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{x^2}} = \frac{\ln x}{x^{-2}},$$

which is now an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} (x^2 \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{-2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \lim_{x \rightarrow 0^+} \frac{x^2}{-2} = 0.$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{x^2} = e^0 = \boxed{1}.$$

53. Because

$$\lim_{x \rightarrow \infty} (x+1) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^{-x} = 0,$$

the expression $(x+1)^{e^{-x}}$ is an indeterminate form at ∞ of the type $\boxed{\infty^0}$. Let $y = (x+1)^{e^{-x}}$. Then

$$\ln y = \ln(x+1)^{e^{-x}} = e^{-x} \ln(x+1),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$e^{-x} \ln(x+1) \quad \text{as} \quad \frac{\ln(1+x)}{e^x},$$

which is now an indeterminate form ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} [e^{-x} \ln(1+x)] = \lim_{x \rightarrow \infty} \frac{\ln(1+x)}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln(1+x)}{\frac{d}{dx} e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x}}{e^x} = 0.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} (x+1)^{e^{-x}} = e^0 = \boxed{1}.$$

54. Because

$$\lim_{x \rightarrow \infty} (1+x^2) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

the expression $(1+x^2)^{1/x}$ is an indeterminate form at ∞ of the type $\boxed{\infty^0}$. Let $y = (1+x^2)^{1/x}$. Then

$$\ln y = \ln(1+x^2)^{1/x} = \frac{1}{x} \ln(1+x^2) = \frac{\ln(1+x^2)}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(1+x^2)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln(1+x^2)}{\frac{d}{dx} x} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{1+x^2}}{1} = 0.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} (1+x^2)^{1/x} = e^0 = \boxed{1}.$$

55. Because

$$\lim_{x \rightarrow 0^+} \csc x = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \sin x = 0,$$

the expression $(\csc x)^{\sin x}$ is an indeterminate form at 0^+ of the type $\boxed{\infty^0}$. Let $y = (\csc x)^{\sin x}$. Then

$$\ln y = \ln(\csc x)^{\sin x} = \sin x \ln(\csc x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\sin x \ln(\csc x) \quad \text{as} \quad \frac{\ln(\csc x)}{\frac{1}{\sin x}} = \frac{\ln(\csc x)}{\csc x},$$

which is now an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} [\sin x \ln(\csc x)] = \lim_{x \rightarrow 0^+} \frac{\ln(\csc x)}{\csc x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(\csc x)}{\frac{d}{dx} \csc x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x \cdot -\csc x \cot x}{-\csc x \cot x} = \lim_{x \rightarrow 0^+} \sin x = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (\csc x)^{\sin x} = e^0 = \boxed{1}.$$

56. Because

$$\lim_{x \rightarrow \infty} x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

the expression $x^{1/x}$ is an indeterminate form at ∞ of the type $\boxed{\infty^0}$. Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} x^{1/x} = e^0 = \boxed{1}.$$

57. Because

$$\lim_{x \rightarrow \pi/2^-} \sin x = 1 \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \tan x = \infty,$$

the expression $(\sin x)^{\tan x}$ is an indeterminate form at $\pi/2^-$ of the type $\boxed{1^\infty}$. Let $y = (\sin x)^{\tan x}$. Then

$$\ln y = \ln(\sin x)^{\tan x} = \tan x \ln(\sin x),$$

which is an indeterminate form at $\pi/2^-$ of the type $0 \cdot \infty$. Rewrite

$$\tan x \ln(\sin x) \quad \text{as} \quad \frac{\ln(\sin x)}{\cot x},$$

which is now an indeterminate form at $\pi/2^-$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \pi/2^-} \ln y &= \lim_{x \rightarrow \pi/2^-} [\tan x \ln(\sin x)] = \lim_{x \rightarrow \pi/2^-} \frac{\ln(\sin x)}{\cot x} \\ &= \lim_{x \rightarrow \pi/2^-} \frac{\frac{d}{dx} \ln(\sin x)}{\frac{d}{dx} \cot x} = \lim_{x \rightarrow \pi/2^-} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x} = \lim_{x \rightarrow \pi/2^-} (-\sin x \cos x) = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \pi/2^-} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \pi/2^-} y = \lim_{x \rightarrow \pi/2^-} (\sin x)^{\tan x} = e^0 = \boxed{1}.$$

58. Because

$$\lim_{x \rightarrow 0^-} \cos x = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty,$$

while

$$\lim_{x \rightarrow 0^+} \cos x = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty,$$

the expression $(\cos x)^{1/x}$ is an indeterminate form at 0 of the type $\boxed{1^\infty}$. Let $y = (\cos x)^{1/x}$. Then

$$\ln y = \ln(\cos x)^{1/x} = \frac{1}{x} \ln(\cos x) = \frac{\ln(\cos x)}{x},$$

which is now an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \ln(\cos x)}{\frac{d}{dx} x} = \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} \cdot -\sin x}{1} = 0.$$

Finally, because $\lim_{x \rightarrow 0} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} (\cos x)^{1/x} = e^0 = \boxed{1}.$$

59. Note that

$$\frac{\cot x}{\cot(2x)} = \frac{\tan(2x)}{\tan x};$$

because

$$\lim_{x \rightarrow 0^+} \tan(2x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \tan x = 0,$$

the expression $\frac{\tan(2x)}{\tan x}$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \frac{\cot x}{\cot(2x)} = \lim_{x \rightarrow 0^+} \frac{\tan(2x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \tan(2x)}{\frac{d}{dx} \tan x} = \lim_{x \rightarrow 0^+} \frac{2 \sec^2(2x)}{\sec^2 x} = \frac{2}{1} = \boxed{2}.$$

60. Because

$$\lim_{x \rightarrow \infty} \ln(\ln x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \ln x = \infty,$$

the expression $\frac{\ln(\ln x)}{\ln x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln(\ln x)}{\frac{d}{dx} \ln x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{1}{\ln x} = \boxed{0}.$$

61. Because

$$\lim_{x \rightarrow 1/2^-} \ln(1 - 2x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1/2^-} \tan(\pi x) = \infty,$$

the expression $\frac{\ln(1 - 2x)}{\tan(\pi x)}$ is an indeterminate form at $1/2^-$ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1/2^-} \frac{\ln(1 - 2x)}{\tan(\pi x)} = \lim_{x \rightarrow 1/2^-} \frac{\frac{d}{dx} \ln(1 - 2x)}{\frac{d}{dx} \tan(\pi x)} = \lim_{x \rightarrow 1/2^-} \frac{\frac{1}{1-2x} \cdot -2}{\pi \sec^2(\pi x)} = \lim_{x \rightarrow 1/2^-} \frac{-2 \cos^2(\pi x)}{\pi(1 - 2x)}.$$

Now,

$$\lim_{x \rightarrow 1/2^-} [-2 \cos^2(\pi x)] = 0 \quad \text{and} \quad \lim_{x \rightarrow 1/2^-} [\pi(1 - 2x)] = 0,$$

so the expression $\frac{-2\cos^2(\pi x)}{\pi(1-2x)}$ is an indeterminate form at $1/2^-$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}\lim_{x \rightarrow 1/2^-} \frac{\ln(1-2x)}{\tan(\pi x)} &= \lim_{x \rightarrow 1/2^-} \frac{-2\cos^2(\pi x)}{\pi(1-2x)} = \lim_{x \rightarrow 1/2^-} \frac{\frac{d}{dx}[-2\cos^2(\pi x)]}{\frac{d}{dx}[\pi(1-2x)]} \\ &= \lim_{x \rightarrow 1/2^-} \frac{-4\pi \cos(\pi x) \cdot -\sin(\pi x)}{-2\pi} = \lim_{x \rightarrow 1/2^-} [-2\sin(\pi x) \cos(\pi x)] = \boxed{0}.\end{aligned}$$

62. Because

$$\lim_{x \rightarrow 1^-} \ln(1-x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \cot(\pi x) = -\infty,$$

the expression $\frac{\ln(1-x)}{\cot(\pi x)}$ is an indeterminate form at 1^- of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 1^-} \frac{\ln(1-x)}{\cot(\pi x)} = \lim_{x \rightarrow 1^-} \frac{\frac{d}{dx} \ln(1-x)}{\frac{d}{dx} \cot(\pi x)} = \lim_{x \rightarrow 1^-} \frac{\frac{1}{1-x} \cdot -1}{-\pi \csc^2(\pi x)} = \lim_{x \rightarrow 1^-} \frac{\sin^2(\pi x)}{\pi(1-x)}.$$

Now,

$$\lim_{x \rightarrow 1^-} \sin^2(\pi x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^-} [\pi(1-x)] = 0,$$

so the expression $\frac{\sin^2(\pi x)}{\pi(1-x)}$ is an indeterminate form at 1^- of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned}\lim_{x \rightarrow 1^-} \frac{\ln(1-x)}{\cot(\pi x)} &= \lim_{x \rightarrow 1^-} \frac{\sin^2(\pi x)}{\pi(1-x)} = \lim_{x \rightarrow 1^-} \frac{\frac{d}{dx} \sin^2(\pi x)}{\frac{d}{dx} [\pi(1-x)]} \\ &= \lim_{x \rightarrow 1^-} \frac{2\pi \sin(\pi x) \cos(\pi x)}{-\pi} = \lim_{x \rightarrow 1^-} [-2\sin(\pi x) \cos(\pi x)] = \boxed{0}.\end{aligned}$$

63. Because

$$\lim_{x \rightarrow \infty} (x^4 + x^3) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (e^x + 1) = \infty,$$

the expression $\frac{x^4 + x^3}{e^x + 1}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x^4 + x^3)}{\frac{d}{dx}(e^x + 1)} = \lim_{x \rightarrow \infty} \frac{4x^3 + 3x^2}{e^x}.$$

Because

$$\lim_{x \rightarrow \infty} (4x^3 + 3x^2) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

the expression $\frac{4x^3 + 3x^2}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{4x^3 + 3x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(4x^3 + 3x^2)}{\frac{d}{dx} e^x} = \lim_{x \rightarrow \infty} \frac{12x^2 + 6x}{e^x}.$$

Now,

$$\lim_{x \rightarrow \infty} (12x^2 + 6x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

so the expression $\frac{12x^2 + 6x}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule a third time,

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{12x^2 + 6x}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(12x^2 + 6x)}{\frac{d}{dx} e^x} = \lim_{x \rightarrow \infty} \frac{24x + 6}{e^x}.$$

Finally, because

$$\lim_{x \rightarrow \infty} (24x + 6) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

the expression $\frac{24x + 6}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule a fourth time,

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{e^x + 1} = \lim_{x \rightarrow \infty} \frac{24x + 6}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(24x + 6)}{\frac{d}{dx}e^x} = \lim_{x \rightarrow \infty} \frac{24}{e^x} = \boxed{0}.$$

64. Because

$$\lim_{x \rightarrow \infty} (x^2 + x - 1) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (e^x + e^{-x}) = \infty,$$

the expression $\frac{x^2 + x - 1}{e^x + e^{-x}}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{x^2 + x - 1}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x^2 + x - 1)}{\frac{d}{dx}(e^x + e^{-x})} = \lim_{x \rightarrow \infty} \frac{2x + 1}{e^x - e^{-x}}.$$

Now,

$$\lim_{x \rightarrow \infty} (2x + 1) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (e^x - e^{-x}) = \infty,$$

so the expression $\frac{2x + 1}{e^x - e^{-x}}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow \infty} \frac{x^2 + x - 1}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{2x + 1}{e^x - e^{-x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(2x + 1)}{\frac{d}{dx}(e^x - e^{-x})} = \lim_{x \rightarrow \infty} \frac{2}{e^x + e^{-x}} = \boxed{0}.$$

65. Because

$$\lim_{x \rightarrow 0} (xe^{4x} - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} [1 - \cos(2x)] = 0,$$

the expression $\frac{xe^{4x} - x}{1 - \cos(2x)}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{xe^{4x} - x}{1 - \cos(2x)} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(xe^{4x} - x)}{\frac{d}{dx}[1 - \cos(2x)]} = \lim_{x \rightarrow 0} \frac{4xe^{4x} + e^{4x} - 1}{2 \sin(2x)}.$$

Now,

$$\lim_{x \rightarrow 0} (4xe^{4x} + e^{4x} - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} [2 \sin(2x)] = 0,$$

so the expression $\frac{4xe^{4x} + e^{4x} - 1}{2 \sin(2x)}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{xe^{4x} - x}{1 - \cos(2x)} &= \lim_{x \rightarrow 0} \frac{4xe^{4x} + e^{4x} - 1}{2 \sin(2x)} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(4xe^{4x} + e^{4x} - 1)}{\frac{d}{dx}[2 \sin(2x)]} \\ &= \lim_{x \rightarrow 0} \frac{16xe^{4x} + 4e^{4x} + 4e^{4x}}{4 \cos(2x)} = \frac{8}{4} = \boxed{2}. \end{aligned}$$

66. Because

$$\lim_{x \rightarrow 0} (x \tan x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 - \cos x) = 0,$$

the expression $\frac{x \tan x}{1 - \cos x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x \tan x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{x \sec^2 x + \tan x}{\sin x}.$$

Now,

$$\lim_{x \rightarrow 0} (x \sec^2 x + \tan x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin x = 0,$$

so the expression $\frac{x \sec^2 x + \tan x}{\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sec^2 x + \tan x}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x \sec^2 x + \tan x)}{\frac{d}{dx} \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2x \sec^2 x \tan x + \sec^2 x + \sec^2 x}{\cos x} = \frac{2}{1} = \boxed{2}. \end{aligned}$$

67. Because

$$\lim_{x \rightarrow 0} \tan^{-1} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $\frac{\tan^{-1} x}{x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tan^{-1} x}{\frac{d}{dx} x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{1} = \boxed{1}.$$

68. Because

$$\lim_{x \rightarrow 0} \tan^{-1} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin^{-1} x = 0,$$

the expression $\frac{\tan^{-1} x}{\sin^{-1} x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{\sin^{-1} x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tan^{-1} x}{\frac{d}{dx} \sin^{-1} x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{1}{1} = \boxed{1}.$$

69. Because

$$\lim_{x \rightarrow 0} (\cos x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} [\cos(2x) - 1] = 0,$$

the expression $\frac{\cos x - 1}{\cos(2x) - 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{\cos(2x) - 1} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}[\cos(2x) - 1]} = \lim_{x \rightarrow 0} \frac{-\sin x}{-2 \sin(2x)} = \lim_{x \rightarrow 0} \frac{\sin x}{2 \sin(2x)}.$$

Now,

$$\lim_{x \rightarrow 0} \sin x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} [2 \sin(2x)] = 0,$$

so the expression $\frac{\sin x}{2 \sin(2x)}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{\cos(2x) - 1} = \lim_{x \rightarrow 0} \frac{\sin x}{2 \sin(2x)} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \sin x}{\frac{d}{dx} [2 \sin(2x)]} = \lim_{x \rightarrow 0} \frac{\cos x}{4 \cos(2x)} = \boxed{\frac{1}{4}}.$$

70. Because

$$\lim_{x \rightarrow 0} (\tan x - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^3 = 0,$$

the expression $\frac{\tan x - \sin x}{x^3}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\tan x - \sin x)}{\frac{d}{dx} x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - \cos x}{3x^2}.$$

Now,

$$\lim_{x \rightarrow 0} (\sec^2 x - \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (3x^2) = 0,$$

so the expression $\frac{\sec^2 x - \cos x}{3x^2}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - \cos x}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sec^2 x - \cos x)}{\frac{d}{dx}(3x^2)} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x + \sin x}{6x}.$$

Finally,

$$\lim_{x \rightarrow 0} (2 \sec^2 x \tan x + \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (6x) = 0,$$

so the expression $\frac{2 \sec^2 x \tan x + \sin x}{6x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule a third time,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x + \sin x}{6x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(2 \sec^2 x \tan x + \sin x)}{\frac{d}{dx}6x} = \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 4 \sec^2 x \tan^2 x + \cos x}{6} = \frac{3}{6} = \boxed{\frac{1}{2}}. \end{aligned}$$

71. Because

$$\lim_{x \rightarrow 0^+} x^{1/2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \ln x = -\infty,$$

the expression $x^{1/2} \ln x$ is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$x^{1/2} \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{x^{1/2}}} = \frac{\ln x}{x^{-1/2}},$$

which is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} (x^{1/2} \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-3/2}} = \lim_{x \rightarrow 0^+} (-2x^{1/2}) = \boxed{0}.$$

72. Because

$$\lim_{x \rightarrow \infty} (x - 1) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} e^{-x^2} = 0,$$

the expression $(x - 1)e^{-x^2}$ is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$(x - 1)e^{-x^2} \quad \text{as} \quad \frac{x - 1}{e^{x^2}},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} [(x - 1)e^{-x^2}] = \lim_{x \rightarrow \infty} \frac{x - 1}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x - 1)}{\frac{d}{dx} e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2xe^{x^2}} = \boxed{0}.$$

73. Because

$$\lim_{x \rightarrow \pi/2^-} \tan x = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \ln(\sin x) = 0,$$

while

$$\lim_{x \rightarrow \pi/2^+} \tan x = -\infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^+} \ln(\sin x) = 0,$$

the expression $\tan x \ln(\sin x)$ is an indeterminate form at $\pi/2$ of the type $0 \cdot \infty$. Rewrite

$$\tan x \ln(\sin x) \quad \text{as} \quad \frac{\ln(\sin x)}{\frac{1}{\tan x}} = \frac{\ln(\sin x)}{\cot x},$$

which is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \pi/2} [\tan x \ln(\sin x)] &= \lim_{x \rightarrow \pi/2} \frac{\ln(\sin x)}{\cot x} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx} \ln(\sin x)}{\frac{d}{dx} \cot x} \\ &= \lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x} = \lim_{x \rightarrow \pi/2} (-\sin x \cos x) = \boxed{0}. \end{aligned}$$

74. Because

$$\lim_{x \rightarrow 0^+} \sin x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \ln(\sin x) = -\infty,$$

the expression $\sin x \ln(\sin x)$ is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\sin x \ln(\sin x) \quad \text{as} \quad \frac{\ln(\sin x)}{\frac{1}{\sin x}} = \frac{\ln(\sin x)}{\csc x},$$

which is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} [\sin x \ln(\sin x)] = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\csc x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(\sin x)}{\frac{d}{dx} \csc x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc x \cot x} = \lim_{x \rightarrow 0^+} (-\sin x) = \boxed{0}.$$

75. Because

$$\lim_{x \rightarrow 0^-} \csc x = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \ln(x+1) = 0,$$

while

$$\lim_{x \rightarrow 0^+} \csc x = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \ln(x+1) = 0,$$

the expression $\csc x \ln(x+1)$ is an indeterminate form at 0 of the type $0 \cdot \infty$. Rewrite

$$\csc x \ln(x+1) \quad \text{as} \quad \frac{\ln(x+1)}{\sin x},$$

which is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} [\csc x \ln(x+1)] = \lim_{x \rightarrow 0} \frac{\ln(x+1)}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \ln(x+1)}{\frac{d}{dx} \sin x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{\cos x} = \frac{1}{1} = \boxed{1}.$$

76. Because

$$\lim_{x \rightarrow \pi/4^-} (1 - \tan x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/4^-} \sec(2x) = \infty,$$

while

$$\lim_{x \rightarrow \pi/4^+} (1 - \tan x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/4^+} \sec(2x) = -\infty,$$

the expression $(1 - \tan x) \sec(2x)$ is an indeterminate form at $\pi/4$ of the type $0 \cdot \infty$. Rewrite

$$(1 - \tan x) \sec(2x) \quad \text{as} \quad \frac{1 - \tan x}{\cos(2x)},$$

which is an indeterminate form at $\pi/4$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \pi/4} [(1 - \tan x) \sec(2x)] = \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\cos(2x)} = \lim_{x \rightarrow \pi/4} \frac{\frac{d}{dx} (1 - \tan x)}{\frac{d}{dx} \cos(2x)} = \lim_{x \rightarrow \pi/4} \frac{-\sec^2 x}{-2 \sin(2x)} = \frac{-2}{-2} = \boxed{1}.$$

77. Because

$$\lim_{x \rightarrow a^-} (a^2 - x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow a^-} \tan\left(\frac{\pi x}{2a}\right) = \infty,$$

while

$$\lim_{x \rightarrow a^+} (a^2 - x^2) = 0 \quad \text{and} \quad \lim_{x \rightarrow a^+} \tan\left(\frac{\pi x}{2a}\right) = -\infty,$$

the expression $(a^2 - x^2) \tan\left(\frac{\pi x}{2a}\right)$ is an indeterminate form at a of the type $0 \cdot \infty$. Rewrite

$$(a^2 - x^2) \tan\left(\frac{\pi x}{2a}\right) \quad \text{as} \quad \frac{a^2 - x^2}{\cot\left(\frac{\pi x}{2a}\right)},$$

which is an indeterminate form at a of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow a} \left[(a^2 - x^2) \tan\left(\frac{\pi x}{2a}\right) \right] = \lim_{x \rightarrow a} \frac{a^2 - x^2}{\cot\left(\frac{\pi x}{2a}\right)} = \lim_{x \rightarrow a} \frac{\frac{d}{dx}(a^2 - x^2)}{\frac{d}{dx} \cot\left(\frac{\pi x}{2a}\right)} = \lim_{x \rightarrow a} \frac{-2x}{-\frac{\pi}{2a} \csc^2\left(\frac{\pi x}{2a}\right)} = \frac{-2a}{-\frac{\pi}{2a}} = \boxed{\frac{4a^2}{\pi}}.$$

78. Because

$$\lim_{x \rightarrow 1^+} (1 - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^+} \tan\left(\frac{1}{2}\pi x\right) = -\infty,$$

the expression $(1 - x) \tan\left(\frac{1}{2}\pi x\right)$ is an indeterminate form at 1^+ of the type $0 \cdot \infty$. Rewrite

$$(1 - x) \tan\left(\frac{1}{2}\pi x\right) \quad \text{as} \quad \frac{1 - x}{\cot\left(\frac{1}{2}\pi x\right)},$$

which is an indeterminate form at 1^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left[(1 - x) \tan\left(\frac{1}{2}\pi x\right) \right] &= \lim_{x \rightarrow 1^+} \frac{1 - x}{\cot\left(\frac{1}{2}\pi x\right)} = \lim_{x \rightarrow 1^+} \frac{\frac{d}{dx}(1 - x)}{\frac{d}{dx} \cot\left(\frac{1}{2}\pi x\right)} \\ &= \lim_{x \rightarrow 1^+} \frac{-1}{-\frac{1}{2}\pi \csc^2\left(\frac{1}{2}\pi x\right)} = \frac{-1}{-\frac{1}{2}\pi} = \boxed{\frac{2}{\pi}}. \end{aligned}$$

79. Because

$$\lim_{x \rightarrow 1^-} \frac{1}{\ln x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x - 1} = -\infty,$$

while

$$\lim_{x \rightarrow 1^+} \frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty,$$

the expression $\frac{1}{\ln x} - \frac{1}{x - 1}$ is an indeterminate form at 1 of the type $\infty - \infty$. Rewrite

$$\frac{1}{\ln x} - \frac{1}{x - 1} \quad \text{as} \quad \frac{x - 1 - \ln x}{(x - 1) \ln x},$$

which is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right) &= \lim_{x \rightarrow 1} \frac{x - 1 - \ln x}{(x - 1) \ln x} \\ &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(x - 1 - \ln x)}{\frac{d}{dx}[(x - 1) \ln x]} = \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{(x - 1) \cdot \frac{1}{x} + \ln x} = \lim_{x \rightarrow 1} \frac{x - 1}{x - 1 + x \ln x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 1} (x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x - 1 + x \ln x) = 0,$$

so the expression $\frac{x-1}{x-1+x \ln x}$ is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} \frac{x-1}{x-1+x \ln x} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x-1+x \ln x)} = \lim_{x \rightarrow 1} \frac{1}{1+x \cdot \frac{1}{x} + \ln x} = \boxed{\frac{1}{2}}.$$

80. Because

$$\lim_{x \rightarrow 1^-} \frac{x}{x-1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{\ln x} = -\infty,$$

while

$$\lim_{x \rightarrow 1^+} \frac{x}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{\ln x} = \infty,$$

the expression $\frac{x}{x-1} - \frac{1}{\ln x}$ is an indeterminate form at 1 of the type $\infty - \infty$. Rewrite

$$\frac{x}{x-1} - \frac{1}{\ln x} \quad \text{as} \quad \frac{x \ln x - (x-1)}{(x-1) \ln x} = \frac{x \ln x - x + 1}{(x-1) \ln x},$$

which is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} \\ &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(x \ln x - x + 1)}{\frac{d}{dx}[(x-1) \ln x]} = \lim_{x \rightarrow 1} \frac{x \cdot \frac{1}{x} + \ln x - 1}{(x-1) \cdot \frac{1}{x} + \ln x} = \lim_{x \rightarrow 1} \frac{x \ln x}{x-1+x \ln x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 1} (x \ln x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x-1+x \ln x) = 0,$$

so the expression $\frac{x \ln x}{x-1+x \ln x}$ is an indeterminate form at 1 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1} \frac{x \ln x}{x-1+x \ln x} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(x \ln x)}{\frac{d}{dx}(x-1+x \ln x)} = \lim_{x \rightarrow 1} \frac{x \cdot \frac{1}{x} + \ln x}{1+x \cdot \frac{1}{x} + \ln x} = \boxed{\frac{1}{2}}.$$

81. Because

$$\lim_{x \rightarrow \pi/2^-} (x \tan x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \frac{\pi}{2} \sec x = \infty,$$

while

$$\lim_{x \rightarrow \pi/2^+} (x \tan x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^+} \frac{\pi}{2} \sec x = -\infty,$$

the expression $x \tan x - \frac{\pi}{2} \sec x$ is an indeterminate form at $\pi/2$ of the type $\infty - \infty$. Rewrite

$$x \tan x - \frac{\pi}{2} \sec x \quad \text{as} \quad x \frac{\sin x}{\cos x} - \frac{\pi}{2} \frac{1}{\cos x} = \frac{x \sin x - \frac{\pi}{2}}{\cos x},$$

which is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x \right) &= \lim_{x \rightarrow \pi/2} \frac{x \sin x - \frac{\pi}{2}}{\cos x} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx}(x \sin x - \frac{\pi}{2})}{\frac{d}{dx} \cos x} \\ &= \lim_{x \rightarrow \pi/2} \frac{x \cos x + \sin x}{-\sin x} = \frac{1}{-1} = \boxed{-1}. \end{aligned}$$

82. Because

$$\lim_{x \rightarrow \pi^-} \cot x = -\infty \quad \text{and} \quad \lim_{x \rightarrow \pi^-} (x \csc x) = \infty,$$

the expression $\cot x - x \csc x$ at π^- is of the form $-\infty - \infty$, which is not an indeterminate form; rather the value of this expression becomes unbounded in the negative direction. On the other hand,

$$\lim_{x \rightarrow \pi^+} \cot x = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi^+} (x \csc x) = -\infty,$$

so the expression $\cot x - x \csc x$ at π^+ is of the form $\infty + \infty$, which is again not an indeterminate form; rather the value of this expression becomes unbounded in the positive direction. Therefore,

$$\lim_{x \rightarrow \pi} (\cot x - x \csc x)$$

does not exist.

83. Because

$$\lim_{x \rightarrow 1^-} (1 - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^-} \tan(\pi x) = 0,$$

the expression $(1 - x)^{\tan(\pi x)}$ is an indeterminate form at 1^- of the type 0^0 . Let $y = (1 - x)^{\tan(\pi x)}$. Then

$$\ln y = \ln(1 - x)^{\tan(\pi x)} = \tan(\pi x) \ln(1 - x),$$

which is an indeterminate form at 1^- of the type $0 \cdot \infty$. Rewrite

$$\tan(\pi x) \ln(1 - x) \quad \text{as} \quad \frac{\ln(1 - x)}{\cot(\pi x)},$$

which is now an indeterminate form at 1^- of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \ln y &= \lim_{x \rightarrow 1^-} [\tan(\pi x) \ln(1 - x)] = \lim_{x \rightarrow 1^-} \frac{\ln(1 - x)}{\cot(\pi x)} \\ &= \lim_{x \rightarrow 1^-} \frac{\frac{d}{dx} \ln(1 - x)}{\frac{d}{dx} \cot(\pi x)} = \lim_{x \rightarrow 1^-} \frac{\frac{-1}{1-x}}{-\pi \csc^2(\pi x)} = \lim_{x \rightarrow 1^-} \frac{\sin^2(\pi x)}{\pi(1 - x)}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 1^-} \sin^2(\pi x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^-} [\pi(1 - x)] = 0,$$

so the expression $\frac{\sin^2(\pi x)}{\pi(1 - x)}$ is an indeterminate form at 1^- of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 1^-} \ln y = \lim_{x \rightarrow 1^-} \frac{\sin^2(\pi x)}{\pi(1 - x)} = \lim_{x \rightarrow 1^-} \frac{\frac{d}{dx} \sin^2(\pi x)}{\frac{d}{dx} [\pi(1 - x)]} = \lim_{x \rightarrow 1^-} \frac{2\pi \sin(\pi x) \cos(\pi x)}{-\pi} = 0.$$

Finally, because $\lim_{x \rightarrow 1^-} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 1^-} y = \lim_{x \rightarrow 1^-} (1 - x)^{\tan(\pi x)} = e^0 = \boxed{1}.$$

84. Because

$$\lim_{x \rightarrow 0^+} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \sqrt{x} = 0,$$

the expression $x^{\sqrt{x}}$ is an indeterminate form at 0^+ of the type 0^0 . Let $y = x^{\sqrt{x}}$. Then

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x,$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\sqrt{x} \ln x \quad \text{as} \quad \frac{\ln x}{\frac{1}{\sqrt{x}}} = \frac{\ln x}{x^{-1/2}},$$

which is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-3/2}} = \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{\sqrt{x}} = e^0 = \boxed{1}.$$

85. Because

$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty,$$

while

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty,$$

the expression $\left(\frac{\sin x}{x}\right)^{1/x}$ is an indeterminate form at 0 of the type 1^∞ . Let $y = \left(\frac{\sin x}{x}\right)^{1/x}$. Then

$$\ln y = \ln \left(\frac{\sin x}{x}\right)^{1/x} = \frac{1}{x} \ln \left(\frac{\sin x}{x}\right) = \frac{\ln \left(\frac{\sin x}{x}\right)}{x},$$

which is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln \left(\frac{\sin x}{x}\right)}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \ln \left(\frac{\sin x}{x}\right)}{\frac{d}{dx} x} = \lim_{x \rightarrow 0} \frac{\frac{x}{\sin x} \cdot \frac{x \cos x - \sin x}{x^2}}{1} = \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x \sin x}.$$

Now,

$$\lim_{x \rightarrow 0} (x \cos x - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x \sin x) = 0,$$

so the expression $\frac{x \cos x - \sin x}{x \sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (x \cos x - \sin x)}{\frac{d}{dx} (x \sin x)} \\ &= \lim_{x \rightarrow 0} \frac{-x \sin x + \cos x - \cos x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{x \cos x + \sin x}. \end{aligned}$$

Continuing,

$$\lim_{x \rightarrow 0} (-x \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x \cos x + \sin x) = 0,$$

so the expression $\frac{-x \sin x}{x \cos x + \sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule a third time,

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{-x \sin x}{x \cos x + \sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (-x \sin x)}{\frac{d}{dx} (x \cos x + \sin x)} \\ &= \lim_{x \rightarrow 0} \frac{-x \cos x - \sin x}{-x \sin x + \cos x + \cos x} = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow 0} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{1/x} = e^0 = \boxed{1}.$$

86. Because

$$\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x} + \frac{3}{x^2} \right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\left(1 + \frac{5}{x} + \frac{3}{x^2} \right)^x$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)^x$.

Then

$$\ln y = \ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)^x = x \ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x \ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right) \quad \text{as} \quad \frac{\ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)}{x^{-1}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)}{x^{-1}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)}{\frac{d}{dx} x^{-1}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{5}{x} + \frac{3}{x^2}} \cdot \left(-\frac{5}{x^2} - \frac{6}{x^3} \right)}{-x^{-2}} = \lim_{x \rightarrow \infty} \frac{5 + \frac{6}{x}}{1 + \frac{5}{x} + \frac{3}{x^2}} = \frac{5}{1} = 5. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 5$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)^x = \boxed{e^5}.$$

87. Because

$$\lim_{x \rightarrow \pi/2^-} \tan x = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \cos x = 0,$$

the expression $(\tan x)^{\cos x}$ is an indeterminate form at $\pi/2^-$ of the type ∞^0 . Let $y = (\tan x)^{\cos x}$. Then

$$\ln y = \ln(\tan x)^{\cos x} = \cos x \ln(\tan x),$$

which is an indeterminate form at $\pi/2^-$ of the type $0 \cdot \infty$. Rewrite

$$\cos x \ln(\tan x) \quad \text{as} \quad \frac{\ln(\tan x)}{\sec x},$$

which is now an indeterminate form at $\pi/2^-$ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \pi/2^-} \ln y &= \lim_{x \rightarrow \pi/2^-} [\cos x \ln(\tan x)] = \lim_{x \rightarrow \pi/2^-} \frac{\ln(\tan x)}{\sec x} \\ &= \lim_{x \rightarrow \pi/2^-} \frac{\frac{d}{dx} \ln(\tan x)}{\frac{d}{dx} \sec x} = \lim_{x \rightarrow \pi/2^-} \frac{\cot x \cdot \sec^2 x}{\sec x \tan x} = \lim_{x \rightarrow \pi/2^-} \frac{\cos x}{\sin^2 x} = 0. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \pi/2^-} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \pi/2^-} y = \lim_{x \rightarrow \pi/2^-} (\tan x)^{\cos x} = e^0 = \boxed{1}.$$

88. Because

$$\lim_{x \rightarrow 0^+} (x^2 + x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} (-\ln x) = \infty,$$

the expression $(x^2 + x)^{-\ln x}$ is of the form 0^∞ , which is not an indeterminate form; rather the value of this expression tends toward 0. Therefore,

$$\lim_{x \rightarrow 0^+} (x^2 + x)^{-\ln x} = \boxed{0}.$$

89. Because

$$\lim_{x \rightarrow 0} \cosh x = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} e^x = 1,$$

the expression $(\cosh x)^{e^x}$ is of the form 1^1 , which is not an indeterminate form; rather the value of this expression tends toward 1. Therefore,

$$\lim_{x \rightarrow 0} (\cosh x)^{e^x} = \boxed{1}.$$

90. Because

$$\lim_{x \rightarrow 0^+} \sinh x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x = 0,$$

the expression $(\sinh x)^x$ is an indeterminate form at 0^+ of the type 0^0 . Let $y = (\sinh x)^x$. Then

$$\ln y = \ln(\sinh x)^x = x \ln(\sinh x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$x \ln(\sinh x) \quad \text{as} \quad \frac{\ln(\sinh x)}{x^{-1}},$$

which is now an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} x \ln(\sinh x) = \lim_{x \rightarrow 0^+} \frac{\ln(\sinh x)}{x^{-1}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(\sinh x)}{\frac{d}{dx} x^{-1}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sinh x} \cdot \cosh x}{-x^{-2}} = \lim_{x \rightarrow 0^+} \frac{-x^2 \cosh x}{\sinh x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 0^+} (-x^2 \cosh x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \sinh x = 0,$$

so the expression $\frac{-x^2 \cosh x}{\sinh x}$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{-x^2 \cosh x}{\sinh x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(-x^2 \cosh x)}{\frac{d}{dx} \sinh x} = \lim_{x \rightarrow 0^+} \frac{-x^2 \sinh x - 2x \cosh x}{\cosh x} = 0.$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (\sinh x)^x = e^0 = \boxed{1}.$$

Applications and Extensions

91. Let

$$w(t) = \frac{K e^{rt}}{\frac{K}{40} + e^{rt} - 1},$$

where $K = 366$, $r = 0.283$, and $t = 0$ represents the year 2000.

(a) Because $r > 0$,

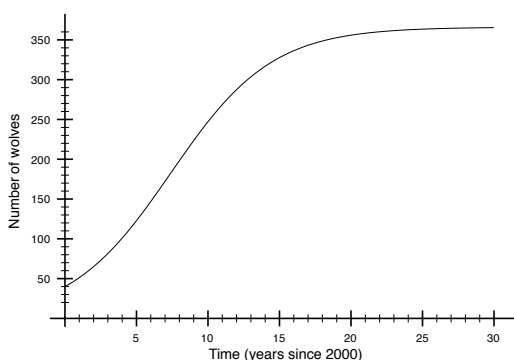
$$\lim_{t \rightarrow \infty} Ke^{rt} = \infty, \quad \text{and} \quad \lim_{t \rightarrow \infty} \left(\frac{K}{40} + e^{rt} - 1 \right) = \infty,$$

$w(t)$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{t \rightarrow \infty} w(t) = \lim_{t \rightarrow \infty} \frac{Ke^{rt}}{\frac{K}{40} + e^{rt} - 1} = \lim_{t \rightarrow \infty} \frac{\frac{d}{dt} Ke^{rt}}{\frac{d}{dt} \left(\frac{K}{40} + e^{rt} - 1 \right)} = \lim_{t \rightarrow \infty} \frac{Kre^{rt}}{re^{rt}} = \lim_{t \rightarrow \infty} K = \boxed{K = 366}.$$

(b) If the population of wolves in Wyoming outside of Yellowstone National Park continues to follow the given logistic curve, in the long run, the population will reach $K = 366$ wolves.

(c) The figure below displays the graph of the wolf population.



92. Let

$$v(t) = -A + RA \frac{e^{Bt+C} - 1}{e^{Bt+C} + 1},$$

where t is the time measured in seconds, and A , B , C , and R are positive constants with $R > 1$.

(a) Because B is positive

$$\lim_{t \rightarrow \infty} (e^{Bt+C} - 1) = \infty, \quad \text{and} \quad \lim_{t \rightarrow \infty} (e^{Bt+C} + 1) = \infty,$$

the expression $\frac{e^{Bt+C} - 1}{e^{Bt+C} + 1}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

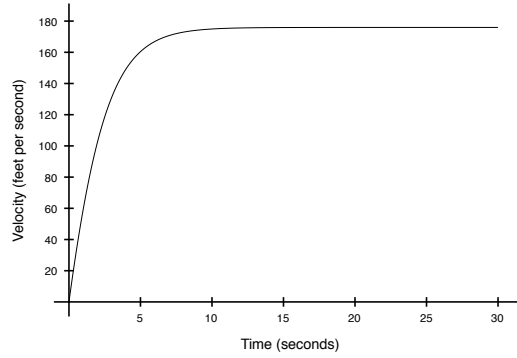
$$\lim_{t \rightarrow \infty} \frac{e^{Bt+C} - 1}{e^{Bt+C} + 1} = \lim_{t \rightarrow \infty} \frac{\frac{d}{dt}(e^{Bt+C} - 1)}{\frac{d}{dt}(e^{Bt+C} + 1)} = \lim_{t \rightarrow \infty} \frac{Be^{Bt+C}}{Be^{Bt+C}} = \lim_{t \rightarrow \infty} 1 = 1.$$

Therefore,

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left(-A + RA \frac{e^{Bt+C} - 1}{e^{Bt+C} + 1} \right) = -A + RA \cdot 1 = \boxed{(R-1)A}.$$

(b) In the long run, the skydiver's downward velocity approaches $(R-1)A$, known as the terminal velocity.

(c) The figure below displays the graph of the skydiver's speed in feet per second, with $A = 108.6$, $B = 0.554$, $C = 0.804$, and $R = 2.62$.



93. (a) Let

$$I = \frac{E}{R}(1 - e^{-Rt/L}).$$

First,

$$\lim_{t \rightarrow \infty} I(t) = \lim_{t \rightarrow \infty} \frac{E}{R}(1 - e^{-Rt/L}) = \frac{E}{R} \cdot 1 = \boxed{\frac{E}{R}}.$$

Next, for the limit as $R \rightarrow 0^+$, note that

$$\lim_{R \rightarrow 0^+} E(1 - e^{-Rt/L}) = 0 \quad \text{and} \quad \lim_{R \rightarrow 0^+} R = 0,$$

so $I(t)$ is an indeterminate form at $R = 0^+$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{R \rightarrow 0^+} I(t) = \lim_{R \rightarrow 0^+} \frac{E(1 - e^{-Rt/L})}{R} = \lim_{R \rightarrow 0^+} \frac{\frac{d}{dR}[E(1 - e^{-Rt/L})]}{\frac{d}{dR}R} = \lim_{R \rightarrow 0^+} \frac{E \frac{t}{L} e^{-Rt/L}}{1} = \boxed{\frac{Et}{L}}.$$

(b) After the circuit has been active for a long time, the current will level out at the value E/R . On the other hand, in the limit as the resistance goes to zero, the current becomes a linear function of time, $I(t) = Et/L$.

94. Because

$$\lim_{x \rightarrow 0} (a^x - b^x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0,$$

the expression $\frac{a^x - b^x}{x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(a^x - b^x)}{\frac{d}{dx}x} = \lim_{x \rightarrow 0} (a^x \ln a - b^x \ln b) = \boxed{\ln a - \ln b}.$$

95. Let $n \geq 1$ be an integer. Because

$$\lim_{x \rightarrow \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x^n = \infty,$$

the expression $\frac{\ln x}{x^n}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^n} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^n} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{nx^{n-1}} = \lim_{x \rightarrow \infty} \frac{1}{nx^n} = 0,$$

because $n \geq 1$.

96. Let $n \geq 1$ be an integer. Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x^n = \infty,$$

the expression $\frac{x^n}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} x^n}{\frac{d}{dx} e^x} = n \lim_{x \rightarrow \infty} \frac{x^{n-1}}{e^x}.$$

Continuing to use L'Hôpital's Rule yields,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^n}{e^x} &= n \lim_{x \rightarrow \infty} \frac{x^{n-1}}{e^x} \\ &= n(n-1) \lim_{x \rightarrow \infty} \frac{x^{n-2}}{e^x} \\ &= n(n-1)(n-2) \lim_{x \rightarrow \infty} \frac{x^{n-3}}{e^x} \\ &\vdots \\ &= n(n-1)(n-2) \cdots 2 \lim_{x \rightarrow \infty} \frac{x}{e^x} \\ &= n(n-1)(n-2) \cdots 2 \cdot 1 \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0. \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

97. Because

$$\lim_{x \rightarrow 0^+} (\cos x + 2 \sin x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \cot x = \infty,$$

the expression $(\cos x + 2 \sin x)^{\cot x}$ is an indeterminate form at 0^+ of the type 1^∞ . Let $y = (\cos x + 2 \sin x)^{\cot x}$. Then

$$\ln y = \ln(\cos x + 2 \sin x)^{\cot x} = \cot x \ln(\cos x + 2 \sin x),$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\cot x \ln(\cos x + 2 \sin x) \quad \text{as} \quad \frac{\ln(\cos x + 2 \sin x)}{\tan x},$$

which is now an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} [\cot x \ln(\cos x + 2 \sin x)] = \lim_{x \rightarrow 0^+} \frac{\ln(\cos x + 2 \sin x)}{\tan x} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(\cos x + 2 \sin x)}{\frac{d}{dx} \tan x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\cos x + 2 \sin x} \cdot (-\sin x + 2 \cos x)}{\sec^2 x} = \frac{1(2)}{1} = 2. \end{aligned}$$

Finally, because $\lim_{x \rightarrow 0^+} \ln y = 2$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (\cos x + 2 \sin x)^{\cot x} = e^2.$$

98. Let P be a polynomial function of degree n with leading coefficient $a_n \neq 0$. Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} P^{(k)}(x) = \pm \infty$$

for $k < n$, the expression $\frac{P^{(k)}(x)}{e^x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$ for $k < n$. Applying L'Hôpital's Rule n times then yields

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{P(x)}{e^x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}P(x)}{\frac{d}{dx}e^x} = \lim_{x \rightarrow \infty} \frac{P'(x)}{e^x} \\ &= \lim_{x \rightarrow \infty} \frac{P''(x)}{e^x} \\ &\vdots \\ &= \lim_{x \rightarrow \infty} \frac{P^{(n)}(x)}{e^x} = \boxed{0},\end{aligned}$$

because

$$\lim_{x \rightarrow \infty} P^{(n)}(x) = n!a_n < \infty.$$

99. Using the properties of the natural logarithm function,

$$\ln(x+1) - \ln(x-1) = \ln \frac{x+1}{x-1},$$

so

$$\lim_{x \rightarrow \infty} [\ln(x+1) - \ln(x-1)] = \lim_{x \rightarrow \infty} \ln \frac{x+1}{x-1} = \ln \left(\lim_{x \rightarrow \infty} \frac{x+1}{x-1} \right).$$

Now,

$$\lim_{x \rightarrow \infty} (x+1) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (x-1) = \infty,$$

so the expression $\frac{x+1}{x-1}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{x+1}{x-1} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(x-1)} = \lim_{x \rightarrow \infty} \frac{1}{1} = 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} [\ln(x+1) - \ln(x-1)] = \ln \left(\lim_{x \rightarrow \infty} \frac{x+1}{x-1} \right) = \ln 1 = \boxed{0}.$$

100. Because

$$\lim_{x \rightarrow 0^+} e^{-1/x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} x = 0,$$

the expression $\frac{e^{-1/x^2}}{x}$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Following the hint in the problem statement, rewrite

$$\frac{e^{-1/x^2}}{x} \quad \text{as} \quad \frac{\frac{1}{x}}{e^{1/x^2}},$$

which is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x} &= \lim_{x \rightarrow 0^+} \frac{x^{-1}}{e^{1/x^2}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}x^{-1}}{\frac{d}{dx}e^{1/x^2}} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^{-2}}{e^{1/x^2} \cdot -2x^{-3}} = \lim_{x \rightarrow 0^+} \frac{x}{2e^{1/x^2}} = \lim_{x \rightarrow 0^+} \frac{xe^{-1/x^2}}{2} = 0.\end{aligned}$$

101. Let n be an integer. If $n < 0$, then $-n > 0$, $\lim_{x \rightarrow 0^+} x^{-n} = 0$, and

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{x \rightarrow 0^+} (x^{-n} e^{-1/x^2}) = 0.$$

If $n = 0$, then

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{x \rightarrow 0^+} e^{-1/x^2} = 0.$$

Finally, suppose $n > 0$. Further suppose that k is the smallest positive integer such that $-n + 2k \geq 0$. Because

$$\lim_{x \rightarrow 0^+} e^{1/x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} x^{-c} = \infty$$

for $c > 0$, the expression $\frac{x^{-c}}{e^{1/x^2}} = \frac{e^{-1/x^2}}{x^c}$ is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$ for all $c > 0$. Using L'Hôpital's Rule k times yields

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x^n} &= \lim_{x \rightarrow 0^+} \frac{x^{-n}}{e^{1/x^2}} = \lim_{x \rightarrow 0^+} \frac{-nx^{-n-1}}{e^{1/x^2} \cdot -\frac{2}{x^3}} = \frac{n}{2} \lim_{x \rightarrow 0^+} \frac{x^{-n+2}}{e^{1/x^2}} \\ &= \frac{n(n-2)}{2^2} \lim_{x \rightarrow 0^+} \frac{x^{-n+4}}{e^{1/x^2}} \\ &\vdots \\ &= \frac{n(n-2) \cdots (n-2k+2)}{2^k} \lim_{x \rightarrow 0^+} \frac{x^{-n+2k}}{e^{1/x^2}}. \end{aligned}$$

If $-n + 2k = 0$, then

$$\lim_{x \rightarrow 0^+} \frac{x^{-n+2k}}{e^{1/x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{e^{1/x^2}} = 0;$$

otherwise,

$$\lim_{x \rightarrow 0^+} \frac{x^{-n+2k}}{e^{1/x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{x^{n-2k} e^{1/x^2}} = 0.$$

Therefore,

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x^n} = 0.$$

102. Because

$$\lim_{x \rightarrow \infty} x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

the expression $\sqrt[x]{x} = x^{1/x}$ is an indeterminate form at ∞ of the type ∞^0 . Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \sqrt[x]{x} = e^0 = 1.$$

103. Because

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\left(1 + \frac{a}{x}\right)^x$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(1 + \frac{a}{x}\right)^x$. Then

$$\ln y = \ln \left(1 + \frac{a}{x}\right)^x = x \ln \left(1 + \frac{a}{x}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x \ln \left(1 + \frac{a}{x}\right) \quad \text{as} \quad \frac{\ln \left(1 + \frac{a}{x}\right)}{\frac{1}{x}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{a}{x}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{a}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(1 + \frac{a}{x}\right)}{\frac{d}{dx} \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{a}{x}} \cdot -\frac{a}{x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{a}{1 + \frac{a}{x}} = \frac{a}{1} = a. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = a$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

104. Because

$$\lim_{x \rightarrow \infty} \frac{x+a}{x-a} = \lim_{x \rightarrow \infty} \frac{x+a}{x-a} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{a}{x}}{1 - \frac{a}{x}} = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\left(\frac{x+a}{x-a}\right)^x$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(\frac{x+a}{x-a}\right)^x$. Then

$$\ln y = \ln \left(\frac{x+a}{x-a}\right)^x = x \ln \left(\frac{x+a}{x-a}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x \ln \left(\frac{x+a}{x-a}\right) \quad \text{as} \quad \frac{\ln \left(\frac{x+a}{x-a}\right)}{\frac{1}{x}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[x \ln \left(\frac{x+a}{x-a}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x+a}{x-a}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(\frac{x+a}{x-a}\right)}{\frac{d}{dx} \frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{x-a}{x+a} \cdot \frac{(x-a)(1) - (x+a)(1)}{(x-a)^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2ax^2}{(x-a)(x+a)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{2a}{\left(1 - \frac{a}{x}\right)\left(1 + \frac{a}{x}\right)} = \frac{2a}{1(1)} = 2a. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 2a$, it follows that

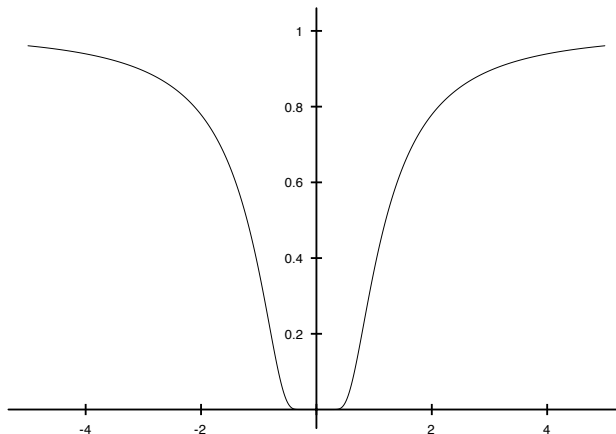
$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a}\right)^x = e^{2a}.$$

105. (a) Note that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x} = 0$$

by Problem 100. Because this limit exists, it follows that f is differentiable at 0; moreover, $f'(0) = 0$.

(b) The figure below displays the graph of f .



106. Let a , $b \neq 0$, and $c > 0$ be real numbers. Because

$$\lim_{x \rightarrow c} (x^a - c^a) = 0 \quad \text{and} \quad \lim_{x \rightarrow c} (x^b - c^b) = 0,$$

the expression $\frac{x^a - c^a}{x^b - c^b}$ is an indeterminate form at c of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow c} \frac{x^a - c^a}{x^b - c^b} = \lim_{x \rightarrow c} \frac{\frac{d}{dx}(x^a - c^a)}{\frac{d}{dx}(x^b - c^b)} = \lim_{x \rightarrow c} \frac{ax^{a-1}}{bx^{b-1}} = \frac{ac^{a-1}}{bc^{b-1}} = \frac{a}{b}c^{a-b}.$$

107. Suppose the expression $\frac{f(x)}{g(x)}$ is an indeterminate form at $-\infty$ of the type $\frac{0}{0}$. Then

$$\lim_{x \rightarrow -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} g(x) = 0.$$

Define

$$F(t) = f\left(\frac{1}{t}\right) \quad \text{and} \quad G(t) = g\left(\frac{1}{t}\right).$$

Now, consider the limit

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}.$$

Make the substitution $t = 1/x$. Then, as $x \rightarrow -\infty$, $t \rightarrow 0^-$, and

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \lim_{t \rightarrow 0^-} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t \rightarrow 0^-} \frac{F(t)}{G(t)}.$$

Because

$$\lim_{t \rightarrow 0^-} F(t) = \lim_{t \rightarrow 0^-} f\left(\frac{1}{t}\right) = \lim_{x \rightarrow -\infty} f(x) = 0$$

and

$$\lim_{t \rightarrow 0^-} G(t) = \lim_{t \rightarrow 0^-} g\left(\frac{1}{t}\right) = \lim_{x \rightarrow -\infty} g(x) = 0,$$

the expression $\frac{F(t)}{G(t)}$ is an indeterminate form at 0^- of the type $\frac{0}{0}$. By the partial proof of L'Hôpital's Rule provided in the text, it follows that

$$\lim_{t \rightarrow 0^-} \frac{F(t)}{G(t)} = \lim_{t \rightarrow 0^-} \frac{F'(t)}{G'(t)}.$$

By the definition of F and G

$$F'(t) = -\frac{1}{t^2} f' \left(\frac{1}{t} \right) \quad \text{and} \quad G'(t) = -\frac{1}{t^2} g' \left(\frac{1}{t} \right),$$

so

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \lim_{t \rightarrow 0^-} \frac{F(t)}{G(t)} = \lim_{t \rightarrow 0^-} \frac{F'(t)}{G'(t)} = \lim_{t \rightarrow 0^-} \frac{-\frac{1}{t^2} f' \left(\frac{1}{t} \right)}{-\frac{1}{t^2} g' \left(\frac{1}{t} \right)} = \lim_{t \rightarrow 0^-} \frac{f' \left(\frac{1}{t} \right)}{g' \left(\frac{1}{t} \right)} = \lim_{x \rightarrow -\infty} \frac{f'(x)}{g'(x)},$$

as desired.

Challenge Problems

108. L'Hôpital's Rule states that, if the expression $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists. Now, it can be shown that

$$\lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

by the Squeeze Theorem, and $\lim_{x \rightarrow 0} \sin x = 0$, so the expression $\frac{x^2 \sin \frac{1}{x}}{\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Moreover,

$$\frac{d}{dx} \left(x^2 \sin \frac{1}{x} \right) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x},$$

so

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx} \left(x^2 \sin \frac{1}{x} \right)}{\frac{d}{dx} \sin x} = \lim_{x \rightarrow 0} \frac{-\cos \frac{1}{x} + 2x \sin \frac{1}{x}}{\cos x}.$$

Although $\lim_{x \rightarrow 0} \left(2x \sin \frac{1}{x} \right) = 0$ by the Squeeze Theorem and $\lim_{x \rightarrow 0} \cos x = 1$,

$$\lim_{x \rightarrow 0} \cos \frac{1}{x}$$

does not exist because the cosine function does not approach a single number. Therefore,

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx} \left(x^2 \sin \frac{1}{x} \right)}{\frac{d}{dx} \sin x}$$

does not exist, so L'Hôpital's Rule does not apply to

$$\lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}.$$

109. (a) Because

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} (-x^2) = -\infty,$$

the expression $\left(1 + \frac{1}{x}\right)^{-x^2}$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(1 + \frac{1}{x}\right)^{-x^2}$. Then

$$\ln y = \ln \left(1 + \frac{1}{x}\right)^{-x^2} = -x^2 \ln \left(1 + \frac{1}{x}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$-x^2 \ln \left(1 + \frac{1}{x}\right) \quad \text{as} \quad \frac{\ln \left(1 + \frac{1}{x}\right)}{-\frac{1}{x^2}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[-x^2 \ln \left(1 + \frac{1}{x}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(1 + \frac{1}{x}\right)}{\frac{d}{dx} \left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{x^2}}{\frac{2}{x^3}} = \lim_{x \rightarrow \infty} \frac{-x}{2 \left(1 + \frac{1}{x}\right)} = -\infty. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = -\infty$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{-x^2} = \lim_{x \rightarrow \infty} e^{\ln y} = \boxed{0}.$$

(b) Because

$$\lim_{x \rightarrow \infty} \left(1 + \frac{\ln a}{x}\right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\left(1 + \frac{\ln a}{x}\right)^x$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(1 + \frac{\ln a}{x}\right)^x$. Then

$$\ln y = \ln \left(1 + \frac{\ln a}{x}\right)^x = x \ln \left(1 + \frac{\ln a}{x}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x \ln \left(1 + \frac{\ln a}{x}\right) \quad \text{as} \quad \frac{\ln \left(1 + \frac{\ln a}{x}\right)}{\frac{1}{x}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{\ln a}{x}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{\ln a}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(1 + \frac{\ln a}{x}\right)}{\frac{d}{dx} \left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{\ln a}{x}} \cdot -\frac{\ln a}{x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{\ln a}{1 + \frac{\ln a}{x}} = \ln a. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = \ln a$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(1 + \frac{\ln a}{x}\right)^x = e^{\ln a} = \boxed{a}.$$

(c) Because

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} x^2 = \infty,$$

the expression $\left(1 + \frac{1}{x}\right)^{x^2}$ is an indeterminate form at ∞ of the type 1^∞ . Let $y = \left(1 + \frac{1}{x}\right)^{x^2}$. Then

$$\ln y = \ln \left(1 + \frac{1}{x}\right)^{x^2} = x^2 \ln \left(1 + \frac{1}{x}\right),$$

which is an indeterminate form at ∞ of the type $0 \cdot \infty$. Rewrite

$$x^2 \ln \left(1 + \frac{1}{x}\right) \quad \text{as} \quad \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x^2}},$$

which is now an indeterminate form at ∞ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \left[x^2 \ln \left(1 + \frac{1}{x}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln \left(1 + \frac{1}{x}\right)}{\frac{d}{dx} \left(\frac{1}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{x^2}}{-\frac{2}{x^3}} = \lim_{x \rightarrow \infty} \frac{x}{2 \left(1 + \frac{1}{x}\right)} = \infty. \end{aligned}$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = \infty$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{x^2} = \lim_{x \rightarrow \infty} e^{\ln y} = \boxed{\infty}.$$

(d) This limit does not exist. To show this, first let $x = n\pi$, where n is a natural number. As $n \rightarrow \infty$, $x \rightarrow \infty$, and

$$\left(1 + \frac{\sin x}{x}\right)^x = \left(1 + \frac{\sin(n\pi)}{n\pi}\right)^{n\pi} = 1^{n\pi} = 1.$$

Next, let $x = \frac{\pi}{2} + 2n\pi$, where n is a natural number. Again, as $n \rightarrow \infty$, $x \rightarrow \infty$, but

$$\left(1 + \frac{\sin x}{x}\right)^x = \left(1 + \frac{\sin(\pi/2 + 2n\pi)}{\pi/2 + 2n\pi}\right)^{\pi/2 + 2n\pi} = \left(1 + \frac{1}{\pi/2 + 2n\pi}\right)^{\pi/2 + 2n\pi} \rightarrow e.$$

Because the value of

$$\left(1 + \frac{\sin x}{x}\right)^x$$

does not approach a single number as $x \rightarrow \infty$, the indicated limit does not exist.

(e) Because

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} -\frac{1}{\ln x} = 0,$$

the expression $(e^x)^{-1/\ln x}$ is an indeterminate form at ∞ of the type ∞^0 . Let $y = (e^x)^{-1/\ln x}$. Then

$$\ln y = \ln(e^x)^{-1/\ln x} = -\frac{1}{\ln x} \ln e^x = -\frac{x}{\ln x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} -\frac{x}{\ln x} = \lim_{x \rightarrow \infty} -\frac{\frac{d}{dx} x}{\frac{d}{dx} \ln x} = \lim_{x \rightarrow \infty} -\frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} -x = -\infty.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = -\infty$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} (e^x)^{1/\ln x} = \lim_{x \rightarrow \infty} e^{\ln y} = \boxed{0}.$$

(f) Because

$$\left[\left(\frac{1}{a} \right)^x \right]^{-1/x} = \left(\frac{1}{a} \right)^{-1} = a$$

for all x ,

$$\lim_{x \rightarrow \infty} \left[\left(\frac{1}{a} \right)^x \right]^{-1/x} = \boxed{a}.$$

(g) Because

$$\lim_{x \rightarrow \infty} x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

the expression $x^{1/x}$ is an indeterminate form at ∞ of the type ∞^0 . Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Finally, because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} x^{1/x} = e^0 = \boxed{1}.$$

(h) Because

$$(a^x)^{1/x} = a$$

for all x ,

$$\lim_{x \rightarrow \infty} (a^x)^{1/x} = \boxed{a}.$$

(i) Because

$$[(2 + \sin x)^x]^{1/x} = (2 + \sin x)$$

for all x ,

$$\lim_{x \rightarrow \infty} [(2 + \sin x)^x]^{1/x} = \lim_{x \rightarrow \infty} (2 + \sin x)$$

which $\boxed{\text{does not exist}}$ because $\sin x$ never approaches a single number as $x \rightarrow \infty$.

(j) For $x > 0$,

$$x^{-1/\ln x} = (e^{\ln x})^{-1/\ln x} = e^{-1},$$

so

$$\lim_{x \rightarrow 0^+} x^{-1/\ln x} = \lim_{x \rightarrow 0^+} e^{-1} = \boxed{e^{-1}}.$$

110. Because

$$\lim_{x \rightarrow 0} (\sin(Ax) + Bx + Cx^2 + Dx^3) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^5 = 0,$$

the expression $\frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sin(Ax) + Bx + Cx^2 + Dx^3)}{\frac{d}{dx}x^5} \\ &= \lim_{x \rightarrow 0} \frac{A \cos(Ax) + B + 2Cx + 3Dx^2}{5x^4}.\end{aligned}$$

Now, $\lim_{x \rightarrow 0} (5x^4) = 0$, so

$$\lim_{x \rightarrow 0} [A \cos(Ax) + B + 2Cx + 3Dx^2] = A + B$$

must be equal to 0 in order for the original limit to exist; therefore, $A + B = 0$. Using L'Hôpital's Rule again,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[A \cos(Ax) + B + 2Cx + 3Dx^2]}{\frac{d}{dx}5x^4} \\ &= \lim_{x \rightarrow 0} \frac{-A^2 \sin(Ax) + 2C + 6Dx}{20x^3}.\end{aligned}$$

As above, because $\lim_{x \rightarrow 0} (20x^3) = 0$,

$$\lim_{x \rightarrow 0} [-A^2 \sin(Ax) + 2C + 6Dx] = 2C$$

must be equal to 0 in order for the original limit to exist; therefore, $C = 0$. Using L'Hôpital's Rule for a third time,

$$\lim_{x \rightarrow 0} \frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[-A^2 \sin(Ax) + 2C + 6Dx]}{\frac{d}{dx}20x^3} = \lim_{x \rightarrow 0} \frac{-A^3 \cos(Ax) + 6D}{60x^2}.$$

The requirement that the original limit exist then yields

$$\lim_{x \rightarrow 0} [-A^3 \cos(Ax) + 6D] = -A^3 + 6D = 0.$$

A fourth application of L'Hôpital's Rule gives

$$\lim_{x \rightarrow 0} \frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[-A^3 \cos(Ax) + 6D]}{\frac{d}{dx}60x^2} = \lim_{x \rightarrow 0} \frac{A^4 \sin(Ax)}{120x},$$

which involves an indeterminate form at 0 of the type $\frac{0}{0}$. One last use of L'Hôpital's Rule produces

$$\lim_{x \rightarrow 0} \frac{\sin(Ax) + Bx + Cx^2 + Dx^3}{x^5} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}A^4 \sin(Ax)}{\frac{d}{dx}120x} = \lim_{x \rightarrow 0} \frac{A^5 \cos(Ax)}{120} = \frac{A^5}{120} = \frac{4}{15}.$$

Finally,

$$\boxed{A = 2, \quad B = -2, \quad C = 0, \quad \text{and} \quad D = \frac{4}{3}}.$$

111. Let f be a function whose derivatives of all orders exist.

(a) Because

$$\lim_{h \rightarrow 0} [f(x + 2h) - 2f(x + h) + f(x)] = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} h^2 = 0,$$

the expression $\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$ is an indeterminate form at $h = 0$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[f(x+2h) - 2f(x+h) + f(x)]}{\frac{d}{dh}h^2} \\ &= \lim_{h \rightarrow 0} \frac{2f'(x+2h) - 2f'(x+h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+2h) - f'(x+h)}{h}. \end{aligned}$$

Now,

$$\lim_{h \rightarrow 0} [f'(x+2h) - f'(x+h)] = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} h = 0,$$

so the expression $\frac{f'(x+2h) - f'(x+h)}{h}$ is an indeterminate form at $h = 0$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+2h) - f'(x+h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[f'(x+2h) - f'(x+h)]}{\frac{d}{dh}h} \\ &= \lim_{h \rightarrow 0} \frac{2f''(x+2h) - f''(x+h)}{1} = \boxed{f''(x)}. \end{aligned}$$

(b) Because

$$\lim_{h \rightarrow 0} [f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)] = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} h^3 = 0,$$

the expression $\frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$ is an indeterminate form at $h = 0$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)]}{\frac{d}{dh}h^3} \\ &= \lim_{h \rightarrow 0} \frac{3f'(x+3h) - 6f'(x+2h) + 3f'(x+h)}{3h^2} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+3h) - 2f'(x+2h) + f'(x+h)}{h^2}. \end{aligned}$$

Now,

$$\lim_{h \rightarrow 0} [f'(x+3h) - 2f'(x+2h) + f'(x+h)] = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} h^2 = 0,$$

so the expression $\frac{f'(x+3h) - 2f'(x+2h) + f'(x+h)}{h^2}$ is an indeterminate form at $h = 0$ of the

type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+3h) - 2f'(x+2h) + f'(x+h)}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[f'(x+3h) - 2f'(x+2h) + f'(x+h)]}{\frac{d}{dh}h^2} \\ &= \lim_{h \rightarrow 0} \frac{3f''(x+3h) - 4f''(x+2h) + f''(x+h)}{2h}. \end{aligned}$$

Finally,

$$\lim_{h \rightarrow 0} [3f''(x+3h) - 4f''(x+2h) + f''(x+h)] = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} (2h) = 0,$$

so the expression $\frac{3f''(x+3h) - 4f''(x+2h) + f''(x+h)}{2h}$ is an indeterminate form at $h = 0$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule a third time

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \\ &= \lim_{h \rightarrow 0} \frac{3f''(x+3h) - 4f''(x+2h) + f''(x+h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[3f''(x+3h) - 4f''(x+2h) + f''(x+h)]}{\frac{d}{dh}(2h)} \\ &= \lim_{h \rightarrow 0} \frac{9f'''(x+3h) - 8f'''(x+2h) + f'''(x+h)}{2} \\ &= \frac{2f'''(x)}{2} = \boxed{f'''(x)}. \end{aligned}$$

- (c) Recognize that in the numerator of each limit, the coefficients on the values of f (1, -2, 1 in part (a) and 1, -3, 3, -1 in part (b)) are the coefficients in $(x-1)^2$ and $(x-1)^3$, respectively; that is, they are binomial coefficients. Therefore, for a positive integer n ,

$$\boxed{\lim_{h \rightarrow 0} \frac{\sum_{k=0}^n (-1)^k \frac{n!}{k!(n-k)!} f(x + (n-k)h)}{h^n} = f^{(n)}(x)}.$$

112. Using the data provided and the results of Problem 111 with $h = 0.1$,

$$\begin{aligned} f'(2) &\approx \frac{f(2.1) - f(2)}{0.1} = \frac{0.7419 - 0.6931}{0.1} = \boxed{0.488}; \\ f''(2) &\approx \frac{f(2.2) - 2f(2.1) + f(2)}{(0.1)^2} = \frac{0.7885 - 2(0.7419) + 0.6931}{0.01} = \boxed{-0.22}; \text{ and} \\ f'''(2) &\approx \frac{f(2.3) - 3f(2.2) + 3f(2.1) - f(2)}{(0.1)^3} = \frac{0.8329 - 3(0.7885) + 3(0.7419) - 0.6931}{0.001} = \boxed{0}. \end{aligned}$$

With $f(x) = \ln x$,

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad \text{and} \quad f'''(x) = \frac{2}{x^3},$$

so

$$f'(2) = \frac{1}{2}, \quad f''(2) = -\frac{1}{4}, \quad \text{and} \quad f'''(2) = \frac{1}{4}.$$

The percentage error in the approximation to the value of the first derivative is $\boxed{2.4\%}$; the percentage error in the approximation to the value of the second derivative is $\boxed{12\%}$; the percentage error in the approximation to the value of the third derivative is $\boxed{100\%}$.

113. Let $f(t, x) = \frac{x^{t+1} - 1}{t + 1}$, where $x > 0$ and $t \neq -1$.

(a) Let $x = x_0$. Because

$$\lim_{t \rightarrow -1} (x_0^{t+1} - 1) = 0 \quad \text{and} \quad \lim_{t \rightarrow -1} (t + 1) = 0,$$

the expression $f(t, x_0) = \frac{x_0^{t+1} - 1}{t + 1}$ is an indeterminate form at $t = -1$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{t \rightarrow -1} f(t, x_0) &= \lim_{t \rightarrow -1} \frac{x_0^{t+1} - 1}{t + 1} = \lim_{t \rightarrow -1} \frac{\frac{d}{dt}(x_0^{t+1} - 1)}{\frac{d}{dt}(t + 1)} \\ &= \lim_{t \rightarrow -1} \frac{x_0^{t+1} \ln x_0}{1} = \ln x_0. \end{aligned}$$

(b) The function $f(t, x)$ is continuous for $t \neq -1$. Based on the result from part (a), define

$$F(t, x) = \begin{cases} f(t, x), & t \neq -1 \\ \ln x, & t = -1. \end{cases}$$

The function $F(t, x)$ is then continuous for all t .

(c) Based on part (b), for t fixed but not equal to -1 ,

$$\frac{d}{dx} F(t, x) = \frac{d}{dx} f(t, x) = \frac{d}{dx} \left(\frac{x^{t+1} - 1}{t + 1} \right) = \frac{1}{t + 1} \cdot (t + 1)x^t = x^t.$$

For $t = -1$,

$$\frac{d}{dx} F(t, x) = \frac{d}{dx} \ln x = \frac{1}{x} = x^{-1} = x^t.$$

4.6: Using Calculus to Graph Functions

Skill Building

1. Let $f(x) = x^4 - 6x^2 + 10$.

Step 1 The polynomial function f has a domain of all real numbers. $f(0) = 10$, so the y -intercept is 10.
To find the x -intercepts, solve the equation $f(x) = 0$. Because

$$x^4 - 6x^2 + 10 = (x^2 - 3)^2 + 1,$$

it follows that there are no real solutions to the equation $f(x) = 0$, so the graph of f has no x -intercepts.

Step 2 The graphs of polynomial functions do not have asymptotes, but the end behavior of the graph of f will resemble the power function $y = x^4$.

Step 3 Now

$$\begin{aligned} f'(x) &= 4x^3 - 12x = 4x(x^2 - 3); \text{ and} \\ f''(x) &= 12x^2 - 12 = 12(x - 1)(x + 1). \end{aligned}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so 0 and $\pm\sqrt{3}$ are the critical numbers. At the points $(-\sqrt{3}, 1)$, $(0, 10)$, and $(\sqrt{3}, 1)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers 0 and $\pm\sqrt{3}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\sqrt{3})$	–	f is decreasing on $(-\infty, -\sqrt{3})$
$(-\sqrt{3}, 0)$	+	f is increasing on $(-\sqrt{3}, 0)$
$(0, \sqrt{3})$	–	f is decreasing on $(0, \sqrt{3})$
$(\sqrt{3}, \infty)$	+	f is increasing on $(\sqrt{3}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\sqrt{3}$, a local maximum value at 0, and a local minimum value at $\sqrt{3}$. The

$$\text{local minimum values are } f(-\sqrt{3}) = 1 \text{ and } f(\sqrt{3}) = 1,$$

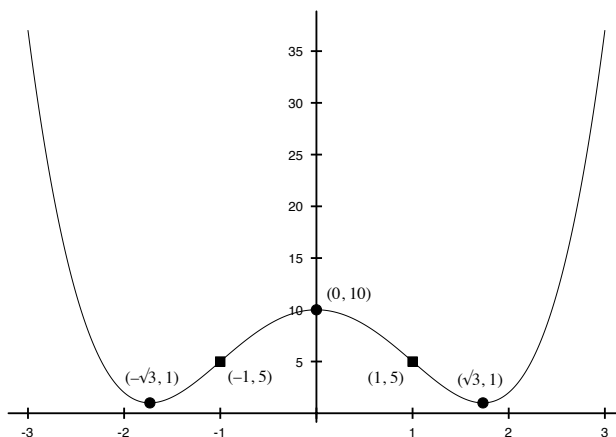
while the local maximum value is $f(0) = 10$.

Step 6 The second derivative is equal to zero at $x = \pm 1$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, 1)$	–	f is concave down on $(-1, 1)$
$(1, \infty)$	+	f is concave up on $(1, \infty)$

The concavity of f changes at ± 1 , so the points $(-1, 5)$ and $(1, 5)$ are points of inflection of f .

Step 7 The figure below displays the graph of f . Local extreme values are highlighted by closed circles, and points of inflection are highlighted by closed squares.



2. Let $f(x) = x^5 - 3x^3 + 4$.

Step 1 The polynomial function f has a domain of all real numbers. $f(0) = 4$, so the y -intercept is 4. To find the x -intercepts, solve the equation $f(x) = 0$. Using the computer algebra system *Maple*, the only real number solution of the equation $f(x) = 0$ is $x \approx -1.894$. Therefore, -1.894 is the approximate x -intercept of f .

Step 2 The graphs of polynomial functions do not have asymptotes, but the end behavior of the graph of f will resemble the power function $y = x^5$.

Step 3 Now

$$\begin{aligned} f'(x) &= 5x^4 - 9x^2 = x^2(5x^2 - 9); \text{ and} \\ f''(x) &= 20x^3 - 18x = 2x(10x^2 - 9). \end{aligned}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so 0 and $\pm \frac{3\sqrt{5}}{5}$ are the critical numbers. At the points

$$\left(-\frac{3\sqrt{5}}{5}, \frac{500 + 162\sqrt{5}}{125}\right), \quad (0, 4), \quad \left(\frac{3\sqrt{5}}{5}, \frac{500 - 162\sqrt{5}}{125}\right),$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers 0 and $\pm \frac{3\sqrt{5}}{5}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{3\sqrt{5}}{5})$	+	f is increasing on $(-\infty, -\frac{3\sqrt{5}}{5})$
$(-\frac{3\sqrt{5}}{5}, 0)$	-	f is decreasing on $(-\frac{3\sqrt{5}}{5}, 0)$
$(0, \frac{3\sqrt{5}}{5})$	-	f is decreasing on $(0, \frac{3\sqrt{5}}{5})$
$(\frac{3\sqrt{5}}{5}, \infty)$	+	f is increasing on $(\frac{3\sqrt{5}}{5}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $-\frac{3\sqrt{5}}{5}$, neither a local maximum value nor a local minimum at 0 , and a local minimum value at $\frac{3\sqrt{5}}{5}$. The

local maximum value is $f\left(-\frac{3\sqrt{5}}{5}\right) = \frac{500 + 162\sqrt{5}}{125}$,

while the

$$\text{local minimum value is } f\left(\frac{3\sqrt{5}}{5}\right) = \frac{500 - 162\sqrt{5}}{125}.$$

Step 6 The second derivative is equal to zero at $x = 0$ and $x = \pm \frac{3\sqrt{10}}{10}$. Use these numbers to divide the number line into four intervals, and determine the sign of f'' on each interval.

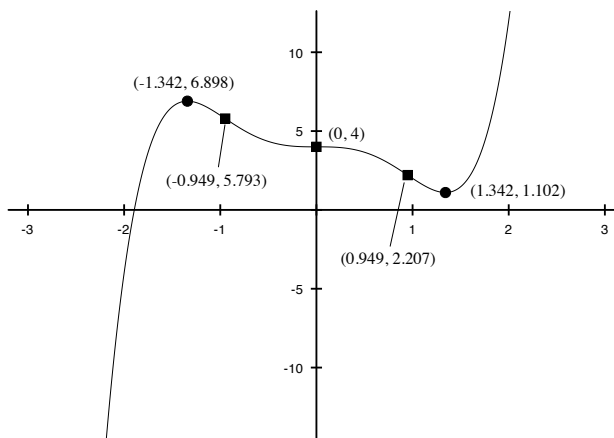
Interval	Sign of f''	Conclusion
$(-\infty, -\frac{3\sqrt{10}}{10})$	−	f is concave down on $(-\infty, -\frac{3\sqrt{10}}{10})$
$(-\frac{3\sqrt{10}}{10}, 0)$	+	f is concave up on $(-\frac{3\sqrt{10}}{10}, 0)$
$(0, \frac{3\sqrt{10}}{10})$	−	f is concave down on $(0, \frac{3\sqrt{10}}{10})$
$(\frac{3\sqrt{10}}{10}, \infty)$	+	f is concave up on $(\frac{3\sqrt{10}}{10}, \infty)$

The concavity of f changes at 0 and $\pm \frac{3\sqrt{10}}{10}$, so the points

$$\left(-\frac{3\sqrt{10}}{10}, \frac{4000 + 567\sqrt{10}}{1000}\right), \quad (0, 4), \quad \left(\frac{3\sqrt{10}}{10}, \frac{4000 - 567\sqrt{10}}{1000}\right),$$

are points of inflection of f .

Step 7 The figure below displays the graph of f . Local extreme values are highlighted by closed circles, and points of inflection are highlighted by closed squares.



3. Let $f(x) = \frac{1}{x-2}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 2\}$. There are no x -intercepts, and the

$$y\text{-intercept is } f(0) = -\frac{1}{2}.$$

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x-2} = \lim_{x \rightarrow \pm\infty} \frac{1}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x}}{1 - \frac{2}{x}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty,$$

the line $x = 2$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x-2)^{-1} = -(x-2)^{-2} = -\frac{1}{(x-2)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx}[-(x-2)^{-2}] = 2(x-2)^{-3} = \frac{2}{(x-2)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Note $f'(x)$ is never equal to 0 and does not exist when $x = 2$. However, 2 is not in the domain of f , so 2 is not a critical number. Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 2 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 2)$	—	f is decreasing on $(-\infty, 2)$
$(2, \infty)$	—	f is decreasing on $(2, \infty)$

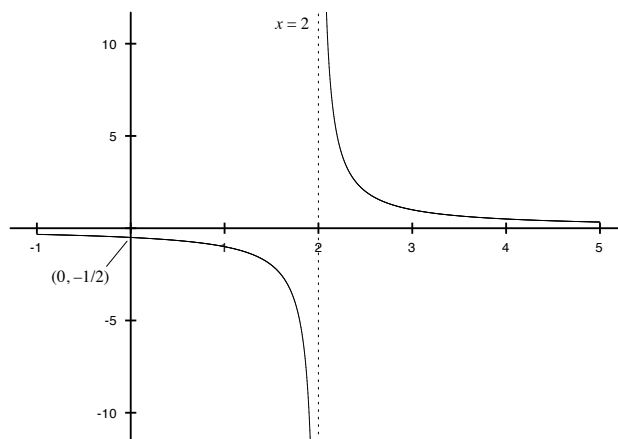
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is never equal to zero and does not exist when $x = 2$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 2)$	—	f is concave down on $(-\infty, 2)$
$(2, \infty)$	+	f is concave up on $(2, \infty)$

Although the concavity of f changes at 2, there is no point of inflection at 2 because 2 is not in the domain of f .

Step 7 The figure below displays the graph of f .



4. Let $f(x) = \frac{2}{x+2}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq -2\}$. There are no x -intercepts, and the y -intercept is $f(0) = 1$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{2}{x+2} = \lim_{x \rightarrow \pm\infty} \frac{2}{x+2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2}{x}}{1 + \frac{2}{x}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -2^-} \frac{2}{x+2} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{2}{x+2} = \infty,$$

the line $x = -2$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} 2(x+2)^{-1} = -2(x+2)^{-2} = -\frac{2}{(x+2)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} [-2(x+2)^{-2}] = 4(x+2)^{-3} = \frac{4}{(x+2)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Note $f'(x)$ is never equal to 0 and does not exist when $x = -2$. However, -2 is not in the domain of f , so -2 is not a critical number. Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the number -2 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	—	f is decreasing on $(-\infty, -2)$
$(-2, \infty)$	—	f is decreasing on $(-2, \infty)$

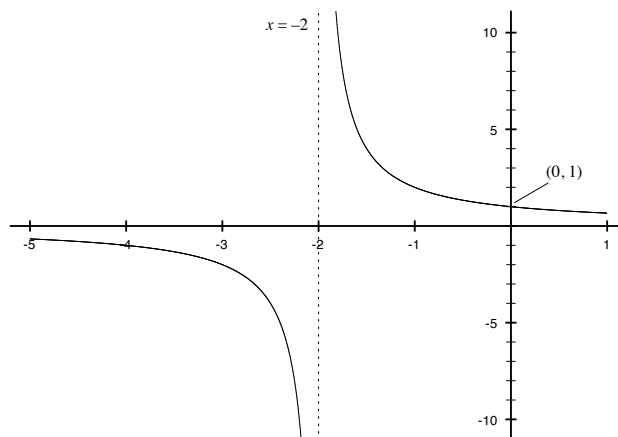
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is never equal to zero and does not exist when $x = -2$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	—	f is concave down on $(-\infty, -2)$
$(-2, \infty)$	+	f is concave up on $(-2, \infty)$

Although the concavity of f changes at -2 , there is no point of inflection at -2 because -2 is not in the domain of f .

Step 7 The figure below displays the graph of f .



5. Let $f(x) = \frac{2}{x^2 - 4}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq \pm 2\}$. There are no x -intercepts, and the y -intercept is $f(0) = -\frac{1}{2}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{2}{x^2 - 4} = \lim_{x \rightarrow \pm\infty} \frac{2}{x^2 - 4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2}{x^2}}{1 - \frac{4}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -2^-} \frac{2}{x^2 - 4} = \infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{2}{x^2 - 4} = -\infty,$$

and

$$\lim_{x \rightarrow 2^-} \frac{2}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{2}{x^2 - 4} = \infty,$$

the lines $x = \pm 2$ are vertical asymptotes.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} 2(x^2 - 4)^{-1} = -2(x^2 - 4)^{-2} \cdot 2x = -\frac{4x}{(x^2 - 4)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[-\frac{4x}{(x^2 - 4)^2} \right] = -\frac{(x^2 - 4)^2 \cdot 4 - 4x \cdot 2(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= -\frac{4(x^2 - 4) - 16x^2}{(x^2 - 4)^3} = \frac{12x^2 + 16}{(x^2 - 4)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and does not exist when $x = \pm 2$. However, ± 2 are not in the domain of f , so ± 2 are not critical numbers. Therefore, 0 is the only critical number of f . The tangent line at the point $\left(0, -\frac{1}{2}\right)$ is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 2 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	+	f is increasing on $(-\infty, -2)$
$(-2, 0)$	+	f is increasing on $(-2, 0)$
$(0, 2)$	-	f is decreasing on $(0, 2)$
$(2, \infty)$	-	f is decreasing on $(2, \infty)$

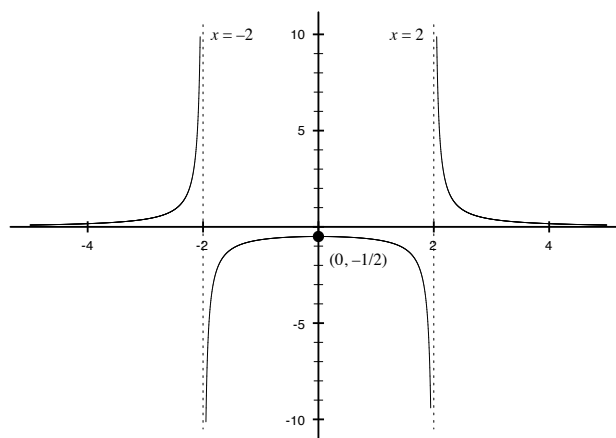
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The local maximum value is $f(0) = -\frac{1}{2}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = \pm 2$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	+	f is concave up on $(-\infty, -2)$
$(-2, 2)$	-	f is concave down on $(-2, 2)$
$(2, \infty)$	+	f is concave up on $(2, \infty)$

Although the concavity of f changes at ± 2 , there is no point of inflection at ± 2 because ± 2 are not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



6. Let $f(x) = \frac{1}{x^2 - 1}$.

Step 1 The domain of the rational function f is the set $\{x | x \neq \pm 1\}$. There are no x -intercepts, and the y -intercept is $f(0) = -1$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 1} = \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x^2}}{1 - \frac{1}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -1^-} \frac{1}{x^2 - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{1}{x^2 - 1} = -\infty,$$

and

$$\lim_{x \rightarrow 1^-} \frac{1}{x^2 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{x^2 - 1} = \infty,$$

the lines $x = \pm 1$ are vertical asymptotes.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2 - 1)^{-1} = -(x^2 - 1)^{-2} \cdot 2x = -\frac{2x}{(x^2 - 1)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[-\frac{2x}{(x^2 - 1)^2} \right] = -\frac{(x^2 - 1)^2 \cdot 2 - 2x \cdot 2(x^2 - 1)(2x)}{(x^2 - 1)^4} \\ &= -\frac{2(x^2 - 1) - 8x^2}{(x^2 - 1)^3} = \frac{6x^2 + 2}{(x^2 - 1)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and does not exist when $x = \pm 1$. However, ± 1 are not in the domain of f , so ± 1 are not critical numbers. Therefore, 0 is the only critical number of f . the tangent line at the point $(0, -1)$ is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	+	f is increasing on $(-\infty, -1)$
$(-1, 0)$	+	f is increasing on $(-1, 0)$
$(0, 1)$	-	f is decreasing on $(0, 1)$
$(1, \infty)$	-	f is decreasing on $(1, \infty)$

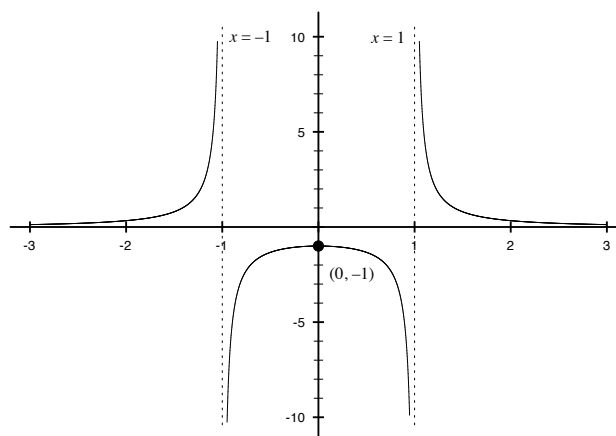
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The local maximum value is $f(0) = -1$.

Step 6 The second derivative is never equal to zero and does not exist when $x = \pm 1$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, 1)$	-	f is concave down on $(-1, 1)$
$(1, \infty)$	+	f is concave up on $(1, \infty)$

Although the concavity of f changes at ± 1 , there is no point of inflection at ± 1 because ± 1 are not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



7. Let $f(x) = \frac{2x-1}{x+1}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq -1\}$. The x -intercept is $\frac{1}{2}$, and the y -intercept is $f(0) = -1$.

Step 2 The degree of the numerator is equal to the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{2x-1}{x+1} = \lim_{x \rightarrow \pm\infty} \frac{2x-1}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \pm\infty} \frac{2 - \frac{1}{x}}{1 + \frac{1}{x}} = 2,$$

the line $y = 2$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -1^-} \frac{2x-1}{x+1} = \infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{2x-1}{x+1} = -\infty,$$

the line $x = -1$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{2x-1}{x+1} \right) = \frac{(x+1) \cdot 2 - (2x-1) \cdot 1}{(x+1)^2} = \frac{3}{(x+1)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} [3(x+1)^{-2}] = -6(x+1)^{-3} = -\frac{6}{(x+1)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = -1$. However, -1 is not in the domain of f , so -1 is not a critical number. Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the number -1 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	+	f is increasing on $(-\infty, -1)$
$(-1, \infty)$	+	f is increasing on $(-1, \infty)$

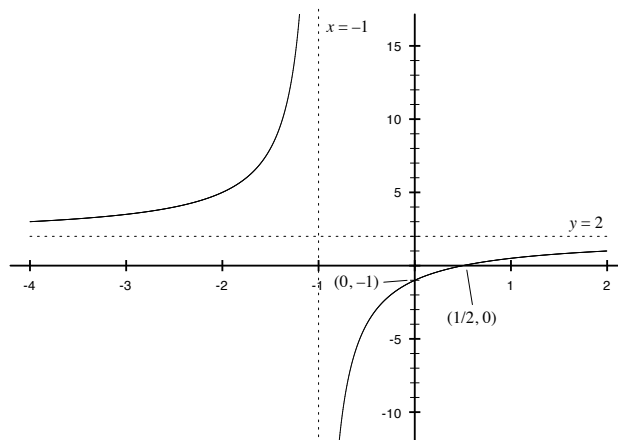
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is never equal to zero and does not exist when $x = -1$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, \infty)$	-	f is concave down on $(-1, \infty)$

Although the concavity of f changes at -1 , there is no point of inflection at -1 because -1 is not in the domain of f .

Step 7 The figure below displays the graph of f .



8. Let $f(x) = \frac{x-2}{x}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. The x -intercept is 2, and there is no y -intercept because f is not defined at 0.

Step 2 The degree of the numerator is equal to the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{x-2}{x} = \lim_{x \rightarrow \pm\infty} \frac{x-2}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{2}{x}}{1} = 1,$$

the line $y = 1$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x-2}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x-2}{x} = -\infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x-2}{x} \right) = \frac{x \cdot 1 - (x-2) \cdot 1}{x^2} = \frac{2}{x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} [2x^{-2}] = -4x^{-3} = -\frac{4}{x^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

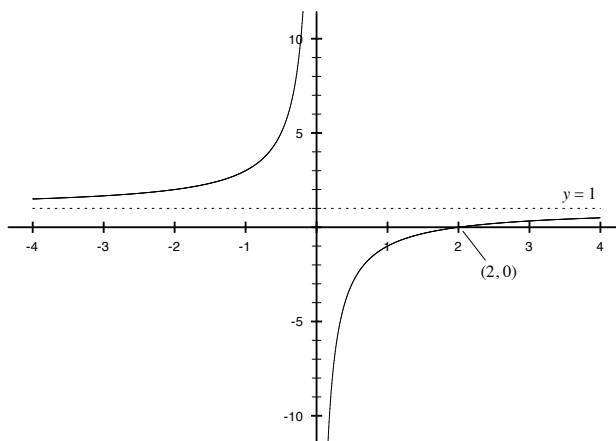
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	+	f is concave up on $(-\infty, 0)$
$(0, \infty)$	-	f is concave down on $(0, \infty)$

Although the concavity of f changes at 0, there is no point of inflection at 0 because 0 is not in the domain of f .

Step 7 The figure below displays the graph of f .



9. Let $f(x) = \frac{x}{x^2 + 1}$.

Step 1 The domain of the rational function f is the set of all real numbers (because $1 + x^2$ is never equal to zero for any real x). The x -intercept is 0, and the y -intercept is $f(0) = 0$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{x}{x^2 + 1} = \lim_{x \rightarrow \pm\infty} \frac{x}{x^2 + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. Because f is defined for all real numbers, there are no vertical asymptotes.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x}{x^2 + 1} \right) = \frac{(x^2 + 1) \cdot 1 - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(\frac{1 - x^2}{(x^2 + 1)^2} \right) = \frac{(x^2 + 1)^2 \cdot (-2x) - (1 - x^2) \cdot 2(x^2 + 1)(2x)}{(x^2 + 1)^4} \\ &= \frac{-2x(x^2 + 1) - 4x(1 - x^2)}{(x^2 + 1)^3} = \frac{2x^3 - 6x}{(x^2 + 1)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ exists everywhere and is equal to 0 when $x = \pm 1$. Therefore, ± 1 are the critical numbers of f . At the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers ± 1 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, 1)$	$+$	f is increasing on $(-1, 1)$
$(1, \infty)$	$-$	f is decreasing on $(0, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1 and a local maximum value at 1 . The local minimum value is $f(-1) = -\frac{1}{2}$, and the

$$\text{local maximum value is } f(1) = \frac{1}{2}.$$

Step 6 The second derivative exists everywhere and is equal to 0 when $x = 0$ and when $x = \pm\sqrt{3}$. Use these numbers to divide the number line into four intervals, and determine the sign of f'' on each interval.

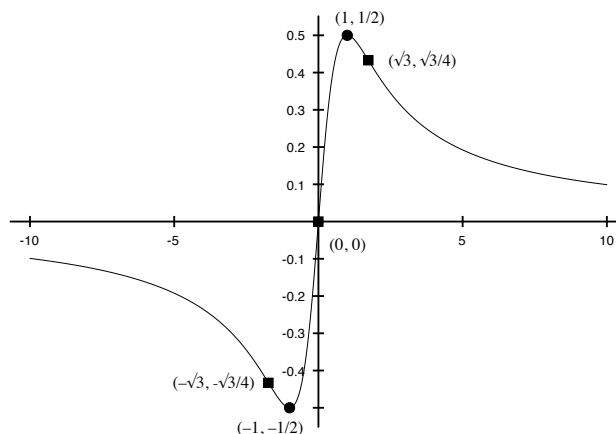
Interval	Sign of f''	Conclusion
$(-\infty, -\sqrt{3})$	$-$	f is concave down on $(-\infty, -\sqrt{3})$
$(-\sqrt{3}, 0)$	$+$	f is concave up on $(-\sqrt{3}, 0)$
$(0, \sqrt{3})$	$-$	f is concave down on $(0, \sqrt{3})$
$(\sqrt{3}, \infty)$	$+$	f is concave up on $(\sqrt{3}, \infty)$

The concavity of f changes at $-\sqrt{3}$, 0 , and $\sqrt{3}$, so the points

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4} \right), \quad (0, 0), \quad \text{and} \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4} \right)$$

are points of inflection of f .

Step 7 The figure below displays the graph of f . Local extreme values are highlighted by closed circles, and points of inflection are highlighted by closed squares.



10. Let $f(x) = \frac{2x}{x^2 - 4}$.

Step 1 The domain of the rational function f is the set $\{x | x \neq \pm 2\}$. The x -intercept is 0 , and the y -intercept is $f(0) = 0$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{2x}{x^2 - 4} = \lim_{x \rightarrow \pm\infty} \frac{2x}{x^2 - 4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2}{x}}{1 - \frac{4}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -2^-} \frac{2x}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{2x}{x^2 - 4} = \infty,$$

and

$$\lim_{x \rightarrow 2^-} \frac{2x}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{2x}{x^2 - 4} = \infty,$$

the lines $x = \pm 2$ are vertical asymptotes.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{2x}{x^2 - 4} \right) = \frac{(x^2 - 4) \cdot 2 - 2x \cdot 2x}{(x^2 - 4)^2} = -\frac{2x^2 + 8}{(x^2 - 4)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[-\frac{2x^2 + 8}{(x^2 - 4)^2} \right] = -\frac{(x^2 - 4)^2 \cdot 4x - (2x^2 + 8) \cdot 2(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= -\frac{4x(x^2 - 4) - 4x(2x^2 + 8)}{(x^2 - 4)^3} = \frac{4x(x^2 + 12)}{(x^2 - 4)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = \pm 2$. However, ± 2 are not in the domain of f , so ± 2 are not critical numbers. Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers ± 2 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	—	f is decreasing on $(-\infty, -2)$
$(-2, 2)$	—	f is decreasing on $(-2, 2)$
$(2, \infty)$	—	f is decreasing on $(2, \infty)$

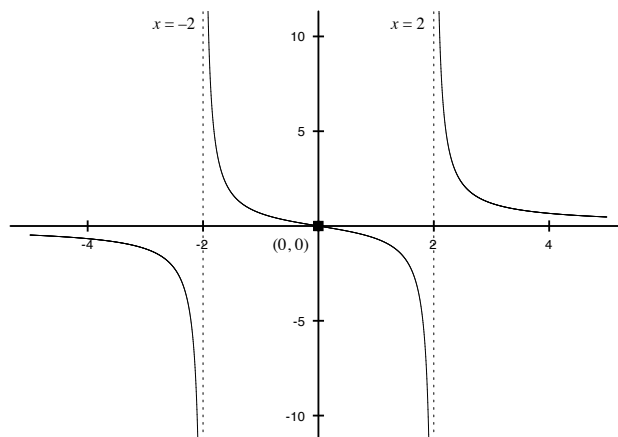
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is equal to zero when $x = 0$ and does not exist when $x = \pm 2$. Use these numbers to divide the number line into four intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	—	f is concave down on $(-\infty, -2)$
$(-2, 0)$	+	f is concave up on $(-2, 0)$
$(0, 2)$	—	f is concave down on $(0, 2)$
$(2, \infty)$	+	f is concave up on $(2, \infty)$

The concavity of f changes at 0, so the point $(0, 0)$ is a point of inflection of f . Although the concavity of f changes at ± 2 , there is no point of inflection at ± 2 because ± 2 are not in the domain of f .

Step 7 The figure below displays the graph of f . The inflection point is highlighted by a closed square.



11. Let $f(x) = \frac{x^2 + 1}{2x}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. There are no x -intercepts because $x^2 + 1$ is never equal to zero for any real x , and there is no y -intercept because f is not defined at 0.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2 + 1}{2x} = \frac{1}{2}x + \frac{1}{2x},$$

it follows that

$$\lim_{x \rightarrow \infty} \left[f(x) - \frac{1}{2}x \right] = \lim_{x \rightarrow \infty} \frac{1}{2x} = 0$$

so the line $y = \frac{1}{2}x$ is an oblique asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^2 + 1}{2x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^2 + 1}{2x} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2 + 1}{2x} \right) = \frac{2x \cdot 2x - (x^2 + 1) \cdot 2}{4x^2} = \frac{2x^2 - 2}{4x^2} = \frac{(x-1)(x+1)}{2x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{x^2 - 1}{2x^2} \right] = \frac{2x^2 \cdot 2x - (x^2 - 1) \cdot 4x}{4x^4} = \frac{4x}{4x^4} = \frac{1}{x^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \pm 1$ and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, ± 1 are the only critical numbers of f . At the points $(-1, -1)$ and $(1, 1)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	+	f is increasing on $(-\infty, -1)$
$(-1, 0)$	-	f is decreasing on $(-1, 0)$
$(0, 1)$	-	f is decreasing on $(0, 1)$
$(1, \infty)$	+	f is increasing on $(1, \infty)$

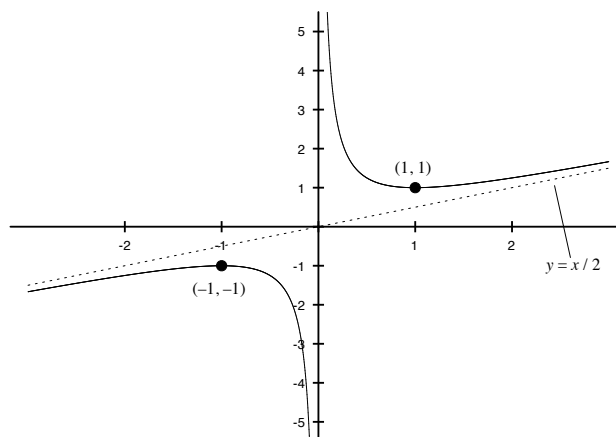
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at -1 and a local minimum value at 1 . The local maximum value is $f(-1) = -1$, and the local minimum value is $f(1) = 1$.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	-	f is concave down on $(-\infty, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

Although the concavity of f changes at 0 , there is no point of inflection at 0 because 0 is not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by a closed circles.



12. Let $f(x) = \frac{x^2 - 1}{2x}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. The x -intercepts are ± 1 , and there is no y -intercept because f is not defined at 0 .

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2 - 1}{2x} = \frac{1}{2}x - \frac{1}{2x},$$

it follows that

$$\lim_{x \rightarrow \infty} \left[f(x) - \frac{1}{2}x \right] = \lim_{x \rightarrow \infty} \left(-\frac{1}{2x} \right) = 0$$

so the line $y = \frac{1}{2}x$ is an oblique asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^2 - 1}{2x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^2 - 1}{2x} = -\infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2 - 1}{2x} \right) = \frac{2x \cdot 2x - (x^2 - 1) \cdot 2}{4x^2} = \frac{2x^2 + 2}{4x^2} = \frac{x^2 + 1}{2x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{x^2 + 1}{2x^2} \right] = \frac{2x^2 \cdot 2x - (x^2 + 1) \cdot 4x}{4x^4} = -\frac{4x}{4x^4} = -\frac{1}{x^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, f has **no critical numbers**.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

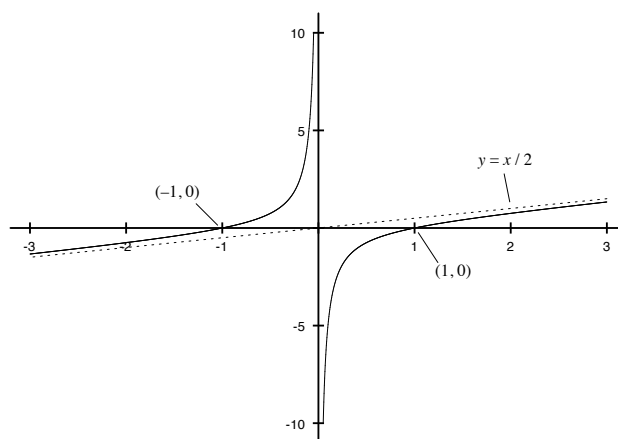
Step 5 Because there are no critical numbers, f has **no local extreme values**.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	+	f is concave up on $(-\infty, 0)$
$(0, \infty)$	-	f is concave down on $(0, \infty)$

Although the concavity of f changes at 0, there is **no point of inflection** at 0 because 0 is not in the domain of f .

Step 7 The figure below displays the graph of f .



13. Let $f(x) = \frac{x^4 + 1}{x^2}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. There are no x -intercepts because $x^4 + 1$ is never equal to zero for any real x , and there is no y -intercept because f is not defined at 0.

Step 2 The degree of the numerator is two more than the degree of the denominator, so the graph of f has no horizontal asymptote, but the end behavior of the graph of f will resemble the power function $y = x^2$. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^4 + 1}{x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^4 + 1}{x^2} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^4 + 1}{x^2} \right) = \frac{x^2 \cdot 4x^3 - (x^4 + 1) \cdot 2x}{x^4} = \frac{2x^5 - 2x}{x^4} = \frac{2(x^4 - 1)}{x^3} \\ &= \frac{2(x-1)(x+1)(x^2+1)}{x^3}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{2(x^4 - 1)}{x^3} \right] = \frac{x^3 \cdot 8x^3 - 2(x^4 - 1) \cdot 3x^2}{x^6} = \frac{2x^6 + 6x^2}{x^6} = \frac{2x^4 + 6}{x^4}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \pm 1$ and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, ± 1 are the only critical numbers of f . At the points $(-1, 2)$ and $(1, 2)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and ± 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	–	f is decreasing on $(-\infty, -1)$
$(-1, 0)$	+	f is increasing on $(-1, 0)$
$(0, 1)$	–	f is decreasing on $(0, 1)$
$(1, \infty)$	+	f is increasing on $(1, \infty)$

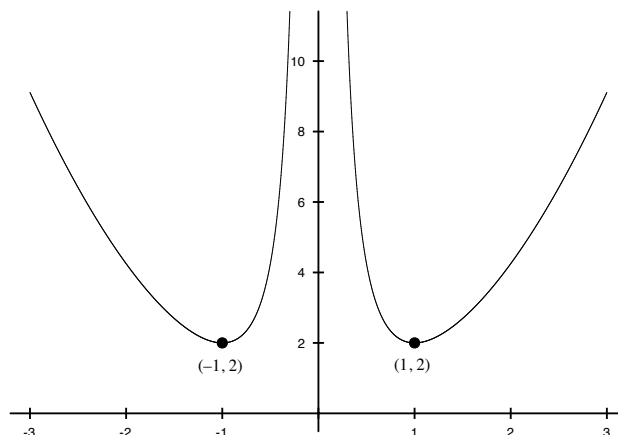
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1 and a local minimum value at 1. The local minimum values are $f(-1) = 2$ and $f(1) = 2$.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	+	f is concave up on $(-\infty, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

Because the concavity of f does not change, f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles.



14. Let $f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x + 1)(x^2 - x + 1)}{x + 1} = x^2 - x + 1$ for $x \neq -1$.

Step 1 The domain of the rational function f is the set $\{x | x \neq -1\}$. There are no x -intercepts because

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

is never equal to zero for any real x , and the y -intercept is $f(0) = 1$.

Step 2 The degree of the numerator is two more than the degree of the denominator, so the graph of f has no horizontal asymptote, but the end behavior of the graph of f will resemble the power function $y = x^2$. To identify vertical asymptotes, check the limit at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} = \lim_{x \rightarrow -1} \frac{(x + 1)(x^2 - x + 1)}{x + 1} = \lim_{x \rightarrow -1} (x^2 - x + 1) = 3,$$

the graph of f does not have a vertical asymptote at $x = -1$, but does have a missing point at $(-1, 3)$.

Step 3 Now, for $x \neq -1$

$$\begin{aligned} f'(x) &= \frac{d}{dx} (x^2 - x + 1) = 2x - 1; \text{ and} \\ f''(x) &= \frac{d}{dx} [2x - 1] = 2. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \frac{1}{2}$ and does not exist when $x = -1$. However, -1 is not in the domain of f , so -1 is

not a critical number. Therefore, $\frac{1}{2}$ is the only critical number of f . At the point $\left(\frac{1}{2}, \frac{3}{4}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -1 and $\frac{1}{2}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, \frac{1}{2})$	$-$	f is decreasing on $(-1, \frac{1}{2})$
$(\frac{1}{2}, \infty)$	$+$	f is increasing on $(\frac{1}{2}, \infty)$

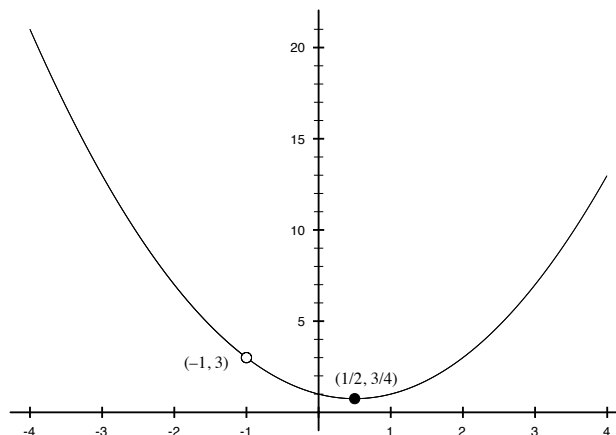
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $\frac{1}{2}$. The local minimum value is $f\left(\frac{1}{2}\right) = \frac{3}{4}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = -1$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, \infty)$	+	f is concave up on $(-1, \infty)$

Because the concavity of f does not change, f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



15. Let $xy = x^2 + 2$, so that $y = f(x) = \frac{x^2 + 2}{x}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. There are no x -intercepts because $x^2 + 2$ is never equal to zero for any real x , and there is no y -intercept because f is not defined at 0.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2 + 2}{x} = x + \frac{2}{x},$$

it follows that

$$\lim_{x \rightarrow \infty} [f(x) - x] = \lim_{x \rightarrow \infty} \frac{2}{x} = 0$$

so the line $y = x$ is an oblique asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^2 + 2}{x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^2 + 2}{x} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2 + 2}{x} \right) = \frac{x \cdot 2x - (x^2 + 2) \cdot 1}{x^2} = \frac{x^2 - 2}{x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{x^2 - 2}{x^2} \right] = \frac{x^2 \cdot 2x - (x^2 - 2) \cdot 2x}{x^4} = \frac{4x}{x^4} = \frac{4}{x^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \pm\sqrt{2}$ and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, $\pm\sqrt{2}$ are the only critical numbers of f . At the points $(-\sqrt{2}, -2\sqrt{2})$ and $(\sqrt{2}, 2\sqrt{2})$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and $\pm\sqrt{2}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\sqrt{2})$	+	f is increasing on $(-\infty, -\sqrt{2})$
$(-\sqrt{2}, 0)$	-	f is decreasing on $(-\sqrt{2}, 0)$
$(0, \sqrt{2})$	-	f is decreasing on $(0, \sqrt{2})$
$(\sqrt{2}, \infty)$	+	f is increasing on $(\sqrt{2}, \infty)$

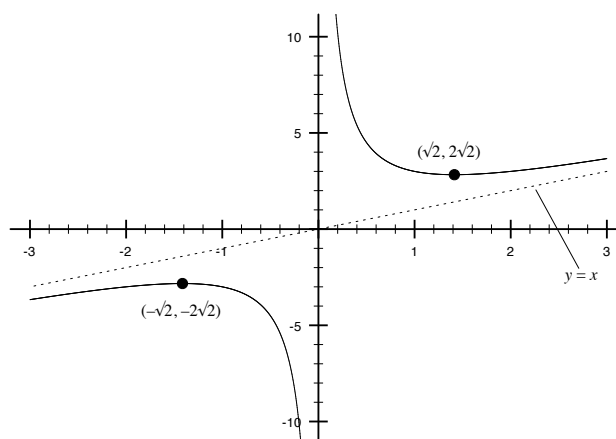
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $-\sqrt{2}$ and a local minimum value at $\sqrt{2}$. The local maximum value is $f(-\sqrt{2}) = -2\sqrt{2}$, and the local minimum value is $f(\sqrt{2}) = 2\sqrt{2}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	-	f is concave down on $(-\infty, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

Although the concavity of f changes at 0, there is no point of inflection at 0 because 0 is not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles.



16. Let $xy = x^2 + x - 1$, so that $y = f(x) = \frac{x^2 + x - 1}{x}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. The x -intercepts are the solutions of the equation $x^2 + x - 1 = 0$. By the quadratic formula

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2},$$

so the x -intercepts are $\frac{-1 \pm \sqrt{5}}{2}$. There is **no y -intercept** because f is not defined at 0.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2 + x - 1}{x} = x + 1 - \frac{1}{x},$$

it follows that

$$\lim_{x \rightarrow \infty} [f(x) - (x + 1)] = \lim_{x \rightarrow \infty} \left(-\frac{1}{x} \right) = 0$$

so the line **$y = x + 1$ is an oblique asymptote**. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^2 + x - 1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^2 + x - 1}{x} = -\infty,$$

the line **$x = 0$ is a vertical asymptote**.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2 + x - 1}{x} \right) = \frac{x \cdot (2x + 1) - (x^2 + x - 1) \cdot 1}{x^2} = \frac{x^2 + 1}{x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{x^2 + 1}{x^2} \right] = \frac{x^2 \cdot 2x - (x^2 + 1) \cdot 2x}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, f has **no critical numbers**.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

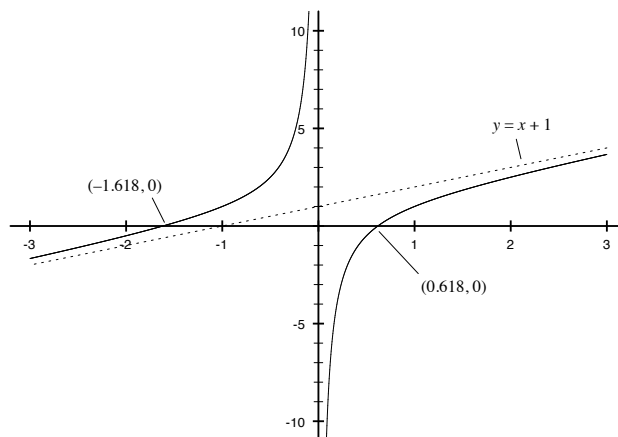
Step 5 Because there are no critical numbers, f has **no local extreme values**.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	+	f is concave up on $(-\infty, 0)$
$(0, \infty)$	-	f is concave down on $(0, \infty)$

Although the concavity of f changes at 0, there is **no point of inflection** at 0 because 0 is not in the domain of f .

Step 7 The figure below displays the graph of f .



17. Let $f(x) = \frac{x^2}{x+3}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq -3\}$. The x -intercept is 0, and the y -intercept is $f(0) = 0$.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{x^2}{x+3} = x - 3 + \frac{9}{x+3},$$

it follows that

$$\lim_{x \rightarrow \infty} [f(x) - (x - 3)] = \lim_{x \rightarrow \infty} \frac{9}{x+3} = 0$$

so the line $y = x - 3$ is an oblique asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -3^-} \frac{x^2}{x+3} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -3^+} \frac{x^2}{x+3} = \infty,$$

the line $x = -3$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2}{x+3} \right) = \frac{(x+3) \cdot 2x - x^2 \cdot 1}{(x+3)^2} = \frac{x^2 + 6x}{(x+3)^2} = \frac{x(x+6)}{(x+3)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{x^2 + 6x}{(x+3)^2} \right] = \frac{(x+3)^2 \cdot (2x+6) - (x^2 + 6x) \cdot 2(x+3)}{(x+3)^4} \\ &= \frac{2(x+3)^2 - 2(x^2 + 6x)}{(x+3)^3} = \frac{18}{(x+3)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and $x = -6$ and does not exist when $x = -3$. However, -3 is not in the domain of f , so -3 is not a critical number. Therefore, -6 and 0 are the only critical numbers of f . At the points $(-6, -12)$ and $(0, 0)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -6 , -3 , and 0 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -6)$	+	f is increasing on $(-\infty, -6)$
$(-6, -3)$	-	f is decreasing on $(-6, -3)$
$(-3, 0)$	-	f is decreasing on $(-3, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

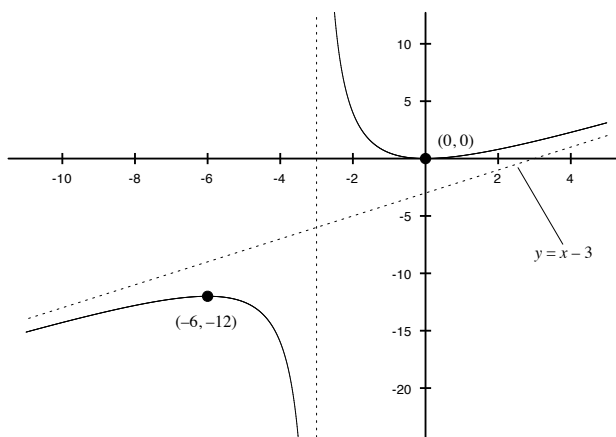
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at -6 and a local minimum value at 0 . The $\boxed{\text{local maximum value is } f(-6) = -12}$, and the $\boxed{\text{local minimum value is } f(0) = 0}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = -3$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -3)$	-	f is concave down on $(-\infty, -3)$
$(-3, \infty)$	+	f is concave up on $(-3, \infty)$

Although the concavity of f changes at -3 , there is $\boxed{\text{no point of inflection}}$ at -3 because -3 is not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles.



18. Let $f(x) = \frac{3x^2 - 1}{x - 1}$.

Step 1 The domain of the rational function f is the set $\boxed{\{x|x \neq 1\}}$. The $\boxed{x\text{-intercepts are } \pm \frac{\sqrt{3}}{3}}$, and the $\boxed{y\text{-intercept is } f(0) = 1}$.

Step 2 The degree of the numerator is one more than the degree of the denominator, so the graph of f has no horizontal asymptote but does have an oblique asymptote. Because

$$f(x) = \frac{3x^2 - 1}{x - 1} = 3x + 3 + \frac{2}{x - 1},$$

it follows that

$$\lim_{x \rightarrow \infty} [f(x) - (3x + 3)] = \lim_{x \rightarrow \infty} \frac{2}{x - 1} = 0$$

so the line $y = 3x + 3$ is an oblique asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 1^-} \frac{3x^2 - 1}{x - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{3x^2 - 1}{x - 1} = \infty,$$

the line $x = 1$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{3x^2 - 1}{x - 1} \right) = \frac{(x - 1) \cdot 6x - (3x^2 - 1) \cdot 1}{(x - 1)^2} = \frac{3x^2 - 6x + 1}{(x - 1)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[\frac{3x^2 - 6x + 1}{(x - 1)^2} \right] = \frac{(x - 1)^2 \cdot (6x - 6) - (3x^2 - 6x + 1) \cdot 2(x - 1)}{(x - 1)^4} \\ &= \frac{6(x - 1)^2 - 2(3x^2 - 6x + 1)}{(x - 1)^3} = \frac{4}{(x - 1)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $3x^2 - 6x + 1 = 0$. By the quadratic formula,

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(1)}}{6} = \frac{6 \pm \sqrt{24}}{6} = 1 \pm \frac{\sqrt{6}}{3},$$

so $1 \pm \frac{\sqrt{6}}{3}$ are critical numbers of f . Although f' does not exist at 1, 1 is not in the domain of f , so 1 is not a critical number. At the points

$$\left(1 - \frac{\sqrt{6}}{3}, 6 - 2\sqrt{6} \right) \quad \text{and} \quad \left(1 + \frac{\sqrt{6}}{3}, 6 + 2\sqrt{6} \right)$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 1 and $1 \pm \frac{\sqrt{6}}{3}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 1 - \frac{\sqrt{6}}{3})$	+	f is increasing on $(-\infty, 1 - \frac{\sqrt{6}}{3})$
$(1 - \frac{\sqrt{6}}{3}, 1)$	-	f is decreasing on $(1 - \frac{\sqrt{6}}{3}, 1)$
$(1, 1 + \frac{\sqrt{6}}{3})$	-	f is decreasing on $(1, 1 + \frac{\sqrt{6}}{3})$
$(1 + \frac{\sqrt{6}}{3}, \infty)$	+	f is increasing on $(1 + \frac{\sqrt{6}}{3}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $1 - \frac{\sqrt{6}}{3}$ and a local minimum value at $1 + \frac{\sqrt{6}}{3}$. The

$$\text{local maximum value is } f\left(1 - \frac{\sqrt{6}}{3}\right) = 6 - 2\sqrt{6},$$

and the

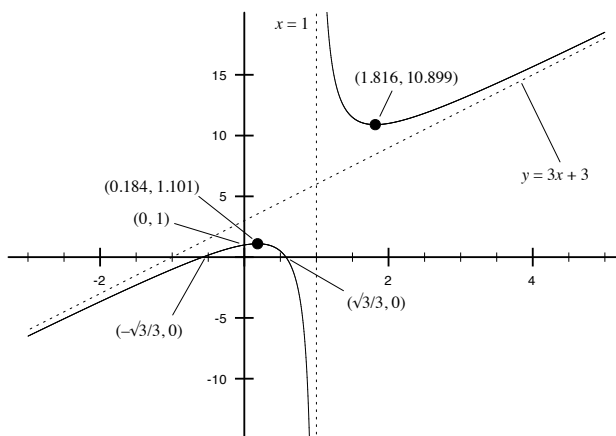
$$\text{local minimum value is } f\left(1 + \frac{\sqrt{6}}{3}\right) = 6 + 2\sqrt{6}.$$

Step 6 The second derivative is never equal to zero and does not exist when $x = 1$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 1)$	$-$	f is concave down on $(-\infty, 1)$
$(1, \infty)$	$+$	f is concave up on $(1, \infty)$

Although the concavity of f changes at 1, there is no point of inflection at 1 because 1 is not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles.



19. Let $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2} = \frac{x^2 + x + 1}{x^2}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. There are no x -intercepts because

$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ is never equal to zero for any real x , and there is no y -intercept because f is not defined at 0.

Step 2 The degree of the numerator is equal to the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = 1,$$

the line $y = 1$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{x^2 + x + 1}{x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{x^2 + x + 1}{x^2} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) = -\frac{1}{x^2} - \frac{2}{x^3} = -\frac{x+2}{x^3}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(-\frac{1}{x^2} - \frac{2}{x^3}\right) = \frac{2}{x^3} + \frac{6}{x^4} = \frac{2x+6}{x^4}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -2$ and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, -2 is the only critical number of f . At the point $\left(-2, \frac{3}{4}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -2 and 0 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$-$	f is decreasing on $(-\infty, -2)$
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, \infty)$	$-$	f is decreasing on $(0, \infty)$

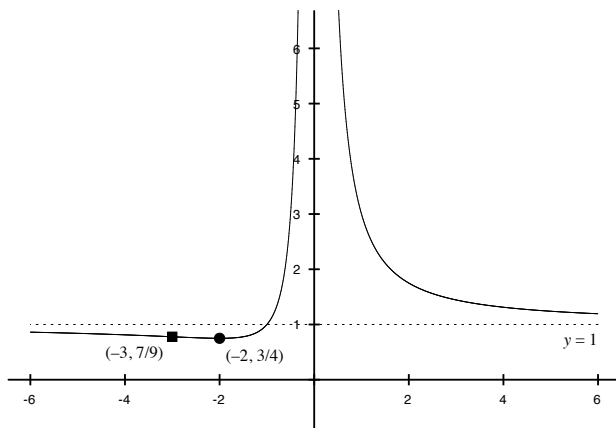
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -2 . The local minimum value is $f(-2) = \frac{3}{4}$.

Step 6 The second derivative is equal to zero when $x = -3$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -3)$	$-$	f is concave down on $(-\infty, -3)$
$(-3, 0)$	$+$	f is concave up on $(-3, 0)$
$(0, \infty)$	$+$	f is concave up on $(0, \infty)$

The concavity of f changes at -3 so the point $\left(-3, \frac{7}{9}\right)$ is a point of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



20. Let $f(x) = \frac{2}{x} + \frac{1}{x^2} = \frac{2x+1}{x^2}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq 0\}$. The x -intercept is $-\frac{1}{2}$, and there is no y -intercept because f is not defined at 0 .

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \left(\frac{2}{x} + \frac{1}{x^2} \right) = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} \frac{2x+1}{x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{2x+1}{x^2} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x} + \frac{1}{x^2} \right) = -\frac{2}{x^2} - \frac{2}{x^3} = -\frac{2x+2}{x^3}; \text{ and}$$

$$f''(x) = \frac{d}{dx} \left(-\frac{2}{x^2} - \frac{2}{x^3} \right) = \frac{4}{x^3} + \frac{6}{x^4} = \frac{4x+6}{x^4}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -1$ and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, -1 is the only critical number of f . At the point $(-1, -1)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -1 and 0 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, 0)$	$+$	f is increasing on $(-1, 0)$
$(0, \infty)$	$-$	f is decreasing on $(0, \infty)$

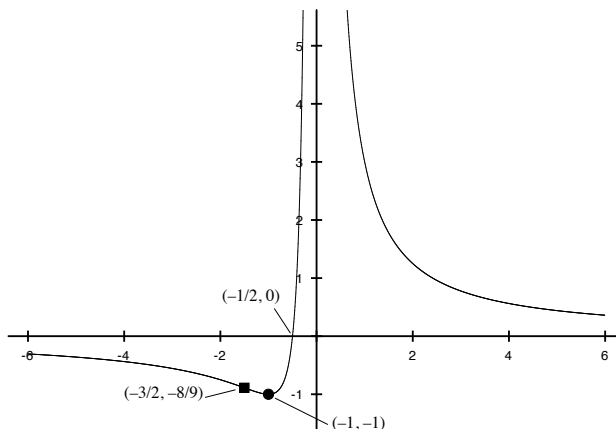
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1 . The local minimum value is $f(-1) = -1$.

Step 6 The second derivative is equal to zero when $x = -\frac{3}{2}$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{3}{2})$	$-$	f is concave down on $(-\infty, -\frac{3}{2})$
$(-\frac{3}{2}, 0)$	$+$	f is concave up on $(-\frac{3}{2}, 0)$
$(0, \infty)$	$+$	f is concave up on $(0, \infty)$

The concavity of f changes at $-\frac{3}{2}$ so the point $(-\frac{3}{2}, -\frac{8}{9})$ is a point of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



21. Let $f(x) = \sqrt{3-x}$.

Step 1 The domain of f is given by the solution to the inequality $3 - x \geq 0$; that is, the set $\{x|x \leq 3\}$.

The x -intercept is 3, and the y -intercept is $f(0) = \sqrt{3}$.

Step 2 Because

$$\lim_{x \rightarrow -\infty} \sqrt{3 - x} = \infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= -\frac{1}{2}(3-x)^{-1/2} = -\frac{1}{2\sqrt{3-x}}; \text{ and} \\ f''(x) &= -\frac{1}{4}(3-x)^{-3/2} = -\frac{1}{4\sqrt{(3-x)^3}}. \end{aligned}$$

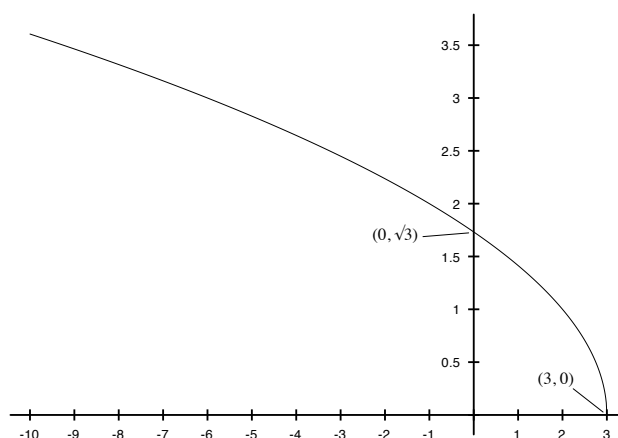
The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 3$. Therefore, 3 is the only critical number of f . At the point $(3, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) < 0$ for all $x < 3$, f is decreasing on the interval $(-\infty, 3)$.

Step 5 Because the only critical number of f is an endpoint of the domain of f , f has no local extreme values.

Step 6 Because $f''(x) < 0$ for all $x < 3$, f is concave down on the interval $(-\infty, 3)$. As the concavity of f never changes, f has no points of inflection.

Step 7 The figure below displays the graph of f .



22. Let $f(x) = x\sqrt{x+2}$.

Step 1 The domain of f is given by the solution to the inequality $x+2 \geq 0$; that is, the set $\{x|x \geq -2\}$.

The x -intercepts are -2 and 0 , and the y -intercept is $f(0) = 0$.

Step 2 Because

$$\lim_{x \rightarrow \infty} x\sqrt{x+2} = \infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x\sqrt{x+2}) = x \cdot \frac{1}{2\sqrt{x+2}} + \sqrt{x+2} = \frac{3x+4}{2\sqrt{x+2}}; \text{ and} \\ f''(x) &= \frac{d}{dx}\left(\frac{3x+4}{2\sqrt{x+2}}\right) = \frac{2\sqrt{x+2} \cdot 3 - (3x+4) \cdot (x+2)^{-1/2}}{4(x+2)} \\ &= \frac{6(x+2) - (3x+4)}{4(x+2)^{3/2}} = \frac{3x+8}{4(x+2)^{3/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -\frac{4}{3}$ and does not exist when $x = -2$. Therefore, -2 and $-\frac{4}{3}$ are the critical numbers of f . At the point $(-2, 0)$, the tangent line is vertical; at the point $\left(-\frac{4}{3}, -\frac{4\sqrt{6}}{9}\right)$, the tangent line is horizontal.

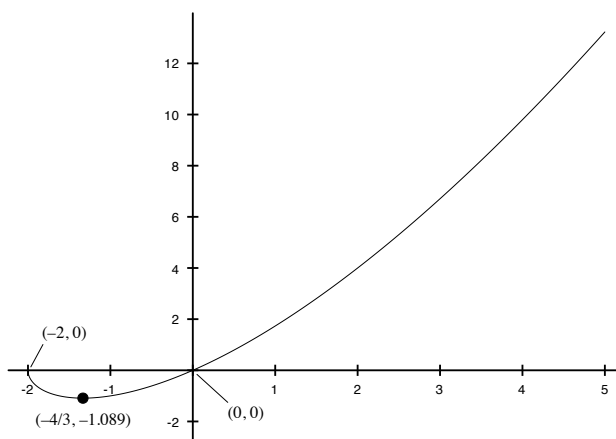
Step 4 To apply the Increasing/Decreasing Function Test, use the number $-\frac{4}{3}$ to divide $[-2, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(-2, -\frac{4}{3})$	$-$	f is decreasing on $(-2, -\frac{4}{3})$
$(-\frac{4}{3}, \infty)$	$+$	f is increasing on $(-\frac{4}{3}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\frac{4}{3}$. The local minimum value is $f\left(-\frac{4}{3}\right) = -\frac{4\sqrt{6}}{9}$.

Step 6 Because $f''(x) > 0$ for all $x > -2$, f is concave up on the interval $(-2, \infty)$. As the concavity of f never changes, f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



23. Let $f(x) = x + \sqrt{x}$.

Step 1 The domain of f is the set $\{x|x \geq 0\}$. The x -intercept is 0, and the y -intercept is $f(0) = 0$.

Step 2 Because

$$\lim_{x \rightarrow \infty} (x + \sqrt{x}) = \infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= 1 + \frac{1}{2\sqrt{x}}; \text{ and} \\ f''(x) &= -\frac{1}{4x^{3/2}}. \end{aligned}$$

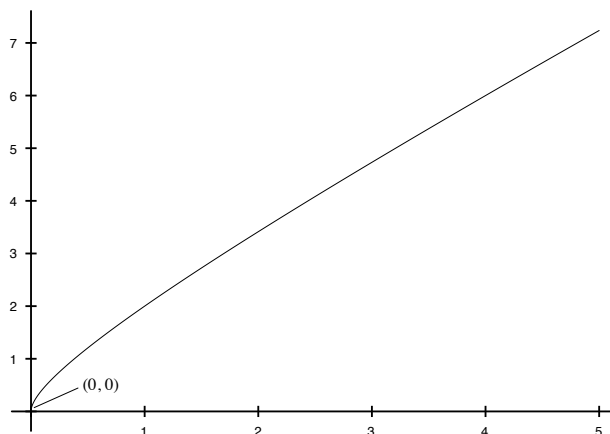
The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = 0$. Therefore, $\boxed{0}$ is the only critical number of f . At the point $(0, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) > 1 > 0$ for all $x > 0$, f is increasing on the interval $(0, \infty)$.

Step 5 Because the only critical number of f is an endpoint of the domain of f , f has no local extreme values.

Step 6 Because $f''(x) < 0$ for all $x > 0$, f is concave down on the interval $(0, \infty)$. As the concavity of f never changes, f has no points of inflection.

Step 7 The figure below displays the graph of f .



24. Let $f(x) = \sqrt{x} - \sqrt{x+1}$.

Step 1 The domain of f is the set $\{x|x \geq 0\}$. There are no x -intercepts, and the y -intercept is $f(0) = -1$.

Step 2 Because

$$\lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x+1}) = \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x+1}) \cdot \frac{\sqrt{x} + \sqrt{x+1}}{\sqrt{x} + \sqrt{x+1}} = \lim_{x \rightarrow \infty} \frac{-1}{\sqrt{x} + \sqrt{x+1}} = 0,$$

the line $y = 0$ is a horizontal asymptote. The graph of f has no vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x+1}}; \text{ and} \\ f''(x) &= -\frac{1}{4x^{3/2}} + \frac{1}{4(x+1)^{3/2}}. \end{aligned}$$

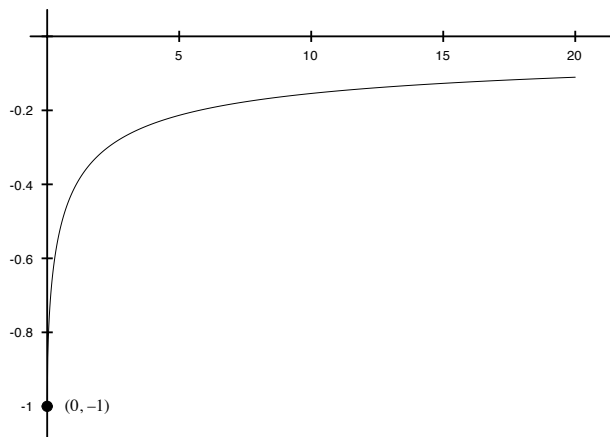
The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and does not exist when $x = -1$ and when $x = 0$. As -1 is not in the domain of f , -1 is not a critical number of f . Therefore, $\boxed{0}$ is the only critical number of f . At the point $(0, -1)$, the tangent line is vertical.

Step 4 Because $f'(x) > 0$ for all $x > 0$, f is increasing on the interval $(0, \infty)$.

Step 5 Because the only critical number of f is an endpoint of the domain of f , f has no local extreme values.

Step 6 Because $f''(x) < 0$ for all $x > 0$, f is concave down on the interval $(0, \infty)$. As the concavity of f never changes, f has no points of inflection.

Step 7 The figure below displays the graph of f .



25. Let $f(x) = \frac{x^2}{\sqrt{x+1}}$.

Step 1 The domain of f is given by the solution to the inequality $x+1 > 0$; that is, the set $\{x|x > -1\}$. The x -intercept is 0, and the y -intercept is $f(0) = 0$.

Step 2 Because

$$\lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x+1}} = \infty,$$

the graph of f does not have a horizontal asymptote. On the other hand,

$$\lim_{x \rightarrow -1^+} \frac{x^2}{\sqrt{x+1}} = \infty,$$

so the line $x = -1$ is a vertical asymptote.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^2}{\sqrt{x+1}} \right) = \frac{\sqrt{x+1} \cdot 2x - x^2 \cdot \frac{1}{2}(x+1)^{-1/2}}{x+1} \\ &= \frac{4x(x+1) - x^2}{2(x+1)^{3/2}} = \frac{3x^2 + 4x}{2(x+1)^{3/2}}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(\frac{3x^2 + 4x}{2(x+1)^{3/2}} \right) = \frac{2(x+1)^{3/2} \cdot (6x+4) - (3x^2 + 4x) \cdot 3(x+1)^{1/2}}{4(x+1)^3} \\ &= \frac{2(x+1)(6x+4) - 3(3x^2 + 4x)}{4(x+1)^{5/2}} = \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 0$ and when $x = -\frac{4}{3}$ and does not exist when $x = -1$. As $-\frac{4}{3}$ and -1 are not in the

domain of f , $-\frac{4}{3}$ and -1 are not critical numbers of f . Therefore, $\boxed{0}$ is the only critical number of f . At the point $(0, 0)$, the tangent line is horizontal.

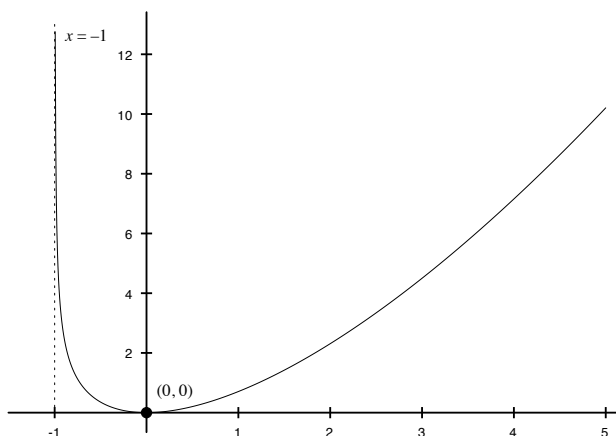
Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide $(-1, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(-1, 0)$	$-$	f is decreasing on $(-1, 0)$
$(0, \infty)$	$+$	f is increasing on $(0, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at 0. The $\boxed{\text{local minimum value is } f(0) = 0}$.

Step 6 Because $f''(x) > 0$ for all $x > -1$, f is $\boxed{\text{concave up on the interval } (-1, \infty)}$. As the concavity of f never changes, f has $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



26. Let $f(x) = \frac{x}{\sqrt{x^2 + 2}}$.

Step 1 Because $x^2 + 2 \geq 2 > 0$ for all x , the domain of f is the set of $\boxed{\text{all real numbers}}$. The $\boxed{x\text{-intercept is } 0}$, and the $\boxed{y\text{-intercept is } f(0) = 0}$.

Step 2 Because

$$\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow -\infty} \frac{x}{|x|\sqrt{1 + \frac{2}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 + \frac{2}{x^2}}} = -1$$

and

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow \infty} \frac{x}{|x|\sqrt{1 + \frac{2}{x^2}}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{2}{x^2}}} = 1,$$

the lines $\boxed{y = \pm 1}$ are horizontal asymptotes. Because f is defined for all real numbers, the graph of f $\boxed{\text{does not have any vertical asymptotes}}$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x}{\sqrt{x^2 + 2}} \right) = \frac{\sqrt{x^2 + 2} \cdot 1 - x \cdot \frac{1}{2}(x^2 + 2)^{-1/2}(2x)}{x^2 + 2} \\ &= \frac{x^2 + 2 - x^2}{(x^2 + 2)^{3/2}} = \frac{2}{(x^2 + 2)^{3/2}}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(\frac{2}{(x^2 + 2)^{3/2}} \right) = -3(x^2 + 2)^{-5/2}(2x) = -\frac{6x}{(x^2 + 2)^{5/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to 0 and exists for all x , so f has no critical numbers.

Step 4 Because $f'(x) > 0$ for all x , f is increasing on the interval $(-\infty, \infty)$.

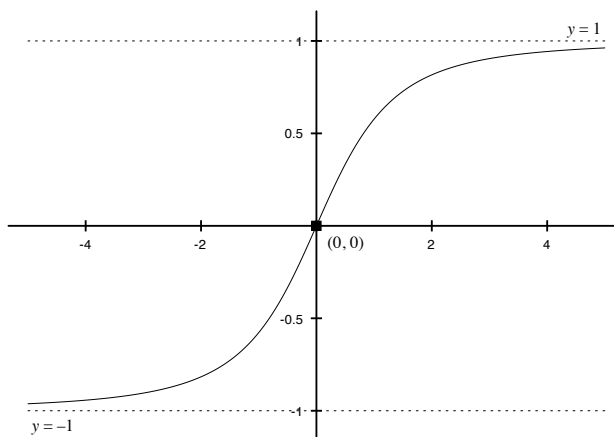
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is equal to zero when $x = 0$ and exists for all x . Use the number 0 to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	+	f is concave up on $(-\infty, 0)$
$(0, \infty)$	-	f is concave down on $(0, \infty)$

The concavity of f changes at 0 so the point $(0, 0)$ is a point of inflection of f .

Step 7 The figure below displays the graph of f . The point of inflection is highlighted by a closed square.



27. Let $f(x) = \frac{1}{(x+1)(x-2)} = \frac{1}{x^2 - x - 2}$.

Step 1 The domain of the rational function f is the set $\{x | x \neq -1, x \neq 2\}$. There are no x -intercepts, and the y -intercept is $f(0) = -\frac{1}{2}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - x - 2} = \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - x - 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -1^-} \frac{1}{(x+1)(x-2)} = \infty \quad \text{and} \quad \lim_{x \rightarrow -1^+} \frac{1}{(x+1)(x-2)} = -\infty,$$

and

$$\lim_{x \rightarrow 2^-} \frac{1}{(x+1)(x-2)} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{1}{(x+1)(x-2)} = \infty,$$

the lines $x = -1$ and $x = 2$ are vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{1}{x^2 - x - 2} \right) = -\frac{2x - 1}{(x^2 - x - 2)^2}; \text{ and} \\ f''(x) &= -\frac{d}{dx} \left(\frac{2x - 1}{(x^2 - x - 2)^2} \right) = -\frac{(x^2 - x - 2)^2 \cdot 2 - (2x - 1) \cdot 2(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4} \\ &= -\frac{2(x^2 - x - 2) - 2(4x^2 - 4x + 1)}{(x^2 - x - 2)^3} = \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \frac{1}{2}$ and does not exist when $x = -1$ and when $x = 2$. As -1 and 2 are not in the

domain of f , -1 and 2 are not critical numbers. Therefore, $\boxed{\frac{1}{2}}$ is the only critical number of f .

At the point $\left(\frac{1}{2}, -\frac{4}{9}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -1 , $\frac{1}{2}$, and 2 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	+	f is increasing on $(-\infty, -1)$
$(-1, \frac{1}{2})$	+	f is increasing on $(-1, \frac{1}{2})$
$(\frac{1}{2}, 2)$	−	f is decreasing on $(\frac{1}{2}, 2)$
$(2, \infty)$	−	f is decreasing on $(2, \infty)$

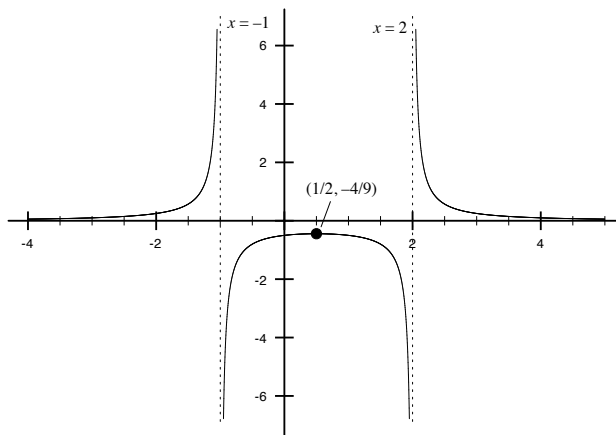
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{1}{2}$. The local maximum value is $f\left(\frac{1}{2}\right) = -\frac{4}{9}$.

Step 6 The second derivative is never equal to zero and does not exist for $x = -1$ and $x = 2$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, 2)$	−	f is concave down on $(-1, 2)$
$(2, \infty)$	+	f is concave up on $(2, \infty)$

Although the concavity of f changes at -1 and 2 , there is no point of inflection at either -1 or 2 because -1 and 2 are not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



28. Let $f(x) = \frac{1}{(x-1)(x+3)} = \frac{1}{x^2 + 2x - 3}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq -3, x \neq 1\}$. There are no x -intercepts,

and the y -intercept is $f(0) = -\frac{1}{3}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 + 2x - 3} = \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 + 2x - 3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x^2}}{1 + \frac{2}{x} - \frac{3}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -3^-} \frac{1}{(x-1)(x+3)} = \infty \quad \text{and} \quad \lim_{x \rightarrow -3^+} \frac{1}{(x-1)(x+3)} = -\infty,$$

and

$$\lim_{x \rightarrow 1^-} \frac{1}{(x-1)(x+3)} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{(x-1)(x+3)} = \infty,$$

the lines $x = -3$ and $x = 1$ are vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{1}{x^2 + 2x - 3} \right) = -\frac{2x + 2}{(x^2 + 2x - 3)^2}; \text{ and} \\ f''(x) &= -\frac{d}{dx} \left(\frac{2x + 2}{(x^2 + 2x - 3)^2} \right) = -\frac{(x^2 + 2x - 3)^2 \cdot 2 - (2x + 2) \cdot 2(x^2 + 2x - 3)(2x + 2)}{(x^2 + 2x - 3)^4} \\ &= -\frac{2(x^2 + 2x - 3) - 2(4x^2 + 8x + 4)}{(x^2 + 2x - 3)^3} = \frac{2(3x^2 + 6x + 7)}{(x^2 + 2x - 3)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -1$ and does not exist when $x = -3$ and when $x = 1$. As -3 and 1 are not in the domain of f , -3 and 1 are not critical numbers. Therefore, -1 is the only critical number of f .

At the point $\left(-1, -\frac{1}{4}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers -3 , -1 , and 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -3)$	+	f is increasing on $(-\infty, -3)$
$(-3, -1)$	+	f is increasing on $(-3, -1)$
$(-1, 1)$	-	f is decreasing on $(-1, 1)$
$(1, \infty)$	-	f is decreasing on $(1, \infty)$

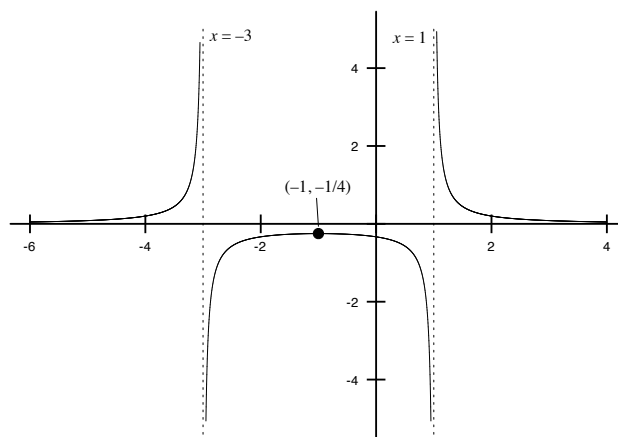
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at -1 . The local maximum value is $f(-1) = -\frac{1}{4}$.

Step 6 The second derivative is never equal to zero and does not exist for $x = -3$ and $x = 1$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -3)$	+	f is concave up on $(-\infty, -3)$
$(-3, 1)$	-	f is concave down on $(-3, 1)$
$(1, \infty)$	+	f is concave up on $(1, \infty)$

Although the concavity of f changes at -3 and 1 , there is no point of inflection at either -3 or 1 because -3 and 1 are not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



29. Let $f(x) = x^{2/3} + 3x^{1/3} + 2$.

Step 1 The domain of f is the set of all real numbers. Solving

$$x^{2/3} + 3x^{1/3} + 2 = (x^{1/3} + 2)(x^{1/3} + 1) = 0,$$

yields -8 and -1 as the x -intercepts; the y -intercept is $f(0) = 2$.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} (x^{2/3} + 3x^{1/3} + 2) = \lim_{x \rightarrow \pm\infty} \left[x^{2/3} \left(1 + \frac{3}{x^{1/3}} + \frac{2}{x^{2/3}} \right) \right] = \infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes because f is defined for all x .

Step 3 Now,

$$f'(x) = \frac{2}{3}x^{-1/3} + x^{-2/3} = \frac{2x^{1/3} + 3}{3x^{2/3}}; \text{ and}$$

$$f''(x) = -\frac{2}{9}x^{-4/3} - \frac{2}{3}x^{-5/3} = -\frac{2x^{1/3} + 6}{9x^{5/3}}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = -\frac{27}{8}$ and does not exist when $x = 0$. Therefore, $-\frac{27}{8}$ and 0 are critical numbers of f . At the point $\left(-\frac{27}{8}, -\frac{1}{4}\right)$, the tangent line is horizontal; at the point $(0, 2)$, the tangent line is vertical.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers $-\frac{27}{8}$ and 0 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{27}{8})$	−	f is decreasing on $(-\infty, -\frac{27}{8})$
$(-\frac{27}{8}, 0)$	+	f is increasing on $(-\frac{27}{8}, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\frac{27}{8}$ and has neither a local maximum value nor a local minimum value at 0. The

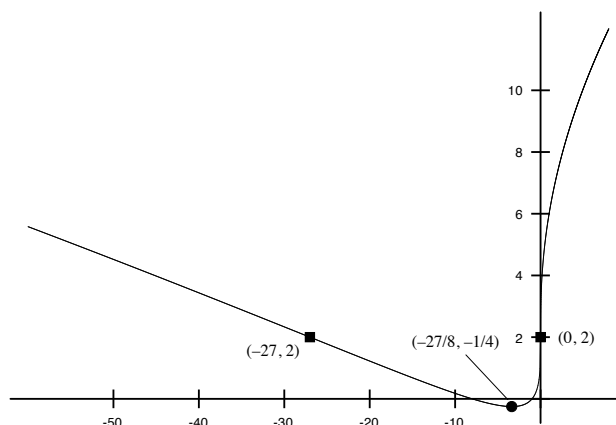
$$\text{local minimum value is } f\left(-\frac{27}{8}\right) = -\frac{1}{4}.$$

Step 6 The second derivative is equal to zero when $x = -27$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -27)$	−	f is concave down on $(-\infty, -27)$
$(-27, 0)$	+	f is concave up on $(-27, 0)$
$(0, \infty)$	−	f is concave down on $(0, \infty)$

The concavity of f changes at -27 and 0, so the points $(-27, 2)$ and $(0, 2)$ are points of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



30. Let $f(x) = x^{5/3} - 5x^{2/3}$.

Step 1 The domain of f is the set of all real numbers. Solving

$$x^{5/3} - 5x^{2/3} = x^{2/3}(x - 5) = 0,$$

yields 0 and 5 as the x -intercepts; the y -intercept is $f(0) = 0$.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} (x^{5/3} - 5x^{2/3}) = \lim_{x \rightarrow \pm\infty} \left[x^{5/3} \left(1 - \frac{5}{x} \right) \right] = \pm\infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes because f is defined for all x .

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5(x-2)}{3x^{1/3}}; \text{ and} \\ f''(x) &= \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10(x+1)}{9x^{4/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 2$ and does not exist when $x = 0$. Therefore, 0 and 2 are critical numbers of f . At the point $(0, 0)$, the tangent line is vertical; at the point $(2, -3\sqrt[3]{4})$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and 2 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, 2)$	−	f is decreasing on $(0, 2)$
$(2, \infty)$	+	f is increasing on $(2, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0 and a local minimum value at 2. The local maximum value is $f(0) = 0$, and the

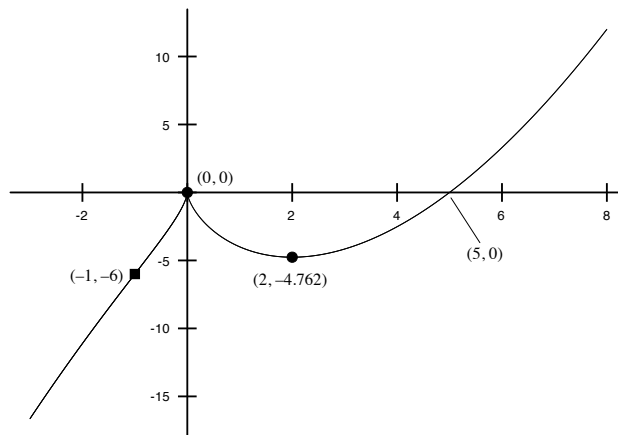
local minimum value is $f(2) = -3\sqrt[3]{4}$.

Step 6 The second derivative is equal to zero when $x = -1$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	−	f is concave down on $(-\infty, -1)$
$(-1, 0)$	+	f is concave up on $(-1, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

The concavity of f changes at -1 , so the point $(-1, -6)$ is a point of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles, and the point of inflection is highlighted by a closed square.



31. Let $f(x) = \sin x - \cos x$. Note the function f is periodic with period 2π .

Step 1 The domain of f is the set of all real numbers. The x -intercepts satisfy the equation

$$\sin x - \cos x = 0 \quad \text{or} \quad \tan x = 1;$$

the solutions to this equation are $\frac{\pi}{4} + k\pi$ for any integer k . The y -intercept is $f(0) = -1$.

Step 2 The function f has no asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \cos x + \sin x; \text{ and} \\ f''(x) &= -\sin x + \cos x. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$, which is where $\cos x = -\sin x$ or $\tan x = -1$. Therefore, the critical numbers of f are $\frac{3\pi}{4} + k\pi$ for any integer k . At the points $\left(\frac{3\pi}{4} + k\pi, (-1)^k\sqrt{2}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π . The results are shown in the table below.

Interval	Sign of f'	Conclusion
$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$	+	f is increasing on $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	-	f is decreasing on $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

Therefore, f is

increasing on intervals of the form $\left(-\frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi\right)$

and is

decreasing on intervals of the form $\left(\frac{3\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi\right)$

for any integer k .

Step 5 By the First Derivative Test and the information above, f has a local maximum value at $\frac{3\pi}{4} + 2k\pi$ and a local minimum value at $\frac{7\pi}{4} + 2k\pi$ for any integer k . The

$$\text{local maximum value is } f\left(\frac{3\pi}{4} + 2k\pi\right) = \sqrt{2},$$

and the

$$\text{local minimum value is } f\left(\frac{7\pi}{4} + 2k\pi\right) = -\sqrt{2}.$$

Step 6 The second derivative exists everywhere and is equal to zero when $\tan x = 1$, which is when $x = \frac{\pi}{4} + k\pi$ for any integer k . Consider the intervals $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$	+	f is concave up on $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	-	f is concave down on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Taking into account the period of the function f , it follows that f is

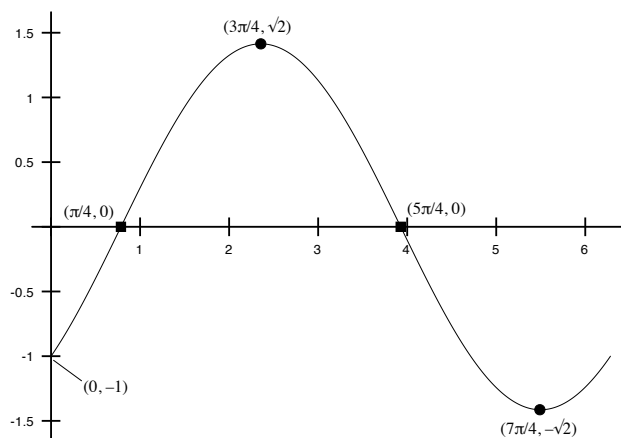
$$\text{concave up on intervals of the form } \left(-\frac{3\pi}{4} + 2k\pi, \frac{\pi}{4} + 2k\pi\right)$$

and is

$$\text{concave down on intervals of the form } \left(\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi\right)$$

for any integer k . The concavity of f changes at $\frac{\pi}{4} + k\pi$ for each integer k , so $\left(\frac{\pi}{4} + k\pi, 0\right)$ is a point of inflection for each integer k .

Step 7 The figure below displays the graph of f over the interval $[0, 2\pi]$. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π .



32. Let $f(x) = \sin x + \tan x$. Note the function f is periodic with period 2π .

Step 1 The domain of f is the set $\left\{x \mid x \neq \frac{2k+1}{2}\pi \text{ for any integer } k\right\}$. The x -intercepts satisfy the equation

$$\sin x + \tan x = 0 \quad \text{or} \quad \sin x \left(1 + \frac{1}{\cos x}\right) = 0;$$

the solutions to this equation are $k\pi$ for any integer k . The y -intercept is $f(0) = 0$.

Step 2 The function f has no horizontal asymptote. Because

$$\lim_{x \rightarrow (2k+1)\pi/2^-} (\sin x + \tan x) = \infty, \quad \text{and} \quad \lim_{x \rightarrow (2k+1)\pi/2^+} (\sin x + \tan x) = -\infty,$$

the lines $x = \frac{2k+1}{2}\pi$ are vertical asymptotes for each integer k .

Step 3 Now,

$$\begin{aligned} f'(x) &= \cos x + \sec^2 x = \frac{\cos^3 x + 1}{\cos^2 x}; \text{ and} \\ f''(x) &= -\sin x + 2\sec^2 x \tan x = \frac{\sin x(2 - \cos^3 x)}{\cos^3 x}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $\cos^3 x = -1$ or $x = (2k+1)\pi$ and does not exist when $\cos x = 0$ or $x = \frac{2k+1}{2}\pi$ where k is any integer. As $\frac{2k+1}{2}\pi$ is not in the domain of f for any integer k , $\frac{2k+1}{2}\pi$ is not a critical number for any integer k . Therefore, the critical numbers of f are $(2k+1)\pi$ for any integer k . At the points $((2k+1)\pi, 0)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, and $\left(\pi, \frac{3\pi}{2}\right)$. The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π . The results are shown in the table below.

Interval	Sign of f'	Conclusion
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	+	f is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\left(\frac{\pi}{2}, \pi\right)$	+	f is increasing on $\left(\frac{\pi}{2}, \pi\right)$
$\left(\pi, \frac{3\pi}{2}\right)$	+	f is increasing on $\left(\pi, \frac{3\pi}{2}\right)$

Therefore f is increasing on intervals of the form $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$ for any integer k .

Step 5 By the First Derivative Test and the information above, f has neither a local maximum value nor a local minimum value at $(2k+1)\pi$ for any integer k . Therefore, f has no local extreme values.

Step 6 The second derivative is equal to zero when $x = k\pi$ and does not exist when $x = \frac{2k+1}{2}\pi$ for any integer k . Consider the intervals $\left(-\frac{\pi}{2}, 0\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, and $\left(\pi, \frac{3\pi}{2}\right)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$\left(-\frac{\pi}{2}, 0\right)$	-	f is concave down on $\left(-\frac{\pi}{2}, 0\right)$
$\left(0, \frac{\pi}{2}\right)$	+	f is concave up on $\left(0, \frac{\pi}{2}\right)$
$\left(\frac{\pi}{2}, \pi\right)$	-	f is concave down on $\left(\frac{\pi}{2}, \pi\right)$
$\left(\pi, \frac{3\pi}{2}\right)$	+	f is concave up on $\left(\pi, \frac{3\pi}{2}\right)$

Taking into account the period of the function f , it follows that f is

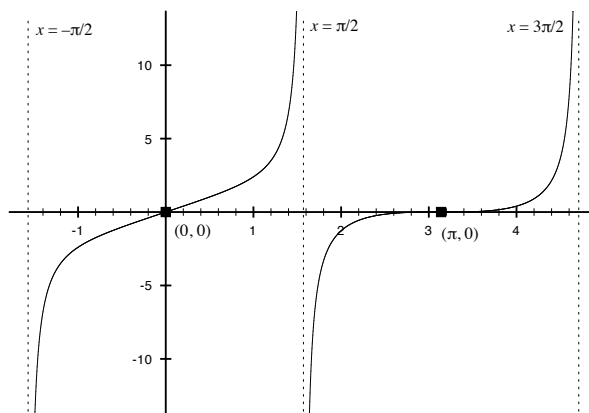
$$\text{concave up on intervals of the form } \left(k\pi, \frac{2k+1}{2}\pi\right)$$

and is

$$\text{concave down on intervals of the form } \left(\frac{2k+1}{2}\pi, (k+1)\pi\right)$$

for any integer k . The concavity of f changes at $k\pi$ for each integer k , so the points $(k\pi, 0)$ are points of inflection for each integer k . Although concavity changes at $\frac{2k+1}{2}\pi$ for each integer k , there is no point of inflection at $\frac{2k+1}{2}\pi$ for any integer k because $\frac{2k+1}{2}\pi$ is not in the domain of f for any integer k .

Step 7 The figure below displays the graph of f over the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. The points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π .



33. Let $f(x) = \sin^2 x - \cos x$. Note the function f is periodic with period 2π .

Step 1 The domain of f is the set of all real numbers. The x -intercepts satisfy the equation

$$\sin^2 x - \cos x = 1 - \cos^2 x - \cos x = 0.$$

This equation is quadratic in form, so by the quadratic formula,

$$\cos x = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)(1)}}{-2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Now, $\frac{-1 - \sqrt{5}}{2} < -1$ and $-1 \leq \cos x \leq 1$, so the x -intercepts are

$$\cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right) + 2k\pi \approx 0.905 + 2k\pi \quad \text{and} \quad 2\pi - \cos^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right) + 2k\pi \approx 5.379 + 2k\pi$$

for any integer k . The y -intercept is $f(0) = -1$.

Step 2 The graph of the function f has no asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= 2 \sin x \cos x + \sin x = \sin x(2 \cos x + 1); \text{ and} \\ f''(x) &= -2 \sin^2 x + \cos x(2 \cos x + 1) = 2 \cos^2 x - 2 \sin^2 x + \cos x \\ &= 4 \cos^2 x + \cos x - 2. \end{aligned}$$

As f is differentiable everywhere, the critical numbers of f occur where $f'(x) = 0$; that is, where $\sin x = 0$ and $2 \cos x + 1 = 0$. Therefore, $k\pi$, $\frac{2\pi}{3} + 2k\pi$, and $\frac{4\pi}{3} + 2k\pi$ are the critical numbers of f for any integer k . At the points $(k\pi, (-1)^{k+1})$, $\left(\frac{2\pi}{3} + 2k\pi, \frac{5}{4}\right)$, and $\left(\frac{4\pi}{3} + 2k\pi, \frac{5}{4}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(0, \frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3}, \pi\right)$, $\left(\pi, \frac{4\pi}{3}\right)$, and $\left(\frac{4\pi}{3}, 2\pi\right)$. The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π . The results are shown in the table below.

Interval	Sign of f'	Conclusion
$\left(0, \frac{2\pi}{3}\right)$	+	f is increasing on $\left(0, \frac{2\pi}{3}\right)$
$\left(\frac{2\pi}{3}, \pi\right)$	-	f is decreasing on $\left(\frac{2\pi}{3}, \pi\right)$
$\left(\pi, \frac{4\pi}{3}\right)$	+	f is increasing on $\left(\pi, \frac{4\pi}{3}\right)$
$\left(\frac{4\pi}{3}, 2\pi\right)$	-	f is decreasing on $\left(\frac{4\pi}{3}, 2\pi\right)$

Therefore, f is

$$\text{increasing on intervals of the form } \left(2k\pi, \frac{2\pi}{3} + 2k\pi\right) \text{ and } \left((2k+1)\pi, \frac{4\pi}{3} + 2k\pi\right)$$

and is

$$\text{decreasing on intervals of the form } \left(\frac{2\pi}{3} + 2k\pi, (2k+1)\pi\right) \text{ and } \left(\frac{4\pi}{3} + 2k\pi, (2k+2)\pi\right)$$

for any integer k .

Step 5 By the First Derivative Test and the information above, f has a local maximum value at $\frac{2\pi}{3} + 2k\pi$ and at $\frac{4\pi}{3} + 2k\pi$ and a local minimum value at $k\pi$ for each integer k . The

$$\text{local maximum values are } f\left(\frac{2\pi}{3} + 2k\pi\right) = \frac{5}{4} \text{ and } f\left(\frac{4\pi}{3} + 2k\pi\right) = \frac{5}{4},$$

and the local minimum values are $f(k\pi) = (-1)^k$.

Step 6 The second derivative exists everywhere and is equal to zero when $4 \cos^2 x + \cos x - 2 = 0$. This equation is quadratic in form, so by the quadratic formula,

$$\cos x = \frac{-1 \pm \sqrt{1^2 - 4(4)(-2)}}{8} = \frac{-1 \pm \sqrt{33}}{8}.$$

Therefore, $f''(x) = 0$ when

$$x = \cos^{-1}\left(\frac{-1 - \sqrt{33}}{8}\right) + 2k\pi \approx 2.574 + 2k\pi, \quad x = 2\pi - \cos^{-1}\left(\frac{-1 - \sqrt{33}}{8}\right) + 2k\pi \approx 3.709 + 2k\pi,$$

$$x = \cos^{-1}\left(\frac{-1 + \sqrt{33}}{8}\right) + 2k\pi \approx 0.936 + 2k\pi, \quad x = 2\pi - \cos^{-1}\left(\frac{-1 + \sqrt{33}}{8}\right) + 2k\pi \approx 5.347 + 2k\pi.$$

Consider the approximate intervals $(0.936, 2.574)$, $(2.574, 3.709)$, $(3.709, 5.347)$, and $(5.347, 7.219)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(0.936, 2.574)$	—	f is concave down on $(0.936, 2.574)$
$(2.574, 3.709)$	+	f is concave up on $(2.574, 3.709)$
$(3.709, 5.347)$	—	f is concave down on $(3.709, 5.347)$
$(5.347, 7.219)$	+	f is concave up on $(5.347, 7.219)$

Taking into account the period of the function f , it follows that f is

concave down on intervals of the form $(0.936 + 2k\pi, 2.574 + 2k\pi)$ and $(3.709 + 2k\pi, 5.347 + 2k\pi)$

and is

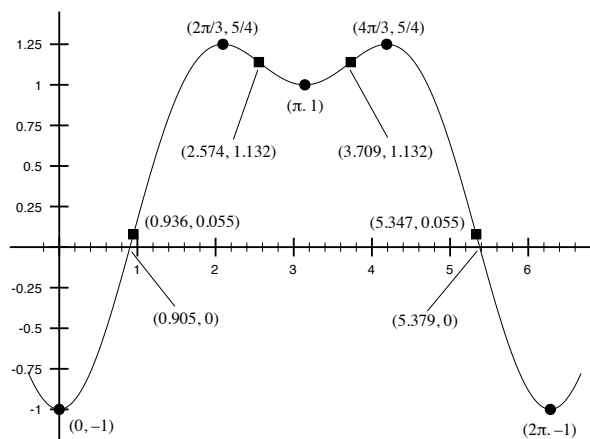
concave up on intervals of the form $(2.574 + 2k\pi, 3.709 + 2k\pi)$ and $(5.347 + 2k\pi, 7.219 + 2k\pi)$

for any integer k . The concavity of f changes at $0.936 + 2k\pi$, $2.574 + 2k\pi$, $3.709 + 2k\pi$, and $5.347 + 2k\pi$ for each integer k , so the points

$(0.936 + 2k\pi, 0.055)$, $(2.574 + 2k\pi, 1.132)$, $(3.709 + 2k\pi, 1.132)$, and $(5.347 + 2k\pi, 0.055)$

are points of inflection of f for each integer k .

Step 7 The figure below displays the graph of f over the interval $[0, 2\pi]$. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π .



34. Let $f(x) = \cos^2 x + \sin x$. Note the function f is periodic with period 2π .

Step 1 The domain of f is the set of all real numbers. The x -intercepts satisfy the equation

$$\cos^2 x + \sin x = 1 - \sin^2 x + \sin x = 0.$$

This equation is quadratic in form, so by the quadratic formula,

$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(1)}}{-2} = \frac{1 \pm \sqrt{5}}{2}.$$

Now, $\frac{1+\sqrt{5}}{2} > 1$ and $-1 \leq \sin x \leq 1$, so the x -intercepts are

$$\sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right) + 2k\pi \approx -0.666 + 2k\pi \quad \text{and} \quad \pi - \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right) + 2k\pi \approx 3.808 + 2k\pi$$

for any integer k . The y -intercept is $f(0) = 1$.

Step 2 The graph of the function f has no asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= -2\sin x \cos x + \cos x = \cos x(1 - 2\sin x); \text{ and} \\ f''(x) &= -2\cos^2 x - \sin x(1 - 2\sin x) = -2\cos^2 x + 2\sin^2 x - \sin x \\ &= 4\sin^2 x - \sin x - 2. \end{aligned}$$

As f is differentiable everywhere, the critical numbers of f occur where $f'(x) = 0$; that is, where $\cos x = 0$ and $1 - 2\sin x = 0$. Therefore, $\frac{\pi}{6} + 2k\pi$, $\frac{2k+1}{2}\pi$, and $\frac{5\pi}{6} + 2k\pi$ are the critical numbers of f for any integer k . At the points $\left(\frac{\pi}{6} + 2k\pi, \frac{5}{4}\right)$, $\left(\frac{2k+1}{2}\pi, (-1)^k\right)$, and $\left(\frac{5\pi}{6} + 2k\pi, \frac{5}{4}\right)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, consider the intervals $\left(-\frac{\pi}{2}, \frac{\pi}{6}\right)$, $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$, and $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$. The increasing/decreasing pattern on these two intervals will repeat indefinitely in either direction in increments of the period 2π . The results are shown in the table below.

Interval	Sign of f'	Conclusion
$\left(-\frac{\pi}{2}, \frac{\pi}{6}\right)$	+	f is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{6}\right)$
$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	-	f is decreasing on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$
$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	+	f is increasing on $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$
$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	-	f is decreasing on $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$

Therefore, f is

$$\text{increasing on intervals of the form } \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{6} + 2k\pi\right) \text{ and } \left(\frac{\pi}{2} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right)$$

and is

$$\text{decreasing on intervals of the form } \left(\frac{\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi\right) \text{ and } \left(\frac{5\pi}{6} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right)$$

for any integer k .

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{\pi}{6} + 2k\pi$ and at $\frac{5\pi}{6} + 2k\pi$ and a local minimum value at $\frac{2k+1}{2}\pi$ for each integer k . The

$$\text{local maximum values are } f\left(\frac{\pi}{6} + 2k\pi\right) = \frac{5}{4} \text{ and } f\left(\frac{5\pi}{6} + 2k\pi\right) = \frac{5}{4},$$

$$\text{and the local minimum values are } f\left(\frac{2k+1}{2}\pi\right) = (-1)^k.$$

Step 6 The second derivative exists everywhere and is equal to zero when $4\sin^2 x - \sin x - 2 = 0$. This equation is quadratic in form, so by the quadratic formula,

$$\sin x = \frac{1 \pm \sqrt{(-1)^2 - 4(4)(-2)}}{8} = \frac{1 \pm \sqrt{33}}{8}.$$

Therefore, $f''(x) = 0$ when

$$x = \sin^{-1}\left(\frac{1 - \sqrt{33}}{8}\right) + 2k\pi \approx -0.635 + 2k\pi, \quad x = \pi - \sin^{-1}\left(\frac{1 - \sqrt{33}}{8}\right) + 2k\pi \approx 3.776 + 2k\pi,$$

$$x = \sin^{-1}\left(\frac{1 + \sqrt{33}}{8}\right) + 2k\pi \approx 1.003 + 2k\pi, \quad x = \pi - \sin^{-1}\left(\frac{1 + \sqrt{33}}{8}\right) + 2k\pi \approx 2.139 + 2k\pi.$$

Consider the intervals $(-0.635, 1.003)$, $(1.003, 2.139)$, $(2.139, 3.776)$, and $(3.776, 5.648)$, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-0.635, 1.003)$	−	f is concave down on $(-0.635, 1.003)$
$(1.003, 2.139)$	+	f is concave up on $(1.003, 2.139)$
$(2.139, 3.776)$	−	f is concave down on $(2.139, 3.776)$
$(3.776, 5.648)$	+	f is concave up on $(3.776, 5.648)$

Taking into account the period of the function f , it follows that f is

concave down on intervals of the form $(-0.635 + 2k\pi, 1.003 + 2k\pi)$ and $(2.139 + 2k\pi, 3.776 + 2k\pi)$

and is

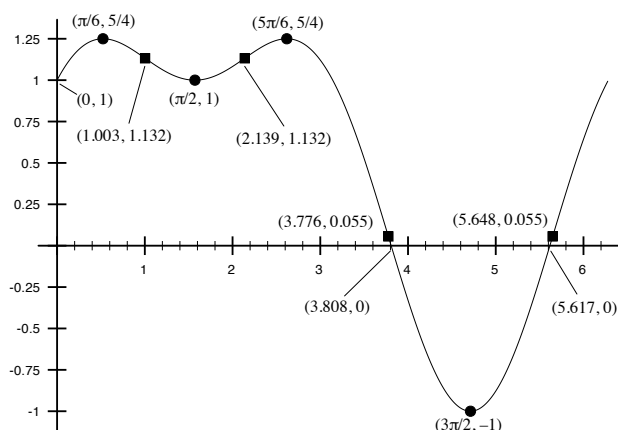
concave up on intervals of the form $(1.003 + 2k\pi, 2.139 + 2k\pi)$ and $(3.776 + 2k\pi, 5.648 + 2k\pi)$

for any integer k . The concavity of f changes at $1.003 + 2k\pi$, $2.139 + 2k\pi$, $3.776 + 2k\pi$, and $5.648 + 2k\pi$ for each integer n , so the points

$(1.003 + 2k\pi, 1.132)$, $(2.139 + 2k\pi, 1.132)$, $(3.776 + 2k\pi, 0.055)$, and $(5.648 + 2k\pi, 0.055)$

are points of inflection of f for any integer k .

Step 7 The figure below displays the graph of f over the interval $[0, 2\pi]$. The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares. The graph repeats indefinitely in either direction with period 2π .



35. Let $y = f(x) = \ln x - x$.

Step 1 The domain of f is the set $\{x|x > 0\}$. There are no x -intercepts as the graphs of $y = \ln x$ and $y = x$ never intersect, and there is also no y -intercept because the function is not defined at 0.

Step 2 Because

$$\lim_{x \rightarrow \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x = \infty,$$

the expression $\ln x - x$ is an indeterminate form at ∞ of the type $\infty - \infty$. Rewrite

$$\ln x - x \quad \text{as} \quad x \left(\frac{\ln x}{x} - 1 \right).$$

Now, the expression $\frac{\ln x}{x}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$; by L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \left(\frac{\ln x}{x} - 1 \right) = -1 \quad \text{and} \quad \lim_{x \rightarrow \infty} (\ln x - x) = \lim_{x \rightarrow \infty} \left[x \left(\frac{\ln x}{x} - 1 \right) \right] = -\infty,$$

so the graph of f has no horizontal asymptote. On the other hand,

$$\lim_{x \rightarrow 0^+} (\ln x - x) = -\infty,$$

so the line $x = 0$ is a vertical asymptote.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{x} - 1 = \frac{1-x}{x}; \text{ and} \\ f''(x) &= -\frac{1}{x^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = 1$ and does not exist when $x = 0$; however, 0 is not a critical number because 0 is not in the domain of f . Therefore, 1 is the only critical number. At the point $(1, -1)$, the tangent line is horizontal.

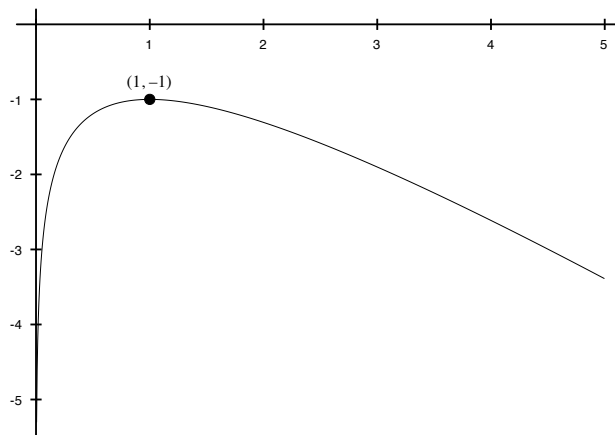
Step 4 To apply the Increasing/Decreasing Function Test, use the number 1 to divide $(0, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(0, 1)$	+	f is increasing on $(0, 1)$
$(1, \infty)$	-	f is decreasing on $(1, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 1. The local maximum value is $f(1) = -1$.

Step 6 Because $f''(x) < 0$ for all $x > 0$, f is concave down on the interval $(0, \infty)$, so f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



36. Let $y = f(x) = x \ln x$.

Step 1 The domain of f is the set $\{x|x > 0\}$. The x -intercept is 1, and there is no y -intercept because the function is not defined at 0.

Step 2 Because

$$\lim_{x \rightarrow \infty} (x \ln x) = \infty,$$

the graph of f has no horizontal asymptote. Additionally, by L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} (x \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \rightarrow 0^+} (-x) = 0,$$

so the graph also has no vertical asymptote.

Step 3 Now,

$$\begin{aligned} f'(x) &= x \cdot \frac{1}{x} + \ln x = 1 + \ln x; \text{ and} \\ f''(x) &= \frac{1}{x}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = e^{-1} = \frac{1}{e}$ and exists everywhere on the domain of f . Therefore, $\frac{1}{e}$ is the only critical number. At the point $\left(\frac{1}{e}, -\frac{1}{e}\right)$, the tangent line is horizontal.

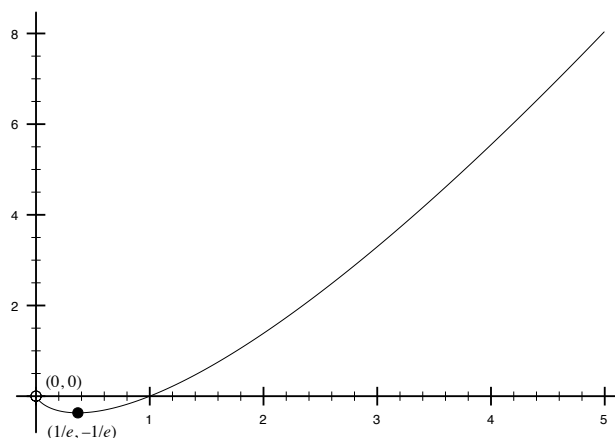
Step 4 To apply the Increasing/Decreasing Function Test, use the number $\frac{1}{e}$ to divide $(0, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(0, \frac{1}{e})$	$-$	f is decreasing on $(0, \frac{1}{e})$
$(\frac{1}{e}, \infty)$	$+$	f is increasing on $(\frac{1}{e}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $\frac{1}{e}$. The local minimum value is $f\left(\frac{1}{e}\right) = -\frac{1}{e}$.

Step 6 Because $f''(x) > 0$ for all $x > 0$, f is concave up on the interval $(0, \infty)$, so f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



37. Let $f(x) = \ln(4 - x^2)$.

Step 1 The domain of f is given by the solution to the inequality $4 - x^2 > 0$; that is, the set $\{x \mid -2 < x < 2\}$.

The x -intercepts are $\pm\sqrt{3}$, and the y -intercept is $f(0) = \ln 4$.

Step 2 Because f is not defined as x becomes unbounded in either direction, the graph of f has no horizontal asymptote. As x approaches the endpoints of the domain

$$\lim_{x \rightarrow -2^+} \ln(4 - x^2) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} \ln(4 - x^2) = -\infty,$$

so the lines $x = \pm 2$ are vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{4 - x^2} \cdot (-2x) = -\frac{2x}{4 - x^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(-\frac{2x}{4 - x^2} \right) = -\frac{(4 - x^2) \cdot 2 - 2x \cdot (-2x)}{(4 - x^2)^2} = -\frac{8 + 2x^2}{(4 - x^2)^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = 0$ and exists everywhere on the domain of f . Therefore, 0 is the only critical number. At the point $(0, \ln 4)$, the tangent line is horizontal.

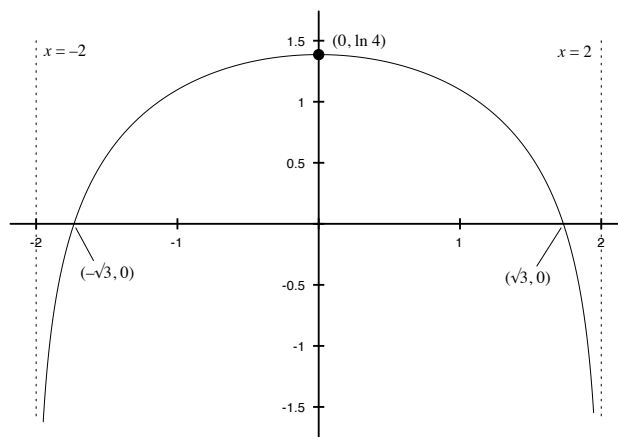
Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide $(-2, 2)$ into two intervals.

Interval	Sign of f'	Conclusion
$(-2, 0)$	+	f is increasing on $(-2, 0)$
$(0, 2)$	-	f is decreasing on $(0, 2)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The local maximum value is $f(0) = \ln 4$.

Step 6 Because $f''(x) < 0$ for all $-2 < x < 2$, f is concave down on the interval $(-2, 2)$, so f has no points of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



38. Let $f(x) = \ln(x^2 + 2)$.

Step 1 The domain of f is the set of all real numbers because $x^2 + 2 \geq 2 > 0$ for all real x . There are no x -intercepts, and the y -intercept is $f(0) = \ln 2$.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} \ln(x^2 + 2) = \infty,$$

the graph of f has no horizontal asymptote. The graph also has no vertical asymptotes because f is defined for all real numbers.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{1}{x^2 + 2} \cdot 2x = \frac{2x}{x^2 + 2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(\frac{2x}{x^2 + 2} \right) = \frac{(x^2 + 2) \cdot 2 - 2x \cdot 2x}{(x^2 + 2)^2} = \frac{4 - 2x^2}{(x^2 + 2)^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = 0$ and exists everywhere. Therefore, 0 is the only critical number. At the point $(0, \ln 2)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	–	f is decreasing on $(-\infty, 0)$
$(0, \infty)$	+	f is increasing on $(0, \infty)$

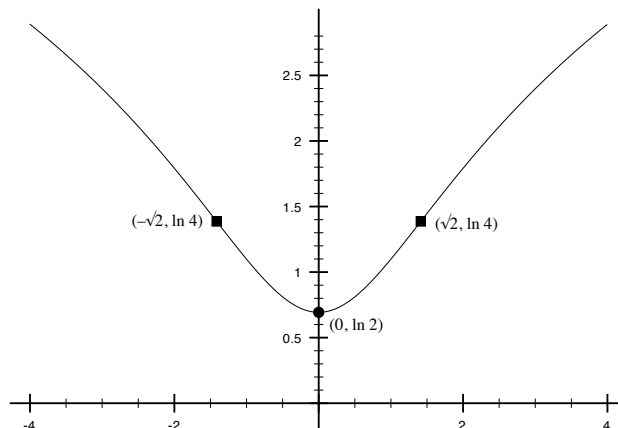
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at 0. The local minimum value is $f(0) = \ln 2$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \pm\sqrt{2}$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\sqrt{2})$	–	f is concave down on $(-\infty, -\sqrt{2})$
$(-\sqrt{2}, \sqrt{2})$	+	f is concave up on $(-\sqrt{2}, \sqrt{2})$
$(\sqrt{2}, \infty)$	–	f is concave down on $(\sqrt{2}, \infty)$

The concavity of f changes at $\pm\sqrt{2}$, so the points $(-\sqrt{2}, \ln 4)$ and $(\sqrt{2}, \ln 4)$ are points of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



39. Let $f(x) = 3e^{3x}(5 - x)$.

Step 1 The domain of f is the set of all real numbers. The x -intercept is 5, and the y -intercept is $f(0) = 15$.

Step 2 Because the domain of f is the set of all real numbers, the graph of f has no vertical asymptotes. To determine if there is a horizontal asymptote, consider the limits at infinity:

$$\lim_{x \rightarrow -\infty} [3e^{3x}(5 - x)] = \lim_{x \rightarrow -\infty} \frac{3(5 - x)}{e^{-3x}} = \lim_{x \rightarrow -\infty} \frac{-3}{-3e^{-3x}} = 0$$

and

$$\lim_{x \rightarrow \infty} [3e^{3x}(5 - x)] = -\infty,$$

where L'Hôpital's Rule was used in the first limit. Therefore, the graph of f has the line $y = 0$ as a horizontal asymptote as $x \rightarrow -\infty$ and no horizontal asymptote as $x \rightarrow \infty$.

Step 3 Now,

$$\begin{aligned} f'(x) &= -3e^{3x} + 9e^{3x}(5 - x) = 3e^{3x}(14 - 3x); \text{ and} \\ f''(x) &= -9e^{3x} + 9e^{3x}(14 - 3x) = 9e^{3x}(13 - 3x). \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = \frac{14}{3}$. At the point $\left(\frac{14}{3}, e^{14}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number $\frac{14}{3}$ to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, \frac{14}{3})$	+	f is increasing on $(-\infty, \frac{14}{3})$
$(\frac{14}{3}, \infty)$	-	f is decreasing on $(\frac{14}{3}, \infty)$

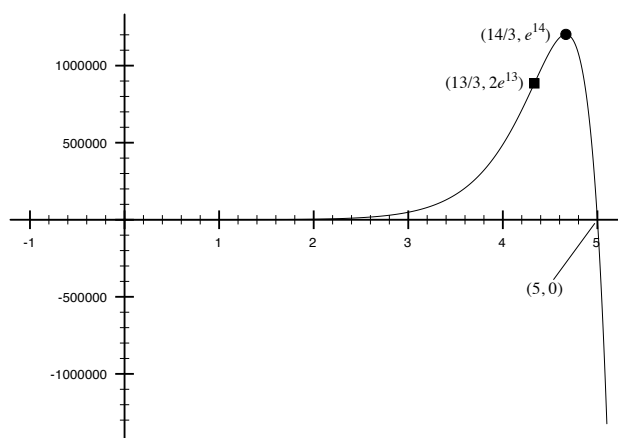
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{14}{3}$. The local maximum value is $f\left(\frac{14}{3}\right) = e^{14}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \frac{13}{3}$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, \frac{13}{3})$	+	f is concave up on $(-\infty, \frac{13}{3})$
$(\frac{13}{3}, \infty)$	-	f is concave down on $(\frac{13}{3}, \infty)$

The concavity of f changes at $\frac{13}{3}$, so the point $\left(\frac{13}{3}, 2e^{13}\right)$ is a point of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



40. Let $f(x) = 3e^{-3x}(x - 4)$.

Step 1 The domain of f is the set of all real numbers. The x -intercept is 4, and the y -intercept is $f(0) = -12$.

Step 2 Because the domain of f is the set of all real numbers, the graph of f has no vertical asymptotes. To determine if there is a horizontal asymptote, consider the limits at infinity:

$$\lim_{x \rightarrow -\infty} [3e^{-3x}(x - 4)] = -\infty$$

and

$$\lim_{x \rightarrow \infty} [3e^{-3x}(x - 4)] = \lim_{x \rightarrow \infty} \frac{3(x - 4)}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{3}{3e^{3x}} = 0$$

where L'Hôpital's Rule was used in the second limit. Therefore, the graph of f has the line $y = 0$ as a horizontal asymptote as $x \rightarrow \infty$ and no horizontal asymptote as $x \rightarrow -\infty$.

Step 3 Now,

$$\begin{aligned} f'(x) &= 3e^{-3x} - 9e^{-3x}(x - 4) = 3e^{-3x}(13 - 3x); \text{ and} \\ f''(x) &= -9e^{-3x} - 9e^{-3x}(13 - 3x) = -9e^{-3x}(14 - 3x). \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = \frac{13}{3}$. At the point $\left(\frac{13}{3}, e^{-13}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number $\frac{13}{3}$ to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, \frac{13}{3})$	+	f is increasing on $(-\infty, \frac{13}{3})$
$(\frac{13}{3}, \infty)$	-	f is decreasing on $(\frac{13}{3}, \infty)$

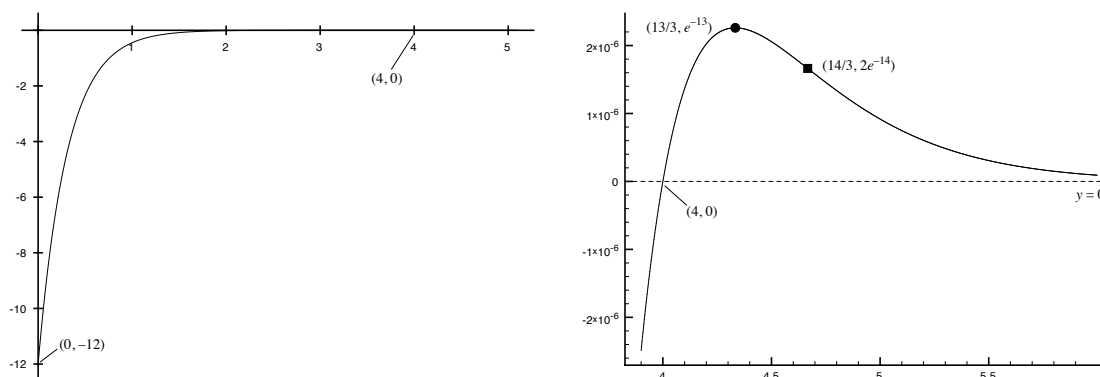
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $\frac{13}{3}$. The local maximum value is $f\left(\frac{13}{3}\right) = e^{-13}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \frac{14}{3}$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, \frac{14}{3})$	-	f is concave down on $(-\infty, \frac{14}{3})$
$(\frac{14}{3}, \infty)$	+	f is concave up on $(\frac{14}{3}, \infty)$

The concavity of f changes at $\frac{14}{3}$, so the point $\left(\frac{14}{3}, 2e^{-14}\right)$ is a point of inflection of f .

Step 7 The figures below display the graph of f . In the figure on the left, the x range is $0 \leq x \leq 5$. The figure on the right zooms in on the x range of $3.9 \leq x \leq 6$; the local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



41. Let $f(x) = e^{-x^2}$.

Step 1 The domain of f is the set of all real numbers. There is no x -intercept, and the y -intercept is $f(0) = 1$.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} e^{-x^2} = 0,$$

the line $y = 0$ is a horizontal asymptote. The graph of f has no vertical asymptotes because f is defined for all real x .

Step 3 Now,

$$\begin{aligned} f'(x) &= -2xe^{-x^2}; \text{ and} \\ f''(x) &= 4x^2e^{-x^2} - 2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}. \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = 0$. At the point $(0, 1)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \infty)$	-	f is decreasing on $(0, \infty)$

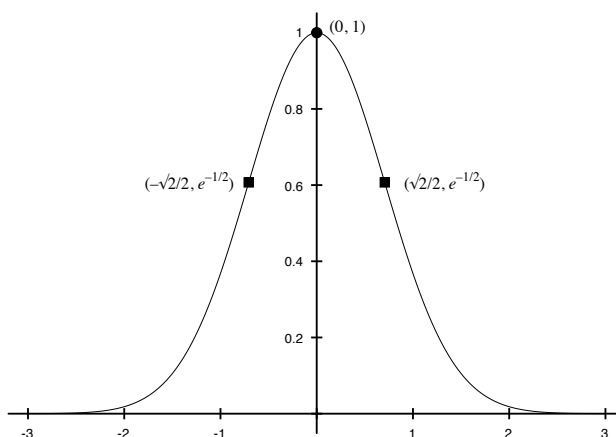
Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The local maximum value is $f(0) = 1$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = \pm \frac{\sqrt{2}}{2}$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{\sqrt{2}}{2})$	+	f is concave up on $(-\infty, -\frac{\sqrt{2}}{2})$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	-	f is concave down on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, \infty)$	+	f is concave up on $(\frac{\sqrt{2}}{2}, \infty)$

The concavity of f changes at $\pm \frac{\sqrt{2}}{2}$, so the points $\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ and $\left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ are points of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



42. Let $f(x) = e^{1/x}$.

Step 1 The domain of f is the set $\{x|x \neq 0\}$. There are no x -intercepts, and there is also no y -intercept because the function is not defined at 0.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} e^{1/x} = 1,$$

the line $y = 1$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow 0^-} e^{1/x} = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} e^{1/x} = \infty,$$

the line $x = 0$ is a vertical asymptote.

Step 3 Now,

$$\begin{aligned}f'(x) &= -\frac{1}{x^2}e^{1/x}; \text{ and} \\f''(x) &= \frac{1}{x^4}e^{1/x} + \frac{2}{x^3}e^{1/x} = \frac{1}{x^4}e^{1/x}(1 + 2x).\end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to zero and does not exist when $x = 0$. However, 0 is not in the domain of f , so 0 is not a critical number. Therefore, f has no critical numbers.

Step 4 Because $f'(x) < 0$ for all $x \neq 0$, f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$.

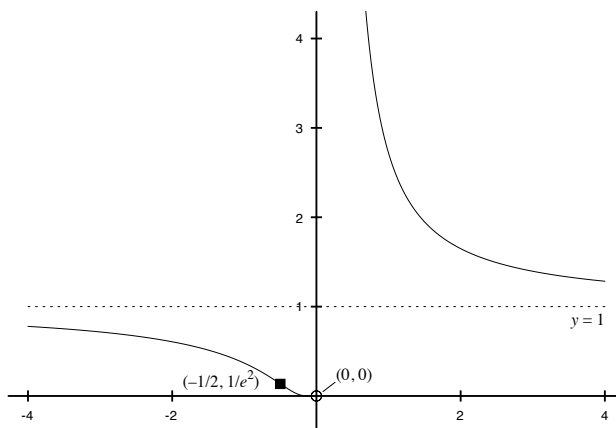
Step 5 Because there are no critical numbers, f has no local extreme values.

Step 6 The second derivative is equal to zero when $x = -\frac{1}{2}$ and does not exist when $x = 0$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{1}{2})$	–	f is concave down on $(-\infty, -\frac{1}{2})$
$(-\frac{1}{2}, 0)$	+	f is concave up on $(-\frac{1}{2}, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

The concavity of f changes at $-\frac{1}{2}$, so the point $(-\frac{1}{2}, \frac{1}{e^2})$ is a point of inflection of f .

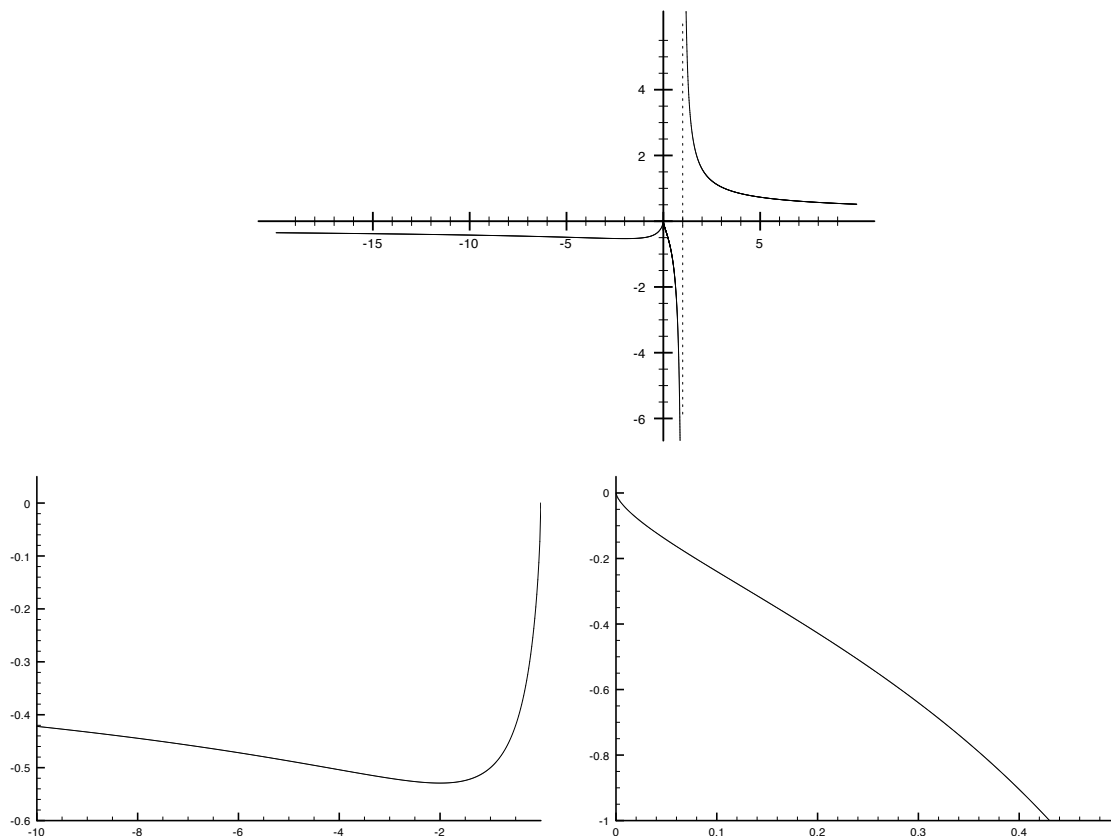
Step 7 The figure below displays the graph of f . The point of inflection is highlighted by a closed square.



Applications and Extensions

43. Let $f(x) = \frac{x^{2/3}}{x-1}$.

- (a) The figures below display the graph of f . In the top figure, the graph is shown for $-20 \leq x \leq 10$. In the figure at the bottom left, attention is focused on $-10 \leq x \leq 0$. Attention is focused on $0 \leq x \leq 0.5$ in the figure at the bottom right.



(b) The graph of f appears to have a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 0$.

(c) The function appears to be

decreasing on the intervals $(-\infty, -2)$, $(0, 1)$, and $(1, \infty)$

and increasing on the interval $(-2, 0)$. Additionally, the function appears to be

concave down, approximately, on the intervals $(-\infty, -4)$ and $(0.1, 1)$

and

concave up, approximately, on the intervals $(-4, 0)$, $(0, 0.1)$, and $(1, \infty)$.

(d) The function appears to have a local minimum value at -2 and a local maximum value at 0 .

(e) Note that

$$f'(x) = \frac{(x-1) \cdot \frac{2}{3}x^{-1/3} - x^{2/3}}{(x-1)^2} = \frac{2x-2-3x}{3x^{1/3}(x-1)^2} = -\frac{x+2}{3x^{1/3}(x-1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x) = 0$ when $x = -2$ and does not exist when $x = 0$ and when $x = 1$; however, 1 is not in the domain of f , so 1 is not a critical number. To apply the Increasing/Decreasing Function Test, use the numbers -2 , 0 , and 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	$-$	f is decreasing on $(-\infty, -2)$
$(-2, 0)$	$+$	f is increasing on $(-2, 0)$
$(0, 1)$	$-$	f is decreasing on $(0, 1)$
$(1, \infty)$	$-$	f is decreasing on $(1, \infty)$

By the First Derivative Test and the information in the table above, f has a local minimum value at -2 and a local maximum value at 0 , in agreement with the approximations given in part (d).

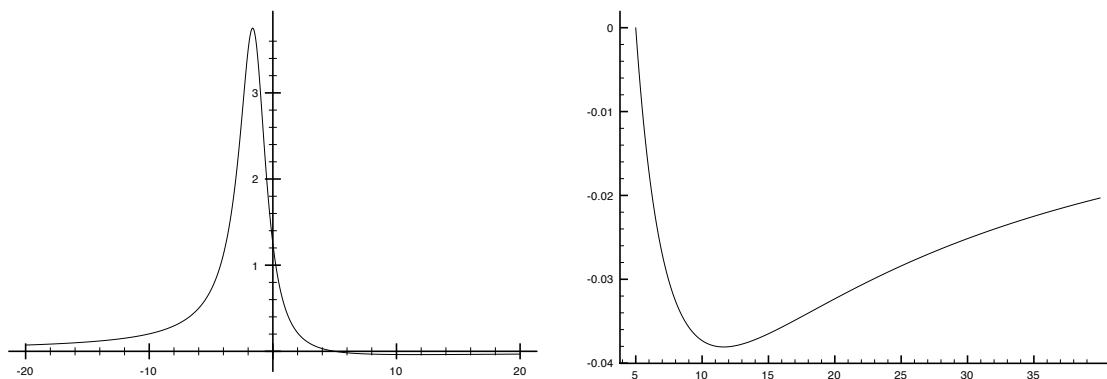
- (f) Concavity appears to change at approximately $x = -4$ and $x = 0.1$; therefore,

$$(-4, f(-4)) = \left(-4, -\frac{2\sqrt[3]{2}}{5}\right) \quad \text{and} \quad (0.1, f(0.1)) \approx (0.1, -0.239)$$

are approximate points of inflection of f .

44. Let $f(x) = \frac{5-x}{x^2+3x+4}$.

- (a) The figures below display the graph of f : on the left over the interval $(-20, 20)$, and on the right over the interval $(5, 40)$.



- (b) The graph appears to have a horizontal asymptote at $y = 0$, but no vertical asymptotes.
- (c) The function appears to be increasing, approximately, on the intervals $(-\infty, -2)$ and $(12, \infty)$ and decreasing, approximately, on the interval $(-2, 12)$. Additionally, f appears to be

concave up, approximately, on the intervals $(-\infty, -3)$ and $(-1, 20)$

and

concave down, approximately, on the intervals $(-3, -1)$ and $(20, \infty)$.

- (d) The function appears to have a local maximum value at approximately -2 and a local minimum value at approximately 12 .
- (e) Note that

$$f'(x) = \frac{(x^2 + 3x + 4) \cdot (-1) - (5 - x) \cdot (2x + 3)}{(x^2 + 3x + 4)^2} = \frac{x^2 - 10x - 19}{(x^2 + 3x + 4)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ exists everywhere but is equal to zero when

$$x = \frac{10 \pm \sqrt{10^2 - 4(1)(-19)}}{2} = \frac{10 \pm 4\sqrt{11}}{2} \approx -1.633, 11.633.$$

To apply the Increasing/Decreasing Function Test, use the numbers -1.633 and 11.633 to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1.633)$	+	f is increasing on $(-\infty, -1.633)$
$(-1.633, 11.633)$	-	f is decreasing on $(-1.633, 11.633)$
$(11.633, \infty)$	+	f is increasing on $(11.633, \infty)$

By the First Derivative Test and the information in the table above, f has a

local maximum value at approximately -1.633

and a

local minimum value at approximately 11.633 .

The approximations given in part (d) are within 0.4 of the exact values.

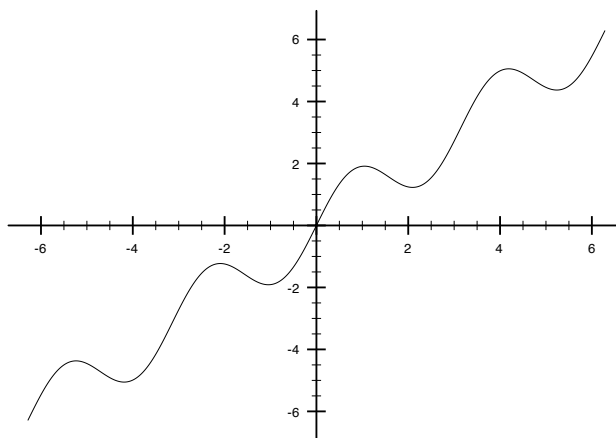
- (f) The concavity of f appears to change at -3 , -1 and 20 ; therefore, the points

$$(-3, f(-3)) = (-3, 2), \quad (-1, f(-1)) = (-1, 2), \quad \text{and} \quad (20, f(20)) = \left(20, -\frac{15}{464}\right)$$

are approximate points of inflection of f .

45. Let $f(x) = x + \sin(2x)$.

- (a) The figure below displays the graph of f . Though the function f is not periodic, observe that the graph consists of one basic shape that is repeated every π units along the x -axis.



- (b) The graph of f does not appear to have any asymptotes.
- (c) The function appears to be increasing on the interval $(-1, 1) \approx \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on the interval $(1, 2) \approx \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$. Given the repetitive structure of the graph, it follows that f is

increasing on intervals of the form $\left(-\frac{\pi}{3} + k\pi, \frac{\pi}{3} + k\pi\right)$

and

decreasing on intervals of the form $\left(\frac{\pi}{3} + k\pi, \frac{2\pi}{3} + k\pi\right)$

for each integer k . Additionally, f appears to be concave up on the interval $(-1.5, 0) \approx \left(-\frac{\pi}{2}, 0\right)$ and concave down on the interval $(0, 1.5) \approx \left(0, \frac{\pi}{2}\right)$. Again taking into account the repetitive

structure of the graph, it follows that f is concave up on intervals of the form $\left(-\frac{\pi}{2} + k\pi, k\pi\right)$

and concave down on intervals of the form $\left(k\pi, \frac{\pi}{2} + k\pi\right)$ for each integer k .

- (d) The graph appears to have a local maximum value at approximately $\frac{\pi}{3} + k\pi$ and a local minimum value at approximately $\frac{2\pi}{3} + k\pi$ for each integer k .

- (e) Note

$$f'(x) = 1 + 2\cos(2x).$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = \frac{\pi}{3} + k\pi$ and when $x = \frac{2\pi}{3} + k\pi$ for each integer k . To apply the Increasing/Decreasing Function Test, use the numbers $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ to divide $(0, \pi)$ into three intervals.

Interval	Sign of f'	Conclusion
$(0, \frac{\pi}{3})$	+	f is increasing on $(0, \frac{\pi}{3})$
$(\frac{\pi}{3}, \frac{2\pi}{3})$	−	f is decreasing on $(\frac{\pi}{3}, \frac{2\pi}{3})$
$(\frac{2\pi}{3}, \pi)$	+	f is increasing on $(\frac{2\pi}{3}, \pi)$

By the First Derivative Test, the information in the table above and the repetitive structure of the graph, f has a local maximum value at $\frac{\pi}{3} + k\pi$ and a local minimum value at $\frac{2\pi}{3} + k\pi$ for each integer k , in agreement with the approximations given in part (d).

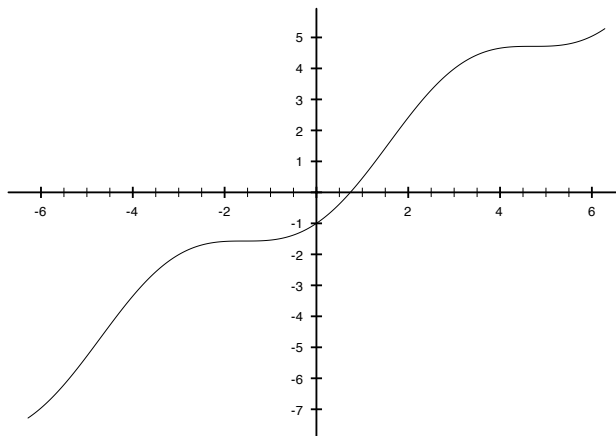
- (f) The concavity of f appears to change at multiples of $\frac{\pi}{2}$; that is, there appear to be points of inflection at

$$\left(\frac{k\pi}{2}, f\left(\frac{k\pi}{2}\right)\right) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$$

for each integer k .

46. Let $f(x) = x - \cos x$.

- (a) The figure below displays the graph of f . Though the function f is not periodic, observe that the graph consists of one basic shape that is repeated every 2π units along the x -axis.



- (b) The graph of f does not appear to have any asymptotes.

- (c) The function appears to be increasing for all real x . Additionally, f appears to be concave up on the interval $(-1.5, 1.5) \approx \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and concave down on the interval $(1.5, 4.5) \approx \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Given the repetitive structure of the graph, it follows that f is

$$\text{concave up on intervals of the form } \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$$

and

$$\text{concave down on intervals of the form } \left(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right)$$

for each integer k .

- (d) The function f does not appear to have any local extreme values.
 (e) Note

$$f'(x) = 1 + \sin x.$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = -\frac{\pi}{2} + 2k\pi$ for each integer k . To apply the Increasing/Decreasing Function Test, use the number $\frac{3\pi}{2}$ to divide $(0, 2\pi)$ into two intervals.

Interval	Sign of f'	Conclusion
$(0, \frac{3\pi}{2})$	+	f is increasing on $(0, \frac{3\pi}{2})$
$(\frac{3\pi}{2}, 2\pi)$	+	f is increasing on $(\frac{3\pi}{2}, 2\pi)$

By the First Derivative Test, the information in the table above and the periodic nature of the derivative, f has neither a local maximum value nor a local minimum value at $\frac{3\pi}{2} + 2k\pi$. Therefore, f has no local extreme values, in agreement with the approximations given in part (d).

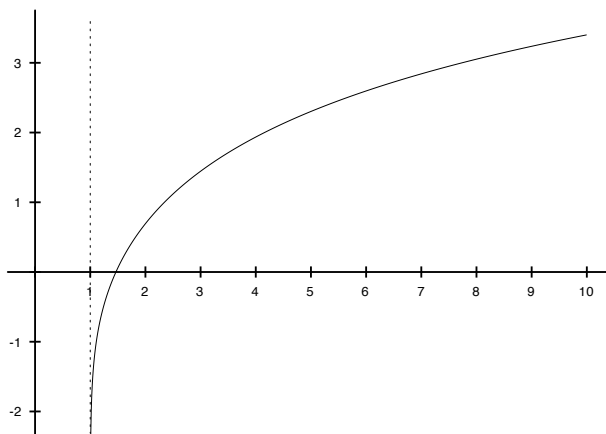
- (f) The concavity of f appears to change at odd multiples of $\frac{\pi}{2}$; that is, there appear to be points of inflection at

$$\left(\frac{(2k+1)\pi}{2}, f\left(\frac{(2k+1)\pi}{2}\right)\right) = \left(\frac{(2k+1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$$

for each integer k .

47. Let $f(x) = \ln(x\sqrt{x-1})$.

- (a) The figure below displays the graph of f .



- (b) The graph of f has no horizontal asymptote, but has a vertical asymptote at $x = 1$.
- (c) The function appears to be increasing and concave down on the interval $(1, \infty)$.
- (d) The function f does not appear to have any local extreme values.
- (e) Note that

$$f(x) = \ln(x\sqrt{x-1}) = \ln x + \ln \sqrt{x-1} = \ln x + \frac{1}{2} \ln(x-1),$$

so

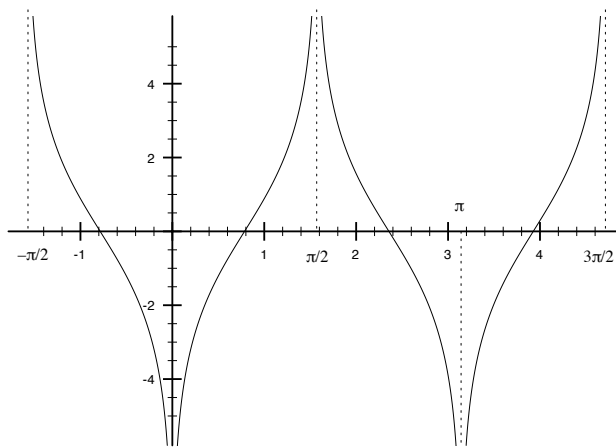
$$f'(x) = \frac{1}{x} + \frac{1}{2(x-1)} = \frac{3x-2}{2x(x-1)}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x) = 0$ when $x = \frac{2}{3}$ and does not exist when $x = 0$ and when $x = 1$; however, none of these numbers is in the domain of f , so none are critical numbers. Because the function has no critical numbers, there are no local extreme values, in agreement with the result of part (d).

- (f) Because the concavity of f never changes, there are no points of inflection.

48. Let $f(x) = \ln(\tan^2 x)$.

- (a) The figure below displays the graph of f . Note that the graph is periodic with period π .



- (b) The graph of f has no horizontal asymptote, but appears to have a vertical asymptote at $x = \frac{k\pi}{2}$ for each integer k .
- (c) Taking into account the period of the function, f appears to be

increasing on intervals of the form $\left(k\pi, k\pi + \frac{\pi}{2}\right)$

and

decreasing on intervals of the form $\left(k\pi - \frac{\pi}{2}, k\pi\right)$

for each integer k . Additionally, f appears to be

concave up on intervals of the form $\left(-\frac{\pi}{2} + k\pi, -\frac{\pi}{4} + k\pi\right)$ and $\left(\frac{\pi}{4} + k\pi, \frac{\pi}{2} + k\pi\right)$

and

concave down on intervals of the form $\left(-\frac{\pi}{4} + k\pi, k\pi\right)$ and $\left(k\pi, \frac{\pi}{4} + k\pi\right)$,

again for each integer k .

(d) The function f does not appear to have any local extreme values.

(e) Note that

$$f(x) = \ln(\tan^2 x) = 2 \ln \tan x,$$

so

$$f'(x) = \frac{2 \sec^2 x}{\tan x}.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is never equal to zero and does not exist when $x = \pm \frac{\pi}{2} + k\pi$ for each integer k ; however, none of these numbers is in the domain of f , so none are critical numbers. Because the function has no critical numbers, there are no local extreme values, in agreement with the result of part (d).

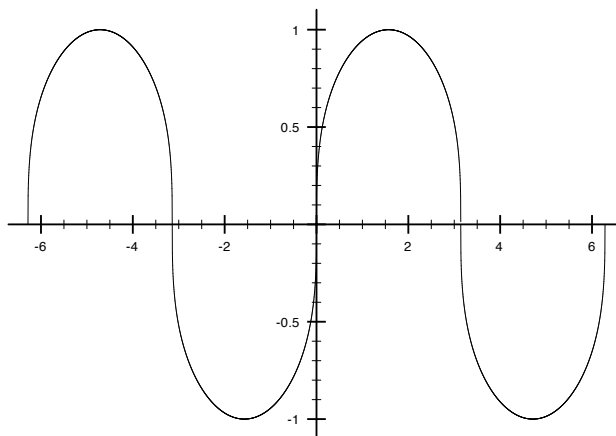
(f) The concavity of f appears to change at odd multiples of $\frac{\pi}{4}$; that is, there appear to be points of inflection at

$$\left(\frac{(2k+1)\pi}{4}, f\left(\frac{(2k+1)\pi}{4}\right) \right) = \left(\frac{(2k+1)\pi}{4}, 0 \right)$$

for each integer k .

49. Let $f(x) = \sqrt[3]{\sin x}$.

(a) The figure below displays the graph of f . Note that the graph is periodic with period 2π .



(b) The graph of f does not appear to have any asymptotes.

(c) Taking into account the period of the function, f appears to be

$$\text{increasing on intervals of the form } \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi \right)$$

and

$$\text{decreasing on intervals of the form } \left(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi \right)$$

for each integer k . Additionally, f appears to be

$$\text{concave up on intervals of the form } (-\pi + 2k\pi, 2k\pi)$$

and

$$\text{concave down on intervals of the form } (2k\pi, \pi + 2k\pi),$$

again for each integer k .

- (d) The graph appears to have a local maximum value at $x = \frac{\pi}{2} + 2k\pi$ and a local minimum value at $x = \frac{3\pi}{2} + 2k\pi$ for each integer k .

(e) Note

$$f'(x) = \frac{1}{3}(\sin x)^{-2/3} \cos x.$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = \frac{\pi}{2} + k\pi$ for each integer k and does not exist when $x = k\pi$ for each integer k . To apply the Increasing/Decreasing Function Test, use the numbers $\frac{\pi}{2}$, π , and $\frac{3\pi}{2}$ to divide $(0, 2\pi)$ into four intervals.

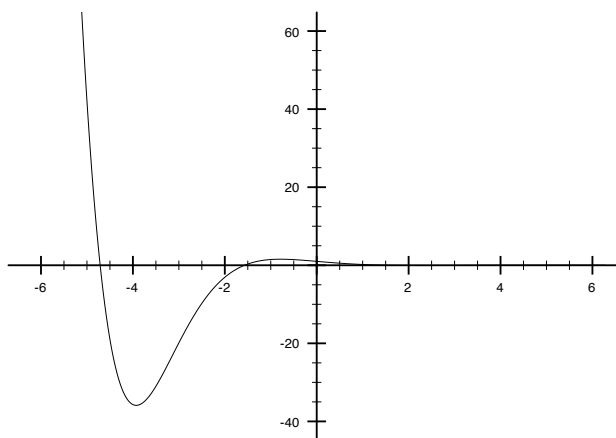
Interval	Sign of f'	Conclusion
$(0, \frac{\pi}{2})$	+	f is increasing on $(0, \frac{\pi}{2})$
$(\frac{\pi}{2}, \pi)$	−	f is decreasing on $(\frac{\pi}{2}, \pi)$
$(\pi, \frac{3\pi}{2})$	−	f is decreasing on $(\pi, \frac{3\pi}{2})$
$(\frac{3\pi}{2}, 2\pi)$	+	f is increasing on $(\frac{3\pi}{2}, 2\pi)$

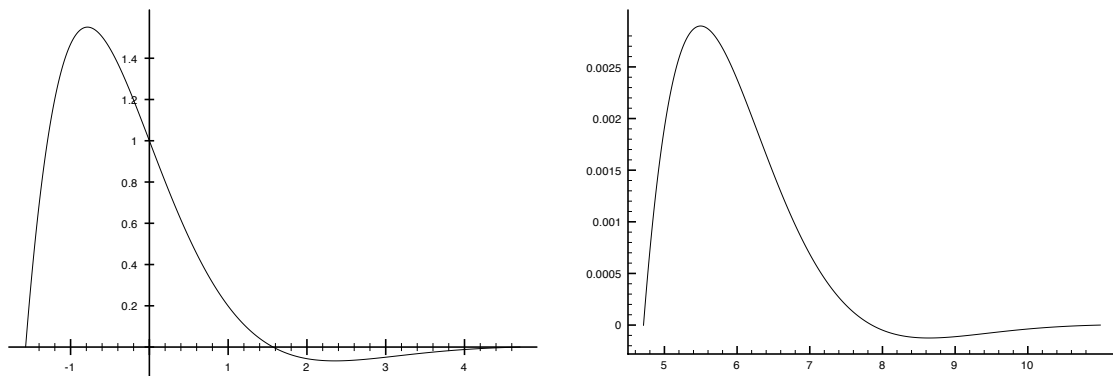
By the First Derivative Test, the information in the table above and the periodic nature of the function, f has a local maximum value at $\frac{\pi}{2} + 2k\pi$ and a local minimum value at $\frac{3\pi}{2} + 2k\pi$ for each integer k , in agreement with the approximations given in part (d).

- (f) The concavity of f appears to change at integer multiples of π ; that is, there appear to be points of inflection at $(k\pi, f(k\pi)) = (k\pi, 0)$ for each integer k .

50. Let $f(x) = e^{-x} \cos x$.

- (a) The figures below display the graph of f . In the top figure, the graph is shown for $-2\pi \leq x \leq 2\pi$. In the figure at the bottom left, attention is focused on $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. Attention is focused on $\frac{3\pi}{2} \leq x \leq \frac{7\pi}{2}$ in the figure at the bottom right. Though the function f is not periodic, observe that the increasing/decreasing and concave up/concave down nature of the function is repeated every 2π units along the x -axis.





(b) The graph appears to have a horizontal asymptote at $y = 0$ for x growing without bound in the positive direction. The graph has no vertical asymptotes.

(c) The function appears to be decreasing on the interval $(-0.8, 2.4) \approx \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and increasing on the interval $(2.4, 5.6) \approx \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. Given the repetitive structure of the graph, it follows that f is

decreasing on intervals of the form $\left(-\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi\right)$

and

increasing on intervals of the form $\left(\frac{3\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi\right)$

for each integer n . Additionally, f appears to be concave up on the interval $(0, 3.2) \approx (0, \pi)$ and concave down on the interval $(3.2, 6.4) \approx (\pi, 2\pi)$. Again taking into account the repetitive structure of the graph, it follows that f is concave up on intervals of the form $(2n\pi, (2n+1)\pi)$ and concave down on intervals of the form $((2n+1)\pi, (2n+2)\pi)$ for each integer n .

(d) The graph appears to have a local maximum value at $x = -\frac{\pi}{4} + 2n\pi$ and a local minimum value at $x = \frac{3\pi}{4} + 2n\pi$ for each integer n .

(e) Note that

$$f'(x) = -e^{-x} \sin x - e^{-x} \cos x = -e^{-x}(\sin x + \cos x).$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = \frac{3\pi}{4} + n\pi$ for each integer n . To apply the Increasing/Decreasing Function Test, use the numbers $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ to divide $(0, 2\pi)$ into three intervals.

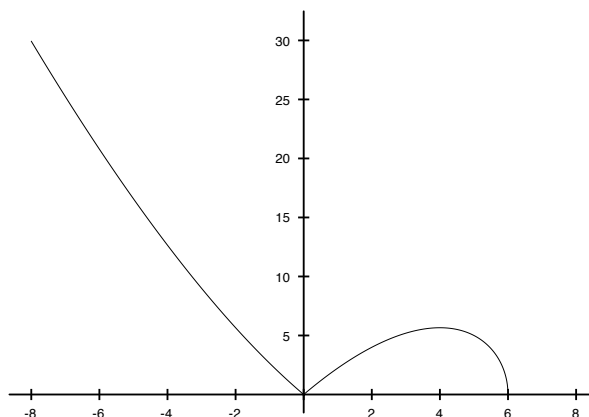
Interval	Sign of f'	Conclusion
$(0, \frac{3\pi}{4})$	−	f is decreasing on $(0, \frac{3\pi}{4})$
$(\frac{3\pi}{4}, \frac{7\pi}{4})$	+	f is increasing on $(\frac{3\pi}{4}, \frac{7\pi}{4})$
$(\frac{7\pi}{4}, 2\pi)$	−	f is decreasing on $(\frac{7\pi}{4}, 2\pi)$

By the First Derivative Test, the information in the table above and the repetitive structure of the graph, f has a local minimum value at $\frac{3\pi}{4} + 2n\pi$ and a local maximum value at $\frac{7\pi}{4} + 2n\pi$ for each integer n , in agreement with the approximations given in part (d).

- (f) The concavity of f appears to change at multiples of π ; that is, there appear to be points of inflection at $\boxed{(n\pi, f(n\pi)) = (n\pi, (-1)^n e^{-n\pi})}$ for each integer n .

51. Let $y^2 = x^2(6 - x)$, and consider $y \geq 0$. Therefore, $y = |x|\sqrt{6 - x}$.

- (a) The figure below displays the graph of $y^2 = x^2(6 - x)$ for $y \geq 0$.



- (b) The graph does not appear to have any asymptotes.

- (c) The function appears to be

decreasing on the intervals $(-\infty, 0)$ and $(4, 6)$

and increasing on the interval $(0, 4)$. Additionally, the function appears to be concave down on the interval $(0, 6)$ and concave up on the interval $(-\infty, 0)$.

- (d) The function appears to have a local minimum value at 0 and a local maximum value at 4.

- (e) Differentiating $y^2 = x^2(6 - x)$ implicitly with respect to x yields

$$2y \frac{dy}{dx} = x^2(-1) + (6 - x)(2x) = -x^2 + 12x - 2x^2 = -3x(x - 4),$$

so that

$$\frac{dy}{dx} = -\frac{3x(x - 4)}{y} = -\frac{3x(x - 4)}{|x|\sqrt{6 - x}}.$$

The critical numbers of y occur where $y'(x) = 0$ and where $y'(x)$ does not exist. $y'(x) = 0$ when $x = 4$ and does not exist when $x = 0$ and when $x = 6$. To apply the Increasing/Decreasing Function Test, use the numbers 0 and 4 to divide the domain of y , the set $\{x | x \leq 6\}$, into three intervals.

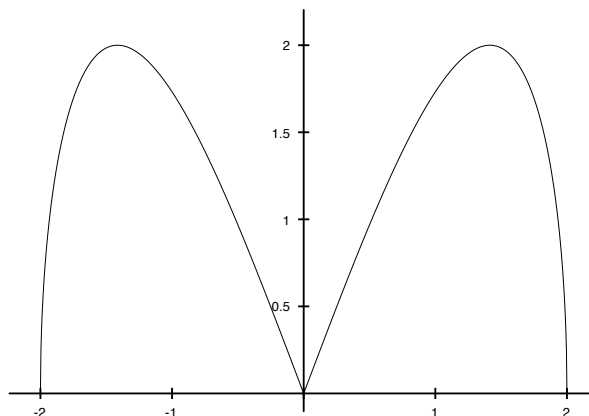
Interval	Sign of y'	Conclusion
$(-\infty, 0)$	−	y is decreasing on $(-\infty, 0)$
$(0, 4)$	+	y is increasing on $(0, 4)$
$(4, 6)$	−	y is decreasing on $(4, 6)$

By the First Derivative Test and the information in the table above, y has a local minimum value at 0 and a local maximum value at 4, in agreement with the approximations given in part (d).

- (f) The concavity of the graph appears to change at 0; that is, there appears to be a point of inflection at $(0, 0)$.

52. Let $y^2 = x^2(4 - x^2)$, and consider $y \geq 0$. Therefore, $y = |x|\sqrt{4 - x^2}$.

(a) The figure below displays the graph of $y^2 = x^2(4 - x^2)$ for $y \geq 0$.



(b) The graph does not appear to have any asymptotes.

(c) The function appears to be

increasing on the intervals $(-2, -1.4)$ and $(0, 1.4)$

and decreasing on the interval $(-1.4, 0)$ and $(1.4, 2)$. Additionally, the function appears to be concave down on the intervals $(-2, 0)$ and $0, 2$.

(d) The function appears to have a local minimum value at 0 and local maximum values at ± 1.4 .

(e) Differentiating $y^2 = x^2(4 - x^2)$ implicitly with respect to x yields

$$2y \frac{dy}{dx} = x^2(-2x) + (4 - x^2)(2x) = 2x(-x^2 + 4 - x^2) = 4x(2 - x^2),$$

so that

$$\frac{dy}{dx} = \frac{4x(2 - x^2)}{y} = \frac{4x(2 - x^2)}{|x|\sqrt{4 - x^2}}.$$

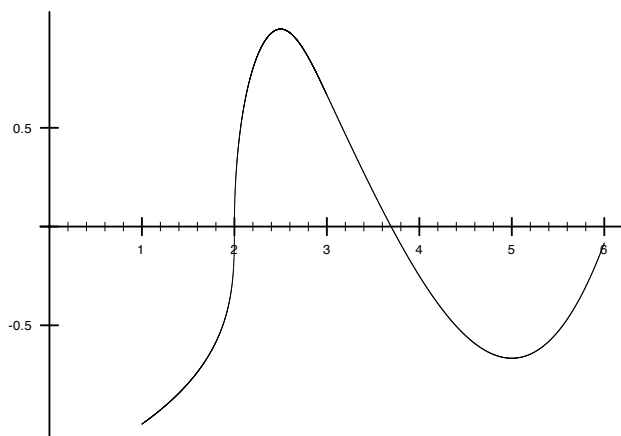
The critical numbers of y occur where $y'(x) = 0$ and where $y'(x)$ does not exist. $y'(x) = 0$ when $x = \pm\sqrt{2}$ and does not exist when $x = 0$ and when $x = \pm 2$. To apply the Increasing/Decreasing Function Test, use the numbers 0 and $\pm\sqrt{2}$ to divide the domain of y , the set $\{x \mid -2 \leq x \leq 2\}$, into four intervals.

Interval	Sign of y'	Conclusion
$(-2, -\sqrt{2})$	+	y is increasing on $(-2, -\sqrt{2})$
$(-\sqrt{2}, 0)$	−	y is decreasing on $(-\sqrt{2}, 0)$
$(0, \sqrt{2})$	+	y is increasing on $(0, \sqrt{2})$
$(\sqrt{2}, 2)$	−	y is decreasing on $(\sqrt{2}, 2)$

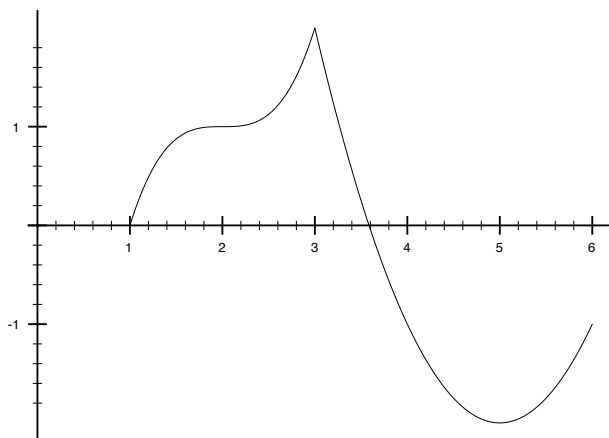
By the First Derivative Test and the information in the table above, y has a local minimum value at 0 and local maximum values at $\pm\sqrt{2}$, in excellent agreement with the approximations given in part (d).

(f) The concavity of the graph does not appear to change, so there appear to be no points of inflection.

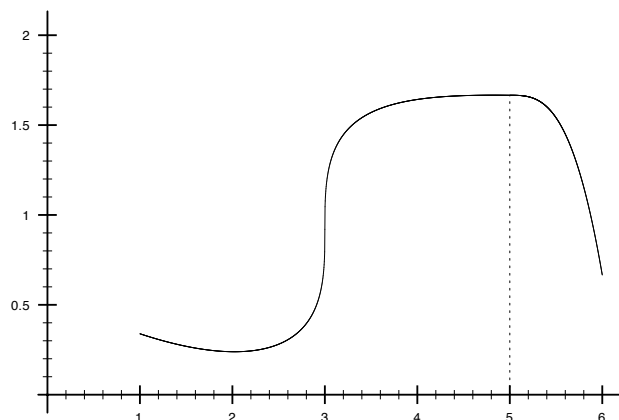
53. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2)$ does not exist, $f'(3) = -1$, $f''(3) = 0$, $f'(5) = 0$, $f''(x) < 0$ for $2 < x < 3$, and $f''(x) > 0$ for $x > 3$.



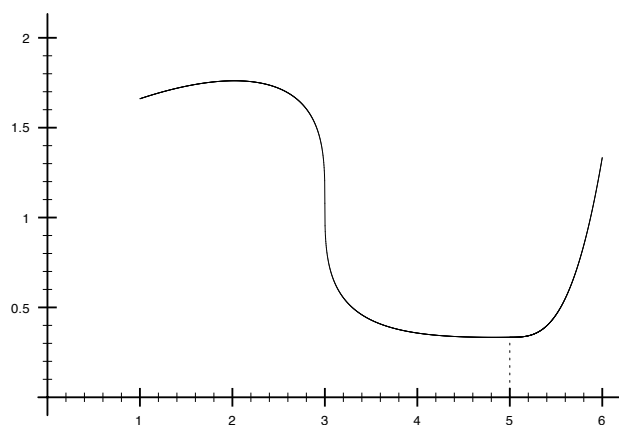
54. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2) = 0$, $f''(2) = 0$, $f'(3)$ does not exist, $f'(5) = 0$, $f''(x) > 0$ for $2 < x < 3$, and $f''(x) < 0$ for $x > 3$.



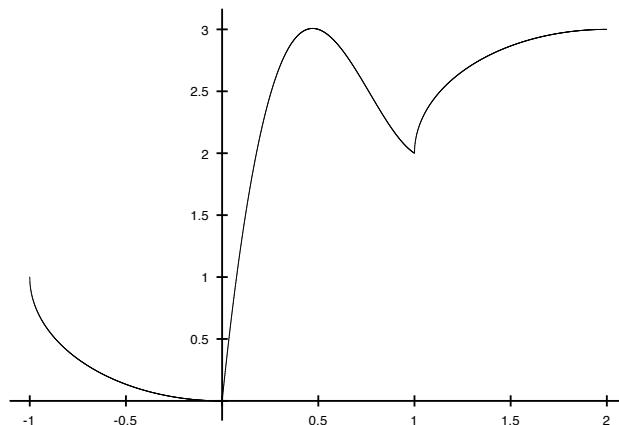
55. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2) = 0$, $\lim_{x \rightarrow 3^-} f'(x) = \infty$, $\lim_{x \rightarrow 3^+} f'(x) = \infty$, $f'(5) = 0$, $f''(x) > 0$ for $x < 3$, and $f''(x) < 0$ for $x > 3$.



56. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[2, 5]$ and satisfies the following conditions: $f'(2) = 0$, $\lim_{x \rightarrow 3^-} f'(x) = -\infty$, $\lim_{x \rightarrow 3^+} f'(x) = -\infty$, $f'(5) = 0$, $f''(x) < 0$ for $x < 3$, and $f''(x) > 0$ for $x > 3$.



57. Answers will vary. The figure below displays the graph of a function f that is continuous on the interval $[-1, 2]$ and satisfies the following conditions: $f(-1) = 1$, $f(1) = 2$, $f(2) = 3$, $f(0) = 0$, $f\left(\frac{1}{2}\right) = 3$, $\lim_{x \rightarrow -1^+} f'(x) = -\infty$, $\lim_{x \rightarrow 1^-} f'(x) = -1$, $\lim_{x \rightarrow 1^+} f'(x) = \infty$, f has a local minimum at 0, and f has a local maximum at $\frac{1}{2}$.



58. Let $y = f(x) = \ln|x^2 - 1|$, and consider the interval $(-1, 1)$. On the interval $(-1, 1)$, $x^2 - 1 < 0$, so $|x^2 - 1| = -(x^2 - 1) = 1 - x^2$ and $f(x) = \ln(1 - x^2)$. Now,

$$\begin{aligned} f'(x) &= -\frac{2x}{1 - x^2}; \text{ and} \\ f''(x) &= \frac{(1 - x^2) \cdot (-2) - (-2x) \cdot (-2x)}{(1 - x^2)^2} = -\frac{2(x^2 + 1)}{(1 - x^2)^2}. \end{aligned}$$

- (a) False. Because $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 1$, it follows that f is increasing on the interval $(-1, 0)$ and decreasing on the interval $(0, 1)$.
- (b) False. By the First Derivative Test, f has a local maximum at $(0, 0)$.
- (c) False. The range of f is the set $\{y | y \leq 0\}$.
- (d) True. Because $f''(x) < 0$ for $-1 < x < 1$, f is concave down on the interval $(-1, 1)$.
- (e) False. $\lim_{x \rightarrow 0} \ln(1 - x^2) = \ln 1 = 0$, so there is no vertical asymptote at $x = 0$. The vertical asymptotes are at $x = \pm 1$.
59. Let $f(x) = \frac{1}{x} + \ln x$, and suppose the function is defined only on the closed interval $\left[\frac{1}{e}, e\right]$.

- (a) Absolute extreme values can only occur at endpoints and critical numbers. The function f is differentiable on the domain $\left[\frac{1}{e}, e\right]$, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x - 1}{x^2},$$

so $f'(x) = 0$ when $x = 1$. Evaluating f at the endpoints of the interval $\left[\frac{1}{e}, e\right]$ and the critical number 1 yields

$$\begin{aligned} f\left(\frac{1}{e}\right) &= e + \ln\left(\frac{1}{e}\right) = e - 1 \approx 1.718 \\ f(1) &= 1 + \ln 1 = 1 \\ f(e) &= \frac{1}{e} + \ln e = \frac{1}{e} + 1 \approx 1.368 \end{aligned}$$

The absolute maximum value of f is $e - 1$, which occurs at $x = \frac{1}{e}$, and the absolute minimum value of f is 1, which occurs at $x = 1$.

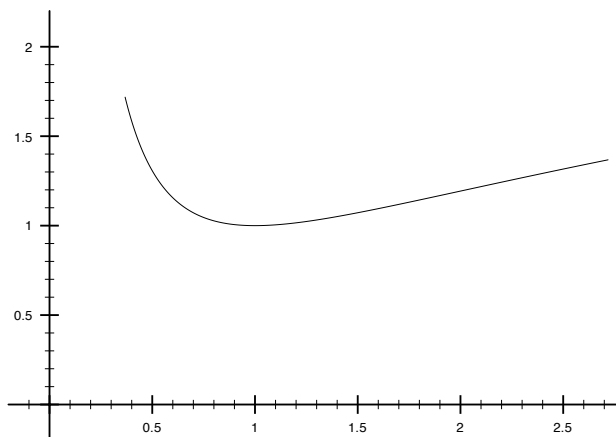
(b) With

$$f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3},$$

it follows that $f''(x) > 0$ for $\frac{1}{e} < x < 2$ and $f''(x) < 0$ for $2 < x < e$. Therefore, f is

concave up on the interval $\left(\frac{1}{e}, 2\right)$ and concave down on the interval $(2, e)$.

(c) The figure below displays the graph of f .



60. Let $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Then

$$\begin{aligned} f'(x) &= -\frac{1}{\sqrt{2\pi}}xe^{-x^2/2}; \text{ and} \\ f''(x) &= \frac{1}{\sqrt{2\pi}}(x^2e^{-x^2/2} - e^{-x^2/2}) = \frac{1}{\sqrt{2\pi}}(x^2 - 1)e^{-x^2/2}. \end{aligned}$$

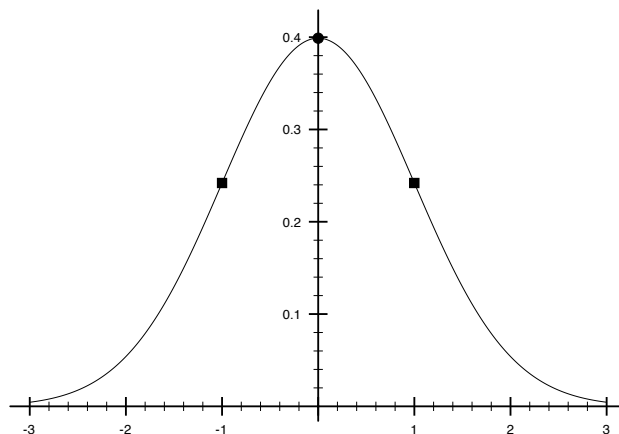
The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = 0$. To apply the Increasing/Decreasing Function Test, use the number 0 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \infty)$	-	f is decreasing on $(0, \infty)$

By the First Derivative Test and the information in the table above, f has a local maximum value at 0. The local maximum value is $f(0) = \frac{1}{\sqrt{2\pi}}$. The second derivative exists everywhere but is equal to zero when $x = \pm 1$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	+	f is concave up on $(-\infty, -1)$
$(-1, 1)$	-	f is concave down on $(-1, 1)$
$(1, \infty)$	+	f is concave up on $(1, \infty)$

The concavity of f changes at ± 1 , so the points $\left(-1, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right)$ and $\left(1, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right)$ are points of inflection. The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



61. Let $f(x) = \frac{\sin(3x)}{x\sqrt{4-x^2}}$.

Step 1 The domain of f is the set $\{x | -2 < x < 0\} \cup \{x | 0 < x < 2\}$. The x -intercepts are $\pm \frac{\pi}{3}$, and there is no y -intercept because $x = 0$ is not in the domain of f .

Step 2 Because

$$\lim_{x \rightarrow -2^+} \frac{\sin(3x)}{x\sqrt{4-x^2}} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} \frac{\sin(3x)}{x\sqrt{4-x^2}} = -\infty,$$

the lines $x = \pm 2$ are vertical asymptotes. On the other hand, by L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x\sqrt{4-x^2}} = \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{x \cdot \left(-\frac{x}{\sqrt{4-x^2}}\right) + \sqrt{4-x^2}} = \lim_{x \rightarrow 0} \frac{3\sqrt{4-x^2} \cos(3x)}{4-2x^2} = \frac{3}{2},$$

so $x = 0$ is not a vertical asymptote; rather, the graph of f has a missing point at $\left(0, \frac{3}{2}\right)$. The graph has no horizontal asymptotes because f is not defined for $|x| \geq 2$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{x\sqrt{4-x^2} \cdot 3 \cos(3x) - \sin(3x) \left(x \cdot \frac{1}{2}(x^2-4)^{-1/2}(-2x) + \sqrt{4-x^2}\right)}{x^2(4-x^2)} \\ &= \frac{3x(4-x^2) \cos(3x) - (4-2x^2) \sin(3x)}{x^2(4-x^2)^{3/2}}; \text{ and} \\ f''(x) &= \frac{(-9x^6 + 78x^4 - 164x^2 + 32) \sin(3x) - 12x(x^4 - 6x^2 + 8) \cos(3x)}{x^3(4-x^2)^{5/2}}, \end{aligned}$$

where the second derivative was obtained using *Mathematica*. The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Using the command

`Solve [ D [ Sin[3x]/(x Sqrt[4-x^2]), x ] == 0, x ]`

in *Wolfram Alpha*, we find that $f'(x) = 0$ when $x \approx \pm 1.642$ and $x \approx \pm 1.908$. Now, $f'(x)$ does not exist when $x = 0$ and when $x = \pm 2$; however, because none of these values are in the domain of f , none are critical numbers. Therefore, ± 1.642 and ± 1.908 are the critical numbers of f . At the points $(\pm 1.642, -0.521)$ and $(\pm 1.908, -0.464)$, the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0, ± 1.642 and ± 1.908 to divide the interval $(-2, 2)$ into six subintervals.

Interval	Sign of f'	Conclusion
$(-2, -1.908)$	+	f is increasing on $(-2, -1.908)$
$(-1.908, -1.642)$	-	f is decreasing on $(-1.908, -1.642)$
$(-1.642, 0)$	+	f is increasing on $(-1.642, 0)$
$(0, 1.642)$	-	f is decreasing on $(0, 1.642)$
$(1.642, 1.908)$	+	f is increasing on $(1.642, 1.908)$
$(1.908, 2)$	-	f is decreasing on $(1.908, 2)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at ± 1.908 and a local minimum value at ± 1.642 . The

$$\text{local maximum values are } f(\pm 1.908) \approx -0.461,$$

and the

$$\text{local minimum values are } f(\pm 1.642) \approx -0.521.$$

Step 6 The second derivative does not exist when $x = 0$ and when $x = \pm 2$ and is equal to zero when $x \approx \pm 0.763$ and when $x \approx \pm 1.826$. These latter values were obtained using the command

$$\text{Solve [D [Sin[3x]/(x Sqrt[4-x^2]) , \{x, 2\}] == 0 , x]}$$

in *Wolfram Alpha*. Use the numbers 0, ± 0.763 and ± 1.826 to divide the interval $(-2, 2)$ into six subintervals, and determine the sign of f'' on each subinterval.

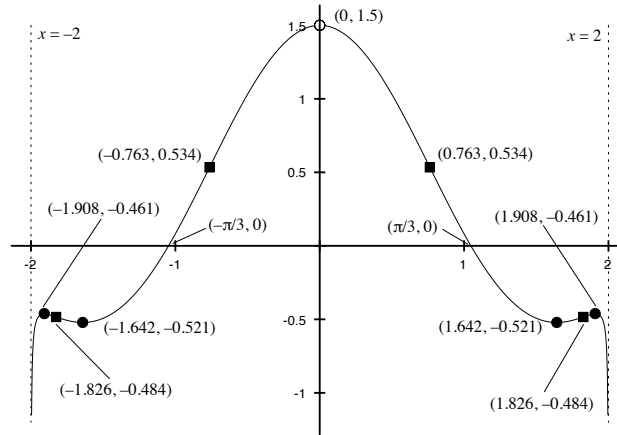
Interval	Sign of f''	Conclusion
$(-2, -1.826)$	-	f is concave down on $(-2, -1.826)$
$(-1.826, -0.763)$	+	f is concave up on $(-1.826, -0.763)$
$(-0.763, 0)$	-	f is concave down on $(-0.763, 0)$
$(0, 0.763)$	-	f is concave down on $(0, 0.763)$
$(0.763, 1.826)$	+	f is concave up on $(0.763, 1.826)$
$(1.826, 2)$	-	f is concave down on $(1.826, 2)$

The concavity of f changes at approximately ± 0.763 and ± 1.826 , so the points

$$(\pm 0.763, 0.534) \quad \text{and} \quad (\pm 1.826, -0.484)$$

are approximate points of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles, and the points of inflection are highlighted by closed squares.



62. Let $f(x) = x^{\sqrt{x}}$.

Step 1 The domain of f is the set $\{x|x > 0\}$. There are no x -intercepts, and there is no y -intercept because $x = 0$ is not in the domain of f .

Step 2 Because

$$\lim_{x \rightarrow \infty} x^{\sqrt{x}} = \infty,$$

the graph of f has no horizontal asymptote. At 0^+ , the expression $x^{\sqrt{x}}$ is an indeterminate form of the type 0^0 . Let $y = x^{\sqrt{x}}$. Then

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x,$$

which is an indeterminate form at 0^+ of the type $0 \cdot \infty$. Rewrite

$$\sqrt{x} \ln x \quad \text{as} \quad \frac{\ln x}{x^{-1/2}},$$

which is now an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} [\sqrt{x} \ln x] = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-\frac{1}{2}x^{-3/2}} = \lim_{x \rightarrow 0^+} (-2x^{1/2}) = 0.$$

Because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{\sqrt{x}} = e^0 = 1,$$

so $x = 0$ is not a vertical asymptote; rather, the graph of f has a missing point at $(0, 1)$.

Step 3 To determine $f'(x)$, first take the natural logarithm of both sides of $f(x) = x^{\sqrt{x}}$ to obtain $\ln f(x) = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$. Now, by implicit differentiation,

$$\frac{1}{f(x)} f'(x) = \sqrt{x} \cdot \frac{1}{x} + \frac{1}{2\sqrt{x}} \ln x = \frac{2 + \ln x}{2\sqrt{x}},$$

so that

$$f'(x) = x^{\sqrt{x}} \frac{2 + \ln x}{2\sqrt{x}}.$$

Next,

$$\begin{aligned}
 f''(x) &= x^{\sqrt{x}} \frac{d}{dx} \left(\frac{2 + \ln x}{2\sqrt{x}} \right) + \frac{2 + \ln x}{2\sqrt{x}} \frac{d}{dx} x^{\sqrt{x}} \\
 &= x^{\sqrt{x}} \frac{2\sqrt{x} \cdot \frac{1}{x} - (2 + \ln x)x^{-1/2}}{4x} + x^{\sqrt{x}} \left(\frac{2 + \ln x}{2\sqrt{x}} \right)^2 \\
 &= x^{\sqrt{x}} \left[-\frac{\ln x}{4x^{3/2}} + \left(\frac{2 + \ln x}{2\sqrt{x}} \right)^2 \right].
 \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x = \frac{1}{e^2}$ and does not exist when $x = 0$. However, $x = 0$ is not in the domain of f , so $x = 0$ is not a critical number of f . Therefore, $\boxed{\frac{1}{e^2}}$ is the only critical number of f . At the point $\left(\frac{1}{e^2}, \frac{1}{e^{2/e}}\right) \approx (0.135, 0.479)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number $\frac{1}{e^2}$ to divide the interval $(0, \infty)$ into two subintervals.

Interval	Sign of f'	Conclusion
$(0, \frac{1}{e^2})$	–	f is decreasing on $(0, \frac{1}{e^2})$
$(\frac{1}{e^2}, \infty)$	+	f is increasing on $(\frac{1}{e^2}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $\frac{1}{e^2}$. The $\boxed{\text{local minimum value is } f\left(\frac{1}{e^2}\right) = \frac{1}{e^{2/e}} \approx 0.479}$.

Step 6 For $0 < x < 1$,

$$-\frac{\ln x}{4x^{3/2}} > 0, \quad \text{so that} \quad -\frac{\ln x}{4x^{3/2}} + \left(\frac{2 + \ln x}{2\sqrt{x}}\right)^2 > 0 \quad \text{and} \quad f''(x) > 0.$$

For $x > 1$,

$$-\frac{\ln x}{4x^{3/2}} > -\frac{\ln x}{4x} > -\frac{8 \ln x}{4x},$$

so that

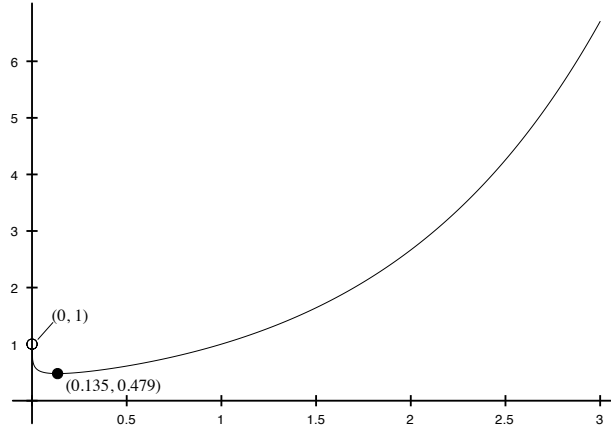
$$-\frac{\ln x}{4x^{3/2}} + \left(\frac{2 + \ln x}{2\sqrt{x}}\right)^2 > -\frac{8 \ln x}{4x} + \left(\frac{2 + \ln x}{2\sqrt{x}}\right)^2 = \left(\frac{2 - \ln x}{2\sqrt{x}}\right)^2 \geq 0 \quad \text{and} \quad f''(x) > 0.$$

Finally,

$$f''(1) = 1^{\sqrt{1}} \left[-\frac{\ln 1}{4(1)^{3/2}} + \left(\frac{2 + \ln 1}{2\sqrt{1}}\right)^2 \right] = 1 > 0.$$

Therefore, $f''(x) > 0$ for all $x > 0$, and f is concave up for all $x > 0$. Because the concavity of f never changes, there are $\boxed{\text{no points of inflection}}$.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



63. Let $f(x) = x^{1/x}$.

Step 1 The domain of f is the set $\{x|x > 0\}$. There are no x -intercepts, and there is no y -intercept because $x = 0$ is not in the domain of f .

Step 2 At ∞ , the expression $x^{1/x}$ is an indeterminate form of the type ∞^0 . Let $y = x^{1/x}$. Then

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x = \frac{\ln x}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0.$$

Because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} x^{1/x} = e^0 = 1,$$

so $y = 1$ is a horizontal asymptote. At 0^+ , the expression $x^{1/x}$ is of the form 0^∞ , which is not an indeterminate form; rather,

$$\lim_{x \rightarrow 0^+} x^{1/x} = 0.$$

The graph of f has a missing point at $(0, 0)$ and not a vertical asymptote at $x = 0$.

Step 3 To determine $f'(x)$, first take the natural logarithm of both sides of $f(x) = x^{1/x}$ to obtain $\ln f(x) = \ln x^{1/x} = \frac{1}{x} \ln x$. Now, by implicit differentiation,

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x = \frac{1 - \ln x}{x^2},$$

so that

$$f'(x) = x^{1/x} \frac{1 - \ln x}{x^2}.$$

Next,

$$\begin{aligned} f''(x) &= x^{1/x} \frac{d}{dx} \left(\frac{1 - \ln x}{x^2} \right) + \frac{1 - \ln x}{x^2} \frac{d}{dx} x^{1/x} \\ &= x^{1/x} \frac{x^2 \left(-\frac{1}{x} \right) - (1 - \ln x)(2x)}{x^4} + x^{1/x} \left(\frac{1 - \ln x}{x^2} \right)^2 \\ &= x^{1/x} \left[\frac{2 \ln x - 3}{x^3} + \left(\frac{1 - \ln x}{x^2} \right)^2 \right]. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. Now, $f'(x) = 0$ when $x = e$ and does not exist when $x = 0$. However, $x = 0$ is not in the domain of f , so $x = 0$ is not a critical number of f . Therefore, $[e]$ is the only critical number of f . At the point $(e, e^{1/e}) \approx (2.718, 1.445)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number e to divide the interval $(0, \infty)$ into two subintervals.

Interval	Sign of f'	Conclusion
$(0, e)$	+	f is increasing on $(0, e)$
(e, ∞)	−	f is decreasing on (e, ∞)

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at e . The local maximum value is $f(e) = e^{1/e} \approx 1.445$.

Step 6 The second derivative exists for all $x > 0$ and is equal to zero when $x \approx 0.582$ and when $x \approx 4.368$. These latter values were obtained using the command

Solve [D [$x^{1/x}$, {x, 2}] == 0, x]

in *Wolfram Alpha*. Use the numbers 0.582 and 4.368 to divide the interval $(0, \infty)$ into three subintervals, and determine the sign of f'' on each subinterval.

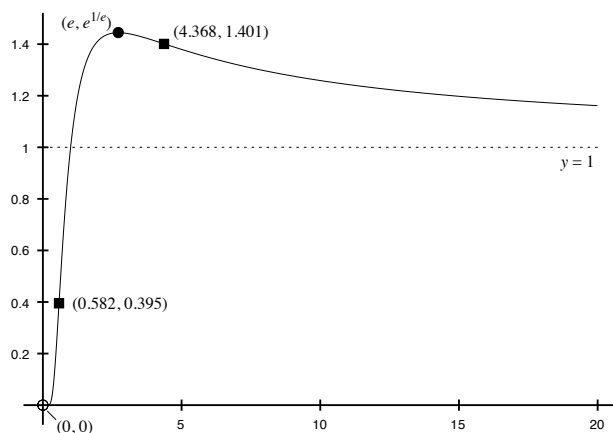
Interval	Sign of f''	Conclusion
$(0, 0.582)$	+	f is concave up on $(0, 0.582)$
$(0.582, 4.368)$	−	f is concave down on $(0.582, 4.368)$
$(4.368, \infty)$	+	f is concave up on $(4.368, \infty)$

The concavity of f changes at approximately 0.582 and 4.368, so the points

$(0.582, 0.395)$ and $(4.368, 1.401)$

are approximate points of inflection of f .

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the points of inflection are highlighted by closed squares.



64. Let $f(x) = \frac{1}{x} \tan x$, and consider the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Step 1 The domain of f is the set $\left\{x \mid -\frac{\pi}{2} < x < 0\right\} \cup \left\{0 < x < \frac{\pi}{2}\right\}$. There are no x -intercepts, and there is no y -intercept because $x = 0$ is not in the domain of f .

Step 2 Because

$$\lim_{x \rightarrow -\pi/2^+} \frac{\tan x}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow \pi/2^-} \frac{\tan x}{x} = \infty,$$

the lines $x = \pm\pi/2$ are vertical asymptotes. On the other hand, by L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = 1,$$

so the graph of f does not have a vertical asymptote at $x = 0$; rather, the graph has a missing point at $(0, 1)$.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{x \sec^2 x - \tan x}{x^2}; \text{ and} \\ f''(x) &= \frac{x^2 (x \cdot 2 \sec x \cdot \sec x \tan x + \sec^2 x - \sec^2 x) - 2x (x \sec^2 x - \tan x)}{x^4} \\ &= \frac{2x^2 \sec^2 x \tan x - 2x \sec^2 x + 2 \tan x}{x^3}. \end{aligned}$$

Note that $x \sec^2 x - \tan x = 0$ when

$$x = \frac{\tan x}{\sec^2 x} = \sin x \cos x = \frac{1}{2} \sin(2x).$$

The only solution of this equation is $x = 0$. However, $x = 0$ is not in the domain of f , so $x = 0$ is not a critical number of f . Therefore, f has no critical numbers.

Step 4 To apply the Increasing/Decreasing Function Test, use the number 0 to divide the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ into two subintervals.

Interval	Sign of f'	Conclusion
$(-\frac{\pi}{2}, 0)$	–	f is decreasing on $(-\frac{\pi}{2}, 0)$
$(0, \frac{\pi}{2})$	+	f is increasing on $(0, \frac{\pi}{2})$

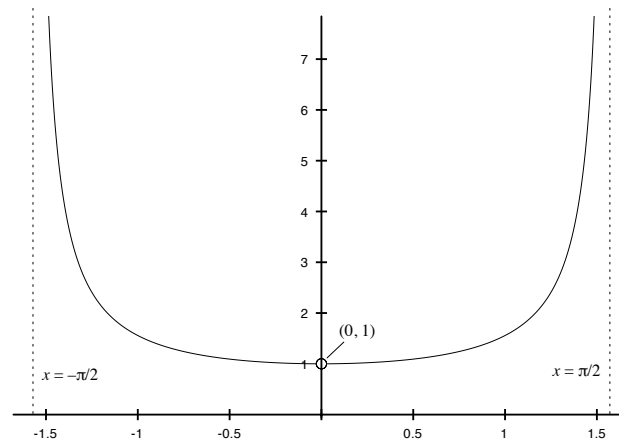
Step 5 Because f has no critical numbers, f has no local extrema.

Step 6 The second derivative is never equal to 0 on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (use *Mathematica*, for example) and does not exist when $x = 0$. Use the number 0 to divide the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ into two subintervals, and determine the sign of f'' on each subinterval.

Interval	Sign of f''	Conclusion
$(-\frac{\pi}{2}, 0)$	+	f is concave up on $(-\frac{\pi}{2}, 0)$
$(0, \frac{\pi}{2})$	+	f is concave up on $(0, \frac{\pi}{2})$

Because the concavity of f does not change, f has no points of inflection.

Step 7 The figure below displays the graph of f .



4.7: Optimization

Applications and Extensions

1. Let x denote the length of fencing used to enclose the plot on each side perpendicular to the highway and y the length of fencing used parallel to the highway (see the diagram below). The area A enclosed by the fencing is then $A = xy$. With 3000 m of fencing available, x and y are related by the equation

$$2x + y = 3000 \quad \text{or} \quad y = 3000 - 2x.$$

Substituting for y in the area formula yields

$$A = x(3000 - 2x) = 3000x - 2x^2.$$

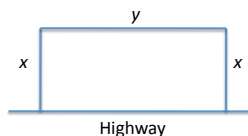
The domain of this function is the closed interval $[0, 1500]$. The function A is differentiable on the open interval $(0, 1500)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 3000 - 4x,$$

so $A'(x) = 0$ when $x = 750$. Evaluating A at the endpoints of the interval $[0, 1500]$ and at the critical number 750 yields

$$A(0) = 0, \quad A(750) = 1,125,000, \quad \text{and} \quad A(1500) = 0.$$

The largest area that can be enclosed is therefore $\boxed{1,125,000 \text{ m}^2}$, achieved by using 1500 m of fencing parallel to the highway and 750 m of fencing on each side perpendicular to the highway.



2. Let x denote the length of fencing used to enclose the plot on each side perpendicular to the highway and y the length of fencing used parallel to the highway (see the diagram below). The area A enclosed by the fencing is then $A = xy$. With 3000 m of fencing available, x and y are related by the equation

$$2x + 2y - 5 = 3000 \quad \text{or} \quad y = \frac{3005}{2} - x.$$

Substituting for y in the area formula yields

$$A = x \left(\frac{3005}{2} - x \right) = \frac{3005}{2}x - x^2.$$

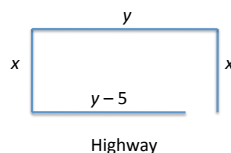
The domain of this function is the closed interval $\left[0, \frac{3005}{2}\right] = [0, 1502.5]$. The function A is differentiable on the open interval $(0, 1502.5)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{3005}{2} - 2x,$$

so $A'(x) = 0$ when $x = \frac{3005}{4} = 751.25$. Evaluating A at the endpoints of the interval $[0, 1502.5]$ and at the critical number 751.25 yields

$$A(0) = 0, \quad A(751.25) = 564,376.5625, \quad \text{and} \quad A(1502.5) = 0.$$

The largest area that can be enclosed is therefore $\boxed{564,376.5625 \text{ m}^2}$, achieved by using 751.25 m of fencing on each side perpendicular to the highway, 751.25 m of fencing parallel to the highway on the side without the access opening, and 746.25 m of fencing parallel to the highway on the side with the access opening.



3. Let x and y denote the width and length of the rectangular plot enclosed on all sides by fencing (see the diagram below). The area A enclosed by the fencing is then $A = xy$. With L m of fencing available, x and y are related by the equation

$$2x + 2y = L \quad \text{or} \quad y = \frac{L}{2} - x.$$

Substituting for y in the area formula yields

$$A = x \left(\frac{L}{2} - x \right) = \frac{L}{2}x - x^2.$$

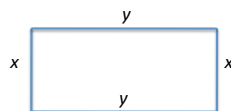
The domain of this function is the closed interval $\left[0, \frac{L}{2}\right]$. The function A is differentiable on the open interval $\left(0, \frac{L}{2}\right)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{L}{2} - 2x,$$

so $A'(x) = 0$ when $x = \frac{L}{4}$. Evaluating A at the endpoints of the interval $\left[0, \frac{L}{2}\right]$ and at the critical number $\frac{L}{4}$ yields

$$A(0) = 0, \quad A\left(\frac{L}{4}\right) = \frac{L^2}{16}, \quad \text{and} \quad A\left(\frac{L}{2}\right) = 0.$$

The largest area that can be enclosed is therefore $\frac{L^2}{16}$ m², achieved by using $L/4$ m of fencing on each side of the rectangle; that is, by making a square plot with dimensions $\boxed{\frac{L}{4} \text{ m} \times \frac{L}{4} \text{ m}}$.



4. Using the diagram in the text, the length of the third side of the triangle is $2x \cos \theta$ and the height of the triangle is $x \sin \theta$. Therefore, the area A of the triangle is $A = x^2 \sin \theta \cos \theta$. Given that the perimeter of the triangle is fixed at L , x and θ are related by the equation

$$2x + 2x \cos \theta = L \quad \text{or} \quad x = \frac{L}{2(1 + \cos \theta)}.$$

Substituting for x in the area formula yields

$$A = \left(\frac{L}{2(1 + \cos \theta)} \right)^2 \sin \theta \cos \theta = \frac{L^2}{4} \frac{\sin \theta \cos \theta}{(1 + \cos \theta)^2}.$$

The domain of this function is the closed interval $\left[0, \frac{\pi}{2}\right]$. Now,

$$\begin{aligned}
 A'(\theta) &= \frac{L^2}{4} \cdot \frac{(1 + \cos \theta)^2 (\cos^2 \theta - \sin^2 \theta) - 2(1 + \cos \theta)(-\sin \theta)(\sin \theta \cos \theta)}{(1 + \cos \theta)^4} \\
 &= \frac{L^2}{4} \cdot \frac{\cos^2 \theta + \cos^3 \theta - \sin^2 \theta - \sin^2 \theta \cos \theta + 2 \sin \theta \cos \theta}{(1 + \cos \theta)^3} \\
 &= \frac{L^2}{4} \cdot \frac{\cos^2 \theta + \cos \theta (\sin^2 \theta + \cos^2 \theta) - 1 + \cos^2 \theta}{(1 + \cos \theta)^3} \\
 &= \frac{L^2}{4} \cdot \frac{2 \cos^2 \theta + \cos \theta - 1}{(1 + \cos \theta)^3} \\
 &= \frac{L^2}{4} \cdot \frac{(2 \cos \theta - 1)(1 + \cos \theta)}{(1 + \cos \theta)^3} = \frac{L^2}{4} \cdot \frac{2 \cos \theta - 1}{(1 + \cos \theta)^2},
 \end{aligned}$$

so the only critical number in the open interval $\left(0, \frac{\pi}{2}\right)$ is $\frac{\pi}{3}$, where $A'\left(\frac{\pi}{3}\right) = 0$. Evaluating A at the endpoints of the domain and at the critical number $\frac{\pi}{3}$ yields

$$A(0) = 0, \quad A\left(\frac{\pi}{3}\right) = \frac{L^2 \sqrt{3}}{36}, \quad \text{and} \quad A\left(\frac{\pi}{2}\right) = 0.$$

The maximum area is therefore enclosed by the triangle with $\theta = \frac{\pi}{3}$. Then

$$x = \frac{L}{2(1 + \cos \theta)} = \frac{L}{2(1 + \frac{1}{2})} = \frac{L}{3},$$

so the resulting triangle has all sides with length equal to $L/3$; that is, the triangle with maximum area is an equilateral triangle with side length $L/3$.

5. Using the diagram in the text, let x denote the length of each of the three vertical sides and y denote the length of each of the two horizontal sides. The area A enclosed by the fencing is then $A = xy$. With 200 m of fencing available, x and y are related by the equation

$$2y + 3x = 200 \quad \text{or} \quad y = 100 - \frac{3}{2}x.$$

Substituting for y in the area formula yields

$$A = x \left(100 - \frac{3}{2}x\right) = 100x - \frac{3}{2}x^2.$$

The domain of this function is the closed interval $\left[0, \frac{200}{3}\right]$. The function A is differentiable on the open interval $\left(0, \frac{200}{3}\right)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 100 - 3x,$$

so $A'(x) = 0$ when $x = \frac{100}{3}$. Evaluating A at the endpoints of the interval $\left[0, \frac{200}{3}\right]$ and at the critical number $\frac{100}{3}$ yields

$$A(0) = 0, \quad A\left(\frac{100}{3}\right) = \frac{5000}{3}, \quad \text{and} \quad A\left(\frac{200}{3}\right) = 0.$$

The largest area that can be enclosed is therefore $\frac{5000}{3} \text{ m}^2$.

6. Let x and y denote the length and width of the rectangular plot of land, and suppose that the extra fencing used to divide the large plot into two pieces has length y (see the diagram below). The amount of fencing F used to enclose the plots is then $F = 2x + 3y$. Given that the area enclosed has to be equal to 600 m^2 , x and y are related by the equation

$$xy = 600 \quad \text{or} \quad y = \frac{600}{x}.$$

Substituting for y in the fencing formula yields

$$F = 2x + 3\left(\frac{600}{x}\right) = 2x + \frac{1800}{x}.$$

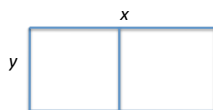
The domain for this function is the open interval $(0, \infty)$. Now,

$$F'(x) = 2 - \frac{1800}{x^2},$$

so the only critical number in the open interval $(0, \infty)$ is 30, where $F'(30) = 0$. Note that $x = -30$ and $x = 0$ are not in the domain of F . Using the Second Derivative Test,

$$F''(x) = \frac{3600}{x^3}, \quad \text{so} \quad F''(30) = \frac{3600}{30^3} > 0$$

and F has a local minimum value at 30. Because $F''(x) > 0$ for all $x > 0$, the local minimum is also an absolute minimum. Therefore, the minimum amount of fencing is used to enclose the plot when $x = 30$ and $y = \frac{600}{30} = 20$. A 30 m $\times$ 20 m enclosure uses the least amount of fencing.



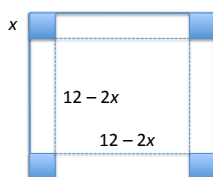
7. Let x denote the side length of the square cut from each corner of the base (see the diagram below). When the sides are turned up, the resulting box will have a square base $12 - 2x$ cm on a side and a height of x cm; the volume will therefore be $V = x(12 - 2x)^2 = 4x(6 - x)^2$. Because both $x \geq 0$ and $12 - 2x \geq 0$, it follows that $x \leq 6$, so the domain of V is the closed interval $[0, 6]$. The function V is differentiable on the open interval $(0, 6)$, so the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = 4x \cdot 2(6 - x)(-1) + 4(6 - x)^2 = (6 - x)(-8x + 24 - 4x) = (6 - x)(24 - 12x),$$

so the only critical number inside the open interval $(0, 6)$ is $x = 2$, where $V'(2) = 0$. Note that $x = 6$ is not in the open interval $(0, 6)$. Evaluating V at the endpoints of the interval $[0, 6]$ and at the critical number 2 yields

$$V(0) = 0, \quad V(2) = 128, \quad \text{and} \quad V(6) = 0.$$

The largest volume is therefore achieved when $x = 2$, and the box with the largest volume has a square base $12 - 2(2) = 8$ cm on a side and a height of 2 cm; that is, the box with the largest possible volume has dimensions 8 cm $\times$ 8 cm $\times$ 2 cm.



8. Let x denote the side length of the square cut from each corner of the base (see the diagram below). When the sides are turned up, the resulting box will have a rectangular base measuring $30 - 2x$ cm by $45 - 2x$ cm and a height of x cm; the volume will therefore be

$$V = x(30 - 2x)(45 - 2x) = 4x^3 - 150x^2 + 1350x.$$

To determine the domain of V , note that the length of each of the three sides of the box must be non-negative. This requires $x \geq 0$, $30 - 2x \geq 0$, and $45 - 2x \geq 0$. The solution of this set of three inequalities is $0 \leq x \leq 15$; therefore, the domain of V is the closed interval $[0, 15]$. The function V is differentiable on the open interval $(0, 15)$, so the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = 12x^2 - 300x + 1350 = 6(2x^2 - 50x + 225),$$

so the only critical number inside the open interval $(0, 15)$ is

$$x = \frac{50 - \sqrt{50^2 - 4(2)(225)}}{4} = \frac{50 - \sqrt{700}}{4} = \frac{25 - 5\sqrt{7}}{2} \approx 5.886.$$

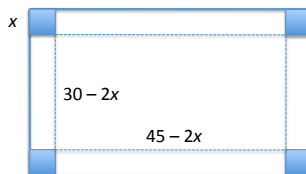
Note that

$$x = \frac{50 + \sqrt{50^2 - 4(2)(225)}}{4} = \frac{50 + \sqrt{700}}{4} = \frac{25 + 5\sqrt{7}}{2} \approx 19.114$$

is not in the domain of V . Evaluating V at the endpoints of the interval $[0, 15]$ and at the approximate critical number 5.886 yields

$$V(0) = 0, \quad V(5.886) \approx 3565.032, \quad \text{and} \quad V(15) = 0.$$

The largest volume is therefore achieved when $x \approx 5.886$, and the box with the largest volume has a rectangular base measuring approximately $30 - 2(5.886) = 18.228$ cm by $45 - 2(5.886) = 33.228$ cm and a height of approximately 5.886 cm; that is, the box with the largest possible volume has dimensions $18.228 \text{ cm} \times 33.228 \text{ cm} \times 5.886 \text{ cm}$.



9. Let s denote the side length of the square base and h denote the height of the open top box. The surface area A of the box is then

$$A = 4sh + s^2,$$

where the first term accounts for the area of the four vertical rectangular sides of the box and the second term accounts for the area of the base. Given that the box is to have a volume of 2000 cm^3 , s and h are related by the equation

$$s^2h = 2000 \quad \text{or} \quad h = \frac{2000}{s^2}.$$

Substituting for h in the area formula yields

$$A = \frac{8000}{s} + s^2.$$

The domain of A is the interval $(0, \infty)$. Now,

$$A'(s) = -\frac{8000}{s^2} + 2s = \frac{2(s^3 - 4000)}{s^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $s = 10\sqrt[3]{4}$, where $A'(s) = 0$. Note that $s = 0$ is not in the domain of A . Using the Second Derivative Test,

$$A''(s) = \frac{16000}{s^3} + 2 \quad \text{so} \quad A''(10\sqrt[3]{4}) = 6 > 0$$

and A has a local minimum value at $10\sqrt[3]{4}$. Because $A''(s) > 0$ for all $s > 0$, the local minimum is also an absolute minimum. Therefore, the minimum amount of material is used to make the box when $s = 10\sqrt[3]{4}$ and

$$h = \frac{2000}{100\sqrt[3]{16}} = \frac{20\sqrt[3]{4}}{4} = 5\sqrt[3]{4}.$$

The box should have a square base measuring $10\sqrt[3]{4} \approx 15.874$ cm on a side and a height of $5\sqrt[3]{4} \approx 7.937$ cm; that is, the dimensions of the box using the minimum amount of material are

$$\boxed{10\sqrt[3]{4} \times 10\sqrt[3]{4} \times 5\sqrt[3]{4} \approx 15.874 \text{ cm} \times 15.874 \text{ cm} \times 7.937 \text{ cm}}.$$

10. Let s denote the side length of the square base and h denote the height of the closed top box. The surface area A of the box is then

$$A = 4sh + 2s^2,$$

where the first term accounts for the area of the four vertical rectangular sides of the box and the second term accounts for the area of the base and the top. Given that the box is to have a volume of 2000 cm^3 , s and h are related by the equation

$$s^2h = 2000 \quad \text{or} \quad h = \frac{2000}{s^2}.$$

Substituting for h in the area formula yields

$$A = \frac{8000}{s} + 2s^2.$$

The domain of A is the interval $(0, \infty)$. Now,

$$A'(s) = -\frac{8000}{s^2} + 4s = \frac{4(s^3 - 2000)}{s^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $s = 10\sqrt[3]{2}$, where $A'(s) = 0$. Note that $s = 0$ is not in the domain of A . Using the Second Derivative Test,

$$A''(s) = \frac{16000}{s^3} + 4 \quad \text{so} \quad A''(10\sqrt[3]{2}) = 12 > 0$$

and A has a local minimum value at $10\sqrt[3]{2}$. Because $A''(s) > 0$ for all $s > 0$, the local minimum is also an absolute minimum. Therefore, the minimum amount of material is used to make the box when $s = 10\sqrt[3]{2}$ and

$$h = \frac{2000}{100\sqrt[3]{4}} = \frac{20\sqrt[3]{2}}{2} = 10\sqrt[3]{2}.$$

The box should have a square base measuring $10\sqrt[3]{2} \approx 12.599$ cm on a side and a height of $10\sqrt[3]{2} \approx 12.599$ cm; that is, the box should be a $\text{cube with side length } 10\sqrt[3]{2} \approx 12.599 \text{ cm}$.

11. Let r denote the radius and h the height of the cylindrical can. The cost C of the material needed to manufacture the can is

$$C = 2\pi r^2(20) + 2\pi rh(15) = 40\pi r^2 + 30\pi rh,$$

where the first term accounts for the cost of the top and bottom of the can and the second term accounts for the lateral surface area of the can. Given that the can is to have a capacity of the 10 m^3 , r and h are related by the equation

$$\pi r^2 h = 10 \quad \text{or} \quad h = \frac{10}{\pi r^2}.$$

Substituting for h in the cost formula yields

$$C = 40\pi r^2 + \frac{300}{r}.$$

The domain of C is the interval $(0, \infty)$. Now,

$$C'(r) = 80\pi r - \frac{300}{r^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $r = \sqrt[3]{\frac{15}{4\pi}}$, where $C'(r) = 0$. Note that $r = 0$ is not in the domain of C . Using the Second Derivative Test,

$$C''(r) = 80\pi + \frac{600}{r^3} \quad \text{so} \quad C''\left(\sqrt[3]{\frac{15}{4\pi}}\right) = 240\pi > 0,$$

and C has a local minimum value at $\sqrt[3]{\frac{15}{4\pi}}$. Because $C''(r) > 0$ for all $r > 0$, the local minimum is also

an absolute minimum. Therefore, the minimum cost of material is achieved when $r = \sqrt[3]{\frac{15}{4\pi}}$ and

$$h = \frac{10}{\pi \sqrt[3]{\frac{225}{16\pi^2}}} = \frac{8}{3} \sqrt[3]{\frac{15}{4\pi}}.$$

The can should have a radius of $\sqrt[3]{\frac{15}{4\pi}} \approx 1.061$ m and a height of $\frac{8}{3} \sqrt[3]{\frac{15}{4\pi}} \approx 2.829$ m.

12. Let x denote the length of fencing used along the front and back of the rectangular plot and y denote the length of fencing used along the other two sides (see the diagram below). The cost C of the fencing is

$$C = x(20) + x(10) + 2y(10) = 30x + 20y.$$

Given that the plot of land enclosed by the fencing is to have an area of 60,000 m², x and y are related by the equation

$$xy = 60,000 \quad \text{or} \quad y = \frac{60,000}{x}.$$

Substituting for y in the cost equation yields

$$C = 30x + \frac{1,200,000}{x}.$$

The domain of C is the interval $(0, \infty)$. Now,

$$C'(x) = 30 - \frac{1,200,000}{x^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $x = 200$, where $C'(x) = 0$. Note that neither $x = -200$ nor $x = 0$ is in the domain of C . Using the Second Derivative Test,

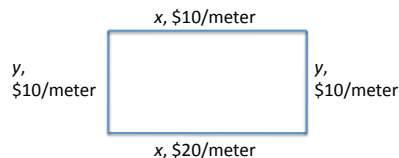
$$C''(x) = \frac{2,400,000}{x^3} \quad \text{so} \quad C''(200) = \frac{3}{10} > 0,$$

and C has a local minimum value at 200. Because $C''(x) > 0$ for all $x > 0$, the local minimum is also an absolute minimum. Therefore, the cost of the fencing is minimized when $x = 200$ and $y = 300$. The builder should purchase 200 m of the \$20 per meter fencing and

$$200 + 300 + 300 = 800 \text{ m of the } \$10 \text{ per meter fencing}$$

at a cost of

$$C = 30(200) + 20(300) = \$12,000.$$



13. Let x denote the number of \$1 increases in the rental price above the original \$18 per day price. The rental price is then $18 + x$ and the number of cars rented is $24 - x$ so the rental income I is

$$I = (18 + x)(24 - x) = 432 + 6x - x^2.$$

The domain of I is the closed interval $[0, 24]$. The function I is differentiable on the open interval $(0, 24)$, so the critical numbers occur where $I'(x) = 0$. Now,

$$I'(x) = 6 - 2x,$$

so $I'(x) = 0$ when $x = 3$. Evaluating I at the endpoints of the interval $[0, 24]$ and at the critical number 3 yields

$$I(0) = 432, \quad I(3) = 21^2 = 441, \quad \text{and} \quad I(24) = 0.$$

Rental income is therefore maximized when $x = 3$, meaning that the agency should charge $18 + 3 = \$21$ per day to maximize income.

14. Let x denote the number of members above 60 in the charter flight club. The number of members is then $60 + x$ and the yearly charge is $200 - 2x$ so the revenue R generated by the dues is

$$R = (60 + x)(200 - 2x) = 12000 + 80x - 2x^2.$$

The domain of R is the closed interval $[0, 100]$. The function R is differentiable on the open interval $(0, 100)$, so the critical numbers occur where $R'(x) = 0$. Now,

$$R'(x) = 80 - 4x,$$

so $R'(x) = 0$ when $x = 20$. Evaluating R at the endpoints of the interval $[0, 100]$ and at the critical number 20 yields

$$R(0) = 12000, \quad R(20) = 80(160) = 12800, \quad \text{and} \quad R(100) = 0.$$

Revenue is therefore maximized when $x = 20$, meaning that $60 + 20 = 80$ members maximizes revenue.

15. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = x^2$ and the point $\left(2, \frac{1}{2}\right)$. Then

$$\begin{aligned} D &= (x - 2)^2 + \left(y - \frac{1}{2}\right)^2 = (x - 2)^2 + \left(x^2 - \frac{1}{2}\right)^2 \\ &= x^2 - 4x + 4 + x^4 - x^2 + \frac{1}{4} \\ &= x^4 - 4x + \frac{17}{4}. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 4x^3 - 4 = 4(x - 1)(x^2 + x + 1),$$

so $x = 1$ is the only critical number. Using the Second Derivative Test,

$$D''(x) = 12x^2 \quad \text{so} \quad D''(1) = 12 > 0$$

and D has a local minimum value at 1. Because $D''(x) \geq 0$ for all x , the local minimum is also an absolute minimum. The point $(1, 1^2) = (1, 1)$ on the graph of the parabola $y = x^2$ is closest to the point $\left(2, \frac{1}{2}\right)$.

16. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = 2x^2$ and the point $(1, 4)$. Then

$$\begin{aligned} D &= (x-1)^2 + (y-4)^2 = (x-1)^2 + (2x^2-4)^2 \\ &= x^2 - 2x + 1 + 4x^4 - 16x^2 + 16 \\ &= 4x^4 - 15x^2 - 2x + 17. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 16x^3 - 30x - 2.$$

Using the computer algebra system *Maple*, the critical numbers are found to be $x \approx -1.335$, $x \approx -0.067$, and $x \approx 1.401$. Evaluating D at each of these values yields

$$D(-1.335) \approx 5.642, \quad D(-0.067) \approx 17.067, \quad \text{and} \quad D(1.401) \approx 0.166.$$

As

$$\lim_{x \rightarrow \pm\infty} D(x) = \infty,$$

it follows that D has an absolute minimum at approximately 1.401. The point $(1.401, 2 \cdot 1.401^2) \approx (1.401, 3.926)$ on the graph of the parabola $y = 2x^2$ is closest to the point $(1, 4)$.

17. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = 4 - x^2$ and the point $(6, 2)$. Then

$$\begin{aligned} D &= (x-6)^2 + (y-2)^2 = (x-6)^2 + (4-x^2-2)^2 \\ &= x^2 - 12x + 36 + 4 - 4x^2 + x^4 \\ &= x^4 - 3x^2 - 12x + 40. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 4x^3 - 6x - 12.$$

Using the computer algebra system *Maple*, the only critical number is $x \approx 1.784$. Using the Second Derivative Test,

$$D''(x) = 12x^2 - 6 \quad \text{so} \quad D''(1.784) \approx 32.192 > 0$$

and D has a local minimum value at 1.784. As

$$\lim_{x \rightarrow \pm\infty} D(x) = \infty,$$

it follows that the local minimum is also an absolute minimum. The point $(1.784, 4 - 1.784^2) \approx (1.784, 0.817)$ on the graph of the parabola $y = 4 - x^2$ is closest to the point $(6, 2)$.

18. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point on the graph of the parabola $y = \sqrt{x}$ and the point $(4, 0)$. Then

$$\begin{aligned} D &= (x-4)^2 + (y-0)^2 = (x-4)^2 + (\sqrt{x}-0)^2 \\ &= x^2 - 8x + 16 + x \\ &= x^2 - 7x + 16. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 2x - 7,$$

so $x = \frac{7}{2}$ is the only critical number. Using the Second Derivative Test,

$$D''(x) = 2 \quad \text{so} \quad D''\left(\frac{7}{2}\right) = 2 > 0$$

and D has a local minimum value at $\frac{7}{2}$. Because $D''(x) > 0$ for all x , the local minimum is also an absolute minimum. The point

$$\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right) = \left(\frac{7}{2}, \frac{\sqrt{14}}{2}\right)$$

on the graph of $y = \sqrt{x}$ is closest to the point $(4, 0)$.

19. Let x denote the traveling speed of a truck and let

$$C(x) = \left(\frac{1600}{x} + x\right)a + \frac{200}{x}b + c$$

be the cost associated with a 200-mile trip where a is the cost per gallon for gasoline, b is the hourly rate paid to the driver, and c is a commission paid to the driver. With a top speed of 75 mph, the domain of C is the interval $(0, 75]$. The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

$$C'(x) = \left(1 - \frac{1600}{x^2}\right)a - \frac{200}{x^2}b,$$

so $C'(x)$ is equal to zero when

$$x = \pm \sqrt{\frac{1600a + 200b}{a}}$$

and does not exist when $x = 0$. Of these numbers, only $\sqrt{\frac{1600a + 200b}{a}}$ lies inside the interval $(0, 75)$. Using the Second Derivative Test,

$$C''(x) = \frac{3200}{x^3}a + \frac{400}{x^3}b > 0$$

for all x in $(0, 75)$, so C has a local and an absolute minimum at

$$x = \sqrt{\frac{1600a + 200b}{a}}.$$

- (a) With $a = \$3.50$, $b = 0$, and $c = 0$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(3.50) + 200(0)}{3.50}} = \sqrt{\frac{5600}{3.50}} = \sqrt{1600} = \boxed{40 \text{ miles per hour}}.$$

(b) With $a = \$3.50$, $b = \$10.00$, and $c = \$500$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(3.50) + 200(10.00)}{3.50}} = \sqrt{\frac{7600}{3.50}} \approx \boxed{46.6 \text{ miles per hour}}.$$

(c) With $a = \$4.00$, $b = \$20.00$, and $c = 0$, the speed that minimizes cost is

$$x = \sqrt{\frac{1600(4.00) + 200(20.00)}{4.00}} = \sqrt{\frac{10400}{4.00}} = \sqrt{2600} \approx \boxed{51.0 \text{ miles per hour}}.$$

20. Using the diagram in the text, the distance from the house to the connection point along the road is $\sqrt{x^2 + 4}$ km and the distance from the connection point to the cable box is $5 - x$ km. The construction cost C is then given by

$$C(x) = 60\sqrt{x^2 + 4} + 50(5 - x),$$

and the domain is the closed interval $[0, 5]$. The critical numbers of C occur where $C'(x) = 0$ or where $C'(x)$ does not exist. Now,

$$C'(x) = 60 \cdot \frac{1}{2}(x^2 + 4)^{-1/2}(2x) - 50 = \frac{60x}{\sqrt{x^2 + 4}} - 50,$$

so $C'(x)$ exists everywhere and is equal to zero when

$$\begin{aligned} \frac{60x}{\sqrt{x^2 + 4}} - 50 &= 0 \\ 60x &= 50\sqrt{x^2 + 4} \\ 3600x^2 &= 2500(x^2 + 4) \\ x^2 &= \frac{100}{11} \\ x &= \frac{10\sqrt{11}}{11}. \end{aligned}$$

Note that $x = -\frac{10\sqrt{11}}{11}$ is not in the domain of C . Evaluating C at the endpoints of the interval $[0, 5]$ and at the critical number $\frac{10\sqrt{11}}{11}$ yields

$$C(0) = 120 + 250 = 370, \quad C\left(\frac{10\sqrt{11}}{11}\right) \approx 316.33, \quad \text{and} \quad C(5) = 60\sqrt{29} \approx 323.11,$$

so the minimum cost occurs when $x = \frac{10\sqrt{11}}{11}$. To minimize construction cost, the connection point

should be $\boxed{5 - \frac{10\sqrt{11}}{11} \approx 1.985 \text{ km}}$ along the road from the cable box.

21. (a) Let x denote the distance from the point on the road closest to house A to the point where the path from house A to house B meets the road. Then the length L of the path from house A to house B is

$$L = \sqrt{q^2 + x^2} + \sqrt{(p - x)^2 + r^2},$$

where the first term accounts for the distance from house A to the road and the second term accounts for the distance from the road to house B . The domain of this function is the closed interval $[0, p]$, and the critical numbers occur where $L'(x) = 0$. Now,

$$L'(x) = \frac{x}{\sqrt{q^2 + x^2}} - \frac{p - x}{\sqrt{(p - x)^2 + r^2}},$$

so $L'(x) = 0$ when

$$\begin{aligned}
 \frac{x}{\sqrt{q^2 + x^2}} &= \frac{p-x}{\sqrt{(p-x)^2 + r^2}} \\
 x^2[(p-x)^2 + r^2] &= (p-x)^2(q^2 + x^2) \\
 x^2 r^2 &= (p-x)^2 q^2 \\
 xr &= |p-x|q = (p-x)q \quad \text{because } p \geq x \\
 x &= \frac{pq}{q+r}.
 \end{aligned}$$

Evaluating L at the endpoints of the interval $[0, p]$ and at the critical number $\frac{pq}{q+r}$ yields

$$\begin{aligned}
 L(0) &= q + \sqrt{r^2 + p^2}, \\
 L\left(\frac{pq}{q+r}\right) &= \sqrt{q^2 + \left(\frac{pq}{q+r}\right)^2} + \sqrt{\left(p - \frac{pq}{q+r}\right)^2 + r^2} \\
 &= \sqrt{q^2 + \left(\frac{pq}{q+r}\right)^2} + \sqrt{\left(\frac{pr}{q+r}\right)^2 + r^2} \\
 &= \sqrt{\frac{q^2}{(q+r)^2}[(q+r)^2 + p^2]} + \sqrt{\frac{r^2}{(q+r)^2}[p^2 + (q+r)^2]} \\
 &= \frac{q}{q+r}\sqrt{p^2 + (q+r)^2} + \frac{r}{q+r}\sqrt{p^2 + (q+r)^2} = \sqrt{p^2 + (q+r)^2}, \quad \text{and} \\
 L(p) &= \sqrt{q^2 + p^2} + r.
 \end{aligned}$$

Because p , q , and r are all positive numbers,

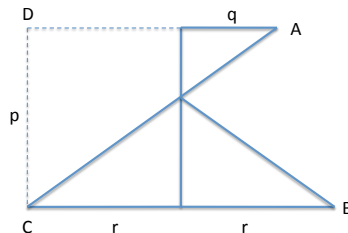
$$q + \sqrt{r^2 + p^2} = \sqrt{q^2 + 2q\sqrt{r^2 + p^2} + r^2 + p^2} > \sqrt{q^2 + 2qr + r^2 + p^2} = \sqrt{(q+r)^2 + p^2},$$

and

$$\sqrt{q^2 + p^2} + r = \sqrt{q^2 + p^2 + 2r\sqrt{q^2 + p^2} + r^2} > \sqrt{q^2 + 2qr + r^2 + p^2} = \sqrt{(q+r)^2 + p^2}.$$

Therefore, the length of the shortest path that goes from house A to the road and then on to house B is $\boxed{\sqrt{(q+r)^2 + p^2}}$.

- (b) Following the hint, reflect the point B across the road to the point C (see the diagram below). Note that any path from A to C is equivalent to (has the same distance as) the path from A to B obtained by reflecting the portion of the path to the left of the road across to the right side of the road. Additionally, the shortest path from A to C is a straight line, and therefore has a distance equal to the hypotenuse of triangle ADC : $\sqrt{(q+r)^2 + p^2}$. Therefore, the length of the shortest path from A to the road and then on to B is $\boxed{\sqrt{(q+r)^2 + p^2}}$, matching the result from part (a).



22. Let x denote the distance along the shore from the point P to the point where the boat lands on the shore. Then the person rows a distance of $\sqrt{x^2 + 9}$ km and walks a distance of $12 - x$ km to reach the town. Taking into account the given rowing and walking speeds, the time T it takes to reach the town is

$$T = \frac{\sqrt{x^2 + 9}}{2.5} + \frac{12 - x}{4} = \frac{2\sqrt{x^2 + 9}}{5} + \frac{12 - x}{4}.$$

The domain of T is the closed interval $[0, 12]$, and the critical numbers occur where $T'(x) = 0$ or where $T'(x)$ does not exist. Now,

$$T'(x) = \frac{2x}{5\sqrt{x^2 + 9}} - \frac{1}{4},$$

so $T'(x)$ exists everywhere and is equal to zero when

$$\begin{aligned} \frac{2x}{5\sqrt{x^2 + 9}} &= \frac{1}{4} \\ 8x &= 5\sqrt{x^2 + 9} \\ 64x^2 &= 25(x^2 + 9) \\ x^2 &= \frac{225}{39} \\ x &= \frac{15\sqrt{39}}{39}. \end{aligned}$$

Note that $x = -\frac{15\sqrt{39}}{39}$ is not in the domain of T . Evaluating T at the endpoints of the interval $[0, 12]$ and at the critical number $\frac{15\sqrt{39}}{39}$ yields

$$T(0) = \frac{6}{5} + 3 = 4.2, \quad T\left(\frac{15\sqrt{39}}{39}\right) \approx 3.937, \quad \text{and} \quad T(12) = \frac{2\sqrt{153}}{5} \approx 4.948,$$

so the minimum occurs when $x = \frac{15\sqrt{39}}{39}$. To minimize the time needed to reach the town, the boat should land on shore $\boxed{\frac{15\sqrt{39}}{39} \approx 2.402 \text{ km from the point } P}.$

23. The shortest beam will simultaneously be in contact with the wall that is being supported, the 2-m high wall in the middle, and the ground. Let x and h be as shown in the diagram below. The length of the ladder is then

$$L = \sqrt{(x + 5)^2 + h^2}.$$

By similar triangles,

$$\frac{x}{2} = \frac{x + 5}{h} \quad \text{so} \quad h = \frac{2(x + 5)}{x}.$$

Substituting for h in the length formula yields

$$L = \sqrt{(x + 5)^2 + \left(\frac{2(x + 5)}{x}\right)^2} = \frac{x + 5}{x} \sqrt{x^2 + 4} = \left(1 + \frac{5}{x}\right) \sqrt{x^2 + 4}.$$

The domain of L is the interval $(0, \infty)$. Now,

$$L'(x) = \left(1 + \frac{5}{x}\right) \cdot \frac{x}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4} \cdot \left(-\frac{5}{x^2}\right) = \frac{x^3 + 5x^2 - 5x^2 - 20}{x^2\sqrt{x^2 + 4}} = \frac{x^3 - 20}{x^2\sqrt{x^2 + 4}},$$

so the only critical number inside the open interval $(0, \infty)$ is $x = \sqrt[3]{20}$, where $L'(x) = 0$. Note that $x = 0$ is not in the domain of L . Using the Second Derivative Test,

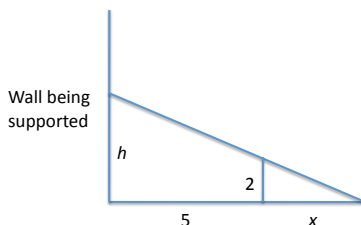
$$\begin{aligned} L''(x) &= \frac{x^2\sqrt{x^2+4} \cdot 3x^2 - (x^3-20) \cdot \left(\frac{x^3}{\sqrt{x^2+4}} + 2x\sqrt{x^2+4}\right)}{x^4(x^2+4)} \\ &= \frac{3x^4(x^2+4) - x^3(x^3-20) - 2x(x^3-20)(x^2+4)}{x^4(x^2+4)^{3/2}} \\ &= \frac{3x^6 + 12x^4 - x^6 + 20x^3 - 2x^6 - 8x^4 + 40x^3 + 160x}{x^4(x^2+4)^{3/2}} \\ &= \frac{4x^4 + 60x^3 + 160x}{x^4(x^2+4)^{3/2}} = \frac{4x^3 + 60x^2 + 160}{x^3(x^2+4)^{3/2}}, \end{aligned}$$

so $L''(x) > 0$ for all $x > 0$ and L has a local and absolute minimum at $\sqrt[3]{20}$. The length of the shortest beam that can be used to brace the wall is therefore

$$L = \left(1 + \frac{5}{\sqrt[3]{20}}\right) \sqrt{\sqrt[3]{400} + 4} \approx \boxed{9.582 \text{ meters}}.$$

The angle of elevation of the beam is

$$\tan^{-1} \frac{2}{\sqrt[3]{20}} \approx 0.635 \text{ rad} \approx \boxed{36.383^\circ}.$$



24. Place the center of the cross section of the log at the origin so that the outer edge of the log corresponds to the equation $x^2 + y^2 = R^2$. With this coordinate system, the width of the beam is $2x$ and the depth of the beam is $2y$ so the strength S of the beam is

$$S = k(2x)(2y)^3 = 16kxy^3,$$

where k is a positive constant of proportionality. Substituting $y = \sqrt{R^2 - x^2}$ into the strength formula yields

$$S = 16kx(R^2 - x^2)^{3/2}.$$

The domain of this function is the closed interval $[0, R]$. Now,

$$\begin{aligned} S'(x) &= 16kx \cdot \frac{3}{2}(R^2 - x^2)^{1/2}(-2x) + 16k(R^2 - x^2)^{3/2} \\ &= 16k\sqrt{R^2 - x^2}(R^2 - x^2 - 3x^2) = 16k\sqrt{R^2 - x^2}(R^2 - 4x^2), \end{aligned}$$

so the only critical number inside the open interval $(0, R)$ is $x = R/2$, where $S'(x) = 0$. Note that neither $x = -R$ nor $x = -R/2$ is in the domain of S , and $x = R$ is not inside the open interval $(0, R)$. Evaluating S at the endpoints of the interval $[0, R]$ and at the critical number $R/2$ yields

$$S(0) = 0, \quad S\left(\frac{R}{2}\right) = 16k \cdot \frac{R}{2} \cdot \frac{3R^3\sqrt{3}}{8} = 3kR^4\sqrt{3}, \quad \text{and} \quad S(R) = 0.$$

Therefore, S achieves its absolute maximum value when $x = R/2$ and

$$y = \sqrt{R^2 - \frac{R^2}{4}} = \frac{R\sqrt{3}}{2}.$$

The strongest beam that can be cut from a log whose cross section has the form of a circle of radius R has a width of R and a depth of $R\sqrt{3}$.

25. Using the hint (that is, choosing the width of the beam to be $2x$ and the depth of the beam to be $2y$) gives

$$S = k(2x)(2y)^2 = 8kxy^2,$$

where k is a positive constant of proportionality. Solve the equation of the cross section, $10x^2 + 9y^2 = 90$, for $y^2 = 10 - \frac{10}{9}x^2$ and then substitute this expression into the formula for S , to obtain

$$S = 8kx \left(10 - \frac{10}{9}x^2 \right) = 80kx - \frac{80}{9}kx^3.$$

The domain of this function is the closed interval $[0, 3]$, and the critical numbers occur where $S'(x) = 0$. Now,

$$S'(x) = 80k - \frac{80}{3}kx^2,$$

so the only critical number inside the open interval $(0, 3)$ is $x = \sqrt{3}$, where $S'(x) = 0$. Note that $x = -\sqrt{3}$ is not in the domain of S . Evaluating S at the endpoints of the interval $[0, 3]$ and at the critical number $\sqrt{3}$ yields

$$S(0) = 0, \quad S(\sqrt{3}) = 80k\sqrt{3} - 80k\frac{\sqrt{3}}{3} = \frac{160k\sqrt{3}}{3}, \quad \text{and} \quad S(3) = 0.$$

Therefore, S achieves its absolute maximum value when $x = \sqrt{3}$ and $y = \sqrt{10 - \frac{10}{3}} = \frac{2\sqrt{15}}{3}$. The strongest beam that can be cut from a log whose cross section has the form of the ellipse $10x^2 + 9y^2 = 90$ has a width of $2\sqrt{3}$ and a depth of $\frac{4\sqrt{15}}{3}$.

26. With the center of the cross section of the log at the origin, as implied by the outer edge of the log having the equation $b^2x^2 + a^2y^2 = a^2b^2$, the width of the beam is $2x$ and the depth of the beam is $2y$ so the strength S of the beam is

$$S = k(2x)(2y)^3 = 16kxy^3,$$

where k is a positive constant of proportionality. Substituting $y = \sqrt{b^2 - \frac{b^2}{a^2}x^2} = \frac{b}{a}\sqrt{a^2 - x^2}$ into the strength formula yields

$$S = \frac{16kb^3}{a^3}x(a^2 - x^2)^{3/2}.$$

The domain of this function is the closed interval $[0, a]$. Now,

$$\begin{aligned} S'(x) &= \frac{16b^3}{a^3}x \cdot \frac{3}{2}(a^2 - x^2)^{1/2}(-2x) + \frac{16b^3}{a^3}(a^2 - x^2)^{3/2} \\ &= \frac{16b^3}{a^3}\sqrt{a^2 - x^2}(a^2 - x^2 - 3x^2) = \frac{16b^3}{a^3}\sqrt{a^2 - x^2}(a^2 - 4x^2), \end{aligned}$$

so the only critical number inside the open interval $(0, a)$ is $x = a/2$, where $S'(x) = 0$. Note that neither $x = -a$ nor $x = -a/2$ is in the domain of S , and $x = a$ is not inside the open interval $(0, a)$. Evaluating S at the endpoints of the interval $[0, a]$ and at the critical number $a/2$ yields

$$S(0) = 0, \quad S\left(\frac{a}{2}\right) = \frac{16kb^3}{a^3} \cdot \frac{a}{2} \cdot \frac{3a^3\sqrt{3}}{8} = 3kab^3\sqrt{3}, \quad \text{and} \quad S(a) = 0.$$

Therefore, S achieves its absolute maximum value when $x = a/2$ and

$$y = \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}} = \frac{b\sqrt{3}}{2}.$$

The strongest beam that can be cut from a log whose cross section has the form of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has a width of a and a depth of $b\sqrt{3}$.

27. Let x denote the number of cases produced and p denote the price of a case. Solve the equation

$$x = 1430 - \frac{11}{6}p \quad \text{for} \quad p = 780 - \frac{6}{11}x,$$

and then substitute for p in the formula

$$P = xp - 132x = x \left(780 - \frac{6}{11}x \right) - 132x = 648x - \frac{6}{11}x^2.$$

The domain for P is the closed interval $[0, 1100]$, and the critical numbers occur where $P'(x) = 0$. Now,

$$P'(x) = 648 - \frac{12}{11}x,$$

so $x = 594$ is the only critical number. Evaluating P at the endpoints of the interval $[0, 1100]$ and at the critical number 594 yields

$$P(0) = 0, \quad P(594) = 192,456, \quad \text{and} \quad P(1100) = 52,800.$$

Therefore, P achieves its absolute maximum when $x = 594$ and

$$p = 780 - \frac{6}{11}(594) = 456.$$

To maximize profit, the winemaker should produce 594 cases of wine and sell them for \$456 per case.

28. Let r denote the radius of the semicircular portion of the window (so that the width of the rectangular portion of the window is $2r$) and h denote the height of the rectangular portion of the window. The total area A of the window is then

$$A = 2rh + \frac{1}{2}\pi r^2.$$

Given that the perimeter of the window is to be 10 m, r and h are related by the equation

$$2r + 2h + \pi r = 10 \quad \text{or} \quad h = 5 - \left(1 + \frac{\pi}{2}\right)r.$$

Substituting for h in the area formula yields

$$A = 2r \left[5 - \left(1 + \frac{\pi}{2}\right)r \right] + \frac{1}{2}\pi r^2 = 10r - \left(2 + \frac{\pi}{2}\right)r^2.$$

In order for h to be non-negative, we must have

$$5 - \left(1 + \frac{\pi}{2}\right)r \geq 0 \quad \text{or} \quad r \leq \frac{10}{2 + \pi}.$$

The domain of A is therefore the closed interval $\left[0, \frac{10}{2 + \pi}\right]$. The critical numbers of A occur where $A'(r) = 0$. Now,

$$A'(r) = 10 - (4 + \pi)r,$$

so $r = \frac{10}{4+\pi}$ is the only critical number. Evaluating A at the endpoints of the interval $\left[0, \frac{10}{2+\pi}\right]$ and at the critical number $\frac{10}{4+\pi}$ yields

$$\begin{aligned} A(0) &= 0; \\ A\left(\frac{10}{4+\pi}\right) &= \left(2 + \frac{1}{2}\pi\right) \left(\frac{10}{4+\pi}\right)^2 \approx 7.001; \text{ and} \\ A\left(\frac{10}{2+\pi}\right) &= \frac{1}{2}\pi \left(\frac{10}{2+\pi}\right)^2 \approx 5.942. \end{aligned}$$

Therefore, A achieves its absolute maximum when $r = \frac{10}{4+\pi}$ and

$$h = 5 - \left(1 + \frac{\pi}{2}\right) \frac{10}{4+\pi} = 5 \frac{4+\pi - (2+\pi)}{4+\pi} = \frac{10}{4+\pi}.$$

The Norman window that admits the most light with a perimeter of 10 m therefore has a

$$\boxed{\text{width of } 2r = \frac{20}{4+\pi} \approx 2.800 \text{ m}} \text{ and a } \boxed{\text{height of } \frac{10}{4+\pi} \approx 1.400 \text{ m}}.$$

29. Maximizing the capacity of the trough will be accomplished by maximizing the cross-sectional area of the trough. Let θ denote the angle between the two sides of the V-shaped trough. The cross-sectional area A of the trough is then

$$A = \frac{1}{2}28^2 \sin \theta = 392 \sin \theta.$$

The domain of A is the closed interval $[0, \pi]$, and the critical numbers occur where $A'(\theta) = 0$. Now,

$$A'(\theta) = 392 \cos \theta,$$

so $\theta = \frac{\pi}{2}$ is the only critical number. Evaluating A at the endpoints of the interval $[0, \pi]$ and at the critical number $\frac{\pi}{2}$ yields

$$A(0) = 0, \quad A\left(\frac{\pi}{2}\right) = 392, \quad \text{and} \quad A(\pi) = 0.$$

Therefore, A achieves its absolute maximum when $\theta = \frac{\pi}{2}$, so the capacity of the trough is maximum

when the angle between the sides is $\boxed{\frac{\pi}{2}}$.

30. Maximizing the carrying capacity of the gutter will be accomplished by maximizing the cross-sectional area of the gutter. Let x be defined as shown in the diagram below. The cross section of the gutter is a trapezoid with bases of length 10 and $10+2x$ and height $\sqrt{100-x^2}$. The area A of the cross section is then

$$A = \frac{1}{2}(10 + 10 + 2x)\sqrt{100-x^2} = (10+x)\sqrt{100-x^2}.$$

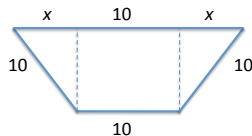
The domain of A is the closed interval $[0, 10]$. Now,

$$\begin{aligned} A'(x) &= (10+x) \frac{-x}{\sqrt{100-x^2}} + \sqrt{100-x^2} = \frac{-10x - x^2 + 100 - x^2}{\sqrt{100-x^2}} \\ &= \frac{-2(x^2 + 5x - 50)}{\sqrt{100-x^2}} = -\frac{2(x+10)(x-5)}{\sqrt{100-x^2}} = -\frac{2(x-5)\sqrt{x+10}}{\sqrt{10-x}}, \end{aligned}$$

so the only critical number inside the open interval $(0, 10)$ is $x = 5$, where $A'(x) = 0$. Note that $x = -10$ is not in the domain of A , and $x = 10$ is not inside the open interval $(0, 10)$. Evaluating A at the endpoints of the interval $[0, 10]$ and at the critical number 5 yields

$$A(0) = 100, \quad A(5) = 75\sqrt{3} \approx 129.904, \quad \text{and} \quad A(10) = 0.$$

Therefore, A achieves its absolute maximum when $x = 5$, so the carrying capacity of the gutter is maximum when the opening across the top is $\boxed{10 + 2(5) = 20 \text{ cm}}$.



31. Let r denote the radius of the semicircular ceiling (so that the width of the floor is $2r$) and h denote the height of the vertical walls. Without loss of generality suppose that the floor and vertical walls cost one unit per square meter so that the ceiling costs three units per square meter. The cost C for each one-meter long section of the tunnel is then

$$C = 2h(1) + 2r(1) + \pi r(3) = 2h + 2r + 3\pi r,$$

where the first term accounts for the cost of the vertical walls, the second term accounts for the cost of the floor, and the final term accounts for the cost of the ceiling. Let A denote the fixed cross-sectional area of the tunnel, so that

$$A = 2rh + \frac{1}{2}\pi r^2 \quad \text{or} \quad h = \frac{A - \frac{1}{2}\pi r^2}{2r} = \frac{A}{2r} - \frac{\pi r}{4}.$$

Substituting for h in the cost formula yields

$$C = 2\left(\frac{A}{2r} - \frac{\pi r}{4}\right) + 2r + 3\pi r = \frac{A}{r} + \left(\frac{5\pi}{2} + 2\right)r.$$

In order for h to be non-negative, we must have

$$\frac{A}{2r} - \frac{\pi r}{4} \geq 0 \quad \text{or} \quad r \leq \sqrt{\frac{2A}{\pi}}.$$

The domain of C is therefore the interval $\left(0, \sqrt{\frac{2A}{\pi}}\right]$. Now,

$$C'(r) = -\frac{A}{r^2} + \left(\frac{5\pi}{2} + 2\right),$$

so the only critical number inside the open interval $\left(0, \sqrt{\frac{2A}{\pi}}\right)$ is $r = \sqrt{\frac{2A}{5\pi + 4}}$, where $C'(r) = 0$. Note that $r = 0$ is not in the domain of C . Using the Second Derivative Test,

$$C''(r) = \frac{2A}{r^3} > 0$$

for all $r > 0$. Therefore, C has both a local and an absolute minimum when $r = \sqrt{\frac{2A}{5\pi + 4}}$. Finally, the most economical ratio of the diameter of the semicircular cylinder to the height of the vertical walls is

$$\frac{2r}{h} = \frac{4r^2}{A - \frac{1}{2}\pi r^2} = \frac{4 \cdot \frac{2A}{5\pi + 4}}{A - \frac{1}{2}\pi \cdot \frac{2A}{5\pi + 4}} = \frac{\frac{8}{5\pi + 4}}{1 - \frac{\pi}{5\pi + 4}} = \frac{8}{4\pi + 4} = \boxed{\frac{2}{\pi + 1}}.$$

32. Let r denote the radius of the hemispherical dome (so that the radius of the right circular cylinder is also r) and h denote the height of the right circular cylinder. Without loss of generality suppose

that the cylindrical wall costs one unit per square meter so that the dome costs three units per square meter. The cost C for the construction of the observatory is then

$$C = 2\pi rh(1) + 2\pi r^2(3) = 2\pi rh + 6\pi r^2,$$

where the first term accounts for the cost of the cylindrical wall and the second term accounts for the cost of the dome. Let V denote the fixed volume of the observatory, so that

$$V = \frac{2}{3}\pi r^3 + \pi r^2 h \quad \text{or} \quad h = \frac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{V}{\pi r^2} - \frac{2}{3}r.$$

Substituting for h in the cost formula yields

$$C = 2\pi r \left(\frac{V}{\pi r^2} - \frac{2}{3}r \right) + 6\pi r^2 = \frac{2V}{r} + \frac{14}{3}\pi r^2.$$

In order for h to be non-negative, we must have

$$\frac{V}{\pi r^2} - \frac{2}{3}r \geq 0 \quad \text{or} \quad r \leq \sqrt[3]{\frac{3V}{2\pi}}.$$

The domain of C is therefore the interval $\left(0, \sqrt[3]{\frac{3V}{2\pi}}\right]$. Now,

$$C'(r) = -\frac{2V}{r^2} + \frac{28}{3}\pi r,$$

so the only critical number inside the open interval $\left(0, \sqrt[3]{\frac{3V}{2\pi}}\right)$ is $r = \sqrt[3]{\frac{3V}{14\pi}}$, where $C'(r) = 0$. Note that $r = 0$ is not in the domain of C . Using the Second Derivative Test,

$$C''(r) = \frac{4V}{r^3} + \frac{28\pi}{3} > 0$$

for all $r > 0$. Therefore, C has both a local and an absolute minimum when $r = \sqrt[3]{\frac{3V}{14\pi}}$ and

$$h = \frac{V - \frac{2}{3}\pi \cdot \frac{3V}{14\pi}}{\pi \sqrt[3]{\left(\frac{3V}{14\pi}\right)^2}} = \frac{\frac{6}{7}V}{\pi \sqrt[3]{\left(\frac{3V}{14\pi}\right)^2}} = \frac{\frac{6}{7}V \cdot \sqrt[3]{\frac{3V}{14\pi}}}{\pi \cdot \frac{3V}{14\pi}} = 4\sqrt[3]{\frac{3V}{14\pi}}.$$

The most economical dimensions for the observatory are

$$\boxed{\text{a radius of } \sqrt[3]{\frac{3V}{14\pi}} \text{ and a height of } 4\sqrt[3]{\frac{3V}{14\pi}}},$$

for a given volume V .

33. Let the weaker light source have intensity I^* and be located at $x = 0$. The other light source has intensity $8I^*$ and is located at $x = 6$. For $0 < x < 6$, the intensity of light I at location x is

$$I = \frac{kI^*}{x^2} + \frac{8kI^*}{(6-x)^2},$$

where k is the constant of proportionality in the inverse square law, the first term accounts for illumination from the weaker light source and the second term accounts for illumination from the stronger source. Now,

$$I'(x) = -\frac{2kI^*}{x^3} + \frac{16kI^*}{(6-x)^3},$$

so $I'(x)$ is equal to zero when

$$\begin{aligned}\frac{2kI^*}{x^3} &= \frac{16kI^*}{(6-x)^3} \\ (6-x)^3 &= 8x^3 \\ 6-x &= 2x \\ x &= 2.\end{aligned}$$

Note that neither $x = 0$ nor $x = 6$ are in the domain of I , so 2 is the only critical number. Using the Second Derivative Test,

$$I''(x) = \frac{6kI^*}{x^4} + \frac{48kI^*}{(6-x)^4} > 0$$

for all x inside the interval $(0, 6)$. Therefore, I has both a local and an absolute minimum when $x = 2$, so the point between the two light sources at which illumination is minimum is 2 meters from the weaker light source.

34. Let $C(t) = \frac{2t}{16+t^2}$, and consider the interval $[0, \infty)$. The critical numbers of C occur where $C'(t) = 0$. Now,

$$C'(t) = \frac{(16+t^2) \cdot 2 - 2t \cdot 2t}{(16+t^2)^2} = \frac{32-2t^2}{(16+t^2)^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $t = 4$, where $C'(t) = 0$. Note that $t = -4$ is not in the domain of C . Using the Second Derivative Test,

$$\begin{aligned}C''(t) &= \frac{(16+t^2)^2 \cdot (-4t) - (32-2t^2) \cdot 2(16+t^2)(2t)}{(16+t^2)^4} \\ &= \frac{-64t - 4t^3 - 128t + 8t^3}{(16+t^2)^3} = \frac{4t^3 - 192t}{(16+t^2)^3},\end{aligned}$$

so

$$C''(4) = \frac{256 - 768}{32^3} = -\frac{1}{64} < 0$$

and C has a local maximum value at 4. The local maximum value is $C(4) = \frac{1}{4}$. Because

$$C(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} C(t) = 0,$$

the local maximum is also an absolute maximum. Therefore, the concentration of the drug in the bloodstream is greatest 4 hours after injection.

35. Let x denote the side length of the square and r denote the radius of the circle made from the two pieces of wire. The length L of the wire is then

$$L = 4x + 2\pi r.$$

Given that the area enclosed by the two pieces of wire is to be 64 cm^2 , x and r are related by the equation

$$x^2 + \pi r^2 = 64 \quad \text{or} \quad x = \sqrt{64 - \pi r^2}.$$

In order to have x defined, we must have

$$\pi r^2 \leq 64 \quad \text{or} \quad r \leq \frac{8}{\sqrt{\pi}}.$$

Substituting for x in the length formula yields

$$L = 4\sqrt{64 - \pi r^2} + 2\pi r,$$

where the domain is the closed interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$. Now,

$$L'(r) = 4 \frac{-\pi r}{\sqrt{64 - \pi r^2}} + 2\pi,$$

so the only critical number inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$ occurs when

$$\begin{aligned} \frac{4\pi r}{\sqrt{64 - \pi r^2}} &= 2\pi \\ 2r &= \sqrt{64 - \pi r^2} \\ 4r^2 &= 64 - \pi r^2 \\ (4 + \pi)r^2 &= 64 \\ r &= \frac{8}{\sqrt{4 + \pi}}. \end{aligned}$$

Note that neither $r = -\frac{8}{\sqrt{\pi}}$ nor $r = -\frac{8}{\sqrt{4 + \pi}}$ is in the domain of L , and $r = \frac{8}{\sqrt{\pi}}$ is not inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$. Evaluating L at the endpoints of the interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$ and at the critical number $\frac{8}{\sqrt{4 + \pi}}$ yields

$$L(0) = 32, \quad L\left(\frac{8}{\sqrt{4 + \pi}}\right) = \frac{64 + 16\pi}{\sqrt{4 + \pi}} \approx 42.758, \quad \text{and} \quad L\left(\frac{8}{\sqrt{\pi}}\right) = 16\sqrt{\pi} \approx 28.359.$$

Therefore, the

minimum length of wire that can be used is approximately 28.359 cm,

and the

maximum length wire that can be used is approximately 42.758 cm.

36. Let s denote the side length of the equilateral triangle and r denote the radius of the circle made from the two pieces of wire. The length L of the wire is then

$$L = 3s + 2\pi r.$$

Given that the area enclosed by the two pieces of wire is to be 64 cm^2 , s and r are related by the equation

$$\frac{s^2\sqrt{3}}{4} + \pi r^2 = 64 \quad \text{or} \quad s = \frac{2}{\sqrt[4]{3}}\sqrt{64 - \pi r^2}.$$

In order to have s defined, we must have

$$\pi r^2 \leq 64 \quad \text{or} \quad r \leq \frac{8}{\sqrt{\pi}}.$$

Substituting for s in the length formula yields

$$L = 2\sqrt[4]{27}\sqrt{64 - \pi r^2} + 2\pi r,$$

where the domain is the closed interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$. Now,

$$L'(r) = 2\sqrt[4]{27} \frac{-\pi r}{\sqrt{64 - \pi r^2}} + 2\pi,$$

so the only critical number inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$ occurs when

$$\begin{aligned}\frac{2\sqrt[4]{27}\pi r}{\sqrt{64 - \pi r^2}} &= 2\pi \\ \sqrt[4]{27}r &= \sqrt{64 - \pi r^2} \\ 3\sqrt{3}r^2 &= 64 - \pi r^2 \\ (3\sqrt{3} + \pi)r^2 &= 64 \\ r &= \frac{8}{\sqrt{3\sqrt{3} + \pi}}.\end{aligned}$$

Note that neither $r = -\frac{8}{\sqrt{\pi}}$ nor $r = -\frac{8}{\sqrt{3\sqrt{3} + \pi}}$ is in the domain of L , and $r = \frac{8}{\sqrt{\pi}}$ is not inside the open interval $\left(0, \frac{8}{\sqrt{\pi}}\right)$. Evaluating L at the endpoints of the interval $\left[0, \frac{8}{\sqrt{\pi}}\right]$ and at the critical number $\frac{8}{\sqrt{3\sqrt{3} + \pi}}$ yields

$$\begin{aligned}L(0) &= 16\sqrt[4]{27} \approx 36.472, \\ L\left(\frac{8}{\sqrt{3\sqrt{3} + \pi}}\right) &= 2\sqrt[4]{27}\sqrt{64 - \pi \cdot \frac{64}{3\sqrt{3} + \pi}} + \frac{16\pi}{\sqrt{3\sqrt{3} + \pi}} \\ &= 16\sqrt[4]{27}\sqrt{\frac{3\sqrt{3}}{3\sqrt{3} + \pi}} + \frac{16\pi}{\sqrt{3\sqrt{3} + \pi}} = \frac{16\sqrt[4]{27} \cdot \sqrt[4]{27}}{\sqrt{3\sqrt{3} + \pi}} + \frac{16\pi}{\sqrt{3\sqrt{3} + \pi}} \\ &= \frac{48\sqrt{3} + 16\pi}{\sqrt{3\sqrt{3} + \pi}} \approx 46.200, \quad \text{and} \\ L\left(\frac{8}{\sqrt{\pi}}\right) &= 16\sqrt{\pi} \approx 28.359.\end{aligned}$$

Therefore, the

$$\boxed{\text{minimum length of wire that can be used is approximately 28.359 cm}},$$

and the

$$\boxed{\text{maximum length wire that can be used is approximately 46.200 cm}}.$$

37. Let x denote the length of wire used to make the square. Then the side length of the square is $x/4$, the length of wire used to make the circle is $35 - x$ and the radius of the resulting circle is $\frac{35 - x}{2\pi}$. The area A enclosed by the two figures is then

$$A = \left(\frac{x}{4}\right)^2 + \pi \left(\frac{35 - x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(35 - x)^2}{4\pi}.$$

The domain of this function is the closed interval $[0, 35]$, and the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{x}{8} - \frac{35 - x}{2\pi},$$

so $A'(x) = 0$ when

$$\begin{aligned}\frac{x}{8} &= \frac{35 - x}{2\pi} \\ 2\pi x &= 280 - 8x \\ x &= \frac{140}{4 + \pi}.\end{aligned}$$

Evaluating A at the endpoints of the interval $[0, 35]$ and at the critical number $\frac{140}{4 + \pi}$ yields

$$\begin{aligned} A(0) &= \frac{35^2}{4\pi} \approx 97.482; \\ A\left(\frac{140}{4 + \pi}\right) &= \frac{1}{16} \left(\frac{140}{4 + \pi}\right)^2 + \frac{1}{4\pi} \left(35 - \frac{140}{4 + \pi}\right)^2 \\ &= \frac{70^2}{4(4 + \pi)^2} + \frac{35^2\pi^2}{4\pi(4 + \pi)^2} = \frac{70^2 + 35^2\pi}{4(4 + \pi)^2} \approx 42.883; \text{ and} \\ A(35) &= \frac{35^2}{16} = 76.5625. \end{aligned}$$

- (a) To enclose the minimum area, the wire should be cut so that

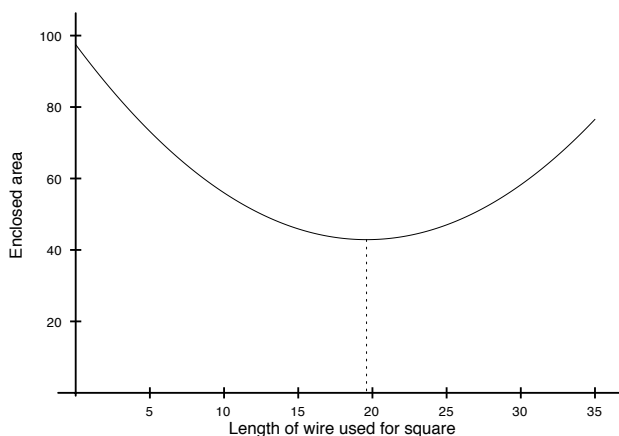
$$\frac{140}{4 + \pi} \approx \boxed{19.603 \text{ cm}}$$

are used for the square and the remaining

$$35 - \frac{140}{4 + \pi} = \frac{35\pi}{4 + \pi} \approx \boxed{15.397 \text{ cm}}$$

are used for the circle.

- (b) To enclose the maximum area, the entire length of wire should be formed into a circle since this corresponds to choosing $x = 0$.
- (c) The figure below displays the graph of the area enclosed as a function of the length of the wire used to make the square. The absolute minimum occurs for x a little less than 20 cm, confirming the result of part (a); the absolute maximum occurs for $x = 0$, confirming the result of part (b).



38. Let x denote the length of wire use to make the equilateral triangle. Then the side length of the equilateral triangle is $x/3$, the length of wire used to make the circle is $35 - x$ and the radius of the resulting circle is $\frac{35 - x}{2\pi}$. The area A enclosed by the two figures is then

$$A = \left(\frac{x}{3}\right)^2 \frac{\sqrt{3}}{4} + \pi \left(\frac{35 - x}{2\pi}\right)^2 = \frac{x^2\sqrt{3}}{36} + \frac{(35 - x)^2}{4\pi}.$$

The domain of this function is the closed interval $[0, 35]$, and the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = \frac{x\sqrt{3}}{18} - \frac{35 - x}{2\pi},$$

so $A'(x) = 0$ when

$$\begin{aligned}\frac{x\sqrt{3}}{18} &= \frac{35-x}{2\pi} \\ 2\pi\sqrt{3}x &= 630 - 18x \\ x &= \frac{315}{9 + \pi\sqrt{3}}.\end{aligned}$$

Evaluating A at the endpoints of the interval $[0, 35]$ and at the critical number $\frac{315}{9 + \pi\sqrt{3}}$ yields

$$\begin{aligned}A(0) &= \frac{35^2}{4\pi} \approx 97.482; \\ A\left(\frac{315}{9 + \pi\sqrt{3}}\right) &= \frac{\sqrt{3}}{36} \left(\frac{315}{9 + \pi\sqrt{3}}\right)^2 + \frac{1}{4\pi} \left(35 - \frac{315}{9 + \pi\sqrt{3}}\right)^2 \\ &= \frac{35^2 \cdot 9\sqrt{3}}{4(9 + \pi\sqrt{3})^2} + \frac{35^2 \cdot 3\pi^2}{4\pi(9 + \pi\sqrt{3})^2} = \frac{35^2(9\sqrt{3} + 3\pi)}{4(9 + \pi\sqrt{3})^2} \approx 36.371; \text{ and} \\ A(35) &= \frac{35^2\sqrt{3}}{36} \approx 58.938.\end{aligned}$$

- (a) To enclose the minimum area, the wire should be cut so that

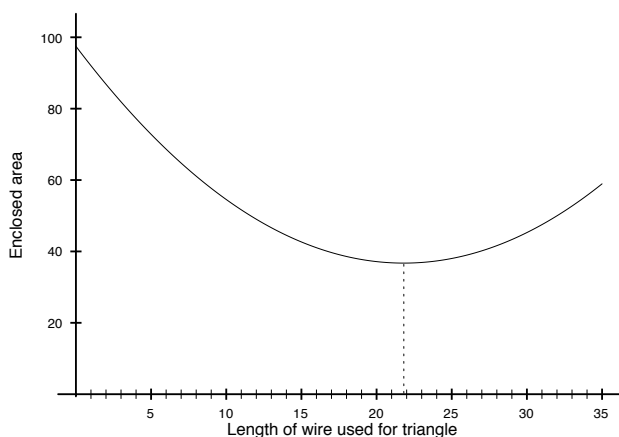
$$\frac{315}{9 + \pi\sqrt{3}} \approx \boxed{21.812 \text{ cm}}$$

are used for the equilateral triangle and the remaining

$$35 - \frac{315}{9 + \pi\sqrt{3}} = \frac{35\pi\sqrt{3}}{9 + \pi\sqrt{3}} \approx \boxed{13.188 \text{ cm}}$$

are used for the circle.

- (b) To enclose the maximum area, the entire length of wire should be formed into a circle since this corresponds to choosing $x = 0$.
- (c) The figure below displays the graph of the area enclosed as a function of the length of the wire used to make the equilateral triangle. The absolute minimum occurs for x a little less than 22 cm, confirming the result of part (a); the absolute maximum occurs for $x = 0$, confirming the result of part (b).



39. Let r denote the radius and h the height of the cylindrical can. The surface area A is then

$$A = 2\pi r^2 + 2\pi rh,$$

where the first term accounts for the area of the top and bottom of the can and the second term accounts for the lateral surface area. Given that the volume of the can is fixed to be V , r and h are related by the equation

$$V = \pi r^2 h \quad \text{or} \quad h = \frac{V}{\pi r^2}.$$

Substituting for h in the area formula yields

$$A = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

The domain for this function is the interval $(0, \infty)$. Now,

$$A'(r) = 4\pi r - \frac{2V}{r^2},$$

so the only critical number inside the open interval $(0, \infty)$ is $r = \sqrt[3]{\frac{V}{2\pi}}$, where $A'(r) = 0$. Note that $r = 0$ is not in the domain of A . Using the Second Derivative Test,

$$A''(r) = 4\pi + \frac{4V}{r^3} > 0$$

for all $r > 0$. Therefore, A has a local and an absolute minimum at $\sqrt[3]{\frac{V}{2\pi}}$. When $r = \sqrt[3]{\frac{V}{2\pi}}$,

$$h = \frac{V}{\pi \sqrt[3]{\left(\frac{V}{2\pi}\right)^2}} = \frac{V \sqrt[3]{\frac{V}{2\pi}}}{\pi \frac{V}{2\pi}} = 2 \sqrt[3]{\frac{V}{2\pi}} = 2r,$$

so the cylindrical container of fixed volume which uses the least material has a height that is twice its radius.

40. Let the point of tangency between the hypotenuse and the graph of $y = 3e^{-x}$ be $(x_0, 3e^{-x_0})$. The equation of the line tangent to the graph of $y = 3e^{-x}$ at x_0 is

$$y - 3e^{-x_0} = -3e^{-x_0}(x - x_0) \quad \text{or} \quad y = -3e^{-x_0}x + 3e^{-x_0}(x_0 + 1).$$

The y -intercept of this line is $3e^{-x_0}(x_0 + 1)$ and the x -intercept of this line is $x_0 + 1$ so the area of the triangle that has been created is

$$A = \frac{1}{2}(x_0 + 1) \cdot 3e^{-x_0}(x_0 + 1) = \frac{3}{2}(x_0 + 1)^2 e^{-x_0}.$$

The domain of this function is the interval $[0, \infty)$. Now,

$$A'(x_0) = -\frac{3}{2}(x_0 + 1)^2 e^{-x_0} + 3(x_0 + 1)e^{-x_0} = \frac{3}{2}(x_0 + 1)e^{-x_0}[2 - (x_0 + 1)] = \frac{3}{2}(1 - x_0^2)e^{-x_0},$$

so the only critical number inside the open interval $(0, \infty)$ is $x_0 = 1$, where $A'(x_0) = 0$. Note that $x_0 = -1$ is not in the domain of A . Using the Second Derivative Test,

$$A''(x_0) = -\frac{3}{2}(1 - x_0^2)e^{-x_0} - 3x_0e^{-x_0},$$

so $A''(1) = -3e^{-1} < 0$ and A has a local maximum at 1. The local maximum value is $A(1) = 6e^{-1} \approx 2.207$. Because $A(0) = \frac{3}{2}$ and

$$\lim_{x_0 \rightarrow \infty} A(t) = \lim_{x_0 \rightarrow \infty} \frac{\frac{3}{2}(x_0 + 1)^2}{e^{x_0}} = \lim_{x_0 \rightarrow \infty} \frac{3(x_0 + 1)}{e^{x_0}} = \lim_{x_0 \rightarrow \infty} \frac{3}{e^{x_0}} = 0,$$

the local maximum is also an absolute maximum. The triangle of largest area that has two sides along the positive coordinate axes and its hypotenuse tangent to the graph of $y = 3e^{-x}$ has vertices at the origin, $(2, 0)$ and $(0, 6e^{-1})$.

41. Let $(x, y) = (x, 9 - x^2)$ be the coordinates of the vertex of the rectangle that lies on the graph of the parabola. Then the width of the rectangle is x , the height is $9 - x^2$, and the area is

$$A = x(9 - x^2) = 9x - x^3.$$

The domain of this function is the closed interval $[0, 3]$, and the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 9 - 3x^2,$$

so the only critical number inside the open interval $(0, 3)$ is $x = \sqrt{3}$, where $A'(x) = 0$. Note that $x = -\sqrt{3}$ is not in the domain of A . Evaluating A at the endpoints of the interval $[0, 3]$ and at the critical number $\sqrt{3}$ yields

$$A(0) = 0, \quad A(\sqrt{3}) = 6\sqrt{3}, \quad \text{and} \quad A(3) = 0.$$

Therefore, A has an absolute maximum value of $6\sqrt{3}$; in other words, the largest area of a rectangle with one vertex on the parabola $y = 9 - x^2$, another at the origin, and the remaining two on the positive coordinate axes is $\boxed{6\sqrt{3} \text{ square units}}$.

42. Using the diagram in the text, if the slant height of the cone is 4, then the radius of the cone is $4 \sin \theta$, the height of the cone is $4 \cos \theta$, and the volume of the cone is

$$V = \frac{1}{3} \pi r^2 h = \frac{64\pi}{3} \sin^2 \theta \cos \theta = \frac{64\pi}{3} (\cos \theta - \cos^3 \theta).$$

The domain of this function is the closed interval $[0, \pi/2]$, and the critical numbers occur where $V'(\theta) = 0$. Now,

$$V'(\theta) = \frac{64\pi}{3} (3 \cos^2 \theta \sin \theta - \sin \theta) = \frac{64\pi}{3} \sin \theta (3 \cos^2 \theta - 1),$$

so the only value of θ inside the interval $(0, \pi/2)$ for which $V'(\theta) = 0$ is $\theta = \cos^{-1} \frac{\sqrt{3}}{3}$. Evaluating V at the endpoints of the interval $[0, \pi/2]$ and at the critical number $\cos^{-1} \frac{\sqrt{3}}{3}$ yields

$$V(0) = 0, \quad V\left(\cos^{-1} \frac{\sqrt{3}}{3}\right) = \frac{128\pi\sqrt{3}}{27}, \quad \text{and} \quad V(\pi/2) = 0.$$

Therefore, V has an absolute maximum at $\cos^{-1} \frac{\sqrt{3}}{3}$. Finally, the right circular cone having a slant height of 4 ft and maximum volume has

$$\boxed{\text{a radius of } \frac{4\sqrt{6}}{3} \text{ ft and a height of } \frac{4\sqrt{3}}{3} \text{ ft} .}$$

43. Let a and b be positive real numbers and let m denote the slope of a line through the point (a, b) .

- Case I: $m > 0$.

Let $m = b/a$. The line $\boxed{y = mx = bx/a}$ passes through the origin, so $x_0 = y_0 = 0$ and the distance between the points $(x_0, 0)$ and $(0, y_0)$ is zero, an absolute minimum.

- Case II: $m < 0$ so that $x_0 > 0$ and $y_0 > 0$.

The equation of the line of slope m which passes through the point (a, b) is

$$y - b = m(x - a) \quad \text{or} \quad y = mx + b - ma.$$

The y -intercept of this line is $y_0 = b - ma$ and the x -intercept is $x_0 = \frac{am - b}{m}$, so the distance D between the points $(x_0, 0)$ and $(0, y_0)$ is

$$D = \sqrt{\left(\frac{am - b}{m}\right)^2 + (b - ma)^2} = |b - ma| \sqrt{1 + \frac{1}{m^2}} = (b - ma) \sqrt{1 + \frac{1}{m^2}},$$

where the absolute value could be removed because $a > 0$, $b > 0$, and $m < 0$ so that $b - ma > 0$. The critical numbers of D occur where $D'(m) = 0$. Now,

$$D'(m) = (b - ma) \cdot \frac{-\frac{2}{m^3}}{2\sqrt{1 + \frac{1}{m^2}}} - a \sqrt{1 + \frac{1}{m^2}} = -\frac{1}{\sqrt{1 + \frac{1}{m^2}}} \left(\frac{b - ma}{m^3} + a + \frac{a}{m^2} \right),$$

so $D'(m) = 0$ when

$$\begin{aligned} \frac{b - ma}{m^3} &= -\left(a + \frac{a}{m^2}\right) \\ b - ma &= -am^3 - ma \\ m &= -\sqrt[3]{\frac{b}{a}}. \end{aligned}$$

As there is just the one critical number and

$$\lim_{m \rightarrow 0^-} D(m) = \infty \quad \text{and} \quad \lim_{m \rightarrow -\infty} D(m) = \infty,$$

D has an absolute minimum at $-\sqrt[3]{\frac{b}{a}}$. The line

$$y = -\sqrt[3]{\frac{b}{a}}x + b + a\sqrt[3]{\frac{b}{a}}$$

produces the minimum distance between the points $(x_0, 0)$ and $(0, y_0)$, where x_0 and y_0 are the x - and y -intercepts, respectively, of the line.

44. Let $v = \frac{\ln t}{t}$, and consider the domain $t \geq 1$. Now,

$$v'(t) = \frac{t \cdot \frac{1}{t} - \ln t}{t^2} = \frac{1 - \ln t}{t^2},$$

so the only critical number inside the open interval $(1, \infty)$ occurs when $1 - \ln t = 0$, or when $t = e$. Note that $t = 0$ is not in the domain of v . For $1 \leq t < e$, $v'(t) > 0$, while for $t > e$, $v'(t) < 0$. Therefore, v is increasing for $1 \leq t < e$ and decreasing for $t > e$. By the First Derivative Test, v has a local maximum at $t = e$. The local maximum value is $v(e) = \frac{1}{e}$. Because

$$v(1) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \frac{\ln t}{t} = \lim_{t \rightarrow \infty} \frac{\frac{1}{t}}{1} = 0,$$

where L'Hôpital's Rule was used to evaluate the limit, it follows that v has an absolute maximum at $t = e$. The object attains its maximum velocity at $t = e \approx 2.782$ seconds.

45. Let

$$F = \frac{cmg}{c \sin \theta + \cos \theta},$$

where the domain is the closed interval $[0, \pi/2]$. The critical numbers of F occur where $F'(\theta) = 0$ or where $F'(\theta)$ does not exist. Now,

$$F'(\theta) = \frac{-cmg}{(c \sin \theta + \cos \theta)^2} \cdot (c \cos \theta - \sin \theta).$$

$F'(\theta)$ exists for each θ in the open interval $(0, \pi/2)$ and is equal to zero when

$$c \cos \theta - \sin \theta = 0 \quad \text{or} \quad \tan \theta = c.$$

Based on the diagram below, when $\tan \theta = c$,

$$\sin \theta = \frac{c}{\sqrt{c^2 + 1}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{c^2 + 1}},$$

so, evaluating F at the endpoints of the interval $[0, \pi/2]$ and at the critical number associated with $\tan \theta = c$ yields

$$F(0) = cmg, \quad F(\tan^{-1} c) = \frac{cmg}{\sqrt{c^2 + 1}}, \quad \text{and} \quad F\left(\frac{\pi}{2}\right) = mg.$$

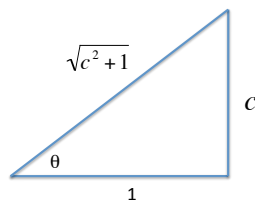
Because

$$\frac{c}{\sqrt{c^2 + 1}} < c \quad \text{its follows that} \quad F(\tan^{-1} c) = mg \cdot \frac{c}{\sqrt{c^2 + 1}} < cmg = F(0),$$

and because

$$\frac{c}{\sqrt{c^2 + 1}} < 1 \quad \text{its follows that} \quad F(\tan^{-1} c) = mg \cdot \frac{c}{\sqrt{c^2 + 1}} < mg = F\left(\frac{\pi}{2}\right),$$

so F has an absolute minimum when $\tan \theta = c$.



46. Let $V = kx(a - x)$ where k is a positive constant. The domain of V is the closed interval $[0, a]$, and the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = ak - 2kx = k(a - 2x),$$

so $x = a/2$ is the only critical number. Evaluating V at the endpoints of the interval $[0, a]$ and at the critical number $a/2$ yields

$$V(0) = 0, \quad V\left(\frac{a}{2}\right) = \frac{ka^2}{4}, \quad \text{and} \quad V(a) = 0.$$

Therefore, V has an absolute maximum when $x = a/2$. The reaction rate is maximum when $\boxed{x = a/2}$.

47. Let x denote the distance from the observer to the wall, and let the angles θ and ψ be as noted in the diagram below. Then

$$\tan \theta = \tan((\theta + \psi) - \psi) = \frac{\tan(\theta + \psi) - \tan \psi}{1 + \tan(\theta + \psi) \tan \psi} = \frac{\frac{7}{x} - \frac{3}{x}}{1 + \frac{7}{x} \cdot \frac{3}{x}} = \frac{4x}{x^2 + 21}$$

and

$$\theta = \tan^{-1} \left(\frac{4x}{x^2 + 21} \right).$$

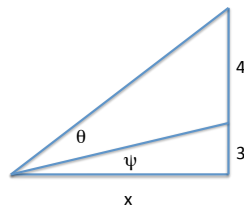
The domain of this function is the interval $(0, \infty)$, and the critical numbers occur where $\theta'(x) = 0$. Now,

$$\begin{aligned}\theta'(x) &= \frac{1}{1 + \left(\frac{4x}{x^2+21}\right)^2} \cdot \frac{(x^2+21) \cdot 4 - 4x \cdot 2x}{(x^2+21)^2} \\ &= \frac{4x^2 + 84 - 8x^2}{(x^2+21)^2 + 16x^2} = 4 \frac{21 - x^2}{(x^2+21)^2 + 16x^2},\end{aligned}$$

so the only critical number inside the open interval $(0, \infty)$ is $x = \sqrt{21}$, where $\theta'(x) = 0$. Note that $x = -\sqrt{21}$ is not in the domain of θ . As there is just this one critical number and

$$\lim_{x \rightarrow 0^+} \theta(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \theta(x) = 0,$$

θ has an absolute maximum at $\sqrt{21}$. Therefore, to obtain the most favorable view of the picture, the observer should stand $\boxed{\sqrt{21} \approx 4.583 \text{ meters}}$ from the wall.



48. Let $v = kx^2 \ln \frac{1}{x} = -kx^2 \ln x$. The domain of this function is the interval $(0, \infty)$. Now,

$$v'(x) = -kx^2 \cdot \frac{1}{x} - 2kx \ln x = -kx(1 + 2 \ln x),$$

so the only critical number inside the open interval $(0, \infty)$ is $x = e^{-1/2}$, where $v'(x) = 0$. Note that $x = 0$ is not in the domain of v .

Suppose the constant k is greater than 0. Then for $0 \leq x < e^{-1/2}$, $v'(x) > 0$, while for $x > e^{-1/2}$, $v'(x) < 0$. Therefore, v is increasing for $0 \leq x < e^{-1/2}$ and decreasing for $x > e^{-1/2}$. By the First Derivative Test, v has a local maximum at $x = e^{-1/2}$. The local maximum value is

$$v(e^{-1/2}) = -ke^{-1} \left(-\frac{1}{2}\right) = \frac{k}{2e} > 0.$$

Because

$$\lim_{x \rightarrow 0^+} v(x) = \lim_{x \rightarrow 0^+} (-kx^2 \ln x) = -k \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-2}} = -k \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = \frac{k}{2} \lim_{x \rightarrow 0^+} x^2 = 0$$

and

$$\lim_{x \rightarrow \infty} v(x) = \lim_{x \rightarrow \infty} (-kx^2 \ln x) = -\infty,$$

where L'Hôpital's Rule was used to evaluate the first limit, it follows that v has an absolute maximum at $\boxed{x = e^{-1/2}}$.

On the other hand, suppose that k is less than 0. Then

$$\lim_{x \rightarrow \infty} -kx^2 \ln x = \infty$$

and v does not have an absolute maximum value.

49. Let a , b , and c be positive constants and consider the function $f(x) = ae^{cx} + be^{-cx}$. Then

$$f'(x) = ace^{cx} - bce^{-cx},$$

and the only critical number occurs when

$$\begin{aligned} ace^{cx} &= bce^{-cx} \\ e^{2cx} &= \frac{b}{a} \\ x &= \frac{1}{2c} \ln \frac{b}{a}. \end{aligned}$$

Next,

$$f''(x) = ac^2e^{cx} + bc^2e^{-cx} \geq 0$$

for all x . Therefore, f achieves an absolute minimum value when $x = \frac{1}{2c} \ln \frac{b}{a}$, and that absolute minimum value is

$$f\left(\frac{1}{2c} \ln \frac{b}{a}\right) = ae^{\frac{1}{2} \ln \frac{b}{a}} + be^{-\frac{1}{2} \ln \frac{b}{a}} = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} = 2\sqrt{ab}.$$

Challenge Problems

50. In order to find the length of the *longest* pipe that can be carried around the corner, we must find the *shortest* length from the right wall to the top wall touching the corner of the inside wall. Any pipe that does not fit in this shortest space cannot be carried around the corner, so an exact fit represents the longest possible pipe. Let θ be as defined in the diagram in the text, and let c_1 be the length of the pipe in the 1-m wide corridor and c_2 be the length of the pipe in the 8-m wide corridor. By the definitions of the sine and cosine functions,

$$c_1 = \frac{1}{\sin \theta} \quad \text{and} \quad c_2 = \frac{8}{\cos \theta},$$

so the total length L of the pipe is

$$L = \frac{1}{\sin \theta} + \frac{8}{\cos \theta}.$$

The domain of this function is the interval $(0, \pi/2)$, and the critical numbers occur where $L'(\theta) = 0$. Now,

$$L'(\theta) = -\frac{1}{\sin^2 \theta} \cdot \cos \theta + \frac{8}{\cos^2 \theta} \cdot \sin \theta = \frac{8 \sin^3 \theta - \cos^3 \theta}{\sin^2 \theta \cos^2 \theta},$$

so $L'(\theta) = 0$ when

$$\begin{aligned} 8 \sin^3 \theta &= \cos^3 \theta \\ \tan^3 \theta &= \frac{1}{8} \\ \theta &= \tan^{-1} \frac{1}{2}. \end{aligned}$$

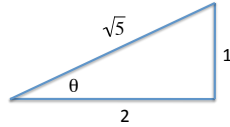
As there is only the one critical number and $L \rightarrow \infty$ as $\theta \rightarrow 0^+$ and as $\theta \rightarrow \pi/2^-$, L has an absolute minimum when $\theta = \tan^{-1} \frac{1}{2}$. Based on the diagram below, when $\theta = \tan^{-1} \frac{1}{2}$,

$$\sin \theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos \theta = \frac{2}{\sqrt{5}},$$

so

$$L(\tan^{-1} 2) = \frac{1}{1/\sqrt{5}} + \frac{8}{2/\sqrt{5}} = \sqrt{5} + 4\sqrt{5} = 5\sqrt{5}.$$

The longest pipe that can be carried around the indicated corner is therefore $5\sqrt{5} \approx 11.180$ meters long.



51. Based on the diagram in the text,

$$\cos \theta = \frac{20}{s} \quad \text{so} \quad s = \frac{20}{\cos \theta}$$

and

$$I = \frac{\sin \theta}{s} = \frac{\sin \theta \cos \theta}{20} = \frac{\sin(2\theta)}{40}.$$

The domain of this function is the interval $[0, \pi/2)$. Note that s is undefined for $\theta = \pi/2$, so this number cannot be included in the domain. Now,

$$I'(\theta) = \frac{1}{20} \cos(2\theta),$$

so $I'(\theta) = 0$ on the open interval $(0, \pi/2)$ when $\theta = \pi/4$. Because

$$I(0) = 0, \quad I\left(\frac{\pi}{4}\right) = \frac{1}{40}, \quad \text{and} \quad \lim_{\theta \rightarrow \pi/2^-} I(\theta) = 0,$$

it follows that I has an absolute maximum when $\theta = \pi/4$. If h is the height of the lamp, then

$$\tan \theta = \frac{h}{20} \quad \text{so} \quad h = 20 \tan \theta.$$

For maximum illumination on the walk, the height of the lamp should be

$$h = 20 \tan \frac{\pi}{4} = \boxed{20 \text{ ft}}.$$

52. Let $y = f(x) = e^{-x^2}$. Then

$$\begin{aligned} f'(x) &= -2xe^{-x^2}; \text{ and} \\ f''(x) &= 4x^2e^{-x^2} - 2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}, \end{aligned}$$

so $f''(x) = 0$ at $x = \pm \frac{\sqrt{2}}{2}$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{\sqrt{2}}{2})$	+	f is concave up on $(-\infty, -\frac{\sqrt{2}}{2})$
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	-	f is concave down on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{2}}{2}, \infty)$	+	f is concave up on $(\frac{\sqrt{2}}{2}, \infty)$

The concavity of f changes at $\pm \frac{\sqrt{2}}{2}$, so the points $\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ and $\left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ are points of inflection.

Now, let (x, e^{-x^2}) and $(-x, e^{-x^2})$ be the coordinates of the upper vertices of the inscribed rectangle, where $x > 0$. The area A of the rectangle is then

$$A = 2xe^{-x^2}.$$

The critical numbers of A occur where $A'(x) = 0$. Next,

$$A'(x) = -4x^2e^{-x^2} + 2e^{-x^2} = -2(2x^2 - 1)e^{-x^2},$$

so $A'(x) = 0$ when $x = \frac{\sqrt{2}}{2}$. Note that $x = -\frac{\sqrt{2}}{2}$ is not in the domain of A . As

$$A\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}e^{-1/2} > 0$$

and

$$\lim_{x \rightarrow 0^+} A(x) = 0, \quad \text{and} \quad \lim_{x \rightarrow \infty} A(x) = \lim_{x \rightarrow \infty} \frac{2x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{xe^{x^2}} = 0,$$

A has an absolute maximum when $x = \frac{\sqrt{2}}{2}$. Observe that this is the x -coordinate of one of the points of inflection of y , so the rectangle of largest area that can be inscribed under the graph of $y = e^{-x^2}$ has two of its vertices at the points of inflection of y .

53. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point (x, y) on the graph of $y = e^{-x/2}$ and the point $(1, 8)$. Then

$$D = (x - 1)^2 + (y - 8)^2 = (x - 1)^2 + (e^{-x/2} - 8)^2.$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 2(x - 1) + 2(e^{-x/2} - 8) \cdot \left(-\frac{1}{2}e^{-x/2}\right) = 2(x - 1) - e^{-x/2}(e^{-x/2} - 8).$$

Using the computer algebra system *Maple*, the only critical number is $x \approx -3.758$. As

$$\lim_{x \rightarrow \pm\infty} D(x) = \infty,$$

it follows that D has an absolute minimum at approximately -3.758 . The point $\boxed{(-3.758, e^{1.879}) \approx (-3.758, 6.547)}$ on the graph of $y = e^{-x/2}$ is closest to the point $(1, 8)$.

4.8: Antiderivatives; Differential Equations

Concepts and Vocabulary

1. A function F is called an **antiderivative** of a function f if $F' = f$.
2. **True**. If F is an antiderivative of f , then $F(x) + C$, where C is a constant, is also an antiderivative of f .
3. All the antiderivatives of $y = x^{-1}$ are $\ln|x| + C$, where C is a constant.
4. **True**. An antiderivative of $\sin x$ is $-\cos x + \pi$ because $\frac{d}{dx}(-\cos x + \pi) = \sin x$.
5. **False**. The general solution of a differential equation $\frac{dy}{dx} = f(x)$ consists of all the antiderivatives of $f(x)$.
6. **True**. Free fall is an example of motion with constant acceleration.
7. **True**. To find a particular solution of a differential equation $\frac{dy}{dx} = f(x)$, we need a boundary condition.
8. **True**. If F_1 and F_2 are both antiderivatives of a function f on an interval I , then $F_1 - F_2 = C$, a constant, on the interval I .

Skill Building

9. All antiderivatives of the function $f(x) = 2$ are $F(x) = 2x + C$, where C is a constant.
10. All antiderivatives of the function $f(x) = \frac{1}{2}$ are $F(x) = \frac{1}{2}x + C$, where C is a constant.
11. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^5$ are

$$F(x) = 4 \cdot \frac{x^{5+1}}{5+1} + C = \frac{2}{3}x^6 + C,$$

where C is a constant.

12. All antiderivatives of the function $f(x) = x$ are

$$F(x) = \frac{x^{1+1}}{1+1} + C = \frac{1}{2}x^2 + C,$$

where C is a constant.

13. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 5x^{3/2}$ are

$$F(x) = 5 \cdot \frac{x^{3/2+1}}{3/2+1} + C = 2x^{5/2} + C,$$

where C is a constant.

14. Using the Sum Rule, all antiderivatives of the function $f(x) = x^{5/2} + 2$ are

$$F(x) = \frac{x^{5/2+1}}{5/2+1} + 2x + C = \frac{2}{7}x^{7/2} + 2x + C,$$

where C is a constant.

15. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 2x^{-2}$ are

$$F(x) = 2 \cdot \frac{x^{-2+1}}{-2+1} + C = -2x^{-1} + C = \boxed{-\frac{2}{x} + C},$$

where C is a constant.

16. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = 3x^{-3}$ are

$$F(x) = 3 \cdot \frac{x^{-3+1}}{-3+1} + C = \boxed{-\frac{3}{2}x^{-2} + C},$$

where C is a constant.

17. All antiderivatives of the function $f(x) = \sqrt{x} = x^{1/2}$ are

$$F(x) = \frac{x^{1/2+1}}{1/2+1} + C = \boxed{\frac{2}{3}x^{3/2} + C},$$

where C is a constant.

18. All antiderivatives of the function $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$ are

$$F(x) = \frac{x^{-1/2+1}}{-1/2+1} + C = 2x^{1/2} + C = \boxed{2\sqrt{x} + C},$$

where C is a constant.

19. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^3 - 3x^2 + 1$ are

$$F(x) = 4 \cdot \frac{x^{3+1}}{3+1} - 3 \cdot \frac{x^{2+1}}{2+1} + x + C = \boxed{x^4 - x^3 + x + C},$$

where C is a constant.

20. Using the Sum Rule, all antiderivatives of the function $f(x) = x^2 - x$ are

$$F(x) = \frac{x^{2+1}}{2+1} - \frac{x^{1+1}}{1+1} + C = \boxed{\frac{1}{3}x^3 - \frac{1}{2}x^2 + C},$$

where C is a constant.

21. Expand

$$(2 - 3x)^2 \quad \text{as} \quad 9x^2 - 12x + 4.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = (2 - 3x)^2 = 9x^2 - 12x + 4$ are

$$F(x) = 9 \cdot \frac{x^{2+1}}{2+1} - 12 \cdot \frac{x^{1+1}}{1+1} + 4x + C = \boxed{3x^3 - 6x^2 + 4x + C},$$

where C is a constant.

22. Expand

$$(3x - 1)^2 \quad \text{as} \quad 9x^2 - 6x + 1.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = (3x - 1)^2 = 9x^2 - 6x + 1$ are

$$F(x) = 9 \cdot \frac{x^{2+1}}{2+1} - 6 \cdot \frac{x^{1+1}}{1+1} + x + C = \boxed{3x^3 - 3x^2 + x + C},$$

where C is a constant.

23. Rewrite

$$\frac{3x-2}{x} \quad \text{as} \quad \frac{3x}{x} - \frac{2}{x} = 3 - \frac{2}{x} = 3 - 2x^{-1}.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = \frac{3x-2}{x} = 3 - 2x^{-1}$ are

$$F(x) = \boxed{3x - 2 \ln |x| + C},$$

where C is a constant.

24. Rewrite

$$\frac{4x^{3/2}-1}{x} \quad \text{as} \quad \frac{4x^{3/2}}{x} - \frac{1}{x} = 4x^{1/2} - x^{-1}.$$

Then, using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = \frac{4x^{3/2}-1}{x} = 4x^{1/2} - x^{-1}$ are

$$F(x) = 4 \cdot \frac{x^{1/2+1}}{1/2+1} - \ln |x| + C = \boxed{\frac{8}{3}x^{3/2} - \ln |x| + C},$$

where C is a constant.

25. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 2x - 3 \cos x$ are

$$F(x) = 2 \cdot \frac{x^{1+1}}{1+1} - 3 \cdot \sin x + C = \boxed{x^2 - 3 \sin x + C},$$

where C is a constant.

26. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 2 \sin x - \cos x$ are

$$F(x) = 2 \cdot (-\cos x) - \sin x + C = \boxed{-2 \cos x - \sin x + C},$$

where C is a constant.

27. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4e^x + x$ are

$$F(x) = 4 \cdot e^x + \frac{x^{1+1}}{1+1} + C = \boxed{4e^x + \frac{1}{2}x^2 + C},$$

where C is a constant.

28. Because

$$\frac{d}{dx}e^{-x} = e^{-x} \cdot (-1) = -e^{-x},$$

it follows that

$$\frac{d}{dx}(-e^{-x}) = e^{-x},$$

so $-e^{-x}$ is an antiderivative of the function e^{-x} . Using the Sum Rule, all antiderivatives of the function $f(x) = e^{-x} + \sec^2 x$ are

$$F(x) = \boxed{-e^{-x} + \tan x + C},$$

where C is a constant.

29. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = \frac{7}{1+x^2}$ are

$$F(x) = 7 \cdot \tan^{-1} x + C = \boxed{7 \tan^{-1} x + C},$$

where C is a constant.

30. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = x + \frac{10}{\sqrt{1-x^2}}$ are

$$F(x) = \frac{x^{1+1}}{1+1} + 10 \cdot \sin^{-1} x + C = \boxed{\frac{1}{2}x^2 + 10 \sin^{-1} x + C},$$

where C is a constant.

31. The general solution of the differential equation $\frac{dy}{dx} = 3x^2 - 2x + 1$ is

$$y = 3 \cdot \frac{x^{2+1}}{2+1} - 2 \cdot \frac{x^{1+1}}{1+1} + x + C = x^3 - x^2 + x + C,$$

where C is a constant. Applying the boundary condition that when $x = 2$, then $y = 1$ yields

$$1 = 2^3 - 2^2 + 2 + C = 6 + C,$$

so that $C = -5$. The particular solution of the differential equation with the boundary condition that when $x = 2$, then $y = 1$, is therefore

$$\boxed{y = x^3 - x^2 + x - 5}.$$

32. The general solution of the differential equation $\frac{dv}{dt} = 3t^2 - 2t + 1$ is

$$v = 3 \cdot \frac{t^{2+1}}{2+1} - 2 \cdot \frac{t^{1+1}}{1+1} + t + C = t^3 - t^2 + t + C,$$

where C is a constant. Applying the boundary condition that when $t = 1$, then $v = 5$ yields

$$5 = 1^3 - 1^2 + 1 + C = 1 + C,$$

so that $C = 4$. The particular solution of the differential equation with the boundary condition that when $t = 1$, then $v = 5$, is therefore

$$\boxed{v = t^3 - t^2 + t + 4}.$$

33. The general solution of the differential equation

$$\frac{dy}{dx} = x^{1/3} + x\sqrt{x} - 2 = x^{1/3} + x^{3/2} - 2$$

is

$$y = \frac{x^{1/3+1}}{1/3+1} + \frac{x^{3/2+1}}{3/2+1} - 2x + C = \frac{3}{4}x^{4/3} + \frac{2}{5}x^{5/2} - 2x + C,$$

where C is a constant. Applying the boundary condition that when $x = 1$, then $y = 2$ yields

$$2 = \frac{3}{4}1^{4/3} + \frac{2}{5}1^{5/2} - 2(1) + C = \frac{3}{4} + \frac{2}{5} - 2 + C = -\frac{17}{20} + C,$$

so that $C = \frac{57}{20}$. The particular solution of the differential equation with the boundary condition that when $x = 1$, then $y = 2$, is therefore

$$\boxed{y = \frac{3}{4}x^{4/3} + \frac{2}{5}x^{5/2} - 2x + \frac{57}{20}}.$$

34. The general solution of the differential equation $s'(t) = t^4 + 4t^3 - 5$ is

$$s(t) = \frac{t^{4+1}}{4+1} + 4 \cdot \frac{t^{3+1}}{3+1} - 5t + C = \frac{1}{5}t^5 + t^4 - 5t + C,$$

where C is a constant. Applying the boundary condition that when $t = 2$, then $s(2) = 5$ yields

$$5 = \frac{1}{5}2^5 + 2^4 - 5(2) + C = \frac{32}{5} + 16 - 10 + C = \frac{62}{5} + C,$$

so that $C = -\frac{37}{5}$. The particular solution of the differential equation with the boundary condition that when $t = 2$, then $s(2) = 5$, is therefore

$$\boxed{s(t) = \frac{1}{5}t^5 + t^4 - 5t - \frac{37}{5}}.$$

35. The general solution of the differential equation

$$\frac{ds}{dt} = t^3 + \frac{1}{t^2} = t^3 + t^{-2}$$

is

$$s = \frac{t^{3+1}}{3+1} + \frac{t^{-2+1}}{-2+1} + C = \frac{1}{4}t^4 - t^{-1} + C = \frac{1}{4}t^4 - \frac{1}{t} + C,$$

where C is a constant. Applying the boundary condition that when $t = 1$, then $s = 2$ yields

$$2 = \frac{1}{4}1^4 - \frac{1}{1} + C = \frac{1}{4} - 1 + C = -\frac{3}{4} + C,$$

so that $C = \frac{11}{4}$. The particular solution of the differential equation with the boundary condition that when $t = 1$, then $s = 2$, is therefore

$$\boxed{s = \frac{1}{4}t^4 - \frac{1}{t} + \frac{11}{4}}.$$

36. The general solution of the differential equation

$$\frac{dy}{dx} = \sqrt{x} - x\sqrt{x} + 1 = x^{1/2} - x^{3/2} + 1$$

is

$$y = \frac{x^{1/2+1}}{1/2+1} - \frac{x^{3/2+1}}{3/2+1} + x + C = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + x + C,$$

where C is a constant. Applying the boundary condition that when $x = 1$, then $y = 0$ yields

$$0 = \frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2} + 1 + C = \frac{2}{3} - \frac{2}{5} + 1 + C = \frac{19}{15} + C,$$

so that $C = -\frac{19}{15}$. The particular solution of the differential equation with the boundary condition that when $x = 1$, then $y = 0$, is therefore

$$\boxed{y = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + x - \frac{19}{15}}.$$

37. The general solution of the differential equation $f'(x) = x - 2 \sin x$ is

$$f(x) = \frac{x^{1+1}}{1+1} - 2 \cdot (-\cos x) + C = \frac{1}{2}x^2 + 2 \cos x + C,$$

where C is a constant. Applying the boundary condition that when $x = \pi$, then $f(\pi) = 0$ yields

$$0 = \frac{1}{2}\pi^2 + 2 \cos \pi + C = \frac{1}{2}\pi^2 - 2 + C,$$

so that $C = 2 - \frac{1}{2}\pi^2$. The particular solution of the differential equation with the boundary condition that when $x = \pi$, then $f(\pi) = 0$, is therefore

$$f(x) = \frac{1}{2}x^2 + 2 \cos x + 2 - \frac{1}{2}\pi^2.$$

38. The general solution of the differential equation $\frac{dy}{dx} = x^2 - 2 \sin x$ is

$$y = \frac{x^{2+1}}{2+1} - 2 \cdot (-\cos x) + C = \frac{1}{3}x^3 + 2 \cos x + C,$$

where C is a constant. Applying the boundary condition that when $x = \pi$, then $y = 0$ yields

$$0 = \frac{1}{3}\pi^3 + 2 \cos \pi + C = \frac{1}{3}\pi^3 - 2 + C,$$

so that $C = 2 - \frac{1}{3}\pi^3$. The particular solution of the differential equation with the boundary condition that when $x = \pi$, then $y = 0$, is therefore

$$y = \frac{1}{3}x^3 + 2 \cos x + 2 - \frac{1}{3}\pi^3.$$

39. All the antiderivatives of $\frac{d^2y}{dx^2} = e^x$ are

$$\frac{dy}{dx} = e^x + C_1,$$

and all the antiderivatives of $\frac{dy}{dx} = e^x + C_1$ are

$$y = e^x + C_1x + C_2,$$

where C_1 and C_2 are constants. Applying the boundary condition that when $x = 0$, then $y = 2$ yields

$$2 = e^0 + C_1(0) + C_2 = 1 + C_2,$$

so that $C_2 = 1$. Next, applying the boundary condition that when $x = 1$, then $y = e$ yields

$$e = e^1 + C_1(1) + 1 = e + 1 + C_1,$$

so that $C_1 = -1$. The particular solution of the differential equation with the given boundary conditions is therefore

$$y = e^x - x + 1.$$

40. All the antiderivatives of $f''(\theta) = \sin \theta + \cos \theta$ are

$$f'(\theta) = -\cos \theta + \sin \theta + C_1,$$

and all antiderivatives of $f'(\theta) = -\cos \theta + \sin \theta + C_1$ are

$$f(\theta) = -\sin \theta - \cos \theta + C_1\theta + C_2,$$

where C_1 and C_2 are constants. Applying the boundary condition that when $\theta = \frac{\pi}{2}$, then $f'(\frac{\pi}{2}) = 2$ yields

$$2 = -\cos \frac{\pi}{2} + \sin \frac{\pi}{2} + C_1 = 0 + 1 + C_1,$$

so that $C_1 = 1$. Next, applying the boundary condition that when $\theta = \pi$, then $f(\pi) = 4$ yields

$$4 = -\sin \pi - \cos \pi + \pi + C_2 = 0 - (-1) + \pi + C_2,$$

so that $C_2 = 3 - \pi$. The particular solution of the differential equation with the given boundary conditions is therefore

$$\boxed{f(\theta) = -\sin \theta - \cos \theta + \theta + 3 - \pi}.$$

41. The general solution of the differential equation $\frac{dv}{dt} = a(t) = -32$ is

$$v(t) = -32t + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 128$ yields

$$128 = -32(0) + C_1 = C_1,$$

so that $v(t) = -32t + 128$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = -32t + 128$ is

$$s(t) = -32 \cdot \frac{t^{1+1}}{1+1} + 128t + C_2 = -16t^2 + 128t + C_2.$$

Applying the initial condition $s(0) = 0$ yields

$$0 = -16(0)^2 + 128(0) + C_2 = C_2,$$

so that $\boxed{s(t) = -16t^2 + 128t}$.

42. The general solution of the differential equation $\frac{dv}{dt} = a(t) = -980$ is

$$v(t) = -980t + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 1980$ yields

$$1980 = -980(0) + C_1 = C_1,$$

so that $v(t) = -980t + 1980$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = -980t + 1980$ is

$$s(t) = -980 \cdot \frac{t^{1+1}}{1+1} + 1980t + C_2 = -490t^2 + 1980t + C_2.$$

Applying the initial condition $s(0) = 5$ yields

$$5 = -490(0)^2 + 1980(0) + C_2 = C_2,$$

so that $\boxed{s(t) = -490t^2 + 1980t + 5}$.

43. The general solution of the differential equation $\frac{dv}{dt} = a(t) = 3t$ is

$$v(t) = 3 \cdot \frac{t^{1+1}}{1+1} + C_1 = \frac{3}{2}t^2 + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 18$ yields

$$18 = \frac{3}{2}0^2 + C_1 = C_1,$$

so that $v(t) = \frac{3}{2}t^2 + 18$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = \frac{3}{2}t^2 + 18$ is

$$s(t) = \frac{3}{2} \cdot \frac{t^{2+1}}{2+1} + 18t + C_2 = \frac{1}{2}t^3 + 18t + C_2.$$

Applying the initial condition $s(0) = 2$ yields

$$2 = \frac{1}{2}(0)^3 + 18(0) + C_2 = C_2,$$

so that $s(t) = \frac{1}{2}t^3 + 18t + 2$.

44. The general solution of the differential equation $\frac{dv}{dt} = a(t) = 5t - 2$ is

$$v(t) = 5 \cdot \frac{t^{1+1}}{1+1} - 2t + C_1 = \frac{5}{2}t^2 - 2t + C_1,$$

where C_1 is a constant. Applying the initial condition $v(0) = 8$ yields

$$8 = \frac{5}{2}0^2 - 2(0) + C_1 = C_1,$$

so that $v(t) = \frac{5}{2}t^2 - 2t + 8$. Next, the general solution of the differential equation $\frac{ds}{dt} = v(t) = \frac{5}{2}t^2 - 2t + 8$ is

$$s(t) = \frac{5}{2} \cdot \frac{t^{2+1}}{2+1} - 2 \cdot \frac{t^{1+1}}{1+1} + 8t + C_2 = \frac{5}{6}t^3 - t^2 + 8t + C_2.$$

Applying the initial condition $s(0) = 0$ yields

$$0 = \frac{5}{6}(0)^3 - (0)^2 + 8(0) + C_2 = C_2,$$

so that $s(t) = \frac{5}{6}t^3 - t^2 + 8t$.

Applications and Extensions

45. Because $u^2 + 10u + 21 = (u + 3)(u + 7)$ and $3u + 9 = 3(u + 3)$,

$$f(u) = \frac{u^2 + 10u + 21}{3u + 9} = \frac{(u + 3)(u + 7)}{3(u + 3)} = \frac{u + 7}{3} = \frac{1}{3}u + \frac{7}{3},$$

for $u \neq -3$. Then, for $u \neq -3$, all of the antiderivatives of f are

$$F(u) = \frac{1}{3} \cdot \frac{u^2}{2} + \frac{7}{3}u + C = \boxed{\frac{1}{6}u^2 + \frac{7}{3}u + C},$$

where C is a constant.

46. Write

$$\frac{t^3 - 5t + 8}{t^5} \quad \text{as} \quad \frac{t^3}{t^5} - \frac{5t}{t^5} + \frac{8}{t^5} = t^{-2} - 5t^{-4} + 8t^{-5}.$$

Then, all of the antiderivatives of f are

$$F(t) = \frac{t^{-1}}{-1} - 5 \cdot \frac{t^{-3}}{-3} + 8 \cdot \frac{t^{-4}}{-4} + C = \boxed{-t^{-1} + \frac{5}{3}t^{-3} - 2t^{-4} + C},$$

where C is a constant.

47. Write

$$\frac{t^4 + 3t - 1}{t} \quad \text{as} \quad \frac{t^4}{t} + \frac{3t}{t} - \frac{1}{t} = t^3 + 3 - \frac{1}{t}.$$

The general solution of the differential equation

$$f'(t) = \frac{t^4 + 3t - 1}{t} = t^3 + 3 - \frac{1}{t}$$

is then

$$f(t) = \frac{1}{4}t^4 + 3t - \ln|t| + C,$$

where C is a constant. Applying the boundary condition that when $t = 1$, then $f(1) = \frac{1}{4}$ yields

$$\frac{1}{4} = \frac{1}{4}1^4 + 3(1) - \ln 1 + C = \frac{13}{4} + C,$$

so that $C = -3$. The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{f(t) = \frac{1}{4}t^4 + 3t - \ln|t| - 3}.$$

48. Write

$$\frac{x^2 - 1}{x^4 - 1} \quad \text{as} \quad \frac{x^2 - 1}{(x^2 - 1)(x^2 + 1)} = \frac{1}{x^2 + 1},$$

which is valid for $x \neq \pm 1$. The general solution of the differential equation

$$g'(x) = \frac{1}{x^2 + 1}$$

is $g(x) = \tan^{-1} x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $g(0) = 0$ yields

$$0 = \tan^{-1} 0 + C = C,$$

so that $C = 0$. The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{g(x) = \tan^{-1} x},$$

provided $x \neq \pm 1$.

49. Given

$$\frac{d}{dx}(x \cos x + \sin x) = -x \sin x + 2 \cos x,$$

it follows that $x \cos x + \sin x$ is an antiderivative of the function $-x \sin x + 2 \cos x$. The general solution of the differential equation

$$\frac{dF}{dx} = -x \sin x + 2 \cos x$$

is $F(x) = x \cos x + \sin x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $F(0) = 1$ yields

$$1 = 0 \cdot \cos 0 + \sin 0 + C = C.$$

The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{F(x) = x \cos x + \sin x + 1}.$$

50. Given

$$\frac{d}{dx} \sin x^2 = 2x \cos x^2,$$

it follows that

$$\frac{d}{dx} \left(\frac{1}{2} \sin x^2 \right) = x \cos x^2$$

and $\frac{1}{2} \sin x^2$ is an antiderivative of the function $x \cos x^2$. The general solution of the differential equation

$$\frac{dh}{dx} = x \cos x^2$$

is $h(x) = \frac{1}{2} \sin x^2 + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $h(0) = 2$ yields

$$2 = \frac{1}{2} \sin 0^2 + C = C.$$

The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{h(x) = \frac{1}{2} \sin x^2 + 2}.$$

51. Let $t = 0$ denote the time the brakes are applied, let v_0 denote the speed of the car when the brakes are applied, and let $s(t)$ represent the distance the car has traveled t seconds after the brakes have been applied. Because the car decelerates at the rate of 10 m/s^2 , its acceleration a is given by

$$a(t) = \frac{dv}{dt} = -10.$$

Solve this differential equation for $v(t)$ to obtain $v(t) = -10t + C_1$, where C_1 is a constant. Applying the initial condition $v(0) = v_0$ yields

$$v_0 = -10(0) + C_1 = C_1,$$

so that $C_1 = v_0$ and $v(t) = -10t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -10t + v_0$$

to obtain

$$s(t) = -10 \cdot \frac{t^2}{2} + v_0 t + C_2 = -5t^2 + v_0 t + C_2,$$

where C_2 is a constant. Applying the initial condition $s(0) = 0$ gives

$$0 = -5(0)^2 + v_0(0) + C_2 = C_2,$$

so that $C_2 = 0$ and $s(t) = -5t^2 + v_0 t$. Now, the car comes to a stop when its speed is equal to zero; that is when

$$-10t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{10}.$$

By the time the car comes to rest, it has traveled

$$s\left(\frac{v_0}{10}\right) = -5\left(\frac{v_0}{10}\right)^2 + v_0 \cdot \frac{v_0}{10} = \frac{v_0^2}{20} \text{ meters}$$

from the moment the brakes were applied. If the car is to stop within 15 meters, then

$$s\left(\frac{v_0}{10}\right) \leq 15 \quad \text{or} \quad \frac{v_0^2}{20} \leq 15.$$

The maximum possible velocity for the car is therefore $\boxed{v_0 = 10\sqrt{3} \text{ m/s} \approx 17.32 \text{ m/s}}$. In miles per hour, this is approximately

$$10\sqrt{3} \text{ m/s} \cdot \frac{3600 \text{ s}}{1 \text{ h}} \cdot \frac{1 \text{ km}}{1000 \text{ m}} \cdot \frac{1 \text{ mi}}{1.60934 \text{ km}} \approx \boxed{38.74 \text{ mi/h}}.$$

52. Let $t = 0$ denote the time that the car begins to accelerate, and let $s(t)$ represent the distance the car has traveled after t seconds. Because the car accelerates at a constant rate from 0 to 60 km/h in 10 s, its acceleration a is given by

$$a = \frac{60 \text{ km/h}}{10 \text{ s}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = \frac{5}{3} \text{ m/s}^2.$$

Solve the differential equation

$$\frac{dv}{dt} = a(t) = \frac{5}{3}$$

for $v(t)$ to obtain $v(t) = \frac{5}{3}t + C_1$, where C_1 is a constant. Apply the initial condition $v(0) = 0$ to determine

$$0 = \frac{5}{3}(0) + C_1 = C_1,$$

so that $v(t) = \frac{5}{3}t$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = \frac{5}{3}t$$

for $s(t)$ to obtain

$$s(t) = \frac{5}{3} \cdot \frac{t^2}{2} + C_2 = \frac{5}{6}t^2 + C_2.$$

The initial condition $s(0) = 0$ determines

$$0 = \frac{5}{6}(0)^2 + C_2 = C_2,$$

so $s(t) = \frac{5}{6}t^2$. Therefore, after 10 seconds, the car has traveled

$$s(10) = \frac{5}{6}(10)^2 = \boxed{\frac{250}{3} \text{ m}}.$$

53. Let $t = 0$ denote the time that the car begins to accelerate, and let $s(t)$ represent the distance the car has traveled after t seconds. Because the car accelerates at a constant rate from 0 to 60 mph in 5 s, its acceleration a is given by

$$a = \frac{60 \text{ mph}}{5 \text{ s}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = 17.6 \text{ ft/s}^2.$$

Solve the differential equation

$$\frac{dv}{dt} = a(t) = 17.6$$

for $v(t)$ to obtain $v(t) = 17.6t + C_1$, where C_1 is a constant. Apply the initial condition $v(0) = 0$ to determine

$$0 = 17.6(0) + C_1 = C_1,$$

so that $v(t) = 17.6t$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = 17.6t$$

for $s(t)$ to obtain

$$s(t) = 17.6 \cdot \frac{t^2}{2} + C_2 = 8.8t^2 + C_2.$$

The initial condition $s(0) = 0$ determines

$$0 = 8.8(0)^2 + C_2 = C_2,$$

so $s(t) = 8.8t^2$. Therefore, after 5 seconds, the car has traveled

$$s(5) = 8.8(5)^2 = \boxed{220 \text{ ft}}.$$

54. First consider the athlete jumping on Earth. Let $t = 0$ denote the time the athlete loses contact with the ground, let v_0 denote the initial upward speed of the athlete, and let $s(t)$ represent the height of the athlete t seconds after initiating the jump. The general solution of the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

is $v(t) = -9.8t + C_1$, where C_1 is a constant. Applying the initial condition $v(0) = v_0$ yields

$$v_0 = -9.8(0) + C_1 = C_1,$$

so that $C_1 = v_0$ and $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

to obtain

$$s(t) = -9.8 \cdot \frac{t^2}{2} + v_0t + C_2 = -4.9t^2 + v_0t + C_2,$$

where C_2 is a constant. Applying the initial condition $s(0) = 0$ gives

$$0 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so that $C_2 = 0$ and $s(t) = -4.9t^2 + v_0t$. Now, the athlete achieves his/her maximum height when $v(t)$ is equal to zero; that is when

$$-9.8t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{9.8}.$$

Substituting this time into $s(t)$ and setting the resulting expression equal to 2 gives

$$2 = -\frac{v_0^2}{19.6} + \frac{v_0^2}{9.8} = \frac{v_0^2}{19.6},$$

so that

$$v_0 = \sqrt{39.2} \text{ m/s}.$$

Now, consider the athlete jumping on the moon. Assume the condition that “the athlete can propel himself or herself with the same force on the moon as on Earth” means that the athlete produces the same initial velocity, $v_0 = \sqrt{39.2}$ m/s, as on Earth. Following the procedure used above, with an acceleration due to gravity on the moon of 1.6 m/s^2 , the athlete’s velocity during the jump on the moon is

$$v(t) = -1.6t + \sqrt{39.2},$$

so the athlete achieves maximum height when

$$-1.6t + \sqrt{39.2} = 0 \quad \text{or} \quad t = \frac{\sqrt{39.2}}{1.6} = \sqrt{15.3125} \text{ s.}$$

The height of the athlete t seconds after initiating the jump is

$$s(t) = -1.6 \frac{t^2}{2} + \sqrt{39.2}t = -0.8t^2 + \sqrt{39.2}t,$$

so

$$\begin{aligned} s(\sqrt{15.3125}) &= -0.8(15.3125) + \sqrt{39.2} \cdot \sqrt{15.3125} \\ &= -12.25 + \sqrt{600.25} = 12.25 \text{ m.} \end{aligned}$$

On the moon, the athlete would attain a height of 12.25 m.

55. First consider Javier Sotomayor jumping on Earth. Let $t = 0$ denote the time Sotomayor loses contact with the ground, let v_0 denote his initial upward speed, and let $s(t)$ represent his height t seconds after initiating the jump. The general solution of the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

is $v(t) = -9.8t + C_1$, where C_1 is a constant. Applying the initial condition $v(0) = v_0$ yields

$$v_0 = -9.8(0) + C_1 = C_1,$$

so that $C_1 = v_0$ and $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

to obtain

$$s(t) = -9.8 \cdot \frac{t^2}{2} + v_0t + C_2 = -4.9t^2 + v_0t + C_2,$$

where C_2 is a constant. Applying the initial condition $s(0) = 0$ gives

$$0 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so that $C_2 = 0$ and $s(t) = -4.9t^2 + v_0t$. Now, Sotomayor achieves his maximum height when $v(t)$ is equal to zero; that is when

$$-9.8t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{9.8}.$$

Substituting this time into $s(t)$ and setting the resulting expression equal to 2.45 gives

$$2.45 = -\frac{v_0^2}{19.6} + \frac{v_0^2}{9.8} = \frac{v_0^2}{19.6},$$

so that

$$v_0 = \sqrt{48.02} \text{ m/s.}$$

Now, consider Javier Sotomayor jumping on the moon. Assume the condition that “he propels himself with the same force on the moon as on Earth” means that Sotomayor produces the same initial velocity,

$v_0 = \sqrt{48.02}$ m/s, as on Earth. Following the procedure used above, with an acceleration due to gravity on the moon of 1.6 m/s^2 , Sotomayor's velocity during the jump on the moon is

$$v(t) = -1.6t + \sqrt{48.02},$$

so that he achieves maximum height when

$$-1.6t + \sqrt{48.02} = 0 \quad \text{or} \quad t = \frac{\sqrt{48.02}}{1.6} \text{ s.}$$

Sotomayor's height t seconds after initiating the jump is

$$s(t) = -1.6 \frac{t^2}{2} + \sqrt{48.02}t = -0.8t^2 + \sqrt{48.02}t,$$

so

$$s\left(\frac{\sqrt{48.02}}{1.6}\right) = -\frac{48.02}{3.2} + \frac{48.02}{1.6} = \frac{48.02}{3.2} = \boxed{15.00625 \text{ m}}.$$

On the moon, Sotomayor would attain a height of a little more than 15 m.

56. Let $t = 0$ denote the time at which the ball is released, and let $v(t)$ and $s(t)$ represent the upward speed of the ball and the height of the ball above ground level, respectively, t seconds after release. Given that the initial upward speed of the ball is 19.6 m/s and the ball is released from ground level, it follows that $v(0) = 19.6$ and $s(0) = 0$. Now, solve the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

to obtain $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = 19.6$ determines

$$19.6 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t + 19.6$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + 19.6$$

for $s(t)$ to obtain

$$s(t) = -9.8 \frac{t^2}{2} + 19.6t + C_2 = -4.9t^2 + 19.6t + C_2.$$

Use the initial condition $s(0) = 0$ to determine

$$0 = -4.9(0)^2 + 19.6(0) + C_2 = C_2,$$

so $s(t) = -4.9t^2 + 19.6t$. The ball reaches its maximum height when $v(t) = 0$; that is, when

$$-9.8t + 19.6 = 0 \quad \text{or} \quad t = 2 \text{ s.}$$

The maximum height is therefore

$$s(2) = -4.9(2)^2 + 19.6(2) = \boxed{19.6 \text{ m}}.$$

The ball is at ground level when $s(t) = 0$; that is, when

$$-4.9t^2 + 19.6t = -4.9t(t - 4) = 0 \quad \text{or} \quad t = 0, 4 \text{ s.}$$

Because $t = 0$ is the time the ball was released, $t = 4$ is the time the ball returns to the ground. The ball returns to ground level after $\boxed{4 \text{ seconds}}$.

57. Let v_0 denote the initial upward velocity of the ball, let $t = 0$ denote the time at which the ball is released, and let $v(t)$ and $s(t)$ represent the upward speed of the ball and the height of the ball above ground level, respectively, t seconds after release. Given that the ball is released at an initial height of 1 m, it follows that $s(0) = 1$. Now, solve the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

to obtain $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = v_0$ determines

$$v_0 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

for $s(t)$ to obtain

$$s(t) = -9.8\frac{t^2}{2} + v_0t + C_2 = -4.9t^2 + v_0t + C_2.$$

Use the initial condition $s(0) = 1$ to determine

$$1 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so $s(t) = -4.9t^2 + v_0t + 1$. The ball reaches its maximum height when $v(t) = 0$; that is, when

$$-9.8t + v_0 = 0 \quad \text{or} \quad t = \frac{v_0}{9.8} \text{ s.}$$

The maximum height is therefore

$$s\left(\frac{v_0}{9.8}\right) = -4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 1 = \frac{v_0^2}{19.6} + 1 \text{ m.}$$

To achieve a maximum height of at least 9.8 m requires

$$\frac{v_0^2}{19.6} + 1 \geq 9.8 \quad \text{or} \quad v_0 \geq \sqrt{19.6(8.8)} \approx \boxed{13.133 \text{ m/s}}.$$

58. Let v_0 denote the initial velocity of the ball, let $t = 0$ denote the time at which the ball is released, and let $v(t)$ and $s(t)$ represent the upward speed of the ball and the height of the ball above ground level, respectively, t seconds after release. Given that the ball is released at an initial height of 49 m, it follows that $s(0) = 49$. Now, solve the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

to obtain $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = v_0$ determines

$$v_0 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t + v_0$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0$$

for $s(t)$ to obtain

$$s(t) = -9.8\frac{t^2}{2} + v_0t + C_2 = -4.9t^2 + v_0t + C_2.$$

Use the initial condition $s(0) = 49$ to determine

$$49 = -4.9(0)^2 + v_0(0) + C_2 = C_2,$$

so $s(t) = -4.9t^2 + v_0t + 49$. Because the ball reaches the ground after 3 seconds, $s(3) = 0$. Therefore,

$$0 = -4.9(3)^2 + v_0(3) + 49 \quad \text{so} \quad v_0 = \frac{44.1 - 49}{3} \approx -1.633 \text{ m/s}.$$

In other words, the ball is released with a downward speed of approximately 1.633 m/s.

59. Note that constant force implies constant acceleration. Given that the object is initially at rest and has a velocity of 12 m/s after 6 s, the acceleration a of the object is

$$a = \frac{12 - 0}{6} = 2 \text{ m/s}^2.$$

By Newton's Second Law, $F = ma$, so the force applied to the object is

$$F = ma = 4(2) = \boxed{8 \text{ N}}.$$

60. Let a denote the constant acceleration of the car, let $t = 0$ denote the time at which the car begins accelerating, and let $v(t)$ and $s(t)$ represent the velocity of the car and the distance traveled by the car, respectively, t minutes after it begins accelerating. Given that the car is initially at rest, it follows that $v(0) = 0$. Additionally, by the definition of $s(t)$, $s(0) = 0$. Now, solve the differential equation

$$\frac{dv}{dt} = a$$

to obtain $v(t) = at + C_1$, where C_1 is a constant. The initial condition $v(0) = 0$ determines

$$0 = a(0) + C_1 = C_1,$$

so $v(t) = at$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = at$$

for $s(t)$ to obtain

$$s(t) = a\frac{t^2}{2} + C_2 = \frac{1}{2}at^2 + C_2.$$

Use the initial condition $s(0) = 0$ to determine

$$0 = \frac{1}{2}a(0)^2 + C_2 = C_2,$$

so $s(t) = \frac{1}{2}at^2$. Because the car travels 2 km in 2 min, $s(2) = 2$. Therefore

$$2 = \frac{1}{2}a(2)^2 \quad \text{or} \quad a = 1 \text{ km/min}^2.$$

In centimeters per second squared, this is

$$a = 1 \frac{\text{km}}{\text{min}^2} \cdot \frac{100000 \text{ cm}}{1 \text{ km}} \left(\frac{1 \text{ min}}{60 \text{ s}} \right)^2 = \boxed{\frac{250}{9} \text{ cm/s}^2}.$$

61. Let $a = g \sin 20^\circ = 9.8 \sin 20^\circ$. The general solution of the differential equation

$$\frac{dv}{dt} = a = 9.8 \sin 20^\circ$$

is

$$v(t) = (9.8 \sin 20^\circ)t + C,$$

where C is a constant. The initial condition $v(0) = 0$ (the skier starts from rest) determines

$$0 = 9.8 \sin 20^\circ(0) + C = C,$$

so

$$v(t) = (9.8 \sin 20^\circ)t.$$

After 5 seconds,

$$v(5) = 9.8 \sin 20^\circ(5) \approx \boxed{16.759 \text{ m/s}}.$$

62. Start with the first rock. Let $t = 0$ denote the time when this rock is released. Because this rock is released from rest at a height of 24 m, $v(0) = 0$ and $s(0) = 24$. The general solution of the differential equation

$$\frac{dv}{dt} = a = -9.8$$

is $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = 0$ determines

$$0 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t$. Solving the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t$$

yields $s(t) = -4.9t^2 + C_2$. The initial condition $s(0) = 24$ determines

$$24 = -4.9(0)^2 + C_2 = C_2,$$

so $s(t) = -4.9t^2 + 24$, and the first rock reaches the ground when

$$-4.9t^2 + 24 = 0 \quad \text{or} \quad t = \sqrt{\frac{24}{4.9}} \text{ s.}$$

Now, the second rock was released at $t = 1$ with $s(1) = 24$ and $v(1) = v_0$. The general solution of the differential equation

$$\frac{dv}{dt} = a = -9.8$$

is $v(t) = -9.8t + C_3$, where C_3 is a constant. The initial condition $v(1) = v_0$ determines

$$v_0 = -9.8(1) + C_3 = C_3 - 9.8,$$

so $v(t) = -9.8t + v_0 + 9.8$. Solving the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + v_0 + 9.8$$

yields $s(t) = -4.9t^2 + (v_0 + 9.8)t + C_4$. The initial condition $s(1) = 24$ determines

$$24 = -4.9(1)^2 + (v_0 + 9.8)(1) + C_4 = C_4 + v_0 + 4.9,$$

so $s(t) = -4.9t^2 + (v_0 + 9.8)t + 19.1 - v_0$. Now,

$$s\left(\sqrt{\frac{24}{4.9}}\right) = -4.9\left(\sqrt{\frac{24}{4.9}}\right)^2 + (v_0 + 9.8)\sqrt{\frac{24}{4.9}} + 19.1 - v_0 = 0,$$

so

$$v_0 = \frac{24 - 9.8\sqrt{\frac{24}{4.9}} - 19.1}{\sqrt{\frac{24}{4.9}} - 1} \approx -13.839.$$

The second rock was thrown downward with an initial velocity of approximately 13.839 m/s.

Challenge Problems

63. (a) Let $I(x)$ denote the intensity of the radiation at the depth x into the tissue. The rate of change of the intensity of the radiation with respect to the depth x is then $\frac{dI}{dx}$. Given that intensity of the radiation decreases with increasing depth (because radiation is absorbed as it passes through the tissue) and that the rate of change is proportional to the intensity with positive constant of proportionality k , it follows that

$$\frac{dI}{dx} = -kI.$$

- (b) Because the constant of proportionality k has been designated to be positive, the negative sign is needed in the differential equation in part (a) to ensure that the intensity of the radiation decreases with increasing depth.
- (c) To solve the differential equation from part (a), we need to find the most general function whose derivative is $-k$ times itself. Note that e^{-kx} is one such function whose derivative is $-k$ times itself; that is

$$\frac{d}{dx}e^{-kx} = e^{-kx} \cdot (-k) = -ke^{-kx}.$$

Now, suppose that f is *any* function for which $f'(x) = -kf(x)$ and consider the function $f(x)e^{kx}$. Because

$$\frac{d}{dx}[f(x)e^{kx}] = f(x) \cdot ke^{kx} + e^{kx} \cdot f'(x) = kf(x)e^{kx} - kf(x)e^{kx} = 0,$$

it follows that $f(x)e^{kx}$ is equal to some constant C . In other words, if f is a function for which $f'(x) = -kf(x)$, then $f(x) = Ce^{-kx}$ for some constant C . It follows that

$$I(x) = Ce^{-kx}$$

for some constant C . Applying the initial condition $I(0) = I_0$ yields

$$I_0 = Ce^{-k(0)} = C;$$

therefore, $I(x) = I_0e^{-kx}$.

- (d) If the intensity is reduced by 90% at the depth of 2.0 cm, then $I(2.0) = 0.1I_0$. Using the formula for $I(x)$ from part (c),

$$0.1I_0 = I_0e^{-2k} \quad \text{so that} \quad k = \left[-\frac{1}{2} \ln 0.1 \text{ cm}^{-1} \right].$$

64. Let $t = 0$ denote the time at which the ball is released, and let $v(t)$ and $s(t)$ represent the upward speed of the ball and the height of the ball above ground level, respectively, t seconds after release.

Given that the initial upward speed of the ball is 19.6 m/s and the ball is released from an initial height of 1 m above ground, it follows that $v(0) = 19.6$ and $s(0) = 1$. Now, solve the differential equation

$$\frac{dv}{dt} = a(t) = -9.8$$

to obtain $v(t) = -9.8t + C_1$, where C_1 is a constant. The initial condition $v(0) = 19.6$ determines

$$19.6 = -9.8(0) + C_1 = C_1,$$

so $v(t) = -9.8t + 19.6$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = -9.8t + 19.6$$

for $s(t)$ to obtain

$$s(t) = -9.8 \frac{t^2}{2} + 19.6t + C_2 = -4.9t^2 + 19.6t + C_2.$$

Use the initial condition $s(0) = 1$ to determine

$$1 = -4.9(0)^2 + 19.6(0) + C_2 = C_2,$$

so $s(t) = -4.9t^2 + 19.6t + 1$.

Now, 3 seconds after the ball is released,

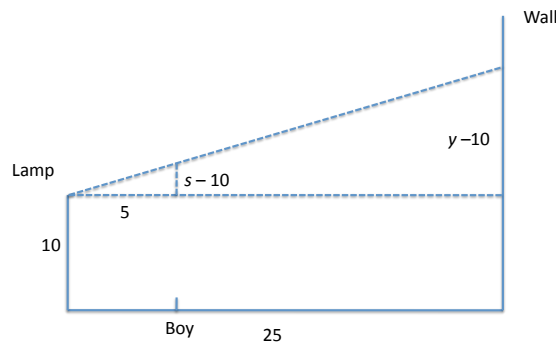
$$v(3) = -9.8(3) + 19.6 = -9.8 \text{ m/s} \quad \text{and} \quad s(3) = -4.9(3)^2 + 19.6(3) + 1 = 15.7 \text{ m}.$$

Let $y(t)$ denote the height of the shadow of the ball on the wall at time t . By similar triangles (see the diagram below),

$$\frac{5}{25} = \frac{s-10}{y-10} \quad \text{so that} \quad y-10 = 5(s-10) \quad \text{or} \quad y = 5s - 40.$$

Therefore,

$$\frac{dy}{dt} = 5 \frac{ds}{dt} \quad \text{so} \quad \left. \frac{dy}{dt} \right|_{t=3} = 5 \left. \frac{ds}{dt} \right|_{t=3} = 5v(3) = -49 \text{ m/s}.$$



Finally, 3 seconds after the ball is released,

- (a) the shadow of the ball is moving down the wall at a rate of 49 m/s;
- (b) the ball has a velocity of -9.8 m/s, so it is moving down; and
- (c) the ball is 15.7 m above the ground.

Chapter 4 Review Exercises

1. The volume and surface area of a sphere of radius r are

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2,$$

respectively. Differentiating with respect to time,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Solving the first equation for $\frac{dr}{dt}$ and substituting into the second equation yields

$$\frac{dS}{dt} = \frac{8\pi r}{4\pi r^2} \frac{dV}{dt} = \frac{2}{r} \frac{dV}{dt}.$$

Given that the snowball is melting at the rate of $2 \text{ cm}^3/\text{min}$,

$$\frac{dV}{dt} = -2.$$

When $r = 5 \text{ cm}$,

$$\frac{dS}{dt} = \frac{2}{5}(-2) = \boxed{-\frac{4}{5} \text{ cm}^2/\text{min}}.$$

2. Let x denote the distance between the point on the shoreline nearest to the lighthouse and the point on the shoreline on which the light beam is shining, and let θ denote the angle between the light beam and the perpendicular to the shoreline (see the diagram below). Then

$$\tan \theta = \frac{x}{3} \quad \text{so that} \quad x = 3 \tan \theta.$$

Differentiating with respect to time yields

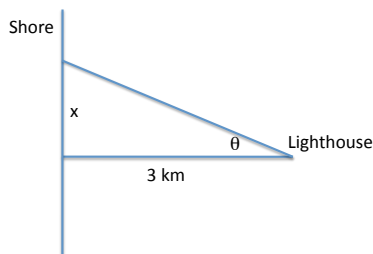
$$\frac{dx}{dt} = 3 \sec^2 \theta \frac{d\theta}{dt}.$$

Given that the light makes one revolution every 8 seconds,

$$\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{8 \text{ s}} = \frac{\pi}{4} \text{ rad/s}.$$

When the light makes an angle of 30° with the shoreline, $\theta = 60^\circ$, and

$$\frac{dx}{dt} = 3 \sec^2 60^\circ \cdot \frac{\pi}{4} = 3(4) \frac{\pi}{4} = \boxed{3\pi \text{ km/s}}.$$



3. Let $x(t)$ denote the distance between the plane approaching the airport from the west and the airport, and let $y(t)$ denote the distance between the plane approaching the airport from the north and the airport. The distance D between the two planes is then

$$D = \sqrt{x^2 + y^2}.$$

Differentiating with respect to time yields

$$\frac{dD}{dt} = \frac{x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

Given that the plane to the west is approaching the airport at 200 mph and the plane to the north is approaching the airport at 250 mph,

$$\frac{dx}{dt} = -200 \quad \text{and} \quad \frac{dy}{dt} = -250.$$

When $x = 20$ miles and $y = 30$ miles,

$$\frac{dD}{dt} = \frac{20 \cdot -200 + 30 \cdot -250}{\sqrt{20^2 + 30^2}} = \frac{-11500}{\sqrt{1300}} \approx -318.953 \text{ mph}.$$

The two planes are approaching one another at approximately 318.953 mph.

4. For the function given in the graph,

$(-10, 0)$:	neither
$(-8, -4)$:	local minimum and absolute minimum
$(-5, 0)$:	neither
$(-2, 6)$:	local maximum
$(0, 0)$:	local minimum
$(2, 10)$:	local maximum and absolute maximum
$(5, 0)$:	neither

5. For the function given in the graph,

$(-8, -9)$:	absolute minimum
$(-5, 0)$:	neither
$(-2, 9)$:	local maximum and absolute maximum
$(1, 0)$:	local minimum
$(3, 4)$:	local maximum
$(5, 0)$:	neither

6. Let $f(x) = \frac{x^2}{2x-1}$.

- (a) The domain of f is the set $\left\{x \mid x \neq \frac{1}{2}\right\}$, and

$$f'(x) = \frac{(2x-1) \cdot 2x - x^2 \cdot 2}{(2x-1)^2} = \frac{4x^2 - 2x - 2x^2}{(2x-1)^2} = \frac{2x(x-1)}{(2x-1)^2}.$$

The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now, $f'(x)$ is equal to zero when $x = 0$ and when $x = 1$ and does not exist when $x = \frac{1}{2}$. Because $\frac{1}{2}$ is not in the domain of f , $\frac{1}{2}$ is not a critical number, so 0 and 1 are the only critical numbers.

- (b) To apply the Increasing/Decreasing Function Test, use the numbers 0, $\frac{1}{2}$, and 1 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, 0)$	+	f is increasing on $(-\infty, 0)$
$(0, \frac{1}{2})$	-	f is decreasing on $(0, \frac{1}{2})$
$(\frac{1}{2}, 1)$	-	f is decreasing on $(\frac{1}{2}, 1)$
$(1, \infty)$	+	f is increasing on $(1, \infty)$

By the First Derivative Test and the information in the table above, f has a local maximum value at 0 and a local minimum value at 1. The local maximum value is $f(0) = 0$, while the local minimum value is $f(1) = 1$.

7. Let $f(x) = \cos(2x)$ and consider the closed interval $[0, \pi]$. Because the trigonometric function f is differentiable everywhere, the critical numbers of f occur where $f'(x) = 0$. Now,

$$f'(x) = -2\sin(2x),$$

so $f'(x) = 0$ when $2x = n\pi$ for any integer n . It follows that $x = \frac{n\pi}{2}$ is a critical number for any integer n . Among these numbers, only 0, $\frac{\pi}{2}$, and π lie on the closed interval $[0, \pi]$. Therefore, 0, $\frac{\pi}{2}$, and π are the critical numbers of f on the closed interval $[0, \pi]$.

8. Let $f(x) = x - \sin(2x)$, and consider the closed interval $[0, 2\pi]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[0, 2\pi]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the open interval. Now, f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 1 - 2\cos(2x) = 0$$

when $\cos(2x) = \frac{1}{2}$. On the interval $0 \leq x \leq 2\pi$, $\cos(2x) = \frac{1}{2}$ when $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{7\pi}{6}$, and $x = \frac{11\pi}{6}$. Evaluating f at the endpoints of the interval $[0, 2\pi]$ and the four critical numbers yields

$$\begin{aligned} f(0) &= 0 - \sin 0 = 0 \\ f\left(\frac{\pi}{6}\right) &= \frac{\pi}{6} - \sin \frac{\pi}{3} = \frac{\pi}{6} - \frac{\sqrt{3}}{2} \approx -0.342 \\ f\left(\frac{5\pi}{6}\right) &= \frac{5\pi}{6} - \sin \frac{5\pi}{3} = \frac{5\pi}{6} + \frac{\sqrt{3}}{2} \approx 3.484 \\ f\left(\frac{7\pi}{6}\right) &= \frac{7\pi}{6} - \sin \frac{7\pi}{3} = \frac{7\pi}{6} - \frac{\sqrt{3}}{2} \approx 2.799 \\ f\left(\frac{11\pi}{6}\right) &= \frac{11\pi}{6} - \sin \frac{11\pi}{3} = \frac{11\pi}{6} + \frac{\sqrt{3}}{2} \approx 6.626 \\ f(2\pi) &= 2\pi - \sin(4\pi) = 2\pi \approx 6.283. \end{aligned}$$

Therefore, the absolute maximum value of f on the interval $[0, 2\pi]$ is $\frac{11\pi}{6} + \frac{\sqrt{3}}{2}$ (and this occurs at the critical number $x = \frac{11\pi}{6}$), while the absolute minimum value of f is $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$ (and this occurs at the critical number $x = \frac{\pi}{6}$).

9. Let $f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 5$, and consider the closed interval $[-2, 3]$. Because f is continuous on this closed interval, the Extreme Value Theorem guarantees that f has an absolute maximum and an absolute minimum on $[-2, 3]$. The absolute maximum and absolute minimum can only occur at the endpoints of the interval or at the critical numbers inside the interval. Now, the polynomial function f is differentiable everywhere, which means that the critical numbers of f occur where $f'(x) = 0$. Moreover,

$$f'(x) = 6x^3 - 6x^2 - 12x = 6x(x^2 - x - 2) = 6x(x - 2)(x + 1) = 0$$

when $x = -1$, $x = 0$, and $x = 2$. Evaluating f at the endpoints of the interval $[-2, 3]$ and at the three critical numbers yields

$$f(-2) = 21, \quad f(-1) = \frac{5}{2}, \quad f(0) = 5, \quad f(2) = -11, \quad f(3) = \frac{37}{2}.$$

Therefore, the absolute maximum value of f on the interval $[-2, 3]$ is $\boxed{21}$ (and this occurs at the endpoint $x = -2$), while the absolute minimum value of f is $\boxed{-11}$ (and this occurs at the critical number $x = 2$).

10. Let $f(x) = x^3 - 4x^2 + 4x$. The polynomial function f is continuous and differentiable everywhere, so it is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$. Additionally,

$$f(0) = 0 \quad \text{and} \quad f(2) = 2^3 - 4(2)^2 + 4(2) = 0,$$

so $f(0) = f(2)$. The function f therefore satisfies all three conditions of Rolle's Theorem on the interval $[0, 2]$. Now, $f'(x) = 3x^2 - 8x + 4 = (3x - 2)(x - 2)$, so $f'(c) = 0$ when $c = \frac{2}{3}$ and when $c = 2$. Both of these values are in the interval $[0, 2]$. There is a horizontal tangent line to the graph of f at the points

$$\boxed{\left(\frac{2}{3}, \frac{32}{27}\right) \text{ and } (2, 0)}.$$

11. Let $f(x) = \frac{2x-1}{x} = 2 - \frac{1}{x}$. The function f is continuous and differentiable on the set $\{x|x \neq 0\}$, so it is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[1, 4]$. Now,

$$f'(x) = \frac{1}{x^2} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{\frac{7}{4} - 1}{3} = \frac{1}{4},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{c^2} = \frac{1}{4}.$$

Therefore, $c = \pm 2$. Of these numbers, only $c = 2$ is in the interval $(1, 4)$. Finally, at the point $\boxed{\left(2, \frac{3}{2}\right)}$,

the graph of f has a tangent line with a slope equal to that of the secant line joining $(1, 1)$ and $\left(4, \frac{7}{4}\right)$.

12. Let $f(x) = \sqrt{x}$. The function f is continuous on the set $\{x|x \geq 0\}$ and differentiable on the set $\{x|x > 0\}$, so it is continuous on the closed interval $[0, 9]$ and differentiable on the open interval $(0, 9)$. The function f therefore satisfies the conditions of the Mean Value Theorem on the interval $[0, 9]$. Now,

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{f(b) - f(a)}{b - a} = \frac{f(9) - f(0)}{9 - 0} = \frac{3 - 0}{9} = \frac{1}{3},$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{when} \quad \frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Therefore, $\boxed{c = \left(\frac{3}{2}\right)^2 = \frac{9}{4}}$.

13. Let $f(x) = x^3 - x^2 - 8x + 1$. The polynomial function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = 3x^2 - 2x - 8 = (3x + 4)(x - 2),$$

so 2 and $-\frac{4}{3}$ are the critical numbers of f .

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 2 and $-\frac{4}{3}$ to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $3x + 4$	Sign of $x - 2$	Sign of $f'(x)$	Conclusion
$(-\infty, -\frac{4}{3})$	—	—	+	f is increasing
$(-\frac{4}{3}, 2)$	+	—	—	f is decreasing
$(2, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-\infty, -\frac{4}{3})$ and $(2, \infty)$ and decreasing on the interval $(-\frac{4}{3}, 2)$. By the First Derivative Test, it follows that f has a local maximum value at $-\frac{4}{3}$ and a local minimum value at 2. The local maximum value is $f\left(-\frac{4}{3}\right) = \frac{203}{27}$, and the

local minimum value is $f(2) = -11$.

- (b) The second derivative of f is

$$f''(x) = 6x - 2.$$

Evaluating the second derivative at the critical numbers yields

$$f''\left(-\frac{4}{3}\right) = -10 < 0 \quad \text{and} \quad f''(2) = 10 > 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at $-\frac{4}{3}$ and a local minimum value at 2. The local maximum value is $f\left(-\frac{4}{3}\right) = \frac{203}{27}$, and the

local minimum value is $f(2) = -11$.

14. Let $f(x) = x^2 - 24x^{2/3}$. The critical numbers of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Now,

$$f'(x) = 2x - 16x^{-1/3} = \frac{2(x^{4/3} - 8)}{x^{1/3}},$$

so $\pm 4\sqrt[4]{2}$ and 0 are critical numbers of f .

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and $\pm 4\sqrt[4]{2}$ to divide the number line into four intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2(x^{4/3} - 8)$	Sign of $x^{1/3}$	Sign of $f'(x)$	Conclusion
$(-\infty, -4\sqrt[4]{2})$	+	—	—	f is decreasing
$(-4\sqrt[4]{2}, 0)$	—	—	+	f is increasing
$(0, 4\sqrt[4]{2})$	—	+	—	f is decreasing
$(4\sqrt[4]{2}, \infty)$	+	+	+	f is increasing

Therefore, f is increasing on the intervals $(-4\sqrt[4]{2}, 0)$ and $(4\sqrt[4]{2}, \infty)$ and decreasing on the intervals $(-\infty, -4\sqrt[4]{2})$ and $(0, 4\sqrt[4]{2})$. By the First Derivative Test, it follows that f has a local maximum value at 0 and local minimum values at $\pm 4\sqrt[4]{2}$. The local maximum value is $f(0) = 0$, and the local minimum values are $f(-4\sqrt[4]{2}) = f(4\sqrt[4]{2}) = -32\sqrt{2}$.

- (b) The second derivative of f is

$$f''(x) = 2 + \frac{16}{3}x^{-4/3}.$$

Evaluating the second derivative at $-4\sqrt[4]{2}$ yields $f''(-4\sqrt[4]{2}) = 8/3 > 0$; by the Second Derivative Test, it follows that f has a local minimum value at $-4\sqrt[4]{2}$. Similarly, the second derivative at $4\sqrt[4]{2}$ yields $f''(4\sqrt[4]{2}) = 8/3 > 0$; by the Second Derivative Test, it follows that f has a local minimum value at $4\sqrt[4]{2}$. The local minimum values are $f(-4\sqrt[4]{2}) = f(4\sqrt[4]{2}) = -32\sqrt{2}$. The Second Derivative Test cannot be applied at $x = 0$ because $f'(x)$ does not exist at 0.

15. Let $f(x) = x^4 e^{-2x}$. The function f is differentiable everywhere, so critical numbers occur where $f'(x) = 0$. Now,

$$f'(x) = x^4 \cdot -2e^{-2x} + 4x^3 e^{-2x} = (4x^3 - 2x^4)e^{-2x} = 2x^3(2 - x)e^{-2x},$$

so 0 and 2 are the critical numbers of f

- (a) To determine where $f'(x) > 0$ and $f'(x) < 0$, use the numbers 0 and 2 to divide the number line into three intervals. The sign of $f'(x)$ is then determined on each interval, as shown in the following table.

Interval	Sign of $2x^3$	Sign of $(2 - x)e^{-2x}$	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	—	+	—	f is decreasing
$(0, 2)$	+	+	+	f is increasing
$(2, \infty)$	+	—	—	f is decreasing

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and increasing on the interval $(0, 2)$. By the First Derivative Test, it follows that f has a local minimum value at 0 and a local maximum value at 2. The local minimum value is $f(0) = 0$, and the local maximum value is $f(2) = 16e^{-4}$.

- (b) The second derivative of f is

$$f''(x) = -2(4x^3 - 2x^4)e^{-2x} + (12x^2 - 8x^3)e^{-2x} = (4x^4 - 16x^3 + 12x^2)e^{-2x}.$$

Evaluating the second derivative at the critical numbers yields

$$f''(0) = 0 \quad \text{and} \quad f''(2) = -16e^{-4} < 0.$$

By the Second Derivative Test, it follows that f has a local maximum value at 2; the local maximum value is $f(2) = 16e^{-4}$. The Second Derivative Test is inconclusive at 0.

16. Let $s = t^4 + 2t^3 - 36t^2$. The object moves to the right when $v(t) = s'(t) > 0$ and moves to the left when $v(t) = s'(t) < 0$. Now,

$$v(t) = s'(t) = 4t^3 + 6t^2 - 72t = 2t(2t^2 + 3t - 36),$$

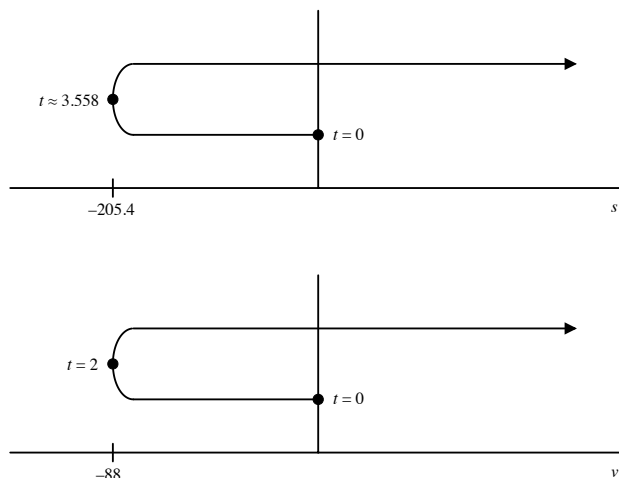
so $v(t) = 0$ when $t = 0$ and when

$$t = \frac{-3 \pm \sqrt{3^2 - 4(2)(-36)}}{4} = \frac{-3 \pm 3\sqrt{33}}{4}.$$

Of these numbers, only $\frac{-3 + 3\sqrt{33}}{4} \approx 3.558$ is greater than zero. On the approximate interval $(0, 3.558)$, $v(t) < 0$ and the object is moving to the left; on the approximate interval $(3.558, \infty)$, $v(t) > 0$ and the object is moving to the right. Additionally, the velocity of the object is increasing when $a(t) = v'(t) > 0$ and is decreasing when $a(t) = v'(t) < 0$. Now

$$a(t) = v'(t) = 12t^2 + 12t - 72 = 12(t^2 + t - 6) = 12(t + 3)(t - 2).$$

On the interval $(0, 2)$, $a(t) < 0$; on the interval $(2, \infty)$, $a(t) > 0$. Therefore, the velocity of the object is decreasing on the interval $(0, 2)$, and the velocity is increasing on the interval $(2, \infty)$. The figures below illustrates the motion (top) and velocity (bottom) of the object.



17. Let $y = f(x) = -x^3 - x^2 + 2x$.

Step 1 The polynomial function f has a domain of all real numbers. $f(0) = 0$, so the y -intercept is 0. To find the x -intercepts, solve the equation $f(x) = 0$. Because

$$-x^3 - x^2 + 2x = -x(x^2 + x - 2) = -x(x - 1)(x + 2),$$

it follows the graph of f has three x -intercepts: -2 , 0 , and 1 .

Step 2 The graphs of polynomial functions do not have asymptotes, but the end behavior of the graph of f will resemble the power function $y = -x^3$.

Step 3 Now

$$\begin{aligned} f'(x) &= -3x^2 - 2x + 2; \text{ and} \\ f''(x) &= -6x - 2 = -2(3x + 1). \end{aligned}$$

The critical numbers of the polynomial function f occur where $f'(x) = 0$, so

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(2)}}{-6} = \frac{2 \pm 2\sqrt{7}}{-6} = \boxed{-\frac{1}{3} \pm \frac{1}{3}\sqrt{7}}$$

are the critical numbers. At the points

$$\left(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{20}{27} - \frac{14}{27}\sqrt{7}\right) \approx (-1.215, -2.113)$$

and

$$\left(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, -\frac{20}{27} + \frac{14}{27}\sqrt{7}\right) \approx (0.549, 0.631),$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the critical numbers $-\frac{1}{3} \pm \frac{1}{3}\sqrt{7}$ to divide the number line into three intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{1}{3} - \frac{1}{3}\sqrt{7})$	$-$	f is decreasing on $(-\infty, -\frac{1}{3} - \frac{1}{3}\sqrt{7})$
$(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{1}{3} + \frac{1}{3}\sqrt{7})$	$+$	f is increasing on $(-\frac{1}{3} - \frac{1}{3}\sqrt{7}, -\frac{1}{3} + \frac{1}{3}\sqrt{7})$
$(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, \infty)$	$-$	f is decreasing on $(-\frac{1}{3} + \frac{1}{3}\sqrt{7}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $-\frac{1}{3} - \frac{1}{3}\sqrt{7}$ and a local maximum value at $-\frac{1}{3} + \frac{1}{3}\sqrt{7}$. The

$$\text{local minimum value is } f\left(-\frac{1}{3} - \frac{1}{3}\sqrt{7}\right) = -\frac{20}{27} - \frac{14}{27}\sqrt{7},$$

while the

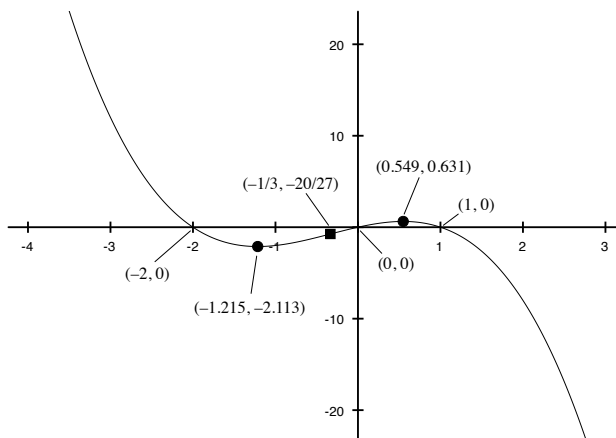
$$\text{local maximum value is } f\left(-\frac{1}{3} + \frac{1}{3}\sqrt{7}\right) = -\frac{20}{27} + \frac{14}{27}\sqrt{7}.$$

Step 6 The second derivative is equal to zero at $x = -\frac{1}{3}$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -\frac{1}{3})$	$+$	f is concave up on $(-\infty, -\frac{1}{3})$
$(-\frac{1}{3}, \infty)$	$-$	f is concave down on $(-\frac{1}{3}, \infty)$

The concavity of f changes at $-\frac{1}{3}$, so the point $\left(-\frac{1}{3}, -\frac{20}{27}\right)$ is a point of inflection.

Step 7 The figure below displays the graph of f . Local extreme values are highlighted by closed circles, and the point of inflection is highlighted by a closed square.



18. Let $y = f(x) = x^{1/3}(x^2 - 9) = x^{7/3} - 9x^{1/3}$.

Step 1 The domain of f is the set of all real numbers. Solving

$$x^{1/3}(x^2 - 9) = 0,$$

yields 0 and ± 3 as the x -intercepts; the y -intercept is $f(0) = 0$.

Step 2 Because

$$\lim_{x \rightarrow \pm\infty} [x^{1/3}(x^2 - 9)] = \lim_{x \rightarrow \pm\infty} \left[x^{7/3} \left(1 - \frac{9}{x^2} \right) \right] = \pm\infty,$$

the graph of f does not have a horizontal asymptote. The graph also has no vertical asymptotes because f is defined for all x .

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{7}{3}x^{4/3} - 3x^{-2/3} = \frac{7x^2 - 9}{3x^{2/3}}; \text{ and} \\ f''(x) &= \frac{28}{9}x^{1/3} + 2x^{-5/3} = \frac{28x^2 + 18}{9x^{5/3}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = \pm \frac{3\sqrt{7}}{7}$ and does not exist when $x = 0$. Therefore, 0 and $\pm \frac{3\sqrt{7}}{7}$ are critical numbers of f . At the point $(0, 0)$, the tangent line is vertical; at the points

$$\left(-\frac{3\sqrt{7}}{7}, \frac{54}{49} \sqrt[6]{151,263} \right) \approx (-1.134, 8.044) \quad \text{and} \quad \left(\frac{3\sqrt{7}}{7}, -\frac{54}{49} \sqrt[6]{151,263} \right) \approx (1.134, -8.044),$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers 0 and $\pm \frac{3\sqrt{7}}{7}$ to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -\frac{3\sqrt{7}}{7})$	+	f is increasing on $(-\infty, -\frac{3\sqrt{7}}{7})$
$(-\frac{3\sqrt{7}}{7}, 0)$	-	f is decreasing on $(-\frac{3\sqrt{7}}{7}, 0)$
$(0, \frac{3\sqrt{7}}{7})$	-	f is decreasing on $(0, \frac{3\sqrt{7}}{7})$
$(\frac{3\sqrt{7}}{7}, \infty)$	+	f is increasing on $(\frac{3\sqrt{7}}{7}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local maximum value at $-\frac{3\sqrt{7}}{7}$ and a local minimum value at $\frac{3\sqrt{7}}{7}$. The

local maximum value is $f\left(-\frac{3\sqrt{7}}{7}\right) = \frac{54}{49} \sqrt[6]{151,263}$,

and the

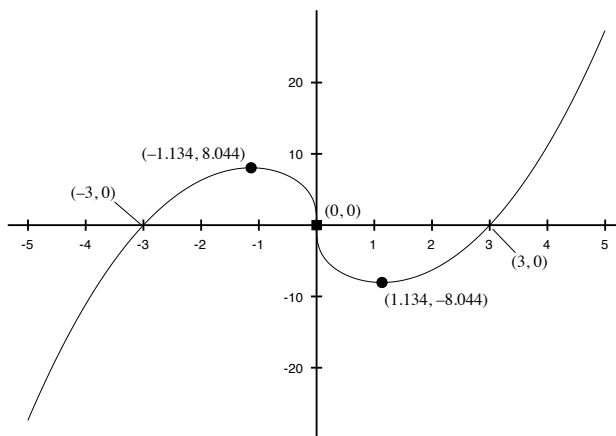
local minimum value is $f\left(\frac{3\sqrt{7}}{7}\right) = -\frac{54}{49} \sqrt[6]{151,263}$.

Step 6 The second derivative is never equal to zero and does not exist when $x = 0$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, 0)$	-	f is concave down on $(-\infty, 0)$
$(0, \infty)$	+	f is concave up on $(0, \infty)$

The concavity of f changes at 0, so the point $(0, 0)$ is a point of inflection.

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles, and the point of inflection is highlighted by a closed square.



19. Let $y = f(x) = xe^x$.

Step 1 The domain of f is the set of all real numbers. The x -intercept is 0, and the y -intercept is $f(0) = 0$.

Step 2 Because the domain of f is the set of all real numbers, the graph of f has no vertical asymptotes. To determine if there is a horizontal asymptote, consider the limits at infinity and at negative infinity:

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = 0$$

and

$$\lim_{x \rightarrow \infty} xe^x = \infty,$$

where L'Hôpital's Rule was used in the first limit. Therefore, the graph of f has the line $y = 0$ as a horizontal asymptote as $x \rightarrow -\infty$ and no horizontal asymptote as $x \rightarrow \infty$.

Step 3 Now,

$$\begin{aligned} f'(x) &= xe^x + e^x = e^x(x+1); \text{ and} \\ f''(x) &= e^x + e^x(x+1) = e^x(x+2). \end{aligned}$$

The function f is differentiable everywhere, so the critical numbers of f occur where $f'(x) = 0$, which is when $x = -1$. At the point $\left(-1, -\frac{1}{e}\right)$, the tangent line is horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the number -1 to divide the number line into two intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1)$	$-$	f is decreasing on $(-\infty, -1)$
$(-1, \infty)$	$+$	f is increasing on $(-1, \infty)$

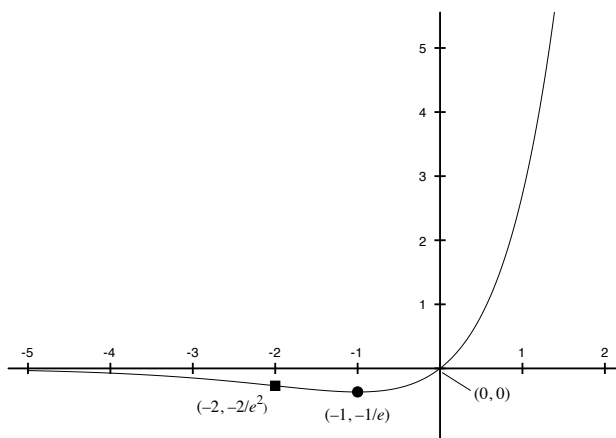
Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at -1 . The local minimum value is $f(-1) = -\frac{1}{e}$.

Step 6 The second derivative exists everywhere and is equal to zero when $x = -2$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	$-$	f is concave down on $(-\infty, -2)$
$(-2, \infty)$	$+$	f is concave up on $(-2, \infty)$

The concavity of f changes at -2 , so the point $\left(-2, -\frac{2}{e^2}\right)$ is a point of inflection.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle, and the point of inflection is highlighted by a closed square.



20. Let $y = f(x) = \frac{x-3}{x^2-4}$.

Step 1 The domain of the rational function f is the set $\{x|x \neq \pm 2\}$. The x -intercept is 3, and the y -intercept is $f(0) = \frac{3}{4}$.

Step 2 The degree of the numerator is less than the degree of the denominator, so the graph of f has a horizontal asymptote. Because

$$\lim_{x \rightarrow \pm\infty} \frac{x-3}{x^2-4} = \lim_{x \rightarrow \pm\infty} \frac{x-3}{x^2-4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x} - \frac{3}{x^2}}{1 - \frac{4}{x^2}} = 0,$$

the line $y = 0$ is a horizontal asymptote. To identify vertical asymptotes, check the one-sided limits at those values for x that are not in the domain of f . As

$$\lim_{x \rightarrow -2^-} \frac{x-3}{x^2-4} = -\infty \quad \text{and} \quad \lim_{x \rightarrow -2^+} \frac{x-3}{x^2-4} = \infty,$$

and

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \infty \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = -\infty,$$

the lines $x = \pm 2$ are vertical asymptotes.

Step 3 Now

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x-3}{x^2-4} \right) = \frac{(x^2-4) \cdot 1 - (x-3) \cdot 2x}{(x^2-4)^2} = -\frac{x^2-6x+4}{(x^2-4)^2}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left[-\frac{x^2-6x+4}{(x^2-4)^2} \right] = -\frac{(x^2-4)^2 \cdot (2x-6) - (x^2-6x+4) \cdot 2(x^2-4)(2x)}{(x^2-4)^4} \\ &= -\frac{(2x-6)(x^2-4) - 4x(x^2-6x+4)}{(x^2-4)^3} = \frac{2(x^3-9x^2+12x-12)}{(x^2-4)^3}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 3 \pm \sqrt{5}$ and does not exist when $x = \pm 2$. However, ± 2 are not in the domain of f , so ± 2 are not critical numbers. Therefore, $\boxed{3 \pm \sqrt{5}}$ are the critical numbers of f . At the points

$$\left(3 - \sqrt{5}, \frac{15 + 5\sqrt{5}}{40}\right) \approx (0.764, 0.655) \quad \text{and} \quad \left(3 + \sqrt{5}, \frac{15 - 5\sqrt{5}}{40}\right) \approx (5.236, 0.095),$$

the tangent lines are horizontal.

Step 4 To apply the Increasing/Decreasing Function Test, use the numbers ± 2 and $3 \pm \sqrt{5}$ to divide the number line into five intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -2)$	—	f is decreasing on $(-\infty, -2)$
$(-2, 3 - \sqrt{5})$	—	f is decreasing on $(-2, 3 - \sqrt{5})$
$(3 - \sqrt{5}, 2)$	+	f is increasing on $(3 - \sqrt{5}, 2)$
$(2, 3 + \sqrt{5})$	+	f is increasing on $(2, 3 + \sqrt{5})$
$(3 + \sqrt{5}, \infty)$	—	f is decreasing on $(3 + \sqrt{5}, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at $3 - \sqrt{5}$ and a local maximum value at $3 + \sqrt{5}$. The

$$\boxed{\text{local minimum value is } f(3 - \sqrt{5}) = \frac{15 + 5\sqrt{5}}{40}},$$

and the

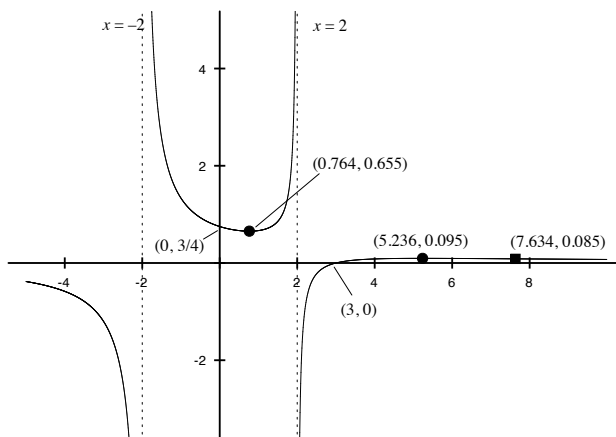
$$\boxed{\text{local maximum value is } f(3 + \sqrt{5}) = \frac{15 - 5\sqrt{5}}{40}}.$$

Step 6 The second derivative is equal to zero when $x \approx 7.634$ (this value was obtained using the computer algebra system *Maple*) and does not exist when $x = \pm 2$. Use these numbers to divide the number line into four intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -2)$	—	f is concave down on $(-\infty, -2)$
$(-2, 2)$	+	f is concave up on $(-2, 2)$
$(2, 7.634)$	—	f is concave down on $(2, 7.634)$
$(7.634, \infty)$	+	f is concave up on $(7.634, \infty)$

The concavity of f changes at approximately 7.634, so the point $\boxed{(7.634, 0.085)}$ is a point of inflection. Although the concavity of f changes at ± 2 , there is no point of inflection at ± 2 because ± 2 are not in the domain of f .

Step 7 The figure below displays the graph of f . The local extreme values are highlighted by closed circles and the inflection point is highlighted by a closed square.



21. Let $y = f(x) = x\sqrt{x-3}$.

Step 1 The domain of f is given by the solution to the inequality $x-3 \geq 0$, that is, the set $\{x|x \geq 3\}$.

The x -intercept is 3, and there is no y -intercept because 0 is not in the domain of f .

Step 2 Because

$$\lim_{x \rightarrow \infty} x\sqrt{x-3} = \infty,$$

the graph of f does not have a horizontal asymptote. The graph also does not have any vertical asymptotes.

Step 3 Now,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x\sqrt{x-3}) = x \cdot \frac{1}{2\sqrt{x-3}} + \sqrt{x-3} = \frac{3x-6}{2\sqrt{x-3}}; \text{ and} \\ f''(x) &= \frac{d}{dx} \left(\frac{3x-6}{2\sqrt{x-3}} \right) = \frac{2\sqrt{x-3} \cdot 3 - (3x-6) \cdot (x-3)^{-1/2}}{4(x-3)} \\ &= \frac{6(x-3) - (3x-6)}{4(x-3)^{3/2}} = \frac{3x-12}{4(x-3)^{3/2}}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to 0 when $x = 2$ and does not exist when $x = 3$; however, 2 is not in the domain of f , so 2 is not a critical number. Therefore, 3 is the only critical number of f . At the point $(3, 0)$, the tangent line is vertical.

Step 4 Because $f'(x) > 0$ for all $x > 3$, f is increasing on the interval $(3, \infty)$.

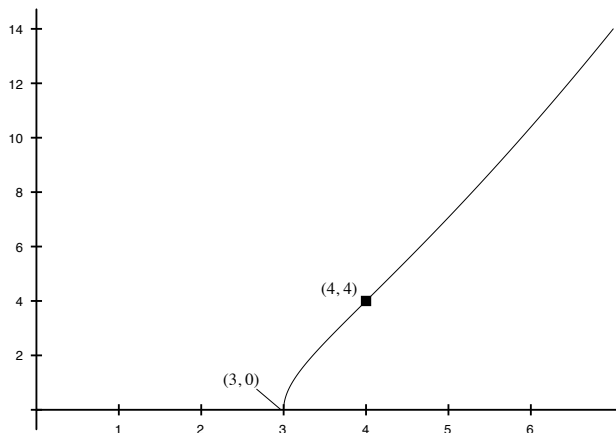
Step 5 Because the only critical number of f is an endpoint of the domain of f , f has no local extreme values.

Step 6 The second derivative exists for $x > 3$ and is equal to zero when $x = 4$. Use this number to divide the number line into two intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(3, 4)$	$-$	f is concave down on $(3, 4)$
$(4, \infty)$	$+$	f is concave up on $(4, \infty)$

The concavity of f changes at 4, so the point $(4, 4)$ is a point of inflection.

Step 7 The figure below displays the graph of f . The point of inflection is highlighted by a closed square.



22. Let $y = f(x) = x^3 - 3 \ln x$.

Step 1 The domain of f is the set $\{x|x > 0\}$. There are no x -intercepts because $x^3 > \ln x^3 = 3 \ln x$ for all $x > 0$, and there is also no y -intercept because the function is not defined at 0.

Step 2 Because

$$\lim_{x \rightarrow \infty} x^3 = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} (3 \ln x) = \infty,$$

the expression $x^3 - 3 \ln x$ is an indeterminate form at ∞ of the type $\infty - \infty$. Rewrite

$$x^3 - 3 \ln x \quad \text{as} \quad x^3 \left(1 - \frac{3 \ln x}{x^3} \right).$$

Now, the expression $\frac{3 \ln x}{x^3}$ is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$; by L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \frac{3 \ln x}{x^3} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(3 \ln x)}{\frac{d}{dx}x^3} = \lim_{x \rightarrow \infty} \frac{\frac{3}{x}}{3x^2} = 0.$$

Therefore,

$$\lim_{x \rightarrow \infty} \left(1 - \frac{3 \ln x}{x^3} \right) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} (x^3 - 3 \ln x) = \lim_{x \rightarrow \infty} \left[x^3 \left(1 - \frac{3 \ln x}{x^3} \right) \right] = \infty,$$

so the graph of f has no horizontal asymptote. On the other hand,

$$\lim_{x \rightarrow 0^+} (x^3 - 3 \ln x) = \infty,$$

so the line $x = 0$ is a vertical asymptote.

Step 3 Now,

$$\begin{aligned} f'(x) &= 3x^2 - \frac{3}{x} = \frac{3(x^3 - 1)}{x}; \text{ and} \\ f''(x) &= 6x + \frac{3}{x^2} = \frac{6x^3 + 3}{x^2}. \end{aligned}$$

The critical numbers of f occur where $f'(x) = 0$ and where $f'(x)$ does not exist. $f'(x)$ is equal to zero when $x = 1$ and does not exist when $x = 0$; however, 0 is not a critical number because 0 is not in the domain of f . Therefore, 1 is the only critical number. At the point (1,1), the tangent line is horizontal.

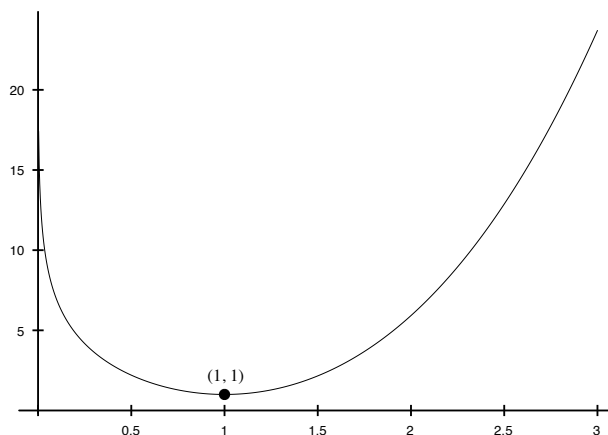
Step 4 To apply the Increasing/Decreasing Function Test, use the number 1 to divide $(0, \infty)$ into two intervals.

Interval	Sign of f'	Conclusion
$(0, 1)$	$-$	f is decreasing on $(0, 1)$
$(1, \infty)$	$+$	f is increasing on $(1, \infty)$

Step 5 By the First Derivative Test and the information in the table above, f has a local minimum value at 1. The local maximum value is $f(1) = 1$.

Step 6 Because $f''(x) > 0$ for all $x > 0$, f is concave up on the interval $(0, \infty)$.

Step 7 The figure below displays the graph of f . The local extreme value is highlighted by a closed circle.



23. Let $f(x) = x^4 + 12x^2 + 36x - 11$. Then

$$\begin{aligned} f'(x) &= 4x^3 + 24x + 36; \text{ and} \\ f''(x) &= 12x^2 + 24. \end{aligned}$$

- (a) Using the computer algebra system *Maple*, it is found that $f'(x) = 0$ for $x \approx -1.207$. For $x < -1.207$, $f'(x) < 0$, while for $x > -1.207$, $f'(x) > 0$. Therefore, f is

decreasing on the approximate interval $(-\infty, -1.207)$

and

increasing on the approximate interval $(-1.207, \infty)$.

- (b) Because $f''(x) \geq 24 > 0$ for all x , f is concave up on the interval $(-\infty, \infty)$.

- (c) Because the concavity of f does not change, f has no points of inflection.

24. Let $f(x) = 3x^4 - 2x^3 - 24x^2 - 7x + 2$. Then

$$\begin{aligned} f'(x) &= 12x^3 - 6x^2 - 48x - 7; \text{ and} \\ f''(x) &= 36x^2 - 12x - 48 = 12(3x^2 - x - 4) = 12(3x - 4)(x + 1). \end{aligned}$$

- (a) Using the computer algebra system *Maple*, it is found that $f'(x) = 0$ for $x \approx -1.677$, $x \approx -0.149$, and $x \approx 2.327$. To apply the Increasing/Decreasing Function Test, use the numbers -1.677 , -0.149 , and 2.327 to divide the number line into four intervals.

Interval	Sign of f'	Conclusion
$(-\infty, -1.677)$	$-$	f is decreasing on $(-\infty, -1.677)$
$(-1.677, -0.149)$	$+$	f is increasing on $(-1.677, -0.149)$
$(-0.149, 2.327)$	$-$	f is decreasing on $(-0.149, 2.327)$
$(2.327, \infty)$	$+$	f is increasing on $(2.327, \infty)$

- (b) The second derivative exists everywhere and is equal to zero when $x = -1$ and $x = \frac{4}{3}$. Use these numbers to divide the number line into three intervals, and determine the sign of f'' on each interval.

Interval	Sign of f''	Conclusion
$(-\infty, -1)$	$+$	f is concave up on $(-\infty, -1)$
$(-1, \frac{4}{3})$	$-$	f is concave down on $(-1, \frac{4}{3})$
$(\frac{4}{3}, \infty)$	$+$	f is concave up on $(\frac{4}{3}, \infty)$

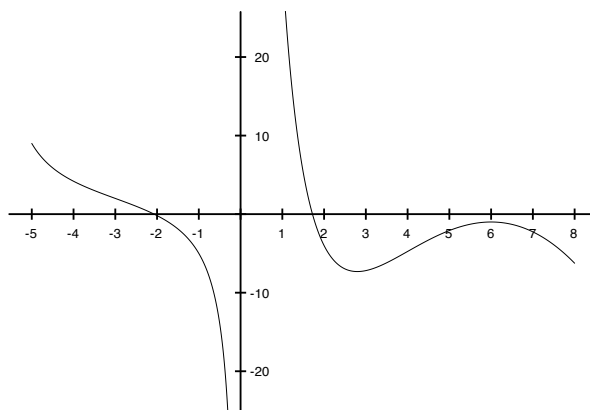
- (c) The concavity of f changes at -1 and $\frac{4}{3}$, so the points $(-1, -10)$ and $(\frac{4}{3}, -\frac{1222}{27})$ are points of inflection.

25. Because $y' > 0$ and $y'' < 0$ for all x , the graph of $y = f(x)$ is always increasing and concave down.

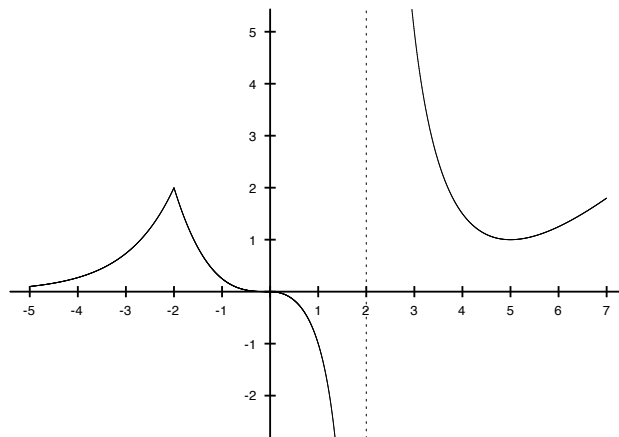
- (A) This cannot be part of the graph of $y = f(x)$ because this graph segment is concave up.
 (B) This could be part of the graph of $y = f(x)$ because this graph segment is increasing and concave down.
 (C) This cannot be part of the graph of $y = f(x)$ because this graph segment is decreasing.
 (D) This cannot be part of the graph of $y = f(x)$ because this graph segment is decreasing.

Therefore, (B) could be a part of the graph of f .

26. Answers will vary. The figure below displays the graph of a function f with the following properties:
 $f(-3) = 2$; $f(-1) = -5$; $f(2) = -4$; $f(6) = -1$; $f'(-3) = f'(6) = 0$; $\lim_{x \rightarrow 0^-} f(x) = -\infty$; $\lim_{x \rightarrow 0^+} f(x) = \infty$;
 $f''(x) > 0$ if $x < -3$ or $0 < x < 4$; and $f''(x) < 0$ if $-3 < x < 0$ or $4 < x$.



27. Answers will vary. The figure below displays the graph of a function f with the following properties:
 $f(-2) = 2$; $f(5) = 1$; $f(0) = 0$; $f'(x) > 0$ if $x < -2$ or $5 < x$ and $f'(x) < 0$ if $-2 < x < 2$ or $2 < x < 5$;
 $f''(x) > 0$ if $x < 0$ or $2 < x$ and $f''(x) < 0$ if $0 < x < 2$; $\lim_{x \rightarrow 2^-} f(x) = -\infty$; $\lim_{x \rightarrow 2^+} f(x) = \infty$.



28. Let $f(x) = x\sqrt{x+1}$. We need to determine b such that

$$\frac{f(b) - f(0)}{b - 0} = f'(3).$$

Now,

$$\frac{f(b) - f(0)}{b - 0} = \frac{b\sqrt{b+1} - 0}{b} = \sqrt{b+1},$$

and

$$f'(x) = \frac{x}{2\sqrt{x+1}} + \sqrt{x+1} \quad \text{so} \quad f'(3) = \frac{3}{4} + 2 = \frac{11}{4}.$$

Therefore,

$$\sqrt{b+1} = \frac{11}{4} \quad \text{or} \quad b = \left(\frac{11}{4}\right)^2 - 1 = \boxed{\frac{105}{16}}.$$

29. Let x denote the side length in inches of the square cut from each corner of the base (see the diagram below, where all lengths are given in inches). When the sides are turned up, the resulting box will have a rectangular base measuring $24 - 2x$ inches by $36 - 2x$ inches and a height of x inches; the volume will therefore be

$$V = x(24 - 2x)(36 - 2x) = 4x^3 - 120x^2 + 864x.$$

To determine the domain of V , note that the length of each of the three sides of the box must be non-negative. This requires $x \geq 0$, $24 - 2x \geq 0$, and $36 - 2x \geq 0$. The solution of this set of three inequalities is $0 \leq x \leq 12$; it follows that the domain of V is the closed interval $[0, 12]$. The function V is differentiable on the open interval $(0, 12)$, so the critical numbers occur where $V'(x) = 0$. Now,

$$V'(x) = 12x^2 - 240x + 864 = 12(x^2 - 20x + 72),$$

so the only critical number inside the open interval $(0, 12)$ is

$$x = \frac{20 - \sqrt{20^2 - 4(1)(72)}}{2} = \frac{20 - 4\sqrt{7}}{2} = 10 - 2\sqrt{7} \approx 4.708.$$

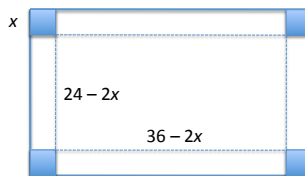
Note that

$$x = \frac{20 + \sqrt{20^2 - 4(1)(72)}}{2} = \frac{20 + 4\sqrt{7}}{2} = 10 + 2\sqrt{7} \approx 15.292$$

is not in the domain of V . Evaluating V at the endpoints of the interval $[0, 12]$ and at the critical number 4.708 yields

$$V(0) = 0, \quad V(4.708) \approx 1825.297, \quad \text{and} \quad V(12) = 0.$$

The largest volume is therefore achieved when $x = 10 - 2\sqrt{7} \approx 4.708$. Squares with a side length of $\boxed{10 - 2\sqrt{7} \approx 4.708 \text{ inches}}$ should be cut out to produce a box with maximum volume.



30. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point (x, y) on the graph of the parabola $y = \frac{1}{2}x^2$ and the point $(4, 1)$. Then

$$\begin{aligned} D &= (x - 4)^2 + (y - 1)^2 = (x - 4)^2 + \left(\frac{1}{2}x^2 - 1\right)^2 \\ &= x^2 - 8x + 16 + \frac{1}{4}x^4 - x^2 + 1 \\ &= \frac{1}{4}x^4 - 8x + 17. \end{aligned}$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = x^3 - 8 = (x - 2)(x^2 + 2x + 4),$$

so $x = 2$ is the only critical number. Using the Second Derivative Test,

$$D''(x) = 3x^2 \quad \text{so} \quad D''(2) = 12 > 0$$

and D has a local minimum value at 2. Because $D''(x) \geq 0$ for all x , the local minimum is also an absolute minimum. The point $\left(2, \frac{1}{2}2^2\right) = (2, 2)$ on the graph of the parabola $y = \frac{1}{2}x^2$ is closest to the point $(4, 1)$.

31. All antiderivatives of the function $f(x) = 0$ are $F(x) = \boxed{C}$, where C is a constant.

32. All antiderivatives of the function $f(x) = x^{1/2}$ are

$$F(x) = \frac{x^{1/2+1}}{1/2+1} + C = \boxed{\frac{2}{3}x^{3/2} + C},$$

where C is a constant.

33. All antiderivatives of the function $f(x) = \cos x$ are $F(x) = \boxed{\sin x + C}$, where C is a constant.

34. All antiderivatives of the function $f(x) = \sec x \tan x$ are $F(x) = \boxed{\sec x + C}$, where C is a constant.

35. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = \frac{2}{x}$ are

$$F(x) = 2 \cdot \ln |x| + C = \boxed{2 \ln |x| + C},$$

where C is a constant.

36. Using the Constant Multiple Rule, all antiderivatives of the function $f(x) = -2x^{-3}$ are

$$F(x) = -2 \cdot \frac{x^{-3+1}}{-3+1} + C = \boxed{x^{-2} + C},$$

where C is a constant.

37. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = 4x^3 - 9x^2 + 10x - 3$ are

$$F(x) = 4 \cdot \frac{x^{3+1}}{3+1} - 9 \cdot \frac{x^{2+1}}{2+1} + 10 \cdot \frac{x^{1+1}}{1+1} - 3x + C = \boxed{x^4 - 3x^3 + 5x^2 - 3x + C},$$

where C is a constant.

38. Using the Sum Rule and the Constant Multiple Rule, all antiderivatives of the function $f(x) = e^x + \frac{4}{x}$ are

$$F(x) = e^x + 4 \cdot \ln |x| + C = \boxed{e^x + 4 \ln |x| + C},$$

where C is a constant.

39. Let $t = 0$ denote the time the box begins to move with a velocity v_0 , and let $v(t)$ and $s(t)$ denote the velocity and distance traveled by the box, respectively, t seconds after it begins to move. The general solution of the differential equation

$$\frac{dv}{dt} = a(t) = t^2(t - 3) = t^3 - 3t$$

is

$$v(t) = \frac{t^4}{4} - 3\frac{t^2}{2} + C_1 = \frac{1}{4}t^4 - \frac{3}{2}t^2 + C_1,$$

where C_1 is a constant. The initial condition $v(0) = v_0$ determines

$$v_0 = \frac{1}{4}0^4 - \frac{3}{2}0^2 + C_1 = C_1.$$

Next, the general solution of the differential equation

$$\frac{ds}{dt} = v(t) = \frac{1}{4}t^4 - \frac{3}{2}t^2 + v_0$$

is

$$s(t) = \frac{1}{4} \cdot \frac{t^5}{5} - \frac{3}{2} \cdot \frac{t^3}{3} + v_0 t + C_2 = \frac{1}{20}t^5 - \frac{1}{2}t^3 + v_0 t + C_2,$$

where C_2 is a constant. The initial condition $s(0) = 0$ determines

$$0 = \frac{1}{20}0^5 - \frac{1}{2}0^3 + v_0(0) + C_2 = C_2.$$

The box travels 10 cm in 2 s, so $s(2) = 10$ and

$$10 = \frac{1}{20}2^5 - \frac{1}{2}2^3 + 2v_0 = \frac{8}{5} - 4 + 2v_0.$$

Therefore,

$$v_0 = 7 - \frac{4}{5} = \boxed{\frac{31}{5} \text{ cm/s}}.$$

40. Let $t = 0$ denote the time the first object begins to fall. Additionally, let $v_1(t)$ and $s_1(t)$ denote the velocity and distance traveled, respectively, by the first object, and let $v_2(t)$ and $s_2(t)$ denote the velocity and distance traveled, respectively, by the second object. The initial conditions are $v_1(0) = 0$, $s_1(0) = 0$, $v_2(1) = 0$, and $s_2(1) = 0$. For the first object, the particular solution of the differential equation

$$\frac{dv_1}{dt} = a = -9.8$$

with the initial condition $v_1(0) = 0$ is $v_1(t) = -9.8t$, and the particular solution of the differential equation

$$\frac{ds_1}{dt} = v_1(t) = -9.8t$$

with the initial condition $s_1(0) = 0$ is $s_1(t) = -4.9t^2$. For the second object, the particular solution of the differential equation

$$\frac{dv_2}{dt} = a = -9.8$$

with the initial condition $v_2(1) = 0$ is $v_2(t) = -9.8t + 9.8$, and the particular solution of the differential equation

$$\frac{ds_2}{dt} = v_2(t) = -9.8t + 9.8$$

with the initial condition $s_2(1) = 0$ is $s_2(t) = -4.9t^2 + 9.8t - 4.9$. The two objects will be 10 m apart when $s_2(t) - s_1(t) = 10$; that is, when

$$-4.9t^2 + 9.8t - 4.9 - (-4.9t^2) = 9.8t - 4.9 = 10,$$

or

$$t = \frac{14.9}{9.8} \approx \boxed{1.52 \text{ s}}.$$

41. Because

$$\lim_{x \rightarrow 0} (xe^{3x} - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} [1 - \cos(2x)] = 0,$$

the expression $\frac{xe^{3x} - x}{1 - \cos(2x)}$ is an $\boxed{\text{indeterminate form at 0 of the type } \frac{0}{0}}$.

42. Because

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \tan x = 0,$$

while

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} \tan x = 0,$$

the expression $\left(\frac{1}{x}\right)^{\tan x}$ is an $\boxed{\text{indeterminate form at 0 of the type } \infty^0}$.

43. Because

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{x^2 \sec x} = \infty,$$

the expression $\frac{1}{x^2} - \frac{1}{x^2 \sec x}$ is an $\boxed{\text{indeterminate form at 0 of the type } \infty - \infty}$.

44. Because

$$\lim_{x \rightarrow 0} (\tan x - x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (x - \sin x) = 0,$$

the expression $\frac{\tan x - x}{x - \sin x}$ is an $\boxed{\text{indeterminate form at 0 of the type } \frac{0}{0}}$.

45. Rewrite

$$\frac{\sec^2 x}{\sec^2(3x)} \quad \text{as} \quad \frac{\cos^2(3x)}{\cos^2 x}.$$

Because

$$\lim_{x \rightarrow \pi/2} [\cos^2(3x)] = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/2} \cos^2 x = 0,$$

the expression $\frac{\cos^2(3x)}{\cos^2 x}$ is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \pi/2} \frac{\cos^2(3x)}{\cos^2 x} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx} \cos^2(3x)}{\frac{d}{dx} \cos^2 x} = \lim_{x \rightarrow \pi/2} \frac{-6 \cos(3x) \sin(3x)}{-2 \cos x \sin x} = \lim_{x \rightarrow \pi/2} \frac{3 \sin(6x)}{\sin(2x)}.$$

Now,

$$\lim_{x \rightarrow \pi/2} [3 \sin(6x)] = 0 \quad \text{and} \quad \lim_{x \rightarrow \pi/2} \sin(2x) = 0,$$

so the expression $\frac{3 \sin(6x)}{\sin(2x)}$ is an indeterminate form at $\pi/2$ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow \pi/2} \frac{\cos^2(3x)}{\cos^2 x} = \lim_{x \rightarrow \pi/2} \frac{3 \sin(6x)}{\sin(2x)} = \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx} [3 \sin(6x)]}{\frac{d}{dx} \sin(2x)} = \lim_{x \rightarrow \pi/2} \frac{18 \cos(6x)}{2 \cos(2x)} = \frac{18(-1)}{2(-1)} = 9.$$

Therefore,

$$\lim_{x \rightarrow \pi/2} \frac{\sec^2 x}{\sec^2(3x)} = \lim_{x \rightarrow \pi/2} \frac{\cos^2(3x)}{\cos^2 x} = \boxed{9}.$$

46. Because

$$\lim_{x \rightarrow 0} \frac{2}{\sin^2 x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{1 - \cos x} = \infty,$$

the expression $\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x}$ is an indeterminate form at 0 of the type $\infty - \infty$. Rewrite

$$\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \quad \text{as} \quad \frac{2 - 2 \cos x - \sin^2 x}{(1 - \cos x) \sin^2 x},$$

which is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \left[\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right] &= \lim_{x \rightarrow 0} \frac{2 - 2 \cos x - \sin^2 x}{(1 - \cos x) \sin^2 x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (2 - 2 \cos x - \sin^2 x)}{\frac{d}{dx} [(1 - \cos x) \sin^2 x]} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x - 2 \sin x \cos x}{(1 - \cos x) \cdot 2 \sin x \cos x + \sin^3 x} = \lim_{x \rightarrow 0} \frac{2 - 2 \cos x}{2(1 - \cos x) \cos x + \sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{2 - 2 \cos x}{1 + 2 \cos x - 3 \cos^2 x}. \end{aligned}$$

Now,

$$\lim_{x \rightarrow 0} (2 - 2 \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 + 2 \cos x - 3 \cos^2 x) = 0,$$

so the expression $\frac{2 - 2 \cos x}{1 + 2 \cos x - 3 \cos^2 x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\begin{aligned} \lim_{x \rightarrow 0} \left[\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right] &= \lim_{x \rightarrow 0} \frac{2 - 2 \cos x}{1 + 2 \cos x - 3 \cos^2 x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (2 - 2 \cos x)}{\frac{d}{dx} (1 + 2 \cos x - 3 \cos^2 x)} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x}{-2 \sin x + 6 \cos x \sin x} = \lim_{x \rightarrow 0} \frac{1}{3 \cos x - 1} = \boxed{\frac{1}{2}}. \end{aligned}$$

47. Because

$$\lim_{x \rightarrow 0} (e^x - e^{-x}) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin x = 0,$$

the expression $\frac{e^x - e^{-x}}{\sin x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} (e^x - e^{-x})}{\frac{d}{dx} \sin x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1 + 1}{1} = \boxed{2}.$$

48. Because

$$\lim_{x \rightarrow 0^-} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^-} \cot(\pi x) = -\infty,$$

the expression $x \cot(\pi x)$ is an indeterminate form at 0^- of the type $0 \cdot \infty$. Rewrite

$$x \cot(\pi x) \quad \text{as} \quad \frac{x}{\tan(\pi x)},$$

which is an indeterminate form at 0^- of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^-} [x \cot(\pi x)] = \lim_{x \rightarrow 0^-} \frac{x}{\tan(\pi x)} = \lim_{x \rightarrow 0^-} \frac{\frac{d}{dx} x}{\frac{d}{dx} \tan(\pi x)} = \lim_{x \rightarrow 0^-} \frac{1}{\pi \sec^2(\pi x)} = \boxed{\frac{1}{\pi}}.$$

49. Because

$$\lim_{x \rightarrow 0} (\tan x + \sec x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (\tan x - \sec x + 1) = 0,$$

the expression $\frac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\tan x + \sec x - 1)}{\frac{d}{dx}(\tan x - \sec x + 1)} = \lim_{x \rightarrow 0} \frac{\sec^2 x + \sec x \tan x}{\sec^2 x - \sec x \tan x} = \frac{1 + 0}{1 - 0} = \boxed{1}.$$

50. If $a = 0$, then

$$\lim_{x \rightarrow a} \frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4} = \lim_{x \rightarrow 0} \frac{-x^2}{-x^4} = \lim_{x \rightarrow 0} \frac{1}{x^2} = \boxed{\infty}.$$

Suppose that $a \neq 0$. Note

$$\lim_{x \rightarrow a} (a^4 - 2a^3x + 2ax^3 - x^4) = 0.$$

Because $a^4 - 2a^3x + 2ax^3 - x^4$ is a polynomial in x , the only way that the limit of this expression as x approaches a can be equal to zero is for the expression to have a factor of $a - x$. Factoring yields

$$a^4 - 2a^3x + 2ax^3 - x^4 = (a - x)(a^3 - a^2x - ax^2 + x^3).$$

Now, substituting $x = a$ into $a^3 - a^2x - ax^2 + x^3$ again produces zero, so this latter expression must also contain a factor of $a - x$. Continuing to factor,

$$\begin{aligned} a^4 - 2a^3x + 2ax^3 - x^4 &= (a - x)(a^3 - a^2x - ax^2 + x^3) \\ &= (a - x)(a - x)(a^2 - x^2) \\ &= (a - x)(a - x)(a - x)(a + x) = (a - x)^3(a + x). \end{aligned}$$

Therefore, for $x \neq a$,

$$\frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4} = \frac{x(a - x)}{(a - x)^3(a + x)} = \frac{x}{(a - x)^2(a + x)}.$$

Finally, because

$$\frac{x}{a + x} \rightarrow \frac{a}{2a} = \frac{1}{2} \quad \text{as} \quad x \rightarrow a,$$

it follows that

$$\lim_{x \rightarrow a} \frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4} = \lim_{x \rightarrow a} \frac{x}{(a - x)^2(a + x)} = \boxed{\infty}.$$

51. Because

$$\lim_{x \rightarrow 0} (x - \sin x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^3 = 0,$$

the expression $\frac{x - \sin x}{x^3}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}.$$

Now,

$$\lim_{x \rightarrow 0} (1 - \cos x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (3x^2) = 0,$$

so the expression $\frac{1 - \cos x}{3x^2}$ is also an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(3x^2)} = \lim_{x \rightarrow 0} \frac{\sin x}{6x} = \frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{6} \cdot 1 = \boxed{\frac{1}{6}}.$$

52. Rewrite

$$\frac{\tan x - \sin x}{\sin^3 x} = \frac{\sec x - 1}{\sin^2 x}.$$

Because

$$\lim_{x \rightarrow 0} (\sec x - 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \sin^2 x = 0,$$

the expression $\frac{\sec x - 1}{\sin^2 x}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x} &= \lim_{x \rightarrow 0} \frac{\sec x - 1}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sec x - 1)}{\frac{d}{dx} \sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{\sec x \tan x}{2 \sin x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{1}{2 \cos^3 x} = \boxed{\frac{1}{2}}. \end{aligned}$$

53. Because

$$\lim_{x \rightarrow \infty} (1 + 4x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{2}{x} = 0,$$

the expression $(1 + 4x)^{2/x}$ is an indeterminate form at ∞ of the type ∞^0 . Let $y = (1 + 4x)^{2/x}$. Then

$$\ln y = \ln(1 + 4x)^{2/x} = \frac{2}{x} \ln(1 + 4x) = \frac{2 \ln(1 + 4x)}{x},$$

which is an indeterminate form at ∞ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(1 + 4x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}[2 \ln(1 + 4x)]}{\frac{d}{dx}x} = \lim_{x \rightarrow \infty} \frac{\frac{8}{1+4x}}{1} = 0.$$

Because $\lim_{x \rightarrow \infty} \ln y = 0$, it follows that

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} (1 + 4x)^{2/x} = e^0 = \boxed{1}.$$

54. Because

$$\lim_{x \rightarrow 1^-} \frac{2}{x^2 - 1} = -\infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x - 1} = -\infty,$$

while

$$\lim_{x \rightarrow 1^+} \frac{2}{x^2 - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty,$$

the expression $\frac{2}{x^2 - 1} - \frac{1}{x - 1}$ is an indeterminate form at 1 of the type $\infty - \infty$. Rewrite

$$\frac{2}{x^2 - 1} - \frac{1}{x - 1} \quad \text{as} \quad \frac{2}{x^2 - 1} - \frac{x + 1}{x^2 - 1} = \frac{1 - x}{x^2 - 1} = -\frac{1}{1 + x}.$$

Therefore,

$$\lim_{x \rightarrow 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x - 1} \right] = \lim_{x \rightarrow 1} \left(-\frac{1}{x + 1} \right) = -\frac{1}{2}.$$

Alternately, note that the expression $\frac{1 - x}{x^2 - 1}$ is an indeterminate form at 1 of the type $\frac{0}{0}$, and then use L'Hôpital's Rule to obtain

$$\lim_{x \rightarrow 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x - 1} \right] = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(1 - x)}{\frac{d}{dx}(x^2 - 1)} = \lim_{x \rightarrow 1} \frac{-1}{2x} = \boxed{-\frac{1}{2}}.$$

55. Because

$$\lim_{x \rightarrow 4} (x^2 - 16) = 0 \quad \text{and} \quad \lim_{x \rightarrow 4} (x^2 + x - 20) = 0,$$

the expression $\frac{x^2 - 16}{x^2 + x - 20}$ is an indeterminate form at 0 of the type $\frac{0}{0}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x^2 + x - 20} = \lim_{x \rightarrow 4} \frac{\frac{d}{dx}(x^2 - 16)}{\frac{d}{dx}(x^2 + x - 20)} = \lim_{x \rightarrow 4} \frac{2x}{2x + 1} = \boxed{\frac{8}{9}}.$$

Alternately,

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x^2 + x - 20} = \lim_{x \rightarrow 4} \frac{(x + 4)(x - 4)}{(x - 4)(x + 5)} = \lim_{x \rightarrow 4} \frac{x + 4}{x + 5} = \boxed{\frac{8}{9}}.$$

56. Because

$$\lim_{x \rightarrow 0^+} \cot x = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} x = 0,$$

the expression $(\cot x)^x$ is an indeterminate form at 0^+ of the type ∞^0 . Let $y = (\cot x)^x$. Then

$$\ln y = \ln(\cot x)^x = x \ln(\cot x) = \frac{\ln(\cot x)}{x^{-1}},$$

which is an indeterminate form at 0^+ of the type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(\cot x)}{x^{-1}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(\cot x)}{\frac{d}{dx} x^{-1}} = \lim_{x \rightarrow 0^+} \frac{\frac{-\csc^2 x}{\cot x}}{-x^{-2}} = \lim_{x \rightarrow 0^+} \frac{x^2}{\sin x \cos x}.$$

Now,

$$\lim_{x \rightarrow 0^+} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} (\sin x \cos x) = 0,$$

so the expression $\frac{x^2}{\sin x \cos x}$ is an indeterminate form at 0^+ of the type $\frac{0}{0}$. Using L'Hôpital's Rule again,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{x^2}{\sin x \cos x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} (\sin x \cos x)} = \lim_{x \rightarrow 0^+} \frac{2x}{-\sin^2 x + \cos^2 x} = 0.$$

Because $\lim_{x \rightarrow 0^+} \ln y = 0$, it follows that

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (\cot x)^x = e^0 = \boxed{1}.$$

57. The general solution of the differential equation

$$\frac{dy}{dx} = e^x$$

is $y = e^x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $y = 2$ yields

$$2 = e^0 + C = 1 + C \quad \text{so that} \quad C = 1.$$

The particular solution of the differential equation with the given boundary condition is therefore $\boxed{y = e^x + 1}$.

58. The general solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{2} \sec x \tan x$$

is $y = \frac{1}{2} \sec x + C$, where C is a constant. Applying the boundary condition that when $x = 0$, then $y = 7$ yields

$$7 = \frac{1}{2} \sec 0 + C = \frac{1}{2} + C \quad \text{so that} \quad C = \frac{13}{2}.$$

The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{y = \frac{1}{2} \sec x + \frac{13}{2}}.$$

59. The general solution of the differential equation

$$\frac{dy}{dx} = \frac{2}{x}$$

is $y = 2 \ln |x| + C$, where C is a constant. Applying the boundary condition that when $x = 1$, then $y = 4$ yields

$$4 = 2 \ln 1 + C = C.$$

The particular solution of the differential equation with the given boundary condition is therefore

$$\boxed{y = 2 \ln |x| + 4}.$$

60. All the antiderivatives of $\frac{d^2y}{dx^2} = x^2 - 4$ are

$$\frac{dy}{dx} = \frac{1}{3}x^3 - 4x + C_1,$$

and all the antiderivatives of $\frac{dy}{dx} = \frac{1}{3}x^3 - 4x + C_1$ are

$$y = \frac{1}{3} \cdot \frac{x^4}{4} - 4 \cdot \frac{x^2}{2} + C_1x + C_2 = \frac{1}{12}x^4 - 2x^2 + C_1x + C_2,$$

where C_1 and C_2 are constants. Applying the boundary condition that when $x = 3$, then $y = 2$ yields

$$2 = \frac{1}{12}3^4 - 2(3)^2 + 3C_1 + C_2 \quad \text{so that} \quad 3C_1 + C_2 = \frac{53}{4}.$$

Next, applying the boundary condition that when $x = 2$, then $y = 2$ yields

$$2 = \frac{1}{12}2^4 - 2(2)^2 + 2C_1 + C_2 \quad \text{so that} \quad 2C_1 + C_2 = \frac{26}{3}.$$

Solving the system of linear equations

$$3C_1 + C_2 = \frac{53}{4} \quad \text{and} \quad 2C_1 + C_2 = \frac{26}{3}$$

gives

$$C_1 = \frac{55}{12} \quad \text{and} \quad C_2 = -\frac{1}{2}.$$

The particular solution of the differential equation with the given boundary conditions is therefore

$$y = \frac{1}{12}x^4 - 2x^2 + \frac{55}{12}x - \frac{1}{2}.$$

61. The cost of producing x items is $C(x) = 200 + 35x + 0.02x^2$ and each item can be sold for \$78, so the revenue produced by selling x items is $R(x) = 78x$. The profit generated by producing and selling x items is then

$$P(x) = R(x) - C(x) = 78x - (200 + 35x + 0.02x^2) = -200 + 43x - 0.02x^2.$$

The polynomial function P is differentiable everywhere, so the critical numbers of P occur where $P'(x) = 0$. Now,

$$P'(x) = 43 - 0.04x,$$

so $x = 1075$ is the only critical number. Using the Second Derivative Test,

$$P''(x) = -0.04 < 0$$

for all x , so P has both a local maximum and an absolute maximum at 1075. Therefore, 1075 items should be produced and sold to maximize profit.

62. Let sales of a new stereo system be given by

$$f(x) = \frac{5000}{1 + 5e^{-x}},$$

for $x \geq 0$, where x is measured in years. The sales *rate* will be maximum when $f'(x)$ is maximum. Now,

$$\begin{aligned} f'(x) &= \frac{25000e^{-x}}{(1 + 5e^{-x})^2}; \text{ and} \\ f''(x) &= \frac{(1 + 5e^{-x})^2(-25000e^{-x}) - (25000e^{-x}) \cdot 2(1 + 5e^{-x})(-5e^{-x})}{(1 + 5e^{-x})^4} \\ &= -25000e^{-x} \frac{1 + 5e^{-x} - 10e^{-x}}{(1 + 5e^{-x})^3} = -25000e^{-x} \frac{1 - 5e^{-x}}{(1 + 5e^{-x})^3}. \end{aligned}$$

$f''(x)$ exists everywhere and is equal to zero when $1 - 5e^{-x} = 0$; that is, when $x = \ln 5$. Because

$$f'(0) = \frac{25000}{36}, \quad f'(\ln 5) = \frac{25000}{20}, \quad \lim_{x \rightarrow \infty} f'(x) = 0,$$

and $\ln 5$ is the only critical number, f' has an absolute maximum at $\ln 5$. The sales rate is a maximum $\ln 5 \approx 1.609$ years after sales began, so the sales rate is a maximum in the second year.

63. Let $(x, \ln x)$ be the coordinates of the vertex of the rectangle on the graph of $y = \ln x$ with $0 < x < 1$. The area A of the rectangle is

$$A = -x \ln x,$$

where the negative sign is included because $\ln x < 0$ for $0 < x < 1$, and

$$A'(x) = -\left(x \cdot \frac{1}{x} + \ln x\right) = -(1 + \ln x).$$

$A'(x)$ exists everywhere on the interval $(0, 1)$ and is equal to zero when $x = e^{-1}$. Using the Second Derivative Test,

$$A''(x) = -\frac{1}{x} \quad \text{so} \quad A''(e^{-1}) = -e < 0,$$

and A has a local maximum at e^{-1} . Because $A''(x) < 0$ for all $0 < x < 1$, the local maximum is also an absolute maximum. Therefore, the area of the largest rectangle in the fourth quadrant that has three vertices on the coordinate axes and the fourth vertex on the graph of $y = \ln x$ is

$$A = -e^{-1} \ln e^{-1} = e^{-1} = \boxed{\frac{1}{e}}.$$

64. Let $t = 0$ denote the time that the motorcycle begins to accelerate, and let $s(t)$ represent the distance the motorcycle has traveled after t seconds. Because the motorcycle accelerates at a constant rate from 0 to 72 km/h in 10 s, its acceleration a is given by

$$a = \frac{72 \text{ km/h}}{10 \text{ s}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} = 2 \text{ m/s}^2.$$

Solve the differential equation

$$\frac{dv}{dt} = a(t) = 2$$

for $v(t)$ to obtain $v(t) = 2t + C_1$, where C_1 is a constant. Apply the initial condition $v(0) = 0$ to determine

$$0 = 2(0) + C_1 = C_1,$$

so that $v(t) = 2t$. Next, solve the differential equation

$$\frac{ds}{dt} = v(t) = 2t$$

for $s(t)$ to obtain

$$s(t) = 2 \cdot \frac{t^2}{2} + C_2 = t^2 + C_2.$$

The initial condition $s(0) = 0$ determines

$$0 = (0)^2 + C_2 = C_2,$$

so $s(t) = t^2$. Therefore, after 10 seconds, the motorcycle has traveled

$$s(10) = (10)^2 = \boxed{100 \text{ m}}.$$

The Integral

5.1 Area

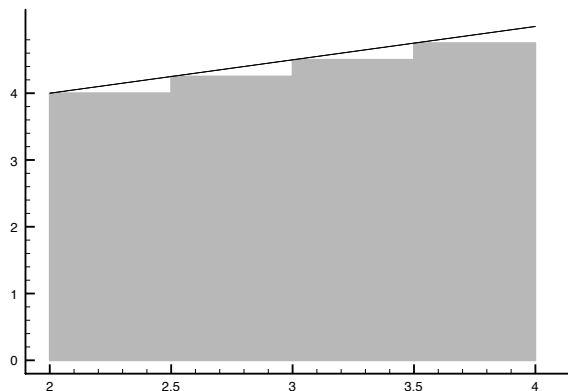
Concepts and Vocabulary

1. Partition the interval $[a, b]$ into n subintervals of length Δx . Then approximate the actual area under the graph of $f(x)$ on each subinterval using a rectangle of height $f(t_i)$, where t_i is some point in the n^{th} subinterval. The area under the graph of $f(x)$ is approximately the sum of the areas of the approximating rectangles: $A \approx \sum_{i=1}^n f(t_i) \Delta x$.
2. False. The length of each subinterval is $\frac{b-a}{n}$.
3. If the closed interval $[-2, 4]$ is partitioned into 12 subintervals, each of the same length, then the length of each subinterval is $\frac{4 - (-2)}{12} = \boxed{\frac{1}{2}}$.
4. True.

Skill Building

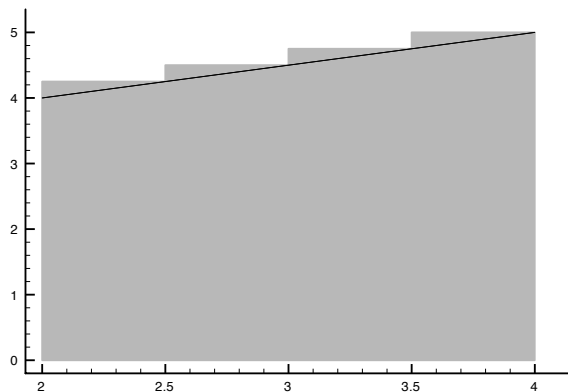
5. Let $f(x) = \frac{1}{2}x + 3$.
 - (a) The four rectangles whose heights are determined by the value of the function at the left endpoint of each subinterval are shown in the figure below. With $\Delta x = \frac{1}{2}$, the sum of the areas of the four rectangles is

$$\begin{aligned} \Delta x \left[f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) \right] \\ &= \left(\frac{1}{2}\right) \left[\left(\frac{1}{2}(2) + 3\right) + \left(\frac{1}{2}\left(\frac{5}{2}\right) + 3\right) + \left(\frac{1}{2}(3) + 3\right) + \left(\frac{1}{2}\left(\frac{7}{2}\right) + 3\right) \right] \\ &= \left(\frac{1}{2}\right) \left(4 + \frac{17}{4} + \frac{9}{2} + \frac{19}{4}\right) = \left(\frac{1}{2}\right) \left(\frac{35}{2}\right) = \boxed{\frac{35}{4}}. \end{aligned}$$



- (b) The four rectangles whose heights are determined by the value of the function at the right endpoint of each subinterval are shown in the figure below. With $\Delta x = \frac{1}{2}$, so the sum of the areas of the four rectangles is

$$\begin{aligned}
 \Delta x \left[f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) \right] \\
 &= \left(\frac{1}{2}\right) \left[\left(\frac{1}{2}\left(\frac{5}{2}\right) + 3\right) + \left(\frac{1}{2}(3) + 3\right) + \left(\frac{1}{2}\left(\frac{7}{2}\right) + 3\right) + \left(\frac{1}{2}(4) + 3\right) \right] \\
 &= \left(\frac{1}{2}\right) \left(\frac{17}{4} + \frac{9}{2} + \frac{19}{4} + 5\right) = \left(\frac{1}{2}\right) \left(\frac{37}{2}\right) = \boxed{\frac{37}{4}}.
 \end{aligned}$$



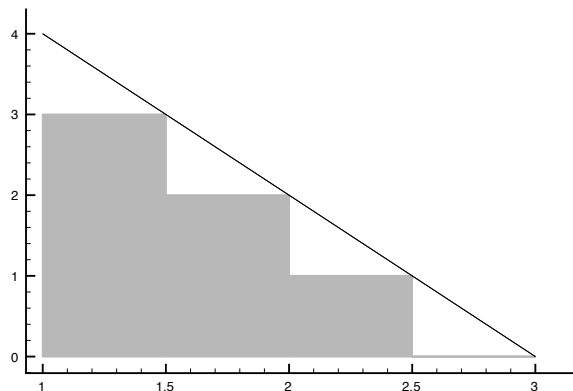
- (c) We have $\frac{35}{4} < 9 < \frac{37}{4}$, as expected. The function $f(x) = \frac{1}{2}x + 3$ is increasing, so the approximation using left endpoints in part (a) is an underestimate, and the approximation using right endpoints in part (b) is an overestimate. Moreover, as noted in Example 1, the error in approximating A is 1 square unit with $n = 1$ and 0.5 square units with $n = 2$. With $n = 4$, the error in approximating A is only 0.25 square units.

6. Let $f(x) = 6 - 2x$.

- (a) The four rectangles whose heights are determined by the value of the function at the right endpoint of each subinterval are shown in the figure below. With $\Delta x = \frac{1}{2}$, the

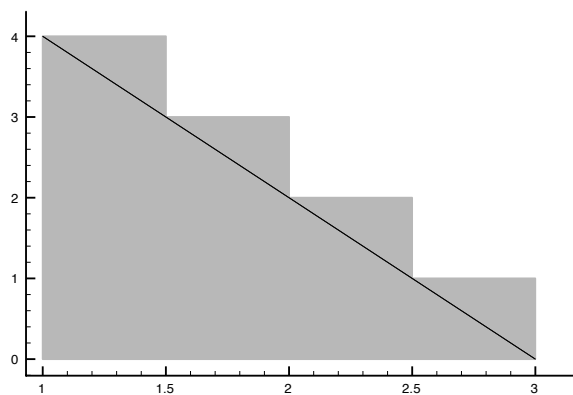
sum of the areas of the four rectangles is

$$\begin{aligned}
 \Delta x \left[f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) \right] \\
 &= \left(\frac{1}{2}\right) \left[\left(6 - 2\left(\frac{3}{2}\right)\right) + (6 - 2(2)) + \left(6 - 2\left(\frac{5}{2}\right)\right) + (6 - 2(3)) \right] \\
 &= \left(\frac{1}{2}\right) (3 + 2 + 1 + 0) = \left(\frac{1}{2}\right) (6) = \boxed{3}.
 \end{aligned}$$



- (b) The four rectangles whose heights are determined by the value of the function at the left endpoint of each subinterval are shown in the figure below. With $\Delta x = \frac{1}{2}$, so the sum of the areas of the four rectangles is

$$\begin{aligned}
 \Delta x \left[f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) \right] \\
 &= \left(\frac{1}{2}\right) \left[(6 - 2(1)) + \left(6 - 2\left(\frac{3}{2}\right)\right) + (6 - 2(2)) + \left(6 - 2\left(\frac{5}{2}\right)\right) \right] \\
 &= \left(\frac{1}{2}\right) (4 + 3 + 2 + 1) = \left(\frac{1}{2}\right) (10) = \boxed{5}.
 \end{aligned}$$



- (c) We have $3 < 4 < 5$, as expected. The function $f(x) = 6 - 2x$ is decreasing, so

the approximation using right endpoints in part (a) is an underestimate, and the approximation using left endpoints in part (b) is an overestimate.

7. From the figure, we see that $\Delta x = 1$ and that the subintervals are $[1, 2]$ and $[2, 3]$.

- (a) By constructing rectangles using the left endpoint of each subinterval, the area under the graph of f is approximately

$$\Delta x [f(1) + f(2)] = 1(1 + 2) = 3.$$

- (b) By constructing rectangles using the right endpoint of each subinterval, the area under the graph of f is approximately

$$\Delta x [f(2) + f(3)] = 1(2 + 4) = 6.$$

8. From the figure, we see that $\Delta x = 1$ and that the subintervals are $[1, 2]$, $[2, 3]$ and $[3, 4]$.

- (a) By constructing rectangles using the left endpoint of each subinterval, the area under the graph of f is approximately

$$\Delta x [f(1) + f(2) + f(3)] = 1 \left(\frac{7}{2} + \frac{5}{2} + 2 \right) = 8.$$

- (b) By constructing rectangles using the right endpoint of each subinterval, the area under the graph of f is approximately

$$\Delta x [f(2) + f(3) + f(4)] = 1 \left(\frac{5}{2} + 2 + \frac{1}{2} \right) = 5.$$

9. The length of the interval $[1, 4]$ is $4 - 1 = 3$, so with $n = 3$ subintervals, the length of each subinterval is

$$\Delta x = \frac{4 - 1}{3} = 1.$$

The coordinates of the endpoints of the subintervals are $x_0 = 1$, $x_1 = 1 + 1 = 2$, $x_2 = 2 + 1 = 3$, and $x_3 = 3 + 1 = 4$; the subintervals are

$$\boxed{[1, 2], [2, 3], \text{ and } [3, 4]}.$$

10. The length of the interval $[0, 9]$ is $9 - 0 = 9$, so with $n = 9$ subintervals, the length of each subinterval is

$$\Delta x = \frac{9 - 0}{9} = 1.$$

The coordinates of the endpoints of the subintervals are $x_0 = 0$, $x_1 = 0 + 1 = 1$, $x_2 = 1 + 1 = 2$, $x_3 = 2 + 1 = 3$, $x_4 = 3 + 1 = 4$, $x_5 = 4 + 1 = 5$, $x_6 = 5 + 1 = 6$, $x_7 = 6 + 1 = 7$, $x_8 = 7 + 1 = 8$, and $x_9 = 8 + 1 = 9$; the subintervals are

$$\boxed{[0, 1], [1, 2], [2, 3], [3, 4], [4, 5], [5, 6], [6, 7], [7, 8], \text{ and } [8, 9]}.$$

11. The length of the interval $[-1, 4]$ is $4 - (-1) = 5$, so with $n = 10$ subintervals, the length of each subinterval is

$$\Delta x = \frac{4 - (-1)}{10} = \frac{5}{10} = \frac{1}{2}.$$

The coordinates of the endpoints of the subintervals are

$$\left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4 \right\};$$

the subintervals are

$$\left[-1, -\frac{1}{2} \right], \left[-\frac{1}{2}, 0 \right], \left[0, \frac{1}{2} \right], \left[\frac{1}{2}, 1 \right], \left[1, \frac{3}{2} \right], \left[\frac{3}{2}, 2 \right], \left[2, \frac{5}{2} \right], \left[\frac{5}{2}, 3 \right], \left[3, \frac{7}{2} \right], \text{ and } \left[\frac{7}{2}, 4 \right].$$

12. The length of the subinterval $[-4, 4]$ is $4 - (-4) = 8$, so with $n = 16$ subintervals, the length of each subinterval is

$$\Delta x = \frac{4 - (-4)}{16} = \frac{8}{16} = \frac{1}{2}.$$

The coordinates of the endpoints of the subintervals are

$$\left\{ -4, -\frac{7}{2}, -3, -\frac{5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4 \right\};$$

the subintervals are

$$\left[-4, -\frac{7}{2} \right], \left[-\frac{7}{2}, -3 \right], \left[-3, -\frac{5}{2} \right], \left[-\frac{5}{2}, -2 \right], \left[-2, -\frac{3}{2} \right], \left[-\frac{3}{2}, -1 \right], \left[-1, -\frac{1}{2} \right], \left[-\frac{1}{2}, 0 \right], \\ \left[0, \frac{1}{2} \right], \left[\frac{1}{2}, 1 \right], \left[1, \frac{3}{2} \right], \left[\frac{3}{2}, 2 \right], \left[2, \frac{5}{2} \right], \left[\frac{5}{2}, 3 \right], \left[3, \frac{7}{2} \right], \text{ and } \left[\frac{7}{2}, 4 \right].$$

13. From the figure, we see that $\Delta x = 2$ and that the subintervals are $[0, 2]$, $[2, 4]$, $[4, 6]$ and $[6, 8]$.

- (a) On the subinterval $[0, 2]$, the absolute minimum value of the function is $f(0) = 1$; on the subinterval $[2, 4]$, the absolute minimum value of the function is $f(4) = 3$; on the subinterval $[4, 6]$, the absolute minimum value of the function is $f(4) = 3$; and on the subinterval $[6, 8]$, the absolute minimum value of the function is $f(8) = 0$. Thus,

$$s_4 = \Delta x [f(0) + f(4) + f(4) + f(8)] = 2(1 + 3 + 3 + 0) = 14.$$

- (b) On the subinterval $[0, 2]$, the absolute maximum value of the function is $f(2) = 5$; on the subinterval $[2, 4]$, the absolute maximum value of the function is $f(2) = 5$; on the subinterval $[4, 6]$, the absolute maximum value of the function is $f(6) = 7$; and on the subinterval $[6, 8]$, the absolute maximum value of the function is $f(6) = 7$. Thus,

$$S_4 = \Delta x [f(2) + f(2) + f(6) + f(6)] = 2(5 + 5 + 7 + 7) = 48.$$

14. From the figure, we see that $\Delta x = 2$ and that the subintervals are $[0, 2]$, $[2, 4]$, $[4, 6]$ and $[6, 8]$.

- (a) On the subinterval $[0, 2]$, the absolute minimum value of the function is $f(2) = 2$; on the subinterval $[2, 4]$, the absolute minimum value of the function is $f(2) = 2$; on the subinterval $[4, 6]$, the absolute minimum value of the function is $f(6) = 2$; and on the subinterval $[6, 8]$, the absolute minimum value of the function is $f(8) = 0$. Thus,

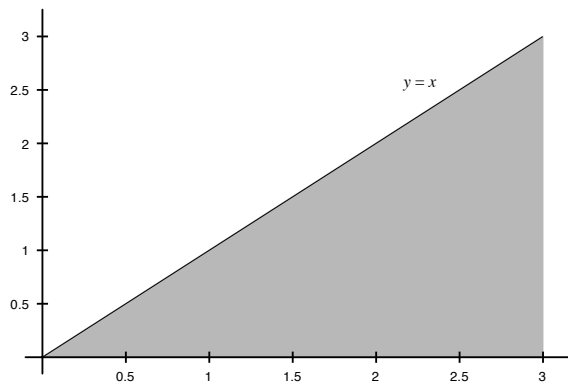
$$s_4 = \Delta x [f(2) + f(2) + f(6) + f(8)] = 2(2 + 2 + 2 + 0) = 12.$$

- (b) On the subinterval $[0, 2]$, the absolute maximum value of the function is $f(0) = 6$; on the subinterval $[2, 4]$, the absolute maximum value of the function is $f(4) = 4$; on the subinterval $[4, 6]$, the absolute maximum value of the function is $f(4) = 4$; and on the subinterval $[6, 8]$, the absolute maximum value of the function is $f(6) = 2$. Thus,

$$S_4 = \Delta x [f(0) + f(4) + f(4) + f(6)] = 2(6 + 4 + 4 + 2) = 32.$$

15. Let $f(x) = x$, and consider the area under the graph of f from 0 to 3.

(a) The graph of f from 0 to 3 is shown below. The area below the graph is shaded.



(b) Partition the interval $[0, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3 - 0}{n} = \frac{3}{n}.$$

The coordinates of the endpoints of each subinterval are

$$x_0 = 0, x_1 = \frac{3}{n}, x_2 = 2\left(\frac{3}{n}\right), \dots, x_{i-1} = (i-1)\left(\frac{3}{n}\right),$$

$$x_i = i\left(\frac{3}{n}\right), \dots, x_n = n\left(\frac{3}{n}\right) = 3.$$

(c) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1)\left(\frac{3}{n}\right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n c_i \Delta x = \sum_{i=1}^n (i-1) \left(\frac{3}{n}\right) \cdot \frac{3}{n} = \sum_{i=1}^n (i-1) \left(\frac{3}{n}\right)^2.$$

(d) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i\left(\frac{3}{n}\right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n C_i \Delta x = \sum_{i=1}^n i \left(\frac{3}{n} \right) \cdot \frac{3}{n} = \sum_{i=1}^n i \left(\frac{3}{n} \right)^2.$$

(e) Using summation properties and the result from part (c),

$$\begin{aligned} s_n &= \sum_{i=1}^n (i-1) \left(\frac{3}{n} \right)^2 = \frac{9}{n^2} \sum_{i=1}^n (i-1) = \frac{9}{n^2} \left(\sum_{i=1}^n i - \sum_{i=1}^n 1 \right) \\ &= \frac{9}{n^2} \left(\frac{n(n+1)}{2} - n \right) = 9 \left(\frac{n+1}{2n} - \frac{1}{n} \right) \\ &= \frac{9}{2} \left(1 - \frac{1}{n} \right); \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{9}{2} \left(1 - \frac{1}{n} \right) = \frac{9}{2} (1 - 0) = \frac{9}{2}.$$

Using summation properties and the result from part (d),

$$\begin{aligned} S_n &= \sum_{i=1}^n i \left(\frac{3}{n} \right)^2 = \frac{9}{n^2} \sum_{i=1}^n i \\ &= \frac{9}{n^2} \cdot \frac{n(n+1)}{2} = \frac{9}{2} \cdot \frac{n+1}{n} \\ &= \frac{9}{2} \left(1 + \frac{1}{n} \right); \end{aligned}$$

it follows that

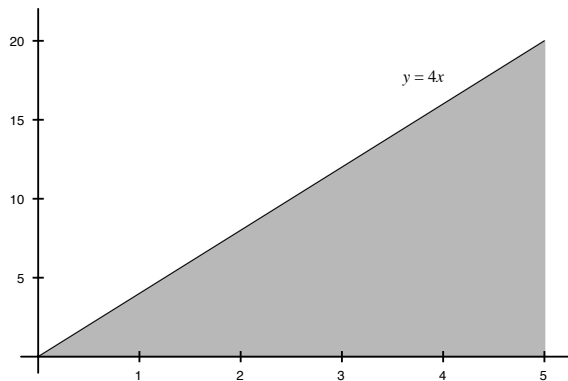
$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{9}{2} \left(1 + \frac{1}{n} \right) = \frac{9}{2} (1 + 0) = \frac{9}{2}.$$

Thus,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n = \frac{9}{2}.$$

16. Let $f(x) = 4x$, and consider the area under the graph of f from 0 to 5.

(a) The graph of f from 0 to 5 is shown below. The area below the graph is shaded.



- (b) Partition the interval $[0, 5]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 5$$

and each subinterval is of length

$$\Delta x = \frac{5 - 0}{n} = \frac{5}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 = 0, x_1 = \frac{5}{n}, x_2 = 2\left(\frac{5}{n}\right), \dots, x_{i-1} = (i-1)\left(\frac{5}{n}\right), \\ x_i = i\left(\frac{5}{n}\right), \dots, x_n = n\left(\frac{5}{n}\right) = 5. \end{aligned}$$

- (c) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1)\left(\frac{5}{n}\right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n 4c_i \Delta x = \sum_{i=1}^n 4(i-1)\left(\frac{5}{n}\right) \cdot \frac{5}{n} = \sum_{i=1}^n (i-1) \frac{100}{n^2}.$$

- (d) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i\left(\frac{5}{n}\right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n 4C_i \Delta x = \sum_{i=1}^n 4i\left(\frac{5}{n}\right) \cdot \frac{5}{n} = \sum_{i=1}^n i \frac{100}{n^2}.$$

- (e) Using summation properties and the result from part (c),

$$\begin{aligned} s_n &= \sum_{i=1}^n (i-1) \frac{100}{n^2} = \frac{100}{n^2} \sum_{i=1}^n (i-1) = \frac{100}{n^2} \left(\sum_{i=1}^n i - \sum_{i=1}^n 1 \right) \\ &= \frac{100}{n^2} \left(\frac{n(n+1)}{2} - n \right) = 100 \left(\frac{n+1}{2n} - \frac{1}{n} \right) \\ &= 50 \left(1 - \frac{1}{n} \right); \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 50 \left(1 - \frac{1}{n} \right) = 50(1 - 0) = 50.$$

Using summation properties and the result from part (d),

$$\begin{aligned} S_n &= \sum_{i=1}^n i \frac{100}{n^2} = \frac{100}{n^2} \sum_{i=1}^n i \\ &= \frac{100}{n^2} \cdot \frac{n(n+1)}{2} = 50 \cdot \frac{n+1}{n} \\ &= 50 \left(1 + \frac{1}{n} \right); \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 50 \left(1 + \frac{1}{n} \right) = 50(1 + 0) = 50.$$

Thus,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n = 50.$$

17. Let $f(x) = -x + 10$. Because f is decreasing on the interval $[0, 8]$, the absolute minimum on any subinterval will occur at the right endpoint of the subinterval, and the absolute maximum on any subinterval will occur at the left endpoint of the subinterval.

(a) With $n = 4$,

$$\Delta x = \frac{8-0}{4} = 2, \quad c_i = 0 + i\Delta x = 2i,$$

and

$$\begin{aligned} s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\ &= (2) [f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\ &= (2) [f(2) + f(4) + f(6) + f(8)] \\ &= (2) (8 + 6 + 4 + 2) \\ &= \boxed{40}. \end{aligned}$$

With $n = 8$, $\Delta x = \frac{8-0}{8} = 1$, $c_i = 0 + i\Delta x = i$, and

$$\begin{aligned} s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\ &= (1) [f(c_1) + \cdots + f(c_8)] \\ &= (1) [f(1) + f(2) + \cdots + f(8)] \\ &= (1) (9 + 8 + 7 + 6 + 5 + 4 + 3 + 2) \\ &= \boxed{44}. \end{aligned}$$

(b) With $n = 4$,

$$\Delta x = \frac{8-0}{4} = 2, \quad C_i = 0 + (i-1)\Delta x = 2(i-1),$$

and

$$\begin{aligned} S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\ &= (2) [f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\ &= (2) [f(0) + f(2) + f(4) + f(6)] \\ &= (2) (10 + 8 + 6 + 4) \\ &= \boxed{56}. \end{aligned}$$

With $n = 8$, $\Delta x = \frac{8-0}{8} = 1$, $C_i = 0 + (i-1)\Delta x = i-1$, and

$$\begin{aligned}
 S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\
 &= (1)[f(C_1) + \cdots + f(C_8)] \\
 &= (1)[f(0) + f(1) + \cdots + f(6) + f(7)] \\
 &= (1)(10 + 9 + 8 + 7 + 6 + 5 + 4 + 3) \\
 &= \boxed{52}.
 \end{aligned}$$

18. Let $f(x) = 2x + 5$. Because f is increasing on the interval $[2, 6]$, the absolute minimum on any subinterval will occur at the left endpoint of the subinterval, and the absolute maximum on any subinterval will occur at the right endpoint of the subinterval.

(a) With $n = 4$, $\Delta x = \frac{6-2}{4} = 1$,

$$c_i = 2 + (i-1)\Delta x = 2 + (i-1)(1) = 1 + i,$$

and

$$\begin{aligned}
 s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\
 &= (1)[f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\
 &= (1)[f(2) + f(3) + f(4) + f(5)] \\
 &= (1)(9 + 11 + 13 + 15) \\
 &= \boxed{48}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{6-2}{8} = \frac{1}{2}, \quad c_i = 2 + (i-1)\Delta x = 2 + (i-1)\left(\frac{1}{2}\right) = \frac{3}{2} + \frac{1}{2}i,$$

and

$$\begin{aligned}
 s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\
 &= \left(\frac{1}{2}\right)[f(c_1) + \cdots + f(c_8)] \\
 &= \left(\frac{1}{2}\right)\left[f(2) + f\left(\frac{5}{2}\right) + \cdots + f(5) + f\left(\frac{11}{2}\right)\right] \\
 &= \left(\frac{1}{2}\right)(9 + 10 + 11 + 12 + 13 + 14 + 15 + 16) \\
 &= \boxed{50}.
 \end{aligned}$$

(b) With $n = 4$, $\Delta x = \frac{6-2}{4} = 1$, $C_i = 2 + i\Delta x = 2 + i$, and

$$\begin{aligned}
 S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\
 &= (1)[f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\
 &= (1)[f(3) + f(4) + f(5) + f(6)] \\
 &= (1)(11 + 13 + 15 + 17) \\
 &= \boxed{56}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{6-2}{8} = \frac{1}{2}, \quad C_i = 2 + i\Delta x = 2 + \frac{1}{2}i,$$

and

$$\begin{aligned} S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\ &= \left(\frac{1}{2}\right) [f(C_1) + \cdots + f(C_8)] \\ &= \left(\frac{1}{2}\right) \left[f\left(\frac{5}{2}\right) + f(3) + \cdots + f\left(\frac{11}{2}\right) + f(6) \right] \\ &= \left(\frac{1}{2}\right) (10 + 11 + 12 + 13 + 14 + 15 + 16 + 17) \\ &= \boxed{54}. \end{aligned}$$

19. Let $f(x) = 16 - x^2$. Because f is decreasing on the interval $[0, 4]$, the absolute minimum on any subinterval will occur at the right endpoint of the subinterval, and the absolute maximum on any subinterval will occur at the left endpoint of the subinterval.

(a) With $n = 4$, $\Delta x = \frac{4-0}{4} = 1$, $c_i = 0 + i\Delta x = i$, and

$$\begin{aligned} s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\ &= (1) [f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\ &= (1) [f(1) + f(2) + f(3) + f(4)] \\ &= (1) (15 + 12 + 7 + 0) \\ &= \boxed{34}. \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{4-0}{8} = \frac{1}{2}, \quad c_i = 0 + i\Delta x = \frac{1}{2}i,$$

and

$$\begin{aligned} s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\ &= \left(\frac{1}{2}\right) [f(c_1) + \cdots + f(c_8)] \\ &= \left(\frac{1}{2}\right) \left(f\left(\frac{1}{2}\right) + f(1) + \cdots + f\left(\frac{7}{2}\right) + f(4) \right) \\ &= \left(\frac{1}{2}\right) \left(\frac{63}{4} + 15 + \frac{55}{4} + 12 + \frac{39}{4} + 7 + \frac{15}{4} + 0 \right) \\ &= \boxed{\frac{77}{2}}. \end{aligned}$$

(b) With $n = 4$, $\Delta x = \frac{4-0}{4} = 1$, $C_i = 0 + (i-1)\Delta x = i-1$, and

$$\begin{aligned}
 S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\
 &= (1)[f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\
 &= (1)[f(0) + f(1) + f(2) + f(3)] \\
 &= (1)(16 + 15 + 12 + 7) \\
 &= \boxed{50}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{4-0}{8} = \frac{1}{2}, \quad C_i = 0 + (i-1)\Delta x = \frac{i-1}{2},$$

and

$$\begin{aligned}
 S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\
 &= \left(\frac{1}{2}\right)[f(C_1) + \cdots + f(C_8)] \\
 &= \left(\frac{1}{2}\right)\left[f(0) + f\left(\frac{1}{2}\right) + \cdots + f(3) + f\left(\frac{7}{2}\right)\right] \\
 &= \left(\frac{1}{2}\right)\left(16 + \frac{63}{4} + 15 + \frac{55}{4} + 12 + \frac{39}{4} + 7 + \frac{15}{4}\right) \\
 &= \boxed{\frac{93}{2}}.
 \end{aligned}$$

20. Let $f(x) = x^3$. Because f is increasing on the interval $[0, 8]$, the absolute minimum on any subinterval will occur at the left endpoint of the subinterval, and the absolute maximum on any subinterval will occur at the right endpoint of the subinterval.

(a) With $n = 4$, $\Delta x = \frac{8-0}{4} = 2$, $c_i = 0 + (i-1)\Delta x = 2(i-1)$, and

$$\begin{aligned}
 s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\
 &= (2)[f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\
 &= (2)[f(0) + f(2) + f(4) + f(6)] \\
 &= (2)(0 + 8 + 64 + 216) \\
 &= \boxed{576}.
 \end{aligned}$$

With $n = 8$, $\Delta x = \frac{8-0}{8} = 1$, $c_i = 0 + (i-1)\Delta x = i-1$, and

$$\begin{aligned}
 s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\
 &= (1)[f(c_1) + \cdots + f(c_8)] \\
 &= (1)[f(0) + f(1) + \cdots + f(6) + f(7)] \\
 &= (1)(0 + 1 + 8 + 27 + 64 + 125 + 216 + 343) \\
 &= \boxed{784}.
 \end{aligned}$$

(b) With $n = 4$, $\Delta x = \frac{8-0}{4} = 2$, $C_i = 0 + i\Delta x = 2i$, and

$$\begin{aligned}
 S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\
 &= (2) [f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\
 &= (2) [f(2) + f(4) + f(6) + f(8)] \\
 &= (2) (8 + 64 + 216 + 512) \\
 &= \boxed{1600}.
 \end{aligned}$$

With $n = 8$, $\Delta x = \frac{8-0}{8} = 1$, $C_i = 0 + i\Delta x = i$, and

$$\begin{aligned}
 S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\
 &= (1) [f(C_1) + \cdots + f(C_8)] \\
 &= (1) [f(1) + f(2) + \cdots + f(7) + f(8)] \\
 &= (1) (1 + 8 + 27 + 64 + 125 + 216 + 343 + 512) \\
 &= \boxed{1296}.
 \end{aligned}$$

21. Let $f(x) = \cos x$.

(a) With $n = 4$,

$$\Delta x = \frac{\pi/2 - (-\pi/2)}{4} = \frac{\pi}{4}.$$

Because f is increasing on $[-\frac{\pi}{2}, 0]$, and decreasing on $[0, \frac{\pi}{2}]$, the absolute minimum on the subintervals determined by the partition points $\{-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\}$ occurs at the points $c_1 = -\frac{\pi}{2}$, $c_2 = -\frac{\pi}{4}$, $c_3 = \frac{\pi}{4}$, and $c_4 = \frac{\pi}{2}$. Thus,

$$\begin{aligned}
 s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\
 &= \left(\frac{\pi}{4}\right) [f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\
 &= \left(\frac{\pi}{4}\right) \left[f\left(-\frac{\pi}{2}\right) + f\left(-\frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right] \\
 &= \left(\frac{\pi}{4}\right) \left(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0\right) \\
 &= \boxed{\frac{\sqrt{2}}{4}\pi \approx 1.111}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{\pi/2 - (-\pi/2)}{8} = \frac{\pi}{8}.$$

The absolute minimum on the subintervals determined by the partition points

$$\left\{-\frac{\pi}{2}, -\frac{3\pi}{8}, -\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$$

occurs at the points $c_1 = -\frac{\pi}{2}$, $c_2 = -\frac{3\pi}{8}$, $c_3 = -\frac{\pi}{4}$, $c_4 = -\frac{\pi}{8}$, $c_5 = \frac{\pi}{8}$, $c_6 = \frac{\pi}{4}$, $c_7 = \frac{3\pi}{8}$, and $c_8 = \frac{\pi}{2}$. Thus,

$$\begin{aligned}
 s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\
 &= \left(\frac{\pi}{8}\right) [f(c_1) + \cdots + f(c_8)] \\
 &= \left(\frac{\pi}{8}\right) \left[f\left(-\frac{\pi}{2}\right) + f\left(-\frac{3\pi}{8}\right) + f\left(-\frac{\pi}{4}\right) + f\left(-\frac{\pi}{8}\right) \right. \\
 &\quad \left. + f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) \right] \\
 &= \left(\frac{\pi}{8}\right) \left(0 + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + 0 \right) \\
 &= \boxed{\frac{\sqrt{2-\sqrt{2}} + \sqrt{\sqrt{2}+2} + \sqrt{2}}{8} \pi \approx 1.582}.
 \end{aligned}$$

(b) With $n = 4$,

$$\Delta x = \frac{\pi/2 - (-\pi/2)}{4} = \frac{\pi}{4}.$$

Because f is increasing on $[-\frac{\pi}{2}, 0]$, and decreasing on $[0, \frac{\pi}{2}]$, the absolute maximum on the subintervals determined by the partition points $\{-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\}$ occurs at the points $C_1 = -\frac{\pi}{4}$, $C_2 = 0$, $C_3 = 0$, and $C_4 = \frac{\pi}{4}$. Thus,

$$\begin{aligned}
 S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\
 &= \left(\frac{\pi}{4}\right) [f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\
 &= \left(\frac{\pi}{4}\right) \left[f\left(-\frac{\pi}{4}\right) + f(0) + f(0) + f\left(\frac{\pi}{4}\right) \right] \\
 &= \left(\frac{\pi}{4}\right) \left(\frac{\sqrt{2}}{2} + 1 + 1 + \frac{\sqrt{2}}{2} \right) \\
 &= \boxed{\frac{\sqrt{2}+2}{4} \pi \approx 2.682}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{\pi/2 - (-\pi/2)}{8} = \frac{\pi}{8}.$$

The absolute maximum on the subintervals determined by the partition points

$$\left\{ -\frac{\pi}{2}, -\frac{3\pi}{8}, -\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2} \right\}$$

occurs at the points $C_1 = -\frac{3\pi}{8}$, $C_2 = -\frac{\pi}{4}$, $C_3 = -\frac{\pi}{8}$, $C_4 = 0$, $C_5 = 0$, $C_6 = \frac{\pi}{8}$,

$C_7 = \frac{\pi}{4}$, and $C_8 = \frac{3\pi}{8}$. Thus,

$$\begin{aligned}
 S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\
 &= \left(\frac{\pi}{8}\right) [f(C_1) + \cdots + f(C_8)] \\
 &= \left(\frac{\pi}{8}\right) \left[f\left(-\frac{3\pi}{8}\right) + f\left(-\frac{\pi}{4}\right) + f\left(-\frac{\pi}{8}\right) + f(0) \right. \\
 &\quad \left. + f(0) + f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) \right] \\
 &= \left(\frac{\pi}{8}\right) \left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + 1 + 1 + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} \right) \\
 &= \boxed{\frac{\sqrt{2-\sqrt{2}} + \sqrt{\sqrt{2}+2} + \sqrt{2} + 2}{8} \pi \approx 2.367}.
 \end{aligned}$$

22. Let $f(x) = \sin x$.

(a) With $n = 4$,

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

Because f is increasing on $[0, \frac{\pi}{2}]$, and decreasing on $[\frac{\pi}{2}, \pi]$, the absolute minimum on the subintervals determined by the partition points $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$ occurs at the points $c_1 = 0$, $c_2 = \frac{\pi}{4}$, $c_3 = \frac{3\pi}{4}$, and $c_4 = \pi$. Thus,

$$\begin{aligned}
 s_4 &= \sum_{i=1}^4 f(c_i) \Delta x \\
 &= \left(\frac{\pi}{4}\right) [f(c_1) + f(c_2) + f(c_3) + f(c_4)] \\
 &= \left(\frac{\pi}{4}\right) \left[f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi) \right] \\
 &= \left(\frac{\pi}{4}\right) \left(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 \right) \\
 &= \boxed{\frac{\sqrt{2}}{4} \pi \approx 1.111}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8}.$$

The absolute minimum on the subintervals determined by the partition points

$$\left\{ 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \pi \right\}$$

occurs at the points $c_1 = 0$, $c_2 = \frac{\pi}{8}$, $c_3 = \frac{\pi}{4}$, $c_4 = \frac{3\pi}{8}$, $c_5 = \frac{5\pi}{8}$, $c_6 = \frac{3\pi}{4}$, $c_7 = \frac{7\pi}{8}$, and

$c_8 = \frac{\pi}{2}$. Thus,

$$\begin{aligned}
 s_8 &= \sum_{i=1}^8 f(c_i) \Delta x \\
 &= \left(\frac{\pi}{8}\right) [f(c_1) + \cdots + f(c_8)] \\
 &= \left(\frac{\pi}{8}\right) \left[f(0) + f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + \right. \\
 &\quad \left. f\left(\frac{5\pi}{8}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{7\pi}{8}\right) + f\left(\frac{\pi}{2}\right) \right] \\
 &= \left(\frac{\pi}{8}\right) \left(0 + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + 0 \right) \\
 &= \boxed{\frac{\sqrt{2-\sqrt{2}} + \sqrt{\sqrt{2}+2} + \sqrt{2}}{8} \pi \approx 1.582}.
 \end{aligned}$$

(b) With $n = 4$,

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

Because f is increasing on $[0, \frac{\pi}{2}]$, and decreasing on $[\frac{\pi}{2}, \pi]$, the absolute maximum on the subintervals determined by the partition points $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$ occurs at the points $C_1 = \frac{\pi}{4}$, $C_2 = \frac{\pi}{2}$, $C_3 = \frac{\pi}{2}$, and $C_4 = \frac{3\pi}{4}$. Thus,

$$\begin{aligned}
 S_4 &= \sum_{i=1}^4 f(C_i) \Delta x \\
 &= \left(\frac{\pi}{4}\right) [f(C_1) + f(C_2) + f(C_3) + f(C_4)] \\
 &= \left(\frac{\pi}{4}\right) \left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) \right] \\
 &= \left(\frac{\pi}{4}\right) \left(\frac{\sqrt{2}}{2} + 1 + 1 + \frac{\sqrt{2}}{2} \right) \\
 &= \boxed{\frac{\sqrt{2}+2}{4} \pi \approx 2.682}.
 \end{aligned}$$

With $n = 8$,

$$\Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8}.$$

The absolute maximum on the subintervals determined by the partition points

$$\left\{ 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \frac{\pi}{2} \right\}$$

occurs at the points $C_1 = \frac{\pi}{8}$, $C_2 = \frac{\pi}{4}$, $C_3 = \frac{3\pi}{8}$, $C_4 = \frac{\pi}{2}$, $C_5 = \frac{\pi}{2}$, $C_6 = \frac{5\pi}{8}$, $C_7 = \frac{3\pi}{4}$,

and $C_8 = \frac{7\pi}{8}$. Thus,

$$\begin{aligned}
 S_8 &= \sum_{i=1}^8 f(C_i) \Delta x \\
 &= \left(\frac{\pi}{8}\right) [f(C_1) + \cdots + f(C_8)] \\
 &= \left(\frac{\pi}{8}\right) \left[f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) + \right. \\
 &\quad \left. f\left(\frac{\pi}{2}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{7\pi}{8}\right) \right] \\
 &= \left(\frac{\pi}{8}\right) \left(\frac{\sqrt{2}-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + 1 + 1 + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}-\sqrt{2}}{2} \right) \\
 &= \boxed{\frac{\sqrt{2}-\sqrt{2}+\sqrt{\sqrt{2}+2}+\sqrt{2}+2}{8}\pi \approx 2.367}.
 \end{aligned}$$

23. Let $f(x) = 3x$. Because f is an increasing function on the interval $[0, 10]$, the absolute minimum on any subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = (i-1) \left(\frac{10}{n} \right)$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n 3c_i \frac{10}{n} = \sum_{i=1}^n \left(3(i-1) \frac{10}{n} \right) \frac{10}{n} = \sum_{i=1}^n \left(\frac{300}{n^2} i - \frac{300}{n^2} \right).$$

Using summation properties,

$$s_n = \frac{300}{n^2} \sum_{i=1}^n i - \frac{300}{n^2} \sum_{i=1}^n 1 = \frac{300}{n^2} \frac{n(n+1)}{2} - \frac{300}{n^2} n = \frac{150(n+1)}{n} - \frac{300}{n} = 150 \left(1 + \frac{1}{n} \right) - \frac{300}{n}.$$

Therefore,

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[150 \left(1 + \frac{1}{n} \right) - \frac{300}{n} \right] = 150 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) - 300 \lim_{n \rightarrow \infty} \frac{1}{n} \\
 &= 150(1) - 300(0) = \boxed{150}.
 \end{aligned}$$

24. Let $f(x) = 16 - x^2$. Because f is a decreasing function on the interval $[0, 4]$, the absolute maximum on any subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = (i-1) \left(\frac{4}{n} \right)$$

and

$$\begin{aligned}
 S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n (16 - C_i^2) \frac{4}{n} = \sum_{i=1}^n \left(16 - \left(\frac{4(i-1)}{n} \right)^2 \right) \frac{4}{n} \\
 &= \sum_{i=1}^n \left(16 - \left(16 \frac{i^2}{n^2} - 32 \frac{i}{n^2} + \frac{16}{n^2} \right) \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{64}{n} - \frac{64}{n^3} + \frac{128i}{n^3} - \frac{64i^2}{n^3} \right).
 \end{aligned}$$

Using summation properties,

$$\begin{aligned}
 S_n &= \left(\frac{64}{n} - \frac{64}{n^3} \right) \sum_{i=1}^n 1 + \frac{128}{n^3} \sum_{i=1}^n i - \frac{64}{n^3} \sum_{i=1}^n i^2 \\
 &= \left(\frac{64}{n} - \frac{64}{n^3} \right) n + \frac{128}{n^3} \frac{n(n+1)}{2} - \frac{64}{n^3} \frac{n(n+1)(2n+1)}{6} \\
 &= 64 - \frac{64}{n^2} + 64 \left(\frac{1}{n^2} + \frac{1}{n} \right) - \frac{32}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[64 - \frac{64}{n^2} + 64 \left(\frac{1}{n^2} + \frac{1}{n} \right) - \frac{32}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\
 &= 64 - 64 \lim_{n \rightarrow \infty} \frac{1}{n^2} + 64 \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{1}{n} \right) - \frac{32}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\
 &= 64 - 64(0) + 64(0) - \frac{32}{3}(2) = \boxed{\frac{128}{3}}.
 \end{aligned}$$

25. Let $f(x) = 2x + 1$, and consider the area under the graph of f from $a = 0$ to $b = 4$. Partition the interval $[0, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 4$$

and each subinterval is of length

$$\Delta x = \frac{4 - 0}{n} = \frac{4}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned}
 x_0 &= 0, x_1 = \frac{4}{n}, x_2 = 2 \left(\frac{4}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{4}{n} \right), \\
 x_i &= i \left(\frac{4}{n} \right), \dots, x_n = n \left(\frac{4}{n} \right) = 4.
 \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1) \left(\frac{4}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(2(i-1) \left(\frac{4}{n} \right) + 1 \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{32i}{n^2} - \frac{32}{n^2} + \frac{4}{n} \right).$$

Using summation properties,

$$\begin{aligned}
 s_n &= \frac{32}{n^2} \sum_{i=1}^n i - \frac{32}{n^2} \sum_{i=1}^n 1 + \frac{4}{n} \sum_{i=1}^n 1 \\
 &= \frac{32}{n^2} \frac{n(n+1)}{2} - \frac{32}{n^2} (n) + \frac{4}{n} (n) = 16 \left(1 + \frac{1}{n} \right) - \frac{32}{n} + 4.
 \end{aligned}$$

Therefore

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[16 \left(1 + \frac{1}{n} \right) - \frac{32}{n} + 4 \right] \\ &= 16 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) - 32 \lim_{n \rightarrow \infty} \frac{1}{n} + 4 = 16(1) - 32(0) + 4 = \boxed{20}. \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i \left(\frac{4}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(2i \left(\frac{4}{n} \right) + 1 \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{32i}{n^2} + \frac{4}{n} \right).$$

Using summation properties,

$$S_n = \frac{32}{n^2} \sum_{i=1}^n i + \frac{4}{n} \sum_{i=1}^n 1 = \frac{32}{n^2} \frac{n(n+1)}{2} + \frac{4}{n}(n) = 16 \left(1 + \frac{1}{n} \right) + 4.$$

Therefore,

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[16 \left(1 + \frac{1}{n} \right) + 4 \right] = 16 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) + 4 = 16(1) + 4 = \boxed{20}.$$

- (c) The work required in part (b) was easier than part (a). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i-1$.
26. Let $f(x) = 3x + 1$, and consider the area under the graph of f from $a = 0$ to $b = 4$. Partition the interval $[0, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 4$$

and each subinterval is of length

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 = 0, x_1 = \frac{4}{n}, x_2 = 2 \left(\frac{4}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{4}{n} \right), \\ x_i = i \left(\frac{4}{n} \right), \dots, x_n = n \left(\frac{4}{n} \right) = 4. \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1) \left(\frac{4}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(3(i-1) \left(\frac{4}{n} \right) + 1 \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{48i}{n^2} - \frac{48}{n^2} + \frac{4}{n} \right).$$

Using summation properties,

$$\begin{aligned} s_n &= \frac{48}{n^2} \sum_{i=1}^n i - \frac{48}{n^2} \sum_{i=1}^n 1 + \frac{4}{n} \sum_{i=1}^n 1 \\ &= \frac{48}{n^2} \frac{n(n+1)}{2} - \frac{48}{n^2}(n) + \frac{4}{n}(n) = 24 \left(1 + \frac{1}{n} \right) - \frac{48}{n} + 4. \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[24 \left(1 + \frac{1}{n} \right) - \frac{48}{n} + 4 \right] = 24 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) - 48 \lim_{n \rightarrow \infty} \frac{1}{n} + 4 \\ &= 24(1) - 48(0) + 4 = \boxed{28}. \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i \left(\frac{4}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(3i \left(\frac{4}{n} \right) + 1 \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{48i}{n^2} + \frac{4}{n} \right).$$

Using summation properties,

$$S_n = \frac{48}{n^2} \sum_{i=1}^n i + \frac{4}{n} \sum_{i=1}^n 1 = \frac{48}{n^2} \frac{n(n+1)}{2} + \frac{4}{n}(n) = 24 \left(1 + \frac{1}{n} \right) + 4.$$

Therefore,

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[24 \left(1 + \frac{1}{n} \right) + 4 \right] = 24 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) + 4 = 24(1) + 4 = \boxed{28}.$$

- (c) The work required in part (b) was easier than part (a). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i-1$.

27. Let $f(x) = 12 - 3x$, and consider the area under the graph of f from $a = 0$ to $b = 4$. Partition the interval $[0, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 4$$

and each subinterval is of length

$$\Delta x = \frac{4 - 0}{n} = \frac{4}{n}.$$

The coordinates of the endpoints of each subinterval are

$$x_0 = 0, x_1 = \frac{4}{n}, x_2 = 2 \left(\frac{4}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{4}{n} \right),$$

$$x_i = i \left(\frac{4}{n} \right), \dots, x_n = n \left(\frac{4}{n} \right) = 4.$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is a decreasing function, the absolute minimum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$c_i = x_i = i \left(\frac{4}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(12 - 3 \left(\frac{4i}{n} \right) \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{48}{n} - \frac{48i}{n^2} \right).$$

Using summation properties,

$$s_n = \frac{48}{n} \sum_{i=1}^n 1 - \frac{48}{n^2} \sum_{i=1}^n i = \frac{48}{n} (n) - \frac{48}{n^2} \frac{n(n+1)}{2} = 48 - 24 \left(1 + \frac{1}{n} \right).$$

Therefore,

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[48 - 24 \left(1 + \frac{1}{n} \right) \right] = 48 - 24 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 48 - 24(1) = \boxed{24}.$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is a decreasing function, the absolute maximum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = x_{i-1} = (i-1) \left(\frac{4}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(12 - 3 \frac{4(i-1)}{n} \right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{48}{n} + \frac{48}{n^2} - \frac{48i}{n^2} \right).$$

Using summation properties,

$$S_n = \left(\frac{48}{n} + \frac{48}{n^2} \right) \sum_{i=1}^n 1 - \frac{48}{n^2} \sum_{i=1}^n i = \left(\frac{48}{n} + \frac{48}{n^2} \right) (n) - \frac{48}{n^2} \frac{n(n+1)}{2} = 48 + \frac{48}{n} - 24 \left(1 + \frac{1}{n} \right).$$

Therefore

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[48 + \frac{48}{n} - 24 \left(1 + \frac{1}{n} \right) \right] \\ &= 48 + 48 \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) - 24 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 48 + 48(0) - 24(1) = \boxed{24}. \end{aligned}$$

- (c) The work required in part (a) was easier than part (b). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i-1$.

28. Let $f(x) = 5 - x$, and consider the area under the graph of f from $a = 0$ to $b = 4$. Partition the interval $[0, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 4$$

and each subinterval is of length

$$\Delta x = \frac{4 - 0}{n} = \frac{4}{n}.$$

The coordinates of the endpoints of each subinterval are

$$x_0 = 0, x_1 = \frac{4}{n}, x_2 = 2\left(\frac{4}{n}\right), \dots, x_{i-1} = (i-1)\left(\frac{4}{n}\right),$$

$$x_i = i\left(\frac{4}{n}\right), \dots, x_n = n\left(\frac{4}{n}\right) = 4.$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is a decreasing function, the absolute minimum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$c_i = x_i = i\left(\frac{4}{n}\right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(5 - \left(\frac{4i}{n}\right)\right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{20}{n} - \frac{16i}{n^2}\right).$$

Using summation properties,

$$s_n = \frac{20}{n} \sum_{i=1}^n 1 - \frac{16}{n^2} \sum_{i=1}^n i = \frac{20}{n} (n) - \frac{16}{n^2} \frac{n(n+1)}{2} = 20 - 8 \left(1 + \frac{1}{n}\right).$$

Therefore,

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[20 - 8 \left(1 + \frac{1}{n}\right)\right] = 20 - 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 20 - 8(1) = \boxed{12}.$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is a decreasing function, the absolute maximum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = x_{i-1} = (i-1)\left(\frac{4}{n}\right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(5 - \frac{4(i-1)}{n}\right) \frac{4}{n} = \sum_{i=1}^n \left(\frac{20}{n} + \frac{16}{n^2} - \frac{16i}{n^2}\right).$$

Using summation properties,

$$\begin{aligned} S_n &= \left(\frac{20}{n} + \frac{16}{n^2}\right) \sum_{i=1}^n 1 - \frac{16}{n^2} \sum_{i=1}^n i = \left(\frac{20}{n} + \frac{16}{n^2}\right) (n) - \frac{16}{n^2} \frac{n(n+1)}{2} \\ &= 20 + \frac{16}{n} - 8 \left(1 + \frac{1}{n}\right). \end{aligned}$$

Therefore

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[20 + \frac{16}{n} - 8 \left(1 + \frac{1}{n} \right) \right] \\
 &= 20 + 16 \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) - 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \\
 &= 20 + 16(0) - 8(1) = \boxed{12}.
 \end{aligned}$$

- (c) The work required in part (a) was easier than part (b). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i - 1$.

29. Let $f(x) = 4x^2$, and consider the area under the graph of f from $a = 0$ to $b = 2$. Partition the interval $[0, 2]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 2$$

and each subinterval is of length

$$\Delta x = \frac{2 - 0}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned}
 x_0 = 0, x_1 = \frac{2}{n}, x_2 = 2 \left(\frac{2}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{2}{n} \right), \\
 x_i = i \left(\frac{2}{n} \right), \dots, x_n = n \left(\frac{2}{n} \right) = 2.
 \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1) \left(\frac{2}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n 4 \left[(i-1) \left(\frac{2}{n} \right) \right]^2 \frac{2}{n} = \sum_{i=1}^n \left(\frac{32}{n^3} i^2 - \frac{64}{n^3} i + \frac{32}{n^3} \right).$$

Using summation properties,

$$\begin{aligned}
 s_n &= \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{64}{n^3} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n 1 \\
 &= \frac{32}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{64}{n^3} \frac{n(n+1)}{2} + \frac{32}{n^3} (n) \\
 &= \frac{16}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - 32 \left(\frac{1}{n} + \frac{1}{n^2} \right) + \frac{32}{n^2}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[\frac{16}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - 32 \left(\frac{1}{n} + \frac{1}{n^2} \right) + \frac{32}{n^2} \right] \\
 &= \frac{16}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - 32 \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) + \lim_{n \rightarrow \infty} \frac{32}{n^2} \\
 &= \frac{16}{3} (2) - 32 (0) + (0) = \boxed{\frac{32}{3}}.
 \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i \left(\frac{2}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n 4 \left[i \left(\frac{2}{n} \right) \right]^2 \frac{2}{n} = \sum_{i=1}^n \frac{32}{n^3} i^2.$$

Using summation properties,

$$S_n = \frac{32}{n^3} \sum_{i=1}^n i^2 = \frac{32}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{16}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right).$$

Therefore,

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{16}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\
 &= \frac{16}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) = \frac{16}{3} (2) = \boxed{\frac{32}{3}}.
 \end{aligned}$$

- (c) The work required in part (b) was easier than part (a). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i - 1$.

30. Let $f(x) = \frac{1}{2}x^2$, and consider the area under the graph of f from $a = 0$ to $b = 3$. Partition the interval $[0, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3 - 0}{n} = \frac{3}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned}
 x_0 &= 0, x_1 = \frac{3}{n}, x_2 = 2 \left(\frac{3}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{3}{n} \right), \\
 x_i &= i \left(\frac{3}{n} \right), \dots, x_n = n \left(\frac{3}{n} \right) = 3.
 \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = (i-1) \left(\frac{3}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \frac{1}{2} \left[(i-1) \left(\frac{3}{n} \right) \right]^2 \frac{3}{n} = \sum_{i=1}^n \left(\frac{27}{2n^3} i^2 - \frac{27}{n^3} i + \frac{27}{2n^3} \right).$$

Using summation properties,

$$\begin{aligned} s_n &= \frac{27}{2n^3} \sum_{i=1}^n i^2 - \frac{27}{n^3} \sum_{i=1}^n i + \frac{27}{2n^3} \sum_{i=1}^n 1 = \frac{27}{2n^3} \frac{n(n+1)(2n+1)}{6} - \frac{27}{n^3} \frac{n(n+1)}{2} + \frac{27}{2n^3} (n) \\ &= \frac{9}{4} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - \frac{27}{2} \left(\frac{1}{n} + \frac{1}{n^2} \right) + \frac{27}{2n^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[\frac{9}{4} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - \frac{27}{2} \left(\frac{1}{n} + \frac{1}{n^2} \right) + \frac{27}{2n^2} \right] \\ &= \frac{9}{4} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - \frac{27}{2} \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) + \frac{27}{2} \lim_{n \rightarrow \infty} \frac{1}{n^2} \\ &= \frac{9}{4} (2) - \frac{27}{2} (0) + \frac{27}{2} (0) = \boxed{\frac{9}{2}}. \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = i \left(\frac{3}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \frac{1}{2} \left[i \left(\frac{3}{n} \right) \right]^2 \frac{3}{n} = \sum_{i=1}^n \left(\frac{27i}{2n^3} i^2 \right).$$

Using summation properties,

$$S_n = \frac{27}{2n^3} \sum_{i=1}^n i^2 = \frac{27}{2n^3} \frac{n(n+1)(2n+1)}{6} = \frac{9}{4} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right).$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{9}{4} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\ &= \frac{9}{4} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) = \frac{9}{4} (2) = \boxed{\frac{9}{2}}. \end{aligned}$$

- (c) The work required in part (b) was easier than part (a). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i-1$.

31. Let $f(x) = 4 - x^2$, and consider the area under the graph of f from $a = 0$ to $b = 2$. Partition the interval $[0, 2]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 2$$

and each subinterval is of length

$$\Delta x = \frac{2 - 0}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 = 0, x_1 = \frac{2}{n}, x_2 = 2 \left(\frac{2}{n} \right), \dots, x_{i-1} = (i-1) \left(\frac{2}{n} \right), \\ x_i = i \left(\frac{2}{n} \right), \dots, x_n = n \left(\frac{2}{n} \right) = 2. \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is a decreasing function, the absolute minimum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$c_i = x_i = i \left(\frac{2}{n} \right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(4 - \left(\frac{2i}{n} \right)^2 \right) \frac{2}{n} = \sum_{i=1}^n \left(\frac{8}{n} - \frac{8}{n^3} i^2 \right).$$

Using summation properties,

$$s_n = \frac{8}{n} \sum_{i=1}^n 1 - \frac{8}{n^3} \sum_{i=1}^n i^2 = \frac{8}{n} (n) - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} = 8 - \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right).$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[8 - \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\ &= 8 - \frac{4}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) = 8 - \frac{4}{3} (1)(2) = \boxed{\frac{16}{3}}. \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is a decreasing function, the absolute maximum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = x_{i-1} = (i-1) \left(\frac{2}{n} \right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(4 - \left(\frac{2(i-1)}{n} \right)^2 \right) \frac{2}{n} = \sum_{i=1}^n \left(\frac{8}{n} - \frac{8}{n^3} + \frac{16}{n^3} i - \frac{8}{n^3} i^2 \right).$$

Using summation properties,

$$\begin{aligned}
 S_n &= \left(\frac{8}{n} - \frac{8}{n^3}\right) \sum_{i=1}^n 1 + \frac{16}{n^3} \sum_{i=1}^n i - \frac{8}{n^3} \sum_{i=1}^n i^2 \\
 &= \left(\frac{8}{n} - \frac{8}{n^3}\right)(n) + \frac{16}{n^3} \frac{n(n+1)}{2} - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \\
 &= 8 - \frac{8}{n^2} + 8\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[8 - \frac{8}{n^2} + 8\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right] \\
 &= 8 - 8 \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) + 8 \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{4}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
 &= 8 - 8(0) + 8(0) - \frac{4}{3}(2) = \boxed{\frac{16}{3}}.
 \end{aligned}$$

- (c) The work required in part (a) was easier than part (b). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i - 1$.
32. Let $f(x) = 12 - x^2$, and consider the area under the graph of f from $a = 0$ to $b = 3$. Partition the interval $[0, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$0 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3 - 0}{n} = \frac{3}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned}
 x_0 = 0, x_1 = \frac{3}{n}, x_2 = 2\left(\frac{3}{n}\right), \dots, x_{i-1} = (i-1)\left(\frac{3}{n}\right), \\
 x_i = i\left(\frac{3}{n}\right), \dots, x_n = n\left(\frac{3}{n}\right) = 3.
 \end{aligned}$$

- (a) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is a decreasing function, the absolute minimum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$c_i = x_i = i\left(\frac{3}{n}\right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n \left(12 - \left(\frac{3i}{n}\right)^2\right) \frac{3}{n} = \sum_{i=1}^n \left(\frac{36}{n} - \frac{27}{n^3} i^2\right).$$

Using summation properties,

$$s_n = \frac{36}{n} \sum_{i=1}^n 1 - \frac{27}{n^3} \sum_{i=1}^n i^2 = \frac{36}{n} (n) - \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} = 36 - \frac{9}{2} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[36 - \frac{9}{2} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right] \\ &= 36 - \frac{9}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = 36 - \frac{9}{2} (1)(2) = \boxed{27}. \end{aligned}$$

- (b) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is a decreasing function, the absolute maximum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = x_{i-1} = (i-1) \left(\frac{3}{n}\right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n \left(12 - \left(\frac{3(i-1)}{n} \right)^2 \right) \frac{3}{n} = \sum_{i=1}^n \left(\frac{36}{n} - \frac{27}{n^3} + \frac{54}{n^3} i - \frac{27}{n^3} i^2 \right).$$

Using summation properties,

$$\begin{aligned} S_n &= \left(\frac{36}{n} - \frac{27}{n^3} \right) \sum_{i=1}^n 1 + \frac{54}{n^3} \sum_{i=1}^n i - \frac{27}{n^3} \sum_{i=1}^n i^2 \\ &= \left(\frac{36}{n} - \frac{27}{n^3} \right) (n) + \frac{54}{n^3} \frac{n(n+1)}{2} - \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= 36 - \frac{27}{n^2} + 27 \left(\frac{1}{n} + \frac{1}{n^2} \right) - \frac{9}{2} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[36 - \frac{27}{n^2} + 27 \left(\frac{1}{n} + \frac{1}{n^2} \right) - \frac{9}{2} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right] \\ &= 36 - 27 \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \right) + 27 \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) - \frac{9}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\ &= 36 + 27(0) - \frac{9}{2} (2) = \boxed{27}. \end{aligned}$$

- (c) The work required in part (a) was easier than part (b). When working with the right endpoint of each subinterval, calculations are done on a simpler expression involving i instead of $i-1$.

Applications and Extensions

33. Let $f(x) = x+3$, and consider the area under the graph of f from $a = 1$ to $b = 3$. Partition the interval $[1, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$1 = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 = 1, x_1 = 1 + \frac{2}{n}, x_2 = 1 + 2\left(\frac{2}{n}\right), \dots, x_{i-1} = 1 + (i-1)\left(\frac{2}{n}\right), \\ x_i = 1 + i\left(\frac{2}{n}\right), \dots, x_n = 1 + n\left(\frac{2}{n}\right) = 3. \end{aligned}$$

Because f is an increasing function on the interval $[1, 3]$, the function value at the right endpoint of each subinterval is the absolute maximum value on that subinterval. Thus,

$$C_i = x_i = 1 + \frac{2i}{n},$$

and

$$\begin{aligned} S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right) + 3\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(\frac{8}{n} + \frac{4i}{n^2}\right) \\ &= \frac{8}{n} \sum_{i=1}^n 1 + \frac{4}{n^2} \sum_{i=1}^n i = \frac{8}{n}(n) + \frac{4}{n^2} \frac{n(n+1)}{2} = 8 + 2\left(1 + \frac{1}{n}\right). \end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[8 + 2\left(1 + \frac{1}{n}\right)\right] = \boxed{10}.$$

34. Let $f(x) = 3 - x$, and consider the area under the graph of f from $a = 1$ to $b = 3$. Partition the interval $[1, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$1 = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 = 1, x_1 = 1 + \frac{2}{n}, x_2 = 1 + 2\left(\frac{2}{n}\right), \dots, x_{i-1} = 1 + (i-1)\left(\frac{2}{n}\right), \\ x_i = 1 + i\left(\frac{2}{n}\right), \dots, x_n = 1 + n\left(\frac{2}{n}\right) = 3. \end{aligned}$$

Because f is a decreasing function on the interval $[1, 3]$, the function value at the right endpoint of each subinterval is the absolute minimum value on that subinterval. Thus,

$$c_i = x_i = 1 + \frac{2i}{n},$$

and

$$\begin{aligned} s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[3 - \left(1 + \frac{2i}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(\frac{4}{n} - \frac{4i}{n^2}\right) \\ &= \frac{4}{n} \sum_{i=1}^n 1 - \frac{4}{n^2} \sum_{i=1}^n i = \frac{4}{n} (n) - \frac{4}{n^2} \frac{n(n+1)}{2} = 4 - 2 \left(1 + \frac{1}{n}\right). \end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[4 - 2 \left(1 + \frac{1}{n}\right)\right] = \boxed{2}.$$

35. Let $f(x) = 2x + 5$, and consider the area under the graph of f from $a = -1$ to $b = 2$. Partition the interval $[-1, 2]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$-1 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 2$$

and each subinterval is of length

$$\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 &= -1, x_1 = -1 + \frac{3}{n}, x_2 = -1 + 2 \left(\frac{3}{n}\right), \dots, x_{i-1} = -1 + (i-1) \left(\frac{3}{n}\right), \\ x_i &= -1 + i \left(\frac{3}{n}\right), \dots, x_n = -1 + n \left(\frac{3}{n}\right) = 2. \end{aligned}$$

Because f is an increasing function on the interval $[-1, 2]$, the function value at the right endpoint of each subinterval is the absolute maximum value on that subinterval. Thus,

$$C_i = x_i = -1 + \frac{3i}{n},$$

and

$$\begin{aligned} S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(-1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) = \sum_{i=1}^n \left[2 \left(-1 + \frac{3i}{n}\right) + 5\right] \left(\frac{3}{n}\right) \\ &= \sum_{i=1}^n \left(\frac{9}{n} + \frac{18i}{n^2}\right) = \frac{9}{n} \sum_{i=1}^n 1 + \frac{18}{n^2} \sum_{i=1}^n i = \frac{9}{n} (n) + \frac{18}{n^2} \frac{n(n+1)}{2} = 9 + 9 \left(1 + \frac{1}{n}\right). \end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[9 + 9 \left(1 + \frac{1}{n}\right)\right] = \boxed{18}.$$

36. Let $f(x) = 2 - 3x$, and consider the area under the graph of f from $a = -2$ to $b = 0$. Partition the interval $[-2, 0]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$-2 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 0$$

and each subinterval is of length

$$\Delta x = \frac{0 - (-2)}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 &= -2, x_1 = -2 + \frac{2}{n}, x_2 = -2 + 2\left(\frac{2}{n}\right), \dots, x_{i-1} = -2 + (i-1)\left(\frac{2}{n}\right), \\ x_i &= -2 + i\left(\frac{2}{n}\right), \dots, x_n = -2 + n\left(\frac{2}{n}\right) = 0. \end{aligned}$$

Because f is a decreasing function on the interval $[-2, 0]$, the function value at the right endpoint of each subinterval is the absolute minimum value on that subinterval. Thus,

$$c_i = x_i = -2 + \frac{2i}{n},$$

and

$$\begin{aligned} s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(-2 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[2 - 3\left(-2 + \frac{2i}{n}\right)\right] \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left(\frac{16}{n} - \frac{12i}{n^2}\right) = \frac{16}{n} \sum_{i=1}^n 1 - \frac{12}{n^2} \sum_{i=1}^n i = \frac{16}{n} (n) - \frac{12}{n^2} \frac{n(n+1)}{2} = 16 - 6\left(1 + \frac{1}{n}\right). \end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[16 - 6\left(1 + \frac{1}{n}\right)\right] = \boxed{10}.$$

37. Let $f(x) = 2x^2 + 1$, and consider the area under the graph of f from $a = 1$ to $b = 3$. Partition the interval $[1, 3]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$1 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 3$$

and each subinterval is of length

$$\Delta x = \frac{3 - 1}{n} = \frac{2}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned} x_0 &= 1, x_1 = 1 + \frac{2}{n}, x_2 = 1 + 2\left(\frac{2}{n}\right), \dots, x_{i-1} = 1 + (i-1)\left(\frac{2}{n}\right), \\ x_i &= 1 + i\left(\frac{2}{n}\right), \dots, x_n = 1 + n\left(\frac{2}{n}\right) = 3. \end{aligned}$$

Because f is an increasing function on the interval $[1, 3]$, the function value at the right endpoint of each subinterval is the absolute maximum value on that subinterval. Thus,

$$C_i = x_i = 1 + \frac{2i}{n},$$

and

$$\begin{aligned}
S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[2 \left(1 + \frac{2i}{n}\right)^2 + 1\right] \left(\frac{2}{n}\right) \\
&= \sum_{i=1}^n \left(\frac{6}{n} + \frac{16}{n^2}i + \frac{16}{n^3}i^2\right) = \frac{6}{n} \sum_{i=1}^n 1 + \frac{16}{n^2} \sum_{i=1}^n i + \frac{16}{n^3} \sum_{i=1}^n i^2 \\
&= \frac{6}{n}(n) + \frac{16}{n^2} \frac{n(n+1)}{2} + \frac{16}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= 6 + 8 \left(1 + \frac{1}{n}\right) + \frac{8}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).
\end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[6 + 8 \left(1 + \frac{1}{n}\right) + \frac{8}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)\right] = \boxed{\frac{58}{3}}.$$

38. Let $f(x) = 4 - x^2$, and consider the area under the graph of f from $a = 1$ to $b = 2$. Partition the interval $[1, 2]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$1 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 2$$

and each subinterval is of length

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}.$$

The coordinates of the endpoints of each subinterval are

$$\begin{aligned}
x_0 &= 1, x_1 = 1 + \frac{1}{n}, x_2 = 1 + 2 \left(\frac{1}{n}\right), \dots, x_{i-1} = 1 + (i-1) \left(\frac{1}{n}\right), \\
x_i &= 1 + i \left(\frac{1}{n}\right), \dots, x_n = 1 + n \left(\frac{1}{n}\right) = 2.
\end{aligned}$$

Because f is a decreasing function on the interval $[1, 2]$, the function value at the right endpoint of each subinterval is the absolute minimum value on that subinterval. Thus,

$$c_i = x_i = 1 + \frac{i}{n},$$

and

$$\begin{aligned}
s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[4 - \left(1 + \frac{i}{n}\right)^2\right] \left(\frac{1}{n}\right) \\
&= \sum_{i=1}^n \left(\frac{3}{n} - \frac{2}{n^2}i - \frac{1}{n^3}i^2\right) = \frac{3}{n} \sum_{i=1}^n 1 - \frac{2}{n^2} \sum_{i=1}^n i - \frac{1}{n^3} \sum_{i=1}^n i^2 \\
&= \frac{3}{n}(n) - \frac{2}{n^2} \frac{n(n+1)}{2} - \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= 3 - \left(1 + \frac{1}{n}\right) - \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).
\end{aligned}$$

Finally,

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[3 - \left(1 + \frac{1}{n}\right) - \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)\right] = \boxed{\frac{5}{3}}.$$

39. Let $f(x) = xe^x$, and consider the area under the graph of f from $a = 0$ to $b = 8$. With $n = 20$,

$$\Delta x = \frac{8-0}{20} = \frac{2}{5} \quad \text{and} \quad x_i = 0 + i\Delta x = \frac{2i}{5}.$$

Because f is increasing on the interval $[0, 8]$, the absolute maximum on any subinterval occurs at the right endpoint of the subinterval; that is,

$$C_i = x_i = \frac{2i}{5}.$$

Therefore,

$$\begin{aligned} A &\approx S_{20} = \sum_{i=1}^{20} f(C_i) \Delta x = \sum_{i=1}^{20} f\left(\frac{2i}{5}\right) \left(\frac{2}{5}\right) = \frac{2}{5} \sum_{i=1}^{20} \left(\frac{2i}{5} e^{\frac{2i}{5}}\right) \\ &= \frac{4}{25} \sum_{i=1}^{20} i e^{\frac{2i}{5}} \approx \boxed{25990}. \end{aligned}$$

40. Let $f(x) = \ln x$, and consider the area under the graph of f from $a = 1$ to $b = 3$. With $n = 20$,

$$\Delta x = \frac{3-1}{20} = \frac{1}{10} \quad \text{and} \quad x_i = 1 + i\Delta x = 1 + \frac{i}{10}.$$

Because f is increasing on the interval $[1, 3]$, the absolute maximum on any subinterval occurs at the right endpoint of the subinterval; that is,

$$C_i = x_i = 1 + \frac{i}{10}.$$

Therefore,

$$A \approx S_{20} = \sum_{i=1}^{20} f(C_i) \Delta x = \sum_{i=1}^{20} f\left(1 + \frac{i}{10}\right) \left(\frac{1}{10}\right) = \frac{1}{10} \sum_{i=1}^{20} \ln\left(1 + \frac{i}{10}\right) \approx \boxed{1.350}.$$

41. Let $f(x) = \frac{1}{x}$, and consider the area under the graph of f from $a = 1$ to $b = 5$. With $n = 20$,

$$\Delta x = \frac{5-1}{20} = \frac{1}{5} \quad \text{and} \quad x_i = 1 + i\Delta x = 1 + \frac{i}{5}.$$

Because f is decreasing on the interval $[1, 5]$, the absolute maximum on any subinterval occurs at the left endpoint of the subinterval; that is,

$$C_i = x_{i-1} = 1 + \frac{i-1}{5}.$$

Therefore,

$$A \approx S_{20} = \sum_{i=1}^{20} f(C_i) \Delta x = \sum_{i=1}^{20} f\left(1 + \frac{i-1}{5}\right) \left(\frac{1}{5}\right) = \frac{1}{5} \sum_{i=1}^{20} \frac{1}{1 + \frac{i-1}{5}} \approx \boxed{1.693}.$$

42. Let $f(x) = \frac{1}{x^2}$, and consider the area under the graph of f from $a = 2$ to $b = 6$. With $n = 20$,

$$\Delta x = \frac{6-2}{20} = \frac{1}{5} \quad \text{and} \quad x_i = 2 + i\Delta x = 2 + \frac{i}{5}.$$

Because f is decreasing on the interval $[2, 6]$, the absolute maximum on any subinterval occurs at the left endpoint of the subinterval; that is,

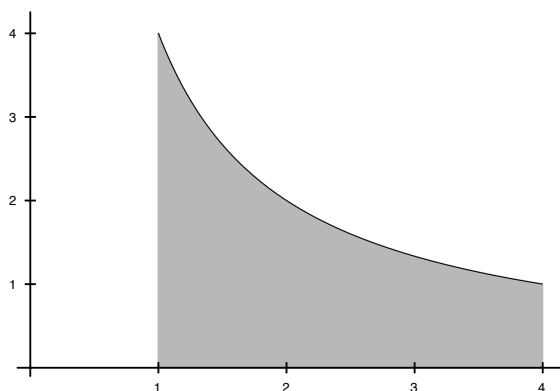
$$C_i = x_{i-1} = 2 + \frac{i-1}{5}.$$

Therefore,

$$A \approx S_{20} = \sum_{i=1}^{20} f(C_i) \Delta x = \sum_{i=1}^{20} f\left(2 + \frac{i-1}{5}\right) \left(\frac{1}{5}\right) = \frac{1}{5} \sum_{i=1}^{20} \frac{1}{\left(2 + \frac{i-1}{5}\right)^2} \approx \boxed{0.356}.$$

43. Let $f(x) = \frac{4}{x}$, and consider the area under the graph of f from 1 to 4.

(a) The graph of f from 1 to 4 is shown below. The area below the graph is shaded.



(b) Partition the interval $[1, 4]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

where

$$1 = x_0 < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < x_n = 4$$

and each subinterval is of length

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}.$$

The coordinates of the endpoints of each subinterval are

$$x_0 = 1, x_1 = 1 + \frac{3}{n}, x_2 = 1 + 2\left(\frac{3}{n}\right), \dots, x_{i-1} = 1 + (i-1)\left(\frac{3}{n}\right),$$

$$x_i = 1 + i\left(\frac{3}{n}\right), \dots, x_n = 1 + n\left(\frac{3}{n}\right) = 4.$$

(c) To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is a decreasing function, the absolute minimum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$c_i = x_i = 1 + \left(\frac{3i}{n}\right),$$

and

$$s_n = \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right) \Delta x = \sum_{i=1}^n \frac{4}{1 + \frac{3i}{n}} \cdot \frac{3}{n}.$$

- (d) To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is a decreasing function, the absolute maximum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$C_i = x_{i-1} = 1 + \left(\frac{3(i-1)}{n}\right),$$

and

$$S_n = \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(1 + \frac{3(i-1)}{n}\right) \Delta x = \sum_{i=1}^n \frac{4}{1 + \frac{3(i-1)}{n}} \cdot \frac{3}{n}.$$

- (e) Using the computer algebra system *Mathematica* to evaluate the sums from parts (c) and (d), we find

n	5	10	50	100
s_n	4.754	5.123	5.456	5.500
S_n	6.554	6.023	5.636	5.590

- (f) Because $s_n \leq A \leq S_n$ for all n , it follows that

$$5.500 \approx s_{100} \leq A \leq S_{100} \approx 5.590.$$

Challenge Problems

44. Let $f(x) = x$, and consider the area under the graph of f from $a \geq 0$ to b . With n subintervals each of length

$$\Delta x = \frac{b-a}{n},$$

it follows that

$$x_i = a + i\Delta x = a + i\frac{b-a}{n}.$$

To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = a + (i-1)\left(\frac{b-a}{n}\right),$$

and

$$\begin{aligned} s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(a + (i-1)\frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) \\ &= \sum_{i=1}^n \left(a + (i-1)\frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) = a\left(\frac{b-a}{n}\right) \sum_{i=1}^n 1 + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^n (i-1) \end{aligned}$$

By shifting the index of summation,

$$\sum_{i=1}^n (i-1) = \sum_{j=0}^{n-1} j = \sum_{j=1}^{n-1} j = \frac{(n-1)n}{2}.$$

Therefore,

$$\begin{aligned}
 s_n &= a \left(\frac{b-a}{n} \right) (n) + \left(\frac{b-a}{n} \right)^2 \frac{(n-1)n}{2} \\
 &= a(b-a) + \frac{(b-a)^2}{2} \left(1 - \frac{1}{n} \right) = a(b-a) + \frac{(b-a)^2}{2} - \frac{(b-a)^2}{2n} \\
 &= (b-a) \left(a + \frac{b-a}{2} \right) - \frac{(b-a)^2}{2n} = (b-a) \left(\frac{b+a}{2} \right) - \frac{(b-a)^2}{2n} \\
 &= \frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n}.
 \end{aligned}$$

To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = a + i \left(\frac{b-a}{n} \right),$$

and

$$\begin{aligned}
 S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f \left(a + i \frac{b-a}{n} \right) \left(\frac{b-a}{n} \right) = \sum_{i=1}^n \left(a + i \frac{b-a}{n} \right) \left(\frac{b-a}{n} \right) \\
 &= a \left(\frac{b-a}{n} \right) \sum_{i=1}^n 1 + \left(\frac{b-a}{n} \right)^2 \sum_{i=1}^n i = a \left(\frac{b-a}{n} \right) (n) + \left(\frac{b-a}{n} \right)^2 \frac{n(n+1)}{2} \\
 &= a(b-a) + \frac{(b-a)^2}{2} \left(1 + \frac{1}{n} \right) = a(b-a) + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n} \\
 &= (b-a) \left(a + \frac{b-a}{2} \right) + \frac{(b-a)^2}{2n} = (b-a) \left(\frac{b+a}{2} \right) + \frac{(b-a)^2}{2n} \\
 &= \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n}.
 \end{aligned}$$

Because $b > a$, $(b-a)^2 > 0$ and

$$s_n = \frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n} < \frac{b^2 - a^2}{2} < \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n} = S_n.$$

45. Let $f(x) = x^2$, and consider the area under the graph of f from $a \geq 0$ to b . With n subintervals each of length

$$\Delta x = \frac{b-a}{n},$$

it follows that

$$x_i = a + i\Delta x = a + i \frac{b-a}{n}.$$

To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because $a \geq 0$, f is an increasing function on the interval $[a, b]$, and the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = a + (i-1) \left(\frac{b-a}{n} \right),$$

and

$$\begin{aligned} s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(a + (i-1) \frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) = \sum_{i=1}^n \left(a + (i-1) \frac{b-a}{n}\right)^2 \left(\frac{b-a}{n}\right) \\ &= a^2 \left(\frac{b-a}{n}\right) \sum_{i=1}^n 1 + 2a \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^n (i-1) + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^n (i-1)^2. \end{aligned}$$

By shifting the index of summation,

$$\sum_{i=1}^n (i-1) = \sum_{j=0}^{n-1} j = \sum_{j=1}^{n-1} j = \frac{(n-1)n}{2}$$

and

$$\sum_{i=1}^n (i-1)^2 = \sum_{j=0}^{n-1} j^2 = \sum_{j=1}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6}.$$

Therefore,

$$\begin{aligned} s_n &= a^2 \left(\frac{b-a}{n}\right) (n) + 2a \left(\frac{b-a}{n}\right)^2 \frac{n(n-1)}{2} + \left(\frac{b-a}{n}\right)^3 \frac{n(n-1)(2n-1)}{6} \\ &= a^2 (b-a) + a (b-a)^2 \left(1 - \frac{1}{n}\right) + \frac{(b-a)^3}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) \\ &= a^2 (b-a) + a (b-a)^2 + \frac{(b-a)^3}{3} - \frac{a (b-a)^2}{n} - \frac{(b-a)^3}{2n} + \frac{(b-a)^3}{6n^2} \\ &= \frac{b-a}{3} \left(3a^2 + 3a(b-a) + (b-a)^2\right) - \frac{(b-a)^2}{2n} (2a + b - a) + \frac{(b-a)^3}{6n^2} \\ &= \frac{b-a}{3} (a^2 + ab + b^2) - \frac{(b-a)^2}{2n} (a + b) + \frac{(b-a)^3}{6n^2} \\ &= \frac{b^3 - a^3}{3} - \frac{(b-a)^2}{2n} (a + b) + \frac{(b-a)^3}{6n^2}. \end{aligned}$$

To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because $a \geq 0$, f is an increasing function on the interval $[a, b]$, and the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = a + i \left(\frac{b-a}{n}\right),$$

and

$$\begin{aligned}
S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(a + i \frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) = \sum_{i=1}^n \left(a + i \frac{b-a}{n}\right)^2 \left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^n a^2 \left(\frac{b-a}{n}\right) + 2a \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^n i + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^n i^2 \\
&= a^2 \left(\frac{b-a}{n}\right) (n) + 2a \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \\
&= a^2(b-a) + a(b-a)^2 \left(1 + \frac{1}{n}\right) + \frac{(b-a)^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \\
&= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} + \frac{a(b-a)^2}{n} + \frac{(b-a)^3}{2n} + \frac{(b-a)^3}{6n^2} \\
&= \frac{b-a}{3} \left(3a^2 + 3a(b-a) + (b-a)^2\right) + \frac{(b-a)^2}{2n} (2a + b - a) + \frac{(b-a)^3}{6n^2} \\
&= \frac{b-a}{3} (a^2 + ab + b^2) + \frac{(b-a)^2}{2n} (a+b) + \frac{(b-a)^3}{6n^2} \\
&= \frac{b^3 - a^3}{3} + \frac{(b-a)^2}{2n} (a+b) + \frac{(b-a)^3}{6n^2}.
\end{aligned}$$

Because $a \geq 0$ and $b > a$,

$$\begin{aligned}
-\frac{(b-a)^2}{2n} (a+b) + \frac{(b-a)^3}{6n^2} &= -\frac{(b-a)^2}{2n} \left[(a+b) - \frac{b-a}{3n} \right] \\
&= -\frac{(b-a)^2}{2n} \left[a \left(1 + \frac{1}{3n}\right) + b \left(1 - \frac{1}{3n}\right) \right] < 0,
\end{aligned}$$

so

$$s_n = \frac{b^3 - a^3}{3} - \frac{(b-a)^2}{2n} (a+b) + \frac{(b-a)^3}{6n^2} < \frac{b^3 - a^3}{3} < \frac{b^3 - a^3}{3} + \frac{(b-a)^2}{2n} (a+b) + \frac{(b-a)^3}{6n^2} = S_n.$$

46. Let $f(x) = \frac{H}{B}x$ on the interval $[0, B]$. The region under the graph of f is then a right triangle with base B and height H . With n subintervals each of length

$$\Delta x = \frac{B-0}{n} = \frac{B}{n},$$

it follows that

$$x_i = a + i\Delta x = \frac{Bi}{n}.$$

To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = \frac{B(i-1)}{n},$$

$$\begin{aligned}
s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(\frac{B(i-1)}{n}\right) \left(\frac{B}{n}\right) = \sum_{i=1}^n \left(\frac{H}{B} \frac{B(i-1)}{n}\right) \left(\frac{B}{n}\right) \\
&= \frac{HB}{n^2} \sum_{i=1}^n (i-1) = \frac{HB}{n^2} \sum_{i=1}^n i - \frac{HB}{n^2} (n) = \frac{HB}{n^2} \frac{n(n+1)}{2} - \frac{HB}{n} \\
&= \frac{1}{2} HB \left(1 + \frac{1}{n}\right) - \frac{HB}{n},
\end{aligned}$$

and

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2}HB \left(1 + \frac{1}{n} \right) - \frac{HB}{n} \right] = \boxed{\frac{1}{2}HB}.$$

To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = \frac{Bi}{n},$$

$$\begin{aligned} S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(\frac{Bi}{n}\right) \left(\frac{B}{n}\right) = \sum_{i=1}^n \left(\frac{H}{B} \frac{Bi}{n}\right) \left(\frac{B}{n}\right) \\ &= \frac{HB}{n^2} \sum_{i=1}^n i = \frac{HB}{n^2} \sum_{i=1}^n i = \frac{HB}{n^2} \frac{n(n+1)}{2} = \frac{1}{2}HB \left(1 + \frac{1}{n} \right), \end{aligned}$$

and

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2}HB \left(1 + \frac{1}{n} \right) \right] = \boxed{\frac{1}{2}HB}.$$

47. Let $f(x) = H_1 + \frac{H_2 - H_1}{B}x$ on the interval $[0, B]$. The region under the graph of f is then a trapezoid of heights H_1 and H_2 and base B . With n subintervals each of length

$$\Delta x = \frac{B - 0}{n} = \frac{B}{n},$$

it follows that

$$x_i = a + i\Delta x = \frac{Bi}{n}.$$

To approximate the area under the graph of f using lower sums, we must find the absolute minimum value of f on each subinterval. Because f is an increasing function, the absolute minimum on each subinterval occurs at the left endpoint of the subinterval. Thus,

$$c_i = x_{i-1} = \frac{B(i-1)}{n},$$

$$\begin{aligned} s_n &= \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^n f\left(\frac{B(i-1)}{n}\right) \left(\frac{B}{n}\right) = \sum_{i=1}^n \left(H_1 + \frac{H_2 - H_1}{B} \frac{B(i-1)}{n} \right) \left(\frac{B}{n}\right) \\ &= \frac{H_1 B}{n} \sum_{i=1}^n 1 + \frac{(H_2 - H_1)B}{n^2} \sum_{i=1}^n (i-1) \\ &= \frac{H_1 B}{n} (n) + \frac{(H_2 - H_1)B}{n^2} \sum_{i=1}^n i - \frac{(H_2 - H_1)B}{n^2} (n) \\ &= H_1 B + \frac{(H_2 - H_1)B}{n^2} \frac{n(n+1)}{2} - \frac{(H_2 - H_1)B}{n} \\ &= H_1 B + \frac{1}{2}(H_2 - H_1)B \left(1 + \frac{1}{n} \right) - \frac{(H_2 - H_1)B}{n}, \end{aligned}$$

and

$$A = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[H_1 B + \frac{1}{2}(H_2 - H_1)B \left(1 + \frac{1}{n} \right) - \frac{(H_2 - H_1)B}{n} \right] = \boxed{\frac{1}{2}(H_2 + H_1)B}.$$

To approximate the area under the graph of f using upper sums, we must find the absolute maximum value of f on each subinterval. Because f is an increasing function, the absolute maximum on each subinterval occurs at the right endpoint of the subinterval. Thus,

$$C_i = x_i = \frac{Bi}{n},$$

$$\begin{aligned} S_n &= \sum_{i=1}^n f(C_i) \Delta x = \sum_{i=1}^n f\left(\frac{Bi}{n}\right) \left(\frac{B}{n}\right) = \sum_{i=1}^n \left(H_1 + \frac{H_2 - H_1}{B} \frac{Bi}{n}\right) \left(\frac{B}{n}\right) \\ &= \frac{H_1 B}{n} \sum_{i=1}^n 1 + \frac{(H_2 - H_1) B}{n^2} \sum_{i=1}^n i \\ &= \frac{H_1 B}{n} (n) + \frac{(H_2 - H_1) B}{n^2} \frac{n(n+1)}{2} \\ &= H_1 B + \frac{1}{2} (H_2 - H_1) B \left(1 + \frac{1}{n}\right), \end{aligned}$$

and

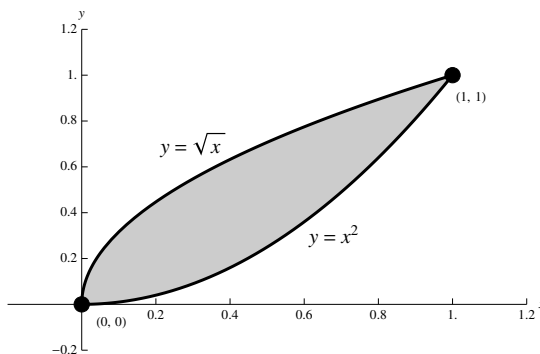
$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[H_1 B + \frac{1}{2} (H_2 - H_1) B \left(1 + \frac{1}{n}\right) \right] = \boxed{\frac{1}{2} (H_2 + H_1) B}.$$

Chapter 6: Applications of the Integral

6.1: Area Between Graphs

Concepts and Vocabulary

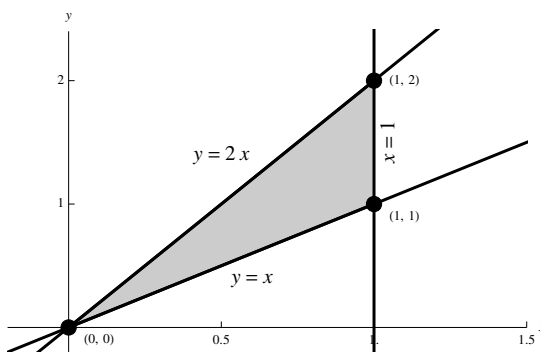
1. The two graphs meet at $x = 0$ and at $x = 1$, and in the interval $[0, 1]$, we have $\sqrt{x} \geq x^2$, so the integral is $\int_0^1 (\sqrt{x} - x^2) dx$. See the figure below:



2. Partitioning the y -axis, the two graphs meet at $y = -1$ and again at $y = 1$; when $-1 \leq y \leq 1$, we have $y^2 \leq 1$, so the integral is $\int_{-1}^1 (1 - y^2) dy$.

Skill Building

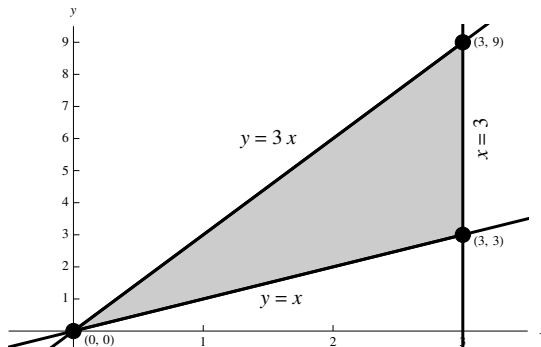
3. The region is shown below:



Partitioning along the x -axis, we have $x \leq 2x$, so the area is

$$\int_0^1 (2x - x) dx = \int_0^1 x dx = \left[\frac{1}{2}x^2 \right]_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}.$$

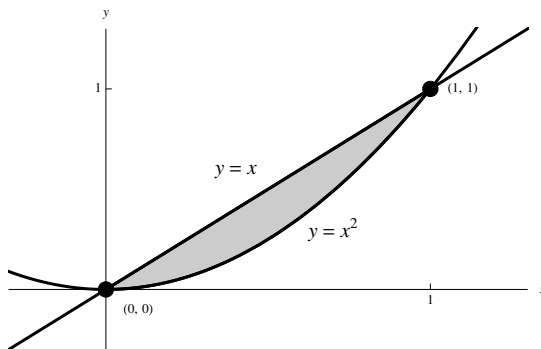
4. The region is shown below:



Partitioning along the x -axis, we have $x \leq 3x$, so the area is

$$\int_0^3 (3x - x) dx = \int_0^3 2x dx = [x^2]_0^3 = 9 - 0 = \boxed{9}.$$

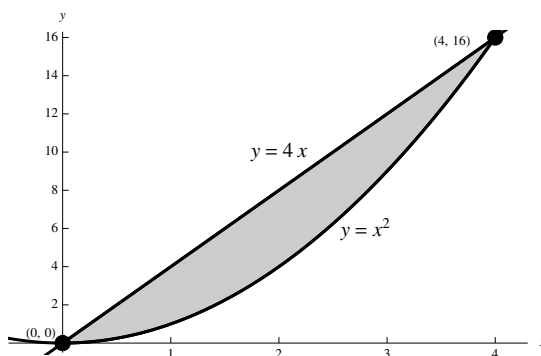
5. The region is shown below:



The two curves intersect when $x^2 = x$, so when $x = 0$ and $x = 1$. Partitioning along the x -axis, we have $x^2 \leq x$, so the area is

$$\int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{1}{6}}.$$

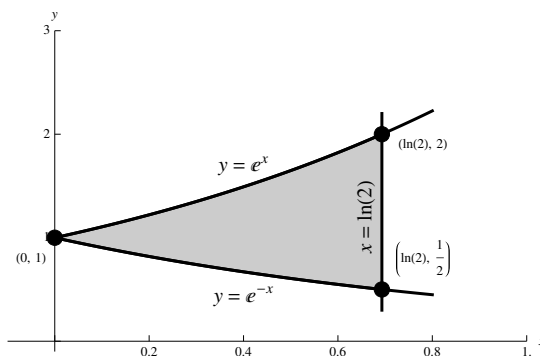
6. The region is shown below:



The two curves intersect when $x^2 = 4x$, so when $x = 0$ and $x = 4$. Partitioning along the x -axis, we have $x^2 \leq 4x$, so the area is

$$\int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 = \left(32 - \frac{64}{3} \right) - (0 - 0) = \boxed{\frac{32}{3}}.$$

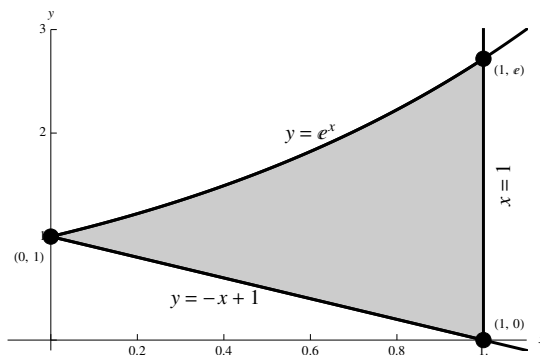
7. The region is shown below:



Partitioning along the x -axis, we have $e^{-x} \leq e^x$, so the area is

$$\int_0^{\ln 2} (e^x - e^{-x}) dx = [e^x + e^{-x}]_0^{\ln 2} = (e^{\ln 2} + e^{-\ln 2}) - (e^0 + e^0) = \left(2 + \frac{1}{2} \right) - (1 + 1) = \boxed{\frac{1}{2}}.$$

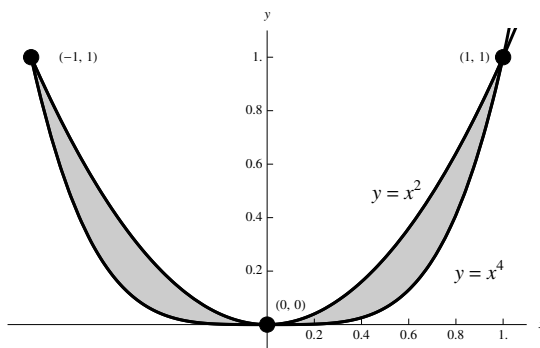
8. The region is shown below:



Partitioning along the x -axis, we have $-x + 1 \leq e^x$, so the area is

$$\begin{aligned} \int_0^1 (e^x - (-x + 1)) dx &= \int_0^1 (e^x + x - 1) dx = \left[e^x + \frac{1}{2}x^2 - x \right]_0^1 \\ &= \left(e^1 + \frac{1}{2} - 1 \right) - (e^0 + 0 - 0) = \boxed{e - \frac{3}{2}}. \end{aligned}$$

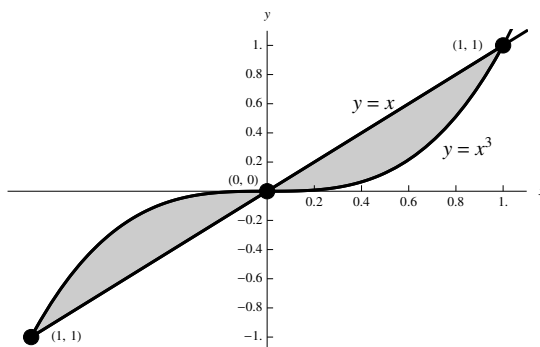
9. The region is shown below:



The two curves intersect when $x^2 = x^4$, so when $x = -1$, $x = 0$, and $x = 1$. Partitioning along the x -axis, we have $x^4 \leq x^2$, so the area is

$$\int_{-1}^1 (x^2 - x^4) dx = \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^1 = \left(\frac{1}{3} - \frac{1}{5} \right) - \left(-\frac{1}{3} + \frac{1}{5} \right) = \boxed{\frac{4}{15}}.$$

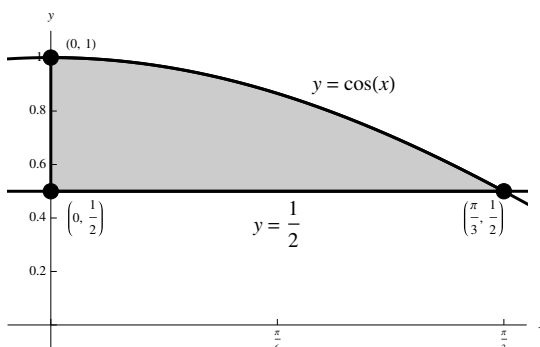
10. The region is shown below:



The two curves intersect when $x = x^3$, so when $x = -1$, $x = 0$, and $x = 1$. Since the region between $x = -1$ and $x = 0$ is below the x -axis, we compute the area from $x = 0$ to $x = 1$ and double it so that we count that area as positive. Partitioning along the x -axis, we have $x^3 \leq x$, so the area is

$$2 \int_0^1 (x - x^3) dx = 2 \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 = 2 \left(\left(\frac{1}{2} - \frac{1}{4} \right) - (0 - 0) \right) = \boxed{\frac{1}{2}}.$$

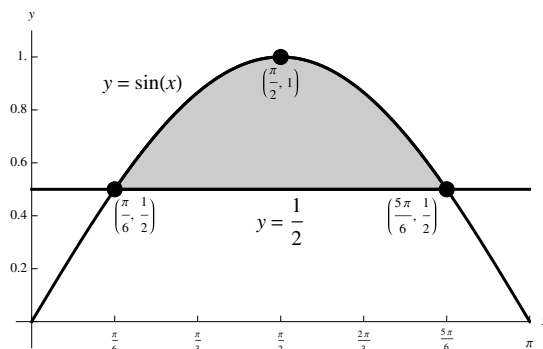
11. The region is shown below:



Partitioning along the x -axis, we have $\frac{1}{2} \leq \cos x$, so the area is

$$\int_0^{\pi/3} \left(\cos x - \frac{1}{2} \right) dx = \left[\sin x - \frac{1}{2}x \right]_0^{\pi/3} = \left(\sin \frac{\pi}{3} - \frac{1}{2} \cdot \frac{\pi}{3} \right) - (\sin 0 - 0) = \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6}}.$$

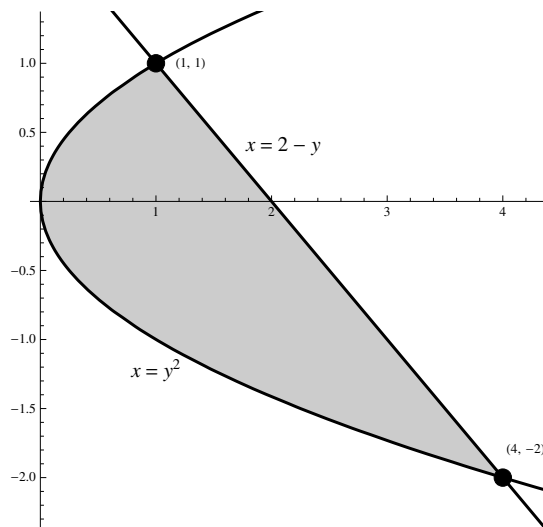
12. The region is shown below:



The two curves intersect when $\sin x = \frac{1}{2}$, so at $(\frac{\pi}{6}, \frac{1}{2})$ and $(\frac{5\pi}{6}, \frac{1}{2})$. Partitioning along the x -axis, we have $\frac{1}{2} \leq \sin x$, so the area is

$$\begin{aligned} \int_{\pi/6}^{5\pi/6} \left(\sin x - \frac{1}{2} \right) dx &= \left[-\cos x - \frac{1}{2}x \right]_{\pi/6}^{5\pi/6} = \left(-\cos \frac{5\pi}{6} - \frac{1}{2} \cdot \frac{5\pi}{6} \right) - \left(-\cos \frac{\pi}{6} - \frac{1}{2} \cdot \frac{\pi}{6} \right) \\ &= \left(\frac{\sqrt{3}}{2} - \frac{5\pi}{12} \right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{12} \right) = \boxed{\sqrt{3} - \frac{\pi}{3}}. \end{aligned}$$

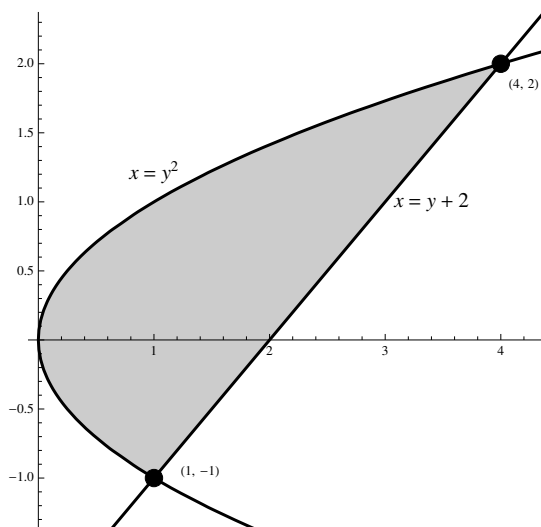
13. The region is shown below:



The two curves intersect when $2 - y = y^2$, so when $y^2 + y - 2 = (y + 2)(y - 1) = 0$; this happens at the points $(1, 1)$ and $(4, -2)$. Partitioning along the y -axis, we have $y^2 \leq 2 - y$, so the area is

$$\int_{-2}^1 (2 - y - y^2) dy = \left[2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-2}^1 = \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) = \boxed{\frac{9}{2}}.$$

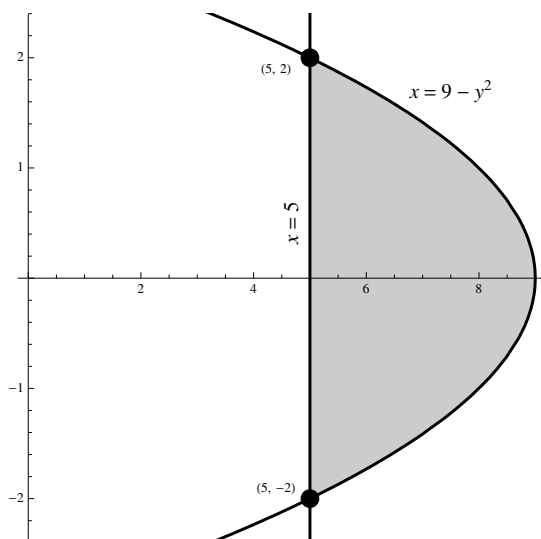
14. The region is shown below:



The two curves intersect when $y + 2 = y^2$, so when $y^2 - y - 2 = (y - 2)(y + 1) = 0$; this happens at the points $(4, 2)$ and $(1, -1)$. Partitioning along the y -axis, we have $y^2 \leq y + 2$, so the area is

$$\int_{-1}^2 (y + 2 - y^2) dy = \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right]_{-1}^2 = \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \boxed{\frac{9}{2}}.$$

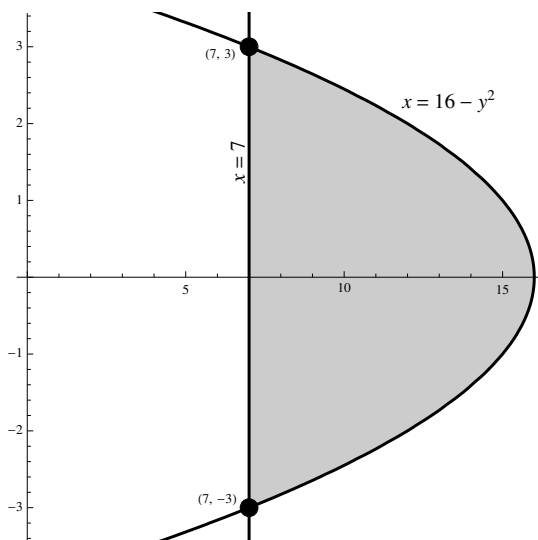
15. The region is shown below:



The two curves intersect where $9 - y^2 = 5$, so when $y = -2$ and $y = 2$. Partitioning along the y -axis, we have $5 \leq 9 - y^2$, so the area is

$$\int_{-2}^2 (9 - y^2 - 5) dy = \int_{-2}^2 (4 - y^2) dy = \left[4y - \frac{1}{3}y^3 \right]_{-2}^2 = \left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) = \boxed{\frac{32}{3}}.$$

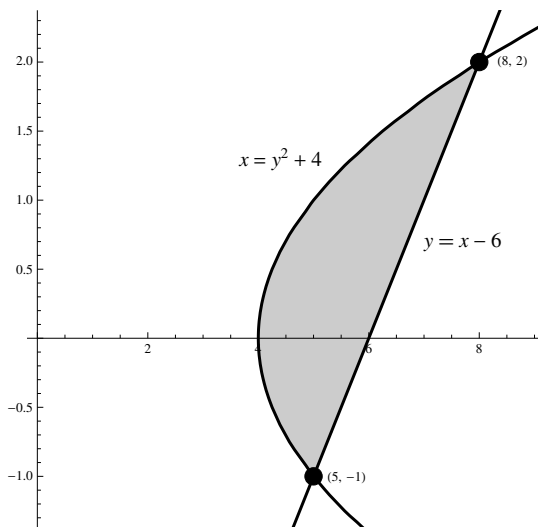
16. The region is shown below:



The two curves intersect where $16 - y^2 = 7$, so when $y = -3$ and $y = 3$. Partitioning along the y -axis, we have $7 \leq 16 - y^2$, so the area is

$$\int_{-3}^3 (16 - y^2 - 7) dy = \int_{-3}^3 (9 - y^2) dy = \left[9y - \frac{1}{3}y^3 \right]_{-3}^3 = (27 - 9) - (-27 + 9) = \boxed{36}.$$

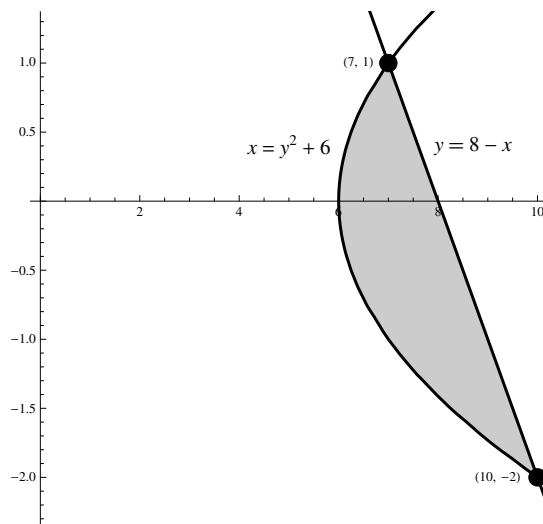
17. The region is shown below:



Solve the second equation for x to get $x = y + 6$. The two curves intersect where $y^2 + 4 = y + 6$, so when $y^2 - y - 2 = (y - 2)(y + 1) = 0$. Therefore the intersection points are $(8, 2)$ and $(5, -1)$. Partitioning along the y -axis, we see that $y^2 + 4 \leq y + 6$, so the area is

$$\begin{aligned} \int_{-1}^2 (y + 6 - (y^2 + 4)) dy &= \int_{-1}^2 (-y^2 + y + 2) dy = \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y \right]_{-1}^2 \\ &= \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) = \boxed{\frac{9}{2}}. \end{aligned}$$

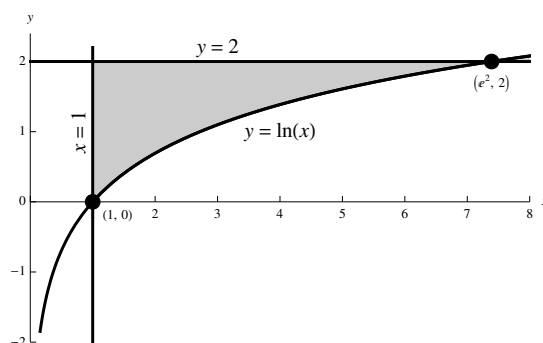
18. The region is shown below:



Solve the second equation for x to get $x = 8 - y$. The two curves intersect where $y^2 + 6 = 8 - y$, so when $y^2 + y - 2 = (y + 2)(y - 1) = 0$. Therefore the intersection points are $(7, 1)$ and $(10, -2)$. Partitioning along the y -axis, we see that $y^2 + 6 \leq 8 - y$, so the area is

$$\begin{aligned} \int_{-2}^1 (8 - y - (y^2 + 6)) \, dy &= \int_{-2}^1 (-y^2 - y + 2) \, dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y \right]_{-2}^1 \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \boxed{\frac{9}{2}}. \end{aligned}$$

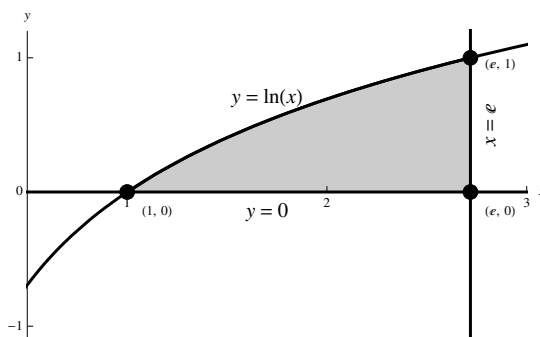
19. The region is shown below:



Solving $y = \ln x$ for x gives $x = e^y$. Then partitioning along the y -axis, we see that $1 \leq e^y$, so the area is

$$\int_0^2 (e^y - 1) \, dy = [e^y - y]_0^2 = (e^2 - 2) - (e^0 - 0) = \boxed{e^2 - 3}.$$

20. The region is shown below:



Solving $y = \ln x$ for x gives $x = e^y$. Then partitioning along the y -axis, we see that $e^y \leq e$, so the area is

$$\int_0^1 (e - e^y) dy = [ey - e^y]_0^1 = (e - e) - (0 - 1) = \boxed{1}.$$

21. This region is bounded above by $y = \cos x$ and below by $y = -\sin x$, and it extends from $x = -\frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. So we integrate by subdividing along the x axis, and we get

$$\begin{aligned} \int_{-\pi/4}^{3\pi/4} (\cos x - (-\sin x)) dx &= \int_{-\pi/4}^{3\pi/4} (\cos x + \sin x) dx = [\sin x - \cos x]_{-\pi/4}^{3\pi/4} \\ &= \left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} \right) \right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = \boxed{2\sqrt{2}}. \end{aligned}$$

22. This region consists of two areas, one below the x axis and one above. For $x < 0$, we have $\sqrt[3]{x} < x^3$, while for $x > 0$ we have $\sqrt[3]{x} > x^3$. So the total area is

$$\begin{aligned} \int_{-1}^0 (x^3 - \sqrt[3]{x}) dx + \int_0^1 (\sqrt[3]{x} - x^3) dx &= \left[\frac{1}{4}x^4 - \frac{3}{4}x^{4/3} \right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} - \frac{1}{4}x^4 \right]_0^1 \\ &= \left((0 - 0) - \left(\frac{1}{4} - \frac{3}{4} \right) \right) + \left(\left(\frac{3}{4} - \frac{1}{4} \right) - (0 - 0) \right) \\ &= \boxed{1}. \end{aligned}$$

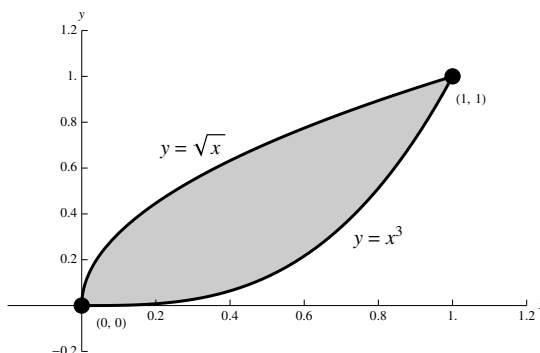
23. If we try to partition along the x -axis, we will require two integrals, one for $-5 \leq x \leq 3$ and the other for $3 \leq x \leq 4$, since the upper bound of the area changes equations at $x = 3$. So we partition along the y -axis. The equation of the parabola is $x = -y^2 + 4$; solving the linear equation for x gives $x = 2y + 1$. Since $-y^2 + 4 \geq 2y + 1$ throughout the region of integration, the area is

$$\begin{aligned} \int_{-3}^1 (-y^2 + 4 - (2y + 1)) dy &= \int_{-3}^1 (-y^2 - 2y + 3) dy = \left[-\frac{1}{3}y^3 - y^2 + 3y \right]_{-3}^1 \\ &= \left(-\frac{1}{3} - 1 + 3 \right) - (9 - 9 - 9) = \boxed{\frac{32}{3}}. \end{aligned}$$

24. Integrate by subdividing along the x -axis. The upper bound is $-x + 3$, while the lower bound is $\frac{4}{x+2}$. So the area is

$$\begin{aligned} \int_{-1}^2 \left(-x + 3 - \frac{4}{x+2} \right) dx &= \left[-\frac{1}{2}x^2 + 3x - 4 \ln|x+2| \right]_{-1}^2 \\ &= (-2 + 6 - 4 \ln 4) - \left(-\frac{1}{2} - 3 - 4 \ln 1 \right) = \boxed{\frac{15}{2} - 4 \ln 4}. \end{aligned}$$

25. The region is shown below:



The two curves intersect when $\sqrt{x} = x^3$, which is at the points $(0, 0)$ and $(1, 1)$. So:

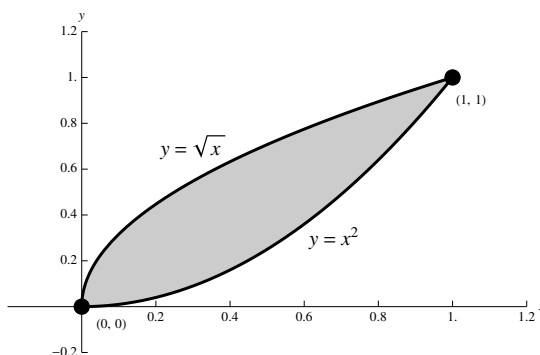
(a) Partitioning the x -axis, the area is

$$\int_0^1 (\sqrt{x} - x^3) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \right]_0^1 = \left(\frac{2}{3} - \frac{1}{4} \right) - (0 - 0) = \boxed{\frac{5}{12}}.$$

(b) Partitioning the y -axis, we must first solve the two equations for x . This gives $x = y^2$ and $x = \sqrt[3]{y}$. Then the area between the curves is

$$\int_0^1 (\sqrt[3]{y} - y^2) dy = \left[\frac{3}{4}y^{4/3} - \frac{1}{3}y^3 \right]_0^1 = \left(\frac{3}{4} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{5}{12}}.$$

26. The region is shown below:



The two curves intersect when $\sqrt{x} = x^2$, which is at the points $(0, 0)$ and $(1, 1)$. So:

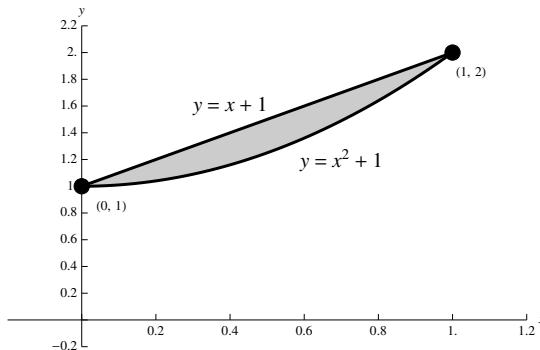
(a) Partitioning the x -axis, the area is

$$\int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{1}{3}}.$$

(b) Partitioning the y -axis, we must first solve the two equations for x . This gives $x = y^2$ and $x = \sqrt{y}$. Then the area between the curves is

$$\int_0^1 (\sqrt{y} - y^2) dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{3}y^3 \right]_0^1 = \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{1}{3}}.$$

27. The region is shown below:



The curves $y = x^2 + 1$ and $y = x + 1$ intersect when $x^2 + 1 = x + 1$, so when $x^2 = x$. As a result, the intersection points are $(0, 1)$ and $(1, 2)$. So:

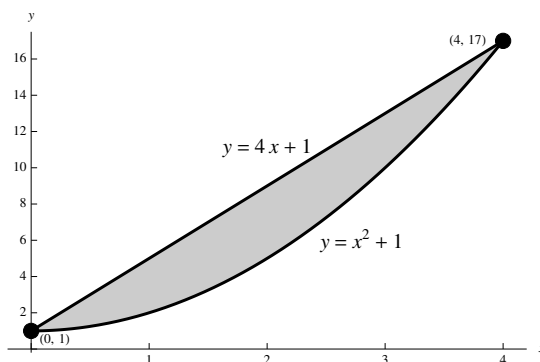
(a) Partitioning the x -axis, the area is

$$\int_0^1 ((x + 1) - (x^2 + 1)) dx = \int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{1}{6}}.$$

(b) Partitioning the y -axis, we must first solve the two equations for x . This gives $x = \sqrt{y - 1}$ and $x = y - 1$. Then the area between the curves is

$$\begin{aligned} \int_1^2 (\sqrt{y - 1} - (y - 1)) dy &= \int_1^2 (1 - y + \sqrt{y - 1}) dy = \left[y - \frac{1}{2}y^2 + \frac{2}{3}(y - 1)^{3/2} \right]_1^2 \\ &= \left(2 - 2 + \frac{2}{3} \right) - \left(1 - \frac{1}{2} + 0 \right) = \boxed{\frac{1}{6}}. \end{aligned}$$

28. The region is shown below:



The curves $y = x^2 + 1$ and $y = 4x + 1$ intersect when $x^2 + 1 = 4x + 1$, so when $x^2 = 4x$, or at the points $(0, 1)$ and $(4, 17)$. So:

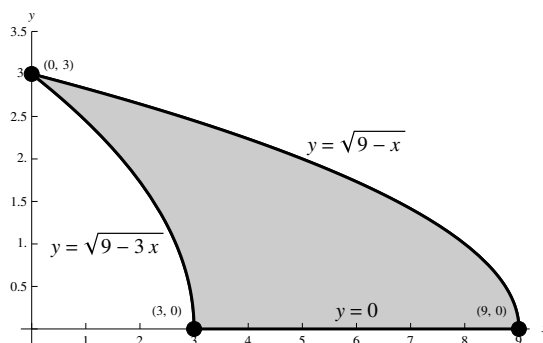
(a) Partitioning the x -axis, the area is

$$\int_0^4 ((4x + 1) - (x^2 + 1)) dx = \int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 = \left(32 - \frac{64}{3} \right) - (0 - 0) = \boxed{\frac{32}{3}}.$$

- (b) Partitioning the y -axis, we must first solve the two equations for x . This gives $x = \sqrt{y-1}$ and $x = \frac{y-1}{4}$. Then the area between the curves is

$$\begin{aligned} \int_1^{17} \left(\sqrt{y-1} - \frac{y-1}{4} \right) dy &= \int_1^{17} \left(\sqrt{y-1} - \frac{1}{4}y + \frac{1}{4} \right) dy \\ &= \left[\frac{2}{3}(y-1)^{3/2} - \frac{1}{8}y^2 + \frac{1}{4}y \right]_1^{17} = \left(\frac{128}{3} - \frac{289}{8} + \frac{17}{4} \right) - \left(0 - \frac{1}{8} + \frac{1}{4} \right) = \boxed{\frac{32}{3}}. \end{aligned}$$

29. The region is shown below:



These curves intersect where $9-x = 9-3x$, so at $(0, 3)$. So:

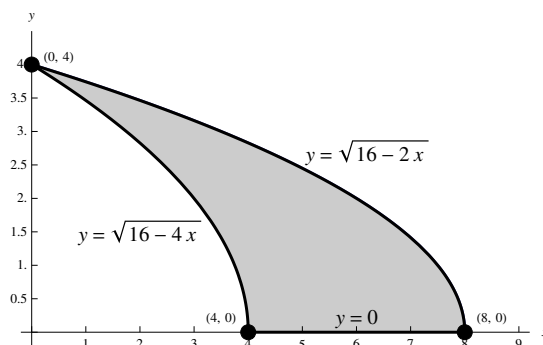
- (a) From $x = 0$ to $x = 3$, the upper bound is $\sqrt{9-x}$ and the lower bound is $\sqrt{9-3x}$, but from $x = 3$ to $x = 9$ the lower bound is 0. Therefore the area is

$$\begin{aligned} \int_0^3 (\sqrt{9-x} - \sqrt{9-3x}) dx + \int_3^9 \sqrt{9-x} dx \\ &= \left[-\frac{2}{3}(9-x)^{3/2} + \frac{2}{9}(9-3x)^{3/2} \right]_0^3 + \left[-\frac{2}{3}(9-x)^{3/2} \right]_3^9 \\ &= \left((-4\sqrt{6} + 0) - (-18 + 6) \right) + \left(0 - (-4\sqrt{6}) \right) = \boxed{12}. \end{aligned}$$

- (b) Solving both equations for x gives $x = 9-y^2$ and $x = 3-\frac{1}{3}y^2$, and the first curve forms the upper bound for integration when subdividing along the y -axis. Then the area is

$$\begin{aligned} \int_0^3 \left((9-y^2) - \left(3 - \frac{1}{3}y^2 \right) \right) dy &= \int_0^3 \left(6 - \frac{2}{3}y^2 \right) dy \\ &= \left[6y - \frac{2}{9}y^3 \right]_0^3 \\ &= (18 - 6) - (0 - 0) = \boxed{12}. \end{aligned}$$

30. The region is shown below:



These curves intersect where $16 - 2x = 16 - 4x$, so at $(0, 4)$.

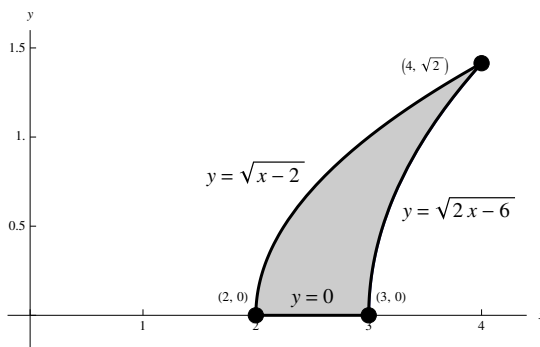
- (a) From $x = 0$ to $x = 4$, the upper bound is $\sqrt{16 - 2x}$ and the lower bound is $\sqrt{16 - 4x}$, but from $x = 4$ to $x = 8$ the lower bound is 0. Therefore the area is

$$\begin{aligned} \int_0^4 (\sqrt{16 - 2x} - \sqrt{16 - 4x}) dx + \int_4^8 \sqrt{16 - 2x} dx \\ = \left[-\frac{1}{3}(16 - 2x)^{3/2} + \frac{1}{6}(16 - 4x)^{3/2} \right]_0^4 + \left[-\frac{1}{3}(16 - 2x)^{3/2} \right]_4^8 \\ = \left(\left(-\frac{16}{3}\sqrt{2} + 0 \right) - \left(-\frac{64}{3} + \frac{32}{3} \right) \right) + 0 - \left(-\frac{16}{3}\sqrt{2} \right) \\ = \boxed{\frac{32}{3}}. \end{aligned}$$

- (b) Solving both equations for x gives $x = 8 - \frac{1}{2}y^2$ and $x = 4 - \frac{1}{4}y^2$, and the first curve forms the upper bound for integration when subdividing along the y -axis. Then the area is

$$\begin{aligned} \int_0^4 \left(\left(8 - \frac{1}{2}y^2 \right) - \left(4 - \frac{1}{4}y^2 \right) \right) dy &= \int_0^4 \left(4 - \frac{1}{4}y^2 \right) dy \\ &= \left[4y - \frac{1}{12}y^3 \right]_0^4 \\ &= 16 - \frac{16}{3} + (0 - 0) = \boxed{\frac{32}{3}}. \end{aligned}$$

31. The region is shown below:



These curves intersect when $\sqrt{x - 2} = \sqrt{2x - 6}$, so at $x = 4$, which is the point $(4, \sqrt{2})$. Therefore:

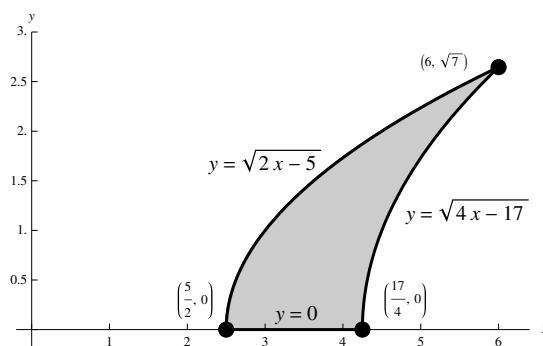
- (a) From $x = 2$ to $x = 3$, the lower bound is zero and the upper bound is $\sqrt{x - 2}$; from $x = 3$ to $x = 4$ the lower bound is $\sqrt{2x - 6}$. Therefore the area is

$$\begin{aligned} \int_2^3 \sqrt{x - 2} dx + \int_3^4 (\sqrt{x - 2} - \sqrt{2x - 6}) dy &= \left[\frac{2}{3}(x - 2)^{3/2} \right]_2^3 + \left[\frac{2}{3}(x - 2)^{3/2} - \frac{1}{3}(2x - 6)^{3/2} \right]_3^4 \\ &= \left(\frac{2}{3} - 0 \right) + \left(\left(\frac{4}{3}\sqrt{2} - \frac{2}{3}\sqrt{2} \right) - \left(\frac{2}{3} - 0 \right) \right) \\ &= \boxed{\frac{2}{3}\sqrt{2}}. \end{aligned}$$

- (b) Solving both equations for x gives $x = 3 + \frac{1}{2}y^2$ and $x = 2 + y^2$; the first curve is the upper bound for integration when subdividing along the y -axis. Then the area is

$$\begin{aligned} \int_0^{\sqrt{2}} \left(\left(3 + \frac{1}{2}y^2 \right) - (2 + y^2) \right) dy &= \int_0^{\sqrt{2}} \left(1 - \frac{1}{2}y^2 \right) dy \\ &= \left[y - \frac{1}{6}y^3 \right]_0^{\sqrt{2}} \\ &= \left(\sqrt{2} - \frac{\sqrt{2}}{3} \right) - (0 - 0) \\ &= \boxed{\frac{2}{3}\sqrt{2}}. \end{aligned}$$

32. The region is shown below:



These curves intersect when $2x - 5 = 4x - 17$, so when $x = 6$, at the point $(6, \sqrt{7})$. Therefore:

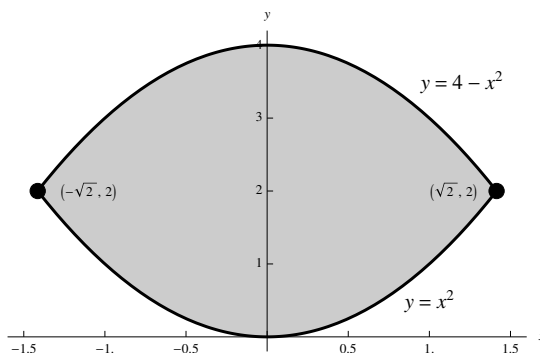
- (a) From $x = \frac{5}{2}$ to $x = \frac{17}{4}$, the upper bound is $\sqrt{2x-5}$ and the lower bound is 0. From $x = \frac{17}{4}$ to $x = 6$ the lower bound is $\sqrt{4x-17}$. So the area is

$$\begin{aligned} \int_{5/2}^{17/4} \sqrt{2x-5} dx + \int_{17/4}^6 (\sqrt{2x-5} - \sqrt{4x-17}) dx \\ &= \left[\frac{1}{3}(2x-5)^{3/2} \right]_{5/2}^{17/4} + \left[\frac{1}{3}(2x-5)^{3/2} - \frac{1}{6}(4x-17)^{3/2} \right]_{17/4}^6 \\ &= \frac{7}{12}\sqrt{14} + \left(\frac{7}{6}\sqrt{7} - \frac{7}{12}\sqrt{14} \right) \\ &= \boxed{\frac{7}{6}\sqrt{7}}. \end{aligned}$$

- (b) Solving both equations for x gives $x = \frac{y^2+5}{2}$ and $x = \frac{y^2+17}{4}$; the second equation forms the upper bound for the integrand. So the area is

$$\begin{aligned} \int_0^{\sqrt{7}} \left(\frac{y^2+17}{4} - \frac{y^2+5}{2} \right) dy &= \int_0^{\sqrt{7}} \left(\frac{7}{4} - \frac{y^2}{4} \right) dy \\ &= \left[\frac{7}{4}y - \frac{y^3}{12} \right]_0^{\sqrt{7}} \\ &= \left(\frac{7}{4}\sqrt{7} - \frac{7}{12}\sqrt{7} \right) - (0 - 0) \\ &= \boxed{\frac{7}{6}\sqrt{7}}. \end{aligned}$$

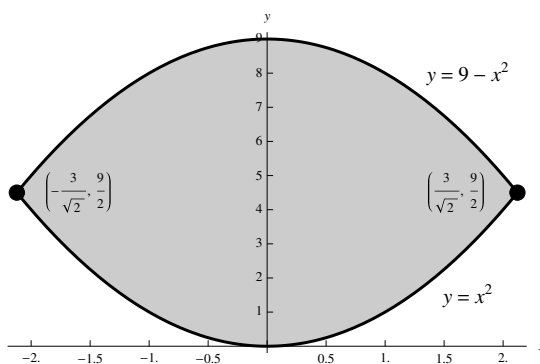
33. The region is shown below:



The two graphs intersect when $4 - x^2 = x^2$, so when $x = \pm\sqrt{2}$. Therefore the area between the curves is

$$\int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^2 - x^2) dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2x^2) dx = \left[4x - \frac{2}{3}x^3 \right]_{-\sqrt{2}}^{\sqrt{2}} = \boxed{\frac{16}{3}\sqrt{2}}.$$

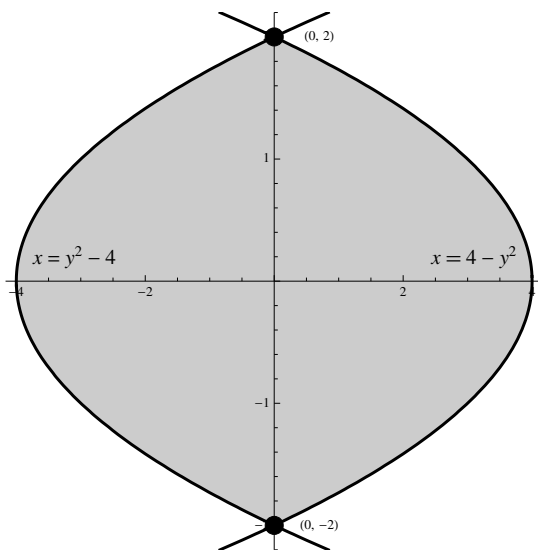
34. The region is shown below:



The graphs intersect when $9 - x^2 = x^2$, so when $x = \pm\frac{3\sqrt{2}}{2}$. Therefore the area between the curves is

$$\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2} (9 - x^2 - x^2) dx = \int_{-3\sqrt{2}/2}^{3\sqrt{2}/2} (9 - 2x^2) dx = \left[9x - \frac{2}{3}x^3 \right]_{-3\sqrt{2}/2}^{3\sqrt{2}/2} = \boxed{18\sqrt{2}}.$$

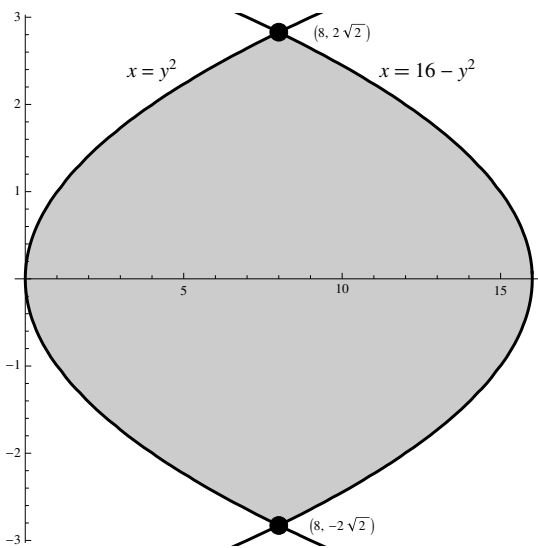
35. The region is shown below:



For these curves, we partition along the y -axis. The graphs intersect when $y^2 - 4 = 4 - y^2$, which is when $y = -2$ and $y = 2$. So the area between the curves is

$$\int_{-2}^2 ((4 - y^2) - (y^2 - 4)) dy = \int_{-2}^2 (8 - 2y^2) dy = \left[8y - \frac{2}{3}y^3 \right]_{-2}^2 = \boxed{\frac{64}{3}}.$$

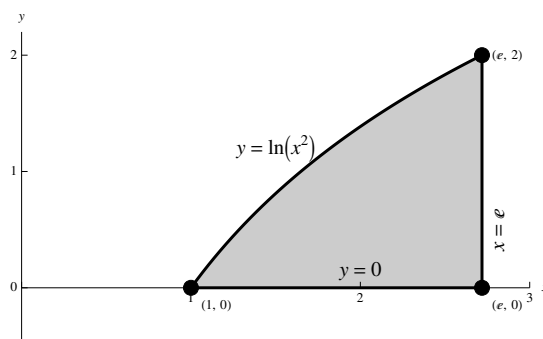
36. The region is shown below:



For these curves, we partition along the y -axis. The graphs intersect when $y^2 = 16 - y^2$, so for $y = \pm\sqrt{8} = \pm 2\sqrt{2}$. Therefore the area between the curves is

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - y^2 - y^2) dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - 2y^2) dy = \left[16y - \frac{2}{3}y^3 \right]_{-2\sqrt{2}}^{2\sqrt{2}} = \boxed{\frac{128}{3}\sqrt{2}}.$$

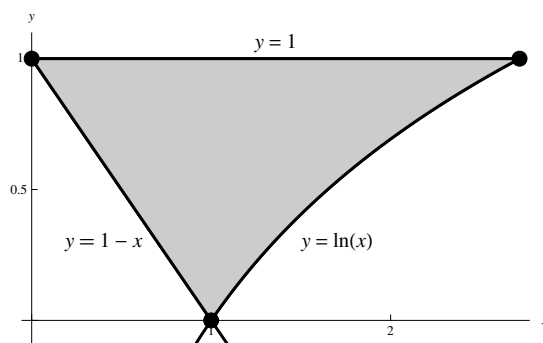
37. The region is shown below:



Note that $\ln x^2 = 2 \ln x$. This area can be computed more easily by subdividing along the y -axis. Solving $y = 2 \ln x$ for x gives $x = e^{y/2}$. The line $x = e$ meets this curve at $y = 2$, so the region extends from $y = 0$ to $y = 2$ and is bounded on the left by $e^{y/2}$ and on the right by e . Therefore the area of the region is

$$\int_0^2 (e - e^{y/2}) dy = [ey - 2e^{y/2}]_0^2 = (2e - 2e) - (0 - 2) = \boxed{2}.$$

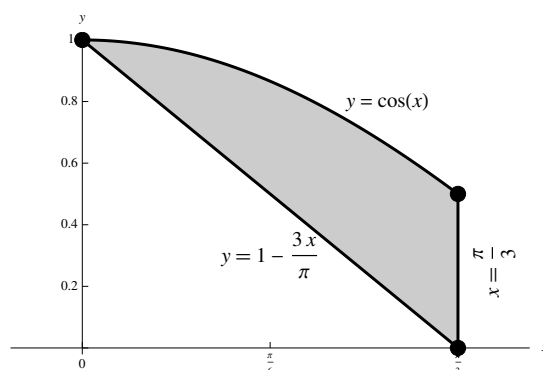
38. The region is shown below:



We partition along the y -axis so that we do not have to split the computation into two integrals. $y = 1 - x$ becomes $x = 1 - y$, and $y = \ln x$ becomes $x = e^y$. The region extends from $y = 0$ to $y = 1$, so the area of the region is

$$\int_0^1 (e^y - (1 - y)) dy = \int_0^1 (y - 1 + e^y) dy = \left[\frac{1}{2}y^2 - y + e^y \right]_0^1 = \boxed{e - \frac{3}{2}}.$$

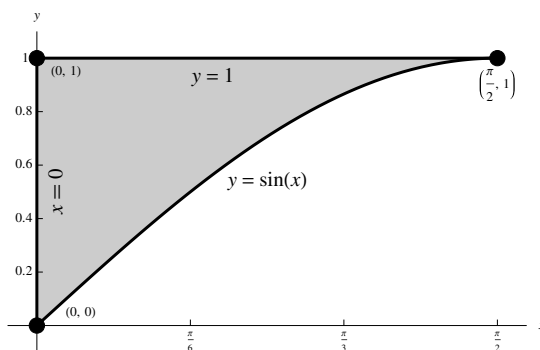
39. The region is shown below:



The area of the region is

$$\int_0^{\pi/3} \left(\cos x - \left(1 - \frac{3}{\pi} x \right) \right) dx = \int_0^{\pi/3} \left(\frac{3}{\pi} x - 1 + \cos x \right) dx = \left[\frac{3}{2\pi} x^2 - x + \sin x \right]_0^{\pi/3} = \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6}}.$$

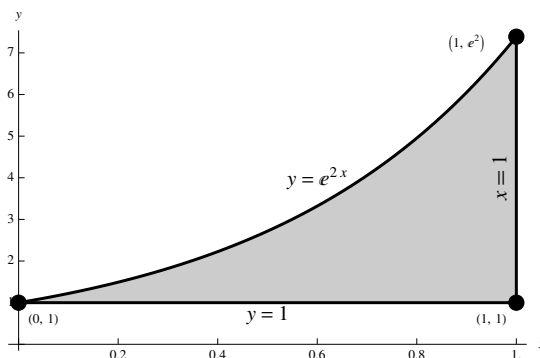
40. The region is shown below:



The region extends from $x = 0$ to $x = \frac{\pi}{2}$, so the area is

$$\int_0^{\pi/2} (1 - \sin x) dx = [x + \cos x]_0^{\pi/2} = \boxed{\frac{\pi}{2} - 1}.$$

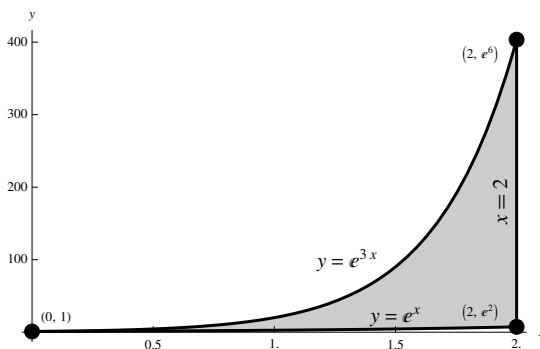
41. The region is shown below:



The area of the region is

$$\int_0^1 (e^{2x} - 1) dx = \left[\frac{1}{2} e^{2x} - x \right]_0^1 = \boxed{\frac{1}{2} e^2 - \frac{3}{2}}.$$

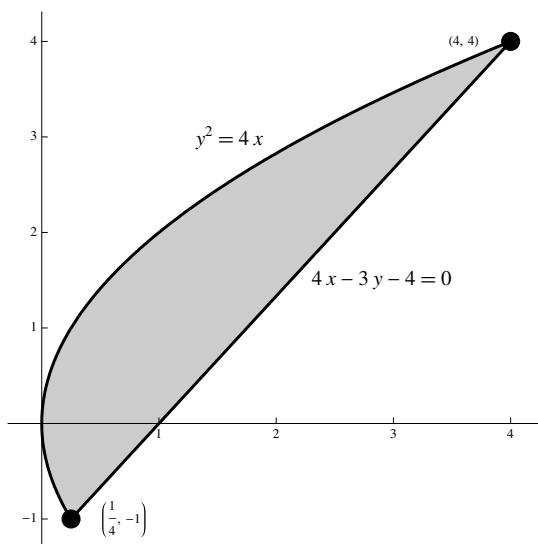
42. The region is shown below:



The area of the region is

$$\int_0^2 (e^{3x} - e^x) dx = \left[\frac{1}{3}e^{3x} - e^x \right]_0^2 = \boxed{\frac{1}{3}e^6 - e^2 + \frac{2}{3}}.$$

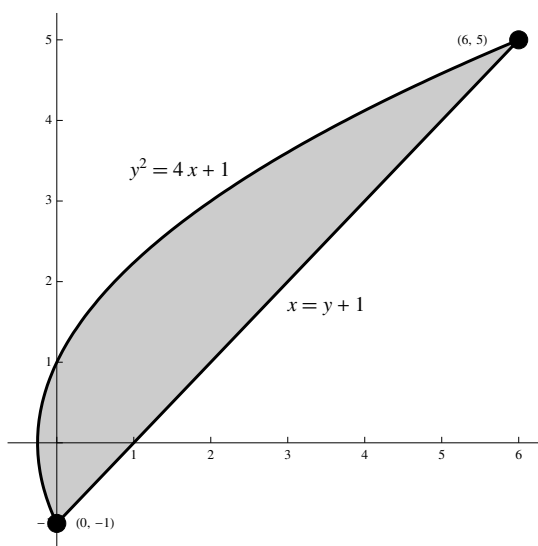
43. The region is shown below:



We partition along the y -axis. Solving these equations for x gives $x = \frac{y^2}{4}$ and $x = 1 + \frac{3}{4}y$. The curves intersect where $y^2 - 3y - 4 = 0$ (since $y^2 = 4x$); this factors as $(y - 4)(y + 1)$, so the y -coordinates of the intersection points are -1 and 4 . Therefore the area of the region is

$$\int_{-1}^4 \left(1 + \frac{3}{4}y - \frac{y^2}{4} \right) dy = \left[y + \frac{3}{8}y^2 - \frac{1}{12}y^3 \right]_{-1}^4 = \boxed{\frac{125}{24}}.$$

44. The region is shown below:

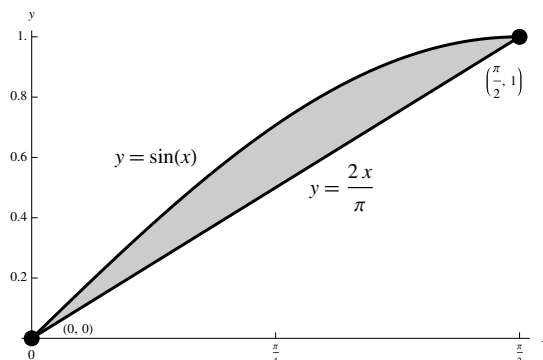


Partition along the y -axis. The two curves have equations $x = \frac{1}{4}(y^2 - 1)$ and $x = y + 1$; these intersect where $y + 1 = \frac{1}{4}(y^2 - 1)$, so when $y^2 - 4y - 5 = 0$. This happens for $y = 5$ and $y = -1$. Therefore the

area of the region is

$$\int_{-1}^5 \left(y + 1 - \frac{1}{4}(y^2 - 1) \right) dy = \int_{-1}^5 \left(-\frac{1}{4}y^2 + y + \frac{5}{4} \right) dy = \left[-\frac{1}{12}y^3 + \frac{1}{2}y^2 + \frac{5}{4}y \right]_{-1}^5 = \boxed{9}.$$

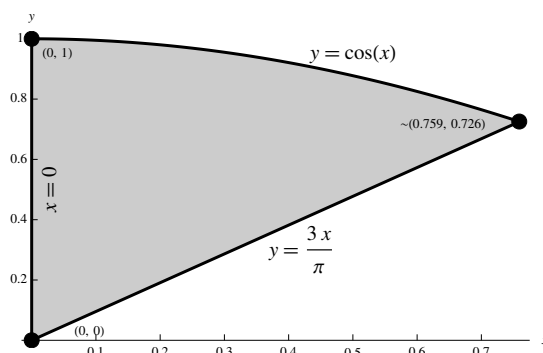
45. The region is shown below:



The graphs intersect at $x = 0$ and again at $x = \frac{\pi}{2}$, where $\sin \frac{\pi}{2} = 1$ and $\frac{2}{\pi}x = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$. So the area of the region is

$$\int_0^{\pi/2} \left(\sin x - \frac{2}{\pi}x \right) dx = \left[-\cos x - \frac{1}{\pi}x^2 \right]_0^{\pi/2} = \boxed{-\frac{\pi}{4} + 1}.$$

46. The region is shown below:



The intersection point of these graphs has no simple exact form, but is approximately equal to 0.759. So the area of the region is approximately

$$\int_0^{0.759} \left(\cos x - \frac{3}{\pi}x \right) dx = \left[\sin x - \frac{3}{2\pi}x^2 \right]_0^{0.759} \approx \boxed{0.413}.$$

Applications and Extensions

47. The slope of BC is $\frac{4-1}{-2-1} = -1$, so the point of tangency of AD with the parabola is a point where the slope of the tangent to the parabola is -1 . With $y = x^2$, we have $y' = 2x$, so the slope is -1 at $x = -\frac{1}{2}$. Therefore the point of tangency is $(-\frac{1}{2}, \frac{1}{4})$, so the line AD has equation

$$y - \frac{1}{4} = -1 \left(x - \left(-\frac{1}{2} \right) \right), \text{ or } y = -x - \frac{1}{4}.$$

Line BC has equation

$$y - 1 = -1(x - 1), \text{ or } y = -x + 2.$$

So the area of the parallelogram is

$$\int_{-2}^1 \left((-x+2) - \left(-x - \frac{1}{4} \right) \right) dx = \int_{-2}^1 \left(\frac{9}{4} \right) dx = \frac{27}{4}.$$

The shaded area, on the other hand, has area

$$\int_{-2}^1 ((-x+2) - x^2) dx = \int_{-2}^1 (-x^2 - x + 2) dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1 = \frac{9}{2}.$$

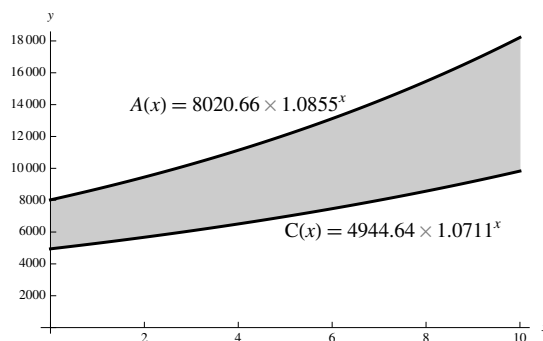
Finally, $\frac{27}{4} \cdot \frac{2}{3} = \frac{9}{2}$, and Archimedes' result follows.

48. Subdividing along the x -axis, we get for the area of the region

$$\int_0^{1/2} (8x - x) dx + \int_{1/2}^1 \left(\frac{1}{x^2} - x \right) dx = \left[\frac{7}{2}x^2 \right]_0^{1/2} + \left[-\frac{1}{x} - \frac{1}{2}x^2 \right]_{1/2}^1 = \frac{7}{8} + \frac{5}{8} = \frac{3}{2}.$$

A triangle with base 1 and height h has area $\frac{1}{2} \cdot 1 \cdot h = \frac{h}{2}$; since we want $\frac{h}{2} = \frac{3}{2}$, we have $\boxed{h = 3}$.

49. The two cost curves, with $A(x)$ the upper curve, are shown below:

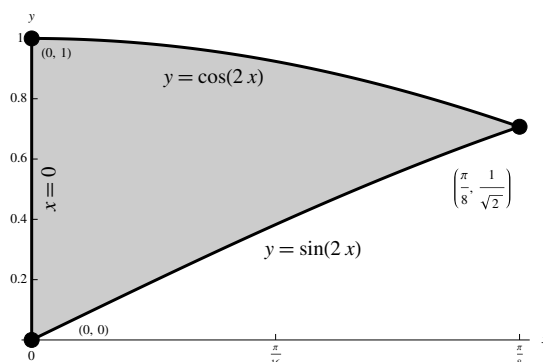


(a) The area between the curves is

$$\begin{aligned} \int_0^{10} (8020.6596 \cdot 1.0855^x - 4944.6424 \cdot 1.0711^x) dx \\ = \left[\frac{8020.6596}{\ln 1.0855} 1.0855^x - \frac{4944.6424}{\ln 1.0711} 1.0711^x \right]_0^{10} \approx \$53,213. \end{aligned}$$

(b) The total average additional amount of money spent by a family of four in the U.S. for health care exceeded by about \$53,213 the amount spent by a family of four in Canada. Answers may vary.

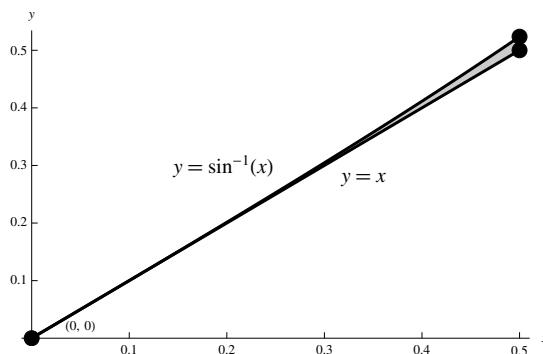
50. The region is shown below:



The region extends from $x = 0$ to $x = \frac{\pi}{8}$, so the area is

$$\int_0^{\pi/8} (\cos 2x - \sin 2x) dx = \left[\frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x \right]_0^{\pi/8} = \boxed{\frac{\sqrt{2}}{2} - \frac{1}{2}}.$$

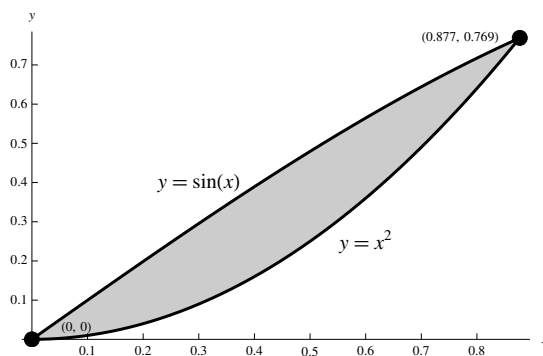
51. The region is shown below:



Since it is easier to integrate $\sin y$ than it is to integrate $\sin^{-1} x$, we choose to partition along the y -axis. The two equations are then $x = \sin y$ and $x = y$. The graph of $x = \sin y$ intersects the line $x = \frac{1}{2}$ at $y = \frac{\pi}{6}$, while the graph of $x = y$ intersects that line at $y = \frac{1}{2}$. So for $0 \leq y \leq \frac{1}{2}$, the region is bounded to the right by $x = y$ and to the left by $x = \sin y$, while for $\frac{1}{2} \leq y \leq \frac{\pi}{6}$, it is bounded to the right by $x = \frac{1}{2}$ and to the left by $x = \sin y$. As a result, the total area is

$$\begin{aligned} \int_0^{1/2} (y - \sin y) dy + \int_{1/2}^{\pi/6} \left(\frac{1}{2} - \sin y \right) dy &= \left[\frac{1}{2} y^2 + \cos y \right]_0^{1/2} + \left[\frac{1}{2} y + \cos y \right]_{1/2}^{\pi/6} \\ &= \left(-\frac{7}{8} + \cos \frac{1}{2} \right) + \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{4} - \cos \frac{1}{2} \right) \\ &= \boxed{-\frac{9}{8} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}}. \end{aligned}$$

52. (a) The two curves are shown below:



(b) The curves intersect when $x^2 = \sin x$. Now, $x = 0$ is such a value of x ; using technology we find the other value to be $x \approx 0.877$. The corresponding y values are $y = 0$ and $y = 0.877^2 \approx 0.769$, so the intersection points are $\boxed{(0, 0) \text{ and } (0.877, 0.769)}$.

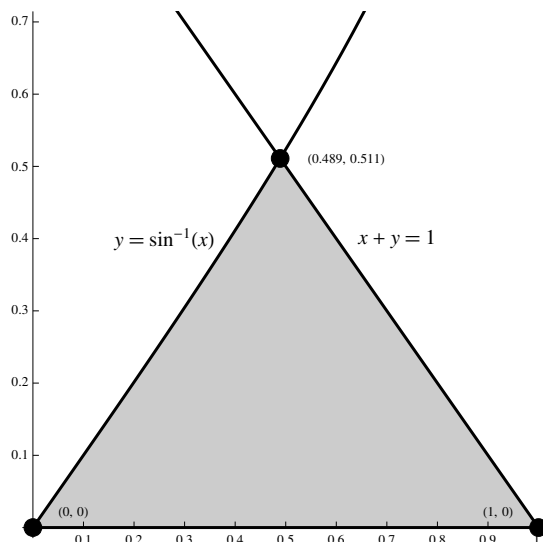
(c) The area is

$$\int_0^{0.877} (\sin x - x^2) dx = \left[-\cos x - \frac{1}{3} x^3 \right]_0^{0.877} = \boxed{\approx 0.136}.$$

(d) Solving the two equations for x gives $x = \sqrt{y}$ and $x = \sin^{-1} y$. So the area between the curves is

$$\int_0^{0.769} (\sqrt{y} - \sin^{-1} y) dy \approx 0.136.$$

53. (a) The three curves are shown below:

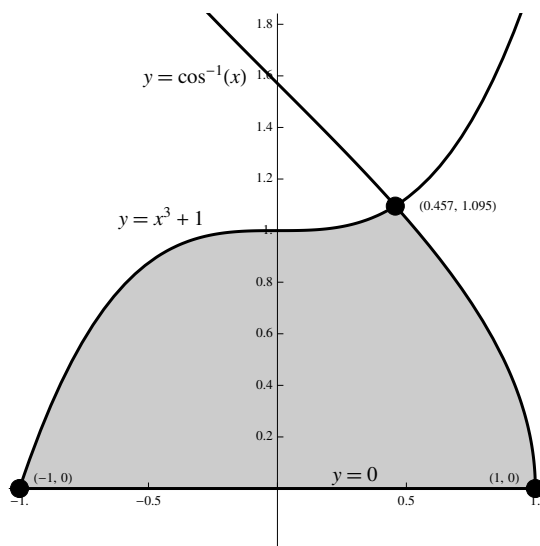


(b) The graphs of $\sin^{-1} x$ and $y = 0$ intersect at the point $(0, 0)$, and $x + y = 1$ and $y = 0$ intersect at the point $(1, 0)$. The third point of intersection is where $y = \sin^{-1} x = 1 - x$; solving $\sin^{-1} x = 1 - x$ using technology gives $x \approx 0.489$, so that $y = 1 - 0.489 = 0.511$, and the intersection point is $\approx (0.489, 0.511)$.

(c) Partitioning along the y -axis allows us to use just one integral, bounded above by $x = 1 - y$ and below by $x = \sin y$, so its area is

$$\int_0^{0.511} (1 - y - \sin y) dy = \left[y - \frac{1}{2}y^2 + \cos y \right]_0^{0.511} \approx 0.253.$$

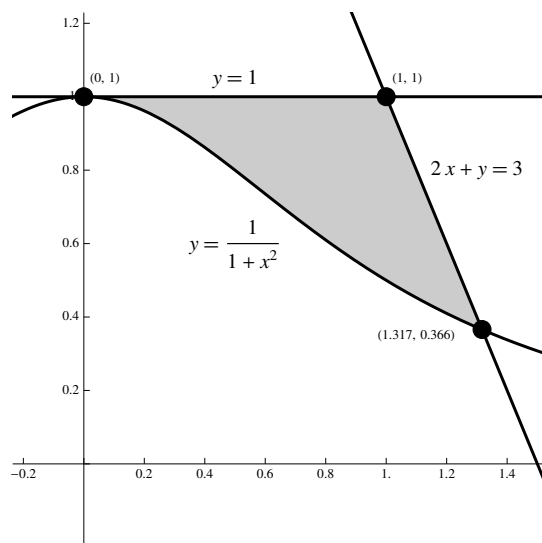
54. (a) The three curves are shown below:



- (b) The graphs of $\cos^{-1} x$ and $y = 0$ intersect on the x -axis, at $(1, 0)$, and $y = x^3 + 1$ and $y = 0$ intersect on the x -axis, at $(-1, 0)$. The third point of intersection is determined by solving $\cos^{-1} x = x^3 + 1$; using technology we get $x \approx 0.457$, so that $y \approx 0.457^3 + 1 \approx 1.095$, and the third point of intersection is $(\approx 0.457, 1.095)$.
- (c) Partitioning along the y -axis allows us to use just one integral, bounded above by $x = \cos y$ and below by $x = \sqrt[3]{y-1}$, so its area is

$$\int_0^{1.095} (\cos y - \sqrt[3]{y-1}) dy = \left[\sin y - \frac{3}{4}(y-1)^{4/3} \right]_0^{1.095} \approx 1.606.$$

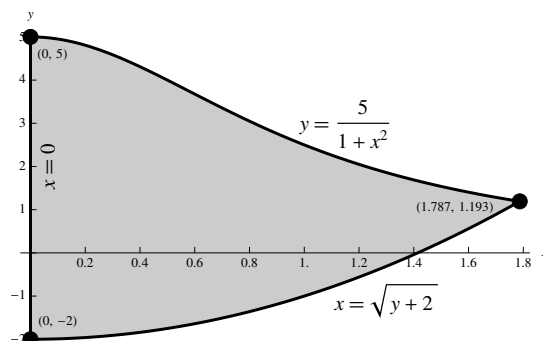
55. (a) The three curves are shown below:



- (b) $2x + y = 3$ and $y = 1$ intersect at $x = 1$, which is the point $(1, 1)$. $y = 1$ intersects $y = \frac{1}{x^2+1}$ when $\frac{1}{x^2+1} = 1$, so when $x = 0$; this point is $(0, 1)$. Finally, $2x + y = 3$ intersects $y = \frac{1}{x^2+1}$ when $3 - 2x = \frac{1}{x^2+1}$. Clearing fractions and simplifying gives $2x^3 - 3x^2 + 2x - 2 = 0$. The only real root of this cubic is, using technology, $x \approx 1.317$. At that value, $y = 3 - 2 \cdot 1.317 \approx 0.366$, so the third point of intersection is $(\approx 1.317, 0.366)$.
- (c) Partitioning along the y -axis allows us to use just one integral, bounded above by $x = \frac{1}{2}(3 - y)$ and below by $x = \sqrt{\frac{1}{y} - 1}$, so its area is

$$\int_{0.366}^1 \left(\frac{1}{2}(3 - y) - \sqrt{\frac{1}{y} - 1} \right) dy \approx 0.295.$$

56. (a) The three curves are shown below:

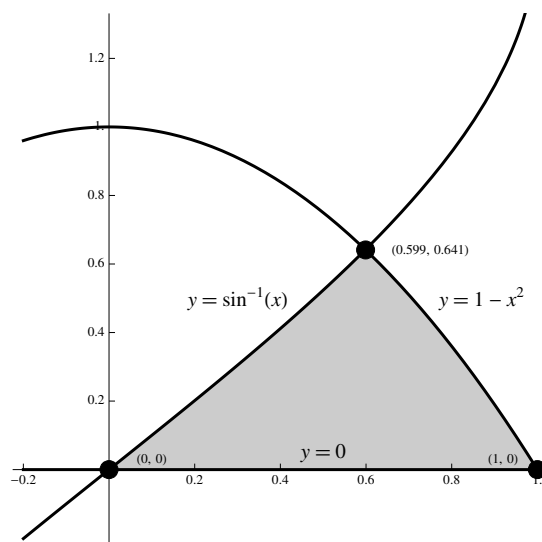


(b) $x = 0$ and $y = \frac{5}{1+x^2}$ intersect at $(0, 5)$, while $x = 0$ and $x = \sqrt{y+2}$ intersect at $(0, -2)$. To find the third intersection point, we must solve the system $y = \frac{5}{1+x^2}$, $x = \sqrt{y+2}$; using technology, we get $(x, y) \approx (1.787, 1.193)$.

(c) Partition along the x -axis; then the second equation becomes $y = x^2 - 2$ and the area of the region is

$$\int_0^{1.787} \left(\frac{5}{1+x^2} - (x^2 - 2) \right) dx \approx 6.975.$$

57. (a) The three curves are shown below:



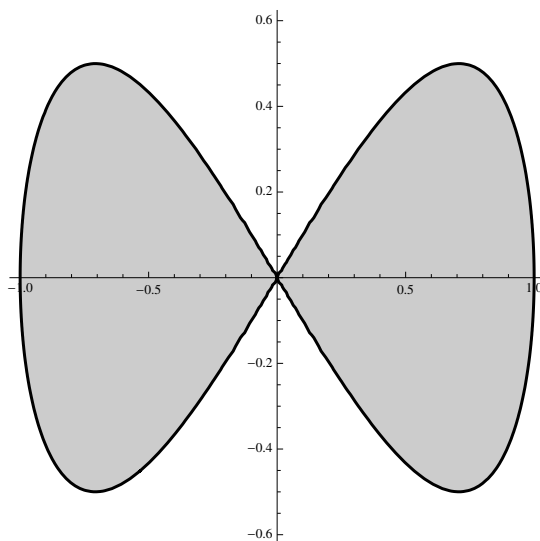
(b) $y = 1 - x^2$ and $y = 0$ intersect at $(-1, 0)$ and at $(1, 0)$; the point we are concerned with here is $(1, 0)$. The graph shows that $y = \sin^{-1} x$ and $y = 0$ intersect at $(0, 0)$. The third point of intersection is found by solving $1 - x^2 = \sin^{-1} x$; using technology we get $x \approx 0.599$, so that $y = 1 - 0.599^2 \approx 0.641$ and the third intersection point is $\approx (0.599, 0.641)$.

(c) Partitioning along the y -axis allows us to use a single integral, bounded above by $x = \sqrt{1 - y}$ and below by $x = \sin y$; the area of the region is then

$$\int_0^{0.641} (\sqrt{1 - y} - \sin y) dy \approx 0.325.$$

Challenge Problems

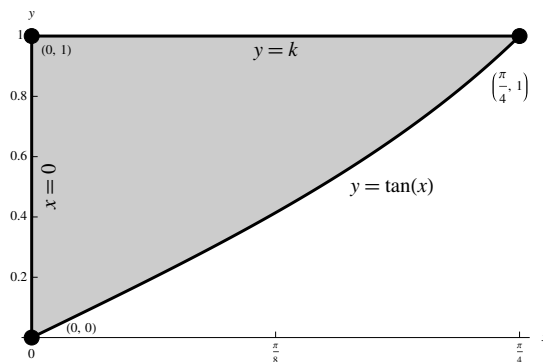
58. The region is shown below:



Since $(-y)^2 = y^2$ and $(-x)^2 - (-x)^4 = x^2 - x^4$, the graph is symmetric about both the x and y -axes. So we can compute its area by computing the portion of the area in the first quadrant and multiplying by 4. In the first quadrant, the equation of the curve is $y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$. Note also that if $x > 1$, then $x^2 - x^4 < 0$, so there are no points of the curve to the right of $x = 1$; therefore, the integral to compute the area will go from $x = 0$ to $x = 1$. Using the substitution $u = 1 - x^2$, $du = -2x dx$, then $x = 0$ corresponds to $u = 1$ and $x = 1$ to $u = 0$, so the total area is

$$4 \int_0^1 x \sqrt{1 - x^2} dx = -4 \cdot \frac{1}{2} \int_1^0 u^{1/2} du = -2 \left[\frac{2}{3} u^{3/2} \right]_1^0 = \boxed{\frac{4}{3}}.$$

59. The region (for $k = 1$) is shown below:



- (a) Partitioning along the x -axis, the upper edge of the region is $y = k$ and the lower edge is $y = \tan x$. The bounds of integration range from $x = 0$ to $x = \tan^{-1} k$, so the area is

$$A = \int_0^{\tan^{-1} k} (k - \tan x) dx = [kx + \ln \cos x]_0^{\tan^{-1} k} = k \tan^{-1} k + \ln \cos \tan^{-1} k.$$

Now, since x is positive, $\tan^{-1} k$ is in the first quadrant, so $\cos \tan^{-1} k = \frac{1}{\sqrt{1+k^2}}$, as can be seen by drawing a right triangle in the first quadrant whose tangent function is equal to k . Therefore

we get for the area

$$A = k \tan^{-1} k + \ln \frac{1}{\sqrt{1+k^2}} = \boxed{k \tan^{-1} k - \frac{1}{2} \ln(1+k^2)}.$$

(b) When $k = 1$,

$$A = 1 \cdot \tan^{-1} 1 - \frac{1}{2} \ln(1+1^2) = \boxed{\frac{\pi}{4} - \frac{1}{2} \ln 2}.$$

(c) Using the Chain Rule,

$$\begin{aligned} \frac{dA}{dt} &= \frac{dA}{dk} \cdot \frac{dk}{dt} \\ &= \left(\tan^{-1} k + \frac{k}{1+k^2} - \frac{1}{2(1+k^2)} \cdot 2k \right) \cdot \frac{dk}{dt} \\ &= \left(\tan^{-1} k + \frac{k}{1+k^2} - \frac{k}{1+k^2} \right) \cdot \frac{dk}{dt} \\ &= \tan^{-1} k \cdot \frac{dk}{dt}. \end{aligned}$$

When $\frac{dk}{dt} = \frac{1}{10}$ and $k = 1$, we get

$$\frac{dA}{dt} = \tan^{-1} 1 \cdot \frac{1}{10} = \boxed{\frac{\pi}{40} \text{ square units per second.}}$$

60. (a) If you reflect the first-quadrant shaded region across the x -axis, you get another region that is the same size as the first; the union of these two regions is a region whose area is A^* in the problem statement. The point where the line $y = -x$ meets the hyperbola in the fourth quadrant is the corner of this region; its coordinates are $(\cosh t, -\sinh t)$. So we get a triangle with vertices $(0, 0)$, $(\cosh t, \sinh t)$, and $(\cosh t, -\sinh t)$. This triangle has base (the vertical) $2 \sinh t$ and height $\cosh t$, so that its area is $\frac{1}{2} \cdot 2 \sinh t \cdot \cosh t = \sinh t \cosh t$. Since $\cosh^2 t - \sinh^2 t = 1$, we can rewrite this as $\cosh t \sqrt{\cosh^2 t - 1}$; since $x = \cosh t$, this becomes $x \sqrt{x^2 - 1}$. To compute the area of A^* , we must subtract from the area of the triangle the area under the hyperbola between $x = 1$ and $x = \cosh t$. We can compute this area by computing the area under the hyperbola in the first quadrant and doubling it, so (using u as a dummy variable), we get

$$2 \int_1^x \sinh u \, du = 2 \int_1^x \sqrt{\cosh^2 u - 1} \, du = 2 \int_1^x \sqrt{u^2 - 1} \, du.$$

Therefore

$$A^* = x \sqrt{x^2 - 1} - 2 \int_1^x \sqrt{u^2 - 1} \, du.$$

(b) Using the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \frac{dA^*}{dx} &= \sqrt{x^2 - 1} + x \cdot \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x - 2\sqrt{x^2 - 1} \\ &= -\sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}} \\ &= \frac{1}{\sqrt{x^2 - 1}}. \end{aligned}$$

(c) Since $\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$, we see from part (b) that $A^* = \cosh^{-1} x + C$ for some constant C .

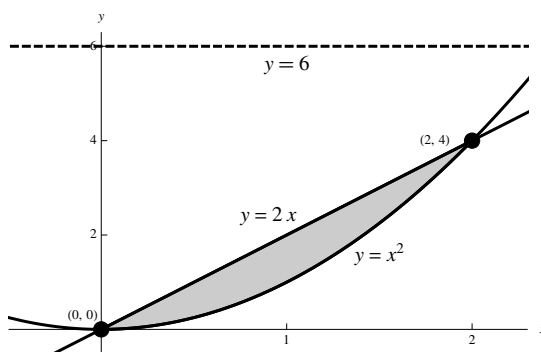
When $x = 1$, the area is empty, so that $A^* = 0$. Therefore we get $0 = \cosh^{-1} 1 + C = 0 + C$, so that $C = 0$. Substituting gives $A^* = \cosh^{-1} x$.

(d) Finally, since $x = \cosh t$ and $t \geq 0$, we get $t = \cosh^{-1} x = A^*$.

6.2: Volume of a Solid of Revolution: Disks and Washers

Concepts and Vocabulary

1. From the boxed formula in subsection 3 preceding Example 4, the formula is $V = \pi \int_a^b f(x)^2 dx$.
2. False. The cross section is indeed the region contained between two concentric circles. The radii of those circles are $f(u_i)$ and $g(u_i)$, so the area between them is the difference of their areas, or $\pi f(u_i)^2 - \pi g(u_i)^2 = \pi[f(u_i)^2 - g(u_i)^2]$, not $\pi[f(u_i) - g(u_i)]^2$.
3. False. From Problem 1, or the boxed formula in subsection 3 preceding Example 4, the volume is $V = \pi \int_a^b [f(x)^2 - g(x)^2] dx$.
4. False. First of all, the two curves intersect at $x = 0$ and $x = 2$, so the integral must go from 0 to 2. See the diagram:



Second, for a given value of x , the outer radius is $6 - x^2$ and the inner radius is $6 - 2x$, so the formula is

$$V = \pi \int_0^2 [(6 - x^2)^2 - (6 - 2x)^2] dx.$$

Skill Building

5. Using the disk method, the radius of each disk is $2\sqrt{x}$, so the volume is

$$V = \pi \int_1^4 (2\sqrt{x})^2 dx = 4\pi \int_1^4 x dx = 4\pi \left[\frac{1}{2}x^2 \right]_1^4 = \boxed{30\pi}.$$

6. Since we are revolving around the y -axis with the disk method, first solve the equation for x to get $x = y^{1/4}$. Then the radius of each disk is $y^{1/4}$, so the volume is

$$V = \pi \int_0^1 (y^{1/4})^2 dy = \pi \int_0^1 y^{1/2} dy = \pi \left[\frac{2}{3}y^{3/2} \right]_0^1 = \boxed{\frac{2}{3}\pi}.$$

7. Since we are revolving around the y -axis with the disk method, first solve the equation for x to get $x = \frac{1}{y}$. Then the radius of each disk is $\frac{1}{y}$, so the volume is

$$V = \pi \int_1^4 \left(\frac{1}{y} \right)^2 dy = \pi \int_1^4 \frac{1}{y^2} dy = \pi \left[-\frac{1}{y} \right]_1^4 = \pi \left(1 - \frac{1}{4} \right) = \boxed{\frac{3}{4}\pi}.$$

8. Since we are revolving around the y -axis with the disk method, first solve the equation for x to get $x = y^{3/2}$. Then the radius of each disk is $y^{3/2}$, so the volume is

$$V = \pi \int_1^4 (y^{3/2})^2 dy = \pi \int_1^4 y^3 dy = \pi \left[\frac{1}{4}y^4 \right]_1^4 = \boxed{\frac{255}{4}\pi}.$$

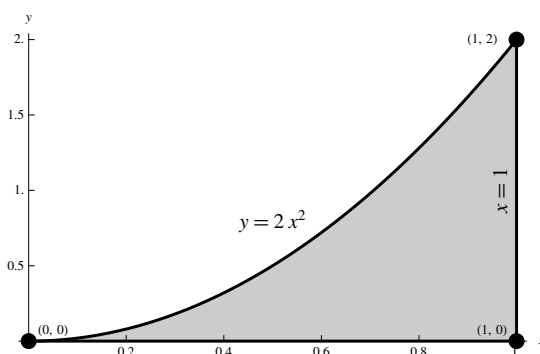
9. Using the washer method along the x -axis, each outer radius is $\sec x$ and the inner radius is 1, so that the volume is

$$V = \pi \int_{-1}^1 (\sec^2 x - 1^2) dx = \pi \int_{-1}^1 (\sec^2 x - 1) dx = \pi [\tan x - x]_{-1}^1 = \boxed{2\pi(\tan 1 - 1)}.$$

10. Since we are revolving around the y -axis, we solve $y = x^2$ for x , giving $x = \sqrt{y}$ (since $x \geq 0$). Then using the washer method, the volume is

$$V = \pi \int_0^4 (2^2 - (\sqrt{y})^2) dy = \pi \int_0^4 (4 - y) dy = \pi \left[4y - \frac{1}{2}y^2 \right]_0^4 = \boxed{8\pi}.$$

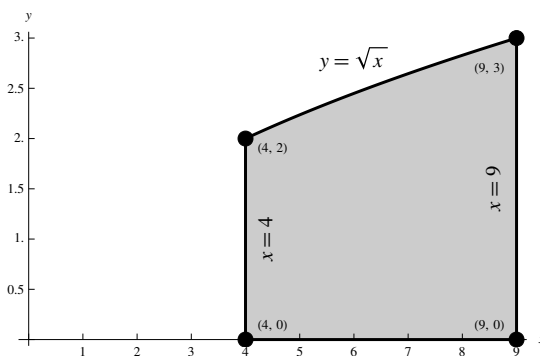
11. The region is shown below:



The radius of each disk is $2x^2$, so the volume is

$$V = \pi \int_0^1 (2x^2)^2 dx = \pi \int_0^1 4x^4 dx = \pi \left[\frac{4}{5}x^5 \right]_0^1 = \boxed{\frac{4}{5}\pi}.$$

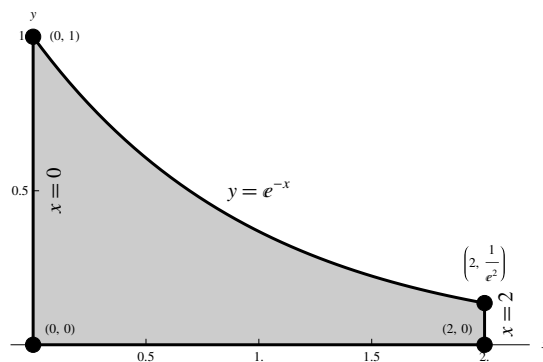
12. The region is shown below:



The radius of each disk is $\sqrt{x}$, so the volume is

$$V = \pi \int_4^9 (\sqrt{x})^2 dx = \pi \int_4^9 x dx = \pi \left[\frac{1}{2}x^2 \right]_4^9 = \boxed{\frac{65}{2}\pi}.$$

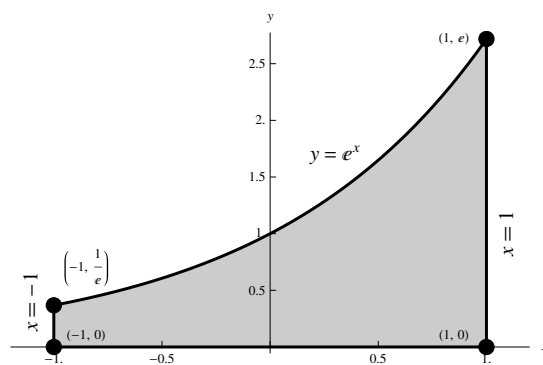
13. The region is shown below:



The radius of each disk is e^{-x} , so the volume is

$$V = \pi \int_0^2 (e^{-x})^2 dx = \pi \int_0^2 e^{-2x} dx = \pi \left[-\frac{1}{2} e^{-2x} \right]_0^2 = \boxed{\frac{1}{2} \pi \left(1 - \frac{1}{e^4} \right)}.$$

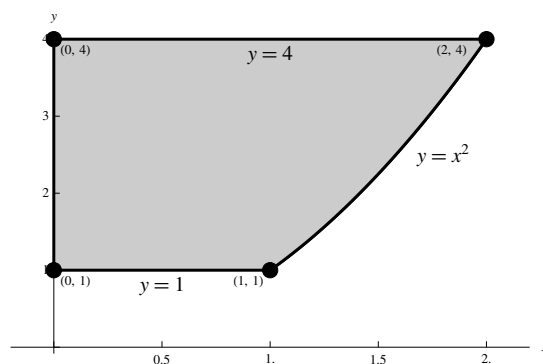
14. The region is shown below:



The radius of each disk is e^x , so the volume is

$$V = \pi \int_{-1}^1 (e^x)^2 dx = \pi \int_{-1}^1 e^{2x} dx = \pi \left[\frac{1}{2} e^{2x} \right]_{-1}^1 = \boxed{\frac{1}{2} \pi \left(e^2 - \frac{1}{e^2} \right)}.$$

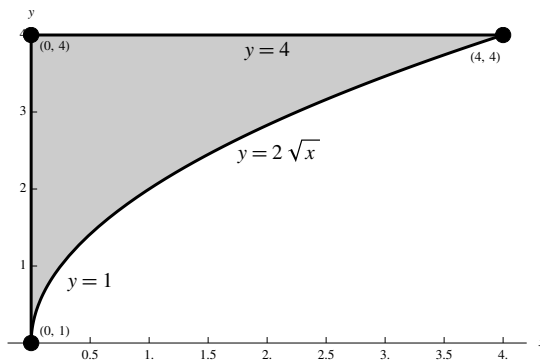
15. The region is shown below:



Solving for x gives $x = \sqrt{y}$ (since $x \geq 0$), so the radius of each disk is $\sqrt{y}$ and the volume is

$$V = \pi \int_1^4 (\sqrt{y})^2 dy = \pi \int_1^4 y dy = \pi \left[\frac{1}{2} y^2 \right]_1^4 = \boxed{\frac{15}{2} \pi}.$$

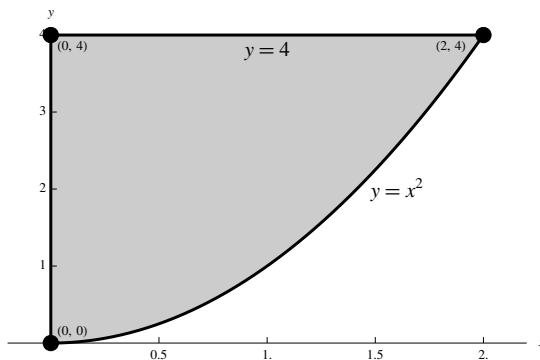
16. The region is shown below:



Solving for x gives $x = \frac{y^2}{4}$, so the radius of each disk is $\frac{y^2}{4}$ and the volume is

$$V = \pi \int_0^4 \left(\frac{y^2}{4} \right)^2 dy = \pi \int_0^4 \frac{1}{16} y^4 dy = \pi \left[\frac{1}{80} y^5 \right]_0^4 = \boxed{\frac{64}{5} \pi}.$$

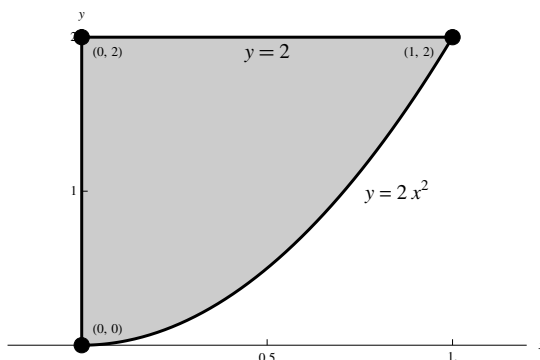
17. The region is shown below:



The outer radius of each washer is 4 and the inner radius is x^2 , so the volume of the solid is

$$V = \pi \int_0^2 (4^2 - (x^2)^2) dx = \pi \int_0^2 (16 - x^4) dx = \pi \left[16x - \frac{1}{5} x^5 \right]_0^2 = \boxed{\frac{128}{5} \pi}.$$

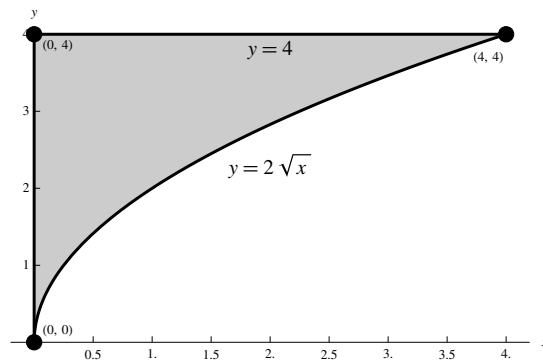
18. The region is shown below:



The outer radius of each washer is 2 and the inner radius is $2x^2$, so the volume of the solid is

$$V = \pi \int_0^1 \left(2^2 - (2x^2)^2 \right) dx = \pi \int_0^1 (4 - 4x^4) dx = \pi \left[4x - \frac{4}{5}x^5 \right]_0^1 = \boxed{\frac{16}{5}\pi}.$$

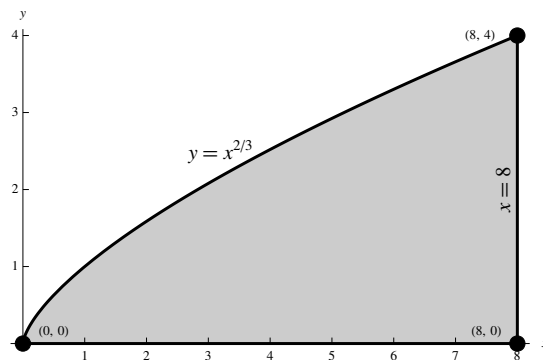
19. The region is shown below:



The outer radius of each washer is 4 and the inner radius is $2\sqrt{x}$, so the volume of the solid is

$$V = \pi \int_0^4 \left(4^2 - (2\sqrt{x})^2 \right) dx = \pi \int_0^4 (16 - 4x) dx = \pi [16x - 2x^2]_0^4 = \boxed{32\pi}.$$

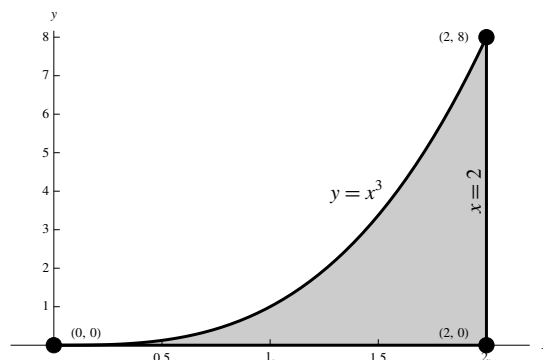
20. The region is shown below:



Solving $y = x^{2/3}$ for x gives $x = y^{3/2}$, so the outer radius of each washer when revolved around the y -axis is 8 and the inner radius is $y^{3/2}$. Therefore the volume is

$$V = \pi \int_0^4 \left(8^2 - (y^{3/2})^2 \right) dy = \pi \int_0^4 (64 - y^3) dy = \pi \left[64y - \frac{1}{4}y^4 \right]_0^4 = \boxed{192\pi}.$$

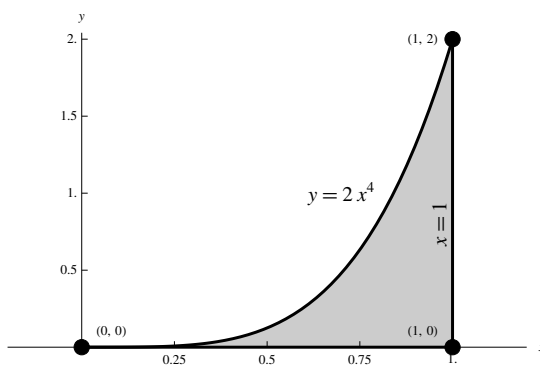
21. The region is shown below:



Solving $y = x^3$ for x gives $x = y^{1/3}$, so the outer radius of each washer when revolved around the y -axis is 2 and the inner radius is $y^{1/3}$. Therefore the volume is

$$V = \pi \int_0^8 \left(2^2 - \left(y^{1/3} \right)^2 \right) dy = \pi \int_0^8 \left(4 - y^{2/3} \right) dy = \pi \left[4y - \frac{3}{5} y^{5/3} \right]_0^8 = \boxed{\frac{64}{5} \pi}.$$

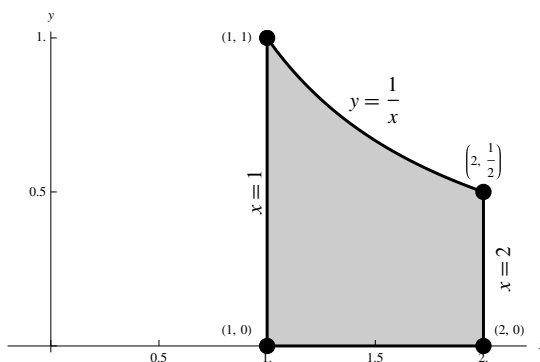
22. The region is shown below:



Solving $y = 2x^4$ for x gives $x = \left(\frac{y}{2}\right)^{1/4}$ (since $x \geq 0$), so the outer radius of each washer when revolved around the y -axis is 1 and the inner radius is $\left(\frac{y}{2}\right)^{1/4}$. Therefore the volume is

$$V = \pi \int_0^2 \left(1^2 - \left(\left(\frac{y}{2} \right)^{1/4} \right)^2 \right) dy = \pi \int_0^2 \left(1 - \frac{1}{\sqrt{2}} y^{1/2} \right) dy = \pi \left[y - \frac{\sqrt{2}}{3} y^{3/2} \right]_0^2 = \boxed{\frac{2}{3} \pi}.$$

23. The region is shown below:



Using the disk method, the radius of each disk around the x -axis is $\frac{1}{x}$, so the volume is

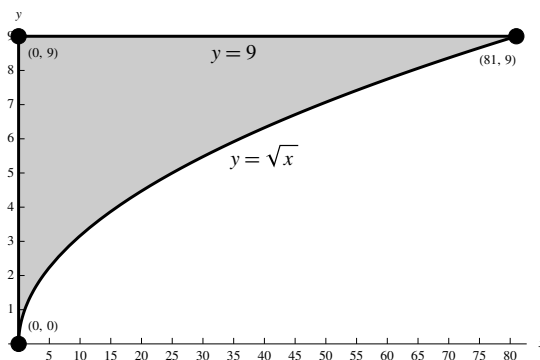
$$V = \pi \int_1^2 \left(\frac{1}{x} \right)^2 dx = \pi \int_1^2 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^2 = \boxed{\frac{\pi}{2}}.$$

24. This is the same region from Problem 23; it is pictured there. To rotate around the y -axis, we use the washer method. Solving $y = \frac{1}{x}$ for x gives $x = \frac{1}{y}$. Now, $x = \frac{1}{y}$ and $x = 2$ intersect when $y = \frac{1}{2}$. So from $y = 0$ to $y = \frac{1}{2}$, the outer radius is 2 while the inner radius is 1. From $y = \frac{1}{2}$ to $y = 1$ the outer

radius is $\frac{1}{y}$ and the inner radius is 1. Therefore the volume is

$$\begin{aligned}
 V &= \pi \int_0^{1/2} (2^2 - 1^2) dy + \pi \int_{1/2}^1 \left(\left(\frac{1}{y} \right)^2 - 1^2 \right) dy \\
 &= \pi \int_0^{1/2} 3 dy + \pi \int_{1/2}^1 \left(\frac{1}{y^2} - 1 \right) dy \\
 &= \pi [3y]_0^{1/2} + \pi \left[-\frac{1}{y} - y \right]_{1/2}^1 \\
 &= \boxed{2\pi}.
 \end{aligned}$$

25. The region is shown below:



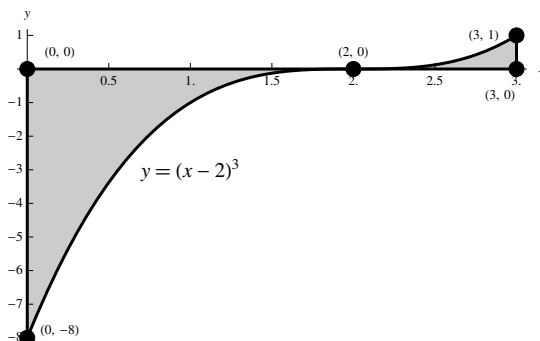
Using the disk method to revolve about the y -axis, first solve $y = \sqrt{x}$ for x to get $x = y^2$. Then the radius of each disk is y^2 , so the volume is

$$V = \pi \int_0^9 (y^2)^2 dy = \pi \int_0^9 y^4 dy = \pi \left[\frac{1}{5} y^5 \right]_0^9 = \boxed{\frac{59049}{5} \pi}.$$

26. This is the same region from Problem 25; it is pictured there. To rotate around the x -axis, we use the disk method. The outer radius of each disk is 9, while the inner radius is $\sqrt{x}$. Therefore the volume is

$$V = \pi \int_0^{81} (9^2 - (\sqrt{x})^2) dx = \pi \int_0^{81} (81 - x) dx = \pi \left[81x - \frac{1}{2} x^2 \right]_0^{81} = \boxed{\frac{6561}{2} \pi}.$$

27. The region is shown below:



When revolving about the x -axis, we use the disk method. Note that the region lies both above and below the x -axis, so from $x = 0$ to $x = 2$, the radius will be $-(x-2)^3$, while from $x = 2$ to $x = 3$ it will

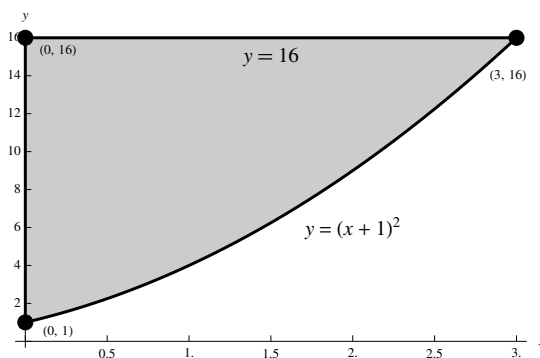
be $(x-2)^3$. However, since we square the radius, the minus sign will disappear in the computation — that is, $((x-2)^3)^2 = (-(x-2)^3)^2$. Therefore the volume is

$$V = \pi \int_0^3 ((x-2)^3)^2 dx = \pi \int_0^3 (x-2)^6 dx = \pi \left[\frac{1}{7}(x-2)^7 \right]_0^3 = \boxed{\frac{129}{7}\pi}.$$

28. This is the same region from Problem 27; it is pictured there. To rotate around the y -axis, first solve $y = (x-2)^3$ for x to get $x = 2 + y^{1/3}$. We use the disk method from $y = -8$ to $y = 0$, where the radius is $2 + y^{1/3}$. From $y = 0$ to $y = 1$ we use the washer method; the outer radius is 3 while the inner radius is $2 + y^{1/3}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_{-8}^0 (2 + y^{1/3})^2 dy + \pi \int_0^1 (3^2 - (2 + y^{1/3})^2) dy \\ &= \pi \int_{-8}^0 (y^{2/3} + 4y^{1/3} + 4) dy + \pi \int_0^1 (5 - 4y^{1/3} - y^{2/3}) dy \\ &= \pi \left[\frac{3}{5}y^{5/3} + 3y^{4/3} + 4y \right]_{-8}^0 + \pi \left[5y - 3y^{4/3} - \frac{3}{5}y^{5/3} \right]_0^1 \\ &= \boxed{\frac{23}{5}\pi}. \end{aligned}$$

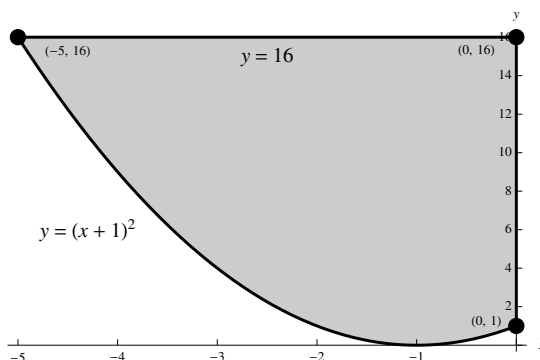
29. The region is shown below:



When revolving about the y -axis, we first solve $y = (x+1)^2$ for x to get $x = \sqrt{y} - 1$. Using the disk method, the radius of each disk is $\sqrt{y} - 1$, so the volume is

$$V = \pi \int_1^{16} (\sqrt{y} - 1)^2 dy = \pi \int_1^{16} (y - 2y^{1/2} + 1) dy = \pi \left[\frac{1}{2}y^2 - \frac{4}{3}y^{3/2} + y \right]_1^{16} = \boxed{\frac{117}{2}\pi}.$$

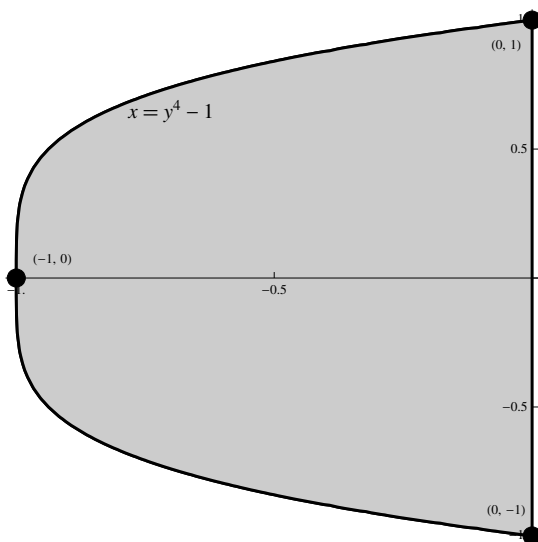
30. The region is shown below:



To revolve around the x -axis, we use the washer method; each outer radius is 16 while the inner radius is $(x+1)^2$. Therefore the volume is

$$V = \pi \int_{-5}^0 \left(16^2 - ((x+1)^2)^2 \right) dx = \pi \int_{-5}^0 (256 - (x+1)^4) dx = \pi \left[256x - \frac{1}{5}(x+1)^5 \right]_{-5}^0 = \boxed{1075\pi}.$$

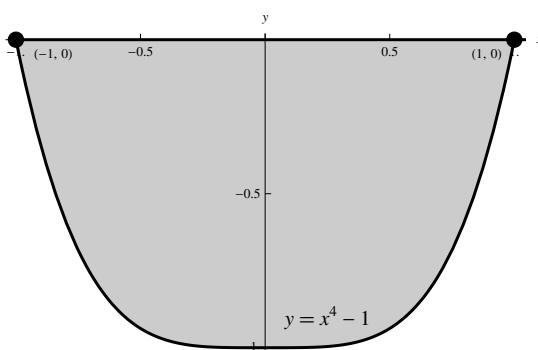
31. The region is shown below:



We use the disk method to revolve about the y -axis; the radius is $1 - y^4$. Therefore the volume is

$$V = \pi \int_{-1}^1 (1 - y^4)^2 dy = \pi \int_{-1}^1 (y^8 - 2y^4 + 1) dy = \pi \left[\frac{1}{9}y^9 - \frac{2}{5}y^5 + y \right]_{-1}^1 = \boxed{\frac{64}{45}\pi}.$$

32. The region is shown below:

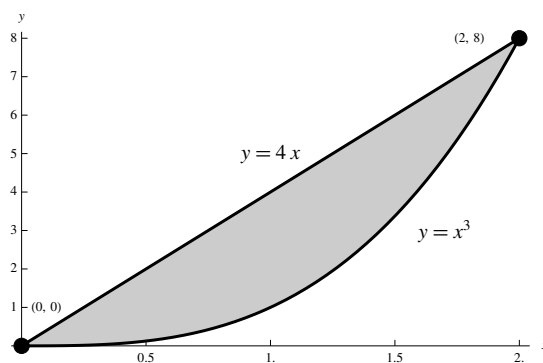


We use the disk method to revolve about the x -axis; the radius is $1 - x^4$ (since $x^4 - 1 < 0$). Therefore the volume is

$$V = \pi \int_{-1}^1 (1 - x^4)^2 dx = \pi \int_{-1}^1 (1 - 2x^4 + x^8) dx = \pi \left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9 \right]_{-1}^1 = \boxed{\frac{64}{45}\pi}.$$

(Note that this is the same region as in Problem 31, rotated 90° , so we would expect to get the same answer.)

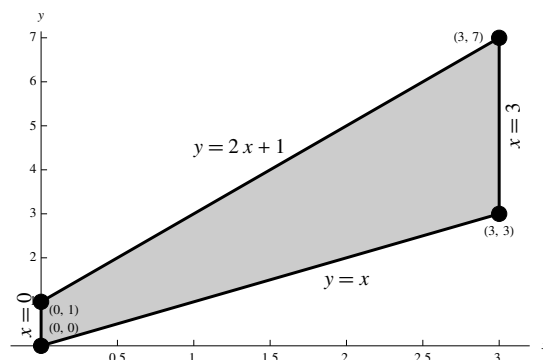
33. The region is shown below:



We use the washer method to revolve about the x -axis. Each outer radius is $4x$, and each inner radius is x^3 , so the volume is

$$V = \pi \int_0^2 \left((4x)^2 - (x^3)^2 \right) dx = \pi \int_0^2 (16x^2 - x^6) dx = \pi \left[\frac{16}{3}x^3 - \frac{1}{7}x^7 \right]_0^2 = \boxed{\frac{512}{21}\pi}.$$

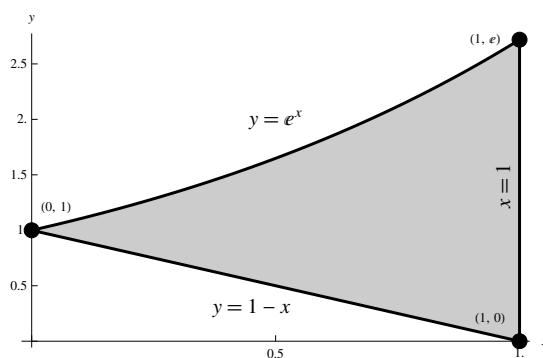
34. The region is shown below:



We use the washer method to revolve about the x -axis. Each outer radius is $2x + 1$, and each inner radius is x , so the volume is

$$V = \pi \int_0^3 \left((2x + 1)^2 - x^2 \right) dx = \pi \int_0^3 (3x^2 + 4x + 1) dx = \pi \left[x^3 + 2x^2 + x \right]_0^3 = \boxed{48\pi}.$$

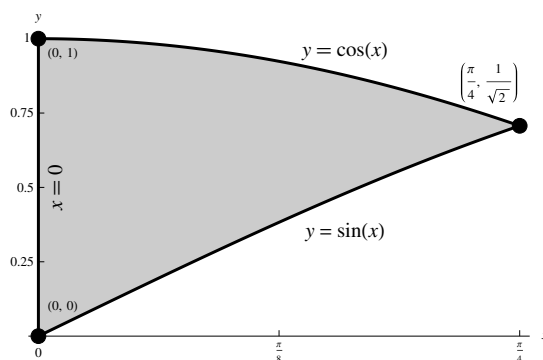
35. The region is shown below:



We use the washer method to revolve about the x -axis. Each outer radius is e^x , and each inner radius is $1 - x$, so the volume is

$$\begin{aligned}
 V &= \pi \int_0^1 \left((e^x)^2 - (1 - x)^2 \right) dx \\
 &= \pi \int_0^1 (e^{2x} - 1 + 2x - x^2) dx \\
 &= \pi \left[\frac{1}{2} e^{2x} - x + x^2 - \frac{1}{3} x^3 \right]_0^1 \\
 &= \boxed{\left(\frac{1}{2} e^2 - \frac{5}{6} \right) \pi}.
 \end{aligned}$$

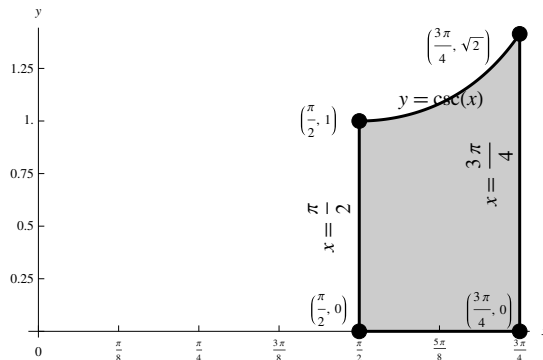
36. The region is shown below:



We use the washer method to revolve about the x -axis. Each outer radius is $\cos x$ and each inner radius is $\sin x$, so the volume is

$$\begin{aligned}
 V &= \pi \int_0^{\pi/4} \left((\cos x)^2 - (\sin x)^2 \right) dx \\
 &= \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx \\
 &= \pi \int_0^{\pi/4} \cos 2x dx \\
 &= \pi \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} \\
 &= \boxed{\frac{\pi}{2}}.
 \end{aligned}$$

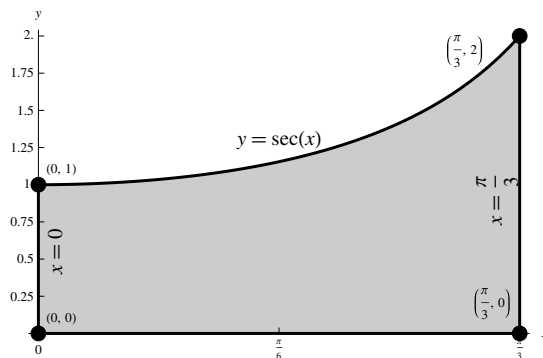
37. The region is shown below:



We use the disk method to revolve about the x -axis. Each radius is $\csc x$, so the volume is

$$V = \pi \int_{\pi/2}^{3\pi/4} (\csc x)^2 dx = \pi \int_{\pi/2}^{3\pi/4} \csc^2 x dx = \pi [-\cot x]_{\pi/2}^{3\pi/4} = \boxed{\pi}.$$

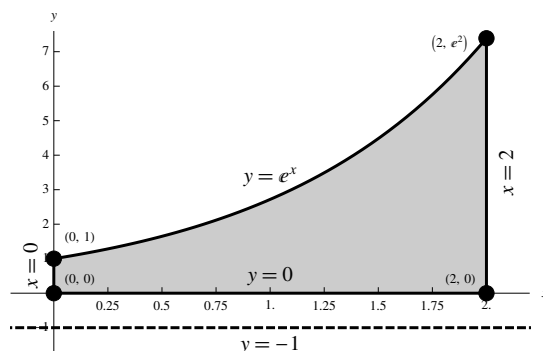
38. The region is shown below:



We use the disk method to revolve about the x -axis. Each radius is $\sec x$, so the volume is

$$V = \pi \int_0^{\pi/3} (\sec x)^2 dx = \pi \int_0^{\pi/3} \sec^2 x dx = \pi [\tan x]_0^{\pi/3} = \boxed{\pi\sqrt{3}}.$$

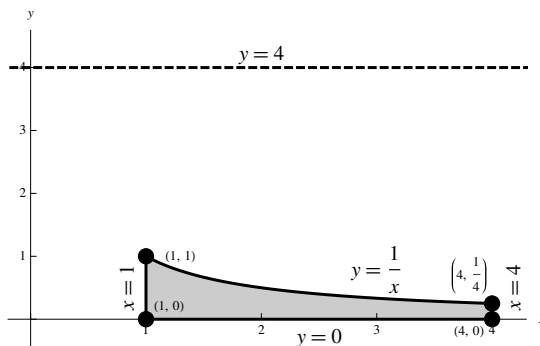
39. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $y = -1$, we use the washer method. Each outer radius is $e^x - (-1) = e^x + 1$, and each inner radius is $0 - (-1) = 1$. Therefore the volume is

$$V = \pi \int_0^2 \left((e^x + 1)^2 - 1^2 \right) dx = \pi \int_0^2 (e^{2x} + 2e^x) dx = \pi \left[\frac{1}{2}e^{2x} + 2e^x \right]_0^2 = \boxed{\pi \left(\frac{1}{2}e^4 + 2e^2 - \frac{5}{2} \right)}.$$

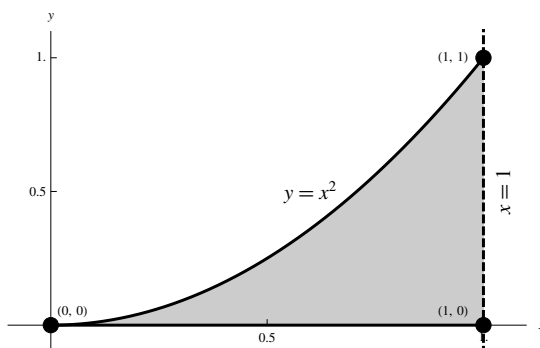
40. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $y = 4$, we use the washer method. Each outer radius is $4 - 0 = 4$, and each inner radius is $4 - \frac{1}{x}$. Therefore the volume is

$$V = \pi \int_1^4 \left(4^2 - \left(4 - \frac{1}{x} \right)^2 \right) dx = \pi \int_1^4 \left(\frac{8}{x} - \frac{1}{x^2} \right) dx = \pi \left[8 \ln x + \frac{1}{x} \right]_1^4 = \boxed{\left(16 \ln 2 - \frac{3}{4} \right) \pi}.$$

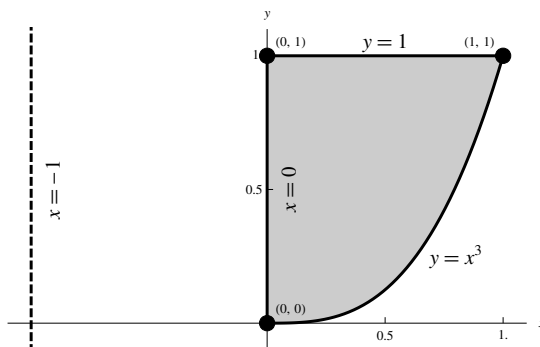
41. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $x = 1$, we first solve $y = x^2$ for x to get $x = \sqrt{y}$. Using the disk method, each radius is $1 - \sqrt{y}$. Therefore the volume is

$$V = \pi \int_0^1 (1 - \sqrt{y})^2 dy = \pi \int_0^1 \left(1 - 2y^{1/2} + y \right) dy = \pi \left[y - \frac{4}{3}y^{3/2} + \frac{1}{2}y^2 \right]_0^1 = \boxed{\frac{\pi}{6}}.$$

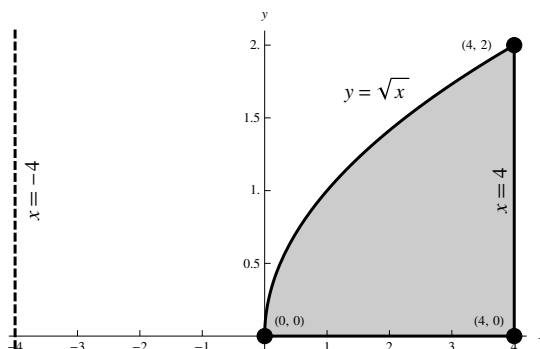
42. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $x = -1$, we first solve $y = x^3$ for x to get $x = y^{1/3}$. Using the washer method, the outer radius is $y^{1/3} - (-1) = y^{1/3} + 1$, while the inner radius is $0 - (-1) = 1$. Therefore the volume is

$$V = \pi \int_0^1 \left((y^{1/3} + 1)^2 - 1^2 \right) dy = \pi \int_0^1 (y^{2/3} + 2y^{1/3}) dy = \pi \left[\frac{3}{5}y^{5/3} + \frac{3}{2}y^{4/3} \right]_0^1 = \boxed{\frac{21}{10}\pi}.$$

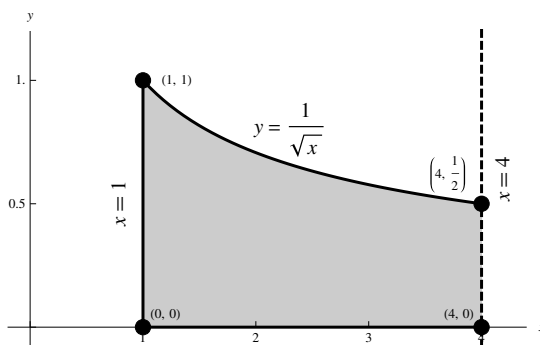
43. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $x = -4$, we first solve $y = \sqrt{x}$ for x to get $x = y^2$. Using the washer method, the outer radius is $4 - (-4) = 8$ and the inner radius is $y^2 - (-4) = y^2 + 4$. Therefore the volume is

$$V = \pi \int_0^2 (8^2 - (y^2 + 4)^2) dy = \pi \int_0^2 (48 - 8y^2 - y^4) dy = \pi \left[48y - \frac{8}{3}y^3 - \frac{1}{5}y^5 \right]_0^2 = \boxed{\frac{1024}{15}\pi}.$$

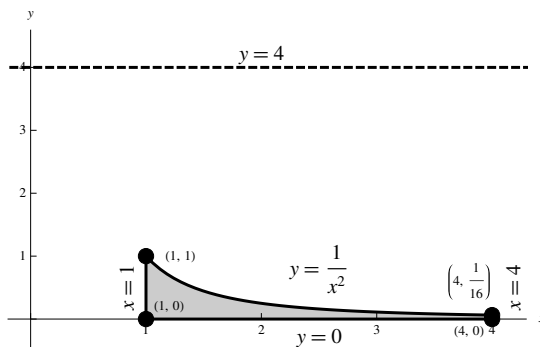
44. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $x = 4$, we first solve $y = \frac{1}{\sqrt{x}}$ for x to get $x = \frac{1}{y^2}$. For $0 \leq y \leq \frac{1}{2}$, we can use the disk method with radius $4 - 1 = 3$. For $\frac{1}{2} \leq y \leq 1$, we use the washer method, with outer radius $4 - 1 = 3$ and inner radius $4 - \frac{1}{y^2}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^{1/2} 3^2 dy + \pi \int_{1/2}^1 \left(3^2 - \left(4 - \frac{1}{y^2} \right)^2 \right) dy \\ &= \pi \int_0^{1/2} 9 dy + \pi \int_{1/2}^1 \left(-7 + \frac{8}{y^2} - \frac{1}{y^4} \right) dy \\ &= \pi [9y]_0^{1/2} + \pi \left[-7y - \frac{8}{y} + \frac{1}{3y^3} \right]_{1/2}^1 \\ &= \boxed{\frac{20}{3}\pi}. \end{aligned}$$

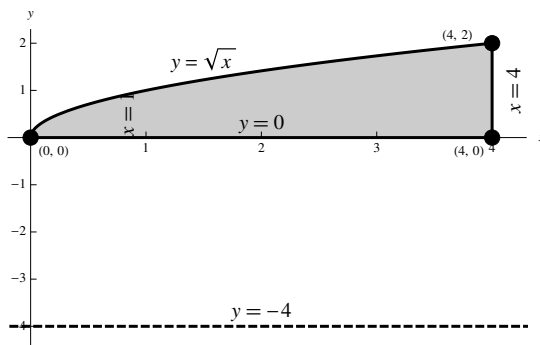
45. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $y = 4$, we use the washer method. The outer radius is $4 - 0 = 4$, while the inner radius is $4 - \frac{1}{x^2}$. Therefore the volume is

$$V = \pi \int_1^4 \left(4^2 - \left(4 - \frac{1}{x^2} \right)^2 \right) dx = \pi \int_1^4 \left(\frac{8}{x^2} - \frac{1}{x^4} \right) dx = \pi \left[-\frac{8}{x} + \frac{1}{3x^3} \right]_1^4 = \boxed{\frac{363}{64}\pi}.$$

46. The region is shown below, with the line of revolution shown as a dashed line:

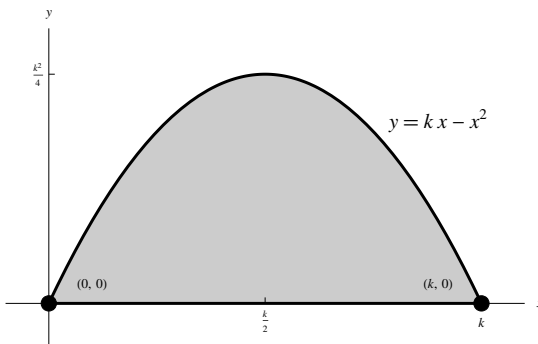


To revolve about $y = -4$, we use the washer method. The outer radius is $\sqrt{x} - (-4) = \sqrt{x} + 4$, while the inner radius is $0 - (-4) = 4$. Therefore the volume is

$$V = \pi \int_0^4 \left((\sqrt{x} + 4)^2 - 4^2 \right) dx = \pi \int_0^4 (8\sqrt{x} + x) dx = \pi \left[\frac{16}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^4 = \boxed{\frac{152}{3}\pi}.$$

Applications and Extensions

47. The graph of $y = kx - x^2$ intersects the x -axis at $x = 0$ and again at $x = k$. A graph of $kx - x^2$, with the region in question shaded, is below:



- (a) When the region is revolved around the x -axis, we use the disk method; the radius is $kx - x^2$, so the volume is

$$V_x = \pi \int_0^k (kx - x^2)^2 dx = \pi \int_0^k (k^2x^2 - 2kx^3 + x^4) dx = \pi \left[\frac{1}{3}k^2x^3 - \frac{1}{2}kx^4 + \frac{1}{5}x^5 \right]_0^k = \boxed{\frac{1}{30}\pi k^5}.$$

- (b) When the region is revolved around the y -axis, we use the washer method. First solve $y = kx - x^2$ for x , giving

$$x = \frac{1}{2}(k \pm \sqrt{k^2 - 4y}).$$

These two roots correspond to the portions of the curve to the left and to the right, respectively, of $x = \frac{k}{2}$. Therefore the inner radius is given by the first root and the outer radius by the second root, so the volume is

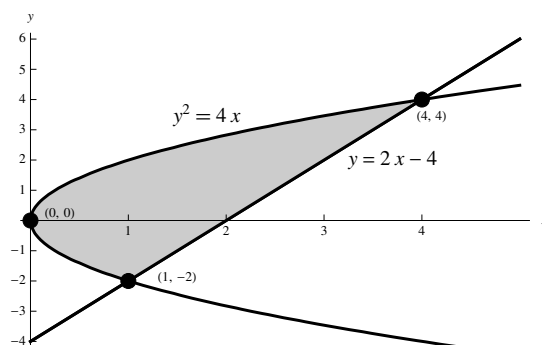
$$\begin{aligned} V_y &= \pi \int_0^{k^2/4} \left(\left(\frac{1}{2}(k + \sqrt{k^2 - 4y}) \right)^2 - \left(\frac{1}{2}(k - \sqrt{k^2 - 4y}) \right)^2 \right) dy \\ &= \frac{\pi}{4} \int_0^{k^2/4} \left(k^2 + 2k\sqrt{k^2 - 4y} + k^2 - 4y - (k^2 - 2k\sqrt{k^2 - 4y} + k^2 - 4y) \right) dy \\ &= \frac{\pi}{4} \int_0^{k^2/4} 4k(k^2 - 4y)^{1/2} dy \\ &= \frac{\pi}{4} \left[-\frac{2}{3}k(k^2 - 4y)^{3/2} \right]_0^{k^2/4} \\ &= \boxed{\frac{1}{6}\pi k^4}. \end{aligned}$$

- (c) The two volumes are equal when $\frac{1}{30}\pi k^5 = \frac{1}{6}\pi k^4$. Since $k > 0$, dividing through by πk^4 gives $\frac{1}{30}k = \frac{1}{6}$, so that $\boxed{k = 5}$.
48. (a) The graphs of $y = 2x + b$ and $y^2 = 4x$ intersect when $(2x + b)^2 = 4x$. Expanding this equation gives $4x^2 + (4b - 4)x + b^2 = 0$; this quadratic equation has roots

$$x = \frac{4 - 4b \pm \sqrt{(4b - 4)^2 - 16b^2}}{8} = \frac{1}{2}(1 - b \pm \sqrt{1 - 2b}).$$

So there are two distinct points of intersection when the roots of this quadratic equation are real and distinct, which occurs when $1 - 2b > 0$, or when $b < \frac{1}{2}$.

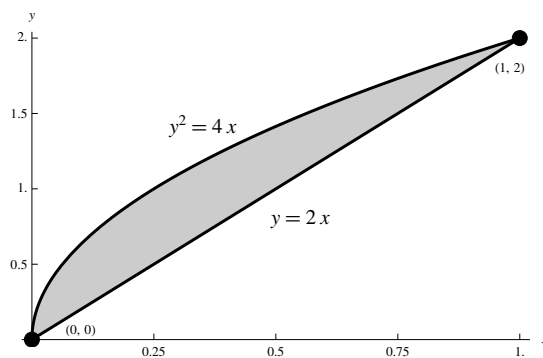
(b) When $b = -4$, the region is shown below:



To find the area of this region, we partition along the y -axis. Solving these two equations for x gives $x = \frac{y^2}{4}$ and $x = \frac{y+4}{2} = \frac{1}{2}y + 2$, so the area of the region is

$$\int_{-2}^4 \left(\frac{1}{2}y + 2 - \frac{y^2}{4} \right) dy = \left[\frac{1}{4}y^2 + 2y - \frac{1}{12}y^3 \right]_{-2}^4 = \boxed{9}.$$

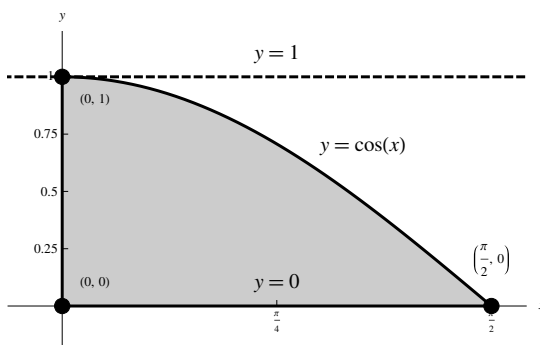
(c) When $b = 0$, the region is shown below:



Revolving this region about the x -axis, we use the washer method; the outer radius is $\sqrt{4x}$ and the inner radius is $2x$, so that the volume is

$$V = \pi \int_0^1 \left((\sqrt{4x})^2 - (2x)^2 \right) dx = \pi \int_0^1 (4x - 4x^2) dx = \pi \left[2x^2 - \frac{4}{3}x^3 \right]_0^1 = \boxed{\frac{2}{3}\pi}.$$

49. The region is shown below, with the line of revolution shown as a dashed line:



Revolving about the line $y = 1$, we use the washer method. The outer radius is $1 - 0 = 1$, and the inner radius is $1 - \cos x$, so the volume is

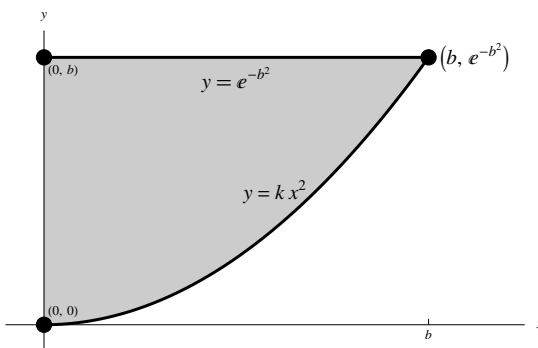
$$\begin{aligned}
 V &= \pi \int_0^{\pi/2} (1^2 - (1 - \cos x)^2) dx = \pi \int_0^{\pi/2} (2 \cos x - \cos^2 x) dx \\
 &= \pi \int_0^{\pi/2} \left(2 \cos x - \left(\frac{1 + \cos 2x}{2} \right) \right) dx \\
 &= \pi \int_0^{\pi/2} \left(2 \cos x - \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\
 &= \pi \left[2 \sin x - \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^{\pi/2} \\
 &= \pi \left(2 - \frac{\pi}{4} \right) = \boxed{2\pi - \frac{\pi^2}{4}}.
 \end{aligned}$$

50. A plot of this region is identical to the plot for Problem 49, except that the line of rotation is $y = -1$ instead of $y = 1$. Again we use the washer method; this time the outer radius is $\cos x - (-1) = 1 + \cos x$ and the inner radius is $0 - (-1) = 1$. Therefore the volume is

$$\begin{aligned}
 V &= \pi \int_0^{\pi/2} ((1 + \cos x)^2 - 1^2) dx = \pi \int_0^{\pi/2} (2 \cos x + \cos^2 x) dx \\
 &= \pi \int_0^{\pi/2} \left(2 \cos x + \frac{1 + \cos 2x}{2} \right) dx \\
 &= \pi \int_0^{\pi/2} \left(2 \cos x + \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \pi \left[2 \sin x + \frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\pi/2} \\
 &= \pi \left(2 + \frac{\pi}{4} \right) = \boxed{2\pi + \frac{\pi^2}{4}}.
 \end{aligned}$$

Challenge Problems

51. $P(x)$ together with the line $y = e^{-b^2}$ is below:



- (a) No matter what k is, $P(0) = 0$, so that $(0, 0)$ is always on the graph. Evaluating at b , we also have $P(b) = kb^2 = e^{-b^2}$, so that $k = \frac{1}{b^2}e^{-b^2}$. Therefore $P(x) = \frac{1}{b^2}e^{-b^2}x^2$.
- (b) The line $y = e^{-b^2}$ intersects the curve at $x = b$. Also, for $0 \leq x \leq b$, we have $\frac{x^2}{b^2} \leq 1$ and so $P(x) = \frac{x^2}{b^2}e^{-b^2} \leq e^{-b^2}$, so that the region is bounded above by $y = e^{-b^2}$ and below by $P(x)$. The

left edge of the region is at $x = 0$ and the right edge is at $x = b$. Using the shell method, we get for the volume

$$\begin{aligned} V &= 2\pi \int_0^b x \left(e^{-b^2} - \frac{1}{b^2} e^{-b^2} x^2 \right) dx = 2\pi e^{-b^2} \int_0^b \left(x - \frac{1}{b^2} x^3 \right) dx \\ &= 2\pi e^{-b^2} \left[\frac{1}{2} x^2 - \frac{1}{4b^2} x^4 \right]_0^b = \boxed{\frac{1}{2} \pi \frac{b^2}{e^{b^2}}}. \end{aligned}$$

- (c) Regarding the volume V as a function of b , it reaches its maximum either at a critical point or at $b = 0$, which is the left end of its interval. However, we need not consider $b = 0$ since we are assuming $b > 0$. Now,

$$V'(b) = \pi b e^{-b^2} + \frac{1}{2} \pi b^2 e^{-b^2} \cdot (-2b) = \pi e^{-b^2} (b - b^3),$$

so the only critical points are when $b - b^3 = 0$, so for $b = 0$ or $b = \pm 1$. Since we are assuming $b > 0$, the only relevant critical point is $b = 1$. Then

$$V(1) = \frac{1}{2} \pi \cdot 1^2 e^{-1^2} = \frac{1}{2} \pi e^{-1/2} \approx 0.578.$$

Since $\lim_{b \rightarrow \infty} V(b) = 0$ as the exponential term dominates, it follows that $\boxed{b = 1}$ is in fact a global maximum.

52. Since $\cosh x \geq 1$ for all values of x , it follows that

$$a \cosh \left(\frac{x}{a} \right) + b - a \geq a + b - a = b,$$

so assuming $b > 0$, we see that $y \geq 0$ for any x . Then using the disk method, the radius of each disk is $a \cosh \left(\frac{x}{a} \right) + b - a$, so the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left(a \cosh \left(\frac{x}{a} \right) + b - a \right)^2 dx \\ &= \pi \int_0^1 \left(a^2 \cosh^2 \left(\frac{x}{a} \right) + 2(b-a)a \cosh \left(\frac{x}{a} \right) + (b-a)^2 \right) dx \\ &= \pi \int_0^1 \left(a^2 \left(\frac{1 + \cosh 2 \left(\frac{x}{a} \right)}{2} \right) + 2(b-a)a \cosh \left(\frac{x}{a} \right) + (b-a)^2 \right) dx \\ &= \pi \int_0^1 \left(\frac{a^2}{2} \cosh 2 \left(\frac{x}{a} \right) + \frac{a^2}{2} + 2(b-a)a \cosh \left(\frac{x}{a} \right) + (b-a)^2 \right) dx \\ &= \pi \left[\frac{a^3}{4} \sinh 2 \left(\frac{x}{a} \right) + \frac{a^2}{2} x + 2(b-a)a^2 \sinh \left(\frac{x}{a} \right) + (b-a)^2 x \right]_0^1 \\ &= \boxed{\pi \left(\frac{a^3}{4} \sinh \frac{2}{a} + \frac{a^2}{2} + 2(b-a)a^2 \sinh \frac{1}{a} + (b-a)^2 \right)}. \end{aligned}$$

6.3: Volume of a Solid of Revolution: Cylindrical Shells

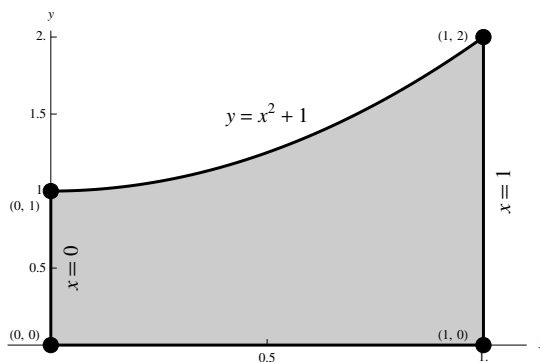
Concepts and Vocabulary

- False. When using the method of shells, the integration occurs along the axis perpendicular to the axis of revolution.
- False. It is given by $\pi R^2 h - \pi r^2 h = \pi(R^2 - r^2)h$.

3. False. Sections 6.2 and 6.3 present methods for determining such volumes under rotation about either the x or y -axis.
4. True. See the discussion in the text. The x represents the radius of each cylinder, $f(x)$ represents the height, and dx represents the thickness.

Skill Building

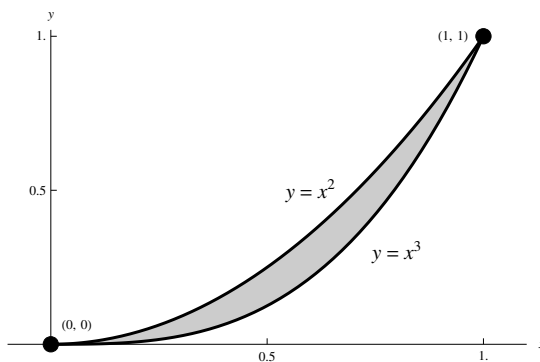
5. The region is shown below:



When revolving about the y -axis, using the shell method, each cylinder has a radius of x and a height of $x^2 + 1$, so the volume is

$$V = 2\pi \int_0^1 x(x^2 + 1) dx = 2\pi \int_0^1 (x^3 + x) dx = 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right]_0^1 = \boxed{\frac{3}{2}\pi}.$$

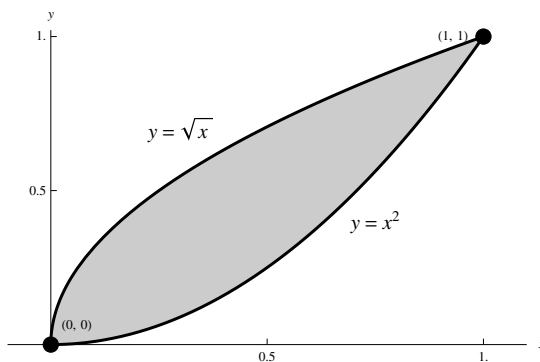
6. The region is shown below:



When revolving about the y -axis using the shell method, each cylinder has a radius of x and a height of $x^2 - x^3$, so the volume is

$$V = 2\pi \int_0^1 x(x^2 - x^3) dx = 2\pi \int_0^1 (x^3 - x^4) dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_0^1 = \boxed{\frac{1}{10}\pi}.$$

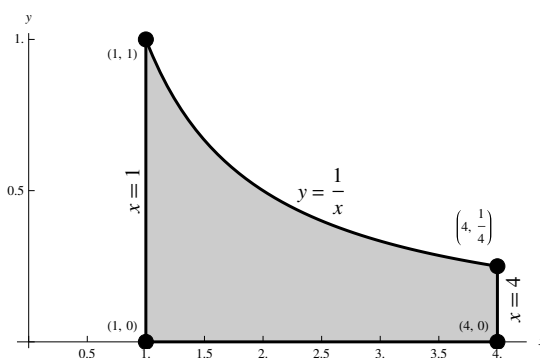
7. The region is shown below:



When revolving about the y -axis using the shell method, each cylinder has a radius of x and a height of $\sqrt{x} - x^2$, so the volume is

$$V = 2\pi \int_0^1 x (\sqrt{x} - x^2) dx = 2\pi \int_0^1 (x^{3/2} - x^3) dx = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right]_0^1 = \boxed{\frac{3}{10}\pi}.$$

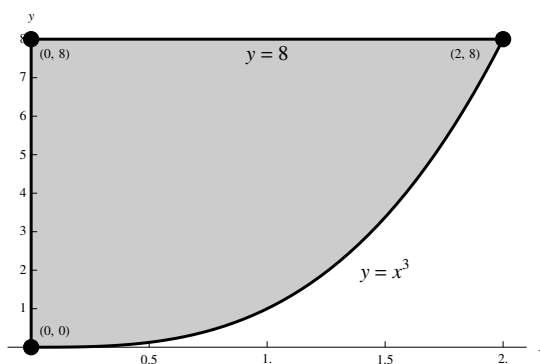
8. The region is shown below:



When revolving about the y -axis using the shell method, each cylinder has a radius of x and a height of $\frac{1}{x}$, so the volume is

$$V = 2\pi \int_1^4 x \cdot \frac{1}{x} dx = 2\pi \int_1^4 1 dx = \boxed{6\pi}.$$

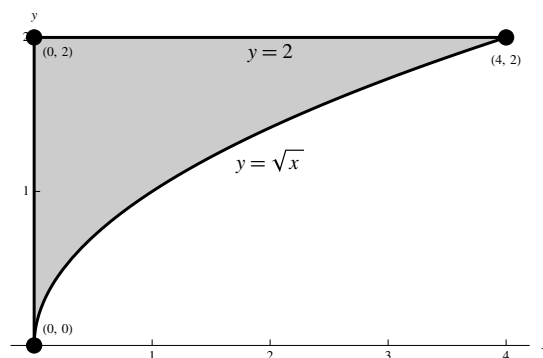
9. The region is shown below:



When revolving about the x -axis using the shell method, we first solve $y = x^3$ for x , giving $x = y^{1/3}$. Then each cylinder has a radius of y and a height of $y^{1/3}$, so the volume is

$$V = 2\pi \int_0^8 y \cdot y^{1/3} dy = 2\pi \int_0^8 y^{4/3} dy = 2\pi \left[\frac{3}{7} y^{7/3} \right]_0^8 = \boxed{\frac{768}{7}\pi}.$$

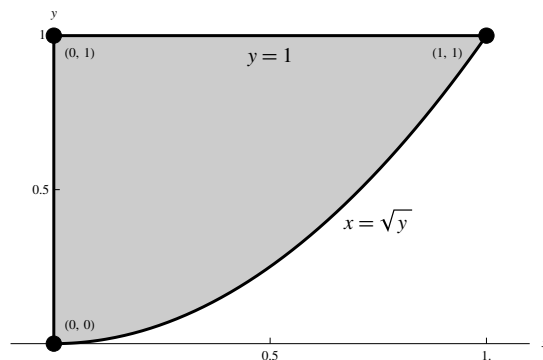
10. The region is shown below:



When revolving about the x -axis using the shell method, we first solve $y = \sqrt{x}$ for x , giving $x = y^2$. Then each cylinder has a radius of y and a height of y^2 , so the volume is

$$V = 2\pi \int_0^2 y \cdot y^2 dy = 2\pi \int_0^2 y^3 dy = 2\pi \left[\frac{1}{4} y^4 \right]_0^2 = \boxed{8\pi}.$$

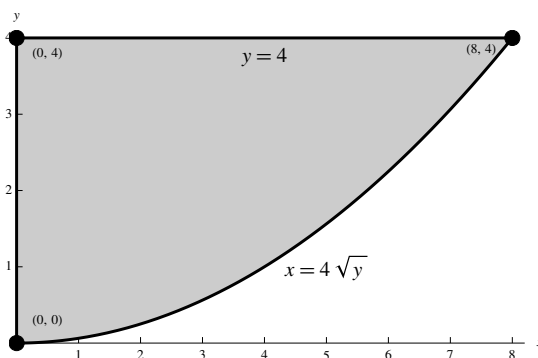
11. The region is shown below:



When revolving about the x -axis using the shell method, each cylinder has a radius of y and a height of $\sqrt{y}$, so the volume is

$$V = 2\pi \int_0^1 y \cdot \sqrt{y} dy = 2\pi \int_0^1 y^{3/2} dy = 2\pi \left[\frac{2}{5} y^{5/2} \right]_0^1 = \boxed{\frac{4}{5}\pi}.$$

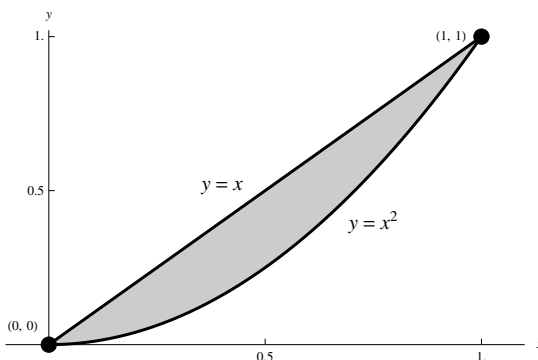
12. The region is shown below:



When revolving about the x -axis using the shell method, each cylinder has a radius of y and a height of $4\sqrt{y}$, so the volume is

$$V = 2\pi \int_0^4 y \cdot 4\sqrt{y} \, dy = 8\pi \int_0^4 y^{3/2} \, dy = 8\pi \left[\frac{2}{5} y^{5/2} \right]_0^4 = \boxed{\frac{512}{5}\pi}.$$

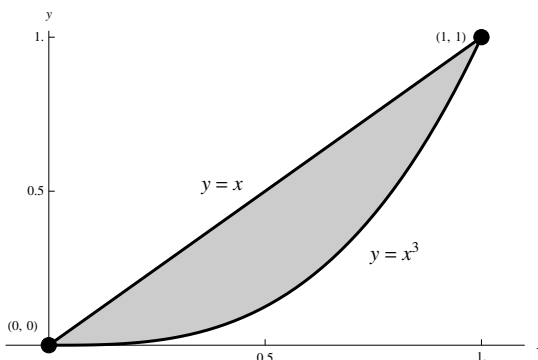
13. The region is shown below:



When revolving about the x -axis using the shell method, we first solve the two equations for x , giving $x = y$ and $x = \sqrt{y}$. Then each shell has a radius of y and a height of $\sqrt{y} - y$, so that the volume is

$$V = 2\pi \int_0^1 y (\sqrt{y} - y) \, dy = 2\pi \int_0^1 (y^{3/2} - y^2) \, dy = 2\pi \left[\frac{2}{5} y^{5/2} - \frac{1}{3} y^3 \right]_0^1 = \boxed{\frac{2}{15}\pi}.$$

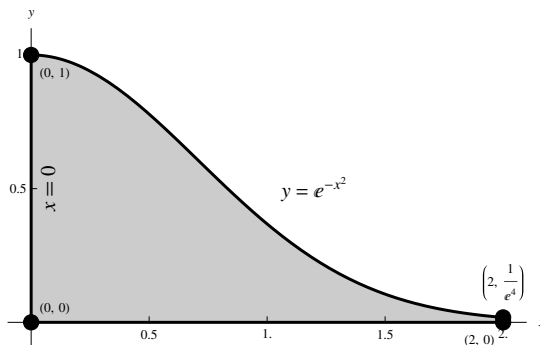
14. The region is shown below:



When revolving about the x -axis using the shell method, we first solve the two equations for x , giving $x = y$ and $x = y^{1/3}$. Then each shell has a radius of y and a height of $y^{1/3} - y$, so that the volume is

$$V = 2\pi \int_0^1 y (y^{1/3} - y) dy = 2\pi \int_0^1 (y^{4/3} - y^2) dy = 2\pi \left[\frac{3}{7} y^{7/3} - \frac{1}{3} y^3 \right]_0^1 = \boxed{\frac{4}{21} \pi}.$$

15. The region is shown below:



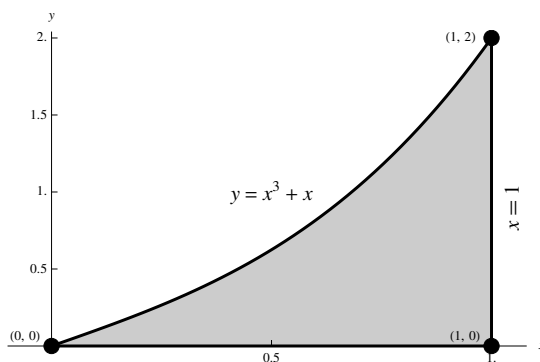
When revolving about the y -axis using the shell method, each shell has a radius of x and a height of e^{-x^2} , so that the volume is

$$V = 2\pi \int_0^2 x e^{-x^2} dx.$$

Now use the substitution $u = -x^2$, so that $du = -2x dx$. Then $x = 0$ corresponds to $u = 0$, and $x = 2$ to $u = -4$, so we get

$$V = -\frac{1}{2} \cdot 2\pi \int_0^{-4} e^u du = -\pi [e^u]_0^{-4} = \boxed{\pi \left(1 - \frac{1}{e^4} \right)}.$$

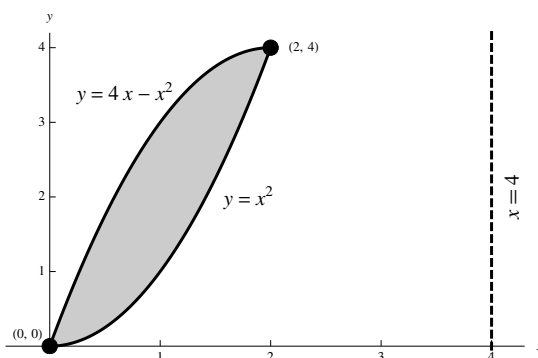
16. The region is shown below:



When revolving about the y -axis using the shell method, each shell has a radius of x and a height of $x^3 + x$, so that the volume is

$$V = 2\pi \int_0^1 x (x^3 + x) dx = 2\pi \int_0^1 (x^4 + x^2) dx = 2\pi \left[\frac{1}{5} x^5 + \frac{1}{3} x^3 \right]_0^1 = \boxed{\frac{16}{15} \pi}.$$

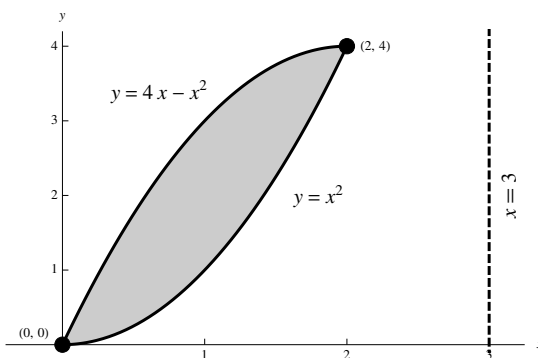
17. The region is shown below:



When revolving about $x = 4$ using the shell method, each shell has a radius of $4 - x$ (the distance from x to 4) and a height of $4x - x^2 - x^2 = 4x - 2x^2$. Therefore the volume is

$$V = 2\pi \int_0^2 (4 - x)(4x - 2x^2) dx = 2\pi \int_0^2 (2x^3 - 12x^2 + 16x) dx = 2\pi \left[\frac{1}{2}x^4 - 4x^3 + 8x^2 \right]_0^2 = \boxed{16\pi}.$$

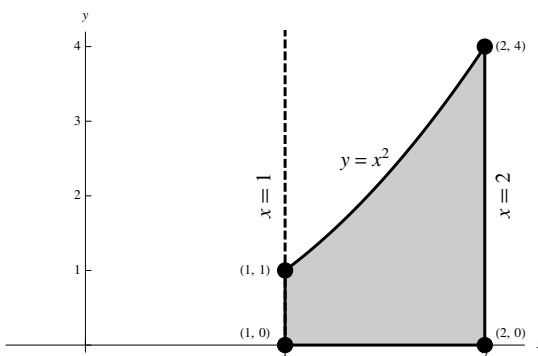
18. The region is shown below:



When revolving about $x = 3$ using the shell method, each shell has a radius of $3 - x$ (the distance from x to 3) and a height of $4x - x^2 - x^2 = 4x - 2x^2$. Therefore the volume is

$$V = 2\pi \int_0^2 (3 - x)(4x - 2x^2) dx = 2\pi \int_0^2 (2x^3 - 10x^2 + 12x) dx = 2\pi \left[\frac{1}{2}x^4 - \frac{10}{3}x^3 + 6x^2 \right]_0^2 = \boxed{\frac{32}{3}\pi}.$$

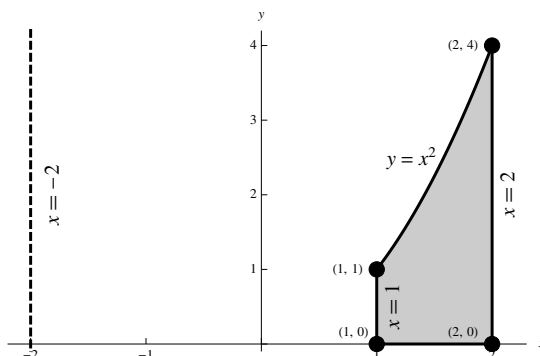
19. The region is shown below:



When revolving about $x = 1$ using the shell method, each shell has a radius of $x - 1$ (the distance from x to 1) and a height of x^2 . Therefore the volume is

$$V = 2\pi \int_1^2 (x-1)x^2 dx = 2\pi \int_1^2 (x^3 - x^2) dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 \right]_1^2 = \boxed{\frac{17}{6}\pi}.$$

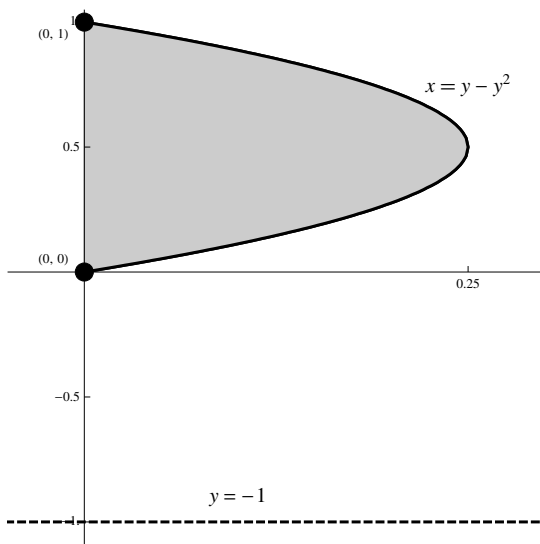
20. The region is shown below:



When revolving about $x = -2$ using the shell method, each shell has a radius of $x - (-2) = x + 2$ (the distance from x to -2) and a height of x^2 . Therefore the volume is

$$V = 2\pi \int_1^2 (x+2)x^2 dx = 2\pi \int_1^2 (x^3 + 2x^2) dx = 2\pi \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_1^2 = \boxed{\frac{43}{6}\pi}.$$

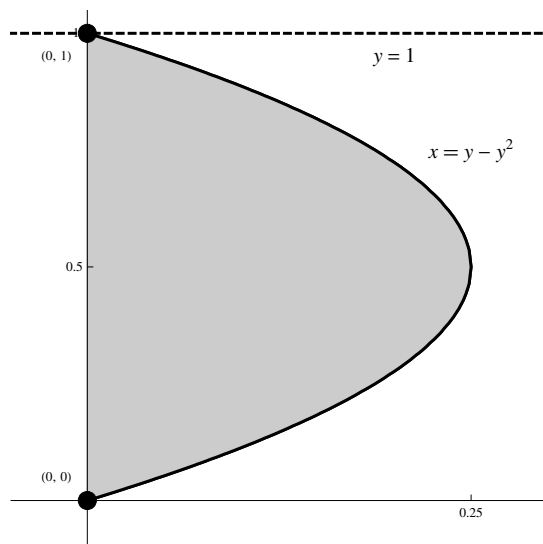
21. The region is shown below:



When revolving about $y = -1$ using the shell method, each shell has a radius of $y - (-1) = y + 1$ (the distance from y to -1) and a height of $y - y^2$. Therefore the volume is

$$V = 2\pi \int_0^1 (y+1)(y-y^2) dy = 2\pi \int_0^1 (-y^3 + y) dy = 2\pi \left[-\frac{1}{4}y^4 + \frac{1}{2}y^2 \right]_0^1 = \boxed{\frac{\pi}{2}}.$$

22. The region is shown below:



When revolving about $y = 1$ using the shell method, each shell has a radius of $1 - y$ (the distance from y to 1) and a height of $y - y^2$. Therefore the volume is

$$V = 2\pi \int_0^1 (1 - y)(y - y^2) dy = 2\pi \int_0^1 (y^3 - 2y^2 + y) dy = 2\pi \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2 \right]_0^1 = \boxed{\frac{\pi}{6}}.$$

23. (a) To rotate about the x -axis, use the disk method; the radius of each disk is $2x^2 - x^3$. Therefore the volume is

$$V = \pi \int_0^2 (2x^2 - x^3)^2 dx = \pi \int_0^2 (4x^4 - 4x^5 + x^6) dx = \pi \left[\frac{4}{5}x^5 - \frac{2}{3}x^6 + \frac{1}{7}x^7 \right]_0^2 = \boxed{\frac{128}{105}\pi}.$$

(b) To rotate about the y -axis, use the shell method; the radius of each shell is x and its height is $2x^2 - x^3$, so the volume is

$$V = 2\pi \int_0^2 x(2x^2 - x^3) dx = 2\pi \int_0^2 (2x^3 - x^4) dx = 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2 = \boxed{\frac{16}{5}\pi}.$$

(c) In each case, we chose to integrate by partitioning along the x -axis, since partitioning along the y -axis would entail solving $y = 2x^2 - x^3$ for x and then using one solution as the lower bound and the other as the upper bound.

24. (a) To rotate about the x -axis, use the washer method; the outer radius of each disk is $-x^2 + 9$ and the inner radius is $2x + 1$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^2 ((-x^2 + 9)^2 - (2x + 1)^2) dx \\ &= \pi \int_0^2 (x^4 - 22x^2 - 4x + 80) dx \\ &= \pi \left[\frac{1}{5}x^5 - \frac{22}{3}x^3 - 2x^2 + 80x \right]_0^2 \\ &= \boxed{\frac{1496}{15}\pi}. \end{aligned}$$

- (b) To rotate about the y -axis, use the shell method. The radius of each shell is x , and its height is $-x^2 + 9 - (2x + 1) = -x^2 - 2x + 8$. Therefore the volume is

$$V = 2\pi \int_0^2 x(-x^2 - 2x + 8) dx = 2\pi \int_0^2 (-x^3 - 2x^2 + 8x) dx = 2\pi \left[-\frac{1}{4}x^4 - \frac{2}{3}x^3 + 4x^2 \right]_0^2 = \boxed{\frac{40}{3}\pi}.$$

- (c) In each case, we chose to integrate by partitioning along the x -axis, since partitioning along the y -axis would entail splitting the integral into two separate integrals, one from $y = 0$ to $y = 5$ and the other from $y = 5$ to $y = 9$.

25. (a) To rotate about the x -axis, use the disk method; the radius of each disk is $x^2 + 1$, so the volume is

$$V = \pi \int_0^2 (x^2 + 1)^2 dx = \pi \int_0^2 (x^4 + 2x^2 + 1) dx = \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_0^2 = \boxed{\frac{206}{15}\pi}.$$

- (b) To rotate about the y -axis, use the shell method; the radius of each shell is x and its height is $x^2 + 1$, so the volume is

$$V = 2\pi \int_0^2 x(x^2 + 1) dx = 2\pi \int_0^2 (x^3 + x) dx = 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right]_0^2 = \boxed{12\pi}.$$

- (c) In each case, we chose to integrate by partitioning along the x -axis, since partitioning along the y -axis would entail splitting the integral into two separate integrals, one from $y = 0$ to $y = 1$ and the other from $y = 1$ to $y = 5$.

26. (a) To rotate about the x -axis, use the disk method; the radius of each disk is $4x - x^2$, so the volume is

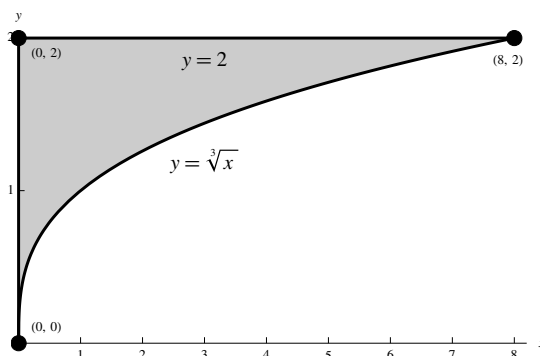
$$V = \pi \int_0^4 (4x - x^2)^2 dx = \pi \int_0^4 (16x^2 - 8x^3 + x^4) dx = \pi \left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5 \right]_0^4 = \boxed{\frac{512}{15}\pi}.$$

- (b) To rotate about the y -axis, use the shell method; the radius of each shell is x and its height is $4x - x^2$, so the volume is

$$V = 2\pi \int_0^4 x(4x - x^2) dx = 2\pi \int_0^4 (4x^2 - x^3) dx = 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 \right]_0^4 = \boxed{\frac{128}{3}\pi}.$$

- (c) In each case, we chose to integrate by partitioning along the x -axis, since partitioning along the y -axis would entail solving $y = 4x - x^2$ for x and then using one solution as the lower bound and the other as the upper bound.

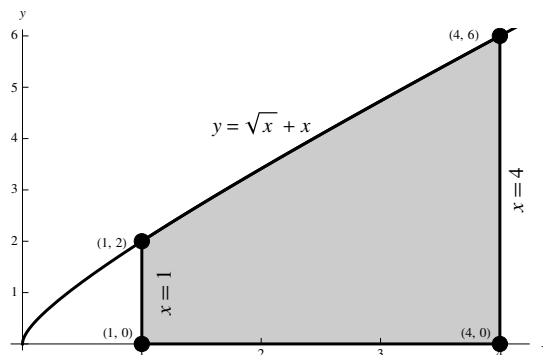
27. The region is shown below:



This could be done using shells along the y -axis or washers along the x -axis. Since $y = x^{1/3}$ becomes $x = y^3$, which is somewhat simpler, we choose the shell method. Then the radius of each shell is y and its height is y^3 , so the volume is

$$V = 2\pi \int_0^2 y \cdot y^3 dy = 2\pi \int_0^2 y^4 dy = 2\pi \left[\frac{1}{5} y^5 \right]_0^2 = \boxed{\frac{64}{5}\pi}.$$

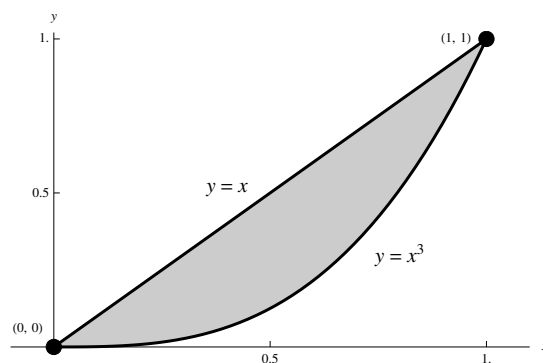
28. The region is shown below:



If we chose disks/washers along the y -axis, we would have to solve the equation for x , and we would have to split the integral up into two separate integrals. So use shells along the x -axis. The radius of each shell is x , and the height is $\sqrt{x} + x$, so the volume is

$$V = 2\pi \int_1^4 x(\sqrt{x} + x) dx = 2\pi \int_1^4 (x^{3/2} + x^2) dx = 2\pi \left[\frac{2}{5} x^{5/2} + \frac{1}{3} x^3 \right]_1^4 = \boxed{\frac{334}{5}\pi}.$$

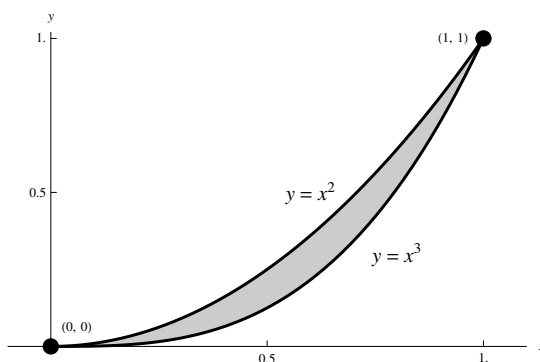
29. The region is shown below:



Either disks/washers along the y -axis or shells along the x -axis could be used. Since we are given y in terms of x , we choose shells. The radius of each shell is x , and the height is $x - x^3$, so the volume is

$$V = 2\pi \int_0^1 x(x - x^3) dx = 2\pi \int_0^1 (x^2 - x^4) dx = 2\pi \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1 = \boxed{\frac{4}{15}\pi}.$$

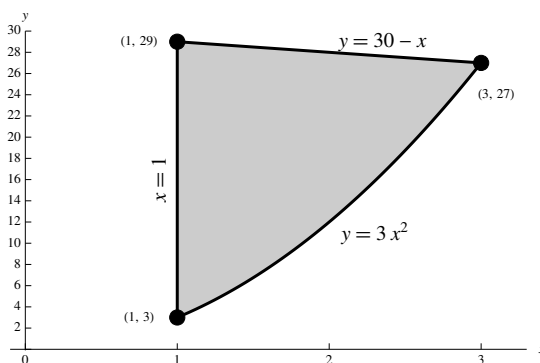
30. The region is shown below:



Either disks/washers along the x -axis or shells along the y -axis could be used. Since we are given y in terms of x , we choose disks/washers. The outer radius of each washer is x^2 and the inner radius is x^3 , so the volume is

$$V = \pi \int_0^1 \left((x^2)^2 - (x^3)^2 \right) dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[\frac{1}{5}x^5 - \frac{1}{7}x^7 \right]_0^1 = \boxed{\frac{2}{35}\pi}.$$

31. The region is shown below:



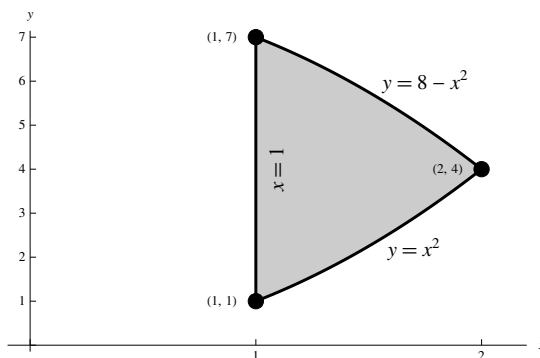
If we use disks along the y -axis, we will need two separate integrals, since the outer radius changes equations at $y = 27$. So use shells along the x -axis. The radius of each shell is x , and the height is $30 - x - 3x^2$, so the volume is

$$V = 2\pi \int_1^3 x (30 - x - 3x^2) dx = 2\pi \int_1^3 (30x - x^2 - 3x^3) dx = 2\pi \left[15x^2 - \frac{1}{3}x^3 - \frac{3}{4}x^4 \right]_1^3 = \boxed{\frac{308}{3}\pi}.$$

32. This is the same region as in Problem 31, but revolved about the x -axis. A similar argument applies: if we use shells along the y -axis, we would require two separate integrals, so we use the washer method along the x -axis. The outer radius of each washer is $30 - x$ and the inner radius is $3x^2$, so the volume is

$$\begin{aligned} V &= \pi \int_1^3 \left((30 - x)^2 - (3x^2)^2 \right) dx \\ &= \pi \int_1^3 (-9x^4 + x^2 - 60x + 900) dx \\ &= \pi \left[-\frac{9}{5}x^5 + \frac{1}{3}x^3 - 30x^2 + 900x \right]_1^3 \\ &= \boxed{\frac{16996}{15}\pi}. \end{aligned}$$

33. The region is shown below:



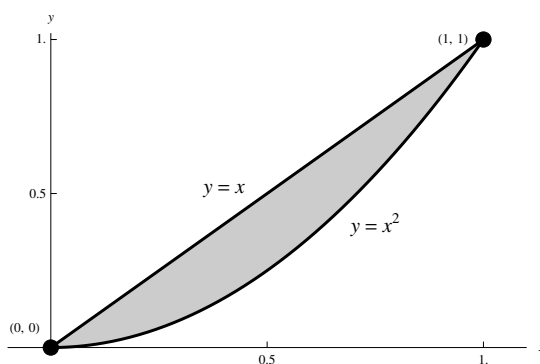
If we use shells along the y -axis, we will need two separate integrals, since the outer radius changes equations at $y = 4$. So use washers along the x -axis. The outer radius is $8 - x^2$ and the inner radius is x^2 , so the volume is

$$V = \pi \int_1^2 \left((8 - x^2)^2 - (x^2)^2 \right) dx = \pi \int_1^2 (64 - 16x^2) dx = \pi \left[64x - \frac{16}{3}x^3 \right]_1^2 = \boxed{\frac{80}{3}\pi}.$$

34. This is the same region as in Problem 33, but revolved about the y -axis. A similar argument applies: if we use disks along the y -axis, we will need two separate integrals, since the outer radius changes equations at $y = 4$. So use shells along the x -axis. The radius of each shell is x , and the height is $8 - x^2 - x^2 = 8 - 2x^2$, so the volume is

$$V = 2\pi \int_1^2 x (8 - 2x^2) dx = 2\pi \int_1^2 (8x - 2x^3) dx = 2\pi \left[4x^2 - \frac{1}{2}x^4 \right]_1^2 = \boxed{9\pi}.$$

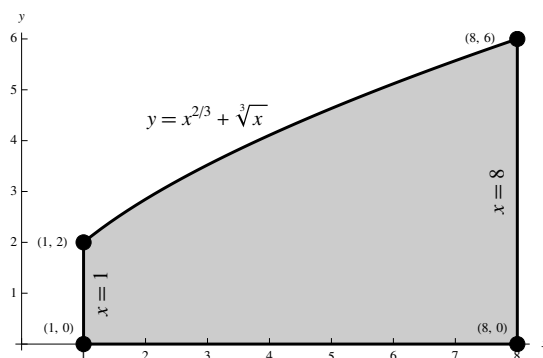
35. The region is shown below:



This can be done either using washers along the y -axis or shells along the x -axis. Since we are given y in terms of x , we choose shells. The radius of each shell is x and the height is $x - x^2$, so the volume is

$$V = 2\pi \int_0^1 x (x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \boxed{\frac{\pi}{6}}.$$

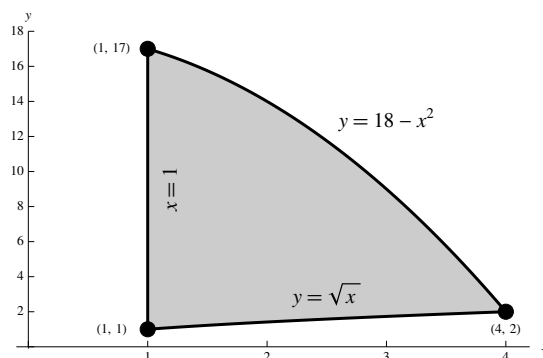
36. The region is shown below:



If we use washers along the y -axis, we will have to solve this equation for x in terms of y , and also split it into two integrals. So using shells along the x -axis is much easier. The radius of each shell is x , and the height is $x^{2/3} + x^{1/3}$, so the volume is

$$V = 2\pi \int_1^8 x \left(x^{2/3} + x^{1/3} \right) dx = 2\pi \int_1^8 \left(x^{5/3} + x^{4/3} \right) dx = 2\pi \left[\frac{3}{8}x^{8/3} + \frac{3}{7}x^{7/3} \right]_1^8 = \boxed{\frac{8403}{28}\pi}.$$

37. The region is shown below:



Revolving about the y -axis, if we use washers along the y -axis we will require two separate integrals. So use shells along the x -axis. The radius of each shell is x , and the height is $18 - x^2 - \sqrt{x}$, so the volume is

$$V = 2\pi \int_1^4 x(18 - x^2 - \sqrt{x}) dx = 2\pi \int_1^4 \left(18x - x^3 - x^{3/2} \right) dx = 2\pi \left[9x^2 - \frac{1}{4}x^4 - \frac{2}{5}x^{5/2} \right]_1^4 = \boxed{\frac{1177}{10}\pi}.$$

38. This is the same region as in Problem 37, but revolved about the x -axis. A similar argument applies: if we use shells along the y -axis, we will need two separate integrals, since the radius changes equations at $y = 2$. So use washers along the x -axis. The outer radius is $18 - x^2$, and the inner radius is $\sqrt{x}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left((18 - x^2)^2 - (\sqrt{x})^2 \right) dx \\ &= \pi \int_1^4 \left(x^4 - 36x^2 - x + 324 \right) dx \\ &= \pi \left[\frac{1}{5}x^5 - 12x^3 - \frac{1}{2}x^2 + 324x \right]_1^4 \\ &= \boxed{\frac{4131}{10}\pi}. \end{aligned}$$

Applications and Extensions

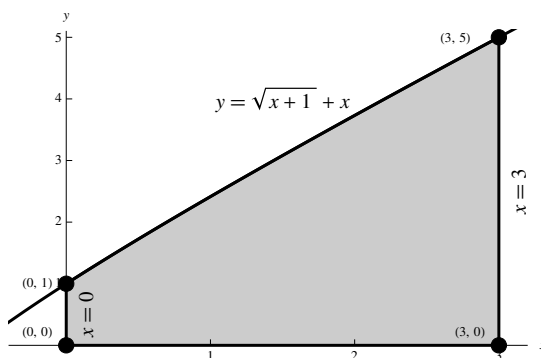
39. Use the shell method along the x -axis. The radius of each shell is x and the height is $\frac{1}{(x^2+1)^2}$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{(x^2+1)^2} dx.$$

Now use the substitution $u = x^2 + 1$, so that $du = 2x dx$. Then $x = 0$ corresponds to $u = 1$, and $x = 1$ to $u = 2$, so that we get

$$V = 2\pi \cdot \frac{1}{2} \int_1^2 \frac{1}{u^2} du = \pi \left[-\frac{1}{u} \right]_1^2 = \boxed{\frac{\pi}{2}}.$$

40. The region is shown below:



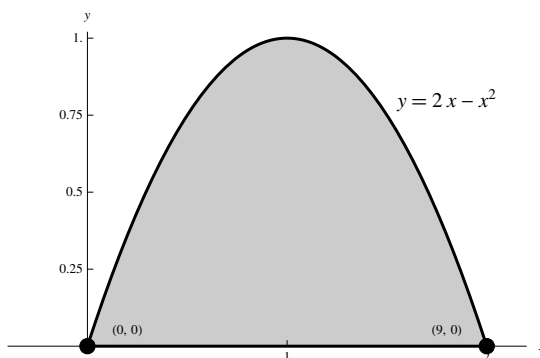
The disk/washer method along the y -axis requires solving the equation for x , which looks difficult. So we use the shell method along the x -axis. The radius of each shell is x , and the height is $\sqrt{x+1} + x$, so the volume is

$$V = 2\pi \int_0^3 x (\sqrt{x+1} + x) dx = 2\pi \left(\int_0^3 x\sqrt{x+1} dx + \int_0^3 x^2 dx \right).$$

For the first integral, use the substitution $u = x + 1$, so that $x = u - 1$ and $du = dx$. Then $x = 0$ corresponds to $u = 1$ while $x = 3$ corresponds to $u = 4$. So we get

$$\begin{aligned} V &= 2\pi \left(\int_1^4 (u-1)\sqrt{u} du + \int_0^3 x^2 dx \right) \\ &= 2\pi \left(\int_1^4 (u^{3/2} - u^{1/2}) du + \int_0^3 x^2 dx \right) \\ &= 2\pi \left(\left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^4 + \left[\frac{1}{3} x^3 \right]_0^3 \right) \\ &= \boxed{\frac{502}{15} \pi}. \end{aligned}$$

41. The region is shown below:



- (a) To revolve about the x -axis, use the disk method. The radius of each disk is $2x - x^2$, so the volume is

$$V = \pi \int_0^2 (2x - x^2)^2 dx = \pi \int_0^2 (4x^2 - 4x^3 + x^4) dx = \pi \left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5 \right]_0^2 = \boxed{\frac{16}{15}\pi}.$$

- (b) To revolve about the y -axis, use the shell method along the x -axis. The radius of each shell is x and its height is $2x - x^2$, so the volume is

$$V = 2\pi \int_0^2 x(2x - x^2) dx = 2\pi \int_0^2 (2x^2 - x^3) dx = 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \boxed{\frac{8}{3}\pi}.$$

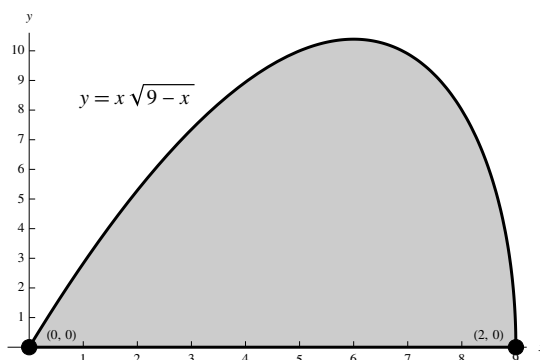
- (c) To revolve about the line $x = 3$, use the shell method along the x -axis. The radius of each shell is $3 - x$ (the distance from x to 3), and the height is $2x - x^2$, so the volume is

$$V = 2\pi \int_0^2 (3 - x)(2x - x^2) dx = 2\pi \int_0^2 (x^3 - 5x^2 + 6x) dx = 2\pi \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 \right]_0^2 = \boxed{\frac{16}{3}\pi}.$$

- (d) To revolve about the line $y = 1$, use the washer method along the x -axis. The outer radius of each shell is $1 - 0 = 1$, and the inner radius is $1 - (2x - x^2) = x^2 - 2x + 1 = (x - 1)^2$. Therefore the volume is

$$V = \pi \int_0^2 \left(1^2 - ((x - 1)^2)^2 \right) dx = \pi \int_0^2 (1 - (x - 1)^4) dx = \pi \left[x - \frac{1}{5}(x - 1)^5 \right]_0^2 = \boxed{\frac{8}{5}\pi}.$$

42. The region is shown below:



- (a) To revolve about the x -axis, use the disk method. The radius of each disk is $x\sqrt{9-x}$, so the volume is

$$V = \pi \int_0^9 (x\sqrt{9-x})^2 dx = \pi \int_0^9 (9x^2 - x^3) dx = \pi \left[3x^3 - \frac{1}{4}x^4 \right]_0^9 = \boxed{\frac{2187}{4}\pi}.$$

- (b) To revolve about the y -axis, use the shell method along the x -axis. The radius of each shell is x and its height is $x\sqrt{9-x}$, so the volume is

$$V = 2\pi \int_0^9 x (x\sqrt{9-x}) dx = 2\pi \int_0^9 x^2 \sqrt{9-x} dx.$$

Now use the substitution $u = 9 - x$, so that $x = 9 - u$ and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= -2\pi \int_9^0 (9-u)^2 \sqrt{u} du \\ &= 2\pi \int_0^9 (u^{5/2} - 18u^{3/2} + 81u^{1/2}) du \\ &= 2\pi \left[\frac{2}{7}u^{7/2} - \frac{36}{5}u^{5/2} + 54u^{3/2} \right]_0^9 \\ &= \boxed{\frac{23328}{35}\pi}. \end{aligned}$$

- (c) To revolve about the line $y = -3$, use the washer method along the x -axis. The outer radius of each shell is $x\sqrt{9-x} - (-3) = x\sqrt{9-x} + 3$, and the inner radius is $0 - (-3) = 3$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^9 ((x\sqrt{9-x} + 3)^2 - 3^2) dx \\ &= \pi \int_0^9 (x^2(9-x) + 6x\sqrt{9-x}) dx \\ &= \pi \left(\int_0^9 (9x^2 - x^3) dx + 6 \int_0^9 x\sqrt{9-x} dx \right). \end{aligned}$$

For the second integral, use the substitution $u = 9 - x$, so that $x = 9 - u$ and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= \pi \left(\int_0^9 (9x^2 - x^3) dx - 6 \int_9^0 (9-u)\sqrt{u} du \right) \\ &= \pi \left(\int_0^9 (9x^2 - x^3) dx + 6 \int_0^9 (9u^{1/2} - u^{3/2}) du \right) \\ &= \pi \left(\left[3x^3 - \frac{1}{4}x^4 \right]_0^9 + 6 \left[6u^{3/2} - \frac{2}{5}u^{5/2} \right]_0^9 \right) \\ &= \boxed{\frac{18711}{20}\pi}. \end{aligned}$$

- (d) To revolve about the line $x = -2$, use the shell method along the x -axis. The radius of each shell is $x - (-2) = x + 2$ (the distance from x to -2), and the height is $x\sqrt{9-x}$, so the volume is

$$V = 2\pi \int_0^9 (x+2)x\sqrt{9-x} dx = 2\pi \int_0^9 (x^2 + 2x)\sqrt{9-x} dx.$$

Again we use the substitution $u = 9 - x$, so that $x = 9 - u$, $x^2 + 2x = u^2 - 20u + 99$, and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= -2\pi \int_9^0 (u^2 - 20u + 99) \sqrt{u} \, du \\ &= 2\pi \int_0^9 (u^{5/2} - 20u^{3/2} + 99u^{1/2}) \, du \\ &= 2\pi \left[\frac{2}{7} u^{7/2} - 8u^{5/2} + 66u^{3/2} \right]_0^9 \\ &= \boxed{\frac{6480}{7}\pi}. \end{aligned}$$

43. Using the disk method to compute the volume gives

$$\pi \int_0^k f(x)^2 \, dx = \frac{1}{5}k^5 + k^4 + \frac{4}{3}k^3.$$

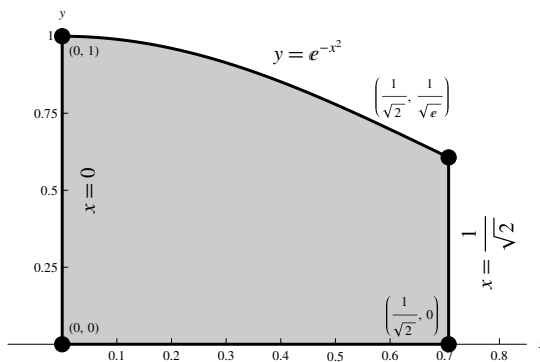
Assuming that the antiderivative of $f(x)^2$, evaluated at 0 is zero, we see that $\frac{1}{5}k^5 + k^4 + \frac{4}{3}k^3$ is π times the antiderivative of $f(x)^2$ evaluated at $x = k$. Therefore $f(x)^2$ has as an antiderivative $\frac{1}{\pi} (\frac{1}{5}x^5 + x^4 + \frac{4}{3}x^3)$, so that $f(x)^2 = \frac{1}{\pi} (x^4 + 4x^3 + 4x^2) = \frac{1}{\pi} x^2 (x^2 + 4x + 4) = \frac{1}{\pi} (x(x+2))^2$. Since

we are given that $f(x) \geq 0$, taking square roots gives $f(x) = \frac{1}{\sqrt{\pi}} x(x+2) = \boxed{\frac{1}{\sqrt{\pi}} (x^2 + 2x)}$.

44. We first determine the inflection point. With $f(x) = e^{-x^2}$, we have $f'(x) = -2xe^{-x^2}$, so that

$$f''(x) = -2x(-2xe^{-x^2}) - 2e^{-x^2} = (4x^2 - 2)e^{-x^2}.$$

Then $f''(x) = 0$ when $x = \pm \frac{1}{\sqrt{2}}$; since we want the positive value, we get $x = \frac{1}{\sqrt{2}}$. Therefore the region in question is the following:



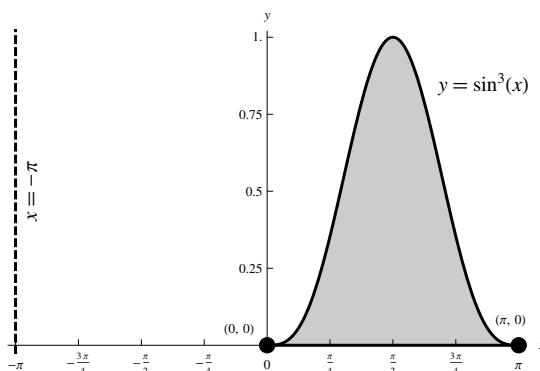
To revolve this about the y -axis, it appears easiest to use the shell method along the x -axis. The radius of each shell is x and its height is e^{-x^2} , so we get for the volume

$$V = 2\pi \int_0^{1/\sqrt{2}} x e^{-x^2} \, dx.$$

Use the substitution $u = -x^2$, so that $du = -2x \, dx$. Then $x = 0$ corresponds to $u = 0$ while $x = \frac{1}{\sqrt{2}}$ corresponds to $u = -\frac{1}{2}$. So we get for the integral

$$V = 2\pi \cdot \left(-\frac{1}{2}\right) \int_0^{-1/2} e^u \, du = -\pi [e^u]_0^{-1/2} = \boxed{\left(1 - \frac{1}{e^{1/2}}\right)\pi}.$$

45. The region is shown below:



We should use the shell method, as the washer method will involve solving $y = \sin^3 x$ for x and then integrating. With the shell method, the radius of each shell is $x - (-\pi) = x + \pi$, and the height is $\sin^3 x$. Therefore the volume is (using a CAS)

$$V = 2\pi \int_0^\pi (x + \pi) \sin^3 x \, dx \approx 12.566\pi \approx 39.478 \approx 4\pi^2.$$

Challenge Problems

46. (a) From the diagram, $f(x)$ is constant and equal to $f(a)$ for $0 \leq x \leq a$, and also $g(x)$ is constant and equal to $g(a)$ for $0 \leq x \leq a$. So using the shell method, we get two separate integrals, one on $[0, a]$ and one on $[a, b]$. For the first, the radius of each shell is x and the height is $f(x) - g(x) = f(a) - g(a)$, while for the second, the radius is x and the height is $f(x) - g(x)$. Therefore the volume of the solid is

$$\begin{aligned} V &= 2\pi \left(\int_0^a x[f(a) - g(a)] \, dx + \int_a^b x[f(x) - g(x)] \, dx \right) \\ &= 2\pi \left(\left[\frac{1}{2}x^2[f(a) - g(a)] \right]_0^a + \int_a^b x[f(x) - g(x)] \, dx \right) \\ &= \pi a^2[f(a) - g(a)] + 2\pi \int_a^b x[f(x) - g(x)] \, dx. \end{aligned}$$

- (b) Using the disk method, we also get two separate integrals, one from $y = g(a)$ to $y = g(b) = f(b)$ and one from $y = f(b)$ to $y = f(a)$. For the first integral, the radius of the disk is $g^{-1}(y)$ and for the second it is $f^{-1}(y)$. Therefore the volume is the sum of these integrals, or

$$V = \pi \int_{g(a)}^{g(b)} [g^{-1}(y)]^2 \, dy + \pi \int_{f(b)}^{f(a)} [f^{-1}(y)]^2 \, dy.$$

47. Consider the region A . Suppose that it is wholly contained between the vertical lines $x = a > 0$ and $x = b > a$. Integrate it using the shell method. At any $x \geq 0$, A consists of one or more line segments. Define $g(x)$ to be the total length of all of those segments at x . Then using the shell method, the volume of the solid of revolution about the y -axis is

$$V = 2\pi \int_a^b xg(x) \, dx.$$

When revolving A about the line $x = -k$, the heights at x are the same, $g(x)$, but the radius in each case is now $x + k$. So using the shell method in this case gives

$$V_{x=-k} = 2\pi \int_a^b (x + k)g(x) \, dx = 2\pi \int_a^b xg(x) \, dx + 2\pi \int_a^b kg(x) \, dx = V + 2\pi k \int_a^b g(x) \, dx = V + 2\pi kA,$$

as desired.

48. For this parabola, the latus rectum is the line $x = 2$; we wish to revolve the area to the left of this line about that line.

- (a) Using the disk method, we must solve the equation for x in terms of y , giving $x = \frac{y^2}{8}$. Then the radius of each disk is $2 - \frac{y^2}{8}$, so that the volume of the solid is

$$V = \pi \int_{-4}^4 \left(2 - \frac{y^2}{8}\right)^2 dy = \pi \int_{-4}^4 \left(4 - \frac{y^2}{2} + \frac{y^4}{64}\right) dy = \pi \left[4y - \frac{y^3}{6} + \frac{y^5}{320}\right]_{-4}^4 = \boxed{\frac{256}{15}\pi}.$$

- (b) Using the shell method, the radius of each shell is $2 - x$, and the height of each shell is then $\sqrt{8x} - (-\sqrt{8x}) = 2\sqrt{8x} = 4x^{1/2}\sqrt{2}$, so that the volume is

$$\begin{aligned} V &= 2\pi \int_0^2 (2 - x) (4x^{1/2}\sqrt{2}) dx \\ &= 8\sqrt{2}\pi \int_0^2 (2x^{1/2} - x^{3/2}) dx \\ &= 8\sqrt{2}\pi \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2 \\ &= \boxed{\frac{256}{15}\pi}. \end{aligned}$$

6.4: Volume of a Solid: Slicing Method

Concepts and Vocabulary

- Answers vary. When computing the volume of a solid whose cross sections have a regular geometric shape whose area we can compute, we can integrate over the area of each slice of the solid along one of its axes.
- False. For instance, Example 3 uses the method of slicing to compute the volume of a square pyramid, which is not a solid of revolution. The slicing method can be used effectively whenever the cross sections have a regular geometric shape whose area we can compute.

Skill Building

- The equation of the circle is $x^2 + y^2 = 4$, so that for a given value of x , y can range from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$. Therefore, for each value of x from -2 to 2 , the cross section is a square of side $2\sqrt{4 - x^2}$, so that its area is $4(4 - x^2)$. Therefore the volume of the solid is

$$V = \int_{-2}^2 4(4 - x^2) dx = 4 \int_{-2}^2 (4 - x^2) dx = 4 \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \boxed{\frac{128}{3}}.$$

- The equation of the circle is $x^2 + y^2 = 4$, so that for a given value of x , y can range from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$. Therefore, for each value of x from -2 to 2 , the cross section is an isosceles right triangle one of whose legs is $2\sqrt{4 - x^2}$, so that its area is $\frac{1}{2}(2\sqrt{4 - x^2})^2 = 2(4 - x^2)$. Therefore the volume of the solid is

$$V = \int_{-2}^2 2(4 - x^2) dx = 2 \int_{-2}^2 (4 - x^2) dx = 2 \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \boxed{\frac{64}{3}}.$$

5. Place the origin at the tip of the pyramid, with the x -axis along the axis of the pyramid. Then at x , the cross section is a square with side length $2x$ meters, so that its area is $4x^2$ square meters. Therefore the volume of the pyramid is

$$V = \int_0^{40} 4x^2 dx = \left[\frac{4}{3}x^3 \right]_0^{40} = \boxed{\frac{256000}{3} \text{ meters}^3}.$$

6. Place the origin at the tip of the pyramid, with the x -axis along the axis of the pyramid. Then at x , the cross section is a rectangle with sides x and $2x$ meters, so that its area is $2x^2$ square meters. Therefore the volume of the pyramid is

$$V = \int_0^{20} 2x^2 dx = \left[\frac{2}{3}x^3 \right]_0^{20} = \boxed{\frac{16000}{3} \text{ meters}^3}.$$

7. From the figure, these two curves intersect at $(0,0)$ and at $(4,2)$, so that we will integrate from 0 to 4. For each value of x , the cross section is a semicircle of diameter $\sqrt{x} - \frac{1}{8}x^2$, so that its area is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \left(\frac{\sqrt{x} - \frac{1}{8}x^2}{2} \right)^2 = \frac{\pi}{512} (8\sqrt{x} - x^2)^2 = \frac{\pi}{512} (64x - 16x^{5/2} + x^4).$$

Therefore the volume is

$$V = \int_0^4 \frac{\pi}{512} (64x - 16x^{5/2} + x^4) dx = \frac{\pi}{512} \left[32x^2 - \frac{32}{7}x^{7/2} + \frac{1}{5}x^5 \right]_0^4 = \boxed{\frac{9}{35}\pi}.$$

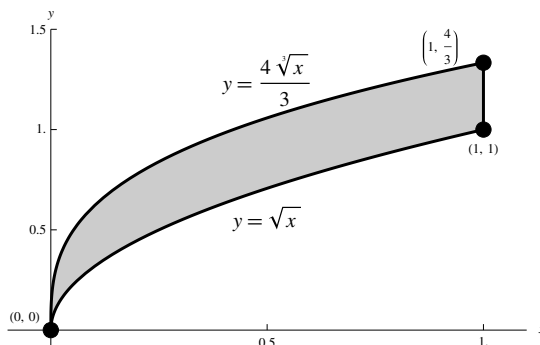
8. From the figure for Problem 7, these two curves intersect at $(0,0)$ and at $(4,2)$, so that we will integrate from 0 to 4. For each value of x , the cross section is a triangle whose base is $\sqrt{x} - \frac{1}{8}x^2$ and whose height is $3(\sqrt{x} - \frac{1}{8}x^2)$. Therefore its area is

$$A = \frac{1}{2}bh = \frac{3}{2} \left(\sqrt{x} - \frac{1}{8}x^2 \right)^2 = \frac{3}{2} \left(x - \frac{1}{4}x^{5/2} + \frac{1}{64}x^4 \right) = \frac{3}{2}x - \frac{3}{8}x^{5/2} + \frac{3}{128}x^4.$$

Therefore the volume is

$$V = \int_0^4 \left(\frac{3}{2}x - \frac{3}{8}x^{5/2} + \frac{3}{128}x^4 \right) dx = \left[\frac{3}{4}x^2 - \frac{3}{28}x^{7/2} + \frac{3}{640}x^5 \right]_0^4 = \boxed{\frac{108}{35}}.$$

9. The region is shown below:



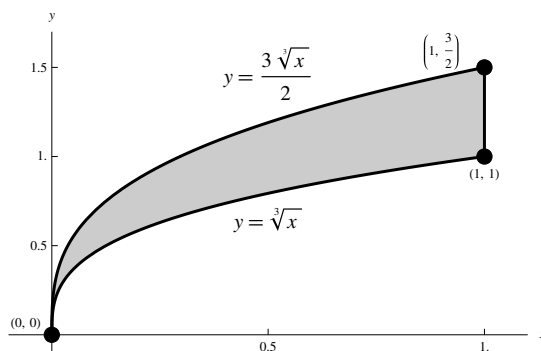
For each x , the cross section is a circle whose diameter is $\frac{4}{3}x^{1/3} - x^{1/2}$, so the area of that circle is

$$A = \pi r^2 = \pi \left(\frac{\frac{4}{3}x^{1/3} - x^{1/2}}{2} \right)^2 = \pi \left(\frac{4}{9}x^{2/3} - \frac{2}{3}x^{5/6} + \frac{1}{4}x \right).$$

Therefore the volume is

$$V = \int_0^1 \pi \left(\frac{4}{9}x^{2/3} - \frac{2}{3}x^{5/6} + \frac{1}{4}x \right) dx = \pi \left[\frac{12}{45}x^{5/3} - \frac{12}{33}x^{11/6} + \frac{1}{8}x^2 \right]_0^1 = \boxed{\frac{37}{1320}\pi}.$$

10. The region is shown below:



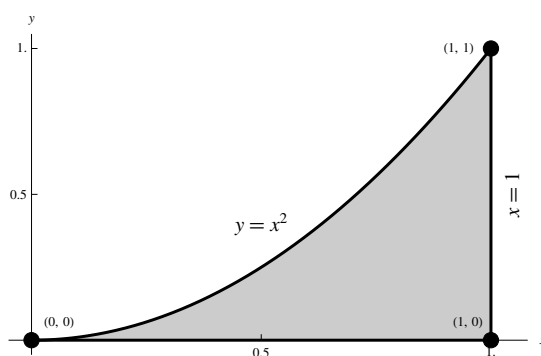
For each x , the cross section is a circle whose diameter is $\frac{3}{2}x^{1/3} - x^{1/3} = \frac{1}{2}x^{1/3}$, so the area of that circle is

$$A = \pi r^2 = \pi \left(\frac{\frac{1}{2}x^{1/3}}{2} \right)^2 = \frac{1}{16}\pi x^{2/3}.$$

Therefore the volume is

$$V = \int_0^1 \left(\frac{1}{16}\pi x^{2/3} \right) dx = \frac{1}{16}\pi \left[\frac{3}{5}x^{5/3} \right]_0^1 = \boxed{\frac{3}{80}\pi}.$$

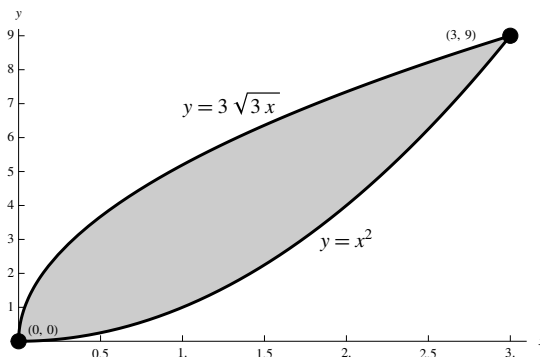
11. The region is shown below:



For each x , the cross section is a semicircle whose diameter is $x^2 - 0 = x^2$, so the area of that semicircle is $\frac{1}{2}\pi \left(\frac{x}{2} \right)^2 = \frac{1}{8}\pi x^2$. Therefore the volume of the solid is

$$V = \int_0^1 \frac{1}{8}\pi x^2 dx = \left[\frac{1}{24}\pi x^3 \right]_0^1 = \boxed{\frac{1}{24}\pi}.$$

12. The region is shown below:



For each x , the cross section is a semicircle whose diameter is $3\sqrt{3x} - x^2$, so the area of that semicircle is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \left(\frac{3\sqrt{3x} - x^2}{2} \right)^2 = \frac{1}{8}\pi (3\sqrt{3x} - x^2)^2 = \frac{1}{8}\pi (27x - 6\sqrt{3}x^{5/2} + x^4).$$

Therefore the volume of the solid is

$$V = \int_0^3 \frac{1}{8}\pi (27x - 6\sqrt{3}x^{5/2} + x^4) dx = \frac{1}{8}\pi \left[\frac{27}{2}x^2 - \frac{12\sqrt{3}}{7}x^{7/2} + \frac{1}{5}x^5 \right]_0^3 = \boxed{\frac{2187}{560}\pi}.$$

Applications and Extensions

13. Place the center of the sphere at the origin. Then x ranges from $-R$ to R , so that for each x , the possible values of y are from $-\sqrt{R^2 - x^2}$ to $\sqrt{R^2 - x^2}$. Therefore the cross section at x is a circle with radius $\sqrt{R^2 - x^2}$, so its area is $\pi(\sqrt{R^2 - x^2})^2 = \pi(R^2 - x^2)$. So the total volume of the sphere is

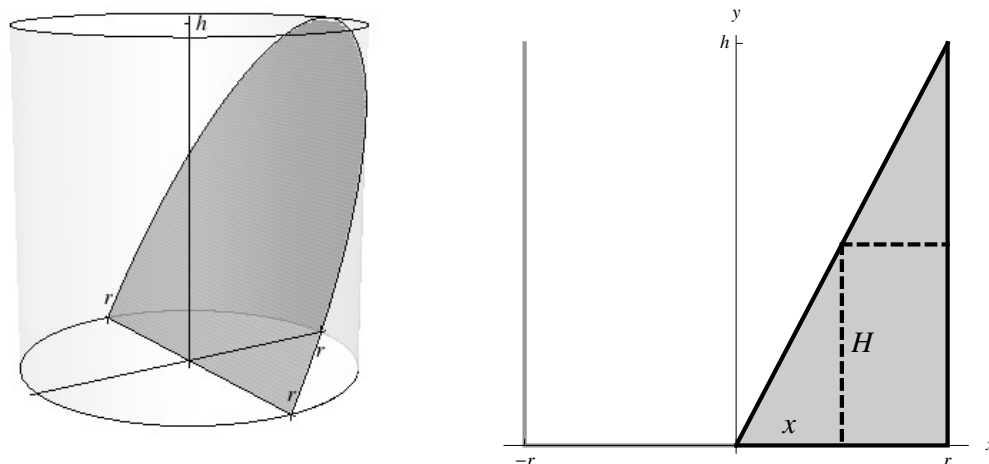
$$V = \int_{-R}^R \pi(R^2 - x^2) dx = \pi \left[R^2x - \frac{1}{3}x^3 \right]_{-R}^R = \pi \cdot \frac{4}{3}R^3 = \boxed{\frac{4}{3}\pi R^3}.$$

14. Place the center of the top of the hemispherical bowl at the origin, with the x -axis pointing down (so the bottom of the bowl is at $(R, 0)$). Then x ranges from $R - h$ to R , and for each x , the possible values of y are from $-\sqrt{R^2 - x^2}$ to $\sqrt{R^2 - x^2}$. Therefore the cross section at x is a circle with radius $\sqrt{R^2 - x^2}$, so its area is $\pi(\sqrt{R^2 - x^2})^2 = \pi(R^2 - x^2)$. So the total volume of the contents is

$$\begin{aligned} V &= \int_{R-h}^R \pi(R^2 - x^2) dx \\ &= \pi \left[R^2x - \frac{1}{3}x^3 \right]_{R-h}^R \\ &= \pi \left(R^3 - \frac{1}{3}R^3 - R^2(R-h) + \frac{1}{3}(R-h)^3 \right) \\ &= \pi \left(R^2h - R^2h + Rh^2 - \frac{1}{3}h^3 \right) \\ &= \boxed{\pi h^2 \left(R - \frac{h}{3} \right)}. \end{aligned}$$

Note that if $h = 0$ (the bowl is empty), this gives 0, while if $h = R$ (the bowl is completely full), it gives $\frac{2}{3}\pi R^3$, which is indeed half the volume of a sphere of radius R .

15. A diagram of the water in the glass, in a three-dimensional view on the left and a two-dimensional projection on the right, is shown below:



For a given value of x , the length of the water line along the base is $2\sqrt{r^2 - x^2}$, since the base is a circle of radius r . To determine its height, use similar triangles (look at the two-dimensional picture): if H is the height, then $\frac{H}{x} = \frac{h}{r}$, so that $H = \frac{h}{r}x$. So the cross section at x is a rectangle with dimensions $2\sqrt{r^2 - x^2} \times \frac{h}{r}x$. Therefore the volume of this amount of water is

$$V = \int_0^r \frac{h}{r}x \cdot 2\sqrt{r^2 - x^2} dx.$$

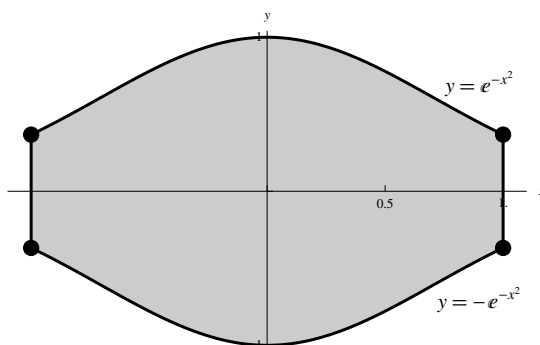
16. Place the origin at the point where the dotted line in the diagram meets the sharp edge of the solid, with the x -axis along that sharp edge. Note that since the cylinder has diameter 10, the wedge penetrates halfway through the cylinder, so that the x -axis coordinates of the wedge range from -5 to 5 . Now, for each value of x , the distance from the x -axis to the outer edge of the wedge is $\sqrt{25 - x^2}$, and the height of the wedge at that point is $\sqrt{25 - x^2} \cdot \tan 30^\circ = \frac{\sqrt{3}}{3}\sqrt{25 - x^2}$. The cross section is therefore a right triangle whose area is

$$A = \frac{1}{2}bh = \frac{1}{2}\sqrt{25 - x^2} \cdot \frac{\sqrt{3}}{3}\sqrt{25 - x^2} = \frac{\sqrt{3}}{6}(25 - x^2).$$

So the volume of the wedge is

$$V = \int_{-5}^5 \frac{\sqrt{3}}{6}(25 - x^2) dx = \frac{\sqrt{3}}{6} \left[25x - \frac{1}{3}x^3 \right]_{-5}^5 = \frac{250\sqrt{3}}{9} \text{ m}^3.$$

17. The region is shown below:



The diameter of each slice is $e^{-x^2} - (-e^{-x^2}) = 2e^{-x^2}$, so the area of each slice is

$$A = \pi r^2 = \pi \left(\frac{1}{2} \cdot 2e^{-x^2} \right)^2 = \pi e^{-2x^2}.$$

Therefore the volume is (using a CAS)

$$V = \pi \int_{-1}^1 e^{-2x^2} dx \approx 3.758.$$

18. Answers may vary. The slicing method uses the area of a slice as the integrand, where that area is described by some “nice” rule. In the case of the slice method applied to a solid of revolution of $f(x)$, each slice is a circle whose radius is $f(x)$, so its area is $\pi f(x)^2$, so that the total volume is $\int_a^b \pi f(x)^2 dx = \pi \int_a^b f(x)^2 dx$, which is exactly the formula for applying the disk method to this solid of revolution.

Challenge Problems

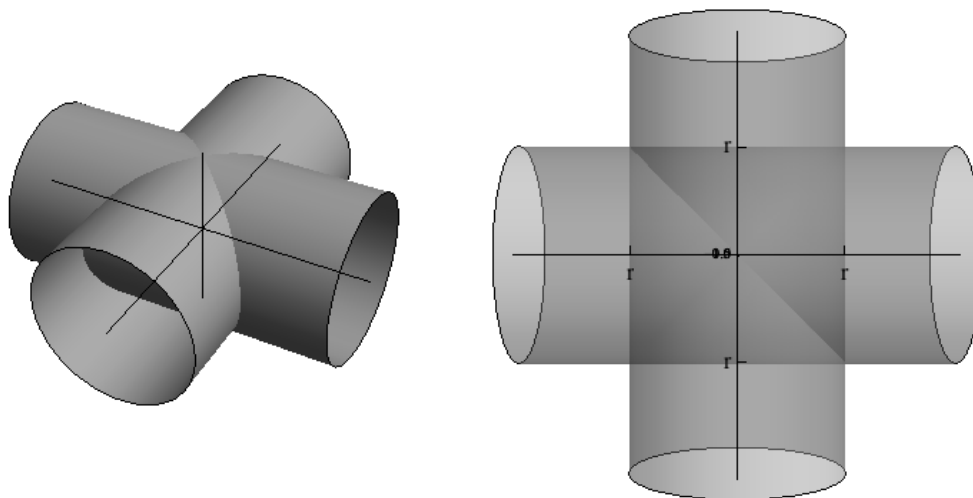
19. The volume removed is almost a cylinder with radius 2, except for the ends, which are spherical caps. Let's first determine the x -coordinate of the point where the bore reaches the edge of the sphere (which is the “edge” in the front of the diagram). We have $x^2 + y^2 = 25$, since the sphere has radius 5. But at this point, $y = 2$, so that $x^2 = 21$ and $x = \sqrt{21}$. Then the volume of metal removed consists of three pieces: a cylinder of height $2\sqrt{21}$ and radius 2, and two spherical “caps” of height $5 - \sqrt{21}$ taken from a sphere of radius 5. We can compute the volume of those spherical caps using Problem 14 with $R = 5$ and $h = 5 - \sqrt{21}$, giving a volume of

$$\pi h^2 \left(R - \frac{h}{3} \right) = \pi (5 - \sqrt{21})^2 \left(5 - \frac{5 - \sqrt{21}}{3} \right) = \pi \left(\frac{250}{3} - 18\sqrt{21} \right).$$

Then the total volume of the bore is

$$\begin{aligned} V &= \text{volume of cylinder} + 2 \text{ volume of cap} \\ &= \pi \cdot 2^2 \cdot 2\sqrt{21} + 2\pi \left(\frac{250}{3} - 18\sqrt{21} \right) = \pi \left(\frac{500}{3} - 28\sqrt{21} \right). \end{aligned}$$

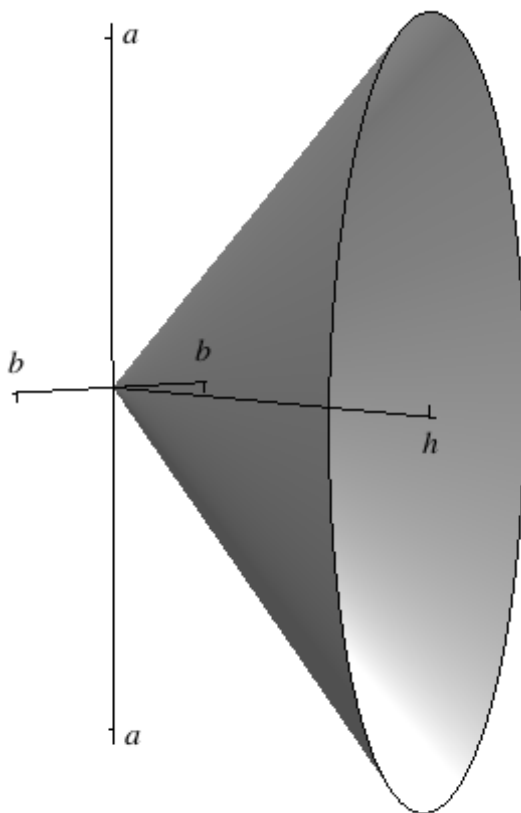
20. Place the pipes so that their axes lie along the x and y -axes. Two plots of the pipes are below; the one on the right is the view from above.



Consider a horizontal slice through the pipes at a distance R from the xy -plane. The cross section of one pipe is the lines $x = \pm\sqrt{r^2 - R^2}$ and the cross section of the other is the lines $y = \pm\sqrt{r^2 - R^2}$. Therefore the slice representing the cross sectional area contained in both pipes at this level is a square of side $2\sqrt{r^2 - R^2}$ (see the right-hand diagram). The possible distance R range from $-r$ to r , so the volume of the solid bounded by both pipes is

$$V = \int_{-r}^r \left(2\sqrt{r^2 - R^2}\right)^2 dR = 4 \int_{-r}^r (r^2 - R^2) dR = 4 \left[r^2 R - \frac{1}{3} R^3 \right]_{-r}^r = \boxed{\frac{16}{3} r^3}.$$

21. Place the origin at the tip of the cone, with the positive x -axis pointing in the direction of the base, so that the base of the cone is an ellipse at $x = h$:



Then for a given value of x , similar triangles tell us that the cross section at x is an ellipse with major axis $2a\frac{x}{h}$ and minor axis $2b\frac{x}{h}$. We must integrate over the area of these ellipses. Now by the hint, the area of each ellipse is

$$\pi \cdot \left(a\frac{x}{h}\right) \cdot \left(b\frac{x}{h}\right) = \pi \frac{ab}{h^2} x^2,$$

so the volume is

$$V = \int_0^h \left(\pi \frac{ab}{h^2} x^2\right) dx = \pi \frac{ab}{h^2} \left[\frac{1}{3} x^3\right]_0^h = \boxed{\frac{1}{3} \pi abh}.$$

22. The parallelepiped is made up of a number of slices, each of which is a parallelogram with sides a and b where the angle between a and b is θ . The area of such a parallelogram is $ab \sin \theta$. The height of the parallelepiped is $c \sin \phi$, so the volume is

$$V = \int_0^{c \sin \phi} ab \sin \theta dx = \boxed{abc \sin \phi \sin \theta}.$$

6.5: Arc Length

Concepts and Vocabulary

- False. The integrand should be $\sqrt{1 + [f'(x)]^2}$, not $\sqrt{1 + [f(x)]^2}$: $s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$.
- True. By regarding f as a piecewise continuous function, we may be able to find the arc length of each continuous piece and add up all the pieces.

Skill Building

- With $f(x) = 3x - 1$, we have $f'(x) = 3$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_1^3 \sqrt{1^2 + 3^2} dx = \int_1^3 \sqrt{10} dx = \boxed{2\sqrt{10}}.$$

Using the Distance Formula, we get $s = \sqrt{(3-1)^2 + (8-2)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$, and the answers are the same.

- With $f(x) = -4x + 1$, we have $f'(x) = -4$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_{-1}^1 \sqrt{1^2 + (-4)^2} dx = \int_{-1}^1 \sqrt{17} dx = \boxed{2\sqrt{17}}.$$

Using the Distance Formula, we get $s = \sqrt{(1 - (-1))^2 + (-3 - 5)^2} = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$, and the answers are the same.

- First solve for y , giving $y = \frac{2}{3}x + \frac{4}{3}$ and $y' = \frac{2}{3}$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_1^4 \sqrt{1^2 + \left(\frac{2}{3}\right)^2} dx = \int_1^4 \frac{\sqrt{13}}{3} dx = \boxed{\sqrt{13}}.$$

Using the Distance Formula, we get $s = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{9+4} = \sqrt{13}$, and the answers are the same.

- First solve for y , giving $y = -\frac{3}{4}x + 3$ and $y' = -\frac{3}{4}$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(-\frac{3}{4}\right)^2} dx = \int_0^4 \frac{5}{4} dx = \boxed{5}.$$

Using the Distance Formula, we get $s = \sqrt{(4-0)^2 + (0-3)^2} = \sqrt{16+9} = 5$, and the answers are the same.

- The derivative of $y = x^{2/3} + 1$ is $y = \frac{2}{3}x^{-1/3}$, so that y has a continuous derivative on an interval containing 1 and 8. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^8 \sqrt{1^2 + \left(\frac{2}{3}x^{-1/3}\right)^2} dx = \frac{1}{3} \int_1^8 \sqrt{9 + 4x^{-2/3}} dx = \frac{1}{3} \int_1^8 x^{-1/3} \sqrt{9x^{2/3} + 4} dx.$$

Now use the substitution $u = 9x^{2/3} + 4$, so that $du = 6x^{-1/3} dx$. Then $x = 1$ corresponds to $u = 13$ while $x = 8$ corresponds to $u = 40$, and we get

$$s = \frac{1}{3} \cdot \frac{1}{6} \int_{13}^{40} \sqrt{u} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_{13}^{40} = \frac{1}{27} (40^{3/2} - 13^{3/2}) = \boxed{\frac{1}{27} (80\sqrt{10} - 13\sqrt{13})}.$$

8. The derivative of $y = x^{2/3} + 6$ is $y' = \frac{2}{3}x^{-1/3}$, so that y has a continuous derivative on an interval containing 1 and 8. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^8 \sqrt{1^2 + \left(\frac{2}{3}x^{-1/3}\right)^2} dx = \frac{1}{3} \int_1^8 \sqrt{9 + 4x^{-2/3}} dx = \frac{1}{3} \int_1^8 x^{-1/3} \sqrt{9x^{2/3} + 4} dx.$$

Now use the substitution $u = 9x^{2/3} + 4$, so that $du = 6x^{-1/3} dx$. Then $x = 1$ corresponds to $u = 13$ while $x = 8$ corresponds to $u = 40$, and we get

$$s = \frac{1}{3} \cdot \frac{1}{6} \int_{13}^{40} \sqrt{u} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_{13}^{40} = \frac{1}{27} (40^{3/2} - 13^{3/2}) = \boxed{\frac{1}{27} (80\sqrt{10} - 13\sqrt{13})}.$$

9. The derivative of $y = x^{3/2}$ is $y' = \frac{3}{2}x^{1/2}$, so that y has a continuous derivative on an interval containing 0 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \frac{1}{2} \int_0^4 \sqrt{4 + 9x} dx.$$

Now substitute $u = 4 + 9x$; then $du = 9 dx$. Further, $x = 0$ corresponds to $u = 4$ while $x = 4$ corresponds to $u = 40$, so we get

$$\frac{1}{2} \int_0^4 \sqrt{4 + 9x} dx = \frac{1}{2} \cdot \frac{1}{9} \int_4^{40} u^{1/2} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{40} = \frac{1}{27} (40\sqrt{40} - 8) = \boxed{\frac{1}{27} (80\sqrt{10} - 8)}.$$

10. The derivative of $y = x^{3/2} + 4$ is $y' = \frac{3}{2}x^{1/2}$, so that y has a continuous derivative on an interval containing 1 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^4 \sqrt{1^2 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \frac{1}{2} \int_1^4 \sqrt{4 + 9x} dx.$$

Now substitute $u = 4 + 9x$; then $du = 9 dx$. Further, $x = 1$ corresponds to $u = 13$ while $x = 4$ corresponds to $u = 40$, so we get

$$\begin{aligned} \frac{1}{2} \int_1^4 \sqrt{4 + 9x} dx &= \frac{1}{2} \cdot \frac{1}{9} \int_{13}^{40} u^{1/2} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_{13}^{40} \\ &= \frac{1}{27} (40\sqrt{40} - 13\sqrt{13}) = \boxed{\frac{1}{27} (80\sqrt{10} - 13\sqrt{13})}. \end{aligned}$$

11. First solve for y , giving (since we are given $y \geq 0$) $y = \frac{2}{3}x^{3/2}$. Then $y' = x^{1/2}$ has a continuous derivative on an interval containing 0 and 1, so that the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + (x^{1/2})^2} dx = \int_0^1 \sqrt{1 + x} dx = \left[\frac{2}{3} (1 + x)^{3/2} \right]_0^1 = \boxed{\frac{2}{3} (2\sqrt{2} - 1)}.$$

12. We have $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned}
 s &= \int_1^3 \sqrt{1^2 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx \\
 &= \int_1^3 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx \\
 &= \int_1^3 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx \\
 &= \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx \\
 &= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx \\
 &= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3 \\
 &= \boxed{\frac{14}{3}}.
 \end{aligned}$$

13. We have $y' = (x^2 + 1)^{1/2} \cdot 2x = 2x\sqrt{x^2 + 1}$, which is continuous on an interval containing 1 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned}
 s &= \int_1^4 \sqrt{1^2 + \left(2x\sqrt{x^2 + 1}\right)^2} dx \\
 &= \int_1^4 \sqrt{4x^4 + 4x^2 + 1} dx \\
 &= \int_1^4 \sqrt{(2x^2 + 1)^2} dx \\
 &= \int_1^4 (2x^2 + 1) dx \\
 &= \left[\frac{2}{3}x^3 + x\right]_1^4 \\
 &= \boxed{45}.
 \end{aligned}$$

14. We have $y' = \frac{1}{2}(x^2 + 2)^{1/2} \cdot 2x = x\sqrt{x^2 + 2}$, which is continuous on an interval containing 2 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned}
 s &= \int_2^4 \sqrt{1^2 + \left(x\sqrt{x^2 + 2}\right)^2} dx \\
 &= \int_2^4 \sqrt{x^4 + 2x^2 + 1} dx \\
 &= \int_2^4 \sqrt{(x^2 + 1)^2} dx \\
 &= \int_2^4 (x^2 + 1) dx \\
 &= \left[\frac{1}{3}x^3 + x \right]_2^4 \\
 &= \boxed{\frac{62}{3}}.
 \end{aligned}$$

15. We have

$$y' = \frac{2}{9}\sqrt{3} \cdot \frac{3}{2}(3x^2 + 1)^{1/2} \cdot 6x = 2\sqrt{3}x\sqrt{3x^2 + 1}.$$

This is continuous on an interval containing -1 and 2 , so the Arc Length Formula applies and the arc length is

$$\begin{aligned}
 s &= \int_{-1}^2 \sqrt{1^2 + \left(2\sqrt{3}x\sqrt{3x^2 + 1}\right)^2} dx \\
 &= \int_{-1}^2 \sqrt{36x^4 + 12x^2 + 1} dx \\
 &= \int_{-1}^2 \sqrt{(6x^2 + 1)^2} dx \\
 &= \int_{-1}^2 (6x^2 + 1) dx \\
 &= [2x^3 + x]_{-1}^2 \\
 &= \boxed{21}.
 \end{aligned}$$

16. We have

$$y' = \frac{3}{2} \left(1 - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}\sqrt{1 - x^{2/3}}.$$

This is continuous on an interval containing $\frac{1}{8}$ and 1 , so that the Arc Length Formula applies and the arc length is

$$\begin{aligned}
 s &= \int_{1/8}^1 \sqrt{1^2 + \left(-x^{-1/3}\sqrt{1 - x^{2/3}}\right)^2} dx \\
 &= \int_{1/8}^1 \sqrt{x^{-2/3}} dx \\
 &= \int_{1/8}^1 x^{-1/3} dx \\
 &= \left[\frac{3}{2}x^{2/3} \right]_{1/8}^1 \\
 &= \boxed{\frac{9}{8}}.
 \end{aligned}$$

17. Solving for y gives $y = \frac{x^4}{8} + \frac{1}{4x^2}$. Then $y' = \frac{x^3}{2} - \frac{1}{2x^3}$, which is continuous on an interval containing 1 and 2. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned}
 s &= \int_1^2 \sqrt{1^2 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2} dx \\
 &= \int_1^2 \sqrt{1 + \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}} dx \\
 &= \int_1^2 \sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}} dx \\
 &= \int_1^2 \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2} dx \\
 &= \int_1^2 \left(\frac{x^3}{2} + \frac{1}{2x^3}\right) dx \\
 &= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2 \\
 &= \boxed{\frac{33}{16}}.
 \end{aligned}$$

18. Solving for y gives $y^2 = \frac{4}{9}(1+x^2)^3$, so that $y = \frac{2}{3}(1+x^2)^{3/2}$ (since $y \geq 0$). Then $y' = (1+x^2)^{1/2} \cdot 2x = 2x\sqrt{1+x^2}$. This is continuous everywhere, so that the Arc Length Formula applies and the arc length is

$$\begin{aligned}
 s &= \int_0^{2\sqrt{2}} \sqrt{1^2 + \left(2x\sqrt{1+x^2}\right)^2} dx \\
 &= \int_0^{2\sqrt{2}} \sqrt{4x^4 + 4x^2 + 1} dx \\
 &= \int_0^{2\sqrt{2}} \sqrt{(2x^2 + 1)^2} dx \\
 &= \int_0^{2\sqrt{2}} (2x^2 + 1) dx \\
 &= \left[\frac{2}{3}x^3 + x\right]_0^{2\sqrt{2}} \\
 &= \boxed{\frac{38}{3}\sqrt{2}}.
 \end{aligned}$$

19. We get $y' = \frac{1}{\sin x} \cdot (-\cos x) = -\cot x$. This is continuous on an interval containing $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so the Arc Length Formula applies. Using the identity $1 + \cot^2 x = \csc^2 x$, the arc length is

$$\begin{aligned} s &= \int_{\pi/6}^{\pi/3} \sqrt{1^2 + (-\cot x)^2} dx \\ &= \int_{\pi/6}^{\pi/3} \sqrt{1 + \cot^2 x} dx \\ &= \int_{\pi/6}^{\pi/3} \csc x dx \\ &= [\ln |\csc x - \cot x|]_{\pi/6}^{\pi/3} \\ &= \ln \left| \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right| - \ln |2 - \sqrt{3}|. \end{aligned}$$

This expression can be simplified to $-\ln(2\sqrt{3} - 3)$.

20. We get $y' = \frac{1}{\cos x} \cdot \sin x = \tan x$. This is continuous on an interval containing $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so the Arc Length Formula applies. Using the identity $1 + \tan^2 x = \sec^2 x$, the arc length is

$$\begin{aligned} s &= \int_{\pi/6}^{\pi/3} \sqrt{1^2 + (\tan x)^2} dx \\ &= \int_{\pi/6}^{\pi/3} \sqrt{1 + \tan^2 x} dx \\ &= \int_{\pi/6}^{\pi/3} \sec x dx \\ &= [\ln |\sec x + \tan x|]_{\pi/6}^{\pi/3} \\ &= \ln |2 + \sqrt{3}| - \ln \left| \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| \\ &= \ln(2 + \sqrt{3}) - \ln(\sqrt{3}) \\ &= \ln \frac{2 + \sqrt{3}}{\sqrt{3}} = \ln \frac{2\sqrt{3} + 3}{3}. \end{aligned}$$

21. Since $y \geq 0$, solving for y gives $y = \frac{1}{2}(x+1)^{3/2}$, so that $y' = \frac{3}{4}\sqrt{x+1}$, which is continuous on an interval containing -1 and 16 . Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_{-1}^{16} \sqrt{1^2 + \left(\frac{3}{4}\sqrt{x+1}\right)^2} dx = \frac{1}{4} \int_{-1}^{16} \sqrt{9x+25} dx.$$

Now use the substitution $u = 9x + 25$, so that $du = 9dx$. Then $x = -1$ corresponds to $u = 16$ and $x = 16$ corresponds to $u = 169$, so we get

$$\frac{1}{4} \int_{-1}^{16} \sqrt{9x+25} dx = \frac{1}{4} \cdot \frac{1}{9} \int_{16}^{169} u^{1/2} du = \frac{1}{36} \left[\frac{2}{3} u^{3/2} \right]_{16}^{169} = \frac{1}{54} (13^3 - 4^3) = \boxed{\frac{79}{2}}.$$

22. The derivative of $x^{3/2} + 8$ is $\frac{3}{2}x^{1/2}$, so that y has a continuous derivative on an interval containing 0 and 4 . Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \frac{1}{2} \int_0^4 \sqrt{4+9x} dx = \frac{1}{2} \left[\frac{2}{27} (4+9x)^{3/2} \right]_0^4 = \boxed{\frac{1}{27} (80\sqrt{10} - 8)}.$$

23. Since $y' = \frac{2}{3}x^{-1/3}$ is not continuous (or even defined) at $x = 0$, we try partitioning the y -axis instead. Solving $y = x^{2/3}$ for x gives $x = y^{3/2}$, so that $x' = \frac{3}{2}\sqrt{y}$. Note that this is the same curve as the one we are interested in, just expressed differently, so that it has the same arc length. Now, $x = 0$ corresponds to $y = 0$, and $x = 1$ to $y = 1$. Since $\frac{3}{2}\sqrt{y}$ is continuous on an interval containing 0 and 1, the Arc Length Formula for partitioning along the y -axis applies, and the arc length is

$$s = \int_0^1 \sqrt{1 + \left(\frac{3}{2}\sqrt{y}\right)^2} dy = \frac{1}{2} \int_0^1 \sqrt{4 + 9y} dy.$$

Now substitute $u = 4 + 9y$; then $du = 9 dy$. Further, $y = 0$ corresponds to $u = 4$ while $y = 1$ corresponds to $u = 13$, so we get

$$\frac{1}{2} \int_0^1 \sqrt{4 + 9y} dy = \frac{1}{2} \cdot \frac{1}{9} \int_4^{13} u^{1/2} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{13} = \boxed{\frac{1}{27}(13\sqrt{13} - 8)}.$$

24. Since $y = x^{2/3}$ is symmetric about the y -axis, its arc length from $x = -1$ to $x = 0$ is the same as its arc length from $x = 0$ to $x = 1$. But this arc length was just computed in Problem 23 to be

$$\boxed{\frac{1}{27}(13\sqrt{13} - 8)}.$$

25. Since we wish to partition along the y -axis, we solve for x , giving $x = 2y^{3/2} - 1$. (Note that we took the positive square root of $4y^3$ since we have $x \geq -1$.) Then $x' = 3y^{1/2} = 3\sqrt{y}$, which is continuous on an interval containing 0 and 1. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + (3\sqrt{y})^2} dy = \int_0^1 \sqrt{9y + 1} dy.$$

Now make the substitution $u = 9y + 1$, so that $du = 9 dy$. Then $y = 0$ corresponds to $u = 1$, and $y = 1$ corresponds to $u = 10$, so we get

$$\int_0^1 \sqrt{9y + 1} dy = \frac{1}{9} \int_1^{10} u^{1/2} du = \frac{1}{9} \left[\frac{2}{3} u^{3/2} \right]_1^{10} = \boxed{\frac{2}{27}(10\sqrt{10} - 1)}.$$

26. We get $x' = (y - 5)^{1/2} = \sqrt{y - 5}$, which is continuous on an interval containing 5 and 6, so the Arc Length Formula applies, and the arc length is

$$s = \int_5^6 \sqrt{1^2 + (\sqrt{y - 5})^2} dy = \int_5^6 \sqrt{y - 4} dy = \left[\frac{2}{3} (y - 4)^{3/2} \right]_5^6 = \boxed{\frac{2}{3}(2\sqrt{2} - 1)}.$$

27. (a) We have $y' = 2x$, which is continuous everywhere, so the Arc Length Formula applies and the arc length is

$$s = \int_0^2 \sqrt{1^2 + (2x)^2} dx = \int_0^2 \sqrt{4x^2 + 1} dx.$$

(b) Using technology, this evaluates to $s \approx 4.64678$.

28. We will determine arc length by subdividing along the y -axis.

(a) $x' = 2y$, which is continuous everywhere, so the Arc Length Formula applies and the arc length is

$$s = \int_1^3 \sqrt{1^2 + (2y)^2} dy = \int_1^3 \sqrt{4y^2 + 1} dy.$$

(b) Using technology, this evaluates to $s \approx 8.26815$.

29. (a) $y' = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = -x(25 - x^2)^{-1/2}$. This is continuous on an interval containing 0 and 4, so the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + (-x(25 - x^2)^{-1/2})^2} dx = \int_0^4 \sqrt{1 + \frac{x^2}{25 - x^2}} dx = \int_0^4 \sqrt{\frac{25}{25 - x^2}} dx.$$

(b) Using technology, this evaluates to $s \approx 4.63648$.

30. We will determine arc length by partitioning along the y -axis.

- (a) $x' = \frac{1}{2}(4 - y^2)^{-1/2} \cdot (-2y) = -y(4 - y^2)^{-1/2}$. This is continuous on an interval containing 0 and 1, so the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + (-y(4 - y^2)^{-1/2})^2} dy = \int_0^1 \sqrt{1 + \frac{y^2}{4 - y^2}} dy = \int_0^1 \sqrt{\frac{4}{4 - y^2}} dy.$$

(b) Using technology, this evaluates to $s \approx 1.0472$.

31. (a) $y' = \cos x$, which is continuous everywhere, so the Arc Length Formula applies. The arc length is

$$s = \int_0^{\pi/2} \sqrt{1^2 + (\cos x)^2} dx = \int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx.$$

(b) Using technology, this evaluates to $s \approx 1.9101$.

32. We will determine arc length by partitioning along the y -axis.

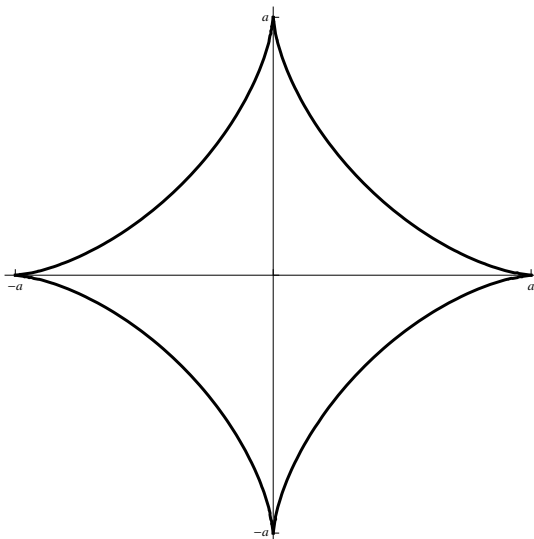
- (a) $x' = 1 + \frac{1}{y}$, which is continuous on an interval containing 1 and 4, so the Arc Length Formula applies. The arc length is

$$s = \int_1^4 \sqrt{1^2 + \left(1 + \frac{1}{y}\right)^2} dy = \int_1^4 \sqrt{\frac{1}{y^2} + \frac{2}{y} + 2} dy = \int_1^4 \frac{\sqrt{2y^2 + 2y + 1}}{y} dy.$$

(b) Using technology, this evaluates to $s \approx 5.32324$.

Applications and Extensions

33. The hypocycloid is shown below:



All four arcs are the same (the graph is symmetric in both x and y , since $(-x)^{2/3} = x^{2/3}$ and $(-y)^{2/3} = y^{2/3}$), and the arc in the first quadrant is symmetric about the line $x = y$, so the total arc length is the length of half of that arc, multiplied by 8. The arc in the first quadrant is divided in half at the point where the x -coordinate and the y -coordinate are equal, which is when $2x^{2/3} = a^{2/3}$, or $x = \frac{a}{2^{3/2}}$. So consider the arc from $x = \frac{a}{2^{3/2}}$ to $x = a$ in the first quadrant. Then $x, y \geq 0$, and solving for y gives

$$y = \left(a^{2/3} - x^{2/3}\right)^{3/2},$$

so that

$$y' = \frac{3}{2} \left(a^{2/3} - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3} \left(x^{2/3} - a^{2/3}\right)^{1/2}.$$

Now, y' is continuous on an interval containing $\frac{a}{2^{3/2}}$ and a , so the Arc Length Formula applies and the total length of the hypocycloid is therefore

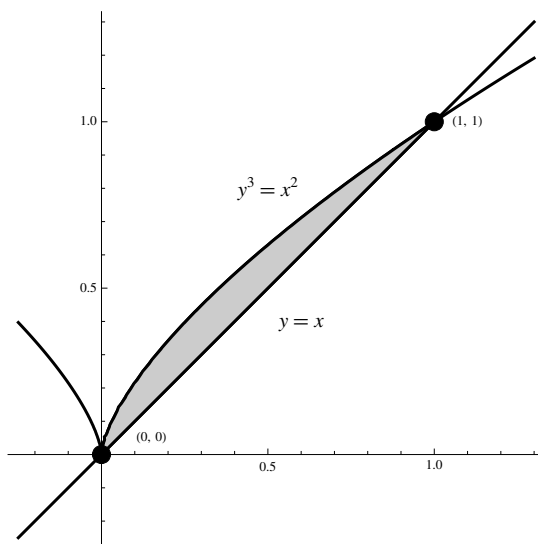
$$\begin{aligned} s &= 8 \int_{a/(2^{3/2})}^a \sqrt{1^2 + \left(-x^{-1/3} \left(a^{2/3} - x^{2/3}\right)^{1/2}\right)^2} dx \\ &= 8 \int_{a/(2^{3/2})}^a \sqrt{1 + x^{-2/3} \left(a^{2/3} - x^{2/3}\right)} dx \\ &= 8 \int_{a/(2^{3/2})}^a \sqrt{a^{2/3} x^{-2/3}} dx \\ &= 8 \int_{a/(2^{3/2})}^a a^{1/3} x^{-1/3} dx \\ &= 8 \left[\frac{3}{2} a^{1/3} x^{2/3} \right]_{a/(2^{3/2})}^a \\ &= 12 \left(a^{1/3} a^{2/3} - a^{1/3} \cdot \frac{a^{2/3}}{(2^{3/2})^{2/3}} \right) \\ &= 12 \left(a - \frac{a}{2} \right) = 6a. \end{aligned}$$

The arc length of the hypocycloid is $\boxed{6a}$.

34. The curve whose arc length we wish to find is in the first quadrant; there, the curve is $y = \sqrt{x^3} = x^{3/2}$, so that $y' = \frac{3}{2}\sqrt{x}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^3 \sqrt{1^2 + \left(\frac{3}{2}\sqrt{x}\right)^2} dx = \frac{1}{2} \int_1^3 \sqrt{4 + 9x} dx = \frac{1}{2} \left[\frac{2}{27} (4 + 9x)^{3/2} \right]_1^3 = \boxed{\frac{1}{27} (31\sqrt{31} - 13\sqrt{13})}.$$

35. The region in question is shown below:



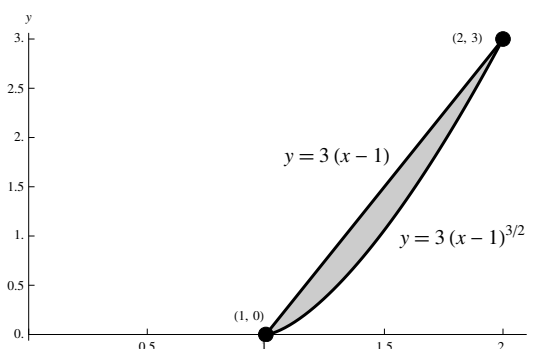
The perimeter of the region is composed of two pieces. The length of the first, along the line $y = x$, can be computed using the Distance Formula: $s = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$. For the second, we use the Arc Length Formula. If we try to partition along the x -axis, we get $y = x^{2/3}$, so that $y' = \frac{2}{3}x^{-1/3}$, which is not continuous at $x = 0$. So partition instead along the y -axis. Solving the equation for x gives $x = y^{3/2}$, so that $x' = \frac{3}{2}\sqrt{y}$, which is continuous on an interval containing 0 and 1. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + \left(\frac{3}{2}\sqrt{y}\right)^2} dy = \frac{1}{2} \int_1^3 \sqrt{4 + 9y} dy = \frac{1}{2} \left[\frac{2}{27}(4 + 9y)^{3/2} \right]_1^3 = \frac{1}{27}(31\sqrt{31} - 13\sqrt{13}).$$

So the total arc length is

$$\boxed{\sqrt{2} + \frac{1}{27}(31\sqrt{31} - 13\sqrt{13})}.$$

36. The region in question is shown below:



The perimeter of the region is composed of two pieces. The length of the first, along the line $y = 3(x-1)$, can be computed using the Distance Formula: $s = \sqrt{(2-1)^2 + (3-0)^2} = \sqrt{10}$. For the second, we use the Arc Length Formula. Partitioning along the x -axis, we have $y' = 3 \cdot \frac{3}{2}(x-1)^{1/2} = \frac{9}{2}\sqrt{x-1}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^3 \sqrt{1^2 + \left(\frac{9}{2}\sqrt{x-1}\right)^2} dx = \frac{1}{2} \int_1^3 \sqrt{81x - 77} dx = \frac{1}{2} \left[\frac{2}{243}(81x - 77)^{3/2} \right]_1^3 = \frac{1}{243}(166\sqrt{166} - 8).$$

Therefore the total arc length is

$$\boxed{\sqrt{10} + \frac{1}{243}(166\sqrt{166} - 8)}.$$

37. Solving for x gives $x = \frac{y^4+3}{6y} = \frac{y^3}{6} + \frac{1}{2y}$. Then $x' = \frac{y^2}{2} - \frac{1}{2y^2}$, which is differentiable on an interval containing 1 and 2. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned} s &= \int_1^2 \sqrt{1^2 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2} dy \\ &= \int_1^2 \sqrt{1 + \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}} dy \\ &= \int_1^2 \sqrt{\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}} dy \\ &= \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2} dy \\ &= \int_1^2 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy \\ &= \left[\frac{y^3}{6} - \frac{1}{2y}\right]_1^2 \\ &= \boxed{\frac{17}{12}}. \end{aligned}$$

38. We have $y' = \sinh x$, which is continuous everywhere, so that the Arc Length Formula applies. Therefore the arc length is

$$s = \int_0^2 \sqrt{1^2 + (\sinh x)^2} dx = \int_0^2 \sqrt{1 + \sinh^2 x} dx.$$

Since $\cosh^2 x - \sinh^2 x = 1$, we see that $1 + \sinh^2 x = \cosh^2 x$. Since $\cosh x > 0$ everywhere, we get for the arc length

$$s = \int_0^2 \sqrt{\cosh^2 x} dx = \int_0^2 \cosh x dx = [\sinh x]_0^2 = \sinh 2 - \sinh 0 = \boxed{\sinh 2}.$$

39. We have $y' = \frac{1}{\csc x} \cdot (-\cot x \csc x) = -\cot x$, which is continuous on an interval containing $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_{\pi/4}^{\pi/2} \sqrt{1^2 + (-\cot x)^2} dx = \int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x} dx.$$

Now use the identity $1 + \cot^2 x = \csc^2 x$; since we are integrating for angles in the first quadrant, $\csc x \geq 0$, so we get for the arc length

$$\begin{aligned} s &= \int_{\pi/4}^{\pi/2} \sqrt{\csc^2 x} dx \\ &= \int_{\pi/4}^{\pi/2} \csc x dx = [\ln |\csc x - \cot x|]_{\pi/4}^{\pi/2} \\ &= \ln 1 - \ln(\sqrt{2} - 1) = \boxed{-\ln(\sqrt{2} - 1)}. \end{aligned}$$

40. Solving the equation for y in the first quadrant (where $y \geq 0$) gives

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Then

$$y' = \frac{b}{a} \cdot \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) = -\frac{bx}{a\sqrt{a^2 - x^2}}.$$

This is continuous on an interval containing 0 and $\frac{a}{2}$, so the Arc Length Formula applies and we get for the arc length

$$\begin{aligned} s &= \int_0^{a/2} \sqrt{1^2 + \left(-\frac{bx}{a\sqrt{a^2 - x^2}}\right)^2} dx \\ &= \int_0^{a/2} \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} dx \\ &= \int_0^{a/2} \sqrt{\frac{b^2 x^2 + a^4 - a^2 x^2}{a^2(a^2 - x^2)}} dx \\ &= \int_0^{a/2} \sqrt{\frac{a^4 + x^2(b^2 - a^2)}{a^2(a^2 - x^2)}} dx \\ &= \int_0^{a/2} \sqrt{\frac{a^4 + a^4\left(\frac{x}{a}\right)^2\left(\left(\frac{b}{a}\right)^2 - 1\right)}{a^4\left(1 - \left(\frac{x}{a}\right)^2\right)}} dx \\ &= \int_0^{a/2} \sqrt{\frac{1 + \left(\frac{x}{a}\right)^2\left(\left(\frac{b}{a}\right)^2 - 1\right)}{1 - \left(\frac{x}{a}\right)^2}} dx. \end{aligned}$$

41. The key point here is that partitioning along either axis works except where the circle meets the axes. Where it meets the x -axis, we must partition along the y -axis since the curve has a vertical tangent line, and where it meets the y -axis, we must partition along the x -axis since the curve has a horizontal tangent line. Solving $x^2 + y^2 = 1$ for y gives $y = \pm\sqrt{1 - x^2}$, so that

$$y' = \pm\frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = \pm x(1 - x^2)^{-1/2}.$$

Similarly, if we solve for x and differentiate, we get $x' = \pm y(1 - y^2)^{-1/2}$. So the arc length integrands when partitioning along each axis are

$$\begin{aligned} x\text{-axis : } & \sqrt{1^2 + (\pm x(1 - x^2)^{-1/2})^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} \\ y\text{-axis : } & \sqrt{1^2 + (\pm y(1 - y^2)^{-1/2})^2} = \sqrt{1 + \frac{y^2}{1 - y^2}} = \frac{1}{\sqrt{1 - y^2}} \end{aligned}$$

Then:

(a) Since P_1 is in quadrant I and P_2 in quadrant II, the only axis crossing occurs at $(0, 1)$, so we can partition along the x -axis for the whole integration, giving

$$s = \int_{x_2}^{x_1} \frac{1}{\sqrt{1 - x^2}} dx.$$

- (b) Since P_1 and P_2 are both in the third quadrant, there are no axis crossings, so we can partition along either axis. Using the y -axis, since $y_1 < y_2$, we get

$$s = \int_{y_1}^{y_2} \frac{1}{\sqrt{1-y^2}} dy.$$

- (c) Since P_1 is in quadrant II and P_2 is in quadrant IV, and we are considering the counterclockwise arc from P_1 to P_2 , there are two axis crossings, at $(-1, 0)$ and at $(0, -1)$. From P_1 to $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ (for example; any other point in the third quadrant would work as well), we can partition along the y -axis, and then from $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ to P_2 we partition along the x -axis. Therefore the arc length is

$$s = \int_{-\sqrt{2}/2}^{y_1} \frac{1}{\sqrt{1-y^2}} dy + \int_{-\sqrt{2}/2}^{x_2} \frac{1}{\sqrt{1-x^2}} dx.$$

42. (a) We want to find the arc length of the curve $y = Ax^{3/2}$ from $y = 0$ to $y = h$. Partitioning along the x -axis, then, we get $y' = \frac{3}{2}Ax^{1/2}$, which is continuous for $x \geq 0$, so the Arc Length Formula applies. Note that $y = h$ gives $h = Ax^{3/2}$, so that $x = (\frac{h}{A})^{2/3}$, and this is the upper bound for integration. Then the arc length is

$$s = \int_0^{(h/A)^{2/3}} \sqrt{1^2 + \left(\frac{3}{2}Ax^{1/2}\right)^2} dx = \boxed{\frac{1}{2} \int_0^{(h/A)^{2/3}} \sqrt{4 + 9A^2x} dx}.$$

- (b) The given numbers mean that $h = 150$ and that $y = h$ corresponds to $x = 250$. Therefore $h = Ax^{3/2}$ means that $150 = A \cdot 250^{3/2}$, so that $A = 150 \cdot 250^{-3/2} \approx 0.0379$.
- (c) Substituting into the formula in part (a), we get

$$\begin{aligned} s &= \frac{1}{2} \int_0^{(h/A)^{2/3}} \sqrt{4 + 9A^2x} dx \approx \frac{1}{2} \int_0^{250} \sqrt{4 + 9 \cdot 0.0379^2 x} dx \\ &\approx \frac{1}{2} \int_0^{250} \sqrt{4 + 0.0129x} dx \approx \boxed{295.099 \text{ m}}. \end{aligned}$$

- (d) If the slope were a straight line, then using the Pythagorean Theorem, its length would be

$$\sqrt{250^2 + 150^2} = \sqrt{85000} \approx 291.55.$$

The length computed in part (c) is slightly longer than the straight-line distance, so it appears reasonable.

43. (a) Since $y' = a \sinh \frac{x}{a} \cdot \frac{1}{a} = \sinh \frac{x}{a}$, and $y'(10) = \frac{3}{4}$, we get $\frac{3}{4} = \sinh \frac{10}{a}$, so that $a = \frac{10}{\sinh^{-1} \frac{3}{4}}$. Now, compute the height of the endpoints above b , the lowest point; this is

$$y - b = a \cosh \frac{10}{a} - a = \frac{10}{\sinh^{-1} \frac{3}{4}} \left(\cosh \sinh^{-1} \frac{3}{4} - 1 \right).$$

Since $\cosh^2 x = 1 + \sinh^2 x$ for all x , we can simplify the above formula as follows:

$$\begin{aligned} \frac{10}{\sinh^{-1} \frac{3}{4}} \left(\cosh \sinh^{-1} \frac{3}{4} - 1 \right) &= \frac{10}{\sinh^{-1} \frac{3}{4}} \left(\sqrt{1 + \sinh^2 \sinh^{-1} \frac{3}{4}} - 1 \right) \\ &= \frac{10}{\sinh^{-1} \frac{3}{4}} \left(\sqrt{1 + \frac{9}{16}} - 1 \right) \\ &= \frac{10}{\sinh^{-1} \frac{3}{4}} \left(\sqrt{\frac{25}{16}} - 1 \right) \\ &= \frac{10}{\sinh^{-1} \frac{3}{4}} \cdot \frac{1}{4} = \frac{5}{2 \sinh^{-1} \frac{3}{4}}. \end{aligned}$$

Therefore the height of the supports is $\boxed{b + \frac{5}{2 \sinh^{-1} \frac{3}{4}}}$.

- (b) We determine the arc length by partitioning along the x -axis. From part (a), we know that $y' = \sinh \frac{x}{a}$, so that the arc length is (using the identity $\cosh^2 x = 1 + \sinh^2 x$ again)

$$\begin{aligned} s &= \int_{-10}^{10} \sqrt{1^2 + \left(\sinh \frac{x}{a}\right)^2} dx \\ &= \int_{-10}^{10} \sqrt{1 + \sinh^2 \frac{x}{a}} dx \\ &= \int_{-10}^{10} \sqrt{\cosh^2 \frac{x}{a}} dx \\ &= \int_{-10}^{10} \cosh \frac{x}{a} dx \\ &= \left[a \sinh \frac{x}{a} \right]_{-10}^{10} \\ &= 2a \sinh \frac{10}{a}. \end{aligned}$$

Now substitute the value of a from part (a), giving

$$s = 2 \cdot \frac{10}{\sinh^{-1} \frac{3}{4}} \sinh \sinh^{-1} \frac{3}{4} = \frac{60}{4 \sinh^{-1} \frac{3}{4}} = \boxed{\frac{15}{\sinh^{-1} \frac{3}{4}} \approx 21.6404 \text{ m}}.$$

44. The length of the rope is the arc length. We have $y' = a \sinh \frac{x}{a} \cdot \frac{1}{a} = \sinh \frac{x}{a}$, so that the arc length is (using the identity $\cosh^2 x = 1 + \sinh^2 x$)

$$\begin{aligned} L &= \int_{-c}^c \sqrt{1^2 + \left(\sinh \frac{x}{a}\right)^2} dx \\ &= \int_{-c}^c \sqrt{1 + \sinh^2 \frac{x}{a}} dx \\ &= \int_{-c}^c \sqrt{\cosh^2 \frac{x}{a}} dx \\ &= \int_{-c}^c \cosh \frac{x}{a} dx \\ &= \left[a \sinh \frac{x}{a} \right]_{-c}^c \\ &= 2a \sinh \frac{c}{a}. \end{aligned}$$

Then since d is the height of the rope when $x = c$, we get

$$(d - b + a)^2 - a^2 = \left(a \cosh \frac{c}{a} + b - a - b + a \right)^2 - a^2 = a^2 \left(\cosh^2 \frac{c}{a} - 1 \right) = a^2 \sinh^2 \frac{c}{a} = \frac{L^2}{4}.$$

Expanding the left-hand side and simplifying gives

$$(d - b)^2 + 2a(d - b) = \frac{L^2}{4}, \text{ so that } a = \frac{\frac{L^2}{4} - (d - b)^2}{2(d - b)} = \frac{L^2 - 4(d - b)^2}{8(d - b)}.$$

(Note that $d \neq b$, so this expression makes sense, since d is the height of the ends of the catenary, while b is its lowest height.)

45. (a) Let the parabola be $h(x) = ax^2 + bx + c$. Since $h(0) = 0$, we must have $c = 0$, so that the equation of motion is $h(x) = ax^2 + bx = x(ax + b)$. Since $h(0) = h(150) = 0$, we see that 150 must be a zero of $ax + b$, so that $h(x) = ax(x - 150) = ax^2 - 150ax$. It follows that $b = -150a$. Finally, $h'(x) = 2ax - 150a$, so that $h'(x) = 0$ when $x = 75$. We are given that $h(75) = 46$, so that

$$46 = h(75) = a \cdot 75 \cdot (75 - 150) = -75^2 a, \text{ so that } a = -\frac{46}{75^2}.$$

Therefore the equation of motion is

$$h(x) = -\frac{46}{75^2}x(x - 150) = \frac{46}{75^2}(150x - x^2).$$

- (b) The total arc length is twice the length of the arc from $x = 0$ to the top of the arc, at $x = -\frac{b}{a}$. Using a CAS to integrate, we get

$$\begin{aligned} s &= 2 \int_0^{-b/a} \sqrt{1^2 + (2ax + b)^2} dx = 2 \int_0^{150} \sqrt{1 + \left(-\frac{92}{75^2}x + \frac{92}{75}\right)^2} dx \\ &= \boxed{2\sqrt{14089} + \frac{5625}{46} \sinh^{-1} \frac{92}{75} \approx 363.704 \text{ m}.} \end{aligned}$$

Challenge Problems

46. This function is not differentiable at $x = 0$. To see this, use the definition of the derivative:

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h}.$$

But as $h \rightarrow 0$, $\frac{1}{h} \rightarrow \infty$, so that $\sin \frac{1}{h}$ oscillates infinitely often between -1 and 1 , so that this limit does not exist. Since it is not differentiable on an interval containing $(-\frac{\pi}{2}, \frac{\pi}{2})$, we cannot apply the Arc Length Formula.

47. (a) The total angle around point O is 360° , and the polygon divides it into 2^n equal angles, each of which must be $\frac{360}{2^n} = \frac{180}{2^{n-1}} = \frac{90}{2^{n-2}}$ degrees. But the dotted ray emanating from O is the perpendicular bisector of the opposite side of an isosceles triangle, so it also bisects its angle. Therefore $\alpha = \frac{1}{2} \cdot \frac{90}{2^{n-2}} = \frac{45^\circ}{2^{n-2}}$.
- (b) The right triangle of which α is one angle has hypotenuse 1. The side opposite α therefore has length $\sin \alpha$. Since there are 2^n sides in the polygon, there are $2 \cdot 2^n = 2^{n+1}$ segments of length $\sin \alpha$, so that the sum of all of these segments has length

$$2^{n+1} \sin \alpha = 2^{n+1} \sin \frac{45^\circ}{2^{n-2}}.$$

- (c) A table showing the value of this expression for several values of n is below. The true circumference of the circle is $2\pi r = 2\pi \approx 6.283185307$:

5	10	50	100	500	1000
6.273096981	6.283175451	6.283185307	6.283185307	6.283185307	6.283185307

48. Note that $a_0 = \frac{\sqrt{2}}{2} = \sin 45^\circ$. Then use the half-angle formula

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - \sqrt{1 - \sin^2 \theta}}{2}}.$$

Since we start with $\theta = 45^\circ$, all angles will be in the first quadrant, so we can always take positive square roots. Then

$$a_1 = \sin \frac{45^\circ}{2}, \quad a_2 = \sin \frac{45^\circ}{4}, \quad \dots, \quad a_n = \sin \frac{45^\circ}{2^n}.$$

Then by Problem 47, we know that $2^{n+3} \sin \frac{45^\circ}{2^n} = 2^{n+3} a_n$ is a good approximation to the circumference of the circle, which is 2π . Therefore $2^{n+2} a_n$ is a good approximation to $\frac{1}{2} \cdot 2\pi = \pi$. A table showing the value of $2^{n+2} a_n$ for several values of n is below. The true value is $\pi \approx 3.141592654$:

5	10	50	100	500	1000
3.141277251	3.141592346	3.141592654	3.141592654	3.141592654	3.141592654

The expression approaches π rather rapidly.

49. Since s is as given, with $y = f(x)$ we must have

$$s = e^x - f(x) = \int_0^x \sqrt{1 + (f'(t))^2} dt.$$

Taking the derivative of both sides and using the Fundamental Theorem gives

$$e^x - f'(x) = \sqrt{1 + (f'(x))^2}; \text{ now square both sides to get } e^{2x} - 2e^x f'(x) + (f'(x))^2 = 1 + (f'(x))^2.$$

Cancel the quadratic terms and solve for $f'(x)$, giving

$$f'(x) = \frac{e^{2x} - 1}{2e^x} = \frac{1}{2}(e^x - e^{-x}).$$

Now integrate to get

$$f(x) = \frac{1}{2}(e^x + e^{-x}) + C = \cosh x + C.$$

Since $f(0) = 1$, we get $C = 0$, so that $f(x) = \cosh x$. As a check, compute the arc length of f from 0 to x (use t as a dummy variable):

$$s = \int_0^x \sqrt{1^2 + \sinh^2 t} dt = \int_0^x \sqrt{\cosh^2 t} dt = \int_0^x \cosh t dt = \sinh x - \sinh 0 = \sinh x.$$

But

$$e^x - f(x) = e^x - \cosh x = e^x - \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x,$$

and the two are equal.

6.6: Work

Concepts and Vocabulary

1. True. See the first sentence of this section.
2. $W = Fx$. See the second sentence of this section.
3. A unit of work is called a *newton-meter*, or *joule*, in SI units and a *foot-pound* in the customary U.S. system of units.
4. The work W done by a continuously varying force $F = F(x)$ acting on an object, which moves the object along a straight line in the direction of F from $x = a$ to $x = b$, is given by the definite integral $\int_a^b F(x) dx$. See subsection 1.
5. A spring is said to be in *equilibrium* when it is neither extended nor compressed. See the discussion in subsection 2 of the text.
6. True. See the discussion of springs in subsection 2.
7. True. See the note in Example 4.
8. True. See the discussion at the end of this section in the text.

Skill Building

9. The work is the integral of the force through the distance, so

$$W = \int_5^{20} (40 - x) dx = \left[40x - \frac{1}{2}x^2 \right]_5^{20} = \boxed{\frac{825}{2} \text{ J}}.$$

10. The work is the integral of the force through the distance, so

$$W = \int_1^2 \frac{1}{x} dx = [\ln x]_1^2 = \boxed{\ln 2 \text{ J}}.$$

11. Following the method of example 1, since the density of the chain is 3 kg/m, and the portion of chain x m below the bridge must be lifted $40 - x$ m, we get (since $g = 9.8 \text{ m/s}^2$)

$$W = 9.8 \int_0^{40} 3(40 - x) dx = 9.8 \left[120x - \frac{3}{2}x^2 \right]_0^{40} = \boxed{23520 \text{ J}}.$$

12. Following the method of Example 1, since the density of the chain is 2 lb/ft, and the portion of chain x ft below the roof must be lifted $120 - x$ feet, we get

$$W = \int_0^{120} 2(120 - x) dx = [240x - x^2]_0^{120} = \boxed{14400 \text{ ft-lbs}}.$$

13. Since the restoring force is -3 N , we have

$$-3 = -k \cdot \frac{1}{4}, \quad \text{so that} \quad k = \boxed{12 \text{ N/m}}.$$

14. Since the restoring force is 6 ft-lb , we have

$$6 = -k \cdot \left(-\frac{1}{2}\right), \quad \text{so that} \quad k = \boxed{12 \text{ lb/ft}}.$$

15. We are stretching the spring 0.6 m beyond equilibrium, so the work done by the spring force is

$$\int_0^{0.6} (-5x) dx = \left[-\frac{5}{2}x^2 \right]_0^{0.6} = \boxed{-0.9 \text{ J}}.$$

16. We are compressing the spring 0.2 m from equilibrium, so the work done by the spring force is

$$\int_0^{-0.2} (-0.3x) dx = [-0.15x^2]_0^{-0.2} = \boxed{-0.006 \text{ J}}.$$

17. Water that is x feet above the bottom of the pool must be lifted $4 - x$ feet. The cross sectional area of the pool at any height is the area of a circle with radius 12 ft, which is 144π . The water at the bottom must be lifted 4 feet, while the water at the top is lifted 0 feet, so the limits of integration are from 0 to 4, and the work performed is

$$\int_0^4 (62.42 \cdot 144\pi \cdot (4 - x)) dx = 8988.48\pi \int_0^4 (4 - x) dx = 8988.48\pi \left[4x - \frac{1}{2}x^2 \right]_0^4 = \boxed{\approx 225905 \text{ ft-lb}}.$$

18. Gasoline that is x meters above the bottom of the tank must be lifted $10 - x$ m. The cross sectional area of the tank at any height is the area of a circle with radius 8, which is 64π . The gasoline at the

bottom must be lifted 10 m, while that at the top is lifted 0 m, so the limits of integration are from 0 to 10, and the work performed is

$$\begin{aligned} 9.8 \int_0^{10} (720 \cdot 64\pi \cdot (10 - x)) \, dx &= 451584\pi \int_0^{10} (10 - x) \, dx \\ &= 451584\pi \left[10x - \frac{1}{2}x^2 \right]_0^{10} = \boxed{\approx 7.093 \times 10^7 \text{ J}}. \end{aligned}$$

19. Each horizontal cross section of the pyramid is a square; the side lengths of this square decrease linearly from 2 at the top to 0 at the bottom, so at a point x meters below the top of the container, the side length is $2 - \frac{2}{5}x$. Therefore the cross sectional area of the slurry at height x is $(2 - \frac{2}{5}x)^2$, and this must be lifted x m. So the work performed (since the container is only filled 4 meters deep) is

$$\begin{aligned} 9.8 \int_1^5 \left(17.9 \cdot \left(2 - \frac{2}{5}x \right)^2 \cdot x \right) \, dx &= 175.42 \int_1^5 \left(4x - \frac{8}{5}x^2 + \frac{4}{25}x^3 \right) \, dx \\ &= 175.42 \left[2x^2 - \frac{8}{15}x^3 + \frac{1}{25}x^4 \right]_1^5 = \boxed{1197.53 \text{ J}}. \end{aligned}$$

20. Each horizontal cross section of the vat is a rectangle; its sides decrease linearly from 2×0.5 at the top to 0×0 at the bottom, so at a point x meters below the top of the vat, the dimensions of the cross section are $(2 - \frac{1}{2}x) \cdot (0.5 - \frac{1}{8}x)$. Therefore the area of this cross section is

$$\left(2 - \frac{1}{2}x \right) \left(\frac{1}{2} - \frac{1}{8}x \right) = 1 - \frac{1}{2}x + \frac{1}{16}x^2.$$

This cross section must be lifted x meters. Finally, the density of olive oil is

$$0.9 \text{ g/cm}^3 \times 0.001 \text{ kg/g} \times 100^3 \text{ cm}^3/\text{m}^3 = 900 \text{ kg/m}^3.$$

The vat is filled to a depth of 2 m, so we must integrate from 2 m below the top to 4 m below the top, and the work is

$$\begin{aligned} 9.8 \int_2^4 \left(900x \left(1 - \frac{1}{2}x + \frac{1}{16}x^2 \right) \right) \, dx &= 8820 \int_2^4 \left(x - \frac{1}{2}x^2 + \frac{1}{16}x^3 \right) \, dx \\ &= 8820 \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{64}x^4 \right]_2^4 = \boxed{3675 \text{ J}}. \end{aligned}$$

Applications and Extensions

21. The elevator itself is lifted 400 ft, so this takes $10,000 \times 400 = 4 \times 10^6$ ft-lbs of work. The remaining work is lifting the cables themselves. For each cable, the portion of the cable x feet from the top must be lifted x feet. The cable weighs 0.36 lbs/in, which is $0.36 \times 12 = 4.32$ lbs/ft, so the work involved in lifting all six cables is

$$6 \int_0^{400} 4.32x \, dx = 6 [2.16x^2]_0^{400} = 2.0736 \times 10^6 \text{ ft-lbs}.$$

The total work is the sum of these two, or $\boxed{6.0736 \times 10^6 \text{ ft-lbs}}$.

22. The 15 m of cable must be lowered another 60 m; since the cable's density is 9 kg/m, this takes $15 \times 9 \times 60 \times 9.8 = 79380$ J. The remainder of the work expended is to lower the additional cable. A bit of cable on the spool that ends up x m below the spool must be lowered x m, so the work expended is

$$9.8 \int_0^{60} 9x \, dx = 9.8 \left[\frac{9}{2}x^2 \right]_0^{60} = 158760 \text{ J}.$$

So the total work expended by gravity is $79380 + 158760 = \boxed{238140 \text{ J}}$.

23. Pulling the bucket itself to the top requires lifting a mass of 75 kg through a distance of 10 m, which takes $75 \times 10 \times 9.8 = 7350$ J. The remainder of the work expended is for the chain itself. The mass density of the chain is $\frac{20}{10} = 2$ kg/m, and the portion of the chain x m below the roof must be lifted x m, so the work expended is

$$9.8 \int_0^{10} 2x \, dx = 9.8 [x^2]_0^{10} = 980 \text{ J.}$$

Therefore the total work is $7350 + 980 = \boxed{8330 \text{ J}}$.

24. Pulling the bucket itself to the top requires lifting a mass of 75 kg through a distance of 10 m, which takes $75 \times 10 \times 9.8 = 7350$ J. The remainder of the work expended is for the chain itself. The mass density of the chain is $\frac{15}{10} = 1.5$ kg/m, and the portion of the chain x m below the roof must be lifted x m, so the work expended is

$$9.8 \int_0^{10} 1.5x \, dx = 9.8 [0.75x^2]_0^{10} = 735 \text{ J.}$$

Therefore the total work is $7350 + 735 = \boxed{8085 \text{ J}}$.

25. The force required to extend the spring from 1 m to 3 m, which is 2 m beyond its equilibrium point, is -3 N, so the spring constant is given by $-3 = -k \cdot 2$, and therefore $k = \frac{3}{2}$. Then the work expended to extend it 1 m from equilibrium (to a length of 2 m) is

$$\int_0^1 \left(-\frac{3}{2}\right) x \, dx = -\left[\frac{3}{4}x^2\right]_0^1 = \boxed{-\frac{3}{4} \text{ J}}.$$

26. The force required to compress the spring by $\frac{3}{2}$ m is 10 N, so the spring constant is given by the equation $10 = -k \cdot \left(-\frac{3}{2}\right)$, and therefore $k = \frac{20}{3}$. Then the work expended to compress the spring to a length of 1 m is

$$\int_0^1 \left(-\frac{20}{3}x\right) \, dx = \left[\frac{10}{3}x^2\right]_0^1 = \boxed{-\frac{10}{3} \text{ J}}.$$

27. The force required to extend the spring by 4 ft is 2 lb, so the spring constant is given by $-2 = -k \cdot 4$, and then $k = \frac{1}{2}$. So

$$-9 \text{ ft-lb} = \int_0^x \left(-\frac{1}{2}t\right) \, dt = -\left[\frac{1}{4}t^2\right]_0^x = -\frac{1}{4}x^2 \text{ ft-lb.}$$

Simplifying gives $x^2 = 36$, so that $x = 6$ (since we are extending the spring, we take the positive solution). Therefore the total length of the spring is $6 + 4 = \boxed{10 \text{ ft}}$.

28. The spring constant is $\frac{1}{2}$, from Problem 27, so using the integral from Problem 27, we get

$$-8 \text{ ft-lb} = -\frac{1}{4}x^2 \text{ ft-lb,}$$

so that $x = \sqrt{32} = 4\sqrt{2}$. So the spring is extended $4\sqrt{2}$ ft, for a total length of $\boxed{4 + 4\sqrt{2} \text{ ft}}$.

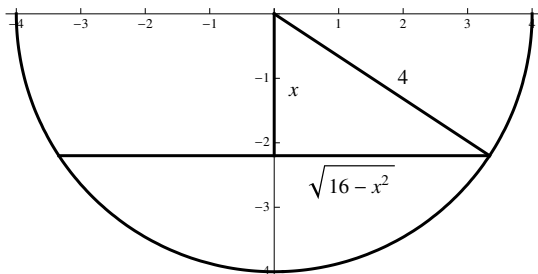
29. The radius of the cone, which is 4 m at the top and 0 m at the bottom, decreases linearly with height, so at a distance of x m from the top, its radius is $4 - x$ m. Therefore the cross sectional area of the cone x m from the top is $\pi(4 - x)^2 \text{ m}^2$. That cross section must be lifted x m, so that the total work required is

$$\begin{aligned} 9.8 \int_0^4 1000 \cdot x \cdot \pi(4 - x)^2 \, dx &= 9800\pi \int_0^4 x(4 - x)^2 \, dx = 9800\pi \int_0^4 (16x - 8x^2 + x^3) \, dx \\ &= 9800\pi \left[8x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4\right]_0^4 = \boxed{\frac{627200}{3}\pi \text{ J}}. \end{aligned}$$

30. All the calculations from Problem 29 are still valid, except that the integration goes from 2 to 4 (since the integration variable x is the distance below the top of the tank). Therefore we get for the work

$$9.8 \cdot 1000\pi \left[8x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4 \right]_2^4 = \boxed{\frac{196000}{3}\pi \text{ J}}.$$

31. To determine the radius of the cross section x m below the top of the bowl, consider the following diagram, which shows a vertical cross section through the center of the bowl:



At a height x m below the top, the radius of the cross section is $\sqrt{16 - x^2}$, so that the area of the cross section at that height is $(16 - x^2)\pi$. That cross section must be lifted x m, so the work performed is

$$9.8 \int_2^4 1000x(16 - x^2)\pi \, dx = 9800\pi \int_2^4 (16x - x^3) \, dx = 9800\pi \left[8x^2 - \frac{1}{4}x^4 \right]_2^4 = \boxed{352800\pi \text{ J}}.$$

32. The area of each cross section is the same as in Problem 31. Now, that cross section must be lifted $x + 2$ m, since we want to lift it 2 m above the top of the tank. The integration bounds are now from 0 to 4, since the tank is full. Then the work performed is

$$\begin{aligned} 9.8 \int_0^4 1000(x + 2)(16 - x^2)\pi \, dx &= 9800\pi \int_0^4 (-x^3 - 2x^2 + 16x + 32) \, dx \\ &= 9800\pi \left[-\frac{1}{4}x^4 - \frac{2}{3}x^3 + 8x^2 + 32x \right]_0^4 = \boxed{\frac{4390400}{3}\pi \text{ J}}. \end{aligned}$$

33. The area of each cross section is 4π , since the tank has a diameter of 4 m. The cross section x m below the top must be lifted x m. We wish to lift only the top half of the water, so the integration bounds are from 0 to 3 and the total work is

$$9.8 \int_0^3 1000x \cdot 4\pi \, dx = 9.8 \int_0^3 4000\pi x \, dx = 9.8 [2000\pi x^2]_0^3 = \boxed{176400\pi \text{ J}}.$$

34. The area of each horizontal cross section is $30 \times 20 = 600 \text{ ft}^2$, and the cross section x ft below the top must be raised x ft. The integration bounds are from 1 to 6 since there is water in the bottom 5 feet of the pool. Therefore the total work is

$$\int_1^6 62.42x \cdot 600 \, dx = [18726x^2]_1^6 = \boxed{655410 \text{ ft-lbs}}.$$

35. The area of each horizontal cross section is $25 \times 15 = 375 \text{ ft}^2$, and the cross section x ft below the top must be raised x ft. The integration bounds are from 1 to 5 since there is water in the bottom 4 feet of the pool. Therefore the total work is

$$\int_1^5 62.42x \cdot 375 \, dx = [11703.8x^2]_1^5 = 280890 \text{ ft-lbs}.$$

Since the motor can do 550 ft-lbs of work per second, the time required to empty the pool is

$$\frac{280890}{550} \text{ s} \approx 511 \text{ s} = \frac{511}{60} \text{ min} \approx \boxed{8.5 \text{ min}}.$$

It takes between 8 and 9 minutes to empty the pool.

36. From the discussion in the text, the work required to move a mass m kg from the surface of the Earth to a distance d meters above the center of the Earth is

$$gRm \left(1 - \frac{R}{R+d} \right) \text{ J},$$

where $g \approx 9.8 \text{ m/s}^2$ is the acceleration due to Earth's gravity and $R \approx 6.37 \times 10^6 \text{ m}$ is the radius of the Earth. So to move a mass of 30 kg a distance of 30 km = 30000 m above the Earth's surface requires

$$9.8 \cdot 6.37 \times 10^6 \cdot 30 \left(1 - \frac{6370000}{6370000 + 30000} \right) \approx \boxed{8.77866 \times 10^6 \text{ J}}.$$

37. From the discussion in the text, the work required to move a mass m kg from the surface of the Earth to a distance d meters above the surface of the Earth is

$$gRm \left(1 - \frac{R}{R+d} \right) \text{ J},$$

where $g \approx 9.8 \text{ m/s}^2$ is the acceleration due to Earth's gravity and $R \approx 6.37 \times 10^6 \text{ m}$ is the radius of the Earth. So to move a mass of 1000 kg a distance of 800 km = 800000 m above the Earth's surface requires

$$9.8 \cdot 6.37 \times 10^6 \cdot 1000 \left(1 - \frac{6370000}{6370000 + 800000} \right) \approx \boxed{6.96524 \times 10^9 \text{ J}}.$$

38. To move the charge from $x = 2a$ to $x = a$, we must apply a force equal to the negative of the electrical force, or $-\frac{m}{x^2}$. Here we are dealing with a unit charge, so $m = 1$, and then the work is

$$\int_{2a}^a \left(-\frac{1}{x^2} \right) dx = \left[\frac{1}{x} \right]_{2a}^a = \boxed{\frac{1}{2a}}.$$

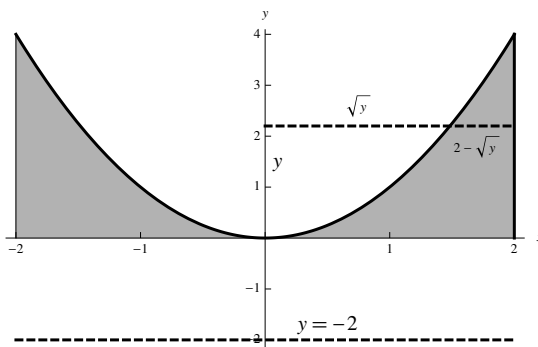
39. Since the water weighs 20 lb at the bottom, and loses a fourth of its weight, it weighs 15 lb at the top. So at a depth x from the top of the well, the weight of the water in the bucket is $15 + 5\frac{x}{25} = 15 + \frac{x}{5}$ lb. Since the bucket weighs 1.5 lb, the total weight being lifted at height x is $16.5 + \frac{x}{5}$ lb. To compute the work, note that there is no need to multiply by x in this integration, since we are lifting this weight "infinitesimally higher", so the expression above is the complete integrand. Another way of looking at this is that in the previous problems, we lifted the cross section all the way to the top, so we needed to multiply by x , the distance to the top, in order to figure out how far to lift it. Here we are lifting it only dx .

$$\int_0^{25} \left(16.5 + \frac{x}{5} \right) dx = \left[16.5x + \frac{1}{10}x^2 \right]_0^{25} = \boxed{475 \text{ ft-lbs}}.$$

40. The force is $F = -300000x$, so the work done is

$$\int_0^{0.1} (-300000x) dx = - \left[150000x^2 \right]_0^{0.1} = \boxed{-1500 \text{ J}}.$$

41. A vertical cross section of the container, with the level of the water source shown, is below:



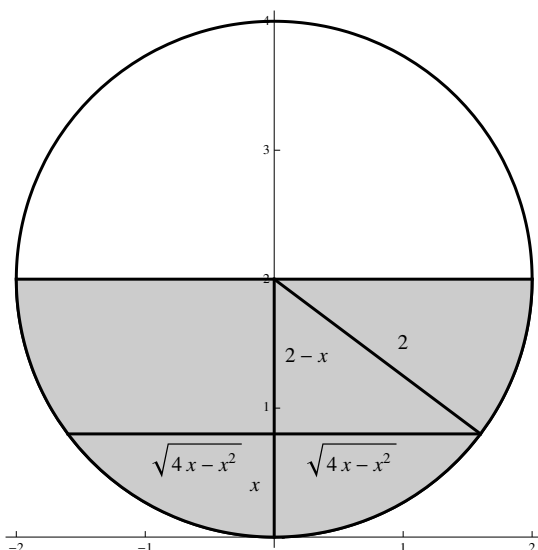
At height y , a horizontal cross section of the container is a washer with inner radius $\sqrt{y}$ and outer radius 2 (see the figure), so its area is $\pi(2^2 - (\sqrt{y})^2) = (4 - y)\pi$. To lift that water to height y , it must be lifted $y + 2$ units since the source is 2 units below the x -axis. Finally, the container is bounded by $x = \pm 2$, so by $y = 2^2 = 4$. Therefore the total work performed is

$$\int_0^4 (y + 2)(4 - y)\pi \, dy = \pi \int_0^4 (-y^2 + 2y + 8) \, dy = \pi \left[-\frac{1}{3}y^3 + y^2 + 8y \right]_0^4 = \boxed{\frac{80}{3}\pi}.$$

42. (a) The force at 0.1 m is $-100 \cdot (-0.1) = 10$ N; at 0.2 m it is $-100 \cdot (-0.2) = 20$ N; and at 0.4 m it is $-100 \cdot (-0.4) = 4$ N.
- (b) The work is given by the integral of $100x$ over the corresponding intervals:

$$\begin{aligned} \int_0^{0.1} (-100x) \, dx &= -[50x^2]_0^{0.1} = -0.5 \text{ J} \\ \int_0^{0.2} (-100x) \, dx &= -[50x^2]_0^{0.2} = -2.0 \text{ J} \\ \int_0^{0.4} (-100x) \, dx &= -[50x^2]_0^{0.4} = -8.0 \text{ J}. \end{aligned}$$

43. An end-on view of the tank is:



- (a) At a height x m from the bottom of the tank, the width of the cross section shown is

$$2\sqrt{4 - (2 - x)^2} = 2\sqrt{4 - (4 - 4x + x^2)} = 2\sqrt{4x - x^2} \text{ m},$$

so the area of the cross sectional layer is (since the tank is 6 m long)

$$2\sqrt{4x - x^2} \cdot 6 = 12\sqrt{4x - x^2} \text{ m}^2.$$

That cross section must be lifted $2 - x + 2 = 4 - x$ m, so the work required is

$$9.8 \cdot 820 \int_0^2 (4 - x) \cdot 12\sqrt{4x - x^2} dx = 96432 \int_0^2 (4 - x)\sqrt{4x - x^2} dx.$$

- (b) Using technology, we get

$$96432 \int_0^2 (4 - x)\sqrt{4x - x^2} dx \approx 863052 \text{ J}.$$

Challenge Problems

44. (a) Substituting into $W = \int_{V_1}^{V_2} p dV$ gives

$$W = \int_{V_1}^{V_2} \frac{nRT}{V} dV = nRT [\ln V]_{V_1}^{V_2} = \boxed{nRT \ln \frac{V_2}{V_1}}.$$

- (b) Again substituting into $W = \int_{V_1}^{V_2} p dV$ gives (since $\gamma \neq 1$)

$$W = \int_{V_1}^{V_2} \frac{K}{V^\gamma} dV = K \left[\frac{1}{1 - \gamma} V^{1-\gamma} \right]_{V_1}^{V_2} = \boxed{\frac{K}{1 - \gamma} \left(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}} \right)}.$$

45. Since the volume is given in cubic feet, and we want cubic inches, we multiply by $12^3 = 1728$ to get 3456 in^3 for the current volume and 6912 in^3 for the desired volume. Since $PV^{1.4} = c$, and $P = 100$ when $V = 3456 \text{ in}^3$, we get $100 \cdot 3456^{1.4} = c$, or $c \approx 8.995 \times 10^6$. Then since we are doubling the volume from 3456 to 6912, the work done is

$$\begin{aligned} W &= \int_{3456}^{6912} P dV \approx \int_{3456}^{6912} \frac{8.995 \times 10^6}{V^{1.4}} dV = 8.995 \times 10^6 \int_{3456}^{6912} V^{-1.4} dV \\ &= 8.995 \times 10^6 \left[-\frac{1}{0.4} V^{-0.4} \right]_{3456}^{6912} \approx 209209 \text{ in-lb}. \end{aligned}$$

46. The work is

$$W = \int_{2.4}^{4.6} P dV = \int_{2.4}^{4.6} \frac{120}{V^{1.2}} dV = 120 \left[-\frac{1}{0.2} V^{-0.2} \right]_{2.4}^{4.6} = 600 \left(\frac{1}{2.4^{0.2}} - \frac{1}{4.6^{0.2}} \right) \approx 61.45 \text{ in-lb}.$$

6.7: Hydrostatic Pressure and Force

Concepts and Vocabulary

1. Pressure is defined as the *force* exerted per unit area.
2. True. The pressure is ρgh , where h is the depth of the plate.
3. (b), Pascal, is correct. See Table 2.
4. True. See the discussion following Table 2.

Skill Building

5. At each height (y -coordinate), the depth of the water above that height is $5 - y$, so the force on a slice at that height is $3\rho g(5 - y)\Delta y$. Therefore the total force is the integral over the possible slices, or

$$\int_0^5 3\rho g(5 - y) dy = 3\rho g \left[5y - \frac{1}{2}y^2 \right]_0^5 = \frac{75}{2}\rho g.$$

Using SI units, we have $\rho g = 9800$, so we get $\frac{75}{2} \cdot 9800 = \boxed{367500 \text{ N}}$.

6. At each height (y -coordinate), the depth of the water above that height is $-y$. At that height, the coordinates of the two edges of the end of the container are, using the Pythagorean Theorem, $\pm\sqrt{4 - y^2}$. So the force on a slice at that height is $2\sqrt{4 - y^2} \cdot \rho g(-y)\Delta y$. Therefore the total force is

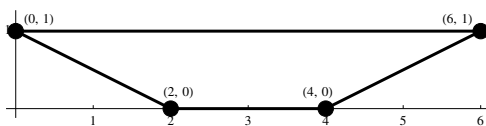
$$\int_{-2}^0 2\sqrt{4 - y^2} \cdot \rho g(-y) dy = -2\rho g \int_{-2}^0 y\sqrt{4 - y^2} dy.$$

Make the substitution $u = 4 - y^2$, so that $du = -2y dy$. Then $y = -2$ corresponds to $u = 0$, and $y = 0$ to $u = 4$, so we get

$$-2\rho g \int_{-2}^0 y\sqrt{4 - y^2} dy = -2\rho g \cdot \left(-\frac{1}{2}\right) \int_0^4 u^{1/2} du = \rho g \left[\frac{2}{3}u^{3/2} \right]_0^4 = \frac{16}{3}\rho g.$$

Using SI units, we have $\rho g = 9800$, so we get $\frac{16}{3} \cdot 9800 = \boxed{\frac{156800}{3} \approx 52266.7 \text{ N}}$.

7. Assuming that the diagonal lines make the same angle with the x -axis, we get the following diagram of the end of the container:



The diagonal line on the left passes through $(2, 0)$ and $(0, 1)$, so it has equation

$$y - 1 = \frac{1 - 0}{0 - 2}(x - 0) = -\frac{1}{2}x, \quad \text{or} \quad y = 1 - \frac{1}{2}x, \quad \text{or} \quad x = 2 - 2y.$$

The diagonal line on the right passes through $(4, 0)$ and $(6, 1)$, so it has equation

$$y = \frac{1 - 0}{6 - 4}(x - 4), \quad \text{or} \quad y = \frac{1}{2}x - 2, \quad \text{or} \quad x = 4 + 2y.$$

At a height y , the distance below the surface of the water is $1 - y$, so the force on a slice at that height is

$$\rho g(1 - y)((4 + 2y) - (2 - 2y)) = \rho g(1 - y)(2 + 4y) = \rho g(-4y^2 + 2y + 2).$$

Therefore the total force on the plate is

$$\int_0^1 \rho g(-4y^2 + 2y + 2) dy = \rho g \left[-\frac{4}{3}y^3 + y^2 + 2y \right]_0^1 = \frac{5}{3}\rho g = \boxed{\frac{49000}{3} \approx 16333.3 \text{ N}}.$$

8. The diagonal side of the container passes through the points $(3, 0)$ and $(0, 4)$, so its equation is given by $4x + 3y = 12$, or $x = 3 - \frac{3}{4}y$. At a given y -coordinate, a cross section is $4 - y$ feet below the surface

of the water. Solving $4x + 3y = 12$ for x , we see that the width of that cross section is $x = 3 - \frac{3}{4}y$. Then the total force on the end of the container is

$$\begin{aligned} \int_0^4 \rho g(4-y) \left(3 - \frac{3}{4}y\right) dy &= \rho g \int_0^4 \left(12 - 6y + \frac{3}{4}y^2\right) dy \\ &= \rho g \left[12y - 3y^2 + \frac{1}{4}y^3\right]_0^4 = 16\rho g = \boxed{1000 \text{ lbs}}. \end{aligned}$$

9. At any depth, the width of the plate is 2 m, so the force on a cross section at height y is $2\rho g y \Delta y$. Therefore the total force on the plate is

$$\int_0^6 2\rho g y dy = [\rho g y^2]_0^6 = 36\rho g = \boxed{352800 \text{ N}}.$$

10. The analysis in the solution to Problem 9 is still valid; the only difference is that the integral goes from $y = 1$ to $y = 7$, since the plate is suspended 1 m below the water surface. Therefore the total force on the plate is

$$\int_1^7 2\rho g y dy = [\rho g y^2]_1^7 = 48\rho g = \boxed{470400 \text{ N}}.$$

Applications and Extensions

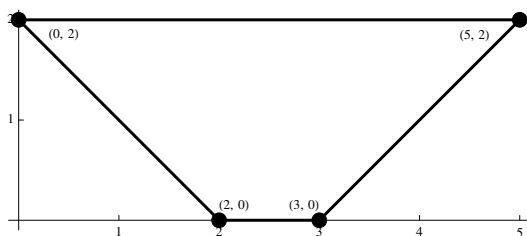
11. At any depth, the width of the pool is 20 ft. Therefore the force on the short side of the pool is (since the pool is 6 ft deep)

$$\int_0^6 \rho g 20y dy = 62.5 [10y^2]_0^6 = \boxed{22500 \text{ lbs}}.$$

12. The long side of the pool is 30 ft long, so, since the pool is again 6 ft deep, the total force is

$$\int_0^6 \rho g 30y dy = 62.5 [15y^2]_0^6 = \boxed{33750 \text{ lbs}}.$$

13. Place the trapezoid on a Cartesian grid:



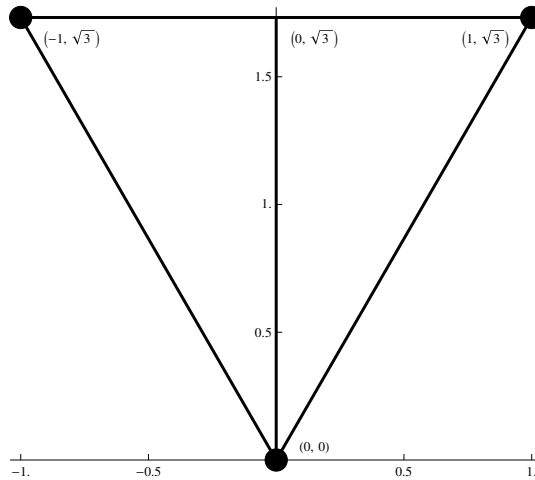
Then the equations of the two diagonal lines are:

$$\begin{aligned} y - 0 &= \frac{0 - 2}{2 - 0}(x - 2) & \text{or} & & y &= 2 - x & \text{or} & & x &= 2 - y \\ y - 0 &= \frac{0 - 2}{3 - 5}(x - 3) & \text{or} & & y &= x - 3 & \text{or} & & x &= y + 3. \end{aligned}$$

So at any height y , the height of the water above a slice at that point is $1 - y$ (since the trough is filled only to a depth of 1 m), and the width of the slice is $(y + 3) - (2 - y) = 2y + 1$, so the force on the end of the trough is

$$\begin{aligned} \int_0^1 \rho g(1-y)(2y+1) dy &= \rho g \int_0^1 (-2y^2 + y + 1) dy = \rho g \left[-\frac{2}{3}y^3 + \frac{1}{2}y^2 + y\right]_0^1 \\ &= 9800 \cdot \frac{5}{6} = \boxed{\frac{24500}{3} \approx 8166.67 \text{ N}}. \end{aligned}$$

14. Placing the bottom vertex of the equilateral triangle at the origin, we get the following picture:



Each of the diagonal lines makes an angle of 60° with the x -axis, so their equations are $\frac{y}{x} = \tan 60^\circ$, or $x = \frac{1}{\sqrt{3}}y$ and $\frac{y}{x} = \tan 120^\circ$, or $x = -\frac{1}{\sqrt{3}}y$. So at a height of y above the bottom the width of the trough is $\frac{1}{\sqrt{3}}y - \left(-\frac{1}{\sqrt{3}}y\right) = \frac{2}{\sqrt{3}}y$. Since the equilateral triangle is 2 m on the side, its height is $\sqrt{3}$, so that a cross section at y is $\sqrt{3} - y$ m below the surface. Therefore the force is

$$\int_0^{\sqrt{3}} \rho g(\sqrt{3} - y) \left(\frac{2}{\sqrt{3}}y\right) dy = \rho g \int_0^{\sqrt{3}} \left(-\frac{2}{\sqrt{3}}y^2 + 2y\right) dy = \rho g \left[-\frac{2}{3\sqrt{3}}y^3 + y^2\right]_0^{\sqrt{3}} = 1\rho g = \boxed{9800 \text{ N}}.$$

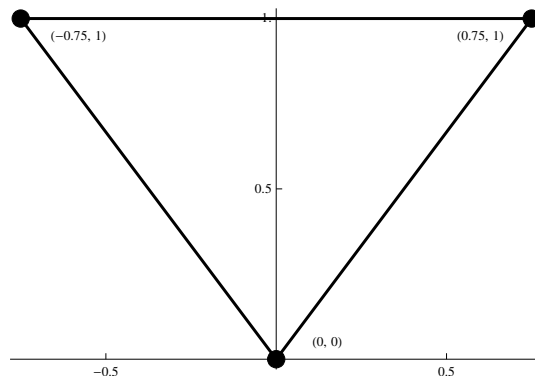
15. With the origin at the dot in the diagram, water at a given value of y has width $2\sqrt{4 - y^2}$, since the left edge of the trough will be at $x = -\sqrt{4 - y^2}$ and the right edge will be at $\sqrt{4 - y^2}$. Since the trough is filled, water at a given value of y is at a depth of $-y$ m. So the total force is

$$\int_{-2}^0 \rho g(-y)2\sqrt{4 - y^2} dy = \rho g \int_{-2}^0 (-2y\sqrt{4 - y^2}) dy.$$

Now make the substitution $u = 4 - y^2$, so that $du = -2y dy$. Then $y = -2$ corresponds to $u = 0$, and $y = 0$ to $u = 4$, so we get

$$\rho g \int_{-2}^0 (-2y\sqrt{4 - y^2}) dy = \rho g \int_0^4 u^{1/2} du = \rho g \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{16}{3}\rho g \approx \boxed{52266 \text{ N}}.$$

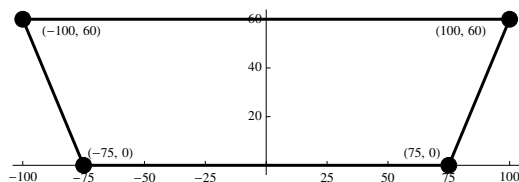
16. The face of the floodgate is shown below:



The diagonal lines have slopes $-\frac{1}{0.75} = -\frac{4}{3}$ and $\frac{1}{0.75} = \frac{4}{3}$, so their equations are $x = -\frac{3}{4}y$ and $x = \frac{3}{4}y$. Therefore for a given value of y , the width of a cross section of the floodgate is $\frac{3}{4}y - (-\frac{3}{4}y) = \frac{3}{2}y$, and this cross section is $1 - y$ m below the surface. Then the total force is

$$\int_0^1 \rho g(1 - y) \left(\frac{3}{2}y \right) dy = \rho g \int_0^1 \left(\frac{3}{2}y - \frac{3}{2}y^2 \right) dy = \rho g \left[\frac{3}{4}y^2 - \frac{1}{2}y^3 \right]_0^1 = \boxed{2450 \text{ lbs}}.$$

17. Placing the origin at the center of the bottom of the dam, the face of the dam is shown below:



The two diagonal lines have equations

$$\begin{aligned} y - 0 &= \frac{60 - 0}{-100 - (-75)}(x - (-75)) \quad \text{or} \quad y = -\frac{12}{5}(x + 75) \quad \text{or} \quad y = -\frac{12}{5}x - 180 \\ y - 0 &= \frac{60 - 0}{100 - 75}(x - 75) \quad \text{or} \quad y = \frac{12}{5}(x - 75) \quad \text{or} \quad y = \frac{12}{5}x - 180. \end{aligned}$$

Solving for x gives

$$x = -\frac{5}{12}y - 75 \quad \text{or} \quad x = \frac{5}{12}y + 75.$$

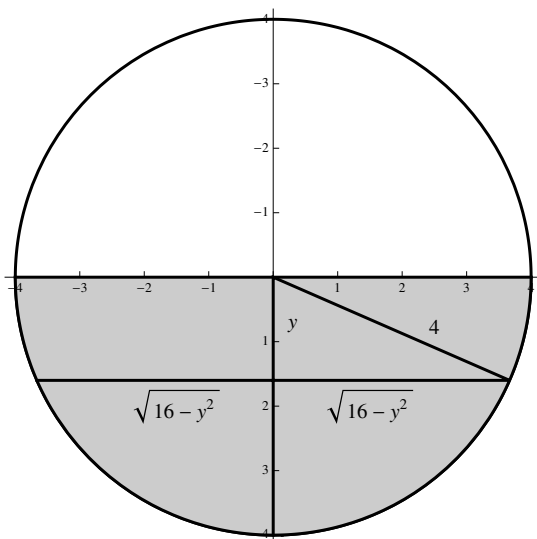
Therefore the width of a cross section at height y is

$$\left(\frac{5}{12}y + 75 \right) - \left(-\frac{5}{12}y - 75 \right) = \frac{5}{6}y + 150,$$

and such a cross section lies $60 - y$ m below the surface. Then the total force is

$$\begin{aligned} \int_0^{60} \rho g(60 - y) \left(\frac{5}{6}y + 150 \right) dy &= \rho g \int_0^{60} \left(9000 - 100y - \frac{5}{6}y^2 \right) dy \\ &= \rho g \left[9000y - 50y^2 - \frac{5}{18}y^3 \right]_0^{60} = \boxed{2.94 \times 10^9 \text{ N}}. \end{aligned}$$

18. Placing the origin at the center of one end of the tank with the positive y -axis pointing down, the face of the tank is shown below:



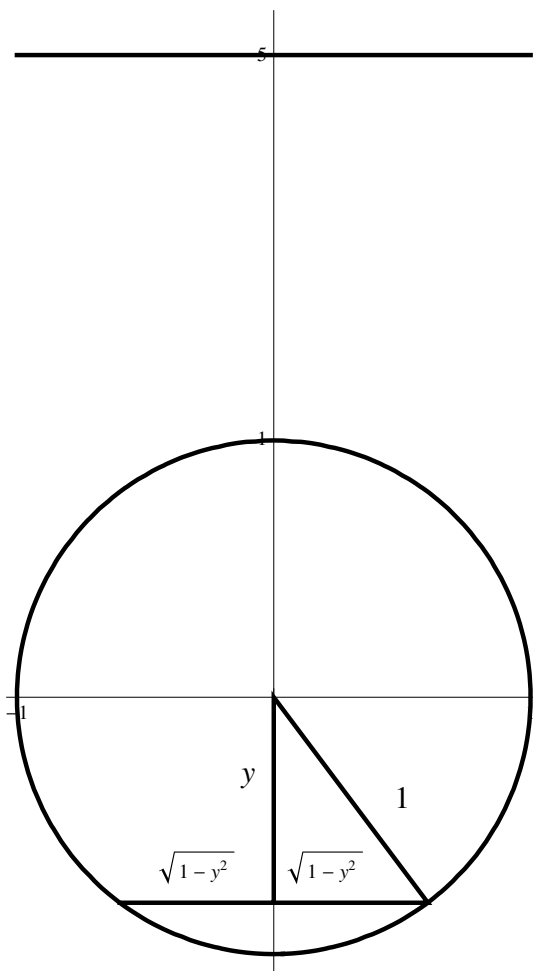
At a given depth y , the distance across that end of the tank is $2\sqrt{16 - y^2}$, and that cross section lies y ft below the surface. Therefore the total force is

$$\int_0^4 \rho g y \cdot 2\sqrt{16 - y^2} dy = 2\rho g \int_0^4 y\sqrt{16 - y^2} dy.$$

Now use the substitution $u = 16 - y^2$, so that $du = -2y dy$. Then $y = 0$ corresponds to $u = 16$, while $y = 4$ corresponds to $u = 0$, so we get

$$2\rho g \int_0^4 y\sqrt{16 - y^2} dy = 2\rho g \cdot \left(-\frac{1}{2}\right) \int_{16}^0 u^{1/2} dy = -\rho g \left[\frac{2}{3}u^{3/2}\right]_{16}^0 = \frac{128}{3}\rho g = 60 \cdot \frac{128}{3} = \boxed{2560 \text{ lbs}}.$$

19. Place the origin at the center of the viewing plate; then the situation is shown below:

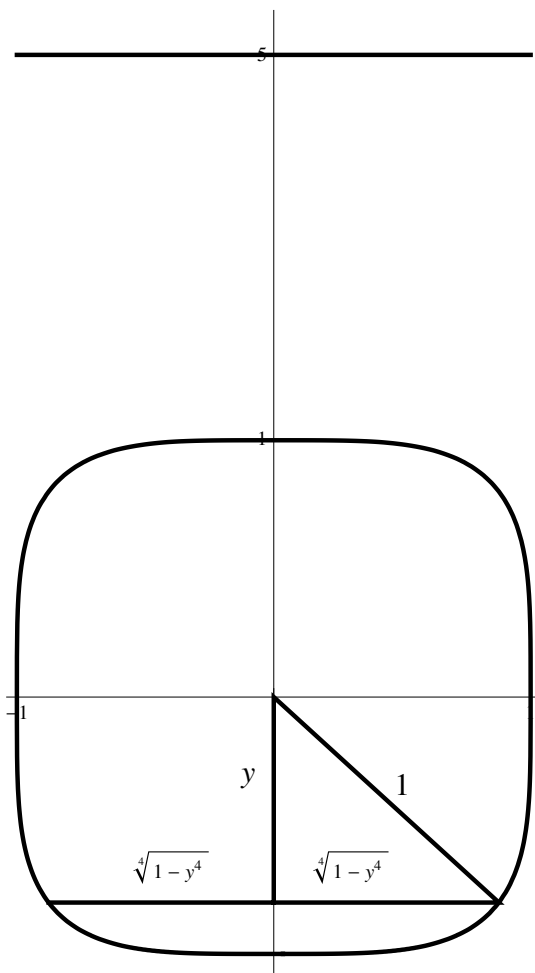


The equation of the viewing plate is $x^2 + y^2 = 1$, so at any y -coordinate, the width of a cross section of the plate is $2\sqrt{1 - y^2}$ (see the figure), and that cross section is $5 - y$ m below the water surface. (Note that if $y < 0$, as drawn, then $5 - y > 5$, which makes sense since this part of the plate is *more* than 5 m below the surface.) Therefore the total force is

$$\int_{-1}^1 \rho g (5 - y) \cdot 2\sqrt{1 - y^2} dy$$

Evaluating numerically, with $\rho g = 1025 \cdot 9.8$, gives $5\rho g\pi \boxed{\approx 157,786 \text{ N}}$.

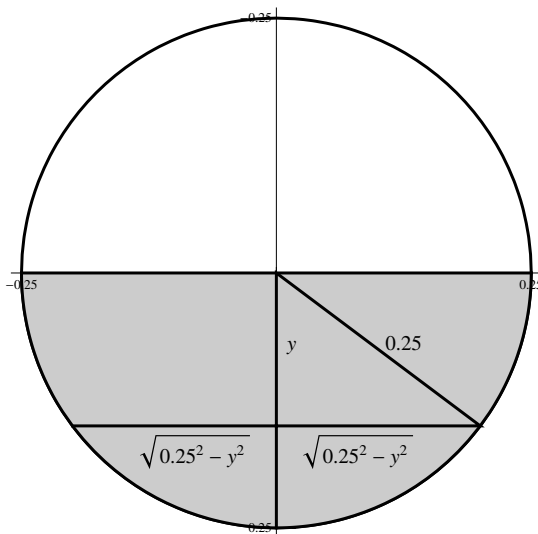
20. Place the origin at the center of the viewing plate; then the situation is shown below:



The equation of the viewing plate is $x^4 + y^4 = 1$, so at any y -coordinate, the width of a cross section of the plate is $2\sqrt[4]{1-y^4}$ (see the figure), and that cross section is $5-y$ m below the water surface. (Note that if $y < 0$, as drawn, then $5-y > 5$, which makes sense since this part of the plate is *more* than 5 m below the surface.) Therefore the total force is (evaluating numerically, with $\rho g = 1025 \cdot 9.8$)

$$\int_{-1}^1 \rho g(5-y) \cdot 2\sqrt[4]{1-y^4} dy \approx 18.5407 \rho g \boxed{\approx 186,241 \text{ N}}.$$

21. Placing the origin at the center of one end of the tank with the positive y -axis pointing down, the face of the tank is shown below:



Since the tank's radius is 0.25 m, it is half-full. At a given y -coordinate (depth), the width of the cross section of that end is $2\sqrt{0.25^2 - y^2}$, and the cross section lies y m below the surface. So the total force is

$$\int_0^{0.25} \rho g y \cdot 2\sqrt{0.25^2 - y^2} dy$$

Now use the substitution $u = 0.25^2 - y^2$, so that $du = -2y dy$. Then $y = 0$ corresponds to $u = 0.25^2$, while $y = 0.25$ corresponds to $y = 0$, and we get

$$\begin{aligned} \int_0^{0.25} \rho g y \cdot 2\sqrt{0.25^2 - y^2} dy &= -\rho g \int_{0.25^2}^0 u^{1/2} du = -\rho g \left[\frac{2}{3} u^{3/2} \right]_{0.25^2}^0 = \frac{2}{3} \rho g \cdot 0.25^3 \\ &= \frac{2}{3} \cdot 690 \cdot 9.8 \cdot 0.015625 = \boxed{70.4375 \text{ N}}. \end{aligned}$$

6.8: Center of Mass; Centroid; the Pappus Theorem

Concepts and Vocabulary

1. (c), center of mass, is correct. See the first few sentences of this chapter.
2. (c), md , is correct. See the discussion in subsection 1.
3. True. See the discussion in subsection 2.
4. (a) is correct; see Example 4.
5. False. See Example 5 for one instance where it does not.
6. (a) is correct. See the discussion in subsection 2.

Skill Building

7. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^2 m_i x_i}{\sum_{i=1}^2 m_i} = \frac{20 \cdot 4 + 50 \cdot 10}{20 + 50} = \boxed{\frac{58}{7}}.$$

8. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^2 m_i x_i}{\sum_{i=1}^2 m_i} = \frac{10 \cdot (-2) + 3 \cdot 3}{10 + 3} = \boxed{-\frac{11}{13}}.$$

9. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^4 m_i x_i}{\sum_{i=1}^4 m_i} = \frac{4 \cdot (-1) + 3 \cdot 2 + 3 \cdot 4 + 5 \cdot 3}{4 + 3 + 3 + 5} = \boxed{\frac{29}{15}}.$$

10. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^4 m_i x_i}{\sum_{i=1}^4 m_i} = \frac{7 \cdot 6 + 3 \cdot (-2) + 2 \cdot (-4) + 4 \cdot (-1)}{7 + 3 + 2 + 4} = \frac{24}{16} = \boxed{\frac{3}{2}}.$$

11. The moments are

$$M_y = \sum_{i=1}^3 m_i x_i = 4 \cdot 0 + 8 \cdot 2 + 1 \cdot 4 = 20$$

$$M_x = \sum_{i=1}^3 m_i y_i = 4 \cdot 2 + 8 \cdot 1 + 1 \cdot 8 = 24$$

and the mass is $M = 4 + 8 + 1 = 13$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{20}{13}, \frac{24}{13} \right)}.$$

12. The moments are

$$M_y = \sum_{i=1}^3 m_i x_i = 6 \cdot (-1) + 2 \cdot 12 + 10 \cdot (-1) = 8$$

$$M_x = \sum_{i=1}^3 m_i y_i = 6 \cdot (-1) + 2 \cdot 6 + 10 \cdot (-2) = -14$$

and the mass is $M = 6 + 2 + 10 = 18$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{8}{18}, -\frac{14}{18} \right) = \boxed{\left(\frac{4}{9}, -\frac{7}{9} \right)}.$$

13. The moments are

$$M_y = \sum_{i=1}^4 m_i x_i = 4 \cdot (-1) + 3 \cdot 2 + 3 \cdot 4 + 5 \cdot 3 = 29$$

$$M_x = \sum_{i=1}^4 m_i y_i = 4 \cdot 2 + 3 \cdot 3 + 3 \cdot 5 + 5 \cdot 6 = 62$$

and the mass is $M = 4 + 3 + 3 + 5 = 15$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{29}{15}, \frac{62}{15} \right)}.$$

14. The moments are

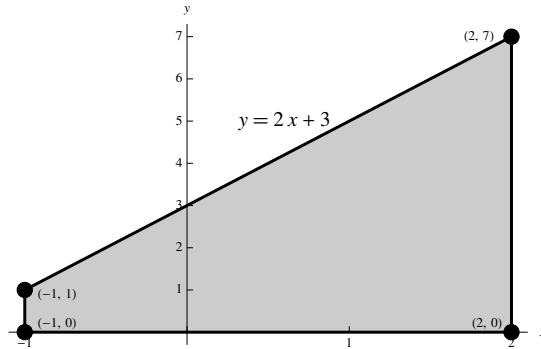
$$M_y = \sum_{i=1}^4 m_i x_i = 8 \cdot (-4) + 6 \cdot 0 + 3 \cdot 6 + 5 \cdot (-3) = -29$$

$$M_x = \sum_{i=1}^4 m_i y_i = 8 \cdot 4 + 6 \cdot 5 + 3 \cdot 4 + 5 \cdot (-5) = 49$$

and the mass is $M = 8 + 6 + 3 + 5 = 22$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(-\frac{29}{22}, \frac{49}{22} \right).$$

15. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \int_{-1}^2 \rho(2x + 3) dx = \rho [x^2 + 3x]_{-1}^2 = 12\rho,$$

and the moments are

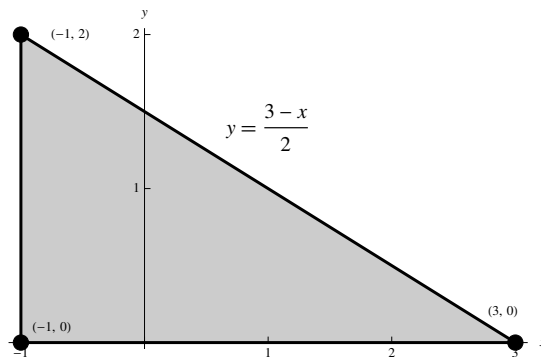
$$M_y = \rho \int_{-1}^2 x(2x + 3) dx = \rho \int_{-1}^2 (2x^2 + 3x) dx = \rho \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 \right]_{-1}^2 = \frac{21}{2}\rho$$

$$M_x = \frac{1}{2}\rho \int_{-1}^2 (2x + 3)^2 dx = \frac{1}{2}\rho \left[\frac{1}{6}(2x + 3)^3 \right]_{-1}^2 = \frac{342}{12}\rho = \frac{57}{2}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(21/2)\rho}{12\rho}, \frac{(57/2)\rho}{12\rho} \right) = \left(\frac{21}{2 \cdot 12}, \frac{57}{2 \cdot 12} \right) = \left(\frac{7}{8}, \frac{19}{8} \right).$$

16. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_{-1}^3 \frac{3-x}{2} dx = \rho \int_{-1}^3 \left(\frac{3}{2} - \frac{1}{2}x \right) dx = \rho \left[\frac{3}{2}x - \frac{1}{4}x^2 \right]_{-1}^3 = 4\rho$$

and the moments are

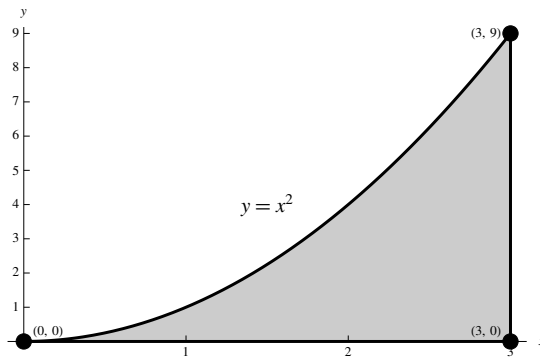
$$M_y = \rho \int_{-1}^3 x \cdot \frac{3-x}{2} dx = \rho \int_{-1}^3 \left(\frac{3}{2}x - \frac{1}{2}x^2 \right) dx = \rho \left[\frac{3}{4}x^2 - \frac{1}{6}x^3 \right]_{-1}^3 = \frac{4}{3}\rho$$

$$M_x = \frac{1}{2}\rho \int_{-1}^3 \left(\frac{3-x}{2} \right)^2 dx = \frac{1}{8}\rho \int_{-1}^3 (3-x)^2 dx = \frac{1}{8}\rho \left[-\frac{1}{3}(3-x)^3 \right]_{-1}^3 = \frac{8}{3}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(4/3)\rho}{4\rho}, \frac{(8/3)\rho}{4\rho} \right) = \left(\frac{4}{3 \cdot 4}, \frac{8}{3 \cdot 4} \right) = \boxed{\left(\frac{1}{3}, \frac{2}{3} \right)}.$$

17. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^3 x^2 dx = \rho \left[\frac{1}{3}x^3 \right]_0^3 = 9\rho$$

and the moments are

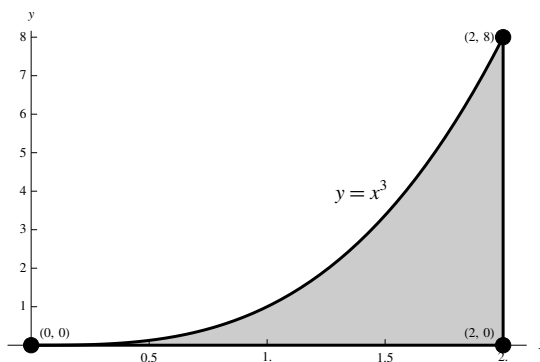
$$M_y = \rho \int_0^3 x \cdot x^2 dx = \rho \int_0^3 x^3 dx = \rho \left[\frac{1}{4}x^4 \right]_0^3 = \frac{81}{4}\rho$$

$$M_x = \frac{1}{2}\rho \int_0^3 (x^2)^2 dx = \frac{1}{2}\rho \int_0^3 x^4 dx = \frac{1}{2}\rho \left[\frac{1}{5}x^5 \right]_0^3 = \frac{243}{10}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(81/4)\rho}{9\rho}, \frac{(243/10)\rho}{9\rho} \right) = \left(\frac{81}{4 \cdot 9}, \frac{243}{10 \cdot 9} \right) = \boxed{\left(\frac{9}{4}, \frac{27}{10} \right)}.$$

18. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^2 x^3 dx = \rho \left[\frac{1}{4} x^4 \right]_0^2 = 4\rho$$

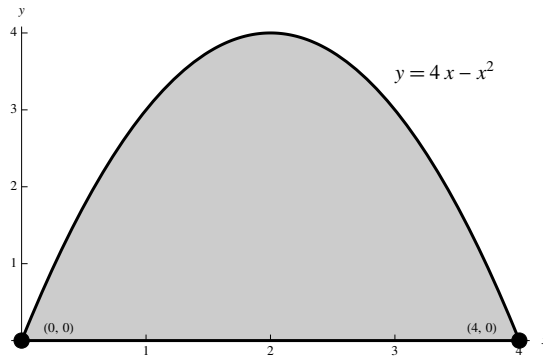
and the moments are

$$\begin{aligned} M_y &= \rho \int_0^2 x \cdot x^3 dx = \rho \int_0^2 x^4 dx = \rho \left[\frac{1}{5} x^5 \right]_0^2 = \frac{32}{5} \rho \\ M_x &= \frac{1}{2} \rho \int_0^2 (x^3)^2 dx = \frac{1}{2} \rho \int_0^2 x^6 dx = \frac{1}{2} \rho \left[\frac{1}{7} x^7 \right]_0^2 = \frac{64}{7} \rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(32/5)\rho}{4\rho}, \frac{(64/7)\rho}{4\rho} \right) = \left(\frac{32}{5 \cdot 4}, \frac{64}{7 \cdot 4} \right) = \boxed{\left(\frac{8}{5}, \frac{16}{7} \right)}.$$

19. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^4 (4x - x^2) dx = \rho \left[2x^2 - \frac{1}{3} x^3 \right]_0^4 = \frac{32}{3} \rho$$

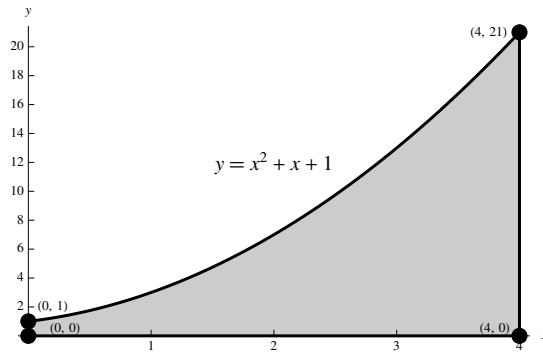
and the moments are

$$\begin{aligned} M_y &= \rho \int_0^4 x \cdot (4x - x^2) dx = \rho \int_0^4 (4x^2 - x^3) dx = \rho \left[\frac{4}{3} x^3 - \frac{1}{4} x^4 \right]_0^4 = \frac{64}{3} \rho \\ M_x &= \frac{1}{2} \rho \int_0^4 (4x - x^2)^2 dx = \frac{1}{2} \rho \int_0^4 (16x^2 - 8x^3 + x^4) dx = \frac{1}{2} \rho \left[\frac{16}{3} x^3 - 2x^4 + \frac{1}{5} x^5 \right]_0^4 = \frac{256}{15} \rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(64/3)\rho}{(32/3)\rho}, \frac{(256/15)\rho}{(32/3)\rho} \right) = \left(2, \frac{256 \cdot 3}{32 \cdot 15} \right) = \boxed{\left(2, \frac{8}{5} \right)}.$$

20. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^4 (x^2 + x + 1) dx = \rho \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x \right]_0^4 = \frac{100}{3}$$

and the moments are

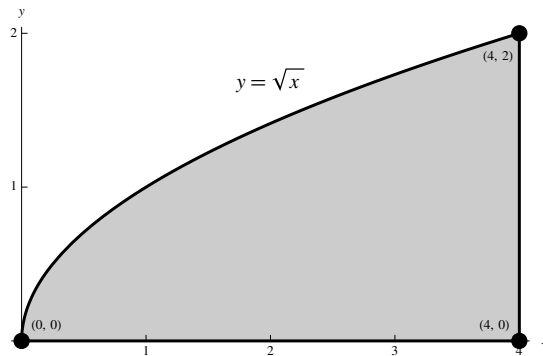
$$M_y = \rho \int_0^4 x \cdot (x^2 + x + 1) dx = \rho \int_0^4 (x^3 + x^2 + x) dx = \rho \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^4 = \frac{280}{3}\rho$$

$$\begin{aligned} M_x &= \frac{1}{2}\rho \int_0^4 (x^2 + x + 1)^2 dx = \frac{1}{2}\rho \int_0^4 (x^4 + 2x^3 + 3x^2 + 2x + 1) dx \\ &= \frac{1}{2}\rho \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 + x^3 + x^2 + x \right]_0^4 = \frac{1042}{5}\rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(280/3)\rho}{(100/3)\rho}, \frac{(1042/5)\rho}{(100/3)\rho} \right) = \left(\frac{280}{100}, \frac{1042 \cdot 3}{100 \cdot 5} \right) = \left(\frac{14}{5}, \frac{1563}{250} \right).$$

21. The lamina is



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^4 x^{1/2} dx = \rho \left[\frac{2}{3}x^{3/2} \right]_0^4 = \frac{16}{3}\rho$$

and the moments are

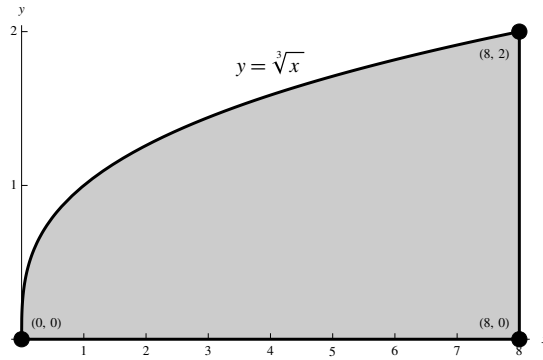
$$M_y = \rho \int_0^4 x \cdot x^{1/2} dx = \rho \int_0^4 x^{3/2} dx = \rho \left[\frac{2}{5}x^{5/2} \right]_0^4 = \frac{64}{5}\rho$$

$$M_x = \frac{1}{2}\rho \int_0^4 (x^{1/2})^2 dx = \frac{1}{2}\rho \int_0^4 x dx = \frac{1}{2}\rho \left[\frac{1}{2}x^2 \right]_0^4 = 4\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(64/5)\rho}{(16/3)\rho}, \frac{4\rho}{(16/3)\rho} \right) = \left(\frac{64 \cdot 3}{16 \cdot 5}, \frac{4 \cdot 3}{16} \right) = \boxed{\left(\frac{12}{5}, \frac{3}{4} \right)}.$$

22. The lamina is shown below:



Let the mass density of the lamina be ρ . Then the mass of the lamina is

$$M = \rho \int_0^8 x^{1/3} dx = \rho \left[\frac{3}{4} x^{4/3} \right]_0^8 = 12\rho$$

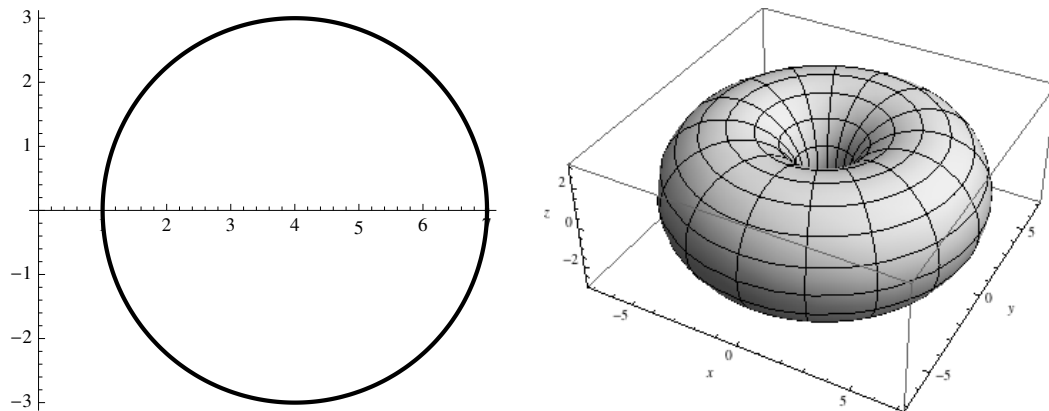
and the moments are

$$\begin{aligned} M_y &= \rho \int_0^8 x \cdot x^{1/3} dx = \rho \int_0^8 x^{4/3} dx = \rho \left[\frac{3}{7} x^{7/3} \right]_0^8 = \frac{384}{7} \rho \\ M_x &= \frac{1}{2} \rho \int_0^8 \left(x^{1/3} \right)^2 dx = \frac{1}{2} \rho \int_0^8 x^{2/3} dx = \frac{1}{2} \rho \left[\frac{3}{5} x^{5/3} \right]_0^8 = \frac{48}{5} \rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(384/7)\rho}{12\rho}, \frac{(48/5)\rho}{12\rho} \right) = \left(\frac{384}{7 \cdot 12}, \frac{48}{5 \cdot 12} \right) = \boxed{\left(\frac{32}{7}, \frac{4}{5} \right)}.$$

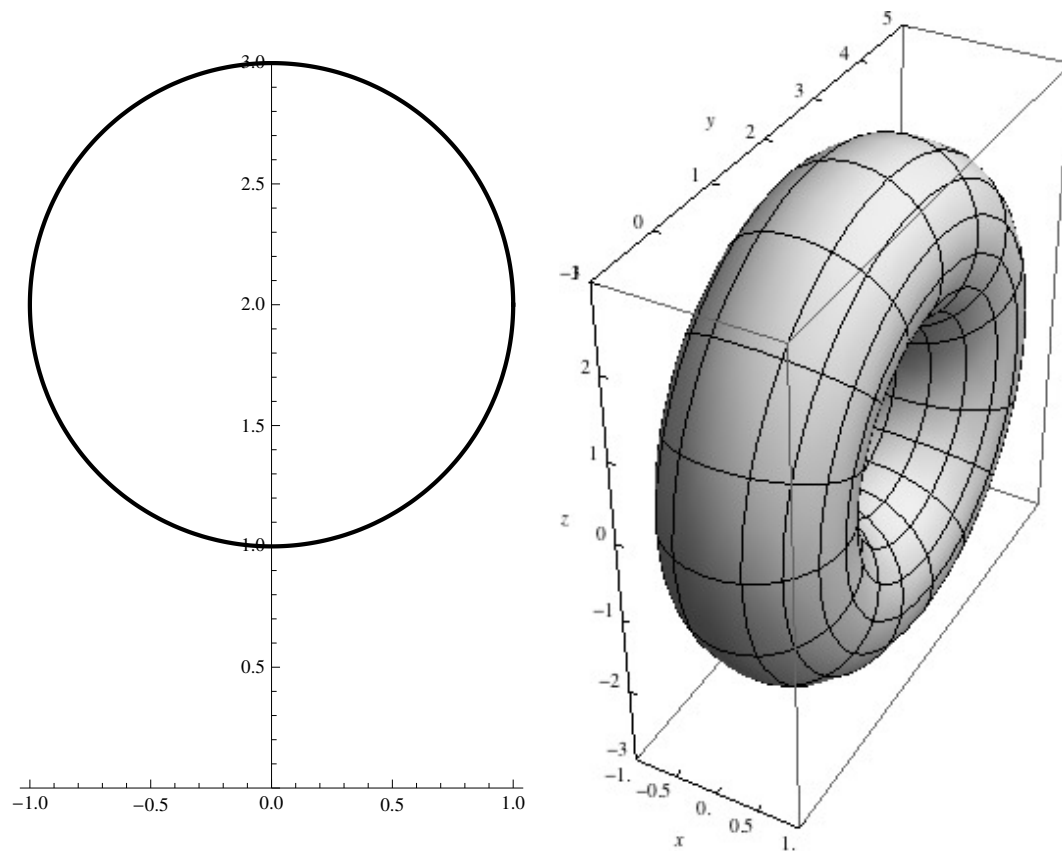
23. The circle and the surface are shown below:



The circle has a center of $(4, 0)$ and a radius of 3. It is symmetric about the point $(4, 0)$, so its centroid is $(\bar{x}, \bar{y}) = (4, 0)$. The distance from $(4, 0)$ to the axis of revolution (the y -axis) is 4, and the area of the circle is $\pi \cdot 3^2 = 9\pi$. The circle does not intersect the y -axis, so by Pappus' Theorem, the volume of the solid of revolution is

$$2\pi Ad = 2\pi \cdot 9\pi \cdot 4 = \boxed{72\pi^2}.$$

24. The circle and the surface are shown below:



The circle has a center of $(0, 2)$ and a radius of 1. It is symmetric about the point $(0, 2)$, so its centroid is $(\bar{x}, \bar{y}) = (0, 2)$. The distance from $(0, 2)$ to the axis of revolution (the x -axis) is 2, and the area of the circle is $\pi \cdot 1^2 = \pi$. The circle does not intersect the x -axis, so by Pappus's Theorem, the volume of the solid of revolution is

$$2\pi Ad = 2\pi \cdot \pi \cdot 2 = \boxed{4\pi^2}.$$

Applications and Extensions

25. If the mass density is ρ , then the mass is the mass of the circle of radius 1 plus the mass of the square of side 2, so it is $(4 + \pi)\rho$. By symmetry, since the figure is symmetric around the line $x = 1$, we have $\bar{x} = 1$. To compute $\bar{y}$, we will need the moment M_x . This figure is the union of a non-overlapping square and circle, so its moment is the sum of the individual moments. The center of the square, at $(1, 1)$, is its centroid, and its mass is 4ρ , so we have

$$1 = \frac{M_1}{4\rho}, \text{ so that } M_1 = 4\rho.$$

The center of the circle, at $(1, 3)$, is its centroid, and its mass is $\pi\rho$, so we have

$$3 = \frac{M_2}{\pi\rho}, \text{ so that } M_2 = 3\pi\rho.$$

So for the figure as a whole, we have $M_x = (4 + 3\pi)\rho$, and then

$$(\bar{x}, \bar{y}) = \left(1, \frac{(4 + 3\pi)\rho}{(4 + \pi)\rho}\right) = \boxed{\left(1, \frac{4 + 3\pi}{4 + \pi}\right)}.$$

26. This triangle has a base of 4 and a height of 4, so its area is $\frac{1}{2} \cdot 4 \cdot 4 = 8$ and then its mass is 8ρ . By symmetry, $\bar{x} = 0$. To compute $\bar{y}$, note that the triangle is the sum of two non-overlapping triangles, one for $x \leq 0$ and one for $x \geq 0$. For $x \leq 0$, the diagonal side of the triangle is the line $y = 4 + 2x$, while for $x \geq 0$, the diagonal side is the line $y = 4 - 2x$. So the moment M_x is the sum of these two moments, or

$$\begin{aligned} M_x &= \frac{1}{2}\rho \int_{-2}^0 (4 + 2x)^2 dx + \frac{1}{2}\rho \int_0^2 (4 - 2x)^2 dx \\ &= \frac{1}{2}\rho \left(\left[\frac{1}{6}(4 + 2x)^3 \right]_{-2}^0 + \left[-\frac{1}{6}(4 - 2x)^3 \right]_0^2 \right) \\ &= \frac{1}{2}\rho \left(\frac{32}{3} + \frac{32}{3} \right) = \frac{32}{3}\rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{(32/3)\rho}{8\rho}\right) = \boxed{\left(0, \frac{4}{3}\right)}.$$

27. This triangle has a base of $2c$ and a height of b , so its area is bc and then its mass is bcp , where ρ is the mass density of the lamina. The triangle is composed of two nonoverlapping triangles, one of which has vertices $(-c, 0)$, $(a, 0)$ and (a, b) and the other of which has vertices $(a, 0)$, $(c, 0)$ and (a, b) . The moments of the given region about either axis are the sum of the moments of these two triangles about that axis. For the first triangle, the equation of the diagonal edge is

$$y - 0 = \frac{b - 0}{a - (-c)}(x - (-c)), \text{ or } y = \frac{b}{a + c}(x + c).$$

Therefore its moments are

$$\begin{aligned}
 M_y &= \rho \int_{-c}^a x \left(\frac{b}{a+c} (x+c) \right) dx \\
 &= \rho \cdot \frac{b}{a+c} \int_{-c}^a (x^2 + cx) dx \\
 &= \rho \cdot \frac{b}{a+c} \left[\frac{1}{3} x^3 + \frac{1}{2} cx^2 \right]_{-c}^a \\
 &= \rho \cdot \frac{b}{a+c} \left(\frac{1}{6} (2a^3 + 3a^2c - c^3) \right) \\
 &= \rho \cdot \frac{b}{a+c} \left(\frac{1}{6} (2a-c)(a+c)^2 \right) \\
 &= \frac{1}{6} b(2a-c)(a+c)\rho
 \end{aligned}$$

$$\begin{aligned}
 M_x &= \frac{1}{2} \rho \int_{-c}^a \left(\frac{b}{a+c} (x+c) \right)^2 dx \\
 &= \frac{1}{2} \rho \left(\frac{b}{a+c} \right)^2 \int_{-c}^a (x+c)^2 dx \\
 &= \frac{1}{2} \rho \left(\frac{b}{a+c} \right)^2 \left[\frac{1}{3} (x+c)^3 \right]_{-c}^a \\
 &= \frac{1}{2} \rho \frac{b^2}{(a+c)^2} \cdot \frac{1}{3} (a+c)^3 \\
 &= \frac{1}{6} b^2 (a+c) \rho.
 \end{aligned}$$

For the second triangle, the equation of the diagonal edge is

$$y - 0 = \frac{b-0}{a-c}(x-c), \text{ or } y = \frac{b}{a-c}(x-c).$$

Therefore its moments are

$$\begin{aligned}
 M_y &= \rho \int_a^c x \left(\frac{b}{a-c} (x-c) \right) dx \\
 &= \rho \cdot \frac{b}{a-c} \int_a^c (x^2 - cx) dx \\
 &= \rho \cdot \frac{b}{a-c} \left[\frac{1}{3} x^3 - \frac{1}{2} cx^2 \right]_a^c \\
 &= \rho \cdot \frac{b}{a-c} \left(-\frac{1}{6} (2a+c)(a-c)^2 \right) \\
 &= -\frac{1}{6} b(2a+c)(a-c)\rho
 \end{aligned}$$

$$\begin{aligned}
M_x &= \frac{1}{2}\rho \int_a^c \left(\frac{b}{a-c}(x-c) \right)^2 dx \\
&= \frac{1}{2}\rho \left(\frac{b}{a-c} \right)^2 \int_a^c (x-c)^2 dx \\
&= \frac{1}{2}\rho \left(\frac{b}{a-c} \right)^2 \left[\frac{1}{3}(x-c)^3 \right]_a^c \\
&= \frac{1}{2}\rho \frac{b^2}{(a-c)^2} \cdot \left(-\frac{1}{3}(a-c)^3 \right) \\
&= -\frac{1}{6}b^2(a-c)\rho.
\end{aligned}$$

So the moments for the figure as a whole are

$$\begin{aligned}
M_y &= \frac{1}{6}b(2a-c)(a+c)\rho - \frac{1}{6}b(2a+c)(a-c)\rho = \frac{1}{3}abc\rho \\
M_x &= \frac{1}{6}b^2(a+c)\rho - \frac{1}{6}b^2(a-c)\rho = \frac{1}{3}b^2c\rho.
\end{aligned}$$

Then the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(abc/3)\rho}{bc\rho}, \frac{(b^2c/3)\rho}{bc\rho} \right) = \boxed{\left(\frac{a}{3}, \frac{b}{3} \right)}.$$

28. If the mass density is ρ , then the mass is the mass of the semicircle of radius 1 plus the mass of the 2×2 square, so it is $(4 + \frac{1}{2}\pi)\rho$. By symmetry, since the figure is symmetric around the line y -axis, we have $\bar{x} = 0$. To compute $\bar{y}$, we will need the moment M_x . This figure is the union of a non-overlapping square and circle, so its moment is the sum of the individual moments. The center of the square, at $(0, -1)$, is its centroid, and its mass is 4ρ , so we have

$$-1 = \frac{M_1}{4\rho}, \text{ so that } M_1 = -4\rho.$$

For the semicircle, we compute

$$M_2 = \frac{1}{2}\rho \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \rho \int_{-1}^1 (1-x^2) dx = \frac{1}{2}\rho \left[x - \frac{1}{3}x^3 \right]_{-1}^1 = \frac{2}{3}\rho.$$

So for the entire lamina,

$$M_x = -4\rho + \frac{2}{3}\rho = -\frac{10}{3}\rho,$$

and the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M} \right) = \left(0, \frac{-(10/3)\rho}{(4 + (1/2)\pi)\rho} \right) = \boxed{\left(0, -\frac{20}{24 + 3\pi} \right)}.$$

29. The mass of the lamina is

$$M = \rho \int_0^a \frac{h}{a^2} x^2 dx = \rho \cdot \frac{h}{a^2} \int_0^a x^2 dx = \rho \cdot \frac{h}{a^2} \left[\frac{1}{3}x^3 \right]_0^a = \rho \cdot \frac{h}{a^2} \cdot \frac{1}{3}a^3 = \frac{1}{3}ah\rho.$$

The moments about the axes are

$$\begin{aligned}
M_y &= \rho \int_0^a x \left(\frac{h}{a^2} x^2 \right) dx = \rho \cdot \frac{h}{a^2} \int_0^a x^3 dx = \rho \cdot \frac{h}{a^2} \left[\frac{1}{4}x^4 \right]_0^a = \frac{1}{4}a^2 h\rho \\
M_x &= \frac{1}{2}\rho \int_0^a \left(\frac{h}{a^2} x^2 \right)^2 dx = \frac{1}{2}\rho \cdot \frac{h^2}{a^4} \int_0^a x^4 dx = \frac{1}{2}\rho \cdot \frac{h^2}{a^4} \left[\frac{1}{5}x^5 \right]_0^a = \frac{1}{10}ah^2.
\end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{(a^2h/4)\rho}{(ah/3)\rho}, \frac{(ah^2/10)\rho}{(ah/3)\rho} \right) = \left(\frac{3a}{4}, \frac{3h}{10} \right).$$

30. By symmetry, $\bar{x} = 0$. Now, this figure is the union of two identical figures, one on each side of the y -axis; each of these figures has a moment about the x -axis that was determined in Problem 29 to be $\frac{1}{10}ah^2\rho$, so the total moment about the x -axis is $\frac{1}{5}ah^2\rho$. The mass of the lamina is twice the mass of the lamina in Problem 29, or $\frac{2}{3}ah\rho$. Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M} \right) = \left(0, \frac{(ah^2/5)\rho}{(2ah/3)\rho} \right) = \left(0, \frac{3h}{10} \right).$$

31. (a) The mass of the bat is

$$M = \int_0^L \lambda dx = \int_0^L kx dx = \left[\frac{1}{2}kx^2 \right]_0^L = \frac{1}{2}kL^2.$$

- (b) Solving the result of part (a) for k gives

$$k = \frac{2M}{L^2}.$$

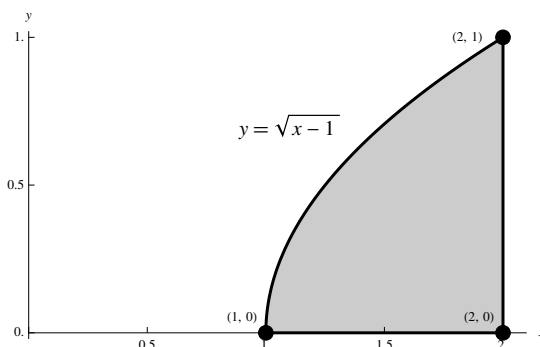
- (c) The center of mass of the bat is

$$\bar{x} = \frac{\int_0^L x\lambda dx}{M} = \frac{2}{kL^2} \int_0^L kx^2 dx = \frac{2}{L^2} \left[\frac{1}{3}x^3 \right]_0^L = \frac{2}{3}L.$$

The center of mass is $\frac{2}{3}$ of the way from the handle to the end.

- (d) Answers will vary. The “sweet spot” is at the center of mass, since at that point the bat is neither underbalanced nor overbalanced, and you will be more likely to make solid contact.
- (e) Answers will vary. A small slice through the bat at any x gives a slice whose density is approximately $\lambda = kx$ and whose width is Δx , so the total mass is the limit of $\sum kx\Delta x$ as $\Delta x \rightarrow 0$; this is a Riemann sum whose limit is $\int_0^L kx dx = \int_0^L \lambda dx$.

32. The region is shown below:



The density is $\rho = 1$, so the mass (area) is

$$\int_1^2 \sqrt{x-1} dx = \left[\frac{2}{3}(x-1)^{3/2} \right]_1^2 = \frac{2}{3}.$$

Since we are revolving about the y -axis, we need to know the distance of the centroid from the y -axis to apply Pappus' Theorem; this distance is $\bar{x}$. So we compute M_y :

$$\begin{aligned} M_y &= \int_1^2 x\sqrt{x-1} \, dx \\ &= \int_1^2 ((x-1)\sqrt{x-1} + \sqrt{x-1}) \, dx \\ &= \int_1^2 ((x-1)^{3/2} dx + (x-1)^{1/2}) \, dx \\ &= \left[\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} \right]_1^2 = \frac{16}{15}. \end{aligned}$$

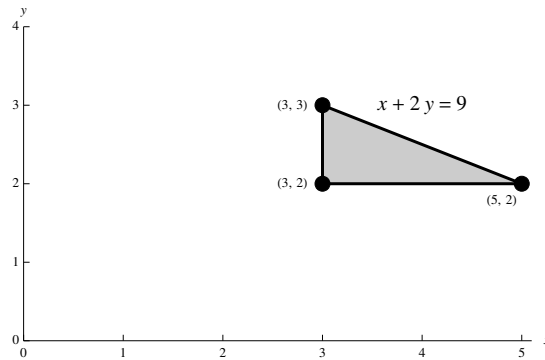
Then

$$\bar{x} = \frac{M_y}{M} = \frac{16/15}{2/3} = \frac{8}{5},$$

which is also the distance from the centroid to the axis of revolution. Therefore by Pappus's Theorem, the volume of the solid of revolution is

$$2\pi Ad = 2\pi \cdot \frac{2}{3} \cdot \frac{8}{5} = \boxed{\frac{32\pi}{15}}.$$

33. The region is shown below:



This triangle has base $5 - 3 = 2$ and height $3 - 2 = 1$, so its area is 1. We are revolving about the y -axis, so to apply Pappus' Theorem we will need to know the distance from the centroid of the region to the y -axis; this distance is $\bar{x}$, so we must compute M_y . This is

$$M_y = \int_3^5 x \left(\frac{9-x}{2} - 2 \right) dx = \int_3^5 \left(\frac{5}{2}x - \frac{1}{2}x^2 \right) dx = \left[\frac{5}{4}x^2 - \frac{1}{6}x^3 \right]_3^5 = \frac{11}{3}.$$

Therefore $\bar{x} = \frac{11}{3}$ (since $M = 1$), so that by Pappus' Theorem the volume of the solid of revolution is

$$2\pi Ad = 2\pi \cdot 1 \cdot \frac{11}{3} = \boxed{\frac{22}{3}\pi}.$$

34. This cone is the solid of revolution about the y -axis of a triangle whose vertices are $(0, 0)$, $(R, 0)$, and $(0, H)$. The equation of the diagonal edge of this triangle is $Hx + Ry = RH$, or $y = H - \frac{H}{R}x$. The area of the triangle is $\frac{1}{2}RH$. We will want the distance of the centroid from the y -axis, so we compute M_y :

$$M_y = \int_0^R x \left(H - \frac{H}{R}x \right) dx = \int_0^R \left(Hx - \frac{H}{R}x^2 \right) dx = \left[\frac{H}{2}x^2 - \frac{H}{3R}x^3 \right]_0^R = \frac{1}{6}HR^2.$$

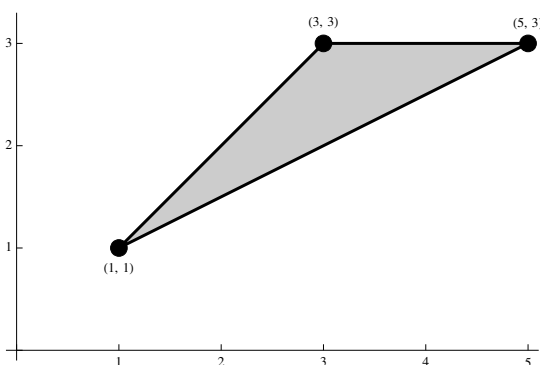
Then

$$\bar{x} = \frac{(1/6)HR^2}{(1/2)RH} = \frac{1}{3}R,$$

so by Pappus' Theorem, the volume of the cone is

$$2\pi Ad = 2\pi A\bar{x} = 2\pi \cdot \frac{1}{2}RH \cdot \frac{1}{3}R = \boxed{\frac{1}{3}\pi R^2 H}.$$

35. The region is shown below:



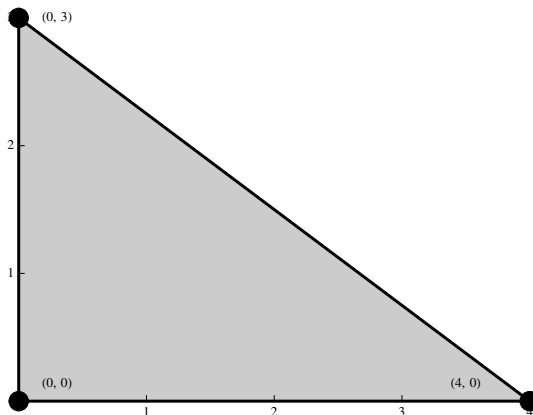
We are revolving this about the x -axis, so using Pappus' Theorem, we will want the y -coordinate of the centroid, $\bar{y}$. We must split this up into two integrals, one from $x = 1$ to $x = 3$ and the other from $x = 3$ to $x = 5$. The equation of the diagonal line from $(1, 1)$ to $(3, 3)$ is $y = x$; the equation of the other diagonal line is $y = \frac{1}{2}(x + 1)$. Then computing M_x gives

$$\begin{aligned} M_x &= \frac{1}{2} \int_1^3 \left(x^2 - \left(\frac{1}{2}(x+1) \right)^2 \right) dx + \frac{1}{2} \int_3^5 \left(3^2 - \left(\frac{1}{2}(x+1) \right)^2 \right) dx \\ &= \frac{1}{2} \int_1^3 \left(\frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{4} \right) dx + \frac{1}{2} \int_3^5 \left(-\frac{1}{4}x^2 - \frac{1}{2}x + \frac{35}{4} \right) dx \\ &= \frac{1}{2} \left[\frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x \right]_1^3 + \frac{1}{2} \left[-\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{35}{4}x \right]_3^5 \\ &= \frac{1}{2} \left(\frac{15}{4} - \left(-\frac{1}{4} \right) \right) + \frac{1}{2} \left(\frac{325}{12} - \frac{87}{4} \right) \\ &= \frac{14}{3}. \end{aligned}$$

Therefore $\bar{y} = \frac{M_x}{M} = \frac{7}{3}$, so by Pappus' Theorem, the volume of the solid is

$$2\pi Ad = 2\pi \cdot 2 \cdot \frac{7}{3} = \boxed{\frac{28}{3}\pi}.$$

36. The region is shown below:



The triangle has a base of 3 and a height of 4, so its area is $\frac{1}{2} \cdot 4 \cdot 3 = 6$. The equation of the diagonal edge of this triangle is $4x + 3y = 12$, or $y = 4 - \frac{4}{3}x$. Therefore the moments are

$$M_y = \int_0^3 x \left(4 - \frac{4}{3}x\right) dx = \int_0^3 \left(4x - \frac{4}{3}x^2\right) dx = \left[2x^2 - \frac{4}{9}x^3\right]_0^3 = 6$$

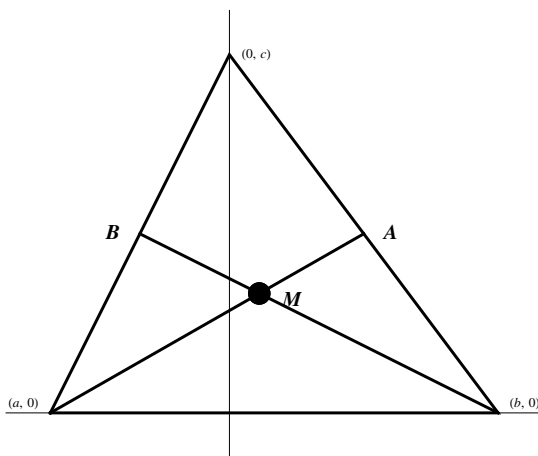
$$M_x = \frac{1}{2} \int_0^3 \left(4 - \frac{4}{3}x\right)^2 dx = \frac{1}{2} \left[-\frac{1}{4} \left(4 - \frac{4}{3}x\right)^3\right]_0^3 = -\frac{1}{8} \left[\left(4 - \frac{4}{3}x\right)^3\right]_0^3 = 8,$$

so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(1, \frac{4}{3}\right)}.$$

Challenge Problems

37. The triangle with two of the medians drawn is shown below:



Here A is the midpoint of the side opposite $(a, 0)$, B is the midpoint of the side opposite $(b, 0)$, and M is the point of intersection of those two medians. So we must show that M is the centroid of the triangle.

Since A is the midpoint of the segment from $(0, c)$ to $(b, 0)$, its coordinates are $\frac{1}{2}((0, c) + (b, 0)) = (\frac{b}{2}, \frac{c}{2})$. Similarly the coordinates of B are $(\frac{a}{2}, \frac{c}{2})$. So the equations of the two medians are

$$y - 0 = \frac{\frac{c}{2} - 0}{\frac{b}{2} - a}(x - a), \text{ or } y = \frac{c}{b - 2a}(x - a)$$

$$y - 0 = \frac{\frac{c}{2} - 0}{\frac{a}{2} - b}(x - b), \text{ or } y = \frac{c}{a - 2b}(x - b).$$

(Note that $b - 2a > 0$ since $b > 0$ and $a < 0$, so that the denominator in the first expression is nonzero. Similarly, $a - 2b < 0$. Therefore the denominators in these equations cannot be zero.) These two lines intersect when

$$\frac{c}{b - 2a}(x - a) = \frac{c}{a - 2b}(x - b).$$

Clearing fractions gives

$$\begin{aligned} c(a - 2b)(x - a) &= c(b - 2a)(x - b) \\ acx - a^2c - 2bcx + 2abc &= bcx - b^2c - 2acx + 2abc \\ (ac - 2bc)x - a^2c &= (bc - 2ac)x - b^2c \\ (ac - 2bc - bc + 2ac)x &= (b^2 - a^2)c \\ 3c(a - b)x &= (b - a)(b + a)c \\ -3x &= -(a + b) \\ x &= \frac{a + b}{3}. \end{aligned}$$

Again note that dividing through by $(a - b)c$ is justified since $a - b < 0$ and $c > 0$, so the product cannot be zero. Then

$$y = \frac{c}{b - 2a}(x - a) = \frac{c}{b - 2a} \cdot \left(\frac{a + b}{3} - a \right) = \frac{c}{b - 2a} \cdot \frac{b - 2a}{3} = \frac{c}{3}.$$

So the point M has coordinates $(\frac{1}{3}(a + b), \frac{1}{3}c)$. Now consider the third median, from $(0, c)$ to the x -axis. If $a = -b$, then the midpoint of the lower side of the triangle is the origin, and the median is the line $x = 0$ from $(0, c)$ to the origin. In this case, the point M has coordinates $(0, \frac{1}{3}c)$ since $a + b = 0$, and so the median from $(0, c)$ passes through M as well. If $a \neq -b$, then the midpoint of the lower side of the triangle is $\frac{a+b}{2}$, and the median has equation

$$y - c = \frac{0 - c}{\frac{a+b}{2} - 0}(x - 0), \text{ or } y = c - \frac{2c}{a + b}x.$$

Substituting the x -coordinate of M for x in this equation gives

$$y = c - \frac{2c}{a + b} \cdot \frac{a + b}{3} = c - \frac{2}{3}c = \frac{1}{3}c,$$

so that again the median passes through M . Therefore M is the point of intersection of all three medians.

It remains to show that M is the centroid of the triangle. The triangle has base $b - a$ and height c , so its area is $\frac{1}{2}c(b - a)$. This triangle is composed of two smaller triangles, one in the first quadrant and one in the second quadrant. The first quadrant triangle is bounded above by the line $y = -\frac{c}{b}x + c$ (note that $b \neq 0$), so its moments are

$$\begin{aligned} M_y &= \int_0^b x \left(-\frac{c}{b}x + c \right) dx = \int_0^b \left(cx - \frac{c}{b}x^2 \right) dx = \left[\frac{1}{2}cx^2 - \frac{c}{3b}x^3 \right]_0^b = \frac{1}{6}cb^2 \\ M_x &= \frac{1}{2} \int_0^b \left(-\frac{c}{b}x + c \right)^2 dx = \frac{1}{2} \left[-\frac{b}{3c} \left(-\frac{c}{b}x + c \right)^3 \right]_0^b = \frac{1}{6}bc^2. \end{aligned}$$

The second quadrant triangle is bounded above by the line $y = -\frac{c}{a}x + c$ (note that $a \neq 0$), so its moments are

$$M_y = \int_a^0 x \left(-\frac{c}{a}x + c \right) dx = \int_a^0 \left(cx - \frac{c}{a}x^2 \right) dx = \left[\frac{1}{2}cx^2 - \frac{c}{3a}x^3 \right]_a^0 = -\frac{1}{6}ca^2$$

$$M_x = \frac{1}{2} \int_a^0 \left(-\frac{c}{a}x + c \right)^2 dx = \frac{1}{2} \left[-\frac{a}{3c} \left(-\frac{c}{a}x + c \right)^3 \right]_a^0 = -\frac{1}{6}ac^2.$$

The moments of the entire figure are the sum of the moments of each figure, so the centroid of this triangle is

$$\begin{aligned} (\bar{x}, \bar{y}) &= \left(\frac{M_y}{M}, \frac{M_x}{M} \right) \\ &= \left(\frac{(1/6)cb^2 - (1/6)ca^2}{(1/2)c(b-a)}, \frac{(1/6)bc^2 - (1/6)ac^2}{(1/2)c(b-a)} \right) \\ &= \left(\frac{b^2 - a^2}{3(b-a)}, \frac{bc - ac}{3(b-a)} \right) \\ &= \left(\frac{1}{3}(a+b), \frac{1}{3}c \right). \end{aligned}$$

These are the coordinates of the point M computed above, so that the centroid of a triangle is the intersection of the medians.

38. Suppose the region is defined as shown in the diagram. As in the text, partition the interval $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b] \quad a = x_0, \quad b = x_n.$$

Each of these has width $\Delta x = \frac{b-a}{n}$. Let u_i be the midpoint of the i^{th} interval for $i = 1, 2, \dots, n$. That is,

$$u_i = \frac{x_{i-1} + x_i}{2}.$$

So we have partitioned the lamina into n nonoverlapping regions R_i for $i = 1, \dots, n$, each of which is roughly rectangular. The centroid of R_i is the point $\left(u_i, \frac{f(u_i) + g(u_i)}{2} \right)$, since that is the center of the rectangle. Further, the height of each rectangle is $f(u_i) - g(u_i)$, and the width is Δx , so the mass m_i of R_i is

$$m_i = \rho A_i = \rho(f(u_i) - g(u_i))\Delta x.$$

The moment of R_i about the y -axis is

$$M_y(R_i) = m_i u_i = \rho u_i (f(u_i) - g(u_i))\Delta x,$$

since the distance of the centroid from the y -axis is u_i . Adding up these moments over all rectangles gives

$$M_y \approx \rho \sum_{i=1}^n u_i (f(u_i) - g(u_i))\Delta x.$$

This is a Riemann sum whose limit as $n \rightarrow \infty$ is

$$M_y = \rho \int_a^b x(f(x) - g(x)) dx.$$

Similarly, the moment of R_i about the x -axis is the product of its mass and the distance of its centroid from the x -axis:

$$M_x(R_i) = m_i \cdot \frac{1}{2}(f(u_i) + g(u_i)) = \rho(f(u_i) - g(u_i))\Delta x \cdot \frac{1}{2}(f(u_i) + g(u_i)) = \frac{1}{2}\rho(f(u_i)^2 - g(u_i)^2)\Delta x.$$

Adding up these moments over all rectangles gives

$$M_x \approx \frac{1}{2} \rho \sum_{i=1}^n (f(u_i)^2 - g(u_i)^2) \Delta x.$$

This is a Riemann sum whose limit as $n \rightarrow \infty$ is

$$M_x = \frac{1}{2} \rho \int_a^b (f(x)^2 - g(x)^2) dx.$$

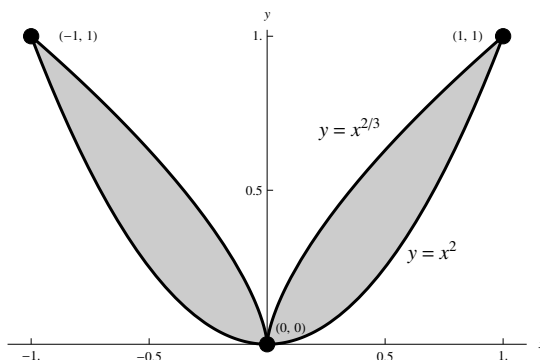
Finally, the mass of the lamina is ρ times the area between the curves, which is

$$M = \rho \int_a^b (f(x) - g(x)) dx.$$

Therefore the centroid of the lamina is

$$\begin{aligned} (\bar{x}, \bar{y}) &= \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{\rho \int_a^b x(f(x) - g(x)) dx}{\rho \int_a^b (f(x) - g(x)) dx}, \frac{\frac{1}{2} \rho \int_a^b (f(x)^2 - g(x)^2) dx}{\rho \int_a^b (f(x) - g(x)) dx} \right) \\ &= \left(\frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx}, \frac{\frac{1}{2} \int_a^b (f(x)^2 - g(x)^2) dx}{\int_a^b (f(x) - g(x)) dx} \right). \end{aligned}$$

39. The region is shown below:



The mass of the lamina is

$$M = \rho \int_{-1}^1 (x^{2/3} - x^2) dx = \rho \left[\frac{3}{5} x^{5/3} - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{8}{15} \rho.$$

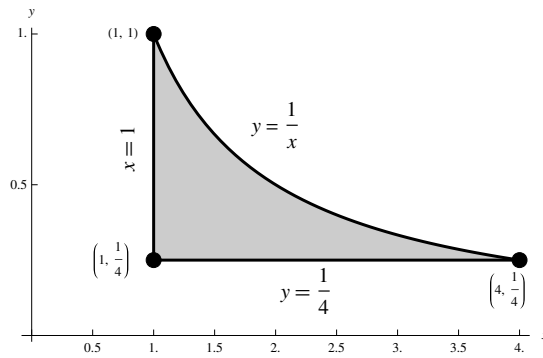
By symmetry, $\bar{x} = 0$, so we need only compute M_x to be able to find $\bar{y}$. From Problem 38, M_x is

$$M_x = \frac{1}{2} \int_{-1}^1 \left((x^{2/3})^2 - (x^2)^2 \right) \rho dx = \frac{1}{2} \rho \int_{-1}^1 (x^{4/3} - x^4) dx = \frac{1}{2} \rho \left[\frac{3}{7} x^{7/3} - \frac{1}{5} x^5 \right]_{-1}^1 = \frac{8}{35} \rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M} \right) = \boxed{\left(0, \frac{3}{7} \right)}.$$

40. The region is shown below:



The mass of the lamina is

$$M = \int_1^4 \left(\frac{1}{x} - \frac{1}{4} \right) \rho \, dx = \rho \left[\ln x - \frac{1}{4}x \right]_1^4 = \left(\ln 4 - \frac{3}{4} \right) \rho.$$

From Problem 38, the moments about the two axes are

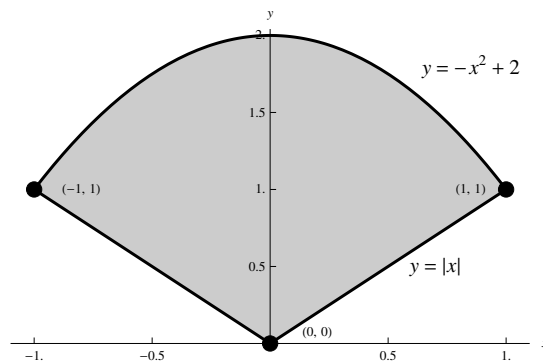
$$M_y = \int_1^4 x \left(\frac{1}{x} - \frac{1}{4} \right) \rho \, dx = \rho \int_1^4 \left(1 - \frac{1}{4}x \right) \, dx = \rho \left[x - \frac{1}{8}x^2 \right]_1^4 = \frac{9}{8}\rho$$

$$M_x = \frac{1}{2} \int_1^4 \left(\left(\frac{1}{x} \right)^2 - \left(\frac{1}{4} \right)^2 \right) \rho \, dx = \frac{1}{2}\rho \int_1^4 \left(\frac{1}{x^2} - \frac{1}{16} \right) \, dx = \frac{1}{2}\rho \left[-\frac{1}{x} - \frac{1}{16}x \right]_1^4 = \frac{9}{32}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{9}{8 \ln 4 - 6}, \frac{9}{32 \ln 4 - 24} \right).$$

41. The region is shown below:



By symmetry, the area of the region is twice the area of that portion of the region lying in the first quadrant. In the first quadrant, $y = |x|$ is the line $y = x$, so we get for the total mass

$$M = 2 \int_0^1 ((-x^2 + 2) - x) \, dx = 2 \int_0^1 (-x^2 - x + 2) \rho \, dx = 2\rho \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_0^1 = \frac{7}{3}\rho.$$

Again by symmetry, $\bar{x} = 0$, so we need only compute $\bar{y}$. Using symmetry yet again, the moment of the first quadrant region and the moment of the second quadrant region about the x -axis are equal,

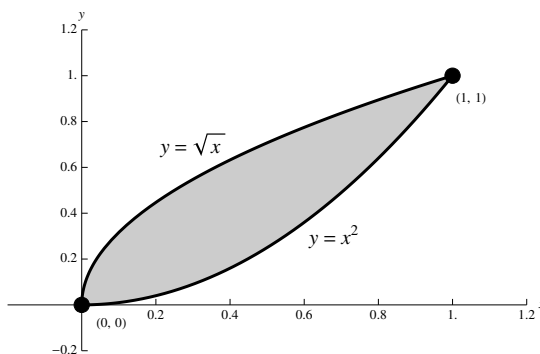
so to compute M_x , we compute it for the first quadrant region and double it (using the hint preceding Problems 25–30). Therefore the moment of the entire region about the x -axis is

$$M_x = 2 \cdot \frac{1}{2} \int_0^1 ((-x^2 + 2)^2 - x^2) \rho \, dx = \rho \int_0^1 (x^4 - 5x^2 + 4) \, dx = \rho \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x \right]_0^1 = \frac{38}{15}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M} \right) = \boxed{\left(0, \frac{38}{35} \right)}.$$

42. The region is shown below:



The mass of the region is

$$M = \int_0^1 (\sqrt{x} - x^2) \rho \, dx = \rho \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{3}\rho.$$

From Problem 38, the moments are

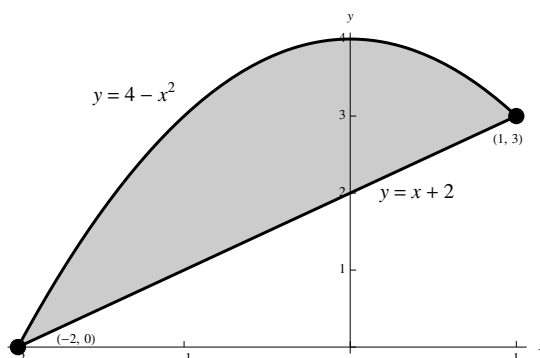
$$M_y = \int_0^1 x (\sqrt{x} - x^2) \rho \, dx = \int_0^1 (x^{3/2} - x^3) \rho \, dx = \rho \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \right]_0^1 = \frac{3}{20}\rho$$

$$M_x = \frac{1}{2} \int_0^1 ((\sqrt{x})^2 - (x^2)^2) \rho \, dx = \frac{1}{2}\rho \int_0^1 (x - x^4) \, dx = \frac{1}{2}\rho \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 = \frac{3}{20}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{9}{20}, \frac{9}{20} \right)}.$$

43. The region is shown below:



The mass of the region is

$$M = \int_{-2}^1 ((4 - x^2) - (x + 2)) \rho \, dx = \rho \int_{-2}^1 (-x^2 - x + 2) \, dx = \rho \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1 = \frac{9}{2}\rho.$$

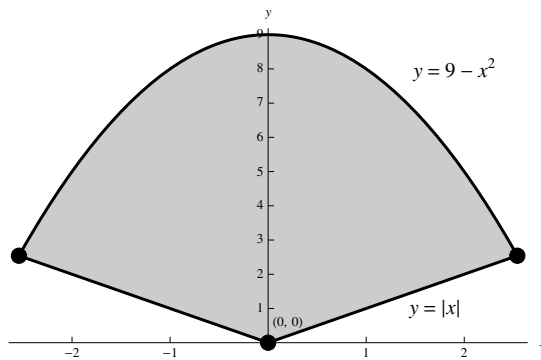
By Problem 38, the moments are

$$\begin{aligned} M_y &= \int_{-2}^1 x ((4 - x^2) - (x + 2)) \rho \, dx = \rho \int_{-2}^1 (-x^3 - x^2 + 2x) \, dx = \rho \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2 \right]_{-2}^1 = -\frac{9}{4}\rho \\ M_x &= \frac{1}{2} \int_{-2}^1 ((4 - x^2)^2 - (x + 2)^2) \rho \, dx = \frac{1}{2}\rho \int_{-2}^1 (x^4 - 9x^2 - 4x + 12) \, dx \\ &= \frac{1}{2}\rho \left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x \right]_{-2}^1 = \frac{54}{5}\rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(-\frac{1}{2}, \frac{12}{5} \right).$$

44. The region is shown below:



The point of intersection in the first quadrant is the point where $9 - x^2 = x$, so that $x^2 + x - 9 = 0$. Using the quadratic formula, this gives

$$x = \frac{1}{2}(-1 + \sqrt{37}), \text{ so that the other intersection point is } x = \frac{1}{2}(1 - \sqrt{37}).$$

The mass of the region is twice the mass of the region in the first quadrant (where $|x| = x$), so it is

$$M = 2 \int_0^{\frac{1}{2}(-1+\sqrt{37})} (9 - x^2 - x) \rho \, dx = 2\rho \left[9x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_0^{\frac{1}{2}(-1+\sqrt{37})} = \frac{1}{6}(37\sqrt{37} - 55)\rho.$$

By symmetry, $\bar{x} = 0$, so we need only compute $\bar{y}$. Using symmetry yet again, the moment of the first quadrant region and the moment of the second quadrant region about the x -axis are equal, so to compute M_x , we compute it for the first quadrant region and double it (using the hint preceding Problems 25–30). Therefore the moment of the entire region about the x -axis is

$$\begin{aligned} M_x &= 2 \cdot \frac{1}{2} \int_0^{\frac{1}{2}(-1+\sqrt{37})} ((9 - x^2)^2 - x^2) \rho \, dx = \rho \int_0^{\frac{1}{2}(-1+\sqrt{37})} (x^4 - 19x^2 + 81) \, dx \\ &= \rho \left[\frac{1}{5}x^5 - \frac{19}{3}x^3 + 81x \right]_0^{\frac{1}{2}(-1+\sqrt{37})} = \frac{2}{15}(148\sqrt{37} + 23)\rho. \end{aligned}$$

Now,

$$\begin{aligned}\frac{M_x}{M} &= \frac{4(148\sqrt{37} + 23)}{5(37\sqrt{37} - 55)} = \frac{4(148\sqrt{37} + 23)(37\sqrt{37} + 55)}{5(37\sqrt{37} - 55)(37\sqrt{37} + 55)} \\ &= \frac{815508 + 35964\sqrt{37}}{238140} = \frac{1}{245}(37\sqrt{37} + 839).\end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{245}(37\sqrt{37} + 839) \right).$$

45. Suppose the region is defined as shown in Figures 69 and 70 in the text. As in the text, partition the interval $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b] \quad a = x_0, \quad b = x_n.$$

Each of these has width $\Delta x = \frac{b-a}{n}$. Let u_i be the midpoint of the i^{th} interval for $i = 1, 2, \dots, n$. That is,

$$u_i = \frac{x_{i-1} + x_i}{2}.$$

So we have partitioned the lamina into n nonoverlapping regions R_i for $i = 1, \dots, n$, each of which is roughly rectangular. Within R_i , the density of the lamina is roughly constant at $\rho(u_i)$, so the centroid of R_i is the point $(u_i, \frac{1}{2}f(u_i))$, since that is the center of R_i . Further, the height of each rectangle is $f(u_i)$, and the width is Δx , so the mass m_i of R_i is

$$m_i = \rho(u_i)A_i = \rho(u_i)f(u_i)\Delta x.$$

The mass of the lamina is therefore approximated by

$$M \approx \sum_{i=1}^n m_i = \sum_{i=1}^n \rho(u_i)f(u_i)\Delta x.$$

This is a Riemann sum whose limit as $n \rightarrow \infty$ is

$$M = \int_a^b \rho(x)f(x) dx.$$

The moment of R_i about the y -axis is

$$M_y(R_i) = m_i u_i = \rho(u_i)u_i f(u_i)\Delta x,$$

since the distance of the centroid from the y -axis is u_i . Adding up these moments over all rectangles gives

$$M_y \approx \sum_{i=1}^n \rho(u_i)u_i f(u_i)\Delta x.$$

This is a Riemann sum whose limit as $n \rightarrow \infty$ is

$$M_y = \int_a^b \rho(x)x f(x) dx.$$

Similarly, the moment of R_i about the x -axis is the product of its mass and the distance of its centroid from the x -axis:

$$M_x(R_i) = m_i \cdot \frac{1}{2}f(u_i) = \rho(u_i)f(u_i)\Delta x \cdot \frac{1}{2}f(u_i) = \frac{1}{2}\rho(u_i)f(u_i)^2\Delta x.$$

Adding up these moments over all rectangles gives

$$M_x \approx \frac{1}{2} \sum_{i=1}^n \rho(u_i) f(u_i)^2 \Delta x.$$

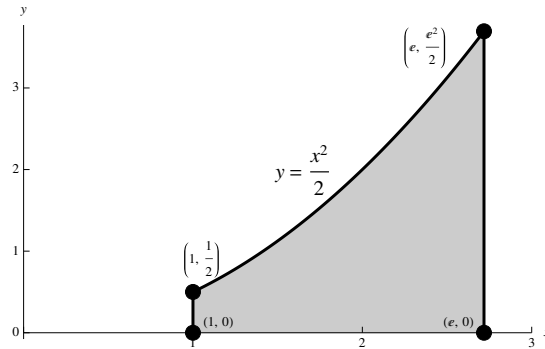
This is a Riemann sum whose limit as $n \rightarrow \infty$ is

$$M_x = \frac{1}{2} \int_a^b \rho(x) f(x)^2 dx.$$

Therefore the centroid of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{\int_a^b \rho(x) x f(x) dx}{\int_a^b \rho(x) f(x) dx}, \frac{\frac{1}{2} \int_a^b \rho(x) f(x)^2 dx}{\int_a^b \rho(x) f(x) dx} \right).$$

46. The region is shown below:



From Problem 45, the mass of the lamina is

$$M = \int_a^b \rho(x) f(x) dx = \int_1^e \frac{1}{x^3} \cdot \frac{1}{2} x^2 dx = \frac{1}{2} \int_1^e \frac{1}{x} dx = \frac{1}{2} [\ln x]_1^e = \frac{1}{2}.$$

The moments about the two axes are

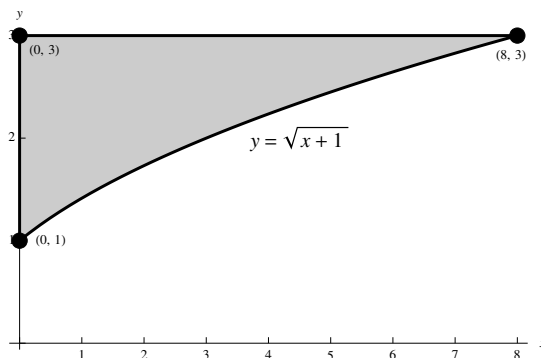
$$M_y = \int_a^b \rho(x) x f(x) dx = \int_1^e \frac{1}{x^3} \cdot x \cdot \frac{1}{2} x^2 dx = \frac{1}{2} \int_1^e 1 dx = \frac{1}{2} (e - 1)$$

$$M_x = \frac{1}{2} \int_a^b \rho(x) f(x)^2 dx = \frac{1}{2} \int_1^e \frac{1}{x^3} \cdot \frac{1}{4} x^4 dx = \frac{1}{8} \int_1^e x dx = \frac{1}{8} \left[\frac{1}{2} x^2 \right]_1^e = \frac{1}{16} (e^2 - 1).$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(e - 1, \frac{1}{8} (e^2 - 1) \right).$$

47. The region is shown below:



From Problems 38 and 45, the mass of the lamina is

$$\begin{aligned}
 M &= \int_0^8 \rho(x)(3 - \sqrt{x+1}) dx \\
 &= \int_0^8 (3x - x\sqrt{x+1}) dx \\
 &= \int_0^8 (3x - (x+1)\sqrt{x+1} + \sqrt{x+1}) dx \\
 &= \int_0^8 \left(3x - (x+1)^{3/2} + (x+1)^{1/2} \right) dx \\
 &= \left[\frac{3}{2}x^2 - \frac{2}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} \right]_0^8 \\
 &= \frac{248}{15}.
 \end{aligned}$$

The moments about the two axes are

$$\begin{aligned}
 M_y &= \int_0^8 \rho(x)x(3 - \sqrt{x+1}) dx \\
 &= \int_0^8 (3x^2 - x^2\sqrt{x+1}) dx \\
 &= \int_0^8 (3x^2 - (x^2 + 2x + 1)\sqrt{x+1} + (2x+1)\sqrt{x+1}) dx \\
 &= \int_0^8 \left(3x^2 - (x+1)^{5/2} + 2(x+1)\sqrt{x+1} - \sqrt{x+1} \right) dx \\
 &= \int_0^8 \left(3x^2 - (x+1)^{5/2} + 2(x+1)^{3/2} - (x+1)^{1/2} \right) dx \\
 &= \left[x^3 - \frac{2}{7}(x+1)^{7/2} + \frac{4}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} \right]_0^8 \\
 &= \frac{6688}{105}
 \end{aligned}$$

and

$$M_x = \frac{1}{2} \int_0^8 \rho(x) \left(3^2 - (\sqrt{x+1})^2 \right) dx = \int_0^8 x(8-x) dx = \int_0^8 (8x - x^2) dx = \left[4x^2 - \frac{1}{3}x^3 \right]_0^8 = \frac{128}{3}.$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{836}{217}, \frac{80}{31} \right).$$

48. From Problem 45, the mass of the lamina is

$$M = \int_0^3 \rho(x)f(x) dx = \int_0^3 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^3 = 9.$$

The moments about the two axes are

$$\begin{aligned}
 M_y &= \int_0^3 \rho(x)xf(x) dx = \int_0^3 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^3 = \frac{81}{4} \\
 M_x &= \frac{1}{2} \int_0^3 \rho(x)f(x)^2 dx = \int_0^3 x^3 dx = \frac{1}{2} \left[\frac{1}{4}x^4 \right]_0^3 = \frac{81}{8}.
 \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{9}{4}, \frac{9}{8} \right)}.$$

49. From Problem 45, the mass of the lamina is

$$M = \int_1^4 \rho(x)f(x) dx = \int_1^4 x(3x-1) dx = \int_1^4 (3x^2 - x) dx = \left[x^3 - \frac{1}{2}x^2 \right]_1^4 = \frac{111}{2}.$$

The moments about the two axes are

$$\begin{aligned} M_y &= \int_1^4 \rho(x)xf(x) dx = \int_1^4 x^2(3x-1) dx = \int_1^4 (3x^3 - x^2) dx = \left[\frac{3}{4}x^4 - \frac{1}{3}x^3 \right]_1^4 = \frac{681}{4} \\ M_x &= \frac{1}{2} \int_1^4 \rho(x)f(x)^2 dx = \int_1^4 x(3x-1)^2 dx = \frac{1}{2} \int_1^4 (9x^3 - 6x^2 + x) dx \\ &= \frac{1}{2} \left[\frac{9}{4}x^4 - 2x^3 + \frac{1}{2}x^2 \right]_1^4 = \frac{1821}{8}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{227}{74}, \frac{607}{148} \right)}.$$

50. From Problem 45, the mass of the lamina is

$$M = \int_0^1 \rho(x)f(x) dx = \int_0^1 (x+1) \cdot 2x dx = \int_0^1 (2x^2 + 2x) dx = \left[\frac{2}{3}x^3 + x^2 \right]_0^1 = \frac{5}{3}.$$

The moments about the two axes are

$$\begin{aligned} M_y &= \int_0^1 \rho(x)xf(x) dx = \int_0^1 (x+1) \cdot x \cdot 2x dx = \int_0^1 (2x^3 + 2x^2) dx = \left[\frac{1}{2}x^4 + \frac{2}{3}x^3 \right]_0^1 = \frac{7}{6} \\ M_x &= \frac{1}{2} \int_0^1 \rho(x)f(x)^2 dx = \frac{1}{2} \int_0^1 (x+1)(2x)^2 dx = \frac{1}{2} \int_0^1 (4x^3 + 4x^2) dx = \frac{1}{2} \left[x^4 + \frac{4}{3}x^3 \right]_0^1 = \frac{7}{6}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{7}{10}, \frac{7}{10} \right)}.$$

51. From Problem 45, the mass of the lamina is

$$M = \int_1^4 \rho(x)f(x) dx = \int_1^4 (x+1)x dx = \int_1^4 (x^2 + x) dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_1^4 = \frac{57}{2}.$$

The moments about the two axes are

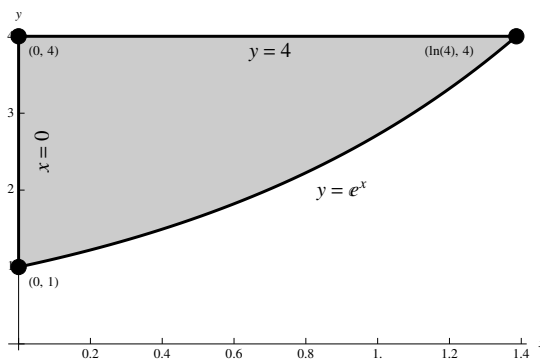
$$\begin{aligned} M_y &= \int_1^4 \rho(x)xf(x) dx = \int_1^4 (x+1) \cdot x \cdot x dx = \int_1^4 (x^3 + x^2) dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 \right]_1^4 = \frac{339}{4} \\ M_x &= \frac{1}{2} \int_1^4 \rho(x)f(x)^2 dx = \frac{1}{2} \int_1^4 (x+1) \cdot x^2 dx = \frac{1}{2} \int_1^4 (x^3 + x^2) dx = \frac{1}{2} \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 \right]_1^4 = \frac{339}{8}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{113}{38}, \frac{113}{76} \right)}.$$

Chapter Review

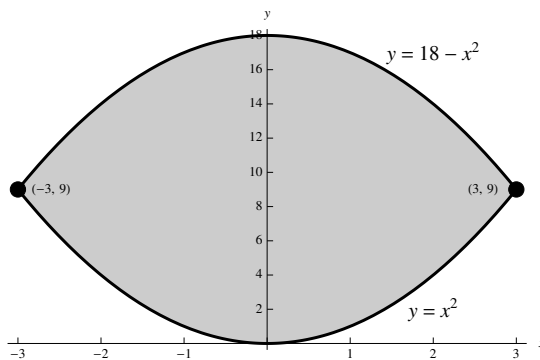
1. The region is shown below:



Partitioning along the x -axis, we have $e^x \leq 4$, so the area is

$$\int_0^{\ln 4} (4 - e^x) dx = [4x - e^x]_0^{\ln 4} = \boxed{4 \ln 4 - 3}.$$

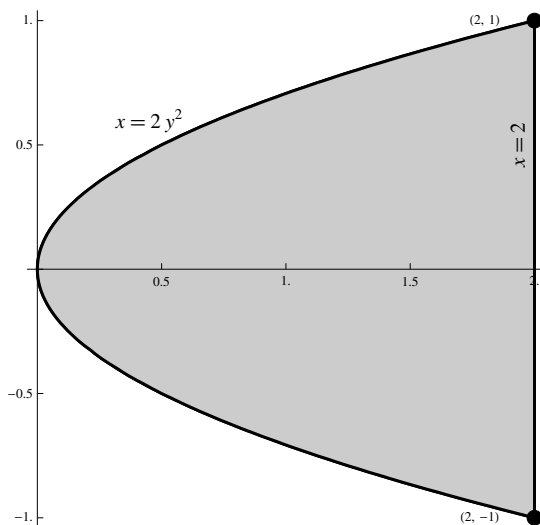
2. The region is shown below:



Partitioning along the x -axis, we have $x^2 \leq 18 - x^2$, so the area is

$$\int_{-3}^3 (18 - x^2 - x^2) dx = \int_{-3}^3 (18 - 2x^2) dx = \left[18x - \frac{2}{3}x^3 \right]_{-3}^3 = \boxed{72}.$$

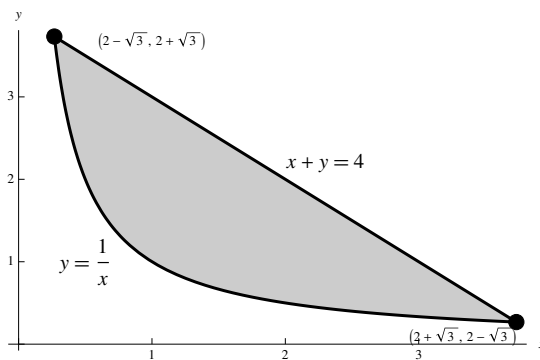
3. The region is shown below:



Since we are given x in terms of y , it is more convenient to partition along the y -axis. There, we have $2y^2 \leq 2$, so the area is

$$\int_{-1}^1 (2 - 2y^2) \, dy = \left[2y - \frac{2}{3}y^3 \right]_{-1}^1 = \boxed{\frac{8}{3}}.$$

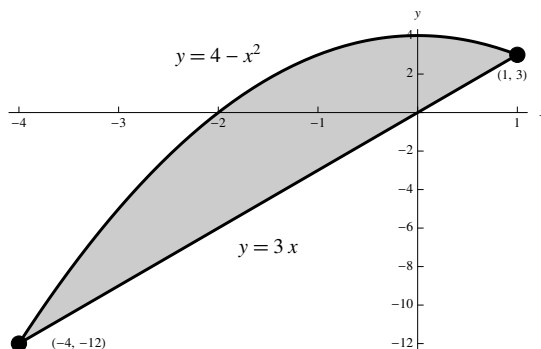
4. The region is shown below:



The equation $x + y = 4$ is the same as $y = 4 - x$. The graphs of $\frac{1}{x}$ and $4 - x$ meet when $\frac{1}{x} = 4 - x$, so when $4x - x^2 = 1$. This gives the quadratic $x^2 - 4x + 1 = 0$, which has roots $2 \pm \sqrt{3}$. Partitioning along the x -axis, we have $\frac{1}{x} \leq 4 - x$, so the area is

$$\begin{aligned} \int_{2-\sqrt{3}}^{2+\sqrt{3}} \left(4 - x - \frac{1}{x} \right) dx &= \left[4x - \frac{1}{2}x^2 - \ln x \right]_{2-\sqrt{3}}^{2+\sqrt{3}} \\ &= \left(8 + 4\sqrt{3} - \left(\frac{7}{2} + 2\sqrt{3} \right) - \ln(2 + \sqrt{3}) \right) \\ &\quad - \left(8 - 4\sqrt{3} - \left(\frac{7}{2} - 2\sqrt{3} \right) - \ln(2 - \sqrt{3}) \right) \\ &= 4\sqrt{3} + \ln \frac{2 - \sqrt{3}}{2 + \sqrt{3}} = \boxed{4\sqrt{3} + \ln(7 - 4\sqrt{3})}. \end{aligned}$$

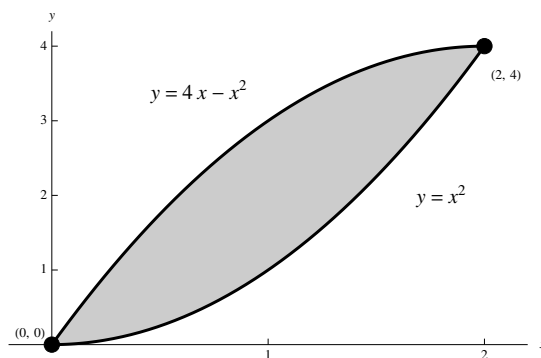
5. The region is shown below:



Partitioning along the x -axis, we have $4 - x^2 \geq 3x$, so the area is

$$\int_{-4}^1 (4 - x^2 - 3x) dx = \left[4x - \frac{1}{3}x^3 - \frac{3}{2}x^2 \right]_{-4}^1 = \boxed{\frac{125}{6}}.$$

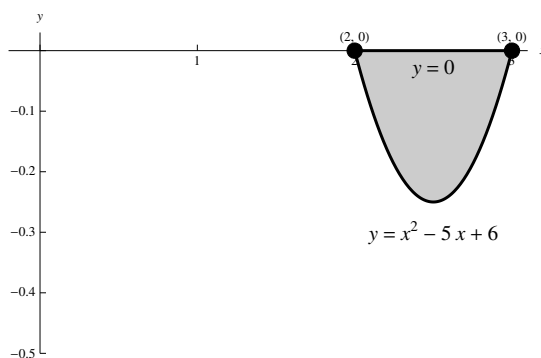
6. The region is shown below:



We use the washer method along the x -axis. Each outer radius is $4x - x^2$ and each inner radius is x^2 , so the volume is

$$V = \pi \int_0^2 \left((4x - x^2)^2 - (x^2)^2 \right) dx = \pi \int_0^2 (16x^2 - 8x^3) dx = \pi \left[\frac{16}{3}x^3 - 2x^4 \right]_0^2 = \boxed{\frac{32}{3}\pi}.$$

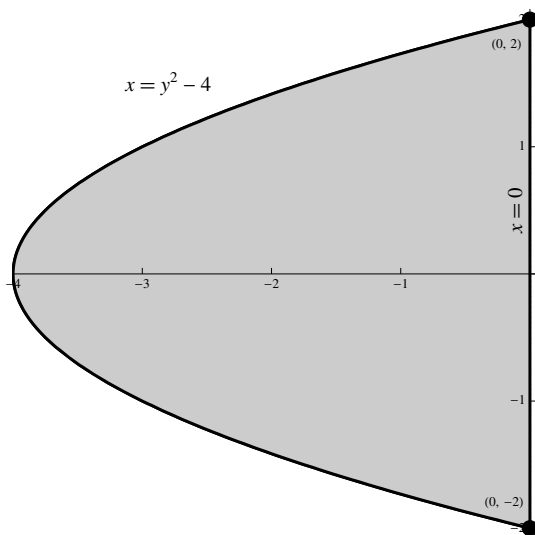
7. The region is shown below:



We use the shell method along the x -axis. Each height is $-x^2 + 5x - 6$, and each radius is x , so the volume is

$$V = 2\pi \int_2^3 x(-x^2 + 5x - 6) dx = 2\pi \int_2^3 (-x^3 + 5x^2 - 6x) dx = 2\pi \left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 3x^2 \right]_2^3 = \boxed{\frac{5}{6}\pi}.$$

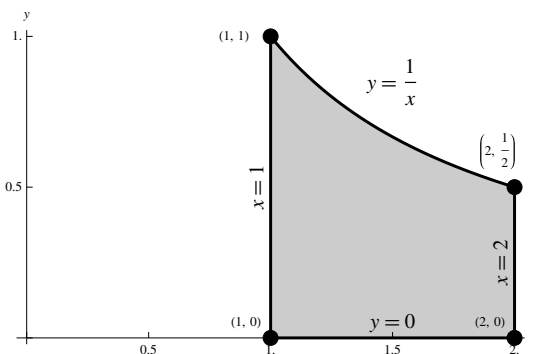
8. The region is shown below:



Since we are given equations expressing x in terms of y , and we are revolving about the y -axis, it is easiest to use the disk method along the y -axis. Each radius is $4 - y^2$, so the volume is

$$V = \pi \int_{-2}^2 (4 - y^2)^2 dy = \pi \int_{-2}^2 (y^4 - 8y^2 + 16) dy = \pi \left[\frac{1}{5}y^5 - \frac{8}{3}y^3 + 16y \right]_{-2}^2 = \boxed{\frac{512}{15}\pi}.$$

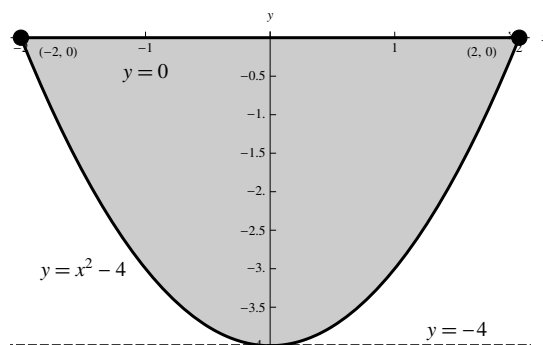
9. The region is shown below:



We use the disk method along the x -axis. Each radius is $\frac{1}{x}$, so the volume is

$$V = \pi \int_1^2 \left(\frac{1}{x} \right)^2 dx = \pi \int_1^2 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^2 = \boxed{\frac{1}{2}\pi}.$$

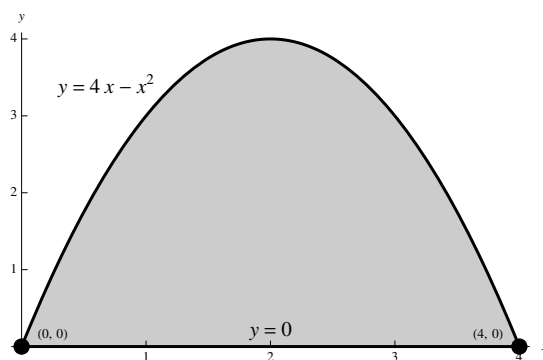
10. The region is shown below:



We use the washer method in the x direction. Each outer radius is $0 - (-4) = 4$, and each inner radius is $x^2 - 4 - (-4) = x^2$. Therefore the volume is

$$V = \pi \int_{-2}^2 \left(4^2 - (x^2)^2 \right) dx = \pi \int_{-2}^2 (16 - x^4) dx = \pi \left[16x - \frac{1}{5}x^5 \right]_{-2}^2 = \boxed{\frac{256}{5}\pi}.$$

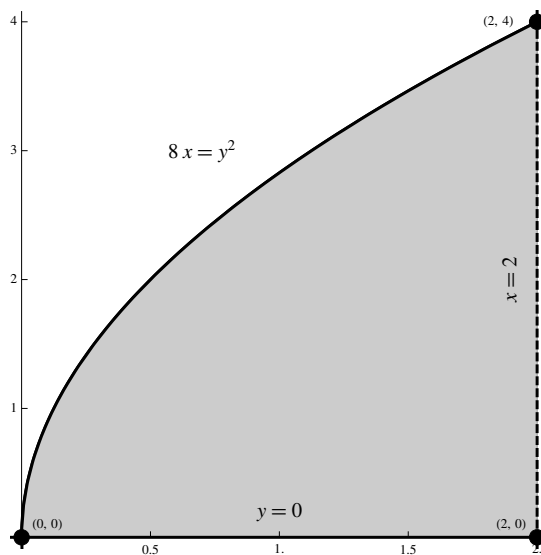
11. The region is shown below:



We use the disk method along the x -axis. Each radius is $4x - x^2$, so the volume is

$$V = \pi \int_0^4 (4x - x^2)^2 dx = \pi \int_0^4 (16x^2 - 8x^3 + x^4) dx = \pi \left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5 \right]_0^4 = \boxed{\frac{512}{15}\pi}.$$

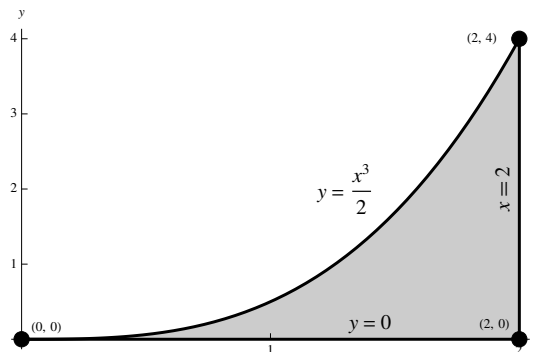
12. The region is shown below:



We use the shell method in the x direction, so we must first solve for y , giving $y = \sqrt{8x}$ (note that from the diagram, we take the positive square root). Revolving about $x = 2$, each radius is $2 - x$, and each height is $\sqrt{8x}$, so that the volume is

$$\begin{aligned} V &= 2\pi \int_0^2 (2-x)\sqrt{8x} \, dx = 4\pi\sqrt{2} \int_0^2 (2x^{1/2} - x^{3/2}) \, dx \\ &= 4\pi\sqrt{2} \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^2 = 4\pi\sqrt{2} \left(\frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2} \right) = \boxed{\frac{128}{15}\pi}. \end{aligned}$$

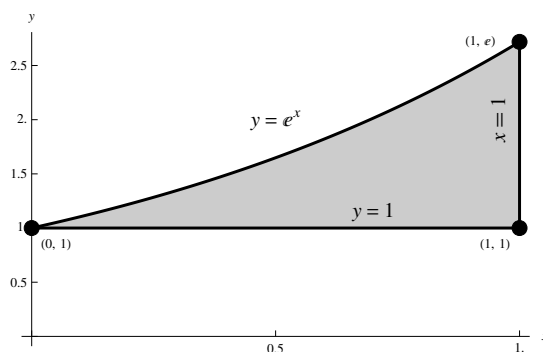
13. The region is shown below:



It is easiest to use the shell method along the x -axis. Each height is $\frac{x^3}{2}$, and each radius is x , so the volume is

$$V = 2\pi \int_0^2 x \cdot \frac{x^3}{2} \, dx = \pi \int_0^2 x^4 \, dx = \pi \left[\frac{1}{5}x^5 \right]_0^2 = \boxed{\frac{32}{5}\pi}.$$

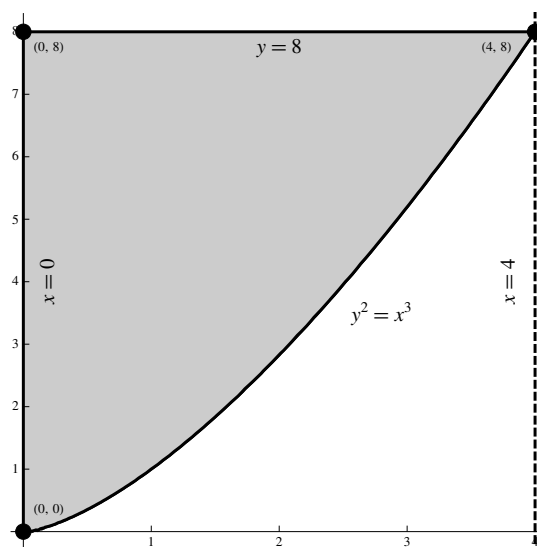
14. The region is shown below:



Use the washer method along the x -axis. Each outer radius is e^x and each inner radius is 1, so the volume is

$$V = \pi \int_0^1 \left((e^x)^2 - 1^2 \right) dx = \pi \int_0^1 (e^{2x} - 1) dx = \pi \left[\frac{1}{2} e^{2x} - x \right]_0^1 = \boxed{\frac{1}{2} \pi (e^2 - 3)}.$$

15. The region is shown below:



To revolve about $x = 4$, we could use either shells along the x -axis or washers in the y direction. We choose to use shells. Solving $y^2 = x^3$ for y gives $y = x^{3/2}$ (we choose the positive square root since this is a first quadrant region). Then each height is $8 - x^{3/2}$, and each radius is $4 - x$. Therefore the volume is

$$\begin{aligned} V &= 2\pi \int_0^4 (4 - x) (8 - x^{3/2}) dx = 2\pi \int_0^4 (32 - 8x - 4x^{3/2} + x^{5/2}) dx \\ &= 2\pi \left[32x - 4x^2 - \frac{8}{5} x^{5/2} + \frac{2}{7} x^{7/2} \right]_0^4 = \boxed{\frac{3456}{35} \pi}. \end{aligned}$$

16. Since $y' = \frac{3}{2}x^{1/2}$, the arc length is

$$L = \int_2^5 \sqrt{(y')^2 + 1} dx = \int_2^5 \sqrt{\frac{9}{4}x + 1} dx = \frac{1}{2} \int_2^5 \sqrt{9x + 4} dx.$$

Now use the substitution $u = 9x + 4$, so that $du = 9 dx$; then $x = 2$ corresponds to $u = 22$ and $x = 5$ to $u = 49$, so that we get

$$L = \frac{1}{2} \int_2^5 \sqrt{9x+4} dx = \frac{1}{18} \int_{22}^{49} u^{1/2} du = \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_{22}^{49} = \boxed{\frac{1}{27} (343 - 22\sqrt{22})}.$$

17. We have $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, so that

$$(y')^2 + 1 = \left(\frac{x^2}{2} - \frac{1}{2x^2} \right)^2 + 1 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^4} + 1 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2.$$

Then the arc length is

$$L = \int_2^6 \sqrt{(y')^2 + 1} dy = \int_2^6 \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx = \left[\frac{x^3}{6} - \frac{1}{2x} \right]_2^6 = \boxed{\frac{209}{6}}.$$

18. Partition along the y -axis. Then $x = g(y) = \sqrt{2} y^{3/2}$, so that $g'(y) = \frac{3}{\sqrt{2}} y^{1/2}$. Then the arc length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(g'(y))^2 + 1} dy \\ &= \int_0^2 \sqrt{\frac{9}{2}x + 1} dy \\ &= \frac{1}{\sqrt{2}} \int_0^2 \sqrt{9x + 2} dy \\ &= \frac{1}{\sqrt{2}} \left[\frac{2}{27} (9x + 2)^{3/2} \right]_0^2 \\ &= \frac{\sqrt{2}}{27} (20\sqrt{20} - 2\sqrt{2}) \\ &= \boxed{\frac{4}{27} (10\sqrt{10} - 1)}. \end{aligned}$$

19. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^4 m_i x_i}{\sum_{i=1}^4 m_i} = \frac{1 \cdot (-1) + 3 \cdot 2 + 8 \cdot 14 + 1 \cdot 0}{1 + 3 + 8 + 1} = \boxed{9}.$$

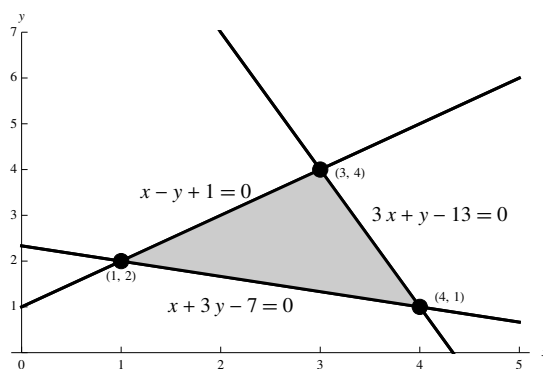
20. The moments are

$$\begin{aligned} M_y &= \sum_{i=1}^4 m_i x_i = 2 \cdot (-4) + 2 \cdot 2 + 3 \cdot 4 + 2 \cdot (-3) = 2 \\ M_x &= \sum_{i=1}^4 m_i y_i = 2 \cdot 4 + 2 \cdot 3 + 3 \cdot 4 + 2 \cdot (-5) = 16 \end{aligned}$$

and the mass is $M = 2 + 2 + 3 + 2 = 9$, so the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \boxed{\left(\frac{2}{9}, \frac{16}{9} \right)}.$$

21. The region is shown below:



The lines $x - y + 1 = 0$ and $3x + y - 13 = 0$ intersect when

$$y = x + 1 = 13 - 3x, \text{ so that } 4x = 12, \text{ or } x = 3, \text{ which is the point } (3, 4).$$

The lines $x - y + 1 = 0$ and $x + 3y - 7 = 0$ intersect when

$$x = y - 1 = 7 - 3y, \text{ so that } 4y = 8, \text{ or } y = 2, \text{ which is the point } (1, 2).$$

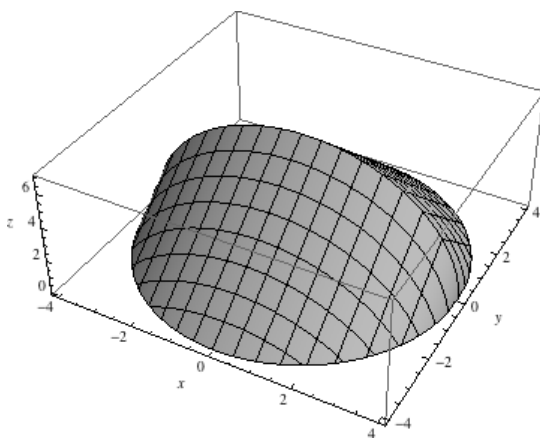
The lines $3x + y - 13 = 0$ and $x + 3y - 7 = 0$ intersect when

$$y = 13 - 3x = \frac{7}{3} - \frac{1}{3}x, \text{ so that } \frac{8}{3}x = \frac{32}{3}, \text{ or } x = 4, \text{ which is the point } (4, 1).$$

We partition along the x -axis, so we end up with two integrals, one from $x = 1$ to $x = 3$ and one from $x = 3$ to $x = 4$. The first has lower bound $y = \frac{7}{3} - \frac{1}{3}x$ and upper bound $y = x + 1$; the second has the same lower bound but has upper bound $y = 13 - 3x$. So the area is

$$\begin{aligned} A &= \int_1^3 \left((x+1) - \left(\frac{7}{3} - \frac{1}{3}x \right) \right) dx + \int_3^4 \left((13-3x) - \left(\frac{7}{3} - \frac{1}{3}x \right) \right) dx \\ &= \int_1^3 \left(\frac{4}{3}x - \frac{4}{3} \right) dx + \int_3^4 \left(-\frac{8}{3}x + \frac{32}{3} \right) dx \\ &= \left[\frac{2}{3}x^2 - \frac{4}{3}x \right]_1^3 + \left[-\frac{4}{3}x^2 + \frac{32}{3}x \right]_3^4 \\ &= \frac{8}{3} + \frac{4}{3} = \boxed{4}. \end{aligned}$$

22. The region is shown below:



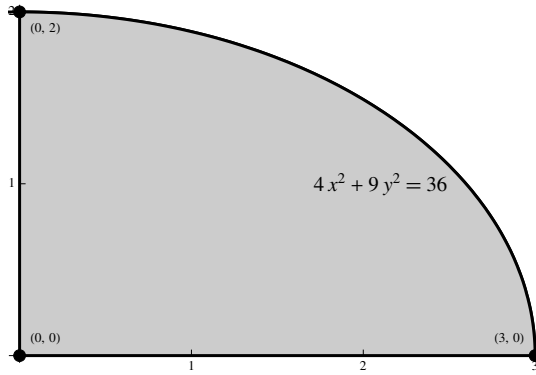
An equilateral triangle with side a has height $a \sin 60^\circ = \frac{\sqrt{3}}{2}a$, so its area is $\frac{1}{2} \cdot a \cdot \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$. With the fixed diameter as shown, along the x -axis, centered at the origin, the side length of the equilateral triangle at a given value of x is $2\sqrt{16 - x^2}$, so the area of the triangle is

$$\frac{\sqrt{3}}{4} \cdot \left(2\sqrt{16 - x^2}\right)^2 = (16 - x^2)\sqrt{3}.$$

Therefore the volume of the solid is

$$V = \int_{-4}^4 (16 - x^2)\sqrt{3} dx = \sqrt{3} \left[16x - \frac{1}{3}x^3 \right]_{-4}^4 = \boxed{\frac{256}{3}\sqrt{3}}.$$

23. The region is shown below:



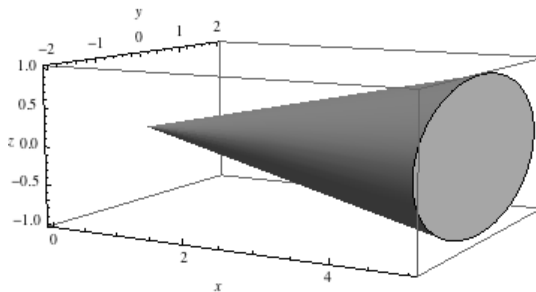
Using the shell method along the y -axis, we solve for x , giving $x = \sqrt{9 - \frac{9}{4}y^2}$. Then the volume is

$$V = 2\pi \int_0^2 y \sqrt{9 - \frac{9}{4}y^2} dy = \pi \int_0^2 y \sqrt{36 - 9y^2} dy.$$

Now use the substitution $u = 36 - 9y^2$, so that $du = -18y dy$. Then $y = 0$ corresponds to $u = 36$, and $y = 2$ to $u = 0$, and the integral becomes

$$V = \pi \int_0^2 y \sqrt{36 - 9y^2} dy = \pi \cdot \left(-\frac{1}{18}\right) \int_{36}^0 u^{1/2} du = -\frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_{36}^0 = \boxed{8\pi}.$$

24. The region is shown below:



Place the origin at the tip of the cone, with the positive x -axis pointing in the direction of the base, so that the base of the cone is an ellipse with semi axes 2 and 1 at $x = 5$. The cross section at each value of x is an ellipse, and since both axes increase linearly, one from 0 to 2 and the other from 0 to 1 as x increases from 0 to 5, it follows that for a given value of x , the cross section at x is an ellipse

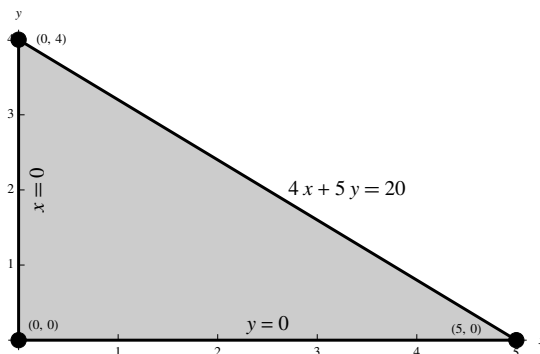
with semi axes $2 \cdot \frac{x}{5} = \frac{2x}{5}$ and $\frac{x}{5}$. We must integrate over the area of these ellipses. Now by the hint, the area of each ellipse is

$$\pi \cdot \left(\frac{2x}{5}\right) \cdot \left(\frac{x}{5}\right) = \frac{2}{25}\pi x^2$$

so the volume is

$$V = \int_0^5 \left(\frac{2}{25}\pi x^2\right) dx = \frac{2}{25}\pi \left[\frac{1}{3}x^3\right]_0^5 = \boxed{\frac{10}{3}\pi}.$$

25. The base is shown below:



Solving $4x + 5y = 20$ for y gives $y = 4 - \frac{4}{5}x$, so for a given value of x , the area of the semicircular cross section is

$$\frac{1}{2}\pi \left(\frac{d}{2}\right)^2 = \frac{1}{2}\pi \left(\frac{4 - \frac{4}{5}x}{2}\right)^2 = \frac{1}{8}\pi \left(16 - \frac{32}{5}x + \frac{16}{25}x^2\right).$$

Therefore the volume of the solid is

$$V = \int_0^5 \frac{1}{8}\pi \left(16 - \frac{32}{5}x + \frac{16}{25}x^2\right) dx = \frac{1}{8}\pi \left[16x - \frac{16}{5}x^2 + \frac{16}{75}x^3\right]_0^5 = \boxed{\frac{10}{3}\pi}.$$

26. We have $y' = x^{1/2}$, which is continuous for $x > 0$. So using the Arc Length Formula, the arc length from 0 to k is

$$L = \int_0^k \sqrt{(y')^2 + 1} dx = \int_0^k \sqrt{x + 1} dx = \left[\frac{2}{3}(x + 1)^{3/2}\right]_0^k = \frac{2}{3}((k + 1)^{3/2} - 1).$$

We want this length to be $\frac{52}{3}$, so setting the two equal and multiplying through by 3 gives

$$2((k + 1)^{3/2} - 1) = 52, \text{ or } (k + 1)^{3/2} - 1 = 26.$$

Add 1 to both sides and simplify to get $k = 8$. Therefore the point P is $(8, \frac{2}{3} \cdot 8^{3/2}) = \boxed{\left(8, \frac{16}{3}\right)}.$

27. The anchor must be lifted 150 ft; since it weighs 800 lb, the work involved in lifting the anchor itself is $800 \cdot 150 = 120000$ ft-lbs. The remainder of the work expended is for the chain itself. The weight density of the chain is 20 lb/ft, and the portion of the chain x ft below the boat must be lifted x ft. Therefore the total work expended in lifting the chain is

$$\int_0^{150} 20x dx = [10x^2]_0^{150} = 225000 \text{ ft-lbs.}$$

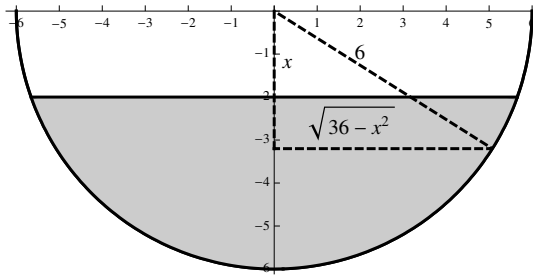
So the work required to lift the anchor and chain is $120000 + 225000 = \boxed{345000 \text{ ft-lbs}}.$

28. The container must be lifted 1200 m; since its mass is 1000 kg, its weight is 9800 N. Then the work involved in lifting the container itself is $1200 \cdot 9800 = 1.176 \times 10^7$ J. The remainder of the work expended is for the cable itself. The density of the cable is 3 kg/m, and the portion of the cable x ft below the surface must be lifted x ft. Therefore the total work expended in lifting the cable is

$$9.8 \int_0^{1200} 3x \, dx = 9.8 \left[\frac{3}{2} x^2 \right]_0^{1200} = 1.0584 \times 10^7 \text{ J.}$$

So the work required to lift the container and the cable is $1.176 \times 10^7 + 1.0584 \times 10^7 = \boxed{2.2344 \times 10^7 \text{ J}}$.

29. To determine the radius of the cross section x m below the top of the bowl, consider the following diagram, which shows a vertical cross section through the center of the bowl:



At a height x m below the top, the radius of the cross section is $\sqrt{36 - x^2}$, so that the area of the cross section at that height is $(36 - x^2)\pi$. That cross section must be lifted x m, so the work performed is

$$9.8 \int_2^6 1000x(36 - x^2)\pi \, dx = 9800\pi \int_2^6 (36h - x^3) \, dx = 9800\pi \left[18x^2 - \frac{1}{4}x^4 \right]_2^6 = \boxed{2.5088 \times 10^6 \pi \text{ J}}.$$

30. To determine k , note that a displacement of 0.2 m requires 4 N, so that $-4 = k \cdot 0.2$, or $k = -20$. So the equation of the spring is $F = -20x$. Then stretching it to 1.4 m, or 0.8 m from its equilibrium point, takes

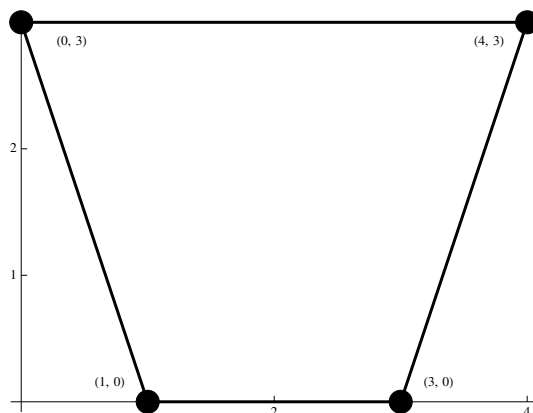
$$\int_0^{0.8} (-20x) \, dx = [-10x^2]_0^{0.8} = \boxed{-6.4 \text{ N}}.$$

31. Let the equilibrium length of the spring be s , and the spring constant be k . Then

$$\begin{aligned} \int_{1-s}^{1.4-s} (-kx) \, dx &= \frac{1}{2} \int_{1.2-s}^{1.8-s} (-kx) \, dx \\ \left[-\frac{1}{2}kx^2 \right]_{1-s}^{1.4-s} &= \frac{1}{2} \left[-\frac{1}{2}kx^2 \right]_{1.2-s}^{1.8-s} \\ [x^2]_{1-s}^{1.4-s} &= \frac{1}{2} [x^2]_{1.2-s}^{1.8-s} \\ 2((1.4-s)^2 - (1-s)^2) &= (1.8-s)^2 - (1.2-s)^2 \\ -1.6s + 1.92 &= -1.2s + 1.8 \\ 0.4s &= 0.12 \\ s &= 0.3. \end{aligned}$$

The equilibrium length of the spring is $\boxed{0.3 \text{ m}}$.

32. Place the trapezoid on a Cartesian grid:



Then the equations of the two diagonal lines are:

$$y - 0 = \frac{3 - 0}{0 - 1}(x - 1), \text{ or } y = -3x + 3, \text{ or } x = -\frac{1}{3}y + 1$$

$$y - 0 = \frac{3 - 0}{4 - 3}(x - 3), \text{ or } y = 3x - 9, \text{ or } x = \frac{1}{3}y + 3.$$

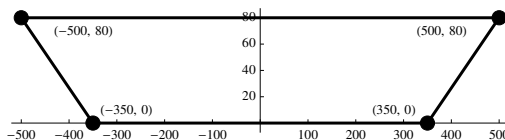
So at any height y , the height of the water above a slice at that point is $3 - y$, and the width of the slice is $(\frac{1}{3}y + 3) - (-\frac{1}{3}y + 1) = \frac{2}{3}y + 2$. For water, $\rho g = 62.5$, so the force on the end of the trough is

$$\int_0^3 \rho g(3 - y) \left(\frac{2}{3}y + 2 \right) dy = \rho g \int_0^3 \left(-\frac{2}{3}y^2 + 6 \right) dy = \rho g \left[-\frac{2}{9}y^3 + 6y \right]_0^3 = 12 \cdot 62.5 = \boxed{750 \text{ lbs}}.$$

33. Since the tank's radius is 5 m, it is half-full. Position the y -axis at the center of one end of the tank, with the positive y -axis pointing down. Then at a given y -coordinate (depth), the width of the cross section of that end is $2\sqrt{5^2 - y^2}$, and the cross section lies y m below the surface. So the total force is

$$\begin{aligned} \int_0^5 \rho g y \cdot 2\sqrt{5^2 - y^2} dy &= 2\rho g \left[-\frac{1}{3}(5^2 - y^2)^{3/2} \right]_0^5 = -\frac{2}{3}\rho g \left((5^2 - 5^2)^{3/2} - (5^2 - 0^2)^{3/2} \right) \\ &= -\frac{2}{3}\rho g \cdot (-125) = \frac{2}{3} \cdot 737 \cdot 9.8 \cdot 125 \approx \boxed{601883 \text{ N}}. \end{aligned}$$

34. Placing the origin at the center of the bottom of the dam, the face of the dam is



Note that for water, $\rho g = 62.5$. The two diagonal lines have equations

$$y - 0 = \frac{80 - 0}{-500 - (-350)}(x - (-350)) \quad \text{or} \quad y = -\frac{8}{15}(x + 350)$$

$$y - 0 = \frac{80 - 0}{500 - 350}(x - 350) \quad \text{or} \quad y = \frac{8}{15}(x - 350).$$

Solving for x gives

$$x = -\frac{15}{8}y - 350, \quad x = \frac{15}{8}y + 350.$$

Therefore the width of a cross section at height y is

$$\left(\frac{15}{8}y + 350\right) - \left(-\frac{15}{8}y - 350\right) = \frac{15}{4}y + 700.$$

(a) If the dam is full, then the cross section at y lies $80 - y$ ft below the surface, and the total force is

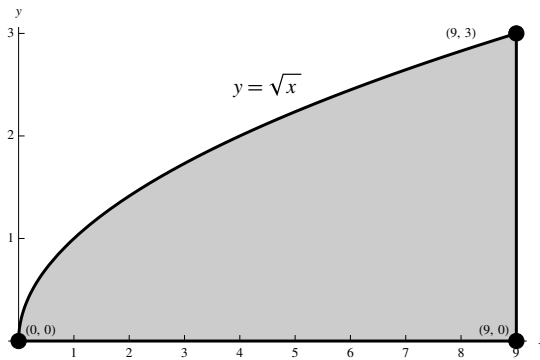
$$\begin{aligned} \int_0^{80} \rho g(80 - y) \left(\frac{15}{4}y + 700\right) dy &= \rho g \int_0^{80} \left(-\frac{15}{4}y^2 - 400y + 56000\right) dy \\ &= \rho g \left[-\frac{5}{4}y^3 - 200y^2 + 56000y\right]_0^{80} \\ &= 2.56 \times 10^6 \rho g = \boxed{1.6 \times 10^8 \text{ lbs}}. \end{aligned}$$

(b) If the water has a depth of 60 ft, then the cross section at y lies only $60 - y$ ft below the surface, and the integral goes from 0 to 60, so the total force is

$$\begin{aligned} \int_0^{60} \rho g(60 - y) \left(\frac{15}{4}y + 700\right) dy &= \rho g \int_0^{60} \left(-\frac{15}{4}y^2 - 475y + 42000\right) dy \\ &= \rho g \left[-\frac{5}{4}y^3 - \frac{475}{2}y^2 + 42000y\right]_0^{60} \\ &= 1.395 \times 10^6 \rho g = \boxed{\approx 8.719 \times 10^7 \text{ lbs}}. \end{aligned}$$

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35. The region is shown below:



The mass of the region is

$$M = \rho \int_0^9 \sqrt{x} dx = \rho \left[\frac{2}{3}x^{3/2}\right]_0^9 = 18\rho,$$

and the moments about the axes are

$$\begin{aligned} M_y &= \rho \int_0^9 x\sqrt{x} dx = \rho \int_0^9 x^{3/2} dx = \rho \left[\frac{2}{5}x^{5/2}\right]_0^9 = \frac{486}{5}\rho \\ M_x &= \rho \cdot \frac{1}{2} \int_0^9 (\sqrt{x})^2 dx = \frac{1}{2}\rho \int_0^9 x dx = \frac{1}{2}\rho \left[\frac{1}{2}x^2\right]_0^9 = \frac{81}{4}\rho. \end{aligned}$$

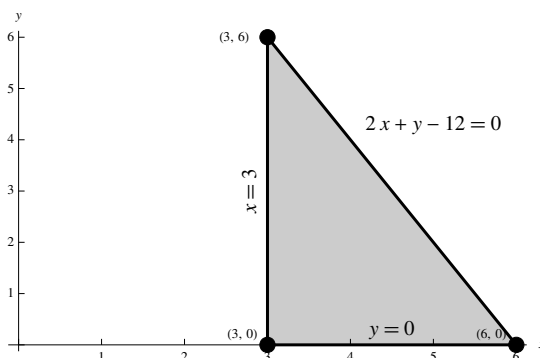
Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{27}{5}, \frac{9}{8}\right)}.$$

36. This torus is the solid of revolution of the circle of radius $\frac{5-2}{2} = \frac{3}{2}$ centered at $(\frac{5+2}{2} = \frac{7}{2}, 0)$ about the y -axis. By symmetry, the centroid of this circle is at $(\frac{7}{2}, 0)$, which is $\frac{7}{2}$ cm from the axis of revolution. So by Pappus' Theorem, the volume of the torus is

$$2\pi Ad = 2\pi \cdot \pi \left(\frac{3}{2}\right)^2 \cdot \frac{7}{2} = \boxed{\frac{63}{4}\pi^2 \text{ cm}^2}.$$

37. The region is shown below:



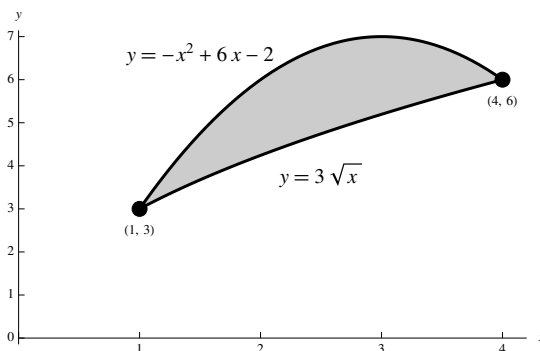
Since we are dealing with areas, we may assume that the region is a homogeneous lamina, and we must compute its centroid. The diagonal line has equation $y = 12 - 2x$, so we have

$$\begin{aligned} A &= \int_3^6 (12 - 2x) dx = [12x - x^2]_3^6 = 36 - 27 = 9 \\ \bar{x} &= \frac{1}{A} \int_3^6 x(12 - 2x) dx = \frac{1}{9} \int_3^6 (12x - 2x^2) dx = \frac{1}{9} \left[6x^2 - \frac{2}{3}x^3 \right]_3^6 = \frac{1}{9}(72 - 36) = 4 \\ \bar{y} &= \frac{1}{2A} \int_3^6 (12 - 2x)^2 dx = \frac{1}{18} \int_3^6 (144 - 48x + 4x^2) dx = \frac{1}{18} \left[144x - 24x^2 + \frac{4}{3}x^3 \right]_3^6 \\ &= \frac{1}{18}(288 - 252) = 2. \end{aligned}$$

Then the centroid is at $(4, 2)$. We are revolving about the y -axis, so the distance from the centroid to the line of revolution is 4. The area of the region is 9. So by Pappus' Theorem, the volume of the solid of revolution is

$$V = 2\pi Ad = 2\pi \cdot 9 \cdot 4 = \boxed{72\pi}.$$

38. The region is shown below:



The two graphs intersect when $3\sqrt{x} = -x^2 + 6x - 2$. Looking at the graph, the intersection points appear to be at $x = 1$ and $x = 4$; substituting those values into the equation above reveals that $3\sqrt{1} = -1^2 + 6 \cdot 1 - 2$ and $3\sqrt{4} = -4^2 + 6 \cdot 4 - 2$, so that those are in fact the correct x -coordinates, and the two points of intersection are $(1, 3)$ and $(4, 6)$.

- (a) To revolve about $y = 0$ (the x -axis), use the washer method. Each outer radius is $-x^2 + 6x - 2$ and each inner radius is $3\sqrt{x}$, so the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left((-x^2 + 6x - 2)^2 - (3\sqrt{x})^2 \right) dx \\ &= \pi \int_1^4 (x^4 - 12x^3 + 40x^2 - 33x + 4) dx \\ &= \pi \left[\frac{1}{5}x^5 - 3x^4 + \frac{40}{3}x^3 - \frac{33}{2}x^2 + 4x \right]_1^4 \\ &= \boxed{\frac{441}{10}\pi}. \end{aligned}$$

- (b) To revolve about $x = 0$ (the y -axis), use the shell method. The radius of each shell is x , and the height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$, so the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 x (-x^2 + 6x - 2 - 3\sqrt{x}) dx \\ &= 2\pi \int_1^4 (-x^3 + 6x^2 - 2x - 3x^{3/2}) dx \\ &= 2\pi \left[-\frac{1}{4}x^4 + 2x^3 - x^2 - \frac{6}{5}x^{5/2} \right]_1^4 \\ &= \boxed{\frac{201}{10}\pi}. \end{aligned}$$

- (c) To revolve about $y = -2$, use the washer method. Outer radii are $-x^2 + 6x - 2 - (-2) = -x^2 + 6x$, and inner radii are $3\sqrt{x} - (-2) = 3\sqrt{x} + 2$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left((-x^2 + 6x)^2 - (3\sqrt{x} + 2)^2 \right) dx \\ &= \pi \int_1^4 (x^4 - 12x^3 + 36x^2 - 9x - 12x^{1/2} - 4) dx \\ &= \pi \left[\frac{1}{5}x^5 - 3x^4 + 12x^3 - \frac{9}{2}x^2 - 8x^{3/2} - 4x \right]_1^4 \\ &= \boxed{\frac{601}{10}\pi}. \end{aligned}$$

- (d) To revolve about $y = 8$, use the washer method. Each outer radius is $8 - 3\sqrt{x}$ and each inner radius is $8 - (-x^2 + 6x - 2) = x^2 - 6x + 10$, so the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left((8 - 3\sqrt{x})^2 - (x^2 - 6x + 10)^2 \right) dx \\ &= \pi \int_1^4 (-x^4 + 12x^3 - 56x^2 + 129x - 48x^{1/2} - 36) dx \\ &= \pi \left[-\frac{1}{5}x^5 + 3x^4 - \frac{56}{3}x^3 + \frac{129}{2}x^2 - 32x^{3/2} - 36x \right]_1^4 \\ &= \boxed{\frac{199}{10}\pi}. \end{aligned}$$

- (e) To revolve about $x = -3$, use the shell method along the x -axis. The height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$ and its radius is $x - (-3) = x + 3$, so the volume is

$$\begin{aligned}
 V &= 2\pi \int_1^4 (x+3) (-x^2 + 6x - 2 - 3\sqrt{x}) \, dx \\
 &= 2\pi \int_1^4 (-x^3 + 3x^2 + 16x - 3x^{3/2} - 9x^{1/2} - 6) \, dx \\
 &= 2\pi \left[-\frac{1}{4}x^4 + x^3 + 8x^2 - \frac{6}{5}x^{5/2} - 6x^{3/2} - 6x \right]_1^4 \\
 &= \boxed{\frac{441}{10}\pi}.
 \end{aligned}$$

- (f) To revolve about $x = 5$, use the shell method along the x -axis. The height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$ and its radius is $5 - x$, so the volume is

$$\begin{aligned}
 V &= 2\pi \int_1^4 (5-x) (-x^2 + 6x - 2 - 3\sqrt{x}) \, dx \\
 &= 2\pi \int_1^4 (x^3 - 11x^2 + 32x + 3x^{3/2} - 15x^{1/2} - 10) \, dx \\
 &= 2\pi \left[\frac{1}{4}x^4 - \frac{11}{3}x^3 + 16x^2 + \frac{6}{5}x^{5/2} - 10x^{3/2} - 10x \right]_1^4 \\
 &= \boxed{\frac{199}{10}\pi}.
 \end{aligned}$$

Techniques of Integration

7.1: Integration by Parts

Concepts and Vocabulary

1. True.
2. $uv - \int v \, du$

Skill Building

3. We use integration by parts with $u = x$ and $dv = e^{2x} \, dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$. We obtain

$$\begin{aligned}\int x e^{2x} \, dx &= x \left(\frac{1}{2} e^{2x} \right) - \int \frac{1}{2} e^{2x} \, dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} \, dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \left(\frac{1}{2} e^{2x} \right) + C \\ &= \boxed{\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C}.\end{aligned}$$

4. We use integration by parts with $u = x$ and $dv = e^{-3x} \, dx$. Then $du = dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}\int x e^{-3x} \, dx &= x \left(-\frac{1}{3} e^{-3x} \right) - \int \left(-\frac{1}{3} e^{-3x} \right) \, dx \\ &= -\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} \, dx \\ &= -\frac{1}{3} x e^{-3x} + \frac{1}{3} \left(-\frac{1}{3} e^{-3x} \right) + C \\ &= \boxed{-\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C}.\end{aligned}$$

5. We use integration by parts with $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \sin x$. We obtain

$$\begin{aligned}\int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x - (-\cos x) + C \\ &= \boxed{x \sin x + \cos x + C}.\end{aligned}$$

6. We use integration by parts with $u = x$ and $dv = \sin(3x) dx$. Then $du = dx$ and $v = \frac{1}{3}(-\cos(3x)) = -\frac{1}{3}\cos(3x)$. We obtain

$$\begin{aligned}\int x \sin(3x) dx &= x \left(-\frac{1}{3} \cos(3x) \right) - \int \left(-\frac{1}{3} \cos(3x) \right) dx \\ &= -\frac{1}{3}x \cos(3x) + \frac{1}{3} \int \cos(3x) dx \\ &= -\frac{1}{3}x \cos(3x) + \frac{1}{3} \left(\frac{1}{3} \sin 3x \right) + C \\ &= \boxed{-\frac{1}{3}x \cos(3x) + \frac{1}{9} \sin(3x) + C}.\end{aligned}$$

7. We use integration by parts with $u = \ln x$ and $dv = \sqrt{x} dx = x^{1/2} dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{2}{3}x^{3/2}$. We obtain

$$\begin{aligned}\int \sqrt{x} \ln x dx &= (\ln x) \left(\frac{2}{3}x^{3/2} \right) - \int \left(\frac{2}{3}x^{3/2} \right) \left(\frac{1}{x} \right) dx \\ &= \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx \\ &= \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \left(\frac{2}{3}x^{3/2} \right) + C \\ &= \boxed{\frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C}.\end{aligned}$$

8. We use integration by parts with $u = \ln x$ and $dv = x^{-2} dx$. Then $du = \frac{1}{x} dx$ and $v = -x^{-1}$. We obtain

$$\begin{aligned}\int x^{-2} \ln x dx &= (\ln x)(-x^{-1}) - \int (-x^{-1}) \left(\frac{1}{x} \right) dx \\ &= -\frac{\ln x}{x} + \int x^{-2} dx \\ &= \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}.\end{aligned}$$

9. We use integration by parts with $u = \cot^{-1} x$ and $dv = dx$. Then $du = \left(-\frac{1}{1+x^2} \right) dx$ and $v = x$. We obtain

$$\begin{aligned}\int \cot^{-1} x dx &= (\cot^{-1} x)(x) - \int \left(-\frac{1}{1+x^2} \right) (x) dx \\ &= x \cot^{-1} x + \int \frac{x}{1+x^2} dx.\end{aligned}$$

Let $u = 1 + x^2$, then $du = 2x dx$, so $x dx = \frac{du}{2}$. We now substitute and obtain

$$\begin{aligned}\int \cot^{-1} x dx &= x \cot^{-1} x + \int \frac{1}{u} \frac{du}{2} \\ &= x \cot^{-1} x + \frac{1}{2} \int \frac{1}{u} du \\ &= x \cot^{-1} x + \frac{1}{2} \ln |u| + C \\ &= x \cot^{-1} x + \frac{1}{2} \ln |1 + x^2| + C \\ &= \boxed{x \cot^{-1} x + \frac{1}{2} \ln(1 + x^2) + C}.\end{aligned}$$

10. We use integration by parts with $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \sin^{-1} x \, dx &= (\sin^{-1} x)(x) - \int \left(\frac{1}{\sqrt{1-x^2}} \right)(x) \, dx \\ &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx.\end{aligned}$$

Let $u = 1 - x^2$, then $du = (-2x) \, dx$, so $x \, dx = \left(-\frac{du}{2}\right)$. We now substitute and obtain

$$\begin{aligned}\int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \frac{1}{\sqrt{u}} \left(-\frac{du}{2}\right) \\ &= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} \, du \\ &= x \sin^{-1} x + \frac{1}{2} (2u^{1/2}) + C \\ &= \boxed{x \sin^{-1} x + \sqrt{1-x^2} + C}.\end{aligned}$$

11. We use integration by parts with $u = (\ln x)^2$ and $dv = dx$. Then $du = \frac{2 \ln x}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^2 \, dx &= (\ln x)^2 x - \int x \left(\frac{2 \ln x}{x} \right) \, dx \\ &= x(\ln x)^2 - 2 \int \ln x \, dx\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^2 \, dx &= x(\ln x)^2 - 2 \left[(\ln x)x - \int x \left(\frac{1}{x} \right) \, dx \right] \\ &= x(\ln x)^2 - 2 \left[(\ln x)x - \int 1 \, dx \right] \\ &= x(\ln x)^2 - 2 [(\ln x)x - x] + C \\ &= \boxed{x(\ln x)^2 - 2x \ln x + 2x + C}.\end{aligned}$$

12. We use integration by parts with $u = (\ln x)^2$ and $dv = x \, dx$. Then $du = \frac{2 \ln x}{x} dx$ and $v = \frac{1}{2} x^2$. We obtain

$$\begin{aligned}\int x(\ln x)^2 \, dx &= (\ln x)^2 \left(\frac{1}{2} x^2 \right) - \int \left(\frac{1}{2} x^2 \right) \left(\frac{2 \ln x}{x} \right) \, dx \\ &= \frac{1}{2} x^2 (\ln x)^2 - \int x \ln x \, dx\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = x \, dx$. Then $du = \frac{1}{x} dx$ and

$v = \frac{1}{2}x^2$. We obtain

$$\begin{aligned}
 \int x(\ln x)^2 dx &= \frac{1}{2}x^2(\ln x)^2 - \left[(\ln x)\left(\frac{1}{2}x^2\right) - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\right) dx \right] \\
 &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2(\ln x) + \frac{1}{2} \int x dx \\
 &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2(\ln x) + \frac{1}{2}\left(\frac{1}{2}x^2\right) + C \\
 &= \boxed{\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2(\ln x) + \frac{1}{4}x^2 + C}.
 \end{aligned}$$

13. We use integration by parts with $u = x^2$ and $dv = \sin x dx$. Then $du = 2x dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}
 \int x^2 \sin x dx &= x^2(-\cos x) - \int (-\cos x)(2x) dx \\
 &= -x^2 \cos x + 2 \int x \cos x dx
 \end{aligned}$$

We use integration by parts again with $u = x$ and $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
 \int x^2 \sin x dx &= -x^2 \cos x + 2 \left[x \sin x - \int \sin x dx \right] \\
 &= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx \\
 &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + C \\
 &= \boxed{-x^2 \cos x + 2x \sin x + 2 \cos x + C}.
 \end{aligned}$$

14. We use integration by parts with $u = x^2$ and $dv = \cos x dx$. Then $du = 2x dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
 \int x^2 \cos x dx &= x^2(\sin x) - \int (\sin x)(2x) dx \\
 &= x^2 \sin x - 2 \int x \sin x dx
 \end{aligned}$$

We use integration by parts again with $u = x$ and $dv = \sin x dx$. Then $du = dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}
 \int x^2 \cos x dx &= x^2 \sin x - 2 \left[x(-\cos x) - \int (-\cos x) dx \right] \\
 &= x^2 \sin x + 2x \cos x - 2 \int \cos x dx \\
 &= \boxed{x^2 \sin x + 2x \cos x - 2 \sin x + C}.
 \end{aligned}$$

15. We use integration by parts with $u = x \cos x$ and $dv = \cos x \, dx$. Then $du = (-x \sin x + \cos x) \, dx$ and $v = \sin x$. We obtain

$$\begin{aligned}
 \int x \cos^2 x \, dx &= (x \cos x)(\sin x) - \int (\sin x)(-x \sin x + \cos x) \, dx \\
 &= x \cos x \sin x + \int x \sin^2 x \, dx - \int \cos x \sin x \, dx \\
 &= x \cos x \sin x + \int x(1 - \cos^2 x) \, dx - \int \cos x \sin x \, dx \\
 &= x \cos x \sin x + \int x \, dx - \int x \cos^2 x \, dx - \int \cos x \sin x \, dx.
 \end{aligned}$$

We add $\int x \cos^2 x \, dx$ to each side, and then let $u = \sin x$, so $du = \cos x \, dx$ in the remaining integral. Upon substitution, we obtain

$$\begin{aligned}
 2 \int x \cos^2 x \, dx &= x \cos x \sin x + \frac{1}{2}x^2 - \int u \, du \\
 &= x \cos x \sin x + \frac{1}{2}x^2 - \frac{1}{2}u^2 + C \\
 &= x \cos x \sin x + \frac{1}{2}x^2 - \frac{1}{2}\sin^2 x + C.
 \end{aligned}$$

Finally, we divide by 2 to obtain

$$\int x \cos^2 x \, dx = \boxed{\frac{1}{2}x \cos x \sin x + \frac{1}{4}x^2 - \frac{1}{4}\sin^2 x + C}.$$

16. We use integration by parts with $u = x \sin x$ and $dv = \sin x \, dx$. Then $du = (x \cos x + \sin x) \, dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}
 \int x \sin^2 x \, dx &= (x \sin x)(-\cos x) - \int (-\cos x)(x \cos x + \sin x) \, dx \\
 &= -x \cos x \sin x + \int x \cos^2 x \, dx + \int \cos x \sin x \, dx \\
 &= -x \cos x \sin x + \int x(1 - \sin^2 x) \, dx + \int \cos x \sin x \, dx \\
 &= -x \cos x \sin x + \int x \, dx - \int x \sin^2 x \, dx + \int \cos x \sin x \, dx.
 \end{aligned}$$

We add $\int x \sin^2 x \, dx$ to each side, and then let $u = \sin x$, so $du = \cos x \, dx$ in the remaining integral. Upon substitution, we obtain

$$\begin{aligned}
 2 \int x \sin^2 x \, dx &= -x \cos x \sin x + \frac{1}{2}x^2 + \int u \, du \\
 &= -x \cos x \sin x + \frac{1}{2}x^2 + \frac{1}{2}u^2 \\
 &= -x \cos x \sin x + \frac{1}{2}x^2 + \frac{1}{2}\sin^2 x.
 \end{aligned}$$

Finally, we divide by 2, add the constant of integration, and obtain

$$\int x \sin^2 x \, dx = \boxed{-\frac{1}{2}x \cos x \sin x + \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C}.$$

17. We use integration by parts with $u = x$ and $dv = \sinh x \, dx$. Then $du = dx$ and $v = \cosh x$. We obtain

$$\begin{aligned}\int x \sinh x \, dx &= x \cosh x - \int \cosh x \, dx \\ &= \boxed{x \cosh x - \sinh x + C}.\end{aligned}$$

18. We use integration by parts with $u = x$ and $dv = \cosh x \, dx$. Then $du = dx$ and $v = \sinh x$. We obtain

$$\begin{aligned}\int x \cosh x \, dx &= x \sinh x - \int \sinh x \, dx \\ &= \boxed{x \sinh x - \cosh x + C}.\end{aligned}$$

19. We use integration by parts with $u = \cosh^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{x^2-1}} \, dx$ and $v = x$. We obtain

$$\int \cosh^{-1} x \, dx = (\cosh^{-1} x)x - \int x \left(\frac{1}{\sqrt{x^2-1}} \right) dx.$$

Let $u = x^2 - 1$, then $du = 2x \, dx$, so $x \, dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned}\int \cosh^{-1} x \, dx &= x \cosh^{-1} x - \int u^{-1/2} \frac{du}{2} \\ &= x \cosh^{-1} x - \frac{1}{2} \int u^{-1/2} \, du \\ &= x \cosh^{-1} x - \frac{1}{2} (2\sqrt{u}) + C \\ &= \boxed{x \cosh^{-1} x - \sqrt{x^2-1} + C}.\end{aligned}$$

20. We use integration by parts with $u = \sinh^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{x^2+1}} \, dx$ and $v = x$. We obtain

$$\int \sinh^{-1} x \, dx = (\sinh^{-1} x)x - \int x \left(\frac{1}{\sqrt{x^2+1}} \right) dx.$$

Let $u = x^2 + 1$, then $du = 2x \, dx$, so $x \, dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned}\int \sinh^{-1} x \, dx &= x \sinh^{-1} x - \int u^{-1/2} \frac{du}{2} \\ &= x \sinh^{-1} x - \frac{1}{2} \int u^{-1/2} \, du \\ &= x \sinh^{-1} x - \frac{1}{2} (2\sqrt{u}) + C \\ &= \boxed{x \sinh^{-1} x - \sqrt{x^2+1} + C}.\end{aligned}$$

21. We use integration by parts with $u = \sin(\ln x)$ and $dv = dx$. Then $du = \frac{\cos(\ln x)}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \sin(\ln x) dx &= (\sin(\ln x))x - \int x \left(\frac{\cos(\ln x)}{x} \right) dx \\ &= x \sin(\ln x) - \int \cos(\ln x) dx.\end{aligned}$$

We use integration by parts again with $u = \cos(\ln x)$ and $dv = dx$. Then $du = \frac{-\sin(\ln x)}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \sin(\ln x) dx &= x \sin(\ln x) - \left[(\cos(\ln x))x - \int x \left(\frac{-\sin(\ln x)}{x} \right) dx \right] \\ &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx.\end{aligned}$$

We add $\int \sin(\ln x) dx$ to each side, divide by 2, and add the constant of integration to obtain

$$\begin{aligned}2 \int \sin(\ln x) dx &= x \sin(\ln x) - x \cos(\ln x) + C \\ \int \sin(\ln x) dx &= \frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x) + C = \boxed{\frac{x}{2} [\sin(\ln x) - \cos(\ln x)] + C}.\end{aligned}$$

22. We use integration by parts with $u = \cos(\ln x)$ and $dv = dx$. Then $du = \frac{-\sin(\ln x)}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \cos(\ln x) dx &= (\cos(\ln x))x - \int x \left(\frac{-\sin(\ln x)}{x} \right) dx \\ &= x \cos(\ln x) + \int \sin(\ln x) dx.\end{aligned}$$

We use integration by parts again with $u = \sin(\ln x)$ and $dv = dx$. Then $du = \frac{\cos(\ln x)}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \cos(\ln x) dx &= x \cos(\ln x) + \left[(\sin(\ln x))x - \int x \left(\frac{\cos(\ln x)}{x} \right) dx \right] \\ &= x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x) dx.\end{aligned}$$

We add $\int \cos(\ln x) dx$ to each side, divide by 2, and add the constant of integration to obtain

$$\begin{aligned}2 \int \cos(\ln x) dx &= x \cos(\ln x) + x \sin(\ln x) + C \\ \int \cos(\ln x) dx &= \boxed{\frac{1}{2}x \cos(\ln x) + \frac{1}{2}x \sin(\ln x) + C}.\end{aligned}$$

23. We use integration by parts with $u = (\ln x)^3$ and $dv = dx$. Then $du = 3\frac{(\ln x)^2}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3 dx &= (\ln x)^3 x - \int x \left(\frac{3(\ln x)^2}{x} \right) dx \\ &= x(\ln x)^3 - 3 \int (\ln x)^2 dx.\end{aligned}$$

We use integration by parts again with $u = (\ln x)^2$ and $dv = dx$. Then $du = 2\frac{\ln x}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3 dx &= x(\ln x)^3 - 3 \left[(\ln x)^2(x) - \int x \left(2\frac{\ln x}{x} \right) dx \right] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6 \int \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^3 dx &= x(\ln x)^3 - 3x(\ln x)^2 + 6 \left[(\ln x)x - \int x \left(\frac{1}{x} \right) dx \right] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6 \int dx \\ &= \boxed{x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C}.\end{aligned}$$

24. We use integration by parts with $u = (\ln x)^4$ and $dv = dx$. Then $du = 4\frac{(\ln x)^3}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^4 dx &= (\ln x)^4 x - \int x \left(\frac{4(\ln x)^3}{x} \right) dx \\ &= x(\ln x)^4 - 4 \int (\ln x)^3 dx.\end{aligned}$$

We use integration by parts again with $u = (\ln x)^3$ and $dv = dx$. Then $du = 3\frac{(\ln x)^2}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^4 dx &= x(\ln x)^4 - 4 \left[(\ln x)^3 x - \int x \left(\frac{3(\ln x)^2}{x} \right) dx \right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12 \int (\ln x)^2 dx.\end{aligned}$$

We use integration by parts again with $u = (\ln x)^2$ and $dv = dx$. Then $du = 2\frac{\ln x}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^4 dx &= x(\ln x)^4 - 4x(\ln x)^3 + 12 \left[(\ln x)^2(x) - \int x \left(2\frac{\ln x}{x} \right) dx \right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24 \int \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int (\ln x)^4 dx &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24 \left[(\ln x)x - \int x \left(\frac{1}{x} \right) dx \right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x \ln x + 24 \int dx \\ &= \boxed{x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x \ln x + 24x + C}.\end{aligned}$$

25. We use integration by parts with $u = (\ln x)^2$ and $dv = x^2 dx$. Then $du = 2\frac{\ln x}{x} dx$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2(\ln x)^2 dx &= (\ln x)^2 \left(\frac{1}{3}x^3 \right) - \int \left(\frac{1}{3}x^3 \right) \left(2\frac{\ln x}{x} \right) dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3} \int x^2 \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = x^2 dx$. Then $du = \frac{dx}{x}$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2(\ln x)^2 dx &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3} \left[(\ln x) \left(\frac{1}{3}x^3 \right) - \int \left(\frac{1}{3}x^3 \right) \frac{dx}{x} \right] \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{9} \int x^2 dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{9} \left(\frac{1}{3}x^3 \right) + C \\ &= \boxed{\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C}.\end{aligned}$$

26. We use integration by parts with $u = (\ln x)^2$ and $dv = x^3 dx$. Then $du = 2\frac{\ln x}{x} dx$ and $v = \frac{1}{4}x^4$. We obtain

$$\begin{aligned}\int x^3(\ln x)^2 dx &= (\ln x)^2 \left(\frac{1}{4}x^4 \right) - \int \left(\frac{1}{4}x^4 \right) \left(2\frac{\ln x}{x} \right) dx \\ &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2} \int x^3 \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = x^3 dx$. Then $du = \frac{dx}{x}$ and $v = \frac{1}{4}x^4$. We obtain

$$\begin{aligned}\int x^3(\ln x)^2 dx &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2} \left[(\ln x) \left(\frac{1}{4}x^4 \right) - \int \left(\frac{1}{4}x^4 \right) \frac{dx}{x} \right] \\ &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4 \ln x + \frac{1}{8} \int x^3 dx \\ &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4 \ln x + \frac{1}{8} \left(\frac{1}{4}x^4 \right) + C \\ &= \boxed{\frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4 \ln x + \frac{1}{32}x^4 + C}.\end{aligned}$$

27. We use integration by parts with $u = \tan^{-1} x$ and $dv = x^2 dx$. Then $du = \frac{1}{1+x^2} dx$ and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2 \tan^{-1} x dx &= (\tan^{-1} x) \left(\frac{1}{3}x^3 \right) - \int \left(\frac{1}{3}x^3 \right) \left(\frac{1}{1+x^2} \right) dx \\ &= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx.\end{aligned}$$

Let $u = 1 + x^2$, then $du = 2x dx$, so $x dx = \frac{du}{2}$. Since $x^2 = u - 1$, we have $x^3 dx = \frac{u-1}{2} du$.

We substitute and obtain

$$\begin{aligned}
 \int x^2 \tan^{-1} x \, dx &= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{u^{-1}}{u} \, du \\
 &= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} \int \left(1 - \frac{1}{u}\right) \, du \\
 &= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} (u - \ln |u|) + C \\
 &= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} ((1 + x^2) - \ln |1 + x^2|) + C \\
 &= \boxed{\frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \ln (1 + x^2) + C},
 \end{aligned}$$

where we have absorbed any constants with the constant C of integration.

28. We use integration by parts with $u = \tan^{-1} x$ and $dv = x \, dx$. Then $du = \frac{1}{1+x^2} \, dx$ and $v = \frac{1}{2} x^2$. We obtain

$$\begin{aligned}
 \int x \tan^{-1} x \, dx &= (\tan^{-1} x) \left(\frac{1}{2} x^2 \right) - \int \left(\frac{1}{2} x^2 \right) \left(\frac{1}{1+x^2} \right) \, dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{(1+x^2) - 1}{1+x^2} \, dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) \, dx \\
 &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + C \\
 &= \boxed{\frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C}.
 \end{aligned}$$

29. We use integration by parts with $u = x$ and $dv = 7^x \, dx$. Then $du = dx$ and $v = \frac{7^x}{\ln 7}$. We obtain

$$\begin{aligned}
 \int 7^x x \, dx &= (x) \left(\frac{7^x}{\ln 7} \right) - \int \frac{7^x}{\ln 7} \, dx \\
 &= \frac{7^x x}{\ln 7} - \frac{1}{\ln 7} \int 7^x \, dx \\
 &= \frac{7^x x}{\ln 7} - \frac{1}{\ln 7} \left(\frac{7^x}{\ln 7} \right) + C \\
 &= \boxed{\frac{7^x x}{\ln 7} - \frac{7^x}{(\ln 7)^2} + C}.
 \end{aligned}$$

30. We use integration by parts with $u = x$ and $dv = 2^{-x} dx$. Then $du = dx$ and $v = -\frac{2^{-x}}{\ln 2}$. We obtain

$$\begin{aligned}\int 2^{-x} x dx &= (x) \left(-\frac{2^{-x}}{\ln 2} \right) - \int \left(-\frac{2^{-x}}{\ln 2} \right) dx \\ &= -\frac{2^{-x} x}{\ln 2} + \frac{1}{\ln 2} \int 2^{-x} dx \\ &= -\frac{2^{-x} x}{\ln 2} + \frac{1}{\ln 2} \left(-\frac{2^{-x}}{\ln 2} \right) + C \\ &= \boxed{-\frac{2^{-x} x}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2} + C}.\end{aligned}$$

31. We use integration by parts with $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x$. We obtain

$$\begin{aligned}\int_0^\pi e^x \cos x dx &= [e^x \sin x]_0^\pi - \int_0^\pi e^x \sin x dx \\ &= [e^\pi \sin \pi - e^0 \sin 0] - \int_0^\pi e^x \sin x dx \\ &= [e^\pi(0) - 1(0)] - \int_0^\pi e^x \sin x dx \\ &= -\int_0^\pi e^x \sin x dx.\end{aligned}$$

We use integration by parts again with $u = e^x$ and $dv = \sin x dx$. Then $du = e^x dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int_0^\pi e^x \cos x dx &= -\left([e^x(-\cos x)]_0^\pi - \int_0^\pi e^x(-\cos x) dx \right) \\ &= -\left([e^\pi(-\cos \pi) - e^0(-\cos 0)] + \int_0^\pi e^x \cos x dx \right) \\ &= -\left([e^\pi(1) - 1(-1)] + \int_0^\pi e^x \cos x dx \right) \\ &= -e^\pi - 1 - \int_0^\pi e^x \cos x dx.\end{aligned}$$

We add $\int_0^\pi e^x \cos x dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2 \int_0^\pi e^x \cos x dx &= -e^\pi - 1 \\ \int_0^\pi e^x \cos x dx &= \frac{-e^\pi - 1}{2} = \boxed{-\frac{e^\pi + 1}{2}}.\end{aligned}$$

32. We use integration by parts with $u = x^2$ and $dv = e^{-x} dx$. Then $du = 2x dx$ and $v = -e^{-x}$. We obtain

$$\begin{aligned}\int_0^1 x^2 e^{-x} dx &= [x^2(-e^{-x})]_0^1 - \int_0^1 (-e^{-x})(2x) dx \\ &= [1^2(-e^{-1}) - 0^2(-e^{-0})] + 2 \int_0^1 x e^{-x} dx \\ &= -e^{-1} + 2 \int_0^1 x e^{-x} dx.\end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = -e^{-x}$. We obtain

$$\begin{aligned}
 \int_0^1 x^2 e^{-x} dx &= -e^{-1} + 2 \left([x(-e^{-x})]_0^1 - \int_0^1 (-e^{-x}) dx \right) \\
 &= -e^{-1} + 2 \left([1(-e^{-1}) - 0(-e^{-0})] + \int_0^1 e^{-x} dx \right) \\
 &= -e^{-1} + 2 \left(-e^{-1} + [-e^{-x}]_0^1 \right) \\
 &= -e^{-1} + 2(-e^{-1} + [-e^{-1} - (-e^{-0})]) \\
 &= \boxed{2 - 5e^{-1}}.
 \end{aligned}$$

33. We use integration by parts with $u = x^2$ and $dv = e^{-3x} dx$. Then $du = 2x dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}
 \int_0^2 x^2 e^{-3x} dx &= \left[x^2 \left(-\frac{1}{3} e^{-3x} \right) \right]_0^2 - \int_0^2 \left(-\frac{1}{3} e^{-3x} \right) (2x) dx \\
 &= \left[2^2 \left(-\frac{1}{3} e^{-3(2)} \right) - 0^2 \left(-\frac{1}{3} e^{-3(0)} \right) \right] + \frac{2}{3} \int_0^2 x e^{-3x} dx \\
 &= -\frac{4}{3} e^{-6} + \frac{2}{3} \int_0^2 x e^{-3x} dx.
 \end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^{-3x} dx$. Then $du = dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}
 \int_0^2 x^2 e^{-3x} dx &= -\frac{4}{3} e^{-6} + \frac{2}{3} \left(\left[x \left(-\frac{1}{3} e^{-3x} \right) \right]_0^2 - \int_0^2 \left(-\frac{1}{3} e^{-3x} \right) dx \right) \\
 &= -\frac{4}{3} e^{-6} + \frac{2}{3} \left(\left[2 \left(-\frac{1}{3} e^{-3(2)} \right) - 0 \left(-\frac{1}{3} e^{-3(0)} \right) \right] + \frac{1}{3} \int_0^2 e^{-3x} dx \right) \\
 &= -\frac{4}{3} e^{-6} + \frac{2}{3} \left(\left(-\frac{2}{3} e^{-6} \right) + \frac{1}{3} \left[-\frac{1}{3} e^{-3x} \right]_0^2 \right) \\
 &= -\frac{4}{3} e^{-6} + \frac{2}{3} \left(\left(-\frac{2}{3} e^{-6} \right) + \frac{1}{3} \left[-\frac{1}{3} e^{-3(2)} - \left(-\frac{1}{3} e^{-3(0)} \right) \right] \right) \\
 &= -\frac{4}{3} e^{-6} + \frac{2}{3} \left(\left(-\frac{2}{3} e^{-6} \right) + \frac{1}{3} \left(\frac{1}{3} - \frac{1}{3} e^{-6} \right) \right) \\
 &= \boxed{\frac{2}{27} - \frac{50}{27} e^{-6}}.
 \end{aligned}$$

34. We use integration by parts with $u = x$ and $dv = \tan^2 x \, dx = (\sec^2 x - 1) \, dx$. Then $du = dx$ and $v = \tan x - x$. We obtain

$$\begin{aligned}
 \int_0^{\pi/4} x \tan^2 x \, dx &= [x(\tan x - x)]_0^{\pi/4} - \int_0^{\pi/4} (\tan x - x) \, dx \\
 &= \left[\frac{\pi}{4} \left(\tan \frac{\pi}{4} - \frac{\pi}{4} \right) - 0(\tan 0 - 0) \right] - \left[\ln |\sec x| - \frac{1}{2} x^2 \right]_0^{\pi/4} \\
 &= \frac{\pi}{4} - \frac{\pi^2}{16} - \left[\ln \left| \sec \frac{\pi}{4} \right| - \frac{1}{2} \left(\frac{\pi}{4} \right)^2 - \left(\ln |\sec 0| - \frac{1}{2} 0^2 \right) \right] \\
 &= \frac{\pi}{4} - \frac{\pi^2}{16} - \left(\ln \sqrt{2} - \frac{\pi^2}{32} \right) \\
 &= \boxed{\frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2} \ln 2}.
 \end{aligned}$$

35. We rewrite $\int_1^9 \ln \sqrt{x} \, dx = \int_1^9 \ln x^{1/2} \, dx = \frac{1}{2} \int_1^9 \ln x \, dx$. We use integration by parts with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} \, dx$ and $v = x$. We obtain

$$\begin{aligned}
 \int_1^9 \ln \sqrt{x} \, dx &= \frac{1}{2} \int_1^9 \ln x \, dx \\
 &= \frac{1}{2} \left([(\ln x)(x)]_1^9 - \int_1^9 x \left(\frac{1}{x} \right) dx \right) \\
 &= \frac{1}{2} \left([(\ln 9)(9) - (\ln 1)(1)] - \int_1^9 dx \right) \\
 &= \frac{1}{2} (9 \ln 9 - 8) \\
 &= \boxed{9 \ln 3 - 4}.
 \end{aligned}$$

36. We use integration by parts with $u = x$ and $dv = \csc^2 x \, dx$. Then $du = dx$ and $v = -\cot x$. We obtain

$$\begin{aligned}
 \int_{\pi/4}^{3\pi/4} x \csc^2 x \, dx &= [x(-\cot x)]_{\pi/4}^{3\pi/4} - \int_{\pi/4}^{3\pi/4} (-\cot x) \, dx \\
 &= \left[\frac{3\pi}{4} \left(-\cot \frac{3\pi}{4} \right) - \frac{\pi}{4} \left(-\cot \frac{\pi}{4} \right) \right] + \int_{\pi/4}^{3\pi/4} \cot x \, dx \\
 &= \left(\frac{3\pi}{4} + \frac{\pi}{4} \right) + [\ln |\sin x|]_{\pi/4}^{3\pi/4} \\
 &= \pi + \left[\ln \left| \sin \frac{3\pi}{4} \right| - \ln \left| \sin \frac{\pi}{4} \right| \right] \\
 &= \pi + \left(\ln \frac{\sqrt{2}}{2} - \ln \frac{\sqrt{2}}{2} \right) \\
 &= \boxed{\pi}.
 \end{aligned}$$

37. We use integration by parts with $u = (\ln x)^2$ and $dv = dx$. Then $du = 2\frac{\ln x}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^e (\ln x)^2 dx &= \left[(\ln x)^2(x) \right]_1^e - \int_1^e x \left(2\frac{\ln x}{x} \right) dx \\ &= (\ln e)^2(e) - (\ln 1)^2(1) - \int_1^e 2 \ln x dx \\ &= e - 2 \int_1^e \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^e (\ln x)^2 dx &= e - 2 \left([(\ln x)x]_1^e - \int_1^e x \frac{1}{x} dx \right) \\ &= e - 2 \left((\ln e)e - (\ln 1)(1) - \int_1^e dx \right) \\ &= e - 2(e - (e - 1)) \\ &= \boxed{e - 2}.\end{aligned}$$

38. We use integration by parts with $u = x$ and $dv = \sec^2 x dx$. Then $du = dx$ and $v = \tan x$. We obtain

$$\begin{aligned}\int_0^{\pi/4} x \sec^2 x dx &= [x(\tan x)]_0^{\pi/4} - \int_0^{\pi/4} \tan x dx \\ &= \left[\frac{\pi}{4} \left(\tan \frac{\pi}{4} \right) - 0(\tan 0) \right] - [\ln |\sec x|]_0^{\pi/4} \\ &= \frac{\pi}{4} - [\ln |\sec \frac{\pi}{4}| - \ln |\sec 0|] \\ &= \frac{\pi}{4} - \ln \sqrt{2} \\ &= \boxed{\frac{\pi}{4} - \frac{1}{2} \ln 2}.\end{aligned}$$

39. To determine where the graphs intersect, set $3 \ln x = x \ln x$, and obtain $x = 3$. Since $\ln x \geq 0$ when $1 \leq x \leq 3$, we have $x \ln x \leq 3 \ln x$ on the interval $1 \leq x \leq 3$ and the area enclosed by the graphs of f and g is

$$\int_1^3 (3 \ln x - x \ln x) dx = \int_1^3 (3 - x) \ln x dx.$$

We use integration by parts with $u = \ln x$ and $dv = (3 - x) dx$. Then $du = \frac{1}{x} dx$ and $v = 3x - \frac{1}{2}x^2$. We obtain

$$\begin{aligned}\int_1^3 (3 - x) \ln x dx &= \left[(\ln x) \left(3x - \frac{1}{2}x^2 \right) \right]_1^3 - \int_1^3 \frac{3x - \frac{1}{2}x^2}{x} dx \\ &= \left[(\ln 3) \left(3(3) - \frac{1}{2}3^2 \right) - (\ln 1) \left(3(1) - \frac{1}{2}1^2 \right) \right] - \int_1^3 \left(3 - \frac{1}{2}x \right) dx \\ &= \frac{9}{2} \ln 3 - \left[3x - \frac{1}{4}x^2 \right]_1^3 \\ &= \frac{9}{2} \ln 3 - \left[\left(3(3) - \frac{1}{4}3^2 \right) - \left(3(1) - \frac{1}{4}1^2 \right) \right] \\ &= \boxed{\frac{9}{2} \ln 3 - 4}.\end{aligned}$$

40. To determine where the graphs intersect, set $4x \ln x = x^2 \ln x$, and obtain $x = 1$ and $x = 4$. Since $\ln x \geq 0$ when $1 \leq x \leq 4$, we have $x^2 \ln x \leq 4x \ln x$ on the interval $1 \leq x \leq 4$, and the area enclosed by the graphs of f and g is

$$\int_1^4 (4x \ln x - x^2 \ln x) dx = \int_1^4 (4x - x^2) \ln x dx.$$

We use integration by parts with $u = \ln x$ and $dv = (4x - x^2) dx$. Then $du = \frac{1}{x} dx$ and $v = 2x^2 - \frac{1}{3}x^3$. We obtain

$$\begin{aligned} \int_1^4 (4x - x^2) \ln x dx &= \left[(\ln x) \left(2x^2 - \frac{1}{3}x^3 \right) \right]_1^4 - \int_1^4 \frac{2x^2 - \frac{1}{3}x^3}{x} dx \\ &= \left[(\ln 4) \left(2(4)^2 - \frac{1}{3}4^3 \right) - (\ln 1) \left(2(1)^2 - \frac{1}{3}1^3 \right) \right] - \int_1^4 \left(2x - \frac{1}{3}x^2 \right) dx \\ &= \frac{32}{3} \ln 4 - \left[x^2 - \frac{1}{9}x^3 \right]_1^4 \\ &= \frac{32}{3} \ln 4 - \left[\left(4^2 - \frac{1}{9}4^3 \right) - \left(1^2 - \frac{1}{9}1^3 \right) \right] \\ &= \boxed{\frac{32}{3} \ln 4 - 8}. \end{aligned}$$

41. The area under the graph of $y = e^x \sin x$ from 0 to π is given by $\int_0^\pi e^x \sin x dx$. We use integration by parts with $u = e^x$ and $dv = \sin x dx$. Then $du = e^x dx$ and $v = -\cos x$. We obtain

$$\begin{aligned} \int_0^\pi e^x \sin x dx &= [e^x(-\cos x)]_0^\pi - \int_0^\pi e^x(-\cos x) dx \\ &= [e^\pi(-\cos \pi) - e^0(-\cos 0)] + \int_0^\pi e^x \cos x dx \\ &= e^\pi + 1 + \int_0^\pi e^x \cos x dx. \end{aligned}$$

We use integration by parts again with $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$ and $v = \sin x$. We obtain

$$\begin{aligned} \int_0^\pi e^x \sin x dx &= e^\pi + 1 + [e^x \sin x]_0^\pi - \int_0^\pi e^x \sin x dx \\ &= e^\pi + 1 + (e^\pi \sin \pi - e^0 \sin 0) - \int_0^\pi e^x \sin x dx \\ &= e^\pi + 1 - \int_0^\pi e^x \sin x dx \end{aligned}$$

We add $\int_0^\pi e^x \sin x dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned} 2 \int_0^\pi e^x \sin x dx &= e^\pi + 1 \\ \int_0^\pi e^x \sin x dx &= \boxed{\frac{e^\pi + 1}{2}}. \end{aligned}$$

42. The volume is given by the integral $\int_0^{\pi/2} 2\pi x \cos x \, dx$. We use integration by parts with $u = 2\pi x$ and $dv = \cos x \, dx$. Then $du = 2\pi \, dx$ and $v = \sin x$. We obtain

$$\begin{aligned} \int_0^{\pi/2} 2\pi x \cos x \, dx &= [(2\pi x)(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} \sin x (2\pi) \, dx \\ &= \left[\left(2\pi \left(\frac{\pi}{2} \right) \right) \left(\sin \frac{\pi}{2} \right) - (2\pi(0))(\sin 0) \right] - 2\pi \int_0^{\pi/2} \sin x \, dx \\ &= \pi^2 - 2\pi [-\cos x]_0^{\pi/2} \\ &= \pi^2 - 2\pi \left[-\cos \frac{\pi}{2} - (-\cos 0) \right] \\ &= \boxed{\pi^2 - 2\pi}. \end{aligned}$$

43. The volume is given by the integral $\int_0^{\pi/2} 2\pi x \sin x \, dx$. We use integration by parts with $u = 2\pi x$ and $dv = \sin x \, dx$. Then $du = 2\pi \, dx$ and $v = -\cos x$. We obtain

$$\begin{aligned} \int_0^{\pi/2} 2\pi x \sin x \, dx &= [(2\pi x)(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(2\pi) \, dx \\ &= \left[\left(2\pi \left(\frac{\pi}{2} \right) \right) \left(-\cos \frac{\pi}{2} \right) - (2\pi(0))(-\cos 0) \right] + 2\pi \int_0^{\pi/2} \cos x \, dx \\ &= 2\pi \int_0^{\pi/2} \cos x \, dx \\ &= 2\pi [\sin x]_0^{\pi/2} \\ &= 2\pi \left[\sin \frac{\pi}{2} - \sin 0 \right] \\ &= \boxed{2\pi}. \end{aligned}$$

44. The volume is given by the integral $\int_0^{\pi/2} \pi \left(x\sqrt{\sin x} \right)^2 \, dx = \int_0^{\pi/2} \pi x^2 \sin x \, dx$. We use integration by parts with $u = \pi x^2$ and $dv = \sin x \, dx$. Then $du = 2\pi x \, dx$ and $v = -\cos x$. We obtain

$$\begin{aligned} \int_0^{\pi/2} \pi x^2 \sin x \, dx &= [(\pi x^2)(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(2\pi x) \, dx \\ &= \left[\left(\pi \left(\frac{\pi}{2} \right)^2 \right) \left(-\cos \left(\frac{\pi}{2} \right) \right) - (\pi 0^2)(-\cos 0) \right] + \int_0^{\pi/2} 2\pi x \cos x \, dx \\ &= \int_0^{\pi/2} 2\pi x \cos x \, dx. \end{aligned}$$

We use integration by parts again with $u = 2\pi x$ and $dv = \cos x \, dx$. Then $du = 2\pi \, dx$ and $v = \sin x$. We obtain

$$\begin{aligned} \int_0^{\pi/2} \pi x^2 \sin x \, dx &= [(2\pi x)(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} (\sin x)(2\pi) \, dx \\ &= \left[\left(2\pi \left(\frac{\pi}{2} \right) \right) \left(\sin \left(\frac{\pi}{2} \right) \right) - (2\pi(0))(\sin 0) \right] - 2\pi \int_0^{\pi/2} \sin x \, dx \\ &= \pi^2 - 2\pi [-\cos x]_0^{\pi/2} \\ &= \pi^2 - 2\pi \left[-\cos \frac{\pi}{2} - (-\cos 0) \right] \\ &= \boxed{\pi^2 - 2\pi}. \end{aligned}$$

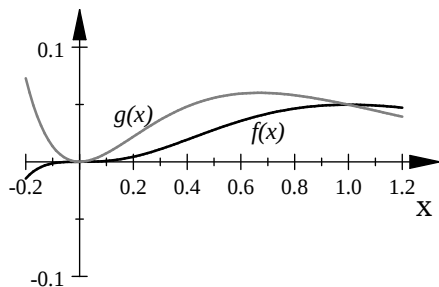
45. The volume is given by the integral $\int_1^e \pi(\ln x)^2 dx$. We use integration by parts with $u = (\ln x)^2$ and $dv = \pi dx$. Then $du = \frac{2\ln x}{x} dx$ and $v = \pi x$. We obtain

$$\begin{aligned}\int_1^e \pi(\ln x)^2 dx &= \left[(\ln x)^2 (\pi x) \right]_1^e - \int_1^e (\pi x) \left(\frac{2\ln x}{x} \right) dx \\ &= \left[(\ln e)^2 (\pi e) - (\ln 1)^2 (\pi 1) \right] - 2\pi \int_1^e \ln x dx \\ &= \pi e - 2\pi \int_1^e \ln x dx.\end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned}\int_1^e \pi(\ln x)^2 dx &= \pi e - 2\pi \left([(\ln x)x]_1^e - \int_1^e x \left(\frac{1}{x} \right) dx \right) \\ &= \pi e - 2\pi \left([(\ln e)e - (\ln 1)1] - \int_1^e dx \right) \\ &= \pi e - 2\pi(e - (e - 1)) \\ &= \boxed{\pi e - 2\pi}.\end{aligned}$$

46. (a)



- (b) The area enclosed by the graphs of f and g is given by $\int_0^1 (x^2 e^{-3x} - x^3 e^{-3x}) dx = \int_0^1 (x^2 - x^3) e^{-3x} dx$. We use integration by parts with $u = x^2 - x^3$ and $dv = e^{-3x} dx$. Then $du = (2x - 3x^2) dx$ and $v = -\frac{1}{3} e^{-3x}$. We obtain

$$\begin{aligned}\int_0^1 (x^2 - x^3) e^{-3x} dx &= \left[(x^2 - x^3) \left(-\frac{1}{3} e^{-3x} \right) \right]_0^1 - \int_0^1 \left(-\frac{1}{3} e^{-3x} \right) (2x - 3x^2) dx \\ &= \left[(1^2 - 1^3) \left(-\frac{1}{3} e^{-3(1)} \right) - (0^2 - 0^3) \left(-\frac{1}{3} e^{-3(0)} \right) \right] \\ &\quad + \frac{1}{3} \int_0^1 (2x - 3x^2) e^{-3x} dx \\ &= \frac{1}{3} \int_0^1 (2x - 3x^2) e^{-3x} dx.\end{aligned}$$

We use integration by parts again with $u = 2x - 3x^2$ and $dv = e^{-3x} dx$. Then

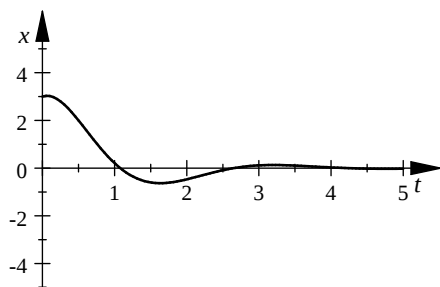
$du = (2 - 6x) dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}\int_0^1 (x^2 - x^3)e^{-3x} dx &= \frac{1}{3} \left(\left[(2x - 3x^2) \left(-\frac{1}{3}e^{-3x} \right) \right]_0^1 - \int_0^1 \left(-\frac{1}{3}e^{-3x} \right) (2 - 6x) dx \right) \\ &= \frac{1}{3} \left[(2(1) - 3(1)^2) \left(-\frac{1}{3}e^{-3(1)} \right) - (2(0) - 3(0)^2) \left(-\frac{1}{3}e^{-3(0)} \right) \right] \\ &\quad + \frac{2}{9} \int_0^1 (1 - 3x)e^{-3x} dx \\ &= \frac{1}{9}e^{-3} + \frac{2}{9} \int_0^1 (1 - 3x)e^{-3x} dx.\end{aligned}$$

We use integration by parts again with $u = 1 - 3x$ and $dv = e^{-3x} dx$. Then $du = -3 dx$ and $v = -\frac{1}{3}e^{-3x}$. We obtain

$$\begin{aligned}\int_0^1 (x^2 - x^3)e^{-3x} dx &= \frac{1}{9}e^{-3} + \frac{2}{9} \left(\left[(1 - 3x) \left(-\frac{e^{-3x}}{3} \right) \right]_0^1 - \int_0^1 \left(-\frac{e^{-3x}}{3} \right) (-3) dx \right) \\ &= \frac{1}{9}e^{-3} + \frac{2}{9} \left[(1 - 3(1)) \left(-\frac{e^{-3(1)}}{3} \right) - (1 - 3(0)) \left(-\frac{e^{-3(0)}}{3} \right) \right] \\ &\quad - \frac{2}{9} \int_0^1 e^{-3x} dx \\ &= \frac{7}{27}e^{-3} + \frac{2}{27} - \frac{2}{9} \left[-\frac{1}{3}e^{-3x} \right]_0^1 \\ &= \frac{7}{27}e^{-3} + \frac{2}{27} - \frac{2}{9} \left[-\frac{1}{3}e^{-3(1)} - \left(-\frac{1}{3}e^{-3(0)} \right) \right] \\ &= \boxed{\frac{1}{3}e^{-3}}.\end{aligned}$$

47. (a)



(b) Set $3e^{-t} \cos(2t) + 2e^{-t} \sin(2t) = 0$ and solve for t . We obtain $\tan(2t) = -\frac{3}{2}$, so $2t = \tan^{-1}\left(-\frac{3}{2}\right) + k\pi$, where k is any integer. So the least positive number t such that $x(t) = 0$ is $t = \boxed{\frac{\pi}{2} + \frac{1}{2} \tan^{-1}\left(-\frac{3}{2}\right)}$.

(c) We integrate $\int_0^{\frac{\pi}{2} - \frac{1}{2} \tan^{-1}\left(\frac{3}{2}\right)} 3e^{-t} \cos(2t) + 2e^{-t} \sin(2t) dt$ using a Computer Algebra System, and we obtain $\int_0^{\frac{\pi}{2} - \frac{1}{2} \tan^{-1}\left(\frac{3}{2}\right)} (3e^{-t} \cos(2t) + 2e^{-t} \sin(2t)) dt \approx \boxed{1.890}$.

48. We use integration by parts, with $u = x$ and $dv = f'(x) dx$. Then $du = dx$ and $v = f(x)$. We obtain

$$\begin{aligned}\int_3^5 x f'(x) dx &= [x f(x)]_3^5 - \int_3^5 f(x) dx \\ &= 5f(5) - 3f(3) - (18) \\ &= 5(11) - 3(8) - 18 \\ &= \boxed{13}.\end{aligned}$$

49. Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}} dx$, so $dx = 2w dw$. We substitute and obtain

$$\int \sin \sqrt{x} dx = \int 2w \sin w dw.$$

We use integration by parts, with $u = 2w$ and $dv = \sin w dw$. Then $du = 2 dw$ and $v = -\cos w$. We obtain

$$\begin{aligned}\int 2w \sin w dw &= (2w)(-\cos w) - \int (-\cos w)(2) dw \\ &= -2w \cos w + 2 \int \cos w dw \\ &= -2w \cos w + 2(\sin w) + C \\ &= \boxed{-2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C}.\end{aligned}$$

50. Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}} dx$, so $dx = 2w dw$. We substitute and obtain

$$\int e^{\sqrt{x}} dx = \int 2w e^w dw.$$

We use integration by parts, with $u = 2w$ and $dv = e^w dw$. Then $du = 2 dw$ and $v = e^w$. We obtain

$$\begin{aligned}\int 2w e^w dw &= (2w)(e^w) - \int (e^w)(2) dw \\ &= 2w e^w - 2 \int e^w dw \\ &= 2w e^w - 2(e^w) + C \\ &= \boxed{2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}.\end{aligned}$$

51. Let $w = \sin x$, then $dw = \cos x dx$. We substitute and obtain

$$\int \cos x \ln(\sin x) dx = \int \ln w dw.$$

We use integration by parts, with $u = \ln w$ and $dv = dw$. Then $du = \frac{1}{w} dw$ and $v = w$. We obtain

$$\begin{aligned}\int \ln w dw &= (\ln w)(w) - \int (w) \left(\frac{1}{w} \right) dw \\ &= w \ln w - \int dw \\ &= w \ln w - w + C \\ &= \boxed{\sin x \ln(\sin x) - \sin x + C}.\end{aligned}$$

52. Let $w = 2 + e^x$, then $dw = e^x dx$. We substitute and obtain

$$\int e^x \ln(2 + e^x) dx = \int \ln w dw.$$

We use integration by parts, with $u = \ln w$ and $dv = dw$. Then $du = \frac{1}{w} dw$ and $v = w$. We obtain

$$\begin{aligned} \int \ln w dw &= (\ln w)(w) - \int (w) \left(\frac{1}{w} \right) dw \\ &= w \ln w - \int dw \\ &= w \ln w - w + C \\ &= (2 + e^x) \ln(2 + e^x) - 2 - e^x + C \\ &= \boxed{(2 + e^x) \ln(2 + e^x) - e^x + C}, \end{aligned}$$

where we have absorbed any constants with the constant C of integration.

53. Let $w = e^{2x}$, then $dw = 2e^{2x} dx$, so $e^{2x} dx = \frac{1}{2} dw$. We substitute and obtain

$$\int e^{4x} \cos(e^{2x}) dx = \int e^{2x} \cos(e^{2x}) e^{2x} dx = \int w \cos w \left(\frac{1}{2} dw \right) = \int \frac{1}{2} w \cos w dw.$$

We use integration by parts, with $u = \frac{1}{2}w$ and $dv = \cos w dw$. Then $du = \frac{1}{2} dw$ and $v = \sin w$. We obtain

$$\begin{aligned} \int \frac{1}{2} w \cos w dw &= \left(\frac{1}{2} w \right) (\sin w) - \int (\sin w) \left(\frac{1}{2} \right) dw \\ &= \frac{1}{2} w \sin w - \frac{1}{2} \int \sin w dw \\ &= \frac{1}{2} w \sin w - \frac{1}{2} (-\cos w) + C \\ &= \frac{1}{2} w \sin w + \frac{1}{2} \cos w + C \\ &= \boxed{\frac{1}{2} e^{2x} \sin(e^{2x}) + \frac{1}{2} \cos(e^{2x}) + C}. \end{aligned}$$

54. Let $w = \sin x$, then $dw = \cos x dx$. We substitute and obtain

$$\int \cos x \tan^{-1}(\sin x) dx = \int \tan^{-1} w dw.$$

We use integration by parts with $u = \tan^{-1} w$ and $dv = dw$. Then $du = \frac{1}{1+w^2} dw$ and $v = w$. We obtain

$$\begin{aligned} \int \tan^{-1} w dw &= (\tan^{-1} w)(w) - \int (w) \left(\frac{1}{1+w^2} \right) dw \\ &= w \tan^{-1} w - \int \frac{w}{1+w^2} dw. \end{aligned}$$

Let $y = 1 + w^2$, then $dy = 2w dw$, so $w dw = \frac{1}{2} dy$. We substitute and obtain

$$\begin{aligned}
 \int \tan^{-1} w dw &= w \tan^{-1} w - \int \frac{1}{y} \left(\frac{1}{2} dy \right) \\
 &= w \tan^{-1} w - \frac{1}{2} \int \frac{1}{y} dy \\
 &= w \tan^{-1} w - \frac{1}{2} \ln |y| + C \\
 &= w \tan^{-1} w - \frac{1}{2} \ln (1 + w^2) + C \\
 &= \boxed{\sin x \tan^{-1} (\sin x) - \frac{1}{2} \ln (1 + \sin^2 x) + C}.
 \end{aligned}$$

55. We use integration by parts with $u = x^2$ and $dv = xe^{x^2} dx$. Then $du = 2x dx$ and $v = \frac{1}{2}e^{x^2}$. We obtain

$$\begin{aligned}
 \int x^3 e^{x^2} dx &= (x^2) \left(\frac{1}{2} e^{x^2} \right) - \int \left(\frac{1}{2} e^{x^2} \right) (2x) dx \\
 &= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx \\
 &= \boxed{\frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C}.
 \end{aligned}$$

56. We use integration by parts with $u = \ln x$ and $dv = x^n dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{1}{n+1} x^{n+1}$. We obtain

$$\begin{aligned}
 \int x^n \ln x dx &= (\ln x) \left(\frac{1}{n+1} x^{n+1} \right) - \int \left(\frac{1}{n+1} x^{n+1} \right) \left(\frac{1}{x} \right) dx \\
 &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n dx \\
 &= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \left(\frac{x^{n+1}}{n+1} \right) + C \\
 &= \boxed{\frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C}.
 \end{aligned}$$

57. We use integration by parts with $u = x \cos x$ and $dv = e^x dx$. Then $du = (-x \sin x + \cos x) dx$ and $v = e^x$. We obtain

$$\int x e^x \cos x dx = (x \cos x) e^x - \int (-x \sin x + \cos x) e^x dx.$$

We use integration by parts again with $u = -x \sin x + \cos x$ and $dv = e^x dx$. Then $du = (-2 \sin x - x \cos x) dx$ and $v = e^x$. We obtain

$$\begin{aligned}
 \int x e^x \cos x dx &= (x \cos x) e^x - \left((-x \sin x + \cos x) e^x - \int (-2 \sin x - x \cos x) e^x dx \right) \\
 &= x \cos x e^x + x \sin x e^x - \cos x e^x - 2 \int \sin x e^x dx - \int x \cos x e^x dx.
 \end{aligned}$$

We add $\int x e^x \cos x dx$ to each side and divide by 2 to obtain

$$\begin{aligned}
 2 \int x e^x \cos x dx &= x \cos x e^x + x \sin x e^x - \cos x e^x - 2 \int \sin x e^x dx \\
 \int x e^x \cos x dx &= \frac{1}{2} (x \cos x e^x + x \sin x e^x - \cos x e^x) - \int \sin x e^x dx.
 \end{aligned}$$

We determine $\int \sin x e^x dx$ using integration by parts. Let $u = \sin x$ and $dv = e^x dx$. Then $du = \cos x dx$ and $v = e^x$. We obtain

$$\int \sin x e^x dx = (\sin x)e^x - \int (\cos x)e^x dx.$$

We integrate by parts again with $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x dx$ and $v = e^x$. We obtain

$$\begin{aligned} \int \sin x e^x dx &= \sin x e^x - \left[(\cos x)e^x - \int (-\sin x)e^x dx \right] \\ &= \sin x e^x - \cos x e^x - \int \sin x e^x dx. \end{aligned}$$

We add $\int (\sin x)e^x dx$ to each side and divide by 2 to obtain

$$\begin{aligned} 2 \int \sin x e^x dx &= \sin x e^x - \cos x e^x \\ \int \sin x e^x dx &= \frac{1}{2} \sin x e^x - \frac{1}{2} \cos x e^x + C. \end{aligned}$$

We substitute into the above to obtain

$$\begin{aligned} \int x e^x \cos x dx &= \frac{1}{2}(x \cos x e^x + x \sin x e^x - \cos x e^x) - \left(\frac{1}{2} \sin x e^x - \frac{1}{2} \cos x e^x \right) + C \\ &= \frac{1}{2} x \cos x e^x + \frac{1}{2} x \sin x e^x - \frac{1}{2} \sin x e^x + C \\ &= \boxed{\frac{1}{2} e^x (x \sin x + x \cos x - \sin x) + C}. \end{aligned}$$

58. We use integration by parts with $u = x \sin x$ and $dv = e^x dx$. Then $du = (x \cos x + \sin x) dx$ and $v = e^x$. We obtain

$$\int x e^x \sin x dx = (x \sin x)e^x - \int (x \cos x + \sin x)e^x dx.$$

We use integration by parts again with $u = x \cos x + \sin x$ and $dv = e^x dx$. Then $du = (-x \sin x + 2 \cos x) dx$ and $v = e^x$. We obtain

$$\begin{aligned} \int x e^x \sin x dx &= x \sin x e^x - \left((x \cos x + \sin x)e^x - \int (-x \sin x + 2 \cos x)e^x dx \right) \\ &= x \sin x e^x - x \cos x e^x - \sin x e^x - \int x \sin x e^x dx + 2 \int \cos x e^x dx. \end{aligned}$$

We add $\int x e^x \sin x dx$ to each side and divide by 2 to obtain

$$\begin{aligned} 2 \int x e^x \sin x dx &= x \sin x e^x - x \cos x e^x - \sin x e^x + 2 \int \cos x e^x dx \\ \int x e^x \sin x dx &= \frac{1}{2}(x \sin x e^x - x \cos x e^x - \sin x e^x) + \int \cos x e^x dx. \end{aligned}$$

We determine $\int \cos x e^x dx$ using integration by parts. Let $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x dx$ and $v = e^x$. We obtain

$$\int \cos x e^x dx = (\cos x)e^x - \int (-\sin x)e^x dx = \cos x e^x + \int \sin x e^x dx.$$

We integrate by parts again with $u = \sin x$ and $dv = e^x dx$. Then $du = \cos x dx$ and $v = e^x$. We obtain

$$\begin{aligned}\int \cos x e^x dx &= \cos x e^x + \left[(\sin x) e^x - \int (\cos x) e^x dx \right] \\ &= \cos x e^x + \sin x e^x - \int \cos x e^x dx.\end{aligned}$$

We add $\int \cos x e^x dx$ to each side and divide by 2 to obtain

$$\begin{aligned}2 \int \cos x e^x dx &= \cos x e^x + \sin x e^x \\ \int \cos x e^x dx &= \frac{1}{2} \cos x e^x + \frac{1}{2} \sin x e^x + C.\end{aligned}$$

We substitute into the above to obtain

$$\begin{aligned}\int x e^x \cos x dx &= \frac{1}{2} (x \sin x e^x - x \cos x e^x - \sin x e^x) + \left(\frac{1}{2} \cos x e^x + \frac{1}{2} \sin x e^x \right) + C \\ &= \boxed{\frac{1}{2} x \sin x e^x - \frac{1}{2} x \cos x e^x + \frac{1}{2} \cos x e^x + C}.\end{aligned}$$

59. We use integration by parts with $u = \sin^{-1} x$ and $dv = x^n dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = \frac{x^{n+1}}{n+1}$. We obtain

$$\int x^n \sin^{-1} x dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx.$$

60. We consider the integral $\int \frac{1}{(x^2+1)^n} dx$ and use integration by parts with $u = \frac{1}{(x^2+1)^n}$ and $dv = dx$. Then $du = \frac{-2nx}{(x^2+1)^{n+1}} dx$ and $v = x$. We obtain

$$\begin{aligned}\int \frac{1}{(x^2+1)^n} dx &= \frac{x}{(x^2+1)^n} - \int (x) \left(\frac{-2nx}{(x^2+1)^{n+1}} \right) dx \\ &= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}} dx \\ &= \frac{x}{(x^2+1)^n} + 2n \int \frac{(x^2+1) - 1}{(x^2+1)^{n+1}} dx \\ &= \frac{x}{(x^2+1)^n} + 2n \int \frac{1}{(x^2+1)^n} dx - 2n \int \frac{1}{(x^2+1)^{n+1}} dx.\end{aligned}$$

We solve for $\int \frac{1}{(x^2+1)^{n+1}} dx$, and obtain

$$\begin{aligned}2n \int \frac{1}{(x^2+1)^{n+1}} dx &= \frac{x}{(x^2+1)^n} + 2n \int \frac{1}{(x^2+1)^n} dx - \int \frac{1}{(x^2+1)^n} dx \\ 2n \int \frac{1}{(x^2+1)^{n+1}} dx &= \frac{x}{(x^2+1)^n} + (2n-1) \int \frac{1}{(x^2+1)^n} dx \\ \int \frac{1}{(x^2+1)^{n+1}} dx &= \frac{x}{2n(x^2+1)^n} + \left(1 - \frac{1}{2n} \right) \int \frac{1}{(x^2+1)^n} dx.\end{aligned}$$

61. We use integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x dx$. Then $du = (n-1) \sin^{n-2} x \cos x dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}
 \int \sin^n x dx &= \sin^{n-1} x (-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \cos^2 x \sin^{n-2} x dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \left(\int \sin^{n-2} x dx - \int \sin^n x dx \right) \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx.
 \end{aligned}$$

We add $(n-1) \int \sin^n x dx$ to each side, and then divide by n to obtain

$$\begin{aligned}
 (1+n-1) \int \sin^n x dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx \\
 \int \sin^n x dx &= -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx.
 \end{aligned}$$

62. We use integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x \cos^m x dx$. Then $du = (n-1) \sin^{n-2} x \cos x dx$ and $v = -\frac{\cos^{m+1} x}{m+1}$. We obtain

$$\begin{aligned}
 \int \sin^n x \cos^m x dx &= \sin^{n-1} x \left(-\frac{\cos^{m+1} x}{m+1} \right) - \int \left(-\frac{\cos^{m+1} x}{m+1} \right) (n-1) \sin^{n-2} x \cos x dx \\
 &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^{m+2} x dx \\
 &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x (1 - \sin^2 x) \cos^m x dx \\
 &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x dx \\
 &\quad - \frac{n-1}{m+1} \int \sin^n x \cos^m x dx.
 \end{aligned}$$

We add $\frac{n-1}{m+1} \int \sin^n x \cos^m x dx$ to each side, and then divide by $\frac{n+m}{m+1}$ to obtain

$$\begin{aligned}
 \left(1 + \frac{n-1}{m+1} \right) \int \sin^n x \cos^m x dx &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x dx \\
 \frac{n+m}{m+1} \int \sin^n x dx &= -\frac{\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x dx \\
 \int \sin^n x dx &= -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} x \cos^m x dx.
 \end{aligned}$$

63. (a) We use integration by parts with $u = x^2$ and $dv = e^{5x} dx$. Then $du = 2x dx$ and $v = \frac{1}{5} e^{5x}$. We obtain

$$\begin{aligned}
 \int x^2 e^{5x} dx &= x^2 \left(\frac{1}{5} e^{5x} \right) - \int \left(\frac{1}{5} e^{5x} \right) (2x) dx \\
 &= \frac{1}{5} x^2 e^{5x} - \frac{2}{5} \int x e^{5x} dx.
 \end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^{5x} dx$. Then $du = dx$ and $v = \frac{1}{5}e^{5x}$. We obtain

$$\begin{aligned}\int x^2 e^{5x} dx &= \frac{1}{5}x^2 e^{5x} - \frac{2}{5} \left[x \left(\frac{1}{5}e^{5x} \right) - \int \frac{1}{5}e^{5x} dx \right] \\ &= \frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{25} \int e^{5x} dx \\ &= \frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{25} \left(\frac{1}{5}e^{5x} \right) + C \\ &= \boxed{\frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{125}e^{5x} + C}.\end{aligned}$$

- (b) We use integration by parts with $u = x^n$ and $dv = e^{kx} dx$. Then $du = nx^{n-1} dx$ and $v = \frac{1}{k}e^{kx}$. We obtain

$$\begin{aligned}\int x^n e^{kx} dx &= x^n \left(\frac{1}{k}e^{kx} \right) - \int \left(\frac{1}{k}e^{kx} \right) (nx^{n-1}) dx \\ &= \boxed{\frac{1}{k}x^n e^{kx} - \frac{n}{k} \int x^{n-1} e^{kx} dx}.\end{aligned}$$

64. (a) If $\int x^3 e^x dx = p(x)e^x$, then $\frac{d}{dx}(p(x)e^x) = x^3 e^x$. So

$$\frac{d}{dx}(p(x)e^x) = p(x)e^x + p'(x)e^x = x^3 e^x.$$

Divide by e^x to obtain $p(x) + p'(x) = x^3$.

- (b) We use integration by parts with $u = x^3$ and $dv = e^x dx$. Then $du = 3x^2 dx$ and $v = e^x$. We obtain

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - \int (e^x)(3x^2) dx \\ &= x^3 e^x - 3 \int x^2 e^x dx.\end{aligned}$$

We use integration by parts again with $u = x^2$ and $dv = e^x dx$. Then $du = 2x dx$ and $v = e^x$. We obtain

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - 3 \left[x^2 e^x - \int (e^x)(2x) dx \right] \\ &= x^3 e^x - 3x^2 e^x + 6 \int x e^x dx.\end{aligned}$$

We use integration by parts again with $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. We obtain

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - 3x^2 e^x + 6 \left[x e^x - \int e^x dx \right] \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\ &= (x^3 - 3x^2 + 6x - 6)e^x + C.\end{aligned}$$

We have found the polynomial $\boxed{p(x) = x^3 - 3x^2 + 6x - 6}$.

65. (a) We use integration by parts with $u = \sin x$ and $dv = \cos x dx$. Then $du = \cos x dx$ and $v = \sin x$. We obtain

$$\begin{aligned}\int \sin x \cos x dx &= (\sin x)(\sin x) - \int (\sin x)(\cos x) dx \\ &= \sin^2 x - \int \sin x \cos x dx.\end{aligned}$$

We add $\int \sin x \cos x dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2 \int \sin x \cos x dx &= \sin^2 x \\ \int \sin x \cos x dx &= \frac{1}{2} \sin^2 x + C_1.\end{aligned}$$

So we obtain $\boxed{f(x) = \frac{1}{2} \sin^2 x}$.

- (b) We use integration by parts with $u = \cos x$ and $dv = \sin x dx$. Then $du = -\sin x dx$ and $v = -\cos x$. We obtain

$$\begin{aligned}\int \sin x \cos x dx &= (\cos x)(-\cos x) - \int (-\cos x)(-\sin x) dx \\ &= -\cos^2 x - \int \sin x \cos x dx.\end{aligned}$$

We add $\int \sin x \cos x dx$ to each side, and then divide by 2 to obtain

$$\begin{aligned}2 \int \sin x \cos x dx &= -\cos^2 x \\ \int \sin x \cos x dx &= -\frac{1}{2} \cos^2 x + C_2.\end{aligned}$$

So we obtain $\boxed{g(x) = -\frac{1}{2} \cos^2 x}$.

- (c) From the identity $\sin(2x) = 2 \sin x \cos x$ we have $\sin x \cos x = \frac{1}{2} \sin(2x)$. So $\int \sin x \cos x dx = \frac{1}{2} \int \sin(2x) dx$. Let $u = 2x$, then $du = 2 dx$, so $dx = \frac{1}{2} du$. We substitute and obtain

$$\begin{aligned}\int \sin x \cos x dx &= \frac{1}{2} \int \sin(2x) dx \\ &= \frac{1}{2} \int \sin u \left(\frac{1}{2} du \right) \\ &= \frac{1}{4} \int \sin u du \\ &= \frac{1}{4} (-\cos u) + C_3 \\ &= -\frac{1}{4} \cos(2x) + C_3.\end{aligned}$$

So we obtain $\boxed{h(x) = -\frac{1}{4} \cos(2x)}$.

- (d) We have $f(x) + C_1 = g(x) + C_2$, so $\frac{1}{2} \sin^2 x + C_1 = -\frac{1}{2} \cos^2 x + C_2$. We obtain $C_2 = \frac{1}{2} \cos^2 x + \frac{1}{2} \sin^2 x + C_1 = \frac{1}{2} + C_1$, so $\boxed{C_2 = \frac{1}{2} + C_1}$.

- (e) We have $f(x) + C_1 = h(x) + C_3$, so $\frac{1}{2}\sin^2 x + C_1 = -\frac{1}{4}\cos(2x) + C_3$. We obtain $C_3 = \frac{1}{4}\cos(2x) + \frac{1}{2}\sin^2 x + C_1 = \frac{1}{4}(1 - 2\sin^2 x) + \frac{1}{2}\sin^2 x + C_1 = \frac{1}{4} + C_1$. So $C_3 = \frac{1}{4} + C_1$.

66. We use integration by parts with $u = \ln(x + \sqrt{x^2 + a^2})$ and $dv = dx$. Then

$$\begin{aligned} du &= \frac{1 + \frac{1}{2}(x^2 + a^2)^{-1/2}(2x)}{x + \sqrt{x^2 + a^2}} dx \\ &= \frac{1 + x(x^2 + a^2)^{-1/2}}{x + \sqrt{x^2 + a^2}} \frac{\sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} dx \\ &= \frac{\sqrt{x^2 + a^2} + x}{(x + \sqrt{x^2 + a^2})\sqrt{x^2 + a^2}} dx \\ &= \frac{1}{\sqrt{x^2 + a^2}} dx \end{aligned}$$

and $v = x$. We obtain

$$\int \ln(x + \sqrt{x^2 + a^2}) dx = x \ln(x + \sqrt{x^2 + a^2}) - \int \frac{x}{\sqrt{x^2 + a^2}} dx.$$

Let $u = x^2 + a^2$, then $du = 2x dx$, so $x dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned} \int \ln(x + \sqrt{x^2 + a^2}) dx &= x \ln(x + \sqrt{x^2 + a^2}) - \frac{1}{2} \int u^{-1/2} du \\ &= x \ln(x + \sqrt{x^2 + a^2}) - \frac{1}{2} (2u^{1/2}) + C \\ &= x \ln(x + \sqrt{x^2 + a^2}) - \sqrt{x^2 + a^2} + C. \end{aligned}$$

67. We use integration by parts with $u = \sin(bx)$ and $dv = e^{ax} dx$. Then $du = b \cos(bx) dx$ and $v = \frac{1}{a}e^{ax}$. We obtain

$$\begin{aligned} \int e^{ax} \sin(bx) dx &= (\sin(bx)) \left(\frac{1}{a}e^{ax} \right) - \int \left(\frac{1}{a}e^{ax} \right) (b \cos(bx)) dx \\ &= \frac{1}{a}e^{ax} \sin(bx) - \frac{b}{a} \int e^{ax} \cos(bx) dx \end{aligned}$$

We integrate by parts again with $u = \cos bx$ and $dv = e^{ax} dx$. Then $du = -b \sin bx dx$ and $v = \frac{1}{a}e^{ax}$. We obtain

$$\begin{aligned} \int e^{ax} \sin(bx) dx &= \frac{1}{a}e^{ax} \sin(bx) - \frac{b}{a} \left[(\cos(bx)) \left(\frac{1}{a}e^{ax} \right) - \int \left(\frac{1}{a}e^{ax} \right) (-b \sin(bx)) dx \right] \\ &= \frac{1}{a}e^{ax} \sin(bx) - \frac{b}{a^2}e^{ax} \cos(bx) - \frac{b^2}{a^2} \int e^{ax} \sin(bx) dx \end{aligned}$$

We add $\frac{b^2}{a^2} \int e^{ax} \sin(bx) dx$ from each side and divide by $\frac{a^2 + b^2}{a^2}$ to obtain

$$\begin{aligned} \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin(bx) dx &= \frac{1}{a}e^{ax} \sin(bx) - \frac{b}{a^2}e^{ax} \cos(bx) \\ \int e^{ax} \sin(bx) dx &= \frac{a}{a^2 + b^2}e^{ax} \sin(bx) - \frac{b}{a^2 + b^2}e^{ax} \cos(bx) + C \\ &= \frac{e^{ax} [a \sin(bx) - b \cos(bx)]}{a^2 + b^2} + C. \end{aligned}$$

68. We use integration by parts with $u = t$ and $dv = g'(t) dt$. Then $du = dt$ and $v = g(t)$. We obtain

$$\begin{aligned} F(x) &= \int_0^x t g'(t) dt = [t g(t)]_0^x - \int_0^x g(t) dt \\ &= x g(x) - 0 g(0) - \int_0^x g(t) dt \\ &= x g(x) - \int_0^x g(t) dt. \end{aligned}$$

69. (a) $\int_0^{\pi/2} \sin^6 x dx = \frac{(5)(3)(1)}{(6)(4)(2)} \left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}.$
 (b) $\int_0^{\pi/2} \sin^5 x dx = \frac{(4)(2)}{(5)(3)(1)} = \boxed{\frac{8}{15}}.$
 (c) $\int_0^{\pi/2} \cos^8 x dx = \int_0^{\pi/2} \sin^8 x dx = \frac{(7)(5)(3)(1)}{(8)(6)(4)(2)} \left(\frac{\pi}{2}\right) = \boxed{\frac{35\pi}{256}}.$
 (d) $\int_0^{\pi/2} \cos^6 x dx = \int_0^{\pi/2} \sin^6 x dx = \frac{(5)(3)(1)}{(6)(4)(2)} \left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}.$

Challenge Problems

70. Suppose first that $n = 2m + 1 > 1$ is odd, so that m is any positive integer. By Problem 61, we have

$$\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \left[-\frac{\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx \\ &= \left[-\frac{\sin^{n-1} \frac{\pi}{2} \cos \frac{\pi}{2}}{n} - \left(-\frac{\sin^{n-1} 0 \cos 0}{n} \right) \right] + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx \\ &= \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx. \end{aligned}$$

We continue, and obtain

$$\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx \\ &= \frac{n-1}{n} \left(\frac{n-3}{n-2} \int_0^{\pi/2} \sin^{n-4} x dx \right) \\ &= \frac{n-1}{n} \frac{n-3}{n-2} \left(\frac{n-5}{n-4} \int_0^{\pi/2} \sin^{n-6} x dx \right) \\ &= \dots \\ &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots \frac{2}{3} \int_0^{\pi/2} \sin x dx \\ &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots \frac{2}{3} [-\cos x]_0^{\pi/2} \\ &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots \frac{2}{3} (1) \\ &= \frac{(2m)(2m-2)\dots(2)}{(2m+1)(2m-1)\dots(3)(1)} \end{aligned}$$

So the formula holds. Now suppose that $n = 2m > 1$ is even, so that m is any positive integer. By the above, we have

$$\begin{aligned}
 \int_0^{\pi/2} \sin^n x \, dx &= \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \\
 &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{3}{4} \int_0^{\pi/2} \sin^2 x \, dx \\
 &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{3}{4} \frac{1}{2} \int_0^{\pi/2} dx \\
 &= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2} \\
 &= \frac{(2m-1)(2m-3)\cdots(3)(1)}{(2m)(2m-2)\cdots(4)(2)} \left(\frac{\pi}{2}\right),
 \end{aligned}$$

and the formula holds.

71. (a) We use integration by parts with $u = x^n$ and $dv = e^x dx$. Then $du = nx^{n-1} dx$ and $v = e^x$. We have

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

Repeating, we obtain for some constants c and k

$$\begin{aligned}
 \int x^n e^x dx &= x^n e^x - n \left(x^{n-1} e^x - (n-1) \int x^{n-2} e^x dx \right) \\
 &= (x^n - nx^{n-1}) e^x + n(n-1) \int x^{n-2} e^x dx \\
 &= \cdots \\
 &= (x^n - nx^{n-1} + \cdots + kx) e^x + c \int e^x dx \\
 &= (x^n - nx^{n-1} + \cdots + kx + c) e^x + C \\
 &= p(x) e^x + C.
 \end{aligned}$$

- (b) Since $\frac{d}{dx}(p(x)e^x) = x^n e^x$, we have $p(x)e^x + p'(x)e^x = x^n e^x$. Divide by e^x to obtain $p(x) + p'(x) = x^n$.

(c) We show $p(x) = \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k}$ satisfies $p(x) + p'(x) = x^n$. We have

$$\begin{aligned}
 p(x) + p'(x) &= \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k} + \frac{d}{dx} \left(\sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k} \right) \\
 &= x^n + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} (n-k) x^{n-k-1} \\
 &= x^n + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} + \sum_{k=1}^n (-1)^{k-1} \frac{n!}{(n-(k-1))!} (n-(k-1)) x^{n-(k-1)-1} \\
 &= x^n + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} + \sum_{k=1}^n (-1)^{k-1} \frac{n!}{(n-k+1)!} (n-k+1) x^{n-k} \\
 &= x^n + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} + \sum_{k=1}^n (-1)^k (-1)^{-1} \frac{n!}{(n-k+1)(n-k)!} (n-k+1) x^{n-k} \\
 &= x^n + \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} - \sum_{k=1}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k} \\
 &= x^n.
 \end{aligned}$$

72. Use integration by parts with $u = e^{x^2}$ and $dv = x^{2n} dx$. Then $du = 2xe^{x^2} dx$ and $v = \frac{1}{2n+1} x^{2n+1}$. We obtain

$$\begin{aligned}
 \int_0^1 x^{2n} e^{x^2} dx &= \left[\frac{1}{2n+1} x^{2n+1} e^{x^2} \right]_0^1 - \int_0^1 \left(\frac{1}{2n+1} x^{2n+1} \right) (2xe^{x^2}) dx \\
 &= e \left(\frac{1}{2n+1} \right) - \frac{2}{2n+1} \int_0^1 x^{2n+2} e^{x^2} dx.
 \end{aligned}$$

Starting with the $n = 0$ case, we repeat the above formula, and obtain

$$\begin{aligned}
 \int_0^1 e^{x^2} dx &= e \left(\frac{1}{1} \right) - \frac{2}{1} \int_0^1 x^2 e^{x^2} dx \\
 &= e(1) - \frac{2}{1} \left(e \left(\frac{1}{3} \right) - \frac{2}{3} \int_0^1 x^4 e^{x^2} dx \right) \\
 &= e \left(1 - \frac{2}{3} \right) + \frac{2^2}{3(1)} \int_0^1 x^6 e^{x^2} dx \\
 &= e \left(1 - \frac{2}{3} \right) + \frac{2^2}{3(1)} \left(e \left(\frac{1}{5} \right) - \frac{2}{5} \int_0^1 x^8 e^{x^2} dx \right) \\
 &= e \left(1 - \frac{2}{3} + \frac{2^2}{5(3)(1)} \right) + \frac{2^3}{5(3)(1)} \int_0^1 x^8 e^{x^2} dx \\
 &= \dots \\
 &= e \left[1 - \frac{2}{3} + \dots + \frac{(-1)^n 2^n}{(2n+1) \dots (3)(1)} \right] + \frac{(-1)^{n+1} 2^{n+1}}{(2n+1) \dots (3)(1)} \int_0^1 x^{2n+2} e^{x^2} dx.
 \end{aligned}$$

So the formula holds for all nonnegative integers n .

73. We use integration by parts with $u = f(x)$ and $dv = e^x dx$. Then $du = f'(x) dx$ and $v = e^x$. We have

$$\int f(x) e^x dx = f(x) e^x - \int f'(x) e^x dx$$

We continue, using $u = f'(x)$, $dv = e^x$ and obtain

$$\begin{aligned}
 \int f(x)e^x dx &= f(x)e^x - \left(f'(x)e^x - \int f''(x)e^x dx \right) \\
 &= (f(x) - f'(x))e^x + \int f''(x)e^x dx \\
 &= \dots \\
 &= \left(f(x) - f'(x) + \dots \pm f^{(n-1)}(x) \right) e^x \pm \int f^{(n)}(x)e^x dx
 \end{aligned}$$

Since $f(x)$ is a polynomial of degree n , $f^{(n)}(x)$ is a constant, k , and we have

$$\begin{aligned}
 \int f(x)e^x dx &= \left(f(x) - f'(x) + \dots \pm f^{(n-1)}(x) \right) e^x \pm \int k e^x dx \\
 &= \left(f(x) - f'(x) + \dots \pm f^{(n-1)}(x) \right) e^x \pm k e^x + C \\
 &= (f(x) - f'(x) + \dots \pm k) e^x + C.
 \end{aligned}$$

We now have $f(x) = g(x)e^x + C$, where $g(x) = f(x) - f'(x) + \dots \pm k$ is a polynomial of degree n .

74. (a) We use integration by parts with $u = f'(t)$ and $dv = dt$. Then $du = f''(t) dt$ and take $v = t - b$. We obtain

$$\begin{aligned}
 f(b) - f(a) &= \int_a^b f'(t) dt \\
 &= [(t - b)f'(t)]_a^b - \int_a^b f''(t)(t - b) dt \\
 &= (b - b)f'(b) - (a - b)f'(a) - \int_a^b f''(t)(t - b) dt \\
 &= f'(a)(b - a) - \int_a^b f''(t)(t - b) dt.
 \end{aligned}$$

- (b) We use integration by parts on $\int_a^b f''(t)(t - b) dt$ with $u = f''(t)$ and $dv = (t - b) dt$. Then $du = f'''(t) dt$ and $v = \frac{1}{2}(t - b)^2$. We obtain

$$\begin{aligned}
 f(b) - f(a) &= f'(a)(b - a) - \int_a^b f''(t)(t - b) dt \\
 &= f'(a)(b - a) - \left(\left[\frac{1}{2}(t - b)^2 f''(t) \right]_a^b - \int_a^b f'''(t) \frac{1}{2}(t - b)^2 dt \right) \\
 &= f'(a)(b - a) \\
 &\quad - \left(\left[\frac{1}{2}(b - b)^2 f''(b) - \frac{1}{2}(a - b)^2 f''(a) \right] - \int_a^b \frac{f'''(t)}{2}(t - b)^2 dt \right) \\
 &= f'(a)(b - a) + \frac{f''(a)}{2}(b - a)^2 + \int_a^b \frac{f'''(t)}{2}(t - b)^2 dt.
 \end{aligned}$$

75. We use integration by parts with $u = f(x)$ and $dv = dx$. Then $du = f'(x) dx$ and take $v = x$. We obtain

$$\begin{aligned}\int_a^b f(x) dx &= [xf(x)]_a^b - \int_a^b xf'(x) dx \\ &= bf(b) - af(a) - \int_a^b xf'(x) dx.\end{aligned}$$

Let $x = f^{-1}(y)$, then $f(x) = y$, so $f'(x) dx = dy$. The limits of integration become $y = f(a)$ and $y = f(b)$. We substitute and obtain

$$\int_a^b f(x) dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) dy,$$

so

$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(y) dy = bf(b) - af(a).$$

76. (a) We use integration by parts with $u = \cosh x$ and $dv = e^x dx$. Then $du = \sinh x dx$ and take $v = e^x$. We obtain

$$\int e^x \cosh x dx = e^x \cosh x - \int e^x \sinh x dx.$$

We use integration by parts again with $u = \sinh x$ and $dv = e^x dx$. Then $du = \cosh x dx$ and take $v = e^x$. We obtain

$$\begin{aligned}\int e^x \cosh x dx &= e^x \cosh x - \left[e^x \sinh x - \int e^x \cosh x dx \right] \\ &= e^x \cosh x - e^x \sinh x + \int e^x \cosh x dx.\end{aligned}$$

The integrals cancel, and this will not allow us to determine the integral. Note that

$$\begin{aligned}e^x \cosh x - e^x \sinh x &= e^x (\cosh x - \sinh x) \\ &= e^x (e^{-x}) \\ &= 1.\end{aligned}$$

This is expected, as any two antiderivatives of $e^x \cosh x$, as denoted by $\int e^x \cosh x dx$ on each side of the equation

$$\int e^x \cosh x dx = e^x \cosh x - e^x \sinh x + \int e^x \cosh x dx,$$

must differ by a constant. So their difference, $e^x \cosh x - e^x \sinh x$, must be constant.

- (b) We rewrite $\cosh x = \frac{e^x + e^{-x}}{2}$, and obtain

$$\begin{aligned}\int e^x \cosh x dx &= \int e^x \left(\frac{e^x + e^{-x}}{2} \right) dx \\ &= \frac{1}{2} \int (e^{2x} + 1) dx \\ &= \frac{1}{2} \left(\frac{1}{2} e^{2x} + x \right) + C \\ &= \boxed{\frac{1}{4} e^{2x} + \frac{1}{2} x + C}.\end{aligned}$$

7.2: Integrals Containing Trigonometric Functions

Concepts and Vocabulary

1. True.

2. True.

Skill Building

3. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (\cos^2 x)^2 \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 \cos x \, dx.\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}\int \cos^5 x \, dx &= \int (1 - u^2)^2 \, du \\ &= \int (u^4 - 2u^2 + 1) \, du \\ &= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C \\ &= \boxed{\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C}.\end{aligned}$$

4. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \sin x \, dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \, dx$, so $\sin x \, dx = -du$. We substitute and obtain

$$\begin{aligned}\int \sin^3 x \, dx &= \int (1 - u^2)(-du) \\ &= \int (u^2 - 1) \, du \\ &= \frac{1}{3}u^3 - u + C \\ &= \boxed{\frac{1}{3}\cos^3 x - \cos x + C}.\end{aligned}$$

5. Use the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int \sin^6 x \, dx &= \int (\sin^2 x)^3 \, dx \\
 &= \int \left(\frac{1 - \cos(2x)}{2} \right)^3 \, dx \\
 &= \frac{1}{8} \int (1 - 3 \cos(2x) + 3 \cos^2(2x) - \cos^3(2x)) \, dx \\
 &= \frac{1}{8} \int dx - \frac{3}{8} \int \cos(2x) \, dx + \frac{3}{8} \int \cos^2(2x) \, dx - \frac{1}{8} \int \cos^3(2x) \, dx \\
 &= \frac{1}{8} x - \frac{3}{16} \sin(2x) + \frac{3}{8} \int \cos^2(2x) \, dx - \frac{1}{8} \int \cos^3(2x) \, dx.
 \end{aligned}$$

We evaluate

$$\begin{aligned}
 \int \cos^2(2x) \, dx &= \int \frac{1 + \cos(4x)}{2} \, dx \\
 &= \frac{1}{2} \int (1 + \cos(4x)) \, dx \\
 &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.
 \end{aligned}$$

And also

$$\begin{aligned}
 \int \cos^3(2x) \, dx &= \int \cos^2(2x) \cos(2x) \, dx \\
 &= \int (1 - \sin^2(2x)) \cos(2x) \, dx.
 \end{aligned}$$

Let $u = \sin(2x)$, then $du = 2 \cos(2x) \, dx$, so $\cos(2x) \, dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned}
 \int \cos^3(2x) \, dx &= \int (1 - u^2) \frac{du}{2} \\
 &= \frac{1}{2} \left(u - \frac{u^3}{3} \right) + C \\
 &= \frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6} + C.
 \end{aligned}$$

We now obtain

$$\begin{aligned}
 \int \sin^6 x \, dx &= \frac{1}{8} x - \frac{3}{16} \sin(2x) + \frac{3}{8} \left(\frac{x}{2} + \frac{\sin(4x)}{8} \right) - \frac{1}{8} \left(\frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6} \right) + C \\
 &= \boxed{\frac{5}{16} x - \frac{1}{4} \sin(2x) + \frac{3}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C}.
 \end{aligned}$$

6. Use the identity $\cos^2 x = \frac{1+\cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx \\
 &= \int \left(\frac{1+\cos(2x)}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 + 2\cos(2x) + \cos^2(2x)) \, dx \\
 &= \frac{1}{4} \int dx + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\
 &= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4} \int \cos^2(2x) \, dx.
 \end{aligned}$$

We evaluate

$$\begin{aligned}
 \int \cos^2(2x) \, dx &= \int \frac{1+\cos(4x)}{2} \, dx \\
 &= \frac{1}{2} \int (1 + \cos(4x)) \, dx \\
 &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.
 \end{aligned}$$

We now obtain

$$\begin{aligned}
 \int \cos^4 x \, dx &= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4} \left(\frac{x}{2} + \frac{\sin(4x)}{8} \right) + C \\
 &= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}.
 \end{aligned}$$

7. Use the identity $\sin^2(\pi x) = \frac{1-\cos(2\pi x)}{2}$ and obtain

$$\begin{aligned}
 \int \sin^2(\pi x) \, dx &= \int \frac{1-\cos(2\pi x)}{2} \, dx \\
 &= \frac{1}{2} \int (1 - \cos(2\pi x)) \, dx \\
 &= \frac{1}{2} \left(x - \frac{1}{2\pi} \sin(2\pi x) \right) + C \\
 &= \boxed{\frac{x}{2} - \frac{1}{4\pi} \sin(2\pi x) + C}.
 \end{aligned}$$

8. Use the identity $\cos^2(2x) = \frac{1+\cos(4x)}{2}$ and obtain

$$\begin{aligned}
 \int \cos^4(2x) \, dx &= \int (\cos^2(2x))^2 \, dx \\
 &= \int \left(\frac{1+\cos(4x)}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 + 2\cos(4x) + \cos^2(4x)) \, dx \\
 &= \frac{1}{4} \int dx + \frac{1}{2} \int \cos(4x) \, dx + \frac{1}{4} \int \cos^2(4x) \, dx \\
 &= \frac{1}{4}x + \frac{1}{8}\sin(4x) + \frac{1}{4} \int \cos^2(4x) \, dx.
 \end{aligned}$$

We evaluate

$$\begin{aligned}\int \cos^2(4x) dx &= \int \frac{1 + \cos(8x)}{2} dx \\ &= \frac{1}{2} \int (1 + \cos(8x)) dx \\ &= \frac{x}{2} + \frac{\sin(8x)}{16} + C.\end{aligned}$$

We now obtain

$$\begin{aligned}\int \cos^4(2x) dx &= \frac{1}{4}x + \frac{1}{8}\sin(4x) + \frac{1}{4}\left(\frac{x}{2} + \frac{\sin(8x)}{16}\right) + C \\ &= \boxed{\frac{3}{8}x + \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + C}.\end{aligned}$$

9. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int_0^\pi \cos^5 x dx &= \int_0^\pi \cos^4 x \cos x dx \\ &= \int_0^\pi (\cos^2 x)^2 \cos x dx \\ &= \int_0^\pi (1 - \sin^2 x)^2 \cos x dx.\end{aligned}$$

Let $u = \sin x$, then $du = \cos x dx$. The lower limit of integration is $u = \sin 0 = 0$ and the upper limit of integration is $u = \sin \pi = 0$. We substitute and obtain

$$\int_0^\pi \cos^5 x dx = \int_0^0 (1 - u)^2 du = \boxed{0}.$$

10. Since $\sin^3 x$ is an odd function on a symmetric interval $[-\frac{\pi}{3}, \frac{\pi}{3}]$ about 0, we have

$$\int_{-\pi/3}^{\pi/3} \sin^3 x dx = \boxed{0}.$$

11. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x dx$, so $\sin x dx = -du$. We substitute and obtain

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int (1 - u^2)(u^2)(-du) \\ &= \int (u^4 - u^2) du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}.\end{aligned}$$

12. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x \cos^2 x \cos x \, dx \\ &= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx.\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}\int \sin^4 x \cos^3 x \, dx &= \int u^4 (1 - u^2) \, du \\ &= \int (u^4 - u^6) \, du \\ &= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C \\ &= \boxed{\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C}.\end{aligned}$$

13. Use the identities $\cos^2 x = \frac{1+\cos(2x)}{2}$ and $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \frac{1 - \cos(2x)}{2} \frac{1 + \cos(2x)}{2} \, dx \\ &= \frac{1}{4} \int (1 - \cos^2(2x)) \, dx \\ &= \frac{1}{4} x - \frac{1}{4} \int \cos^2(2x) \, dx.\end{aligned}$$

We evaluate

$$\begin{aligned}\int \cos^2(2x) \, dx &= \int \frac{1 + \cos(4x)}{2} \, dx \\ &= \frac{1}{2} \int (1 + \cos(4x)) \, dx \\ &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.\end{aligned}$$

We now obtain

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \frac{1}{4} x - \frac{1}{4} \left(\frac{x}{2} + \frac{\sin(4x)}{8} \right) + C \\ &= \boxed{\frac{1}{8} x - \frac{1}{32} \sin(4x) + C}.\end{aligned}$$

14. Use the identities $\cos^2 x = \frac{1+\cos(2x)}{2}$ and $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}\int \sin^4 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \, dx \\ &= \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \frac{1 + \cos(2x)}{2} \, dx \\ &= \frac{1}{8} \int (1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)) \, dx \\ &= \frac{x}{8} - \frac{\sin(2x)}{16} - \frac{1}{8} \int \cos^2(2x) \, dx + \frac{1}{8} \int \cos^3(2x) \, dx.\end{aligned}$$

We evaluate

$$\begin{aligned}\int \cos^2(2x) dx &= \int \frac{1 + \cos(4x)}{2} dx \\ &= \frac{1}{2} \int (1 + \cos(4x)) dx \\ &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.\end{aligned}$$

And also

$$\begin{aligned}\int \cos^3(2x) dx &= \int \cos^2(2x) \cos(2x) dx \\ &= \int (1 - \sin^2(2x)) \cos(2x) dx.\end{aligned}$$

Let $u = \sin(2x)$, then $du = 2 \cos(2x) dx$, so $\cos(2x) dx = \frac{du}{2}$. We substitute and obtain

$$\begin{aligned}\int \cos^3(2x) dx &= \int (1 - u^2) \frac{du}{2} \\ &= \frac{1}{2} \left(u - \frac{u^3}{3} \right) + C \\ &= \frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6} + C.\end{aligned}$$

We now obtain

$$\begin{aligned}\int \sin^4 x \cos^2 x dx &= \frac{x}{8} - \frac{\sin(2x)}{16} - \frac{1}{8} \left(\frac{x}{2} + \frac{\sin(4x)}{8} \right) + \frac{1}{8} \left(\frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6} \right) + C \\ &= \boxed{\frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C}.\end{aligned}$$

15. Let $u = \cos x$, then $du = -\sin x dx$, so $\sin x dx = -du$. We substitute and obtain

$$\begin{aligned}\int \sin x \cos^{1/3} x dx &= \int u^{1/3}(-du) \\ &= -\int u^{1/3} du \\ &= -\frac{3}{4}u^{4/3} + C \\ &= \boxed{-\frac{3}{4}\cos^{4/3} x + C}.\end{aligned}$$

16. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int \cos^3 x \sin^{1/2} x dx &= \int \cos^2 x \sin^{1/2} x \cos x dx \\ &= \int (1 - \sin^2 x) \sin^{1/2} x \cos x dx.\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}\int \cos^3 x \sin^{1/2} x \, dx &= \int (1 - u^2) u^{1/2} \, du \\ &= \int (u^{1/2} - u^{5/2}) \, du \\ &= \frac{2}{3} u^{3/2} - \frac{2}{7} u^{7/2} + C \\ &= \boxed{\frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x + C}.\end{aligned}$$

17. Factor out $\cos\left(\frac{x}{2}\right)$ and use the identity $\cos^2\left(\frac{x}{2}\right) = 1 - \sin^2\left(\frac{x}{2}\right)$.

$$\begin{aligned}\int \sin^2\left(\frac{x}{2}\right) \cos^3\left(\frac{x}{2}\right) \, dx &= \int \cos^2\left(\frac{x}{2}\right) \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \, dx \\ &= \int \left(1 - \sin^2\left(\frac{x}{2}\right)\right) \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \, dx.\end{aligned}$$

Let $u = \sin\left(\frac{x}{2}\right)$, then $du = \frac{1}{2} \cos\left(\frac{x}{2}\right) \, dx$, so $\cos\left(\frac{x}{2}\right) \, dx = 2 \, du$. We substitute and obtain

$$\begin{aligned}\int \sin^2\left(\frac{x}{2}\right) \cos^3\left(\frac{x}{2}\right) \, dx &= \int (1 - u^2)(u^2)(2 \, du) \\ &= 2 \int (u^2 - u^4) \, du \\ &= 2\left(\frac{1}{3} u^3 - \frac{1}{5} u^5\right) + C \\ &= \boxed{\frac{2}{3} \sin^3\left(\frac{x}{2}\right) - \frac{2}{5} \sin^5\left(\frac{x}{2}\right) + C}.\end{aligned}$$

18. Factor out $\cos(4x)$ and use the identity $\cos^2(4x) = 1 - \sin^2(4x)$.

$$\begin{aligned}\int \sin^3(4x) \cos^3(4x) \, dx &= \int \cos^2(4x) \sin^3(4x) \cos(4x) \, dx \\ &= \int (1 - \sin^2(4x)) \sin^3(4x) \cos(4x) \, dx.\end{aligned}$$

Let $u = \sin(4x)$, then $du = 4 \cos(4x) \, dx$, so $\cos(4x) \, dx = \frac{du}{4}$. We substitute and obtain

$$\begin{aligned}\int \sin^3(4x) \cos^3(4x) \, dx &= \int (1 - u^2)(u^3) \frac{du}{4} \\ &= \frac{1}{4} \int (u^3 - u^5) \, du \\ &= \frac{1}{4} \left(\frac{1}{4} u^4 - \frac{1}{6} u^6\right) + C \\ &= \boxed{\frac{1}{16} \sin^4(4x) - \frac{1}{24} \sin^6(4x) + C}.\end{aligned}$$

19. Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned}\int \tan^3 x \sec^2 x \, dx &= \int u^3 \, du \\ &= \frac{1}{4} u^4 + C \\ &= \boxed{\frac{1}{4} \tan^4 x + C}.\end{aligned}$$

20. Factor out $\sec x \tan x$. Then let $u = \sec x$, so $du = \sec x \tan x \, dx$. We substitute and obtain

$$\begin{aligned}\int \tan x \sec^5 x \, dx &= \int \sec^4 x \sec x \tan x \, dx \\ &= \int u^4 \, du \\ &= \frac{1}{5} u^5 + C \\ &= \boxed{\frac{1}{5} \sec^5 x + C}.\end{aligned}$$

21. Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned}\int \tan^2 x \sec^2 x \, dx &= \int u^2 \, du \\ &= \frac{1}{3} u^3 + C \\ &= \boxed{\frac{1}{3} \tan^3 x + C}.\end{aligned}$$

22. Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned}\int \tan^5 x \sec^2 x \, dx &= \int u^5 \, du \\ &= \frac{1}{6} u^6 + C \\ &= \boxed{\frac{1}{6} \tan^6 x + C}.\end{aligned}$$

23. We first use the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\int \tan^2 x \sec^3 x \, dx &= \int (\sec^2 x - 1) \sec^3 x \, dx \\ &= \int \sec^5 x \, dx - \int \sec^3 x \, dx.\end{aligned}$$

We use integration by parts for $\int \sec^5 x \, dx$ with $u = \sec^3 x$ and $dv = \sec^2 x \, dx$. Then $du = 3 \sec^3 x \tan x \, dx$ and $v = \tan x$. We obtain

$$\begin{aligned}\int \sec^5 x \, dx &= \sec^3 x \tan x - 3 \int (\sec^3 x \tan x) \tan x \, dx \\ &= \sec^3 x \tan x - 3 \int \sec^3 x \tan^2 x \, dx \\ &= \sec^3 x \tan x - 3 \int \sec^3 x (\sec^2 x - 1) \, dx \\ &= \sec^3 x \tan x - 3 \int \sec^5 x \, dx + 3 \int \sec^3 x \, dx.\end{aligned}$$

We add $3 \int \sec^5 x \, dx$ to both sides, and divide by 4 and obtain

$$\begin{aligned}4 \int \sec^5 x \, dx &= \sec^3 x \tan x + 3 \int \sec^3 x \, dx \\ \int \sec^5 x \, dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx.\end{aligned}$$

We now have

$$\begin{aligned}
 \int \tan^2 x \sec^3 x \, dx &= \int \sec^5 x \, dx - \int \sec^3 x \, dx \\
 &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx - \int \sec^3 x \, dx \\
 &= \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \int \sec^3 x \, dx.
 \end{aligned}$$

From Example 7 we have $\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$, so

$$\begin{aligned}
 \int \tan^2 x \sec^3 x \, dx &= \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \left(\frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] \right) + C \\
 &= \boxed{\frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x - \frac{1}{8} \ln |\sec x + \tan x| + C}.
 \end{aligned}$$

24. We first use the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}
 \int \tan^4 x \sec x \, dx &= \int (\tan^2 x)^2 \sec x \, dx \\
 &= \int (\sec^2 x - 1)^2 \sec x \, dx \\
 &= \int (\sec^4 x - 2 \sec^2 x + 1) \sec x \, dx \\
 &= \int \sec^5 x \, dx - 2 \int \sec^3 x \, dx + \int \sec x \, dx \\
 &= \int \sec^5 x \, dx - 2 \int \sec^3 x \, dx + \ln |\sec x + \tan x|
 \end{aligned}$$

We use integration by parts for $\int \sec^5 x \, dx$ with $u = \sec^3 x$ and $dv = \sec^2 x \, dx$. Then $du = 3 \sec^3 x \tan x \, dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
 \int \sec^5 x \, dx &= \sec^3 x \tan x - 3 \int (\sec^3 x \tan x) \tan x \, dx \\
 &= \sec^3 x \tan x - 3 \int \sec^3 x \tan^2 x \, dx \\
 &= \sec^3 x \tan x - 3 \int \sec^3 x (\sec^2 x - 1) \, dx \\
 &= \sec^3 x \tan x - 3 \int \sec^5 x \, dx + 3 \int \sec^3 x \, dx.
 \end{aligned}$$

We add $3 \int \sec^5 x \, dx$ to both sides, and divide by 4 and obtain

$$\begin{aligned}
 4 \int \sec^5 x \, dx &= \sec^3 x \tan x + 3 \int \sec^3 x \, dx \\
 \int \sec^5 x \, dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx.
 \end{aligned}$$

We now have

$$\begin{aligned}
 \int \tan^4 x \sec x \, dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx - 2 \int \sec^3 x \, dx + \ln |\sec x + \tan x| \\
 &= \frac{1}{4} \sec^3 x \tan x - \frac{5}{4} \int \sec^3 x \, dx + \ln |\sec x + \tan x|.
 \end{aligned}$$

From Example 7 we have $\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$, so

$$\begin{aligned} \int \tan^4 x \sec x \, dx &= \frac{1}{4} \sec^3 x \tan x - \frac{5}{4} \left(\frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] \right) \\ &\quad + \ln |\sec x + \tan x| + C \\ &= \boxed{\frac{1}{4} \sec^3 x \tan x - \frac{5}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C}. \end{aligned}$$

25. Factor out $\csc x \cot x$, and use the identity $\cot^2 x = \csc^2 x - 1$.

$$\begin{aligned} \int \cot^3 x \csc x \, dx &= \int \cot^2 x \csc x \cot x \, dx \\ &= \int (\csc^2 x - 1) \csc x \cot x \, dx. \end{aligned}$$

Let $u = \csc x$, then $du = -\csc x \cot x \, dx$, so $\csc x \cot x \, dx = -du$. We substitute and obtain

$$\begin{aligned} \int \cot^3 x \csc x \, dx &= \int (\csc^2 x - 1) \csc x \cot x \, dx \\ &= \int (u^2 - 1)(-du) \\ &= \int (1 - u^2) \, du \\ &= u - \frac{1}{3} u^3 + C \\ &= \boxed{\csc x - \frac{1}{3} \csc^3 x + C}. \end{aligned}$$

26. Let $u = \cot x$, then $du = -\csc^2 x \, dx$, so $\csc^2 x \, dx = -du$. We substitute and obtain

$$\begin{aligned} \int \cot^3 x \csc^2 x \, dx &= \int u^3 (-du) \\ &= -\int u^3 \, du \\ &= -\frac{1}{4} u^4 + C \\ &= \boxed{-\frac{1}{4} \cot^4 x + C}. \end{aligned}$$

27. We use the identity $\sin A \cos B = \frac{1}{2} (\sin(A+B) + \sin(A-B))$ to obtain

$$\begin{aligned} \int \sin(3x) \cos x \, dx &= \frac{1}{2} \int (\sin(4x) + \sin(2x)) \, dx \\ &= \frac{1}{2} \left(-\frac{1}{4} \cos(4x) - \frac{1}{2} \cos(2x) \right) + C \\ &= \boxed{-\frac{1}{8} \cos(4x) - \frac{1}{4} \cos(2x) + C}. \end{aligned}$$

28. We use the identity $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ to obtain

$$\begin{aligned}\int \sin x \cos(3x) dx &= \frac{1}{2} \int (\sin(4x) + \sin(-2x)) dx \\ &= \frac{1}{2} \int (\sin(4x) - \sin(2x)) dx \\ &= \frac{1}{2} \left(-\frac{1}{4} \cos(4x) - \left(-\frac{1}{2} \cos(2x) \right) \right) + C \\ &= \boxed{-\frac{1}{8} \cos(4x) + \frac{1}{4} \cos(2x) + C}.\end{aligned}$$

29. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to obtain

$$\begin{aligned}\int \cos x \cos(3x) dx &= \frac{1}{2} \int (\cos(4x) + \cos(-2x)) dx \\ &= \frac{1}{2} \int (\cos(4x) + \cos(2x)) dx \\ &= \frac{1}{2} \left(\frac{1}{4} \sin(4x) + \frac{1}{2} \sin(2x) \right) + C \\ &= \boxed{\frac{1}{8} \sin(4x) + \frac{1}{4} \sin(2x) + C}.\end{aligned}$$

30. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to obtain

$$\begin{aligned}\int \cos(2x) \cos x dx &= \frac{1}{2} \int (\cos(3x) + \cos(x)) dx \\ &= \frac{1}{2} \left(\frac{1}{3} \sin(3x) + \sin x \right) + C \\ &= \boxed{\frac{1}{6} \sin(3x) + \frac{1}{2} \sin x + C}.\end{aligned}$$

31. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}\int \sin(2x) \sin(4x) dx &= \frac{1}{2} \int (\cos(-2x) - \cos(6x)) dx \\ &= \frac{1}{2} \int (\cos(2x) - \cos(6x)) dx \\ &= \frac{1}{2} \left(\frac{1}{2} \sin(2x) - \frac{1}{6} \sin(6x) \right) + C \\ &= \boxed{\frac{1}{4} \sin(2x) - \frac{1}{12} \sin(6x) + C}.\end{aligned}$$

32. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}\int \sin(3x) \sin(x) dx &= \frac{1}{2} \int (\cos(3x-x) - \cos(3x+x)) dx \\ &= \frac{1}{2} \int (\cos(2x) - \cos(4x)) dx \\ &= \frac{1}{2} \left(\frac{1}{2} \sin(2x) - \frac{1}{4} \sin(4x) \right) + C \\ &= \boxed{\frac{1}{4} \sin(2x) - \frac{1}{8} \sin(4x) + C}.\end{aligned}$$

33. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$ to obtain

$$\begin{aligned}
 \int_0^{\pi/2} \sin(2x) \sin x \, dx &= \frac{1}{2} \int_0^{\pi/2} (\cos(x) - \cos(3x)) \, dx \\
 &= \frac{1}{2} \left[\sin x - \frac{1}{3} \sin(3x) \right]_0^{\pi/2} \\
 &= \frac{1}{2} \left[\left(\sin \frac{\pi}{2} - \frac{1}{3} \sin \left(\frac{3\pi}{2} \right) \right) - \left(\sin 0 - \frac{1}{3} \sin(3(0)) \right) \right] \\
 &= \frac{1}{2} \left(\frac{4}{3} - 0 \right) \\
 &= \boxed{\frac{2}{3}}.
 \end{aligned}$$

34. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$ to obtain

$$\begin{aligned}
 \int_0^{\pi} \cos x \cos(4x) \, dx &= \frac{1}{2} \int_0^{\pi} (\cos(5x) + \cos(-3x)) \, dx \\
 &= \frac{1}{2} \int_0^{\pi} (\cos(5x) + \cos(3x)) \, dx \\
 &= \frac{1}{2} \left[\frac{1}{5} \sin(5x) + \frac{1}{3} \sin(3x) \right]_0^{\pi} \\
 &= \frac{1}{2} \left[\frac{1}{5} \sin(5\pi) + \frac{1}{3} \sin(3\pi) - \left(\frac{1}{5} \sin(5(0)) + \frac{1}{3} \sin(3(0)) \right) \right] \\
 &= \boxed{0}.
 \end{aligned}$$

35. Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}
 \int \sin^2 x \cos x \, dx &= \int u^2 \, du \\
 &= \frac{1}{3} u^3 + C \\
 &= \boxed{\frac{1}{3} \sin^3 x + C}.
 \end{aligned}$$

36. Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}
 \int \sin^3 x \cos x \, dx &= \int u^3 \, du \\
 &= \frac{1}{4} u^4 + C \\
 &= \boxed{\frac{1}{4} \sin^4 x + C}.
 \end{aligned}$$

37. We rewrite the integral, and obtain

$$\begin{aligned}
 \int \frac{\sin x}{\cos^2 x} \, dx &= \int \frac{1}{\cos x} \frac{\sin x}{\cos x} \, dx \\
 &= \int \sec x \tan x \, dx \\
 &= \boxed{\sec x + C}.
 \end{aligned}$$

38. We rewrite the integral, and obtain

$$\begin{aligned}\int \frac{\cos x}{\sin^4 x} dx &= \int \frac{1}{\sin^3 x} \frac{\cos x}{\sin x} dx \\ &= \int \csc^3 x \cot x dx \\ &= \int \csc^2 x (\csc x \cot x) dx.\end{aligned}$$

Let $u = \csc x$, then $du = -\csc x \cot x dx$, so $\csc x \cot x dx = -du$. We substitute and obtain

$$\begin{aligned}\int \frac{\cos x}{\sin^4 x} dx &= \int u^2 (-du) \\ &= -\int u^2 du \\ &= -\frac{1}{3}u^3 + C \\ &= \boxed{-\frac{1}{3}\csc^3 x + C}.\end{aligned}$$

39. Factor out $\cos(3x)$ and use the identity $\cos^2(3x) = 1 - \sin^2(3x)$.

$$\begin{aligned}\int \cos^3(3x) dx &= \int \cos^2(3x) \cos(3x) dx \\ &= \int (1 - \sin^2(3x)) \cos(3x) dx.\end{aligned}$$

Let $u = \sin(3x)$, then $du = 3 \cos(3x) dx$, so $\cos(3x) dx = \frac{du}{3}$. We substitute and obtain

$$\begin{aligned}\int \cos^3(3x) dx &= \int (1 - u^2) \frac{du}{3} \\ &= \frac{1}{3} \int (1 - u^2) du \\ &= \frac{1}{3} \left(u - \frac{1}{3}u^3 \right) + C \\ &= \boxed{\frac{1}{3} \sin(3x) - \frac{1}{9} \sin^3(3x) + C}.\end{aligned}$$

40. Factor out $\sin(3x)$ and use the identity $\sin^2(3x) = 1 - \cos^2(3x)$.

$$\begin{aligned}\int \sin^5(3x) dx &= \int \sin^4(3x) \sin(3x) dx \\ &= \int (\sin^2(3x))^2 \sin(3x) dx \\ &= \int (1 - \cos^2(3x))^2 \sin(3x) dx.\end{aligned}$$

Let $u = \cos(3x)$, then $du = -3 \sin(3x) dx$, so $\sin x dx = -\frac{du}{3}$. We substitute and obtain

$$\begin{aligned}\int \sin^5(3x) dx &= \int (1 - u^2)^2 \left(-\frac{du}{3}\right) \\&= -\frac{1}{3} \int (u^4 - 2u^2 + 1) du \\&= -\frac{1}{3} \left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C \\&= \boxed{-\frac{1}{15} \cos^5(3x) + \frac{2}{9} \cos^3(3x) - \frac{1}{3} \cos(3x) + C}.\end{aligned}$$

41. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int_0^\pi \sin^3 x \cos^5 x dx &= \int_0^\pi \sin^2 x \cos^5 x \sin x dx \\&= \int_0^\pi (1 - \cos^2 x) \cos^5 x \sin x dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x dx$, so $\sin x dx = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos \pi = -1$. We substitute and obtain

$$\begin{aligned}\int_0^\pi \sin^3 x \cos^5 x dx &= \int_1^{-1} (1 - u^2) u^5 (-du) \\&= \int_{-1}^1 (u^5 - u^7) du \\&= \boxed{0},\end{aligned}$$

since we are integrating an odd function on a symmetric interval about 0.

42. Factor out $\sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x \cos^3 x dx &= \int_0^{\pi/2} \sin^2 x \cos^3 x \sin x dx \\&= \int_0^{\pi/2} (1 - \cos^2 x) \cos^3 x \sin x dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x dx$, so $\sin x dx = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos \frac{\pi}{2} = 0$. We substitute and obtain

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x \cos^3 x dx &= \int_1^0 (1 - u^2) u^3 (-du) \\&= \int_0^1 (u^3 - u^5) du \\&= \left[\frac{1}{4}u^4 - \frac{1}{6}u^6\right]_0^1 \\&= \left(\frac{1}{4}1^4 - \frac{1}{6}1^6\right) - \left(\frac{1}{4}0^4 - \frac{1}{6}0^6\right) \\&= \boxed{\frac{1}{12}}.\end{aligned}$$

43. Rewrite the integral, factor out $\sin x$, and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \, dx \\ &= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, dx \\ &= \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, dx.\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x \, dx$, so $\sin x \, dx = -du$. We substitute and obtain

$$\begin{aligned}\int \tan^3 x \, dx &= \int \frac{1 - u^2}{u^3} (-du) \\ &= \int \left(\frac{1}{u} - u^{-3} \right) du \\ &= \ln |u| - \frac{u^{-2}}{-2} + C \\ &= \boxed{\ln |\cos x| + \frac{1}{2} \sec^2 x + C}.\end{aligned}$$

44. Rewrite the integral, factor out $\cos x$, and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\begin{aligned}\int \cot^5 x \, dx &= \int \frac{\cos^5 x}{\sin^5 x} \, dx \\ &= \int \frac{\cos^4 x}{\sin^5 x} \cos x \, dx \\ &= \int \frac{(\cos^2 x)^2}{\sin^5 x} \cos x \, dx \\ &= \int \frac{(1 - \sin^2 x)^2}{\sin^5 x} \cos x \, dx\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}\int \cot^5 x \, dx &= \int \frac{(1 - u^2)^2}{u^5} \, du \\ &= \int \frac{1 - 2u^2 + u^4}{u^5} \, du \\ &= \int \left(u^{-5} - 2u^{-3} + \frac{1}{u} \right) du \\ &= -\frac{1}{4}u^{-4} + u^{-2} + \ln |u| + C \\ &= \boxed{-\frac{1}{4} \csc^4 x + \csc^2 x + \ln |\sin x| + C}.\end{aligned}$$

45. Factor out $\sec^2 x$ and use the identity $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned}\int \frac{\sec^6 x}{\tan^3 x} \, dx &= \int \frac{\sec^4 x}{\tan^3 x} \sec^2 x \, dx \\ &= \int \frac{(\sec^2 x)^2}{\tan^3 x} \sec^2 x \, dx \\ &= \int \frac{(1 + \tan^2 x)^2}{\tan^3 x} \sec^2 x \, dx.\end{aligned}$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned}
 \int \frac{\sec^6 x}{\tan^3 x} \, dx &= \int \frac{(1+u^2)^2}{u^3} \, du \\
 &= \int \frac{1+2u^2+u^4}{u^3} \, du \\
 &= \int \left(u^{-3} + \frac{2}{u} + u \right) \, du \\
 &= \frac{u^{-2}}{-2} + 2 \ln |u| + \frac{1}{2} u^2 + C \\
 &= \boxed{-\frac{1}{2} \cot^2 x + 2 \ln |\tan x| + \frac{1}{2} \tan^2 x + C}.
 \end{aligned}$$

46. Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned}
 \int \tan^{1/2} x \sec^2 x \, dx &= \int u^{1/2} \, du \\
 &= \frac{2}{3} u^{3/2} + C \\
 &= \boxed{\frac{2}{3} \tan^{3/2} x + C}.
 \end{aligned}$$

47. Let $u = \cot x$, then $du = -\csc^2 x \, dx$, so $\csc^2 x \, dx = -du$. We substitute and obtain

$$\begin{aligned}
 \int \csc^2 x \cot^5 x \, dx &= \int u^5 (-du) \\
 &= -\int u^5 \, du \\
 &= -\frac{1}{6} u^6 + C \\
 &= \boxed{-\frac{1}{6} \cot^6 x + C}.
 \end{aligned}$$

48. Let $u = \cot x$, then $du = -\csc^2 x \, dx$, so $\csc^2 x \, dx = -du$. We substitute and obtain

$$\begin{aligned}
 \int \cot x \csc^2 x \, dx &= \int u (-du) \\
 &= -\int u \, du \\
 &= -\frac{1}{2} u^2 + C \\
 &= \boxed{-\frac{1}{2} \cot^2 x + C}.
 \end{aligned}$$

49. Factor out $\csc(2x) \cot(2x)$ to obtain

$$\int \cot(2x) \csc^4(2x) \, dx = \int \csc^3(2x) \csc(2x) \cot(2x) \, dx.$$

Let $u = \csc(2x)$, then $du = -2 \csc(2x) \cot(2x) \, dx$, so $\csc x \cot x \, dx = -\frac{du}{2}$. We substitute

and obtain

$$\begin{aligned}
 \int \cot(2x) \csc^4(2x) dx &= \int u^3 \left(-\frac{du}{2} \right) \\
 &= -\frac{1}{2} \int u^3 du \\
 &= -\frac{1}{2} \left(\frac{1}{4} u^4 \right) + C \\
 &= \boxed{-\frac{1}{8} \csc^4(2x) + C}.
 \end{aligned}$$

50. We first use the identity $\cot^2(2x) = \csc^2(2x) - 1$.

$$\begin{aligned}
 \int \cot^2(2x) \csc^3(2x) dx &= \int (\csc^2(2x) - 1) \csc^3(2x) dx \\
 &= \int \csc^5(2x) dx - \int \csc^3(2x) dx.
 \end{aligned}$$

We use integration by parts for $\int \csc^5(2x) dx$ with $u = \csc^3(2x)$ and $dv = \csc^2(2x) dx$. Then $du = -6 \csc^3(2x) \cot(2x) dx$ and $v = -\frac{1}{2} \cot(2x)$. We obtain

$$\begin{aligned}
 \int \csc^5(2x) dx &= -\frac{1}{2} \csc^3(2x) \cot(2x) - 3 \int (\csc^3(2x) \cot(2x)) \cot(2x) dx \\
 &= -\frac{1}{2} \csc^3(2x) \cot(2x) - 3 \int \csc^3(2x) \cot^2(2x) dx \\
 &= -\frac{1}{2} \csc^3(2x) \cot(2x) - 3 \int \csc^3(2x) (\csc^2(2x) - 1) dx \\
 &= -\frac{1}{2} \csc^3(2x) \cot(2x) - 3 \int \csc^5(2x) dx + 3 \int \csc^3(2x) dx.
 \end{aligned}$$

We add $3 \int \csc^5(2x) dx$ to both sides, and divide by 4 and obtain

$$\begin{aligned}
 4 \int \csc^5(2x) dx &= -\frac{1}{2} \csc^3(2x) \cot(2x) + 3 \int \csc^3(2x) dx \\
 \int \csc^5(2x) dx &= -\frac{1}{8} \csc^3(2x) \cot(2x) + \frac{3}{4} \int \csc^3(2x) dx.
 \end{aligned}$$

We now have

$$\begin{aligned}
 \int \cot^2(2x) \csc^3(2x) dx &= \int \csc^5(2x) dx - \int \csc^3(2x) dx \\
 &= -\frac{1}{8} \csc^3(2x) \cot(2x) + \frac{3}{4} \int \csc^3(2x) dx - \int \csc^3(2x) dx \\
 &= -\frac{1}{8} \csc^3(2x) \cot(2x) - \frac{1}{4} \int \csc^3(2x) dx.
 \end{aligned}$$

We use integration by parts for $\int \csc^3(2x) dx$ with $u = \csc(2x)$ and $dv = \csc^2(2x) dx$.

Then $du = -2 \csc(2x) \cot(2x) dx$ and $v = -\frac{1}{2} \cot(2x)$. We obtain

$$\begin{aligned}
 \int \csc^3(2x) dx &= -\frac{1}{2} \csc(2x) \cot(2x) - \int (\csc(2x) \cot(2x)) \cot(2x) dx \\
 &= -\frac{1}{2} \csc(2x) \cot(2x) - \int \csc(2x) \cot^2(2x) dx \\
 &= -\frac{1}{2} \csc(2x) \cot(2x) - \int \csc(2x) (\csc^2(2x) - 1) dx \\
 &= -\frac{1}{2} \csc(2x) \cot(2x) - \int \csc^3(2x) dx + \int \csc(2x) dx \\
 &= -\frac{1}{2} \csc(2x) \cot(2x) - \int \csc^3(2x) dx - \frac{1}{2} \ln |\csc(2x) + \cot(2x)|.
 \end{aligned}$$

We add $\int \csc^3(2x) dx$ to both sides, and divide by 2 and obtain

$$\begin{aligned}
 2 \int \csc^3(2x) dx &= -\frac{1}{2} \csc(2x) \cot(2x) - \frac{1}{2} \ln |\csc(2x) + \cot(2x)| \\
 \int \csc^3(2x) dx &= -\frac{1}{4} \csc(2x) \cot(2x) - \frac{1}{4} \ln |\csc(2x) + \cot(2x)| + C.
 \end{aligned}$$

We now have

$$\begin{aligned}
 \int \cot^2(2x) \csc^3(2x) dx &= \frac{-\csc^3(2x) \cot(2x)}{8} - \frac{1}{4} \left(\frac{-\csc(2x) \cot(2x)}{4} - \frac{\ln |\csc(2x) + \cot(2x)|}{4} \right) + C \\
 &= \boxed{-\frac{\csc^3(2x) \cot(2x)}{8} + \frac{\csc(2x) \cot(2x)}{16} + \frac{\ln |\csc(2x) + \cot(2x)|}{16} + C}.
 \end{aligned}$$

51. We rewrite the integral using the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}
 \int_0^{\pi/4} \tan^4 x \sec^3 x dx &= \int_0^{\pi/4} (\tan^2 x)^2 \sec^3 x dx \\
 &= \int_0^{\pi/4} (\sec^2 x - 1)^2 \sec^3 x dx \\
 &= \int_0^{\pi/4} (\sec^4 x - 2 \sec^2 x + 1) \sec^3 x dx \\
 &= \int_0^{\pi/4} \sec^7 x dx - 2 \int_0^{\pi/4} \sec^5 x dx + \int_0^{\pi/4} \sec^3 x dx.
 \end{aligned}$$

From Example 7 we have $\int \sec^3 x dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$, so

$$\begin{aligned}
 \int_0^{\pi/4} \sec^3 x dx &= \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|]_0^{\pi/4} \\
 &= \frac{1}{2} \left[\sec \frac{\pi}{4} \tan \frac{\pi}{4} + \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - (\sec 0 \tan 0 + \ln |\sec 0 + \tan 0|) \right] \\
 &= \frac{1}{2} \ln (\sqrt{2} + 1) + \frac{\sqrt{2}}{2}.
 \end{aligned}$$

We use integration by parts for $\int_0^{\pi/4} \sec^5 x dx$ with $u = \sec^3 x$ and $dv = \sec^2 x dx$. Then

$du = 3 \sec^3 x \tan x \, dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
\int_0^{\pi/4} \sec^5 x \, dx &= [\sec^3 x \tan x]_0^{\pi/4} - 3 \int_0^{\pi/4} (\sec^3 x \tan x) \tan x \, dx \\
&= \sec^3 \frac{\pi}{4} \tan \frac{\pi}{4} - (\sec^3 0 \tan 0) - 3 \int_0^{\pi/4} \sec^3 x \tan^2 x \, dx \\
&= 2\sqrt{2} - 3 \int_0^{\pi/4} \sec^3 x (\sec^2 x - 1) \, dx \\
&= 2\sqrt{2} - 3 \left(\int_0^{\pi/4} \sec^5 x \, dx - \int_0^{\pi/4} \sec^3 x \, dx \right) \\
&= 2\sqrt{2} - 3 \int_0^{\pi/4} \sec^5 x \, dx + 3 \left(\frac{1}{2} \ln(\sqrt{2} + 1) + \frac{\sqrt{2}}{2} \right) \\
&= \frac{3}{2} \ln(\sqrt{2} + 1) + \frac{7}{2} \sqrt{2} - 3 \int_0^{\pi/4} \sec^5 x \, dx.
\end{aligned}$$

We add $3 \int_0^{\pi/4} \sec^5 x \, dx$ to each side and divide by 4 to obtain

$$\begin{aligned}
4 \int_0^{\pi/4} \sec^5 x \, dx &= \frac{3}{2} \ln(\sqrt{2} + 1) + \frac{7}{2} \sqrt{2} \\
\int_0^{\pi/4} \sec^5 x \, dx &= \frac{3}{8} \ln(\sqrt{2} + 1) + \frac{7}{8} \sqrt{2}.
\end{aligned}$$

We use integration by parts for $\int_0^{\pi/4} \sec^7 x \, dx$ with $u = \sec^5 x$ and $dv = \sec^2 x \, dx$. Then $du = 5 \sec^5 x \tan x \, dx$ and $v = \tan x$. We obtain

$$\begin{aligned}
\int_0^{\pi/4} \sec^7 x \, dx &= [\sec^5 x \tan x]_0^{\pi/4} - 5 \int_0^{\pi/4} (\sec^5 x \tan x) \tan x \, dx \\
&= \sec^5 \frac{\pi}{4} \tan \frac{\pi}{4} - (\sec^5 0 \tan 0) - 5 \int_0^{\pi/4} \sec^5 x \tan^2 x \, dx \\
&= 4\sqrt{2} - 5 \int_0^{\pi/4} \sec^5 x (\sec^2 x - 1) \, dx \\
&= 4\sqrt{2} - 5 \left(\int_0^{\pi/4} \sec^7 x \, dx - \int_0^{\pi/4} \sec^5 x \, dx \right) \\
&= 4\sqrt{2} - 5 \int_0^{\pi/4} \sec^7 x \, dx + 5 \left(\frac{3}{8} \ln(\sqrt{2} + 1) + \frac{7}{8} \sqrt{2} \right) \\
&= \frac{15}{8} \ln(\sqrt{2} + 1) + \frac{67}{8} \sqrt{2} - 5 \int_0^{\pi/4} \sec^7 x \, dx.
\end{aligned}$$

We add $5 \int_0^{\pi/4} \sec^7 x \, dx$ to each side and divide by 6 to obtain

$$\begin{aligned}
6 \int_0^{\pi/4} \sec^7 x \, dx &= \frac{15}{8} \ln(\sqrt{2} + 1) + \frac{67}{8} \sqrt{2} \\
\int_0^{\pi/4} \sec^7 x \, dx &= \frac{5}{16} \ln(\sqrt{2} + 1) + \frac{67}{48} \sqrt{2}.
\end{aligned}$$

We now obtain

$$\begin{aligned}
 \int_0^{\pi/4} \tan^4 x \sec^3 x \, dx &= \int_0^{\pi/4} \sec^7 x \, dx - 2 \int_0^{\pi/4} \sec^5 x \, dx + \int_0^{\pi/4} \sec^3 x \, dx \\
 &= \left(\frac{5}{16} \ln(\sqrt{2} + 1) + \frac{67}{48} \sqrt{2} \right) - 2 \left(\frac{3}{8} \ln(\sqrt{2} + 1) + \frac{7}{8} \sqrt{2} \right) \\
 &\quad + \left(\frac{1}{2} \ln(\sqrt{2} + 1) + \frac{\sqrt{2}}{2} \right) \\
 &= \boxed{\frac{7}{48} \sqrt{2} + \frac{1}{16} \ln(\sqrt{2} + 1)}.
 \end{aligned}$$

52. We rewrite the integral using the identity $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}
 \int_0^{\pi/4} \tan^2 x \sec x \, dx &= \int_0^{\pi/4} (\sec^2 x - 1) \sec x \, dx \\
 &= \int_0^{\pi/4} \sec^3 x \, dx - \int_0^{\pi/4} \sec x \, dx \\
 &= \int_0^{\pi/4} \sec^3 x \, dx - [\ln |\sec x + \tan x|]_0^{\pi/4} \\
 &= \int_0^{\pi/4} \sec^3 x \, dx - \left[\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln |\sec 0 + \tan 0| \right] \\
 &= \int_0^{\pi/4} \sec^3 x \, dx - \ln(\sqrt{2} + 1)
 \end{aligned}$$

From Example 7 we have $\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$, so

$$\begin{aligned}
 \int_0^{\pi/4} \sec^3 x \, dx &= \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|]_0^{\pi/4} \\
 &= \frac{1}{2} \left[\sec \frac{\pi}{4} \tan \frac{\pi}{4} + \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - (\sec 0 \tan 0 + \ln |\sec 0 + \tan 0|) \right] \\
 &= \frac{1}{2} \ln(\sqrt{2} + 1) + \frac{\sqrt{2}}{2}.
 \end{aligned}$$

We now obtain

$$\begin{aligned}
 \int_0^{\pi/4} \tan^2 x \sec x \, dx &= \int_0^{\pi/4} \sec^3 x \, dx - \ln(\sqrt{2} + 1) \\
 &= \left(\frac{1}{2} \ln(\sqrt{2} + 1) + \frac{\sqrt{2}}{2} \right) - \ln(\sqrt{2} + 1) \\
 &= \frac{1}{2} \sqrt{2} - \frac{1}{2} \ln(\sqrt{2} + 1).
 \end{aligned}$$

53. We use the identity $\sin A \cos B = \frac{1}{2} (\sin(A + B) + \sin(A - B))$ to obtain

$$\begin{aligned}
 \int \sin\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) \, dx &= \frac{1}{2} \int (\sin(2x) + \sin(-x)) \, dx \\
 &= \frac{1}{2} \int (\sin(2x) - \sin(x)) \, dx \\
 &= \frac{1}{2} \left(-\frac{1}{2} \cos(2x) - (-\cos(x)) \right) + C \\
 &= \boxed{-\frac{1}{4} \cos(2x) + \frac{1}{2} \cos(x) + C}.
 \end{aligned}$$

54. We use the identity $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ to obtain

$$\begin{aligned}
 \int \cos(-x) \sin(4x) dx &= \int \sin(4x) \cos(-x) dx = \\
 &= \frac{1}{2} \int (\sin(3x) + \sin(5x)) dx \\
 &= \frac{1}{2} \left(-\frac{1}{3} \cos(3x) - \frac{1}{5} \cos(5x) \right) + C \\
 &= \boxed{-\frac{1}{6} \cos(3x) - \frac{1}{10} \cos(5x) + C}.
 \end{aligned}$$

55. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ to obtain

$$\begin{aligned}
 \int \sin\left(\frac{x}{2}\right) \sin\left(\frac{3x}{2}\right) dx &= \frac{1}{2} \int (\cos(-x) - \cos(2x)) dx \\
 &= \frac{1}{2} \int (\cos(x) - \cos(2x)) dx \\
 &= \frac{1}{2} \left(\sin x - \frac{1}{2} \sin(2x) \right) + C \\
 &= \boxed{\frac{1}{2} \sin x - \frac{1}{4} \sin(2x) + C}.
 \end{aligned}$$

56. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ to obtain

$$\begin{aligned}
 \int \cos(\pi x) \cos(3\pi x) dx &= \frac{1}{2} \int (\cos(4\pi x) + \cos(-2\pi x)) dx \\
 &= \frac{1}{2} \int (\cos(4\pi x) + \cos(2\pi x)) dx \\
 &= \frac{1}{2} \left(\frac{1}{4\pi} \sin(4\pi x) + \frac{1}{2\pi} \sin(2\pi x) \right) + C \\
 &= \boxed{\frac{1}{8\pi} \sin(4\pi x) + \frac{1}{4\pi} \sin(2\pi x) + C}.
 \end{aligned}$$

57. The volume is given by the integral $\int_0^\pi \pi(\sin x)^2 dx$. We write $\sin^2 x = \frac{1-\cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int_0^\pi \pi(\sin x)^2 dx &= \int_0^\pi \pi \left(\frac{1-\cos(2x)}{2} \right) dx \\
 &= \frac{\pi}{2} \int_0^\pi (1-\cos(2x)) dx \\
 &= \frac{\pi}{2} \left[x - \frac{1}{2} \sin(2x) \right]_0^\pi \\
 &= \frac{\pi}{2} \left[\pi - \frac{1}{2} \sin(2\pi) - \left(0 - \frac{1}{2} \sin(2(0)) \right) \right] \\
 &= \boxed{\frac{1}{2} \pi^2}.
 \end{aligned}$$

58. The volume is given by the integral $\int_0^{\pi/4} \pi [(\cos x)^2 - (\sin x)^2] dx$. We write $\cos^2 x - \sin^2 x = \cos(2x)$ and obtain

$$\begin{aligned} \int_0^{\pi/4} \pi [(\cos x)^2 - (\sin x)^2] dx &= \int_0^{\pi/4} \pi \cos(2x) dx \\ &= \pi \int_0^{\pi/4} \cos(2x) dx \\ &= \pi \left[\frac{1}{2} \sin(2x) \right]_0^{\pi/4} \\ &= \pi \left[\frac{1}{2} \sin\left(2\left(\frac{\pi}{4}\right)\right) - \frac{1}{2} \sin(2(0)) \right] \\ &= \boxed{\frac{1}{2}\pi}. \end{aligned}$$

59. (a) The average value is given by $\frac{1}{\pi-0} \int_0^{\pi} \sin x \cos^4 x dx$. Let $u = \cos x$, then $du = -\sin x dx$, so $\sin x dx = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos \pi = -1$. We substitute and obtain

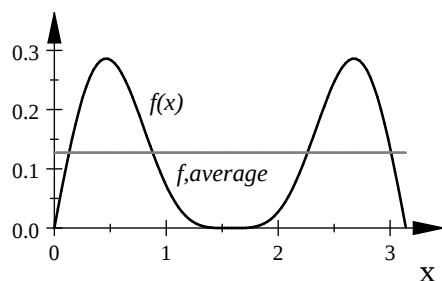
$$\begin{aligned} \frac{1}{\pi-0} \int_0^{\pi} \sin x \cos^4 x dx &= \frac{1}{\pi} \int_1^{-1} u^4 (-du) \\ &= \frac{1}{\pi} \int_{-1}^1 u^4 du \\ &= \frac{2}{\pi} \int_0^1 u^4 du \end{aligned}$$

using that we have an even function on an interval symmetric about 0. So

$$\begin{aligned} \frac{1}{\pi-0} \int_0^{\pi} \sin x \cos^4 x dx &= \frac{2}{\pi} \left[\frac{1}{5} u^5 \right]_0^1 \\ &= \frac{2}{\pi} \left[\frac{1}{5} 1^5 - \frac{1}{5} 0^5 \right] \\ &= \boxed{\frac{2}{5\pi}}. \end{aligned}$$

- (b) The area under the graph of the function f is the same as the area of the rectangle with height $\frac{2}{5\pi} \approx 0.1273$ and width $\pi - 0 = \pi$.

(c)



60. The velocity is given by

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(w) dw \\ &= 5 + \int_0^t \cos^2 w \sin w dw. \end{aligned}$$

Let $u = \cos w$, then $du = -\sin w dw$, so $\sin w dw = -du$. The lower limit of integration is $u = \cos 0 = 1$, and the upper limit of integration is $u = \cos t$. We substitute and obtain

$$\begin{aligned} v(t) &= 5 + \int_1^{\cos t} u^2 (-du) \\ &= 5 - \int_1^{\cos t} u^2 du \\ &= 5 - \left[\frac{1}{3} u^3 \right]_1^{\cos t} \\ &= 5 - \left[\frac{1}{3} \cos^3 t - \frac{1}{3} \cos^3 0 \right] \\ &= \frac{16}{3} - \frac{1}{3} \cos^3 t. \end{aligned}$$

The distance is given by

$$\begin{aligned} s(t) &= s(0) + \int_0^t v(w) dw \\ &= 0 + \int_0^t \left(\frac{16}{3} - \frac{1}{3} \cos^3 w \right) dw \\ &= \frac{16}{3}t - \frac{1}{3} \int_0^t \cos^3 w dw. \end{aligned}$$

We factor out $\cos w$ and use the identity $\cos^2 w = 1 - \sin^2 w$. Let $u = \sin w$, then $du = \cos w dw$, the lower limit of integration is $u = \sin 0 = 0$, and the upper limit of integration is $u = \sin t$. We substitute and obtain

$$\begin{aligned} s(t) &= \frac{16}{3}t - \frac{1}{3} \int_0^t \cos^2 w \cos w dw \\ &= \frac{16}{3}t - \frac{1}{3} \int_0^t (1 - \sin^2 w) \cos w dw \\ &= \frac{16}{3}t - \frac{1}{3} \int_0^{\sin t} (1 - u^2) du \\ &= \frac{16}{3}t - \frac{1}{3} \left[u - \frac{1}{3} u^3 \right]_0^{\sin t} \\ &= \frac{16}{3}t - \frac{1}{3} \left[\sin t - \frac{1}{3} \sin^3 t - \left(\sin 0 - \frac{1}{3} \sin^3 0 \right) \right] \\ &= \boxed{\frac{16}{3}t - \frac{1}{3} \sin t + \frac{1}{9} \sin^3 t}. \end{aligned}$$

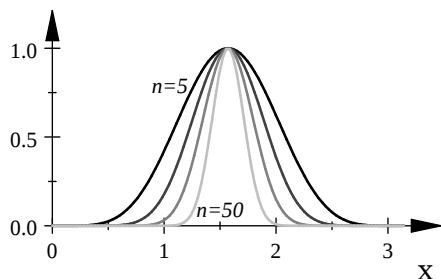
61. (a) Since $\sec x > 2 \sin x$ on the interval $[0, \frac{\pi}{4}]$, the area A is determined by

$$\begin{aligned}
 A &= \int_0^{\pi/4} (\sec x - 2 \sin x) dx \\
 &= [\ln |\sec x + \tan x| - (-2 \cos x)]_0^{\pi/4} \\
 &= \left(\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| + 2 \cos \frac{\pi}{4} \right) - (\ln |\sec 0 + \tan 0| + 2 \cos 0) \\
 &= \boxed{\ln(\sqrt{2} + 1) + \sqrt{2} - 2}.
 \end{aligned}$$

- (b) The volume is given by the integral $\int_0^{\pi/4} \pi [(\sec x)^2 - (2 \sin x)^2] dx$. We use the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int_0^{\pi/4} \pi [(\sec x)^2 - (2 \sin x)^2] dx &= \int_0^{\pi/4} \pi \left[\sec^2 x - 4 \frac{1 - \cos(2x)}{2} \right] dx \\
 &= \pi \int_0^{\pi/4} [\sec^2 x - 2 + 2 \cos(2x)] dx \\
 &= \pi [\tan x - 2x + \sin(2x)]_0^{\pi/4} \\
 &= \pi \left[\tan \frac{\pi}{4} - 2\left(\frac{\pi}{4}\right) + \sin\left(2\left(\frac{\pi}{4}\right)\right) \right. \\
 &\quad \left. - [\tan 0 - 2(0) + \sin(2 \cdot 0)] \right] \\
 &= \boxed{\frac{4\pi - \pi^2}{2}}.
 \end{aligned}$$

62. (a)



- (b) Using a Computer Algebra System, we find

$$\int_0^{\pi} \sin^5 x dx \approx 1.067, \quad \int_0^{\pi} \sin^{10} x dx \approx 0.773, \quad \int_0^{\pi} \sin^{20} x dx \approx 0.554, \quad \text{and} \quad \int_0^{\pi} \sin^{50} x dx \approx 0.353.$$

- (c) The shape of the graph appears to contract towards $x = \frac{\pi}{2}$, becoming very narrow, as $n \rightarrow \infty$.
- (d) We continue to evaluate, and obtain $\int_0^{\pi} \sin^{100} x dx \approx 0.250$, $\int_0^{\pi} \sin^{1000} x dx \approx 0.0792$, and $\int_0^{\pi} \sin^{10000} x dx \approx 0.0251$. We surmise that

$$\lim_{n \rightarrow \infty} \int_0^{\pi} \sin^n x dx = \boxed{0}.$$

63. (a) Use the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx \\
 &= \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 - 2\cos(2x) + \cos^2(2x)) \, dx \\
 &= \frac{1}{4} \int dx - \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\
 &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4} \int \cos^2(2x) \, dx.
 \end{aligned}$$

We evaluate

$$\begin{aligned}
 \int \cos^2(2x) \, dx &= \int \frac{1 + \cos(4x)}{2} \, dx \\
 &= \frac{1}{2} \int (1 + \cos(4x)) \, dx \\
 &= \frac{x}{2} + \frac{\sin(4x)}{8} + C.
 \end{aligned}$$

We now obtain

$$\begin{aligned}
 \int \cos^4 x \, dx &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4} \left(\frac{x}{2} + \frac{\sin(4x)}{8} \right) + C \\
 &= \boxed{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}.
 \end{aligned}$$

(b) Use $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$ with $n = 4$ to obtain

$$\begin{aligned}
 \int \sin^4 x \, dx &= -\frac{\sin^3 x \cos x}{4} + \frac{4-1}{4} \int \sin^{4-2} x \, dx \\
 &= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx.
 \end{aligned}$$

And again with $n = 2$,

$$\begin{aligned}
 \int \sin^4 x \, dx &= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \left(-\frac{\sin^{2-1} x \cos x}{2} + \frac{2-1}{2} \int \sin^{2-2} x \, dx \right) \\
 &= -\frac{\sin^3 x \cos x}{4} - \frac{3 \sin x \cos x}{8} + \frac{3}{8} \int dx \\
 &= \boxed{-\frac{\sin^3 x \cos x}{4} - \frac{3 \sin x \cos x}{8} + \frac{3}{8}x + C}.
 \end{aligned}$$

(c) We have

$$\begin{aligned}
 \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) &= \frac{3}{8}x - \frac{1}{4}(2\sin x \cos x) + \frac{1}{32}(2\sin(2x) \cos(2x)) \\
 &= \frac{3}{8}x - \frac{1}{2}\sin x \cos x + \frac{1}{16}(2\sin x \cos x)(1 - 2\sin^2 x) \\
 &= \frac{3}{8}x - \frac{1}{2}\sin x \cos x + \frac{1}{8}\sin x \cos x - \frac{1}{4}\sin^3 x \cos x \\
 &= -\frac{\sin^3 x \cos x}{4} - \frac{3 \sin x \cos x}{8} + \frac{3}{8}x.
 \end{aligned}$$

So the two antiderivatives are equal.

(d) Using a Computer Algebra System, we obtain

$$\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C,$$

which agrees with the result obtained in part (a).

64. (a) Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned} \int \sin x \cos x \, dx &= \int u \, du \\ &= \frac{1}{2}u^2 + C_1 \\ &= \frac{1}{2}\sin^2 x + C_1. \end{aligned}$$

So a function is $\boxed{f(x) = \frac{1}{2}\sin^2 x}$.

(b) Let $u = \cos x$, then $du = -\sin x \, dx$, so $\sin x \, dx = -du$. We substitute and obtain

$$\begin{aligned} \int \sin x \cos x \, dx &= \int u(-du) \\ &= -\frac{1}{2}u^2 + C_2 \\ &= -\frac{1}{2}\cos^2 x + C_2. \end{aligned}$$

So a function is $\boxed{g(x) = -\frac{1}{2}\cos^2 x}$.

(c) From the identity $\sin(2x) = 2\sin x \cos x$ we have $\sin x \cos x = \frac{1}{2}\sin(2x)$. So $\int \sin x \cos x \, dx = \frac{1}{2}\int \sin(2x) \, dx$. Let $u = 2x$, then $du = 2 \, dx$, so $dx = \frac{1}{2} \, du$. We substitute and obtain

$$\begin{aligned} \int \sin x \cos x \, dx &= \frac{1}{2} \int \sin(2x) \, dx \\ &= \frac{1}{2} \int \sin u \left(\frac{1}{2} \, du\right) \\ &= \frac{1}{4} \int \sin u \, du \\ &= \frac{1}{4}(-\cos u) + C_3 \\ &= -\frac{1}{4}\cos(2x) + C_3. \end{aligned}$$

So the function is $\boxed{h(x) = -\frac{1}{4}\cos(2x)}$.

(d) We have $f(x) - g(x) = \frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$.

(e) We have $f(x) - h(x) = \frac{1}{2}\sin^2 x - (-\frac{1}{4}\cos(2x)) = \frac{1}{2}\sin^2 x + \frac{1}{4}(2\cos^2 x - 1) = \frac{1}{2}(\sin^2 x + \cos^2 x) - \frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

65. We use the identity $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$ to obtain

$$\begin{aligned}\int \sin(mx) \sin(nx) dx &= \frac{1}{2} \int (\cos((m-n)x) - \cos((m+n)x)) dx \\ &= \frac{1}{2} \left(\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n} \right) + C \\ &= \boxed{\frac{\sin((m-n)x)}{2(m-n)} - \frac{\sin((m+n)x)}{2(m+n)} + C}.\end{aligned}$$

66. We use the identity $\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$ to obtain

$$\begin{aligned}\int \sin(mx) \cos(nx) dx &= \frac{1}{2} \int (\sin((m+n)x) + \sin((m-n)x)) dx \\ &= \frac{1}{2} \left(-\frac{\cos((m+n)x)}{m+n} - \frac{\cos((m-n)x)}{m-n} \right) + C \\ &= \boxed{-\frac{\cos((m+n)x)}{2(m+n)} - \frac{\cos((m-n)x)}{2(m-n)} + C}.\end{aligned}$$

67. We use the identity $\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$ to obtain

$$\begin{aligned}\int \cos(mx) \cos(nx) dx &= \frac{1}{2} \int (\cos((m+n)x) + \cos((m-n)x)) dx \\ &= \frac{1}{2} \left(\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n} \right) + C \\ &= \boxed{\frac{\sin((m+n)x)}{2(m+n)} + \frac{\sin((m-n)x)}{2(m-n)} + C}.\end{aligned}$$

Challenge Problems

68. Let $\sqrt{x} = \sin y$, so $x = \sin^2 y$, and $dx = 2 \sin y \cos y dy$. The lower limit of integration is $y = \sin^{-1} 0 = 0$, and the upper limit of integration is $y = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$. We substitute and obtain

$$\begin{aligned}\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}} dx &= \int_0^{\pi/4} \frac{\sin y}{\sqrt{1-\sin^2 y}} (2 \sin y \cos y) dy \\ &= \int_0^{\pi/4} \frac{\sin y}{\sqrt{\cos^2 y}} (2 \sin y \cos y) dy \\ &= \int_0^{\pi/4} \frac{\sin y}{\cos y} (2 \sin y \cos y) dy \\ &= 2 \int_0^{\pi/4} \sin^2 y dy.\end{aligned}$$

Use the identity $\sin^2 y = \frac{1 - \cos(2y)}{2}$ and obtain

$$\begin{aligned}
 \int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}} dx &= 2 \int_0^{\pi/4} \frac{1 - \cos(2y)}{2} dy \\
 &= \int_0^{\pi/4} (1 - \cos(2y)) dy \\
 &= \left[y - \frac{1}{2} \sin(2y) \right]_0^{\pi/4} \\
 &= \left(\frac{\pi}{4} - \frac{1}{2} \sin\left(2\left(\frac{\pi}{4}\right)\right) \right) - \left(0 - \frac{1}{2} \sin(2(0)) \right) \\
 &= \boxed{\frac{1}{4}\pi - \frac{1}{2}}.
 \end{aligned}$$

69. Let $u = \frac{\pi}{2} - \theta$, then $du = -d\theta$, so $d\theta = -du$. Also, $\theta = \frac{\pi}{2} - u$. The lower limit of integration is $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$, and the upper limit of integration is $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$. We substitute and obtain

$$\begin{aligned}
 \int_0^{\pi/2} \sin^n \theta d\theta &= \int_{\pi/2}^0 \sin^n \left(\frac{\pi}{2} - u \right) (-du) \\
 &= - \int_{\pi/2}^0 \left[\sin \left(\frac{\pi}{2} - u \right) \right]^n du \\
 &= \int_0^{\pi/2} [\cos u]^n du \\
 &= \int_0^{\pi/2} \cos^n \theta d\theta
 \end{aligned}$$

where we used the identity $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

70. (a) When writing $(\cos^2 x)^{3/2} = (\cos x)^3$, a mistake is made. When $\frac{\pi}{2} < x \leq \pi$, $(\cos x)^3 < 0$, but $(\cos^2 x)^{3/2} > 0$. The correct equality would be $(\cos^2 x)^{3/2} = |\cos x|^3$.
- (b) Use the identity $\cos^2 x = \frac{1 + \cos(2x)}{2}$ and obtain

$$\begin{aligned}
 \int_0^\pi \cos^4 x dx &= \int_0^\pi (\cos^2 x)^2 dx \\
 &= \int_0^\pi \left(\frac{1 + \cos(2x)}{2} \right)^2 dx \\
 &= \frac{1}{4} \int_0^\pi (1 + 2\cos(2x) + \cos^2(2x)) dx \\
 &= \frac{1}{4} \int_0^\pi dx + \frac{1}{2} \int_0^\pi \cos(2x) dx + \frac{1}{4} \int_0^\pi \cos^2(2x) dx \\
 &= \left[\frac{1}{4}x \right]_0^\pi + \left[\frac{1}{4} \sin(2x) \right]_0^\pi + \frac{1}{4} \int_0^\pi \cos^2(2x) dx \\
 &= \left[\frac{1}{4}\pi - \frac{1}{4}(0) \right] + \left[\frac{1}{4} \sin(2\pi) - \frac{1}{4} \sin(2(0)) \right] + \frac{1}{4} \int_0^\pi \cos^2(2x) dx \\
 &= \frac{\pi}{4} + \frac{1}{4} \int_0^\pi \cos^2(2x) dx.
 \end{aligned}$$

We evaluate

$$\begin{aligned}\int_0^\pi \cos^2(2x) \, dx &= \int_0^\pi \frac{1 + \cos(4x)}{2} \, dx \\&= \frac{1}{2} \int_0^\pi (1 + \cos(4x)) \, dx \\&= \left[\frac{x}{2} + \frac{\sin(4x)}{8} \right]_0^\pi \\&= \frac{\pi}{2} + \frac{\sin(4\pi)}{8} - \left(\frac{0}{2} + \frac{\sin(4(0))}{8} \right) \\&= \frac{\pi}{2}.\end{aligned}$$

We now obtain

$$\int_0^\pi \cos^4 x \, dx = \frac{\pi}{4} + \frac{1}{4} \left(\frac{\pi}{2} \right) = \boxed{\frac{3\pi}{8}}.$$

7.3: Integration Using Trigonometric Substitution

Concepts and Vocabulary

1. True.
2. (d), $x = 4 \tan \theta$
3. (c), $x = 3 \sec \theta$
4. (b), $x = \frac{5}{2} \sin \theta$

Skill Building

5. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{4-x^2} dx &= \int \sqrt{4-(2 \sin \theta)^2} (2 \cos \theta) d\theta \\&= 2 \int \sqrt{4-4 \sin^2 \theta} \cos \theta d\theta \\&= 2 \int \sqrt{4(1-\sin^2 \theta)} \cos \theta d\theta \\&= 2 \int \sqrt{4 \cos^2 \theta} \cos \theta d\theta \\&= 2 \int (2 \cos \theta) \cos \theta d\theta \\&= 4 \int \cos^2 \theta d\theta \\&= 4 \int \frac{1+\cos(2\theta)}{2} d\theta \\&= 2 \int (1+\cos(2\theta)) d\theta \\&= 2\left(\theta + \frac{1}{2} \sin(2\theta)\right) + C \\&= 2\theta + \sin(2\theta) + C \\&= 2\theta + 2 \sin \theta \cos \theta + C.\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos \theta = \sqrt{1-\sin^2 \theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. We obtain

$$\begin{aligned}\int \sqrt{4-x^2} dx &= 2 \sin^{-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right) + C \\&= \boxed{2 \sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C}.\end{aligned}$$

6. Let $x = 4 \sin \theta$, then $dx = 4 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{16 - x^2} dx &= \int \sqrt{16 - (4 \sin \theta)^2} (4 \cos \theta) d\theta \\
 &= 4 \int \sqrt{16 - 16 \sin^2 \theta} \cos \theta d\theta \\
 &= 4 \int (4 \cos \theta) \cos \theta d\theta \\
 &= 16 \int \cos^2 \theta d\theta \\
 &= 16 \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= 8 \int (1 + \cos(2\theta)) d\theta \\
 &= 8 \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= 8\theta + 4 \sin(2\theta) + C \\
 &= 8\theta + 8 \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{x}{4} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/4)^2} = \frac{1}{4} \sqrt{16 - x^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{16 - x^2} dx &= 8 \sin^{-1} \left(\frac{x}{4} \right) + 8 \left(\frac{x}{4} \right) \left(\frac{1}{4} \sqrt{16 - x^2} \right) + C \\
 &= \boxed{8 \sin^{-1} \left(\frac{x}{4} \right) + \frac{1}{2} x \sqrt{16 - x^2} + C}.
 \end{aligned}$$

7. Let $x = 4 \sin \theta$, then $dx = 4 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{16 - x^2}} dx &= \int \frac{(4 \sin \theta)^2}{\sqrt{16 - (4 \sin \theta)^2}} (4 \cos \theta) d\theta \\
 &= 64 \int \frac{\sin^2 \theta}{\sqrt{16 - 16 \sin^2 \theta}} \cos \theta d\theta \\
 &= 64 \int \frac{\sin^2 \theta}{\sqrt{16(1 - \sin^2 \theta)}} \cos \theta d\theta \\
 &= 64 \int \frac{\sin^2 \theta}{\sqrt{16 \cos^2 \theta}} \cos \theta d\theta \\
 &= 16 \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\
 &= 16 \int \sin^2 \theta d\theta \\
 &= 16 \int \frac{1 - \cos(2\theta)}{2} d\theta \\
 &= 8 \int (1 - \cos(2\theta)) d\theta \\
 &= 8 \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\
 &= 8\theta - 8 \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{4}\right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/4)^2} = \frac{1}{4}\sqrt{16 - x^2}$. We obtain

$$\begin{aligned}\int \frac{x^2}{\sqrt{16 - x^2}} dx &= 8 \sin^{-1}\left(\frac{x}{4}\right) - 8\left(\frac{x}{4}\right)\left(\frac{1}{4}\sqrt{16 - x^2}\right) + C \\ &= \boxed{8 \sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16 - x^2} + C}.\end{aligned}$$

8. Let $x = 6 \sin \theta$, then $dx = 6 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{x^2}{\sqrt{36 - x^2}} dx &= \int \frac{(6 \sin \theta)^2}{\sqrt{36 - (6 \sin \theta)^2}} (6 \cos \theta) d\theta \\ &= 216 \int \frac{\sin^2 \theta}{\sqrt{36 - 36 \sin^2 \theta}} \cos \theta d\theta \\ &= 216 \int \frac{\sin^2 \theta}{\sqrt{36(1 - \sin^2 \theta)}} \cos \theta d\theta \\ &= 216 \int \frac{\sin^2 \theta}{\sqrt{36 \cos^2 \theta}} \cos \theta d\theta \\ &= 36 \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\ &= 36 \int \sin^2 \theta d\theta \\ &= 36 \int \frac{1 - \cos(2\theta)}{2} d\theta \\ &= 18 \int (1 - \cos(2\theta)) d\theta \\ &= 18\left(\theta - \frac{1}{2} \sin 2\theta\right) + C \\ &= 18\theta - 18 \sin \theta \cos \theta + C.\end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{6}\right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/6)^2} = \frac{1}{6}\sqrt{36 - x^2}$. We obtain

$$\begin{aligned}\int \frac{x^2}{\sqrt{36 - x^2}} dx &= 18 \sin^{-1}\left(\frac{x}{6}\right) - 18\left(\frac{x}{6}\right)\left(\frac{1}{6}\sqrt{36 - x^2}\right) + C \\ &= \boxed{18 \sin^{-1}\left(\frac{x}{6}\right) - \frac{1}{2}x\sqrt{36 - x^2} + C}.\end{aligned}$$

9. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{\sqrt{4-x^2}}{x^2} dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{(2\sin\theta)^2} (2\cos\theta) d\theta \\
 &= \frac{1}{2} \int \frac{\sqrt{4-4\sin^2\theta}}{\sin^2\theta} \cos\theta d\theta \\
 &= \frac{1}{2} \int \frac{\sqrt{4(1-\sin^2\theta)}}{\sin^2\theta} \cos\theta d\theta \\
 &= \frac{1}{2} \int \frac{\sqrt{4\cos^2\theta}}{\sin^2\theta} \cos\theta d\theta \\
 &= \int \frac{\cos\theta \cos\theta}{\sin^2\theta} d\theta \\
 &= \int \cot^2\theta d\theta \\
 &= \int (\csc^2\theta - 1) d\theta \\
 &= -\cot\theta - \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/2)^2} = \frac{1}{2}\sqrt{4-x^2}$. So $\cot\theta = \frac{\frac{1}{2}\sqrt{4-x^2}}{x/2} = \frac{\sqrt{4-x^2}}{x}$. We obtain

$$\int \frac{\sqrt{4-x^2}}{x^2} dx = \boxed{-\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C}.$$

10. Let $x = 3 \sin \theta$, then $dx = 3 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{\sqrt{9-x^2}}{x^2} dx &= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} (3\cos\theta) d\theta \\
 &= \frac{1}{3} \int \frac{\sqrt{9-9\sin^2\theta}}{\sin^2\theta} \cos\theta d\theta \\
 &= \frac{1}{3} \int \frac{\sqrt{9(1-\sin^2\theta)}}{\sin^2\theta} \cos\theta d\theta \\
 &= \frac{1}{3} \int \frac{\sqrt{9\cos^2\theta}}{\sin^2\theta} \cos\theta d\theta \\
 &= \int \frac{\cos\theta \cos\theta}{\sin^2\theta} d\theta \\
 &= \int \cot^2\theta d\theta \\
 &= \int (\csc^2\theta - 1) d\theta \\
 &= -\cot\theta - \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{3}\right)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(x/3)^2} = \frac{1}{3}\sqrt{9-x^2}$. So $\cot\theta = \frac{\frac{1}{3}\sqrt{9-x^2}}{x/3} = \frac{\sqrt{9-x^2}}{x}$. We obtain

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}.$$

11. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int x^2 \sqrt{4 - x^2} dx &= \int (2 \sin \theta)^2 \sqrt{4 - (2 \sin \theta)^2} (2 \cos \theta) d\theta \\
 &= 8 \int (\sin^2 \theta) \sqrt{4 - 4 \sin^2 \theta} \cos \theta d\theta \\
 &= 8 \int (\sin^2 \theta) \sqrt{4(1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= 8 \int (\sin^2 \theta) \sqrt{4 \cos^2 \theta} \cos \theta d\theta \\
 &= 8 \int (\sin^2 \theta) (2 \cos \theta) \cos \theta d\theta \\
 &= 16 \int \sin^2 \theta \cos^2 \theta d\theta \\
 &= 16 \int \left(\frac{\sin 2\theta}{2} \right)^2 d\theta \\
 &= 4 \int \sin^2 (2\theta) d\theta \\
 &= 4 \int \frac{1 - \cos (4\theta)}{2} d\theta \\
 &= 2 \int (1 - \cos (4\theta)) d\theta \\
 &= 2 \left(\theta - \frac{1}{4} \sin (4\theta) \right) + C \\
 &= 2\theta - \frac{1}{2} \sin (4\theta) + C \\
 &= 2\theta - \sin 2\theta \cos 2\theta + C \\
 &= 2\theta - (2 \sin \theta \cos \theta) (1 - 2 \sin^2 \theta) + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{x}{2} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/2)^2} = \frac{1}{2} \sqrt{4 - x^2}$. We obtain

$$\begin{aligned}
 \int x^2 \sqrt{4 - x^2} dx &= 2 \sin^{-1} \left(\frac{x}{2} \right) - 2 \left(\frac{x}{2} \right) \left(\frac{1}{2} \sqrt{4 - x^2} \right) \left(1 - 2 \left(\frac{x}{2} \right)^2 \right) + C \\
 &= \boxed{\frac{x(x^2 - 2)}{4} \sqrt{4 - x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) + C}.
 \end{aligned}$$

12. Let $x = 4 \sin \theta$, then $dx = 4 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int x^2 \sqrt{16 - x^2} dx &= \int (4 \sin \theta)^2 \sqrt{16 - (4 \sin \theta)^2} (4 \cos \theta) d\theta \\
 &= 64 \int (\sin^2 \theta) \sqrt{16 - 16 \sin^2 \theta} \cos \theta d\theta \\
 &= 64 \int (\sin^2 \theta) \sqrt{16(1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= 64 \int (\sin^2 \theta) \sqrt{16 \cos^2 \theta} \cos \theta d\theta \\
 &= 64 \int (\sin^2 \theta) (4 \cos \theta) \cos \theta d\theta \\
 &= 256 \int \sin^2 \theta \cos^2 \theta d\theta \\
 &= 256 \int \left(\frac{\sin 2\theta}{2} \right)^2 d\theta \\
 &= 64 \int \sin^2 (2\theta) d\theta \\
 &= 64 \int \frac{1 - \cos (4\theta)}{2} d\theta \\
 &= 32 \int (1 - \cos (4\theta)) d\theta \\
 &= 32 \left(\theta - \frac{1}{4} \sin (4\theta) \right) + C \\
 &= 32\theta - 8 \sin (4\theta) + C \\
 &= 32\theta - 16 \sin 2\theta \cos 2\theta + C \\
 &= 32\theta - 16(2 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta) + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{x}{4} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/4)^2} = \frac{1}{4} \sqrt{16 - x^2}$. We obtain

$$\begin{aligned}
 \int x^2 \sqrt{16 - x^2} dx &= 32 \sin^{-1} \left(\frac{x}{4} \right) - 32 \left(\frac{x}{4} \right) \left(\frac{1}{4} \sqrt{16 - x^2} \right) \left(1 - 2 \left(\frac{x}{4} \right)^2 \right) + C \\
 &= \boxed{32 \sin^{-1} \left(\frac{x}{4} \right) - \frac{1}{4} x \sqrt{16 - x^2} (8 - x^2) + C}.
 \end{aligned}$$

13. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{1}{(4 - x^2)^{3/2}} dx &= \int \frac{1}{(4 - 4 \sin^2 \theta)^{3/2}} (2 \cos \theta) d\theta \\
 &= 2 \int \frac{1}{4^{3/2} (\cos^2 \theta)^{3/2}} \cos \theta d\theta \\
 &= \frac{1}{4} \int \frac{\cos \theta}{\cos^3 \theta} d\theta \\
 &= \frac{1}{4} \int \sec^2 \theta d\theta \\
 &= \frac{1}{4} \tan \theta + C.
 \end{aligned}$$

We have $\sin \theta = \frac{x}{2}$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/2)^2} = \frac{1}{2}\sqrt{4 - x^2}$. So $\tan \theta = \frac{x/2}{\frac{1}{2}\sqrt{4 - x^2}} = \frac{x}{\sqrt{4 - x^2}}$. We obtain

$$\begin{aligned} \int \frac{1}{(4 - x^2)^{3/2}} dx &= \frac{1}{4} \tan \theta + C \\ &= \boxed{\frac{x}{4\sqrt{4 - x^2}} + C}. \end{aligned}$$

14. Let $x = \sin \theta$, then $dx = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{1}{(1 - x^2)^{3/2}} dx &= \int \frac{1}{(1 - \sin^2 \theta)^{3/2}} (\cos \theta) d\theta \\ &= \int \frac{1}{(\cos^2 \theta)^{3/2}} \cos \theta d\theta \\ &= \int \frac{\cos \theta}{\cos^3 \theta} d\theta \\ &= \int \sec^2 \theta d\theta \\ &= \tan \theta + C. \end{aligned}$$

We have $\sin \theta = x$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}$. So $\tan \theta = \frac{x}{\sqrt{1 - x^2}}$. We obtain

$$\int \frac{1}{(1 - x^2)^{3/2}} dx = \boxed{\frac{x}{\sqrt{1 - x^2}} + C}.$$

15. Let $x = 2 \tan \theta$, then $dx = 2 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \sqrt{4 + x^2} dx &= \int \sqrt{4 + (2 \tan \theta)^2} (2 \sec^2 \theta) d\theta \\ &= 2 \int \sqrt{4 + 4 \tan^2 \theta} \sec^2 \theta d\theta \\ &= 2 \int (2 \sec \theta) \sec^2 \theta d\theta \\ &= 4 \int \sec^3 \theta d\theta \\ &= 4 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\ &= 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

We have $\tan \theta = \frac{x}{2}$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + (x/2)^2} = \frac{1}{2}\sqrt{4 + x^2}$. We obtain

$$\begin{aligned} \int \sqrt{4 + x^2} dx &= 2 \left(\frac{1}{2} \sqrt{4 + x^2} \right) \left(\frac{x}{2} \right) + 2 \ln \left| \frac{1}{2} \sqrt{4 + x^2} + \left(\frac{x}{2} \right) \right| + C \\ &= \boxed{\frac{1}{2} x \sqrt{4 + x^2} + 2 \ln \left| \frac{\sqrt{4 + x^2} + x}{2} \right| + C} \end{aligned}$$

16. Let $x = \tan \theta$, then $dx = \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{1+x^2} dx &= \int \sqrt{1+(\tan \theta)^2} (\sec^2 \theta) d\theta \\
 &= \int (\sec \theta) \sec^2 \theta d\theta \\
 &= \int \sec^3 \theta d\theta \\
 &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = x$, and $\sec \theta = \sqrt{1+\tan^2 \theta} = \sqrt{1+x^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{1+x^2} dx &= \frac{1}{2} \sqrt{1+x^2}(x) + \frac{1}{2} \ln |\sqrt{1+x^2} + x| + C \\
 &= \boxed{\frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \ln |\sqrt{1+x^2} + x| + C}.
 \end{aligned}$$

17. Let $x = 4 \tan \theta$, then $dx = 4 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{1}{\sqrt{(4 \tan \theta)^2 + 16}} (4 \sec^2 \theta) d\theta \\
 &= 4 \int \frac{1}{\sqrt{16 \tan^2 \theta + 16}} (\sec^2 \theta) d\theta \\
 &= 4 \int \frac{1}{\sqrt{16(\tan^2 \theta + 1)}} (\sec^2 \theta) d\theta \\
 &= 4 \int \frac{1}{\sqrt{16 \sec^2 \theta}} (\sec^2 \theta) d\theta \\
 &= \int \frac{1}{\sec \theta} (\sec^2 \theta) d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = \frac{x}{4}$, and $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{(x/4)^2 + 1} = \frac{1}{4} \sqrt{x^2 + 16}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2+16}} &= \ln \left| \frac{1}{4} \sqrt{x^2+16} + \left(\frac{x}{4} \right) \right| + C \\
 &= \boxed{\ln \left| \frac{\sqrt{x^2+16}+x}{4} \right| + C}.
 \end{aligned}$$

18. Let $x = 5 \tan \theta$, then $dx = 5 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + 25}} &= \int \frac{1}{\sqrt{(5 \tan \theta)^2 + 25}} (5 \sec^2 \theta) d\theta \\
 &= 5 \int \frac{1}{\sqrt{25 \tan^2 \theta + 25}} (\sec^2 \theta) d\theta \\
 &= 5 \int \frac{1}{\sqrt{25(\tan^2 \theta + 1)}} (\sec^2 \theta) d\theta \\
 &= 5 \int \frac{1}{\sqrt{25 \sec^2 \theta}} (\sec^2 \theta) d\theta \\
 &= \int \frac{1}{\sec \theta} (\sec^2 \theta) d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = \frac{x}{5}$, and $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{(x/5)^2 + 1} = \frac{1}{5} \sqrt{x^2 + 25}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + 25}} &= \ln \left| \frac{1}{5} \sqrt{x^2 + 25} + \left(\frac{x}{5} \right) \right| + C \\
 &= \boxed{\ln \left| \frac{\sqrt{x^2 + 25} + x}{5} \right| + C}.
 \end{aligned}$$

19. Let $x = \frac{1}{3} \tan \theta$, then $dx = \frac{1}{3} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{1 + 9x^2} dx &= \int \sqrt{1 + 9 \left(\frac{1}{3} \tan \theta \right)^2} \left(\frac{1}{3} \sec^2 \theta \right) d\theta \\
 &= \frac{1}{3} \int \sqrt{1 + \tan^2 \theta} \sec^2 \theta d\theta \\
 &= \frac{1}{3} \int (\sec \theta) \sec^2 \theta d\theta \\
 &= \frac{1}{3} \int \sec^3 \theta d\theta \\
 &= \frac{1}{3} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\
 &= \frac{1}{6} \sec \theta \tan \theta + \frac{1}{6} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = 3x$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + (3x)^2} = \sqrt{1 + 9x^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{1 + 9x^2} dx &= \frac{1}{6} (\sqrt{1 + 9x^2}) (3x) + \frac{1}{6} \ln |\sqrt{1 + 9x^2} + (3x)| + C \\
 &= \boxed{\frac{1}{2} x \sqrt{1 + 9x^2} + \frac{1}{6} \ln (\sqrt{1 + 9x^2} + 3x) + C}.
 \end{aligned}$$

20. Let $x = \frac{3}{2} \tan \theta$, then $dx = \frac{3}{2} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{9+4x^2} dx &= \int \sqrt{9+4\left(\frac{3}{2} \tan \theta\right)^2} \left(\frac{3}{2} \sec^2 \theta\right) d\theta \\
 &= \frac{3}{2} \int \sqrt{9+9 \tan^2 \theta} \sec^2 \theta d\theta \\
 &= \frac{3}{2} \int \sqrt{9(1+\tan^2 \theta)} \sec^2 \theta d\theta \\
 &= \frac{3}{2} \int \sqrt{9 \sec^2 \theta} \sec^2 \theta d\theta \\
 &= \frac{3}{2} \int (3 \sec \theta) \sec^2 \theta d\theta \\
 &= \frac{9}{2} \int \sec^3 \theta d\theta \\
 &= \frac{9}{2} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\
 &= \frac{9}{4} \sec \theta \tan \theta + \frac{9}{4} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = \frac{2x}{3}$, and $\sec \theta = \sqrt{1+\tan^2 \theta} = \sqrt{1+\left(\frac{2x}{3}\right)^2} = \frac{1}{3} \sqrt{9+4x^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{9+4x^2} dx &= \frac{9}{4} \left(\frac{1}{3} \sqrt{9+4x^2} \right) \left(\frac{2x}{3} \right) + \frac{9}{4} \ln \left| \frac{1}{3} \sqrt{9+4x^2} + \frac{2x}{3} \right| + C \\
 &= \boxed{\frac{1}{2} x \sqrt{9+4x^2} + \frac{9}{4} \ln \left| \frac{\sqrt{9+4x^2}+2x}{3} \right| + C}.
 \end{aligned}$$

21. Let $x = \frac{2}{3} \tan \theta$, then $dx = \frac{2}{3} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{4+9x^2}} dx &= \int \frac{\left(\frac{2}{3} \tan \theta\right)^2}{\sqrt{4+9\left(\frac{2}{3} \tan \theta\right)^2}} \left(\frac{2}{3} \sec^2 \theta\right) d\theta \\
 &= \frac{8}{27} \int \frac{\tan^2 \theta}{\sqrt{4+4 \tan^2 \theta}} \sec^2 \theta d\theta \\
 &= \frac{8}{27} \int \frac{\tan^2 \theta}{\sqrt{4(1+\tan^2 \theta)}} \sec^2 \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
&= \frac{8}{27} \int \frac{\tan^2 \theta}{\sqrt{4 \sec^2 \theta}} \sec^2 \theta d\theta \\
&= \frac{4}{27} \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta d\theta \\
&= \frac{4}{27} \int \tan^2 \theta \sec \theta d\theta \\
&= \frac{4}{27} \int (\sec^2 \theta - 1) \sec \theta d\theta \\
&= \frac{4}{27} \int (\sec^3 \theta - \sec \theta) d\theta \\
&= \frac{4}{27} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right] + C \\
&= \frac{2}{27} \sec \theta \tan \theta - \frac{2}{27} \ln |\sec \theta + \tan \theta| + C.
\end{aligned}$$

We have $\tan \theta = \frac{3x}{2}$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{3x}{2}\right)^2} = \frac{1}{2} \sqrt{4 + 9x^2}$. We obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4 + 9x^2}} dx &= \frac{2}{27} \left(\frac{1}{2} \sqrt{4 + 9x^2} \right) \left(\frac{3x}{2} \right) - \frac{2}{27} \ln \left| \frac{1}{2} \sqrt{4 + 9x^2} + \frac{3x}{2} \right| + C \\
&= \boxed{\frac{1}{18} x \sqrt{4 + 9x^2} - \frac{2}{27} \ln \left(\frac{\sqrt{4 + 9x^2} + 3x}{2} \right) + C}.
\end{aligned}$$

22. Let $x = 4 \tan \theta$, then $dx = 4 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2 + 16}} dx &= \int \frac{(4 \tan \theta)^2}{\sqrt{(4 \tan \theta)^2 + 16}} (4 \sec^2 \theta) d\theta \\
&= 64 \int \frac{\tan^2 \theta}{\sqrt{16 \tan^2 \theta + 16}} \sec^2 \theta d\theta \\
&= 64 \int \frac{\tan^2 \theta}{\sqrt{16(\tan^2 \theta + 1)}} \sec^2 \theta d\theta \\
&= 16 \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta d\theta \\
&= 16 \int \tan^2 \theta \sec \theta d\theta \\
&= 16 \int (\sec^2 \theta - 1) \sec \theta d\theta \\
&= 16 \int (\sec^3 \theta - \sec \theta) d\theta \\
&= 16 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right] + C \\
&= 8 \sec \theta \tan \theta - 8 \ln |\sec \theta + \tan \theta| + C.
\end{aligned}$$

We have $\tan \theta = \frac{x}{4}$, and $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{x}{4}\right)^2 + 1} = \frac{1}{4}\sqrt{x^2 + 16}$. We obtain

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 + 16}} dx &= 8 \sec \theta \tan \theta - 8 \ln |\sec \theta + \tan \theta| + C \\ &= 8 \left(\frac{1}{4} \sqrt{x^2 + 16} \right) \left(\frac{x}{4} \right) - 8 \ln \left| \frac{1}{4} \sqrt{x^2 + 16} + \frac{x}{4} \right| + C \\ &= \boxed{\frac{1}{2} x \sqrt{x^2 + 16} - 8 \ln \left(\frac{\sqrt{x^2 + 16} + x}{4} \right) + C}. \end{aligned}$$

23. Let $x = 2 \tan \theta$, then $dx = 2 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 4}} &= \int \frac{2 \sec^2 \theta}{(2 \tan \theta)^2 \sqrt{(2 \tan \theta)^2 + 4}} d\theta \\ &= \frac{1}{2} \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{4 \tan^2 \theta + 4}} d\theta \\ &= \frac{1}{2} \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{4(\tan^2 \theta + 1)}} d\theta \\ &= \frac{1}{4} \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta \\ &= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta. \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 4}} &= \frac{1}{4} \int \frac{1}{u^2} du \\ &= \frac{1}{4} \left(-\frac{1}{u} \right) + C \\ &= -\frac{1}{4} \csc \theta + C. \end{aligned}$$

We have $\tan \theta = \frac{x}{2}$, so $\cot \theta = \frac{2}{x}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{2}{x}\right)^2} = \frac{1}{x} \sqrt{x^2 + 4}$. We obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 4}} &= -\frac{1}{4} \frac{1}{x} \sqrt{x^2 + 4} + C \\ &= \boxed{-\frac{\sqrt{x^2 + 4}}{4x} + C}. \end{aligned}$$

24. Let $x = \frac{1}{2} \tan \theta$, then $dx = \frac{1}{2} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{4x^2 + 1}} &= \int \frac{\frac{1}{2} \sec^2 \theta}{\left(\frac{1}{2} \tan \theta\right)^2 \sqrt{4\left(\frac{1}{2} \tan \theta\right)^2 + 1}} d\theta \\ &= 2 \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{\tan^2 \theta + 1}} d\theta \\ &= 2 \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta \\ &= 2 \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= 2 \int \frac{\cos \theta}{\sin^2 \theta} d\theta. \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{4x^2 + 1}} &= 2 \int \frac{1}{u^2} du \\ &= 2 \left(-\frac{1}{u} \right) + C \\ &= -2 \csc \theta + C. \end{aligned}$$

We have $\tan \theta = 2x$, so $\cot \theta = \frac{1}{2x}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{1}{2x}\right)^2} = \frac{1}{2x} \sqrt{4x^2 + 1}$. We obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{4x^2 + 1}} &= -2 \left(\frac{1}{2x} \sqrt{4x^2 + 1} \right) + C \\ &= \boxed{-\frac{\sqrt{4x^2 + 1}}{x} + C}. \end{aligned}$$

25. Let $x = 2 \tan \theta$, then $dx = 2 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{(x^2 + 4)^{3/2}} &= \int \frac{2 \sec^2 \theta}{\left((2 \tan \theta)^2 + 4\right)^{3/2}} d\theta \\ &= 2 \int \frac{\sec^2 \theta}{(4 \tan^2 \theta + 4)^{3/2}} d\theta \\ &= 2 \int \frac{\sec^2 \theta}{(4(\tan^2 \theta + 1))^{3/2}} d\theta \\ &= \frac{1}{4} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\ &= \frac{1}{4} \int \cos \theta d\theta \\ &= \frac{1}{4} \sin \theta + C. \end{aligned}$$

We have $\tan \theta = \frac{x}{2}$, so $\cot \theta = \frac{2}{x}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{2}{x}\right)^2} = \frac{1}{x} \sqrt{x^2 + 4}$. So $\sin \theta = \frac{x}{\sqrt{x^2 + 4}}$ and we obtain

$$\begin{aligned} \int \frac{dx}{(x^2 + 4)^{3/2}} &= \frac{1}{4} \frac{x}{\sqrt{x^2 + 4}} + C \\ &= \boxed{\frac{x}{4\sqrt{x^2 + 4}} + C}. \end{aligned}$$

26. Let $x = \tan \theta$, then $dx = \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{(x^2 + 1)^{3/2}} &= \int \frac{\sec^2 \theta}{\left((\tan \theta)^2 + 1\right)^{3/2}} d\theta \\ &= \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\ &= \int \cos \theta d\theta \\ &= \sin \theta + C. \end{aligned}$$

We have $\tan \theta = x$, so $\cot \theta = \frac{1}{x}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{1}{x}\right)^2} = \frac{1}{x} \sqrt{x^2 + 1}$. So $\sin \theta = \frac{x}{\sqrt{x^2 + 1}}$ and we obtain

$$\int \frac{dx}{(x^2 + 1)^{3/2}} = \boxed{\frac{x}{\sqrt{x^2 + 1}} + C}.$$

27. Let $x = 5 \sec \theta$, then $dx = 5 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 25}} dx &= \int \frac{(5 \sec \theta)^2}{\sqrt{(5 \sec \theta)^2 - 25}} (5 \sec \theta \tan \theta) d\theta \\ &= 125 \int \frac{\sec^2 \theta}{\sqrt{25 \sec^2 \theta - 25}} \sec \theta \tan \theta d\theta \\ &= 25 \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\ &= 25 \int \frac{\sec^2 \theta}{\tan \theta} \sec \theta \tan \theta d\theta \\ &= 25 \int \sec^3 \theta d\theta \\ &= 25 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C \\ &= \frac{25}{2} \sec \theta \tan \theta + \frac{25}{2} \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

We have $\sec \theta = \frac{x}{5}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{5}\right)^2 - 1} = \frac{1}{5} \sqrt{x^2 - 25}$. We obtain

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 25}} dx &= \frac{25}{2} \left(\frac{x}{5} \right) \left(\frac{1}{5} \sqrt{x^2 - 25} \right) + \frac{25}{2} \ln \left| \frac{x}{5} + \frac{1}{5} \sqrt{x^2 - 25} \right| + C \\ &= \boxed{\frac{1}{2} x \sqrt{x^2 - 25} + \frac{25}{2} \ln \left| \frac{x + \sqrt{x^2 - 25}}{5} \right| + C}. \end{aligned}$$

28. Let $x = 4 \sec \theta$, then $dx = 4 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{x^2 - 16}} dx &= \int \frac{(4 \sec \theta)^2}{\sqrt{(4 \sec \theta)^2 - 16}} (4 \sec \theta \tan \theta) d\theta \\
 &= 64 \int \frac{\sec^2 \theta}{\sqrt{16 \sec^2 \theta - 16}} \sec \theta \tan \theta d\theta \\
 &= 16 \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\
 &= 16 \int \frac{\sec^2 \theta}{\tan \theta} \sec \theta \tan \theta d\theta \\
 &= 16 \int \sec^3 \theta d\theta \\
 &= 16 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C \\
 &= 8 \sec \theta \tan \theta + 8 \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{4}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{4}\right)^2 - 1} = \frac{1}{4} \sqrt{x^2 - 16}$. We obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{x^2 - 16}} dx &= 8 \left(\frac{x}{4} \right) \left(\frac{1}{4} \sqrt{x^2 - 16} \right) + 8 \ln \left| \frac{x}{4} + \frac{1}{4} \sqrt{x^2 - 16} \right| + C \\
 &= \boxed{\frac{1}{2} x \sqrt{x^2 - 16} + 8 \ln \left| \frac{x + \sqrt{x^2 - 16}}{4} \right| + C}.
 \end{aligned}$$

29. Let $x = \sec \theta$, then $dx = \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\sqrt{(\sec \theta)^2 - 1}}{\sec \theta} (\sec \theta \tan \theta) d\theta \\
 &= \int \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta \\
 &= \int \tan^2 \theta d\theta \\
 &= \int (\sec^2 \theta - 1) d\theta \\
 &= \tan \theta - \theta + C.
 \end{aligned}$$

We have $\theta = \sec^{-1} x$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{x^2 - 1}$. We obtain

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \boxed{\sqrt{x^2 - 1} - \sec^{-1} x + C}.$$

30. Let $x = \sec \theta$, then $dx = \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{\sqrt{x^2 - 1}}{x^2} dx &= \int \frac{\sqrt{(\sec \theta)^2 - 1}}{\sec^2 \theta} (\sec \theta \tan \theta) d\theta \\
 &= \int \frac{\tan \theta}{\sec^2 \theta} \sec \theta \tan \theta d\theta \\
 &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\
 &= \int \frac{\sec^2 \theta - 1}{\sec \theta} d\theta \\
 &= \int (\sec \theta - \cos \theta) d\theta \\
 &= \ln |\sec \theta + \tan \theta| - \sin \theta + C.
 \end{aligned}$$

We have $\sec \theta = x$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{x^2 - 1}$. And $\cos \theta = \frac{1}{x}$, so $\sin \theta = \sqrt{1 - \left(\frac{1}{x}\right)^2} = \frac{\sqrt{x^2 - 1}}{x}$. We obtain

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \boxed{\ln |x + \sqrt{x^2 - 1}| - \frac{\sqrt{x^2 - 1}}{x} + C}.$$

31. Let $x = 6 \sec \theta$, then $dx = 6 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{x^2 \sqrt{x^2 - 36}} &= \int \frac{6 \sec \theta \tan \theta}{(6 \sec \theta)^2 \sqrt{(6 \sec \theta)^2 - 36}} d\theta \\
 &= \frac{1}{6} \int \frac{\sec \theta \tan \theta}{\sec^2 \theta \sqrt{36 \sec^2 \theta - 36}} d\theta \\
 &= \frac{1}{36} \int \frac{\tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \frac{1}{36} \int \frac{\tan \theta}{\sec \theta \tan \theta} d\theta \\
 &= \frac{1}{36} \int \cos \theta d\theta \\
 &= \frac{1}{36} \sin \theta + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{6}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{6}\right)^2 - 1} = \frac{1}{6} \sqrt{x^2 - 36}$. So $\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{\frac{1}{6} \sqrt{x^2 - 36}}{\frac{x}{6}} = \frac{\sqrt{x^2 - 36}}{x}$. We obtain

$$\int \frac{dx}{x^2 \sqrt{x^2 - 36}} = \boxed{\frac{1}{36} \frac{\sqrt{x^2 - 36}}{x} + C}.$$

32. Let $x = 3 \sec \theta$, then $dx = 3 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{x^2 \sqrt{x^2 - 9}} &= \int \frac{3 \sec \theta \tan \theta}{(3 \sec \theta)^2 \sqrt{(3 \sec \theta)^2 - 9}} d\theta \\
 &= \frac{1}{3} \int \frac{\sec \theta \tan \theta}{\sec^2 \theta \sqrt{9 \sec^2 \theta - 9}} d\theta \\
 &= \frac{1}{9} \int \frac{\tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \frac{1}{9} \int \frac{\tan \theta}{\sec \theta \tan \theta} d\theta \\
 &= \frac{1}{9} \int \cos \theta d\theta \\
 &= \frac{1}{9} \sin \theta + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{3}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3} \sqrt{x^2 - 9}$. So $\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{\frac{1}{3} \sqrt{x^2 - 9}}{\frac{x}{3}} = \frac{\sqrt{x^2 - 9}}{x}$. We obtain

$$\int \frac{dx}{x^2 \sqrt{x^2 - 9}} = \boxed{\frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C}.$$

33. Let $x = \frac{3}{2} \sec \theta$, then $dx = \frac{3}{2} \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4x^2 - 9}} &= \int \frac{\frac{3}{2} \sec \theta \tan \theta}{\sqrt{4\left(\frac{3}{2} \sec \theta\right)^2 - 9}} d\theta \\
 &= \frac{3}{2} \int \frac{\sec \theta \tan \theta}{\sqrt{9 \sec^2 \theta - 9}} d\theta \\
 &= \frac{3}{2} \int \frac{\sec \theta \tan \theta}{\sqrt{9(\sec^2 \theta - 1)}} d\theta \\
 &= \frac{1}{2} \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\
 &= \frac{1}{2} \int \sec \theta d\theta \\
 &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\sec \theta = \frac{2x}{3}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{2x}{3}\right)^2 - 1} = \frac{1}{3} \sqrt{4x^2 - 9}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4x^2 - 9}} &= \frac{1}{2} \ln \left| \frac{2x}{3} + \frac{1}{3} \sqrt{4x^2 - 9} \right| + C \\
 &= \boxed{\frac{1}{2} \ln \left| \frac{2x + \sqrt{4x^2 - 9}}{3} \right| + C}.
 \end{aligned}$$

34. Let $x = \frac{2}{3} \sec \theta$, then $dx = \frac{2}{3} \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{9x^2 - 4}} &= \int \frac{\frac{2}{3} \sec \theta \tan \theta}{\sqrt{9\left(\frac{2}{3} \sec \theta\right)^2 - 4}} d\theta \\
 &= \frac{2}{3} \int \frac{\sec \theta \tan \theta}{\sqrt{4 \sec^2 \theta - 4}} d\theta \\
 &= \frac{2}{3} \int \frac{\sec \theta \tan \theta}{\sqrt{4(\sec^2 \theta - 1)}} d\theta \\
 &= \frac{1}{3} \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\
 &= \frac{1}{3} \int \sec \theta d\theta \\
 &= \frac{1}{3} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\sec \theta = \frac{3x}{2}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{3x}{2}\right)^2 - 1} = \frac{1}{2} \sqrt{9x^2 - 4}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{9x^2 - 4}} &= \frac{1}{3} \ln \left| \frac{3x}{2} + \frac{1}{2} \sqrt{9x^2 - 4} \right| + C \\
 &= \boxed{\frac{1}{3} \ln \left| \frac{3x + \sqrt{9x^2 - 4}}{2} \right| + C}.
 \end{aligned}$$

35. Let $x = 3 \sec \theta$, then $dx = 3 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{(x^2 - 9)^{3/2}} &= \int \frac{3 \sec \theta \tan \theta}{\left((3 \sec \theta)^2 - 9\right)^{3/2}} d\theta \\
 &= 3 \int \frac{\sec \theta \tan \theta}{(9 \sec^2 \theta - 9)^{3/2}} d\theta \\
 &= 3 \int \frac{\sec \theta \tan \theta}{(9(\sec^2 \theta - 1))^{3/2}} d\theta \\
 &= \frac{1}{9} \int \frac{\sec \theta \tan \theta}{\tan^3 \theta} d\theta \\
 &= \frac{1}{9} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\
 &= \frac{1}{9} \int \frac{\cos \theta}{\sin^2 \theta} d\theta.
 \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{(x^2 - 9)^{3/2}} &= \frac{1}{9} \int \frac{du}{u^2} \\
 &= \frac{1}{9} \left(-\frac{1}{u} \right) + C \\
 &= -\frac{1}{9} \csc \theta + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{3}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3} \sqrt{x^2 - 9}$. Then $\csc \theta = \frac{\sec \theta}{\tan \theta} =$

$\frac{x/3}{\frac{1}{3}\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$. We obtain

$$\int \frac{dx}{(x^2-9)^{3/2}} = \boxed{-\frac{1}{9} \frac{x}{\sqrt{x^2-9}} + C}.$$

36. Let $x = \frac{1}{5} \sec \theta$, then $dx = \frac{1}{5} \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{(25x^2-1)^{3/2}} &= \int \frac{\frac{1}{5} \sec \theta \tan \theta}{\left(25\left(\frac{1}{5} \sec \theta\right)^2-1\right)^{3/2}} d\theta \\ &= \frac{1}{5} \int \frac{\sec \theta \tan \theta}{(\sec^2 \theta - 1)^{3/2}} d\theta \\ &= \frac{1}{5} \int \frac{\sec \theta \tan \theta}{\tan^3 \theta} d\theta \\ &= \frac{1}{5} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \frac{1}{5} \int \frac{\cos \theta}{\sin^2 \theta} d\theta. \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{(25x^2-1)^{3/2}} &= \frac{1}{5} \int \frac{du}{u^2} \\ &= \frac{1}{5} \left(-\frac{1}{u} \right) + C \\ &= -\frac{1}{5} \csc \theta + C. \end{aligned}$$

We have $\sec \theta = 5x$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{(5x)^2 - 1} = \sqrt{25x^2 - 1}$. Then $\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{5x}{\sqrt{25x^2-1}}$. We obtain

$$\begin{aligned} \int \frac{dx}{(25x^2-1)^{3/2}} &= -\frac{1}{5} \frac{5x}{\sqrt{25x^2-1}} + C \\ &= \boxed{-\frac{x}{\sqrt{25x^2-1}} + C}. \end{aligned}$$

37. Let $x = 3 \sec \theta$, then $dx = 3 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 9)^{3/2}} &= \int \frac{(3 \sec \theta)^2 (3 \sec \theta \tan \theta)}{\left((3 \sec \theta)^2 - 9\right)^{3/2}} d\theta \\
 &= 27 \int \frac{\sec^3 \theta \tan \theta}{(9 \sec^2 \theta - 9)^{3/2}} d\theta \\
 &= 27 \int \frac{\sec^3 \theta \tan \theta}{(9(\sec^2 \theta - 1))^{3/2}} d\theta \\
 &= \int \frac{\sec^3 \theta \tan \theta}{\tan^3 \theta} d\theta \\
 &= \int \frac{\sec^3 \theta}{\tan^2 \theta} d\theta \\
 &= \int \frac{\sec \theta (\tan^2 \theta + 1)}{\tan^2 \theta} d\theta \\
 &= \int \left(\sec \theta + \frac{\sec \theta}{\tan^2 \theta} \right) d\theta \\
 &= \ln |\sec \theta + \tan \theta| + \int \frac{\cos \theta}{\sin^2 \theta} d\theta.
 \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 9)^{3/2}} &= \ln |\sec \theta + \tan \theta| + \int \frac{du}{u^2} \\
 &= \ln |\sec \theta + \tan \theta| + \left(-\frac{1}{u} \right) + C \\
 &= \ln |\sec \theta + \tan \theta| - \csc \theta + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{3}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3} \sqrt{x^2 - 9}$. Then $\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\frac{x}{3}}{\frac{1}{3} \sqrt{x^2 - 9}} = \frac{x}{\sqrt{x^2 - 9}}$. We obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 9)^{3/2}} &= \ln \left| \frac{x}{3} + \frac{1}{3} \sqrt{x^2 - 9} \right| - \frac{x}{\sqrt{x^2 - 9}} + C \\
 &= \boxed{\ln \left| \frac{x + \sqrt{x^2 - 9}}{3} \right| - \frac{x}{\sqrt{x^2 - 9}} + C}.
 \end{aligned}$$

38. Let $x = 2 \sec \theta$, then $dx = 2 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 4)^{3/2}} &= \int \frac{(2 \sec \theta)^2 (2 \sec \theta \tan \theta)}{\left((2 \sec \theta)^2 - 4\right)^{3/2}} d\theta \\
 &= 8 \int \frac{\sec^3 \theta \tan \theta}{(4 \sec^2 \theta - 4)^{3/2}} d\theta \\
 &= 8 \int \frac{\sec^3 \theta \tan \theta}{(4(\sec^2 \theta - 1))^{3/2}} d\theta \\
 &= \int \frac{\sec^3 \theta \tan \theta}{\tan^3 \theta} d\theta \\
 &= \int \frac{\sec^3 \theta}{\tan^2 \theta} d\theta \\
 &= \int \frac{\sec \theta (\tan^2 \theta + 1)}{\tan^2 \theta} d\theta \\
 &= \int \left(\sec \theta + \frac{\sec \theta}{\tan^2 \theta} \right) d\theta \\
 &= \ln |\sec \theta + \tan \theta| + \int \frac{\cos \theta}{\sin^2 \theta} d\theta.
 \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 4)^{3/2}} &= \ln |\sec \theta + \tan \theta| + \int \frac{du}{u^2} \\
 &= \ln |\sec \theta + \tan \theta| + \left(-\frac{1}{u} \right) + C \\
 &= \ln |\sec \theta + \tan \theta| - \csc \theta + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{2}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{2}\right)^2 - 1} = \frac{1}{2}\sqrt{x^2 - 4}$. Then $\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\frac{x}{2}}{\frac{1}{2}\sqrt{x^2 - 4}} = \frac{x}{\sqrt{x^2 - 4}}$. We obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 - 4)^{3/2}} &= \ln \left| \frac{x}{2} + \frac{1}{2}\sqrt{x^2 - 4} \right| - \frac{x}{\sqrt{x^2 - 4}} + C \\
 &= \boxed{\ln \left| \frac{x + \sqrt{x^2 - 4}}{2} \right| - \frac{x}{\sqrt{x^2 - 4}} + C}.
 \end{aligned}$$

39. Let $x = 4 \tan \theta$, then $dx = 4 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{16 + x^2} dx &= \int \frac{(4 \tan \theta)^2}{16 + (4 \tan \theta)^2} (4 \sec^2 \theta) d\theta \\
 &= 64 \int \frac{\tan^2 \theta}{16 + 16 \tan^2 \theta} \sec^2 \theta d\theta \\
 &= 4 \int \frac{\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta \\
 &= 4 \int \tan^2 \theta d\theta \\
 &= 4 \int (\sec^2 \theta - 1) d\theta \\
 &= 4(\tan \theta - \theta) + C \\
 &= 4\left(\frac{x}{4}\right) - 4 \tan^{-1} \left(\frac{x}{4}\right) + C \\
 &= \boxed{x - 4 \tan^{-1} \left(\frac{x}{4}\right) + C}.
 \end{aligned}$$

40. Let $x = \frac{1}{4} \tan \theta$, then $dx = \frac{1}{4} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{1 + 16x^2} dx &= \int \frac{\left(\frac{1}{4} \tan \theta\right)^2}{1 + 16\left(\frac{1}{4} \tan \theta\right)^2} \left(\frac{1}{4} \sec^2 \theta\right) d\theta \\
 &= \frac{1}{64} \int \frac{\tan^2 \theta}{1 + \tan^2 \theta} \sec^2 \theta d\theta \\
 &= \frac{1}{64} \int \frac{\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta \\
 &= \frac{1}{64} \int \tan^2 \theta d\theta \\
 &= \frac{1}{64} \int (\sec^2 \theta - 1) d\theta \\
 &= \frac{1}{64} (\tan \theta - \theta) + C \\
 &= \frac{1}{64} (4x) - \frac{1}{64} \tan^{-1} (4x) + C \\
 &= \boxed{\frac{x}{16} - \frac{1}{64} \tan^{-1} (4x) + C}.
 \end{aligned}$$

41. Let $x = \frac{2}{5} \sin \theta$, then $dx = \frac{2}{5} \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{4 - 25x^2} dx &= \int \sqrt{4 - 25\left(\frac{2}{5} \sin \theta\right)^2} \left(\frac{2}{5} \cos \theta\right) d\theta \\
 &= \frac{2}{5} \int \sqrt{4 - 4 \sin^2 \theta} \cos \theta d\theta \\
 &= \frac{2}{5} \int \sqrt{4(1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= \frac{4}{5} \int \cos^2 \theta d\theta \\
 &= \frac{4}{5} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{2}{5} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{2}{5} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{2}{5} \theta + \frac{1}{5} \sin(2\theta) + C \\
 &= \frac{2}{5} \theta + \frac{2}{5} \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{5x}{2} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{5x}{2} \right)^2} = \frac{1}{2} \sqrt{4 - 25x^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{4 - 25x^2} dx &= \frac{2}{5} \sin^{-1} \left(\frac{5x}{2} \right) + \frac{2}{5} \left(\frac{5x}{2} \right) \left(\frac{1}{2} \sqrt{4 - 25x^2} \right) + C \\
 &= \boxed{\frac{2}{5} \sin^{-1} \left(\frac{5x}{2} \right) + \frac{1}{2} x \sqrt{4 - 25x^2} + C}.
 \end{aligned}$$

42. Let $x = \frac{3}{4} \sin \theta$, then $dx = \frac{3}{4} \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{9 - 16x^2} dx &= \int \sqrt{9 - 16\left(\frac{3}{4} \sin \theta\right)^2} \left(\frac{3}{4} \cos \theta\right) d\theta \\
 &= \frac{3}{4} \int \sqrt{9 - 9 \sin^2 \theta} \cos \theta d\theta \\
 &= \frac{3}{4} \int \sqrt{9(1 - \sin^2 \theta)} \cos \theta d\theta \\
 &= \frac{9}{4} \int \cos^2 \theta d\theta \\
 &= \frac{9}{4} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{9}{8} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{9}{8} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{9}{8} \theta + \frac{9}{16} \sin(2\theta) + C \\
 &= \frac{9}{8} \theta + \frac{9}{8} \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{4x}{3} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{4x}{3} \right)^2} = \frac{1}{3} \sqrt{9 - 16x^2}$. We obtain

$$\begin{aligned} \int \sqrt{9 - 16x^2} \, dx &= \frac{9}{8} \sin^{-1} \left(\frac{4x}{3} \right) + \frac{9}{8} \left(\frac{4x}{3} \right) \left(\frac{1}{3} \sqrt{9 - 16x^2} \right) + C \\ &= \boxed{\frac{9}{8} \sin^{-1} \left(\frac{4x}{3} \right) + \frac{1}{2} x \sqrt{9 - 16x^2} + C}.. \end{aligned}$$

43. Let $x = \frac{2}{5} \sin \theta$, then $dx = \frac{2}{5} \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{1}{(4 - 25x^2)^{3/2}} \, dx &= \int \frac{1}{\left(4 - 25 \left(\frac{2}{5} \sin \theta \right)^2 \right)^{3/2}} \left(\frac{2}{5} \cos \theta \right) d\theta \\ &= \frac{2}{5} \int \frac{1}{4^{3/2} (\cos^2 \theta)^{3/2}} \cos \theta \, d\theta \\ &= \frac{1}{20} \int \frac{\cos \theta}{\cos^3 \theta} \, d\theta \\ &= \frac{1}{20} \int \sec^2 \theta \, d\theta \\ &= \frac{1}{20} \tan \theta + C. \end{aligned}$$

We have $\sin \theta = \frac{5x}{2}$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{5x}{2} \right)^2} = \frac{1}{2} \sqrt{4 - 25x^2}$. So $\tan \theta = \frac{5x/2}{\frac{1}{2} \sqrt{4 - 25x^2}} = \frac{5x}{\sqrt{4 - 25x^2}}$. We obtain

$$\begin{aligned} \int \frac{1}{(4 - 25x^2)^{3/2}} \, dx &= \frac{1}{20} \left(\frac{5x}{\sqrt{4 - 25x^2}} \right) + C \\ &= \boxed{\frac{x}{4\sqrt{4 - 25x^2}} + C}. \end{aligned}$$

44. Let $x = \frac{1}{3} \sin \theta$, then $dx = \frac{1}{3} \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{1}{(1 - 9x^2)^{3/2}} \, dx &= \int \frac{1}{\left(1 - 9 \left(\frac{1}{3} \sin \theta \right)^2 \right)^{3/2}} \left(\frac{1}{3} \cos \theta \right) d\theta \\ &= \int \frac{1}{(1 - \sin^2 \theta)^{3/2}} \left(\frac{1}{3} \cos \theta \right) d\theta \\ &= \frac{1}{3} \int \frac{1}{(\cos^2 \theta)^{3/2}} \cos \theta \, d\theta \\ &= \frac{1}{3} \int \frac{\cos \theta}{\cos^3 \theta} \, d\theta \\ &= \frac{1}{3} \int \sec^2 \theta \, d\theta \\ &= \frac{1}{3} \tan \theta + C. \end{aligned}$$

We have $\sin \theta = 3x$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (3x)^2} = \sqrt{1 - 9x^2}$. So $\tan \theta =$

$\frac{3x}{\sqrt{1-9x^2}}$. We obtain

$$\begin{aligned}\int \frac{1}{(1-9x^2)^{3/2}} dx &= \frac{1}{3} \left(\frac{3x}{\sqrt{1-9x^2}} \right) + C \\ &= \boxed{\frac{x}{\sqrt{1-9x^2}} + C}.\end{aligned}$$

45. Let $x = \frac{2}{5} \tan \theta$, then $dx = \frac{2}{5} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{4+25x^2} dx &= \int \sqrt{4+25\left(\frac{2}{5} \tan \theta\right)^2} \left(\frac{2}{5} \sec^2 \theta\right) d\theta \\ &= \frac{2}{5} \int \sqrt{4+4 \tan^2 \theta} \sec^2 \theta d\theta \\ &= \frac{2}{5} \int \sqrt{4(1+\tan^2 \theta)} \sec^2 \theta d\theta \\ &= \frac{2}{5} \int (2 \sec \theta) \sec^2 \theta d\theta \\ &= \frac{4}{5} \int \sec^3 \theta d\theta \\ &= \frac{4}{5} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\ &= \frac{2}{5} \sec \theta \tan \theta + \frac{2}{5} \ln |\sec \theta + \tan \theta| + C.\end{aligned}$$

We have $\tan \theta = \frac{5x}{2}$, and $\sec \theta = \sqrt{1+\tan^2 \theta} = \sqrt{1+\left(\frac{5x}{2}\right)^2} = \frac{1}{2} \sqrt{4+25x^2}$. We obtain

$$\begin{aligned}\int \sqrt{4+25x^2} dx &= \frac{2}{5} \left(\frac{1}{2} \sqrt{4+25x^2} \right) \left(\frac{5x}{2} \right) + \frac{2}{5} \ln \left| \frac{1}{2} \sqrt{4+25x^2} + \frac{5x}{2} \right| + C \\ &= \boxed{\frac{1}{2} x \sqrt{4+25x^2} + \frac{2}{5} \ln \left(\frac{\sqrt{4+25x^2}+5x}{2} \right) + C}.\end{aligned}$$

46. Let $x = \frac{3}{4} \tan \theta$, then $dx = \frac{3}{4} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{9+16x^2} dx &= \int \sqrt{9+16\left(\frac{3}{4} \tan \theta\right)^2} \left(\frac{3}{4} \sec^2 \theta\right) d\theta \\ &= \frac{3}{4} \int \sqrt{9+9 \tan^2 \theta} \sec^2 \theta d\theta \\ &= \frac{3}{4} \int \sqrt{9(1+\tan^2 \theta)} \sec^2 \theta d\theta \\ &= \frac{3}{4} \int (3 \sec \theta) \sec^2 \theta d\theta \\ &= \frac{9}{4} \int \sec^3 \theta d\theta \\ &= \frac{9}{4} \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\ &= \frac{9}{8} \sec \theta \tan \theta + \frac{9}{8} \ln |\sec \theta + \tan \theta| + C.\end{aligned}$$

We have $\tan \theta = \frac{4x}{3}$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{4x}{3}\right)^2} = \frac{1}{3}\sqrt{9 + 16x^2}$. We obtain

$$\begin{aligned} \int \sqrt{9 + 16x^2} \, dx &= \frac{9}{8} \left(\frac{1}{3} \sqrt{9 + 16x^2} \right) \left(\frac{4x}{3} \right) + \frac{9}{8} \ln \left| \frac{1}{3} \sqrt{9 + 16x^2} + \frac{4x}{3} \right| + C \\ &= \boxed{\frac{1}{2}x\sqrt{9 + 16x^2} + \frac{9}{8} \ln \left(\frac{\sqrt{9 + 16x^2} + 4x}{3} \right) + C}. \end{aligned}$$

47. Let $x = 4 \sec \theta$, then $dx = 4 \sec \theta \tan \theta \, d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2 - 16}} &= \int \frac{4 \sec \theta \tan \theta}{(4 \sec \theta)^3 \sqrt{(4 \sec \theta)^2 - 16}} \, d\theta \\ &= \frac{1}{16} \int \frac{\sec \theta \tan \theta}{\sec^3 \theta \sqrt{16 \sec^2 \theta - 16}} \, d\theta \\ &= \frac{1}{64} \int \frac{\tan \theta}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}} \, d\theta \\ &= \frac{1}{64} \int \frac{\tan \theta}{\sec^2 \theta \tan \theta} \, d\theta \\ &= \frac{1}{64} \int \cos^2 \theta \, d\theta \\ &= \frac{1}{64} \int \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \frac{1}{128} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\ &= \frac{1}{128} \theta + \frac{1}{256} (2 \sin \theta \cos \theta) + C \\ &= \frac{1}{128} \theta + \frac{1}{128} \sin \theta \cos \theta + C. \end{aligned}$$

We have $\theta = \sec^{-1} \left(\frac{x}{4} \right)$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{4} \right)^2 - 1} = \frac{1}{4} \sqrt{x^2 - 16}$. So $\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{\frac{1}{4} \sqrt{x^2 - 16}}{\frac{x}{4}} = \frac{\sqrt{x^2 - 16}}{x}$, and $\cos \theta = \frac{4}{x}$. We obtain

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2 - 16}} &= \frac{1}{128} \sec^{-1} \left(\frac{x}{4} \right) + \frac{1}{128} \left(\frac{\sqrt{x^2 - 16}}{x} \right) \left(\frac{4}{x} \right) + C \\ &= \boxed{\frac{1}{128} \sec^{-1} \left(\frac{x}{4} \right) + \frac{\sqrt{x^2 - 16}}{32x^2} + C}. \end{aligned}$$

48. Let $x = \sec \theta$, then $dx = \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{x^3 \sqrt{x^2 - 1}} &= \int \frac{\sec \theta \tan \theta}{(\sec \theta)^3 \sqrt{(\sec \theta)^2 - 1}} d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\sec^3 \theta \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \int \frac{\tan \theta}{\sec^2 \theta \tan \theta} d\theta \\
 &= \int \cos^2 \theta d\theta \\
 &= \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{1}{2} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{1}{2} \theta + \frac{1}{4} (2 \sin \theta \cos \theta) + C \\
 &= \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sec^{-1} x$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{x^2 - 1}$. So $\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{\sqrt{x^2 - 1}}{x}$, and $\cos \theta = \frac{1}{x}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{x^3 \sqrt{x^2 - 1}} &= \frac{1}{2} \sec^{-1} x + \frac{1}{2} \left(\frac{\sqrt{x^2 - 1}}{x} \right) \left(\frac{1}{x} \right) + C \\
 &= \boxed{\frac{1}{2} \sec^{-1} x + \frac{\sqrt{x^2 - 1}}{2x^2} + C}.
 \end{aligned}$$

49. Let $x = \sin \theta$, then $dx = \cos \theta d\theta$. The lower limit of integration is $\theta = \sin^{-1} 0 = 0$, and the upper limit of integration is $\theta = \sin^{-1} 1 = \frac{\pi}{2}$. We substitute and obtain

$$\begin{aligned}
 \int_0^1 \sqrt{1 - x^2} dx &= \int_0^{\pi/2} \sqrt{1 - \sin^2 \theta} (\cos \theta) d\theta \\
 &= \int_0^{\pi/2} \cos^2 \theta d\theta \\
 &= \int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{1}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/2} \\
 &= \frac{1}{2} \left[\left(\frac{\pi}{2} + \frac{1}{2} \sin \left(2 \left(\frac{\pi}{2} \right) \right) \right) - \left(0 + \frac{1}{2} \sin(2(0)) \right) \right] \\
 &= \boxed{\frac{1}{4} \pi}.
 \end{aligned}$$

50. Let $x = \frac{1}{2} \sin \theta$, then $dx = \frac{1}{2} \cos \theta d\theta$. The lower limit of integration is $\theta = \sin^{-1}(2(0)) = 0$, and the upper limit of integration is $\theta = \sin^{-1}(2(\frac{1}{2})) = \frac{\pi}{2}$. We substitute and obtain

$$\begin{aligned}
 \int_0^{1/2} \sqrt{1-4x^2} dx &= \int_0^{\pi/2} \sqrt{1-4\left(\frac{1}{2} \sin \theta\right)^2} \left(\frac{1}{2} \cos \theta\right) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \sqrt{1-\sin^2 \theta} (\cos \theta) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta \\
 &= \frac{1}{4} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/2} \\
 &= \frac{1}{4} \left[\left(\frac{\pi}{2} + \frac{1}{2} \sin\left(2\left(\frac{\pi}{2}\right)\right) \right) - \left(0 + \frac{1}{2} \sin(2(0)) \right) \right] \\
 &= \boxed{\frac{1}{8}\pi}.
 \end{aligned}$$

51. Let $x = \tan \theta$, then $dx = \sec^2 \theta d\theta$. The lower limit of integration is $\theta = \tan^{-1} 0 = 0$, and the upper limit of integration is $\theta = \tan^{-1} 1 = \frac{\pi}{4}$. We substitute and obtain

$$\begin{aligned}
 \int_0^1 \sqrt{1+x^2} dx &= \int_0^{\pi/4} \sqrt{1+(\tan \theta)^2} (\sec^2 \theta) d\theta \\
 &= \int_0^{\pi/4} (\sec \theta) \sec^2 \theta d\theta \\
 &= \int_0^{\pi/4} \sec^3 \theta d\theta \\
 &= \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \\
 &= \frac{1}{2} \sec \frac{\pi}{4} \tan \frac{\pi}{4} + \frac{1}{2} \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \left(\frac{1}{2} \sec 0 \tan 0 + \frac{1}{2} \ln |\sec 0 + \tan 0| \right) \\
 &= \boxed{\frac{\sqrt{2} + \ln(\sqrt{2}+1)}{2}}.
 \end{aligned}$$

52. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3 \tan \theta$, then $dx = 3 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{9+x^2}} dx &= \int \frac{(3 \tan \theta)^2}{\sqrt{9+(3 \tan \theta)^2}} (3 \sec^2 \theta) d\theta \\
 &= 27 \int \frac{\tan^2 \theta}{\sqrt{9 \tan^2 \theta + 9}} \sec^2 \theta d\theta \\
 &= 9 \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta d\theta \\
 &= 9 \int \tan^2 \theta \sec \theta d\theta \\
 &= 9 \int (\sec^2 \theta - 1) \sec \theta d\theta \\
 &= 9 \int (\sec^3 \theta - \sec \theta) d\theta \\
 &= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

Since $\tan \theta = \frac{x}{3}$, $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{x}{3}\right)^2} = \frac{1}{3} \sqrt{9 + x^2}$. So

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{9+x^2}} dx &= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| + C \\
 &= \frac{9}{2} \left(\frac{1}{3} \sqrt{9+x^2} \right) \left(\frac{x}{3} \right) - \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{9+x^2} + \frac{x}{3} \right| + C \\
 &= \frac{1}{2} x \sqrt{x^2+9} - \frac{9}{2} \ln \left| \frac{1}{3} x + \frac{1}{3} \sqrt{x^2+9} \right| + C
 \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^2 \frac{x^2}{\sqrt{9+x^2}} dx &= \left[\frac{1}{2} x \sqrt{x^2+9} - \frac{9}{2} \ln \left| \frac{1}{3} x + \frac{1}{3} \sqrt{x^2+9} \right| \right]_0^2 \\
 &= \left(\frac{1}{2} 2 \sqrt{2^2+9} - \frac{9}{2} \ln \left| \frac{1}{3} 2 + \frac{1}{3} \sqrt{2^2+9} \right| \right) \\
 &\quad - \left(\frac{1}{2} 0 \sqrt{0^2+9} - \frac{9}{2} \ln \left| \frac{1}{3} 0 + \frac{1}{3} \sqrt{0^2+9} \right| \right) \\
 &= \left(\sqrt{13} - \frac{9}{2} \ln \left(\frac{1}{3} \sqrt{13} + \frac{2}{3} \right) \right) - 0 \\
 &= \boxed{\sqrt{13} - \frac{9}{2} \ln \left(\frac{1}{3} \sqrt{13} + \frac{2}{3} \right)}.
 \end{aligned}$$

53. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3 \sec \theta$, then $dx = 3 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{x^2-9}} dx &= \int \frac{(3 \sec \theta)^2}{\sqrt{(3 \sec \theta)^2-9}} (3 \sec \theta \tan \theta) d\theta \\
 &= 27 \int \frac{\sec^2 \theta}{\sqrt{9 \sec^2 \theta-9}} \sec \theta \tan \theta d\theta \\
 &= 9 \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta-1}} \sec \theta \tan \theta d\theta \\
 &= 9 \int \frac{\sec^2 \theta}{\tan \theta} \sec \theta \tan \theta d\theta \\
 &= 9 \int \sec^3 \theta d\theta \\
 &= \frac{9}{2} \sec \theta \tan \theta + \frac{9}{2} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

Since $\sec \theta = \frac{x}{3}$, $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{3}\right)^2 - 1} = \frac{1}{3} \sqrt{x^2 - 9}$. So

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{x^2-9}} dx &= \frac{9}{2} \left(\frac{1}{3} \sqrt{x^2-9} \right) \frac{x}{3} + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{x^2-9} + \frac{x}{3} \right| + C \\
 &= \frac{1}{2} x \sqrt{x^2-9} + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{x^2-9} + \frac{x}{3} \right| + C.
 \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_4^5 \frac{x^2}{\sqrt{x^2-9}} dx &= \left[\frac{1}{2} x \sqrt{x^2-9} + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{x^2-9} + \frac{x}{3} \right| \right]_4^5 \\
 &= \left(\frac{1}{2} 5 \sqrt{5^2-9} + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{5^2-9} + \frac{5}{3} \right| \right) - \left(\frac{1}{2} 4 \sqrt{4^2-9} + \frac{9}{2} \ln \left| \frac{1}{3} \sqrt{4^2-9} + \frac{4}{3} \right| \right) \\
 &= \frac{9}{2} \ln 3 + 10 - \left(\frac{9}{2} \ln \left(\frac{1}{3} \sqrt{7} + \frac{4}{3} \right) + 2\sqrt{7} \right) \\
 &= \frac{9}{2} \ln 3 + 10 - \frac{9}{2} \ln (\sqrt{7} + 4) + \frac{9}{2} \ln 3 - 2\sqrt{7} \\
 &= \boxed{10 - 2\sqrt{7} + 9 \ln 3 - \frac{9}{2} \ln (4 + \sqrt{7})}.
 \end{aligned}$$

54. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \frac{1}{2} \sec \theta$, then $dx = \frac{1}{2} \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{4x^2 - 1}} dx &= \int \frac{\left(\frac{1}{2} \sec \theta\right)^2}{\sqrt{4\left(\frac{1}{2} \sec \theta\right)^2 - 1}} \left(\frac{1}{2} \sec \theta \tan \theta\right) d\theta \\
 &= \frac{1}{8} \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{8} \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta - 1}} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{8} \int \frac{\sec^2 \theta}{\tan \theta} \sec \theta \tan \theta d\theta \\
 &= \frac{1}{8} \int \sec^3 \theta d\theta \\
 &= \frac{1}{16} \sec \theta \tan \theta + \frac{1}{16} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

Since $\sec \theta = 2x$, $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{(2x)^2 - 1} = \sqrt{4x^2 - 1}$. So

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{4x^2 - 1}} dx &= \frac{1}{16} (2x) \sqrt{4x^2 - 1} + \frac{1}{16} \ln |2x + \sqrt{4x^2 - 1}| + C \\
 &= \frac{1}{8} x \sqrt{4x^2 - 1} + \frac{1}{16} \ln |2x + \sqrt{4x^2 - 1}| + C.
 \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_1^2 \frac{x^2}{\sqrt{4x^2 - 1}} dx &= \left[\frac{1}{8} x \sqrt{4x^2 - 1} + \frac{1}{16} \ln |2x + \sqrt{4x^2 - 1}| \right]_1^2 \\
 &= \left(\frac{1}{8} 2 \sqrt{4(2)^2 - 1} + \frac{1}{16} \ln |2(2) + \sqrt{4(2)^2 - 1}| \right) \\
 &\quad - \left(\frac{1}{8} (1) \sqrt{4(1)^2 - 1} + \frac{1}{16} \ln |2(1) + \sqrt{4(1)^2 - 1}| \right) \\
 &= \left(\frac{1}{16} \ln (\sqrt{15} + 4) + \frac{1}{4} \sqrt{15} \right) - \left(\frac{1}{16} \ln (\sqrt{3} + 2) + \frac{1}{8} \sqrt{3} \right) \\
 &= \boxed{\frac{1}{16} \ln (\sqrt{15} + 4) - \frac{1}{16} \ln (\sqrt{3} + 2) - \frac{1}{8} \sqrt{3} + \frac{1}{4} \sqrt{15}}.
 \end{aligned}$$

55. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 4 \sin \theta$, then $dx = 4 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{(16-x^2)^{3/2}} dx &= \int \frac{(4 \sin \theta)^2}{(16 - (4 \sin \theta)^2)^{3/2}} (4 \cos \theta) d\theta \\
 &= 64 \int \frac{\sin^2 \theta}{(16 \cos^2 \theta)^{3/2}} \cos \theta d\theta \\
 &= \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta \\
 &= \int \tan^2 \theta d\theta \\
 &= \int (\sec^2 \theta - 1) d\theta \\
 &= \tan \theta - \theta + C \\
 &= \frac{\sin \theta}{\cos \theta} - \theta + C.
 \end{aligned}$$

Since $\sin \theta = \frac{x}{4}$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{x}{4}\right)^2} = \frac{1}{4} \sqrt{16 - x^2}$. So we have

$$\begin{aligned}
 \int \frac{x^2}{(16-x^2)^{3/2}} dx &= \frac{\frac{x}{4}}{\frac{1}{4} \sqrt{16-x^2}} - \sin^{-1} \frac{x}{4} + C \\
 &= \frac{x}{\sqrt{16-x^2}} - \sin^{-1} \frac{x}{4} + C.
 \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^2 \frac{x^2}{(16-x^2)^{3/2}} dx &= \left[\frac{x}{\sqrt{16-x^2}} - \sin^{-1} \frac{x}{4} \right]_0^2 \\
 &= \left(\frac{2}{\sqrt{16-2^2}} - \sin^{-1} \frac{2}{4} \right) - \left(\frac{0}{\sqrt{4-0^2}} - \sin^{-1} \frac{0}{4} \right) \\
 &= \frac{\sqrt{3}}{3} - \frac{\pi}{6} - 0 \\
 &= \boxed{\frac{\sqrt{3}}{3} - \frac{\pi}{6}}.
 \end{aligned}$$

56. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 5 \sin \theta$, then $dx = 5 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{(25-x^2)^{3/2}} dx &= \int \frac{(5 \sin \theta)^2}{(25 - (5 \sin \theta)^2)^{3/2}} (5 \cos \theta) d\theta \\
 &= 125 \int \frac{\sin^2 \theta}{(25 \cos^2 \theta)^{3/2}} \cos \theta d\theta \\
 &= \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta \\
 &= \int \tan^2 \theta d\theta \\
 &= \int (\sec^2 \theta - 1) d\theta \\
 &= \tan \theta - \theta + C.
 \end{aligned}$$

Since $\sin \theta = \frac{x}{5}$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{x}{5}\right)^2} = \frac{1}{5}\sqrt{25 - x^2}$. So $\tan \theta = \frac{\frac{x}{5}}{\frac{1}{5}\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$, and $\theta = \sin^{-1} \frac{x}{5}$. So

$$\int \frac{x^2}{(25 - x^2)^{3/2}} dx = \frac{x}{\sqrt{25 - x^2}} - \sin^{-1} \frac{x}{5} + C.$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_0^1 \frac{x^2}{(25 - x^2)^{3/2}} dx &= \left[\frac{x}{\sqrt{25 - x^2}} - \sin^{-1} \frac{x}{5} \right]_0^1 \\ &= \left(\frac{1}{\sqrt{25 - 1^2}} - \sin^{-1} \frac{1}{5} \right) - \left(\frac{0}{\sqrt{25 - 0^2}} - \sin^{-1} \frac{0}{5} \right) \\ &= \left(\frac{1}{12}\sqrt{6} - \sin^{-1} \frac{1}{5} \right) - (0) \\ &= \boxed{\frac{1}{12}\sqrt{6} - \sin^{-1} \frac{1}{5}}. \end{aligned}$$

57. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3 \tan \theta$, then $dx = 3 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{x^2}{9 + x^2} dx &= \int \frac{(3 \tan \theta)^2}{9 + (3 \tan \theta)^2} (3 \sec^2 \theta) d\theta \\ &= 27 \int \frac{\tan^2 \theta}{9 \tan^2 \theta + 9} \sec^2 \theta d\theta \\ &= 3 \int \frac{\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta \\ &= 3 \int \tan^2 \theta d\theta \\ &= 3 \int (\sec^2 \theta - 1) d\theta \\ &= 3(\tan \theta - \theta) + C. \end{aligned}$$

Since $\tan \theta = \frac{x}{3}$, and $\theta = \tan^{-1} \frac{x}{3}$. So

$$\begin{aligned} \int \frac{x^2}{9 + x^2} dx &= 3 \left(\frac{x}{3} - \tan^{-1} \frac{x}{3} \right) + C \\ &= x - 3 \tan^{-1} \frac{x}{3} + C. \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_0^3 \frac{x^2}{9 + x^2} dx &= \left[x - 3 \tan^{-1} \frac{x}{3} \right]_0^3 \\ &= \left(3 - 3 \tan^{-1} \frac{3}{3} \right) - \left(0 - 3 \tan^{-1} \frac{0}{3} \right) \\ &= \left(3 - \frac{3}{4}\pi \right) - (0) \\ &= \boxed{3 - \frac{3}{4}\pi}. \end{aligned}$$

58. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 5 \tan \theta$, then $dx = 5 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^2}{25 + x^2} dx &= \int \frac{(5 \tan \theta)^2}{25 + (5 \tan \theta)^2} (5 \sec^2 \theta) d\theta \\
 &= 125 \int \frac{\tan^2 \theta}{25 \tan^2 \theta + 25} \sec^2 \theta d\theta \\
 &= 5 \int \frac{\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta \\
 &= 5 \int \tan^2 \theta d\theta \\
 &= 5 \int (\sec^2 \theta - 1) \sec \theta d\theta \\
 &= 5 (\tan \theta - \theta) + C.
 \end{aligned}$$

Since $\tan \theta = \frac{x}{5}$ and $\theta = \tan^{-1} \frac{x}{5}$, we have

$$\begin{aligned}
 \int \frac{x^2}{25 + x^2} dx &= 5 \left(\frac{x}{5} - \tan^{-1} \frac{x}{5} \right) + C \\
 &= x - 5 \tan^{-1} \frac{x}{5} + C.
 \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^1 \frac{x^2}{25 + x^2} dx &= \left[x - 5 \tan^{-1} \frac{x}{5} \right]_0^1 \\
 &= 1 - 5 \tan^{-1} \frac{1}{5} - \left(0 - 5 \tan^{-1} \frac{0}{5} \right) \\
 &= \boxed{1 - 5 \tan^{-1} \frac{1}{5}}.
 \end{aligned}$$

Applications and Extensions

59. Let $x = a \sin \theta$, then $dx = a \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 - (a \sin \theta)^2} (a \cos \theta) d\theta \\
 &= a \int \sqrt{a^2 - a^2 \sin^2 \theta} \cos \theta d\theta \\
 &= a^2 \int \cos^2 \theta d\theta \\
 &= a^2 \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{a^2}{2} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{x}{a} \right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/a)^2} = \frac{1}{a} \sqrt{a^2 - x^2}$. We obtain

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \frac{a^2}{2} \left(\sin^{-1} \left(\frac{x}{a} \right) + \left(\frac{x}{a} \right) \left(\frac{1}{a} \sqrt{a^2 - x^2} \right) \right) + C \\ &= \frac{1}{2} a^2 \sin^{-1} \left(\frac{x}{a} \right) + \frac{1}{2} x \sqrt{a^2 - x^2} + C. \end{aligned}$$

The area of the ellipse is given by $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$, so we obtain

$$\begin{aligned} A &= 4 \left(\frac{b}{a} \right) \left[\frac{1}{2} a^2 \sin^{-1} \left(\frac{x}{a} \right) + \frac{1}{2} x \sqrt{a^2 - x^2} \right]_0^a \\ &= 4 \left(\frac{b}{a} \right) \left[\frac{1}{2} a^2 \sin^{-1} \left(\frac{a}{a} \right) + \frac{1}{2} a \sqrt{a^2 - a^2} - \left(\frac{1}{2} a^2 \sin^{-1} \left(\frac{0}{a} \right) + \frac{1}{2} (0) \sqrt{a^2 - 0^2} \right) \right] \\ &= \boxed{\pi ab}. \end{aligned}$$

60. (a) We recognize $\int_0^2 \sqrt{4 - x^2} dx$ as the area of the quarter-circle in the first quadrant, centered at $(0, 0)$, with radius 2. So $\int_0^2 \sqrt{4 - x^2} dx = \left(\frac{1}{4} \right) \pi (2)^2 = \boxed{\pi}$.

(b) Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. The lower limit of integration is $\theta = \sin^{-1} \left(\frac{0}{2} \right) = 0$, and the upper limit of integration is $\theta = \sin^{-1} \left(\frac{2}{2} \right) = \frac{\pi}{2}$. We substitute and obtain

$$\begin{aligned} \int_0^2 \sqrt{4 - x^2} dx &= \int_0^{\pi/2} \sqrt{4 - (2 \sin \theta)^2} (2 \cos \theta) d\theta \\ &= 2 \int_0^{\pi/2} \sqrt{4 - 4 \sin^2 \theta} \cos \theta d\theta \\ &= 4 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 4 \int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta \\ &= 2 \int_0^{\pi/2} (1 + \cos(2\theta)) d\theta \\ &= 2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} \\ &= 2 \left(\frac{\pi}{2} + \frac{1}{2} \sin \left(2 \left(\frac{\pi}{2} \right) \right) \right) - 2 \left(0 + \frac{1}{2} \sin(2(0)) \right) \\ &= \boxed{\pi}. \end{aligned}$$

(c) The area is given by $\int_{-a}^a \sqrt{a^2 - x^2} dx = 2 \int_0^a \sqrt{a^2 - x^2} dx$. Let $x = a \sin \theta$, then $dx = a \cos \theta d\theta$. The lower limit of integration is $\theta = \sin^{-1} \left(\frac{0}{a} \right) = 0$, and the upper

limit of integration is $\theta = \sin^{-1}\left(\frac{a}{a}\right) = \frac{\pi}{2}$. We substitute and obtain

$$\begin{aligned}
 2 \int_0^a \sqrt{a^2 - x^2} \, dx &= 2 \int_0^{\pi/2} \sqrt{a^2 - (a \sin \theta)^2} (a \cos \theta) \, d\theta \\
 &= 2a \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cos \theta \, d\theta \\
 &= 2a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\
 &= 2a^2 \int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} \, d\theta \\
 &= a^2 \int_0^{\pi/2} (1 + \cos(2\theta)) \, d\theta \\
 &= a^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} \\
 &= a^2 \left(\frac{\pi}{2} + \frac{1}{2} \sin \left(2 \left(\frac{\pi}{2} \right) \right) \right) - a^2 \left(0 + \frac{1}{2} \sin(2(0)) \right) \\
 &= \boxed{\frac{1}{2} \pi a^2}.
 \end{aligned}$$

61. The average value is given by $\frac{1}{1-0} \int_0^1 \frac{1}{\sqrt{9-4x^2}} \, dx = \int_0^1 \frac{1}{\sqrt{9-4x^2}} \, dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \frac{3}{2} \sin \theta$, then $dx = \frac{3}{2} \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{1}{\sqrt{9-4x^2}} \, dx &= \int \frac{1}{\sqrt{9-4\left(\frac{3}{2} \sin \theta\right)^2}} \left(\frac{3}{2} \cos \theta \right) \, d\theta \\
 &= \frac{3}{2} \int \frac{1}{\sqrt{9-9 \sin^2 \theta}} (\cos \theta) \, d\theta \\
 &= \frac{1}{2} \int \frac{\cos \theta}{\cos \theta} \, d\theta \\
 &= \frac{1}{2} \int d\theta \\
 &= \frac{1}{2} \theta + C \\
 &= \frac{1}{2} \sin^{-1} \frac{2x}{3} + C.
 \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{9-4x^2}} \, dx &= \left[\frac{1}{2} \sin^{-1} \frac{2x}{3} \right]_0^1 \\
 &= \left(\frac{1}{2} \sin^{-1} \frac{2(1)}{3} \right) - \left(\frac{1}{2} \sin^{-1} \frac{2(0)}{3} \right) \\
 &= \left(\frac{1}{2} \sin^{-1} \frac{2}{3} \right) - (0) \\
 &= \boxed{\frac{1}{2} \sin^{-1} \left(\frac{2}{3} \right)}.
 \end{aligned}$$

62. The average value is given by $\frac{1}{7-2} \int_2^7 \sqrt{x^2-4} dx = \frac{1}{5} \int_2^7 \sqrt{x^2-4} dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 2 \sec \theta$, then $dx = 2 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned} \frac{1}{5} \int \sqrt{x^2-4} dx &= \frac{1}{5} \int \sqrt{(2 \sec \theta)^2 - 4} (2 \sec \theta \tan \theta) d\theta \\ &= \frac{2}{5} \int \sqrt{4 \sec^2 \theta - 4} \sec \theta \tan \theta d\theta \\ &= \frac{4}{5} \int \sec \theta \tan^2 \theta d\theta \\ &= \frac{4}{5} \int (\sec^3 \theta - \sec \theta) d\theta \\ &= \frac{4}{5} \left(\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C. \end{aligned}$$

Since $\sec \theta = \frac{x}{2}$, $\tan \theta = \sqrt{\left(\frac{x}{2}\right)^2 - 1} = \frac{1}{2} \sqrt{x^2 - 4}$. So

$$\begin{aligned} \frac{1}{5} \int \sqrt{x^2-4} dx &= \frac{4}{5} \left(\frac{1}{2} \left(\frac{x}{2} \right) \left(\frac{1}{2} \sqrt{x^2-4} \right) - \frac{1}{2} \ln \left| \frac{x}{2} + \frac{1}{2} \sqrt{x^2-4} \right| \right) + C \\ &= \frac{1}{10} x \sqrt{x^2-4} - \frac{2}{5} \ln \left| \frac{1}{2} x + \frac{1}{2} \sqrt{x^2-4} \right| + C. \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned} \frac{1}{5} \int_2^7 \sqrt{x^2-4} dx &= \left[\frac{1}{10} x \sqrt{x^2-4} - \frac{2}{5} \ln \left| \frac{1}{2} x + \frac{1}{2} \sqrt{x^2-4} \right| \right]_2^7 \\ &= \left(\frac{1}{10} 7 \sqrt{7^2-4} - \frac{2}{5} \ln \left| \frac{1}{2} 7 + \frac{1}{2} \sqrt{7^2-4} \right| \right) \\ &\quad - \left(\frac{1}{10} 2 \sqrt{2^2-4} - \frac{2}{5} \ln \left| \frac{1}{2} 2 + \frac{1}{2} \sqrt{2^2-4} \right| \right) \\ &= \frac{21}{10} \sqrt{5} - \frac{2}{5} \ln \left(\frac{3}{2} \sqrt{5} + \frac{7}{2} \right) - (0) \\ &= \boxed{\frac{21}{10} \sqrt{5} - \frac{2}{5} \ln \left(\frac{3}{2} \sqrt{5} + \frac{7}{2} \right)}. \end{aligned}$$

63. The area under the graph is given by $\int_0^2 \frac{x^3}{\sqrt{9-x^2}} dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3 \sin \theta$, then

$dx = 3 \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{9-x^2}} \, dx &= \int \frac{(3 \sin \theta)^3}{\sqrt{9-(3 \sin \theta)^2}} (3 \cos \theta) \, d\theta \\
 &= 81 \int \frac{\sin^3 \theta}{\sqrt{9-9 \sin^2 \theta}} \cos \theta \, d\theta \\
 &= 27 \int \frac{\sin^3 \theta}{\cos \theta} \cos \theta \, d\theta \\
 &= 27 \int (1 - \cos^2 \theta) \sin \theta \, d\theta \\
 &= 27 \int (\sin \theta - \sin \theta \cos^2 \theta) \sin \theta \, d\theta \\
 &= 27 \left(-\cos \theta + \frac{1}{3} \cos^3 \theta \right) + C.
 \end{aligned}$$

Since $\sin \theta = \frac{x}{3}$, $\cos \theta = \sqrt{1 - \left(\frac{x}{3}\right)^2} = \frac{1}{3} \sqrt{9-x^2}$. So

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{9-x^2}} \, dx &= 27 \left(-\frac{1}{3} \sqrt{9-x^2} + \frac{1}{3} \left(\frac{1}{3} \sqrt{9-x^2} \right)^3 \right) + C \\
 &= \frac{1}{3} (9-x^2)^{\frac{3}{2}} - 9\sqrt{9-x^2} + C.
 \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^2 \frac{x^3}{\sqrt{9-x^2}} \, dx &= \left[\frac{1}{3} (9-x^2)^{\frac{3}{2}} - 9\sqrt{9-x^2} \right]_0^2 \\
 &= \left(\frac{1}{3} (9-2^2)^{\frac{3}{2}} - 9\sqrt{9-2^2} \right) - \left(\frac{1}{3} (9-0^2)^{\frac{3}{2}} - 9\sqrt{9-0^2} \right) \\
 &= -\frac{22}{3} \sqrt{5} - (-18) \\
 &= \boxed{18 - \frac{22}{3} \sqrt{5}}.
 \end{aligned}$$

64. The area under the graph is given by $\int_0^4 x \sqrt{16-x^2} \, dx$. Let $u = 16-x^2$, then $du = -2x \, dx$, so $x \, dx = (-\frac{1}{2}) du$. The lower limit of integration is $u = 16-0^2 = 16$, and the upper limit of integration is $u = 16-(4)^2 = 0$. We substitute and obtain

$$\begin{aligned}
 \int_0^4 x \sqrt{16-x^2} \, dx &= \int_{16}^0 \sqrt{u} \left(-\frac{1}{2} \right) du \\
 &= \frac{1}{2} \int_0^{16} u^{1/2} \, du \\
 &= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^{16} \\
 &= \frac{1}{2} \left[\frac{2}{3} (16)^{\frac{3}{2}} - \frac{2}{3} (0)^{\frac{3}{2}} \right] \\
 &= \boxed{\frac{64}{3}}.
 \end{aligned}$$

65. The area under the graph is given by $\int_3^5 \frac{x^2}{\sqrt{x^2-1}} dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \sec \theta$, then $dx = \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2-1}} dx &= \int \frac{(\sec \theta)^2}{\sqrt{(\sec \theta)^2-1}} (\sec \theta \tan \theta) d\theta \\ &= \int \frac{\sec^2 \theta}{\tan \theta} \sec \theta \tan \theta d\theta \\ &= \int \sec^3 \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Since $\sec \theta = x$, $\tan \theta = \sqrt{x^2-1}$. So

$$\int \frac{x^2}{\sqrt{x^2-1}} dx = \frac{1}{2} x \sqrt{x^2-1} + \frac{1}{2} \ln |x + \sqrt{x^2-1}| + C.$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_3^5 \frac{x^2}{\sqrt{x^2-1}} dx &= \left(\frac{1}{2} 5 \sqrt{5^2-1} + \frac{1}{2} \ln |5 + \sqrt{5^2-1}| \right) - \left(\frac{1}{2} 3 \sqrt{3^2-1} + \frac{1}{2} \ln |3 + \sqrt{3^2-1}| \right) \\ &= \frac{1}{2} \ln (2\sqrt{6} + 5) + 5\sqrt{6} - \left(\frac{1}{2} \ln (2\sqrt{2} + 3) + 3\sqrt{2} \right) \\ &= \boxed{\frac{1}{2} \ln (2\sqrt{6} + 5) - \frac{1}{2} \ln (2\sqrt{2} + 3) - 3\sqrt{2} + 5\sqrt{6}}. \end{aligned}$$

66. The force due to hydrostatic pressure is given by the integral $\int_{-2}^2 (1000(9.8)(3-y))(2\sqrt{4-y^2}) dy$. We expand, and recognize the final integral as the area of a quarter-circle to obtain

$$\begin{aligned}\int_{-2}^2 (9800(3-y))(2\sqrt{4-y^2}) dy &= 58800 \int_{-2}^2 \sqrt{4-y^2} dy - 196000 \int_{-2}^2 y\sqrt{4-y^2} dy \\ &= 117600 \int_0^2 \sqrt{4-y^2} dy - 0 \\ &= 117600 \left(\frac{1}{4} \pi (2)^2 \right) \\ &= \boxed{117600\pi \text{ N}}.\end{aligned}$$

67. (a) We equate $x^2 = 4 - y^2$ with $x^2 = 1 - (y-2)^2$ to get the points of intersection. We obtain

$$\begin{aligned}4 - y^2 &= 1 - (y-2)^2 \\ 4 - y^2 &= -y^2 + 4y - 3 \\ 4 &= 4y - 3 \\ \frac{7}{4} &= y.\end{aligned}$$

So $x^2 = 4 - \left(\frac{7}{4}\right)^2 = \frac{15}{16}$, and $x = \pm \frac{\sqrt{15}}{4}$. The area is given by

$$A = \int_{-\frac{\sqrt{15}}{4}}^{\frac{\sqrt{15}}{4}} \left[\sqrt{4-x^2} - \left(2 - \sqrt{1-x^2}\right) \right] dx = 2 \int_0^{\frac{\sqrt{15}}{4}} \left[\sqrt{4-x^2} - \left(2 - \sqrt{1-x^2}\right) \right] dx.$$

We expand to obtain

$$\begin{aligned}A &= 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{4-x^2} dx + 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{1-x^2} dx - 2 \int_0^{\frac{\sqrt{15}}{4}} 2 dx \\ &= 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{4-x^2} dx + 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{1-x^2} dx - \sqrt{15}\end{aligned}$$

For the first integral, we evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}2 \int \sqrt{4-x^2} dx &= 2 \int \sqrt{4-(2 \sin \theta)^2} (2 \cos \theta) d\theta \\ &= 4 \int \sqrt{4-4 \sin^2 \theta} \cos \theta d\theta \\ &= 8 \int \cos^2 \theta d\theta \\ &= 8 \int \frac{1 + \cos(2\theta)}{2} d\theta \\ &= 4 \int (1 + \cos(2\theta)) d\theta \\ &= 4 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= 4\theta + 4 \cos \theta \sin \theta + C.\end{aligned}$$

Since $\sin \theta = \frac{x}{2}$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{x}{2}\right)^2} = \frac{1}{2}\sqrt{4 - x^2}$, and $\theta = \sin^{-1} \frac{x}{2}$. We obtain

$$\begin{aligned} 2 \int \sqrt{4 - x^2} \, dx &= 4 \sin^{-1} \frac{x}{2} + 4 \left(\frac{1}{2} \sqrt{4 - x^2} \right) \left(\frac{x}{2} \right) + C \\ &= 4 \sin^{-1} \frac{x}{2} + x \sqrt{4 - x^2} + C. \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{4 - x^2} \, dx &= \left[4 \sin^{-1} \frac{x}{2} + x \sqrt{4 - x^2} \right]_0^{\frac{\sqrt{15}}{4}} \\ &= \left(4 \sin^{-1} \left(\frac{\frac{\sqrt{15}}{4}}{2} \right) + \frac{\sqrt{15}}{4} \sqrt{4 - \left(\frac{\sqrt{15}}{4} \right)^2} \right) - \left(4 \sin^{-1} \frac{0}{2} + 0 \sqrt{4 - 0^2} \right) \\ &= \left(4 \sin^{-1} \left(\frac{\sqrt{15}}{8} \right) + \frac{7\sqrt{15}}{16} \right) - 0 \\ &= 4 \sin^{-1} \left(\frac{\sqrt{15}}{8} \right) + \frac{7\sqrt{15}}{16}. \end{aligned}$$

For the second integral, we evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = \sin \theta$, then $dx = \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned} 2 \int \sqrt{1 - x^2} \, dx &= 2 \int \sqrt{1 - \sin^2 \theta} (\cos \theta) \, d\theta \\ &= 2 \int \cos^2 \theta \, d\theta \\ &= 2 \int \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \int (1 + \cos(2\theta)) \, d\theta \\ &= \theta + \frac{1}{2} \sin 2\theta + C \\ &= \theta + \cos \theta \sin \theta + C. \end{aligned}$$

Since $\sin \theta = x$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}$, and $\theta = \sin^{-1} x$. We obtain

$$\begin{aligned} 2 \int \sqrt{1 - x^2} \, dx &= \sin^{-1} x + \sqrt{1 - x^2} (x) + C \\ &= \sin^{-1} x + x \sqrt{1 - x^2} + C. \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{1-x^2} dx &= \left[\sin^{-1} x + x\sqrt{1-x^2} \right]_0^{\frac{\sqrt{15}}{4}} \\
 &= \left(\sin^{-1} \frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{4} \sqrt{1 - \left(\frac{\sqrt{15}}{4} \right)^2} \right) - \left(\sin^{-1} 0 + 0\sqrt{1-0^2} \right) \\
 &= \left(\sin^{-1} \frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16} \right) - 0 \\
 &= \sin^{-1} \frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16}.
 \end{aligned}$$

So we obtain $A = 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{4-x^2} dx + 2 \int_0^{\frac{\sqrt{15}}{4}} \sqrt{1-x^2} dx - \sqrt{15} = \left(4 \sin^{-1} \frac{\sqrt{15}}{8} + \frac{7\sqrt{15}}{16} \right) + \left(\sin^{-1} \frac{\sqrt{15}}{4} + \frac{\sqrt{15}}{16} \right) - \sqrt{15} = \boxed{\sin^{-1} \frac{\sqrt{15}}{4} + 4 \sin^{-1} \frac{\sqrt{15}}{8} - \frac{1}{2}\sqrt{15}}.$

(b) Here we subtract the area of the smaller lune from the area of the upper circle. We obtain

$$\begin{aligned}
 A &= \pi(1)^2 - \left(\sin^{-1} \frac{\sqrt{15}}{4} + 4 \sin^{-1} \frac{\sqrt{15}}{8} - \frac{1}{2}\sqrt{15} \right) \\
 &= \boxed{\pi - \sin^{-1} \frac{\sqrt{15}}{4} - 4 \sin^{-1} \frac{\sqrt{15}}{8} + \frac{1}{2}\sqrt{15}}.
 \end{aligned}$$

68. (a) The points of intersection are determined by $y^2 = 16\left(\frac{6^2}{9} - 1\right) = 48$, so $y = \pm 4\sqrt{3}$.

The area is given by $A = \int_{-4\sqrt{3}}^{4\sqrt{3}} \left(6 - \frac{3}{4}\sqrt{y^2 + 16} \right) dy$. We expand to obtain

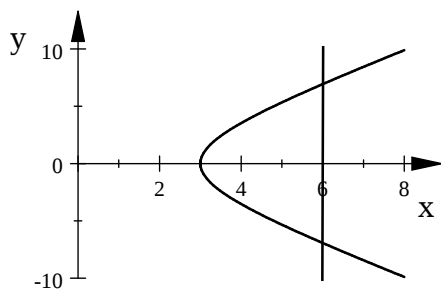
$$\begin{aligned}
 A &= 2 \int_0^{4\sqrt{3}} \left(6 - \frac{3}{4}\sqrt{y^2 + 16} \right) dy \\
 &= 12 \int_0^{4\sqrt{3}} dy - \frac{3}{2} \int_0^{4\sqrt{3}} \sqrt{y^2 + 16} dy \\
 &= 48\sqrt{3} - \frac{3}{2} \int_0^{4\sqrt{3}} \sqrt{y^2 + 16} dy.
 \end{aligned}$$

Let $y = 4 \tan \theta$, then $dy = 4 \sec^2 \theta d\theta$. The lower limit of integration is $\theta = \tan^{-1} \left(\frac{0}{4} \right) = 0$, and the upper limit of integration is $\theta = \tan^{-1} \left(\frac{4\sqrt{3}}{4} \right) = \frac{\pi}{3}$. We substitute and

obtain

$$\begin{aligned}
 A &= 48\sqrt{3} - \frac{3}{2} \int_0^{\pi/3} \sqrt{(4 \tan \theta)^2 + 16} (4 \sec^2 \theta) d\theta \\
 &= 48\sqrt{3} - 24 \int_0^{\pi/3} \sec^3 \theta d\theta \\
 &= 48\sqrt{3} - 24 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right]_0^{\pi/3} \\
 &= 48\sqrt{3} - 24 \left[\frac{1}{2} \sec \frac{\pi}{3} \tan \frac{\pi}{3} + \frac{1}{2} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| \right] \\
 &\quad + 24 \left[\frac{1}{2} \sec 0 \tan 0 + \frac{1}{2} \ln |\sec 0 + \tan 0| \right] \\
 &= \boxed{24\sqrt{3} - 12 \ln (\sqrt{3} + 2)}.
 \end{aligned}$$

(b)



69. The length of the graph is given by $L = \int_0^5 \sqrt{1 + \left[\frac{d}{dx}(5x - x^2)\right]^2} dx = \int_0^5 \sqrt{1 + (5 - 2x)^2} dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $u = 5 - 2x$, so $du = -2 dx$ and $dx = -\frac{1}{2} du$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{1 + (5 - 2x)^2} dx &= \int \sqrt{1 + u^2} \left(-\frac{1}{2} du\right) \\
 &= -\frac{1}{2} \int \sqrt{1 + u^2} du.
 \end{aligned}$$

Let $u = \tan \theta$, then $du = \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{1 + (5 - 2x)^2} dx &= -\frac{1}{2} \int \sqrt{1 + \tan^2 \theta} \sec^2 \theta d\theta \\
 &= -\frac{1}{2} \int \sec^3 \theta d\theta \\
 &= -\frac{1}{4} \sec \theta \tan \theta - \frac{1}{4} \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

Since $\tan \theta = u$, $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + u^2}$, and also substituting $u = 5 - 2x$, we obtain

$$\begin{aligned}
 \int \sqrt{1 + (5 - 2x)^2} dx &= -\frac{1}{4} \sqrt{1 + u^2} (u) - \frac{1}{4} \ln |\sqrt{1 + u^2} + u| + C \\
 &= -\frac{1}{4} (5 - 2x) \sqrt{1 + (5 - 2x)^2} - \frac{1}{4} \ln \left| \sqrt{1 + (5 - 2x)^2} + (5 - 2x) \right| + C.
 \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^5 \sqrt{1+(5-2x)^2} dx &= \left[-\frac{1}{4}(5-2x)\sqrt{1+(5-2x)^2} - \frac{1}{4} \ln \left| \sqrt{1+(5-2x)^2} + (5-2x) \right| \right]_0^5 \\
 &= \left(-\frac{(5-2(5))\sqrt{1+(5-2(5))^2}}{4} - \frac{\ln \left| \sqrt{1+(5-2(5))^2} + 5-2(5) \right|}{4} \right) \\
 &\quad - \left(-\frac{(5-2(0))\sqrt{1+(5-2(0))^2}}{4} - \frac{\ln \left| \sqrt{1+(5-2(0))^2} + 5-2(0) \right|}{4} \right) \\
 &= \frac{5}{4}\sqrt{26} - \frac{1}{4} \ln(\sqrt{26}-5) - \left(-\frac{1}{4} \ln(\sqrt{26}+5) - \frac{5}{4}\sqrt{26} \right) \\
 &= \boxed{\frac{1}{4} \ln(\sqrt{26}+5) - \frac{1}{4} \ln(\sqrt{26}-5) + \frac{5}{2}\sqrt{26}}.
 \end{aligned}$$

70. The length of the graph is given by $L = \int_{\sqrt{3}/3}^{\sqrt{3}} \sqrt{1 + \left[\frac{d}{dx}(\ln x) \right]^2} dx = \int_{\sqrt{3}/3}^{\sqrt{3}} \sqrt{1 + \left(\frac{1}{x} \right)^2} dx = \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x} dx$. Let $x = \tan \theta$, then $dx = \sec^2 \theta d\theta$. The lower limit of integration is $\theta = \tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6}$ and the upper limit of integration is $u = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$. We substitute and obtain

$$\begin{aligned}
 L &= \int_{\pi/6}^{\pi/3} \frac{\sqrt{1+\tan^2 \theta}}{\tan \theta} \sec^2 \theta d\theta \\
 &= \int_{\pi/6}^{\pi/3} \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta \\
 &= \int_{\pi/6}^{\pi/3} \frac{\sec^3 \theta}{\tan \theta} d\theta \\
 &= \int_{\pi/6}^{\pi/3} (\sec \theta \tan \theta + \csc \theta) d\theta \\
 &= [\sec \theta - \ln |\csc \theta + \cot \theta|]_{\pi/6}^{\pi/3} \\
 &= \left[\sec \frac{\pi}{3} - \ln \left| \csc \frac{\pi}{3} + \cot \frac{\pi}{3} \right| \right] - \left[\sec \frac{\pi}{6} - \ln \left| \csc \frac{\pi}{6} + \cot \frac{\pi}{6} \right| \right] \\
 &= \left(2 - \ln \sqrt{3} \right) - \left(\frac{2}{3}\sqrt{3} - \ln(\sqrt{3}+2) \right) \\
 &= \boxed{\ln(\sqrt{3}+2) - \ln \sqrt{3} - \frac{2}{3}\sqrt{3} + 2}.
 \end{aligned}$$

71. The volume is given by $\int_0^1 \pi \left(\frac{1}{x^2+4} \right)^2 dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 2 \tan \theta$, then

$dx = 2 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \pi \left(\frac{1}{x^2 + 4} \right)^2 dx &= \pi \int \left(\frac{1}{(2 \tan \theta)^2 + 4} \right)^2 (2 \sec^2 \theta) d\theta \\
 &= 2\pi \int \left(\frac{1}{4 \sec^2 \theta} \right)^2 \sec^2 \theta d\theta \\
 &= \frac{\pi}{8} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\
 &= \frac{\pi}{8} \int \cos^2 \theta d\theta \\
 &= \frac{\pi}{8} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{\pi}{16} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{\pi}{16} \left(\theta + \frac{1}{2} \sin(2\theta) \right) \\
 &= \frac{\pi}{16} (\theta + \sin \theta \cos \theta) + C \\
 &= \frac{\pi}{16} \left(\theta + \frac{\tan \theta}{\sec^2 \theta} \right) + C.
 \end{aligned}$$

Since $\tan \theta = \frac{x}{2}$, $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{x}{2}\right)^2 + 1} = \frac{1}{2}\sqrt{x^2 + 4}$, and $\theta = \tan^{-1} \frac{x}{2}$. We have

$$\begin{aligned}
 \int \pi \left(\frac{1}{x^2 + 4} \right)^2 dx &= \frac{\pi}{16} \left(\tan^{-1} \frac{x}{2} + \frac{\frac{x}{2}}{\left(\frac{1}{2}\sqrt{x^2 + 4}\right)^2} \right) + C \\
 &= \frac{\pi}{16} \left(\tan^{-1} \frac{x}{2} + \frac{2x}{x^2 + 4} \right) + C.
 \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_0^1 \pi \left(\frac{1}{x^2 + 4} \right)^2 dx &= \left[\frac{\pi}{16} \left(\tan^{-1} \frac{x}{2} + \frac{2x}{x^2 + 4} \right) \right]_0^1 \\
 &= \left(\frac{\pi}{16} \left(\tan^{-1} \frac{1}{2} + \frac{2(1)}{(1)^2 + 4} \right) \right) - \left(\frac{\pi}{16} \left(\tan^{-1} \frac{0}{2} + \frac{2(0)}{(0)^2 + 4} \right) \right) \\
 &= \frac{\pi}{16} \left(\tan^{-1} \frac{1}{2} + \frac{2}{5} \right) - 0 \\
 &= \boxed{\frac{\pi}{40} + \frac{\pi}{16} \tan^{-1} \frac{1}{2}}.
 \end{aligned}$$

72. The volume is given by $\int_0^2 \pi \left(\frac{1}{\sqrt{9-x^2}} \right)^2 dx = \pi \int_0^2 \frac{1}{9-x^2} dx$. We evaluate the corresponding indefinite integral, and then apply the Fundamental Theorem of Calculus. Let $x = 3 \sin \theta$,

then $dx = 3 \cos \theta$. We substitute and obtain

$$\begin{aligned}
 \pi \int \frac{1}{9-x^2} dx &= \pi \int \frac{1}{9-(3 \sin \theta)^2} (3 \cos \theta) d\theta \\
 &= 3\pi \int \frac{\cos \theta}{9-9 \sin^2 \theta} d\theta \\
 &= \frac{\pi}{3} \int \frac{\cos \theta}{\cos^2 \theta} d\theta \\
 &= \frac{\pi}{3} \int \sec \theta d\theta \\
 &= \frac{\pi}{3} \ln |\sec \theta + \tan \theta| + C \\
 &= \frac{\pi}{3} \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| + C
 \end{aligned}$$

Since $\sin \theta = \frac{x}{3}$, $\cos \theta = \sqrt{1 - \left(\frac{x}{3}\right)^2} = \frac{1}{3}\sqrt{9-x^2}$. So

$$\begin{aligned}
 \pi \int \frac{1}{9-x^2} dx &= \frac{\pi}{3} \ln \left| \frac{1 + \frac{x}{3}}{\frac{1}{3}\sqrt{9-x^2}} \right| + C \\
 &= \frac{\pi}{3} \ln \left| \frac{x+3}{\sqrt{9-x^2}} \right| + C.
 \end{aligned}$$

So by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \pi \int_0^2 \frac{1}{9-x^2} dx &= \left[\frac{\pi}{3} \ln \left| \frac{x+3}{\sqrt{9-x^2}} \right| \right]_0^2 \\
 &= \left(\frac{\pi}{3} \ln \left| \frac{2+3}{\sqrt{9-2^2}} \right| \right) - \left(\frac{\pi}{3} \ln \left| \frac{0+3}{\sqrt{9-0^2}} \right| \right) \\
 &= \left(\frac{\pi}{3} \ln |\sqrt{5}| \right) - \left(\frac{\pi}{3} \ln |1| \right) \\
 &= \boxed{\frac{\pi}{6} \ln 5}.
 \end{aligned}$$

73. Let $u = x - 2$, then $du = dx$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{1-(x-2)^2}} &= \int \frac{du}{\sqrt{1-u^2}} \\
 &= \sin^{-1} u + C \\
 &= \boxed{\sin^{-1}(x-2) + C}.
 \end{aligned}$$

74. Let $u = x + 2$, then $du = dx$. We substitute and obtain

$$\int \sqrt{4-(x+2)^2} dx = \int \sqrt{4-u^2} du.$$

Let $u = 2 \sin \theta$, then $du = 2 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{4 - (x + 2)^2} dx &= \int \sqrt{4 - (2 \sin \theta)^2} (2 \cos \theta) d\theta \\
 &= 2 \int \sqrt{4 - 4 \sin^2 \theta} \cos \theta d\theta \\
 &= 4 \int \cos^2 \theta d\theta \\
 &= 4 \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= 2 \int (1 + \cos(2\theta)) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= 2\theta + 2 \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{u}{2} \right)$, so $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{u}{2} \right)^2} = \frac{1}{2} \sqrt{4 - u^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{4 - (x + 2)^2} dx &= 2 \sin^{-1} \left(\frac{u}{2} \right) + 2 \left(\frac{u}{2} \right) \left(\frac{1}{2} \sqrt{4 - u^2} \right) + C \\
 &= 2 \sin^{-1} \left(\frac{u}{2} \right) + \frac{1}{2} u \sqrt{4 - u^2} + C.
 \end{aligned}$$

And since $u = x + 2$, we have

$$\int \sqrt{4 - (x + 2)^2} dx = \boxed{2 \sin^{-1} \left(\frac{x+2}{2} \right) + \frac{1}{2} (x + 2) \sqrt{4 - (x + 2)^2} + C}.$$

75. Let $u = x - 1$, then $du = dx$. We substitute and obtain

$$\int \frac{dx}{\sqrt{(x - 1)^2 - 4}} = \int \frac{du}{\sqrt{u^2 - 4}}.$$

Let $u = 2 \sec \theta$, then $du = 2 \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{(x - 1)^2 - 4}} &= \int \frac{2 \sec \theta \tan \theta}{\sqrt{(2 \sec \theta)^2 - 4}} d\theta \\
 &= 2 \int \frac{\sec \theta \tan \theta}{\sqrt{4 \sec^2 \theta - 4}} d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\sec \theta = \frac{u}{2}$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{u}{2} \right)^2 - 1} = \frac{1}{2} \sqrt{u^2 - 4}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{(x - 1)^2 - 4}} &= \ln \left| \frac{u}{2} + \frac{1}{2} \sqrt{u^2 - 4} \right| + C \\
 &= \ln \left| u + \sqrt{u^2 - 4} \right| - \ln 2 + C \\
 &= \ln \left| u + \sqrt{u^2 - 4} \right| + C.
 \end{aligned}$$

And since $u = x - 1$, we have

$$\int \frac{dx}{\sqrt{(x-1)^2 - 4}} = \boxed{\ln \left| x - 1 + \sqrt{(x-1)^2 - 4} \right| + C}.$$

76. Let $u = x - 2$, then $du = dx$. We substitute and obtain

$$\int \frac{dx}{(x-2)\sqrt{(x-2)^2 + 9}} = \int \frac{du}{u\sqrt{u^2 + 9}}.$$

Let $u = 3 \tan \theta$, then $du = 3 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{(x-2)\sqrt{(x-2)^2 + 9}} &= \int \frac{3 \sec^2 \theta}{(3 \tan \theta)\sqrt{(3 \tan \theta)^2 + 9}} d\theta \\ &= \int \frac{\sec^2 \theta}{\tan \theta \sqrt{9 \tan^2 \theta + 9}} d\theta \\ &= \frac{1}{3} \int \frac{\sec^2 \theta}{\tan \theta \sec \theta} d\theta \\ &= \frac{1}{3} \int \frac{\sec \theta}{\tan \theta} d\theta \\ &= \frac{1}{3} \int \csc \theta d\theta \\ &= -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C. \end{aligned}$$

We have $\tan \theta = \frac{u}{3}$, so $\cot \theta = \frac{3}{u}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{3}{u}\right)^2} = \frac{1}{u} \sqrt{u^2 + 9}$. We obtain

$$\int \frac{dx}{(x-2)\sqrt{(x-2)^2 + 9}} = -\frac{1}{3} \ln \left| \frac{\sqrt{u^2 + 9}}{u} + \frac{3}{u} \right| + C.$$

And since $u = x - 2$, we have

$$\int \frac{dx}{(x-2)\sqrt{(x-2)^2 + 9}} = -\frac{1}{3} \ln \left| \frac{\sqrt{(x-2)^2 + 9}}{x-2} + \frac{3}{x-2} \right| + C.$$

77. Let $u = e^x$, then $du = e^x dx$. We substitute and obtain

$$\int e^x \sqrt{25 - e^{2x}} dx = \int \sqrt{25 - u^2} du.$$

Let $u = 5 \sin \theta$, then $du = 5 \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int e^x \sqrt{25 - e^{2x}} dx &= \int \sqrt{25 - (5 \sin \theta)^2} (5 \cos \theta) d\theta \\
 &= 5 \int \sqrt{25 - 25 \sin^2 \theta} \cos \theta d\theta \\
 &= 25 \int \cos^2 \theta d\theta \\
 &= 25 \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{25}{2} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{25}{2} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{25}{2} \theta + \frac{25}{2} \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} \left(\frac{u}{5} \right)$, so $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{u}{5} \right)^2} = \frac{1}{5} \sqrt{25 - u^2}$. We obtain

$$\begin{aligned}
 \int e^x \sqrt{25 - e^{2x}} dx &= \frac{25}{2} \sin^{-1} \left(\frac{u}{5} \right) + \frac{25}{2} \left(\frac{u}{5} \right) \left(\frac{1}{5} \sqrt{25 - u^2} \right) + C \\
 &= \frac{25}{2} \sin^{-1} \left(\frac{u}{5} \right) + \frac{1}{2} u \sqrt{25 - u^2} + C.
 \end{aligned}$$

And since $u = e^x$, we have

$$\int e^x \sqrt{25 - e^{2x}} dx = \boxed{\frac{25}{2} \sin^{-1} \left(\frac{e^x}{5} \right) + \frac{1}{2} e^x \sqrt{25 - e^{2x}} + C}.$$

78. Let $u = e^x$, then $du = e^x dx$. We substitute and obtain

$$\int e^x \sqrt{4 + e^{2x}} dx = \int \sqrt{4 + u^2} du.$$

Let $u = 2 \tan \theta$, then $du = 2 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int e^x \sqrt{4 + e^{2x}} dx &= \int \sqrt{4 + (2 \tan \theta)^2} (2 \sec^2 \theta) d\theta \\
 &= 2 \int \sqrt{4 + 4 \tan^2 \theta} \sec^2 \theta d\theta \\
 &= 2 \int (2 \sec \theta) \sec^2 \theta d\theta \\
 &= 4 \int \sec^3 \theta d\theta \\
 &= 4 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C \\
 &= 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\tan \theta = \frac{u}{2}$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + (u/2)^2} = \frac{1}{2}\sqrt{4 + u^2}$. We obtain

$$\begin{aligned}\int e^x \sqrt{4 + e^{2x}} dx &= 2 \left(\frac{1}{2} \sqrt{4 + u^2} \right) \left(\frac{u}{2} \right) + 2 \ln \left| \frac{1}{2} \sqrt{4 + u^2} + \left(\frac{u}{2} \right) \right| + C \\ &= \frac{1}{2} u \sqrt{4 + u^2} + 2 \ln \left| \sqrt{4 + u^2} + u \right| - 2 \ln 2 + C \\ &= \frac{1}{2} u \sqrt{4 + u^2} + 2 \ln \left| \sqrt{4 + u^2} + u \right| + C.\end{aligned}$$

And since $u = e^x$, we have

$$\int e^x \sqrt{4 + e^{2x}} dx = \boxed{\frac{1}{2} e^x \sqrt{4 + e^{2x}} + 2 \ln \left| \sqrt{4 + e^{2x}} + e^x \right| + C}.$$

79. Let $u = \sin^{-1} x$ and $dv = x dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = \frac{1}{2}x^2$. We use integration by parts and obtain

$$\begin{aligned}\int x \sin^{-1} x dx &= \frac{1}{2} x^2 \sin^{-1} x - \int \left(\frac{1}{2} x^2 \right) \left(\frac{1}{\sqrt{1-x^2}} \right) dx \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx.\end{aligned}$$

Let $x = \sin \theta$, then $dx = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int x \sin^{-1} x dx &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{1 - \sin^2 \theta}} (\cos \theta) d\theta \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta d\theta \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{1 - \cos(2\theta)}{2} d\theta \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \int (1 - \cos(2\theta)) d\theta \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + C.\end{aligned}$$

We have $\theta = \sin^{-1} x$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}$. We obtain

$$\int x \sin^{-1} x dx = \boxed{\frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C}.$$

80. Let $u = \cos^{-1} x$ and $dv = x dx$. Then $du = -\frac{1}{\sqrt{1-x^2}} dx$ and $v = \frac{1}{2}x^2$. We use integration by parts and obtain

$$\begin{aligned}\int x \cos^{-1} x dx &= \frac{1}{2} x^2 \cos^{-1} x - \int \left(\frac{1}{2} x^2 \right) \left(-\frac{1}{\sqrt{1-x^2}} \right) dx \\ &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx.\end{aligned}$$

Let $x = \sin \theta$, then $dx = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int x \cos^{-1} x \, dx &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{1 - \sin^2 \theta}} (\cos \theta) d\theta \\
 &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \sin^2 \theta d\theta \\
 &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \frac{1 - \cos(2\theta)}{2} d\theta \\
 &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \int (1 - \cos(2\theta)) d\theta \\
 &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\
 &= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \theta - \frac{1}{4} \sin \theta \cos \theta + C.
 \end{aligned}$$

We have $\theta = \sin^{-1} x$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}$. We obtain

$$\int x \cos^{-1} x \, dx = \boxed{\frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \sin^{-1} x - \frac{1}{4} x \sqrt{1 - x^2} + C}.$$

81. (a) Let $x = a \tan \theta$, then $dx = a \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{x^2 + a^2} \, dx &= \int \sqrt{(a \tan \theta)^2 + a^2} (a \sec^2 \theta) d\theta \\
 &= a^2 \int \sec^3 \theta d\theta \\
 &= a^2 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C \\
 &= \frac{a^2}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C.
 \end{aligned}$$

We have $\tan \theta = \frac{x}{a}$, and $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{x}{a}\right)^2 + 1} = \frac{1}{a} \sqrt{x^2 + a^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{x^2 + a^2} \, dx &= \frac{a^2}{2} \left(\frac{1}{a} \sqrt{x^2 + a^2} \left(\frac{x}{a} \right) + \ln \left| \frac{1}{a} \sqrt{x^2 + a^2} + \frac{x}{a} \right| \right) + C \\
 &= \boxed{\frac{1}{2} a^2 \ln \left| \frac{x + \sqrt{a^2 + x^2}}{a} \right| + \frac{1}{2} x \sqrt{a^2 + x^2} + C}.
 \end{aligned}$$

(b) Let $x = a \sinh \theta$, then $dx = a \cosh \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{x^2 + a^2} dx &= \int \sqrt{(a \sinh \theta)^2 + a^2} (a \cosh \theta) d\theta \\
 &= a^2 \int \sqrt{\sinh^2 \theta + 1} \cosh \theta d\theta \\
 &= a^2 \int \cosh^2 \theta d\theta \\
 &= a^2 \int \left(\frac{e^\theta + e^{-\theta}}{2} \right)^2 d\theta \\
 &= \frac{a^2}{4} \int (e^{2\theta} + 2 + e^{-2\theta}) d\theta \\
 &= \frac{a^2}{4} \left(\frac{1}{2} e^{2\theta} + 2\theta - \frac{1}{2} e^{-2\theta} \right) + C \\
 &= \frac{1}{2} a^2 \theta + \frac{a^2}{8} (e^{2\theta} - e^{-2\theta}) + C \\
 &= \frac{1}{2} a^2 \theta + \frac{a^2}{2} \left(\frac{e^\theta - e^{-\theta}}{2} \right) \left(\frac{e^\theta + e^{-\theta}}{2} \right) + C \\
 &= \frac{1}{2} a^2 \theta + \frac{a^2}{2} \sinh \theta \cosh \theta + C.
 \end{aligned}$$

Since $\sinh \theta = \frac{x}{a}$, $\cosh \theta = \sqrt{\sinh^2 \theta + 1} = \sqrt{(x/a)^2 + 1} = \frac{1}{a} \sqrt{x^2 + a^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{x^2 + a^2} dx &= \frac{1}{2} a^2 \sinh^{-1} \left(\frac{x}{a} \right) + \frac{a^2}{2} \left(\frac{x}{a} \right) \left(\frac{1}{a} \sqrt{x^2 + a^2} \right) + C \\
 &= \boxed{\frac{1}{2} a^2 \sinh^{-1} \left(\frac{x}{a} \right) + \frac{1}{2} x \sqrt{a^2 + x^2} + C}.
 \end{aligned}$$

82. Let $x = a \sin \theta$, then $dx = a \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a \cos \theta}{\sqrt{a^2 - (a \sin \theta)^2}} d\theta \\
 &= \int \frac{a \cos \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} d\theta \\
 &= \frac{1}{a} \int \frac{a \cos \theta}{\sqrt{1 - \sin^2 \theta}} d\theta \\
 &= \int \frac{\cos \theta}{\cos \theta} d\theta \\
 &= \theta + C \\
 &= \boxed{\sin^{-1} \left(\frac{x}{a} \right) + C}.
 \end{aligned}$$

83. Let $x = a \tan \theta$, then $dx = a \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{a^2 + x^2} &= \int \frac{a \sec^2 \theta}{a^2 + (a \tan \theta)^2} d\theta \\
 &= \int \frac{a \sec^2 \theta}{a^2 + a^2 \tan^2 \theta} d\theta \\
 &= \frac{1}{a} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta \\
 &= \frac{1}{a} \theta + C \\
 &= \boxed{\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C}.
 \end{aligned}$$

84. Let $x = a \sec \theta$, then $dx = a \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{x^2 - a^2}} &= \int \frac{a \sec \theta \tan \theta}{a \sec \theta \sqrt{(a \sec \theta)^2 - a^2}} d\theta \\
 &= \frac{1}{a} \int \frac{\tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \frac{1}{a} \int \frac{\tan \theta}{\tan \theta} d\theta \\
 &= \frac{1}{a} \theta + C \\
 &= \boxed{\frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + C}.
 \end{aligned}$$

85. Let $x = a \sec \theta$, then $dx = a \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{1}{\sqrt{(a \sec \theta)^2 - a^2}} (a \sec \theta \tan \theta) d\theta \\
 &= \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{a}$, and $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{1}{a} \sqrt{x^2 - a^2}$. We obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 - a^2}} &= \ln \left| \frac{x}{a} + \frac{1}{a} \sqrt{x^2 - a^2} \right| + C \\
 &= \boxed{\ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + C}.
 \end{aligned}$$

86. Let $x = a \tan \theta$, then $dx = a \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \int \frac{1}{\sqrt{(a \tan \theta)^2 + a^2}} (a \sec^2 \theta) d\theta \\ &= \int \frac{\sec^2 \theta}{\sec \theta} d\theta \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C_1. \end{aligned}$$

We have $\tan \theta = \frac{x}{a}$, and $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{x}{a}\right)^2 + 1} = \frac{1}{a} \sqrt{x^2 + a^2}$. We obtain

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \ln \left| \frac{1}{a} \sqrt{x^2 + a^2} + \frac{x}{a} \right| + C_1 \\ &= \ln \left| x + \sqrt{x^2 + a^2} \right| - \ln |a| + C_1 \\ &= \boxed{\ln (x + \sqrt{x^2 + a^2}) + C}, \end{aligned}$$

where $C = -\ln |a| + C_1$.

Challenge Problems

87. We complete the square: $3x - x^2 = \frac{9}{4} - \left(x^2 - 3x + \frac{9}{4}\right) = \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$. Then

$$\int \frac{dx}{\sqrt{3x - x^2}} = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}}.$$

Let $u = x - \frac{3}{2}$, then $du = dx$. We substitute, use the result of exercise 92, and obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{3x - x^2}} &= \int \frac{du}{\sqrt{\left(\frac{3}{2}\right)^2 - u^2}} \\ &= \sin^{-1} \left(\frac{u}{(3/2)} \right) + C \\ &= \sin^{-1} \left(\frac{2(x - \frac{3}{2})}{3} \right) + C \\ &= \boxed{\sin^{-1} \left(\frac{2x-3}{3} \right) + C}. \end{aligned}$$

88. Let $x = a \sec \theta$, then $dx = a \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned}
 \int \sqrt{x^2 - a^2} dx &= \int \sqrt{(a \sec \theta)^2 - a^2} (a \sec \theta \tan \theta) d\theta \\
 &= a^2 \int (\tan \theta)(\sec \theta \tan \theta) d\theta \\
 &= a^2 \int \tan^2 \theta \sec \theta d\theta \\
 &= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta \\
 &= a^2 \int (\sec^3 \theta - \sec \theta) d\theta \\
 &= a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right] + C \\
 &= a^2 \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C.
 \end{aligned}$$

We have $\sec \theta = \frac{x}{a}$, and $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{1}{a} \sqrt{x^2 - a^2}$. We obtain

$$\begin{aligned}
 \int \sqrt{x^2 - a^2} dx &= a^2 \left[\frac{1}{2} \left(\frac{x}{a}\right) \left(\frac{1}{a} \sqrt{x^2 - a^2}\right) - \frac{1}{2} \ln \left| \frac{x}{a} + \frac{1}{a} \sqrt{x^2 - a^2} \right| \right] + C \\
 &= \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + \frac{1}{2} a^2 \ln |a| + C \\
 &= \boxed{\frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 - a^2} \right| + C}.
 \end{aligned}$$

89. Let $u = \sinh^{-1} \left(\frac{x}{a}\right)$, so $x = a \sinh u$, and $dx = a \cosh u du$. We substitute and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a \cosh u}{\sqrt{(a \sinh u)^2 + a^2}} du \\
 &= \int \frac{\cosh u}{\sqrt{\sinh^2 u + 1}} du \\
 &= \int \frac{\cosh u}{\cosh u} du \\
 &= u + C \\
 &= \sinh^{-1} \left(\frac{x}{a}\right) + C.
 \end{aligned}$$

From the identity $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + a^2}} &= \ln \left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1} \right) + C \\
 &= \ln \left(\frac{x}{a} + \frac{1}{a} \sqrt{x^2 + a^2} \right) + C \\
 &= \boxed{\ln \left(\frac{x + \sqrt{x^2 + a^2}}{a} \right) + C}.
 \end{aligned}$$

90. Let $u = \tan x$, then $du = \sec^2 x \, dx$. We substitute and obtain

$$\begin{aligned} \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6 \tan x + 8}} dx &= \int \frac{du}{\sqrt{u^2 - 6u + 8}} \\ &= \int \frac{du}{\sqrt{u^2 - 6u + 9 - 1}} \\ &= \int \frac{du}{\sqrt{(u-3)^2 - 1}}. \end{aligned}$$

Let $y = u - 3$, then $dy = du$. We substitute and obtain

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6 \tan x + 8}} dx = \int \frac{dy}{\sqrt{y^2 - 1}}.$$

Let $y = \sec \theta$, then $dy = \sec \theta \tan \theta \, d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6 \tan x + 8}} dx &= \int \frac{\sec \theta \tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta \\ &= \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta \\ &= \int \sec \theta \, d\theta \\ &= \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

We have $\sec \theta = y$, and $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{y^2 - 1}$. We obtain

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6 \tan x + 8}} dx = \ln |y + \sqrt{y^2 - 1}| + C.$$

Since $y = u - 3 = \tan x - 3$, we now have

$$\begin{aligned} \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 6 \tan x + 8}} dx &= \ln \left| \tan x - 3 + \sqrt{(\tan x - 3)^2 - 1} \right| + C \\ &= \boxed{\ln \left| \tan x - 3 + \sqrt{\tan^2 x - 6 \tan x + 8} \right| + C}. \end{aligned}$$

7.4: Substitution: Integrands Containing $ax^2 + bx + c$

Skill Building

1. We complete the square: $x^2 + 4x + 5 = (x + 2)^2 + 1$. Then we let $u = x + 2$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{x^2 + 4x + 5} &= \int \frac{dx}{(x + 2)^2 + 1} \\ &= \int \frac{du}{u^2 + 1} \\ &= \tan^{-1} u + C \\ &= \boxed{\tan^{-1}(x + 2) + C}.\end{aligned}$$

2. We complete the square: $x^2 + 2x + 5 = (x + 1)^2 + 4$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{x^2 + 2x + 5} &= \int \frac{dx}{(x + 1)^2 + 4} \\ &= \int \frac{du}{u^2 + 2^2} \\ &= \frac{1}{2} \tan^{-1} \frac{u}{2} + C \\ &= \boxed{\frac{1}{2} \tan^{-1} \left(\frac{x+1}{2} \right) + C}.\end{aligned}$$

3. We complete the square: $x^2 + 4x + 8 = (x + 2)^2 + 4$. Then we let $u = x + 2$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{x^2 + 4x + 8} &= \int \frac{dx}{(x + 2)^2 + 4} \\ &= \int \frac{du}{u^2 + 2^2} \\ &= \frac{1}{2} \tan^{-1} \frac{u}{2} + C \\ &= \boxed{\frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + C}.\end{aligned}$$

4. We complete the square: $x^2 - 6x + 10 = (x - 3)^2 + 1$. Then we let $u = x - 3$, so $du = dx$, and we obtain

$$\begin{aligned}\int \frac{dx}{x^2 - 6x + 10} &= \int \frac{dx}{(x - 3)^2 + 1} \\ &= \int \frac{du}{u^2 + 1} \\ &= \tan^{-1} u + C \\ &= \boxed{\tan^{-1}(x - 3) + C}.\end{aligned}$$

5. We complete the square: $3 + 2x + 2x^2 = 2\left(x + \frac{1}{2}\right)^2 + \frac{5}{2}$. Then we let $u = x + \frac{1}{2}$ to obtain

$$\begin{aligned}
 \int \frac{2 \, dx}{3 + 2x + 2x^2} &= \int \frac{2 \, dx}{2\left(x + \frac{1}{2}\right)^2 + \frac{5}{2}} \\
 &= 2 \int \frac{du}{2u^2 + \frac{5}{2}} \\
 &= \int \frac{du}{u^2 + \left(\frac{\sqrt{5}}{2}\right)^2} \\
 &= \frac{2}{\sqrt{5}} \tan^{-1} \left(\frac{u}{\sqrt{5}/2} \right) + C \\
 &= \frac{2\sqrt{5}}{5} \tan^{-1} \left(\frac{2\left(x + \frac{1}{2}\right)}{\sqrt{5}} \right) + C \\
 &= \boxed{\frac{2\sqrt{5}}{5} \tan^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + C}.
 \end{aligned}$$

6. We complete the square: $x^2 + 6x + 10 = (x + 3)^2 + 1$. Then we let $u = x + 3$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{3 \, dx}{x^2 + 6x + 10} &= 3 \int \frac{dx}{(x + 3)^2 + 1} \\
 &= 3 \int \frac{du}{u^2 + 1} \\
 &= 3 \tan^{-1} u + C \\
 &= \boxed{3 \tan^{-1} (x + 3) + C}.
 \end{aligned}$$

7. We force the the derivative of the denominator to appear in the numerator. We then complete the square with $2x^2 + 2x + 3 = 2\left(x + \frac{1}{2}\right)^2 + \frac{5}{2}$, and let $u = x + \frac{1}{2}$, so $du = dx$. We obtain

$$\begin{aligned}
 \int \frac{x \, dx}{2x^2 + 2x + 3} &= \int \frac{\frac{1}{4}(4x + 2) - \frac{1}{2}}{2x^2 + 2x + 3} \, dx \\
 &= \frac{1}{4} \int \frac{4x + 2}{2x^2 + 2x + 3} \, dx - \frac{1}{2} \int \frac{dx}{2x^2 + 2x + 3} \\
 &= \frac{1}{4} \ln |2x^2 + 2x + 3| - \frac{1}{2} \int \frac{dx}{2x^2 + 2x + 3} \\
 &= \frac{1}{4} \ln (2x^2 + 2x + 3) - \frac{1}{2} \int \frac{dx}{2\left(x + \frac{1}{2}\right)^2 + \frac{5}{2}} \\
 &= \frac{1}{4} \ln (2x^2 + 2x + 3) - \frac{1}{4} \int \frac{du}{u^2 + \left(\frac{\sqrt{5}}{2}\right)^2} \\
 &= \frac{1}{4} \ln (2x^2 + 2x + 3) - \frac{1}{4} \frac{2}{\sqrt{5}} \tan^{-1} \left(\frac{u}{\sqrt{5}/2} \right) + C \\
 &= \frac{1}{4} \ln (2x^2 + 2x + 3) - \frac{\sqrt{5}}{10} \tan^{-1} \left(\frac{2\left(x + \frac{1}{2}\right)}{\sqrt{5}} \right) + C \\
 &= \boxed{\frac{1}{4} \ln (2x^2 + 2x + 3) - \frac{\sqrt{5}}{10} \tan^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + C}.
 \end{aligned}$$

8. We force the the derivative of the denominator to appear in the numerator. We then complete the square with $x^2 + 6x + 10 = (x + 3)^2 + 1$, and let $u = x + 3$, so $du = dx$. We obtain

$$\begin{aligned}
 \int \frac{3x \, dx}{x^2 + 6x + 10} &= \int \frac{\frac{3}{2}(2x + 6) - 9}{x^2 + 6x + 10} \, dx \\
 &= \frac{3}{2} \int \frac{(2x + 6)}{x^2 + 6x + 10} \, dx - 9 \int \frac{dx}{x^2 + 6x + 10} \\
 &= \frac{3}{2} \ln |x^2 + 6x + 10| - 9 \int \frac{dx}{(x + 3)^2 + 1} \\
 &= \frac{3}{2} \ln (x^2 + 6x + 10) - 9 \int \frac{du}{u^2 + 1} \\
 &= \frac{3}{2} \ln (x^2 + 6x + 10) - 9 \tan^{-1} u + C \\
 &= \boxed{\frac{3}{2} \ln (x^2 + 6x + 10) - 9 \tan^{-1} (x + 3) + C}.
 \end{aligned}$$

9. We complete the square: $8 + 2x - x^2 = 9 - (x - 1)^2$. Then we let $u = x - 1$ to obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{8 + 2x - x^2}} &= \int \frac{dx}{\sqrt{9 - (x - 1)^2}} \\
 &= \int \frac{du}{\sqrt{3^2 - u^2}} \\
 &= \sin^{-1} \left(\frac{u}{3} \right) + C \\
 &= \boxed{\sin^{-1} \left(\frac{x-1}{3} \right) + C}.
 \end{aligned}$$

10. We complete the square: $5 - 4x - 2x^2 = 7 - 2(x + 1)^2$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{5 - 4x - 2x^2}} &= \int \frac{dx}{\sqrt{7 - 2(x + 1)^2}} \\
 &= \int \frac{du}{\sqrt{7 - 2u^2}} \\
 &= \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{\left(\frac{\sqrt{14}}{2}\right)^2 - u^2}} \\
 &= \frac{\sqrt{2}}{2} \sin^{-1} \left(\frac{u}{\sqrt{14}/2} \right) + C \\
 &= \boxed{\frac{\sqrt{2}}{2} \sin^{-1} \left(\frac{\sqrt{14}(x+1)}{7} \right) + C}.
 \end{aligned}$$

11. We complete the square: $4x - x^2 = 4 - (x - 2)^2$. Then we let $u = x - 2$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x - 2)^2}} \\
 &= \int \frac{du}{\sqrt{2^2 - u^2}} \\
 &= \sin^{-1} \left(\frac{u}{2} \right) + C \\
 &= \boxed{\sin^{-1} \left(\frac{x-2}{2} \right) + C}.
 \end{aligned}$$

12. We complete the square: $x^2 - 6x - 10 = (x - 3)^2 - 19$. Then we let $u = x - 3$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 - 6x - 10}} &= \int \frac{dx}{\sqrt{(x - 3)^2 - 19}} \\
 &= \int \frac{du}{\sqrt{u^2 - (\sqrt{19})^2}} \\
 &= \ln \left| u + \sqrt{u^2 - 19} \right| + C \quad (\text{from Exercise 89, Section 7.3}) \\
 &= \ln \left| (x - 3) + \sqrt{(x - 3)^2 - 19} \right| + C \\
 &= \boxed{\ln \left| x - 3 + \sqrt{x^2 - 6x - 10} \right| + C}.
 \end{aligned}$$

13. We complete the square: $x^2 + 2x + 2 = (x + 1)^2 + 1$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{(x + 1)\sqrt{x^2 + 2x + 2}} &= \int \frac{dx}{(x + 1)\sqrt{(x + 1)^2 + 1}} \\
 &= \int \frac{du}{u\sqrt{u^2 + 1^2}} \\
 &= \ln \left| \frac{\sqrt{u^2 + 1} - 1}{u} \right| + C \\
 &= \ln \left| \frac{\sqrt{(x + 1)^2 + 1} - 1}{x + 1} \right| + C \\
 &= \boxed{\ln \left| \frac{\sqrt{x^2 + 2x + 2} - 1}{x + 1} \right| + C}.
 \end{aligned}$$

14. We complete the square: $x^2 - 8x + 17 = (x - 4)^2 + 1$. Then we let $u = x - 4$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{(x-4)\sqrt{x^2-8x+17}} &= \int \frac{dx}{(x-4)\sqrt{(x-4)^2+1}} \\
 &= \int \frac{du}{u\sqrt{u^2+1^2}} \\
 &= \ln \left| \frac{\sqrt{u^2+1}-1}{u} \right| + C \\
 &= \ln \left| \frac{\sqrt{(x-4)^2+1}-1}{x-4} \right| + C \\
 &= \boxed{\ln \left| \frac{\sqrt{x^2-8x+17}-1}{x-4} \right| + C}.
 \end{aligned}$$

15. We complete the square: $24 - 2x - x^2 = 25 - (x + 1)^2$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{24-2x-x^2}} &= \int \frac{dx}{\sqrt{25-(x+1)^2}} \\
 &= \int \frac{du}{\sqrt{5^2-u^2}} \\
 &= \sin^{-1} \left(\frac{u}{5} \right) + C \\
 &= \boxed{\sin^{-1} \left(\frac{x+1}{5} \right) + C}.
 \end{aligned}$$

16. We complete the square: $9x^2 + 6x + 10 = 9\left(x + \frac{1}{3}\right)^2 + 9$. Then we let $u = x + \frac{1}{3}$ to obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{9x^2+6x+10}} &= \int \frac{dx}{\sqrt{9\left(x+\frac{1}{3}\right)^2+9}} \\
 &= \frac{1}{3} \int \frac{du}{\sqrt{u^2+1^2}} \\
 &= \frac{1}{3} \ln \left(u + \sqrt{u^2+1} \right) + C \\
 &= \frac{1}{3} \ln \left(x + \frac{1}{3} + \sqrt{\left(x + \frac{1}{3}\right)^2 + 1} \right) + C \\
 &= \frac{1}{3} \ln \left(x + \frac{1}{3} + \sqrt{x^2 + \frac{2}{3}x + \frac{10}{9}} \right) + C \\
 &= \boxed{\frac{1}{3} \ln \left(3x + 1 + \sqrt{9x^2 + 6x + 10} \right) + C}.
 \end{aligned}$$

17. We force the the derivative of the denominator to appear in the numerator, and obtain two integrals.

$$\begin{aligned}\int \frac{x-5}{\sqrt{x^2-2x+5}} dx &= \int \frac{\frac{1}{2}(2x-2)-4}{\sqrt{x^2-2x+5}} dx \\ &= \frac{1}{2} \int \frac{(2x-2)}{\sqrt{x^2-2x+5}} dx - 4 \int \frac{dx}{\sqrt{(x-1)^2+4}}.\end{aligned}$$

Now for the first integral let $w = x^2 - 2x + 5$, so $dw = (2x - 2) dx$. In the second integral, complete the square with $x^2 - 2x + 5 = (x - 1)^2 + 4$. ; and let $u = x - 1$, so $du = dx$. We obtain

$$\begin{aligned}&= \frac{1}{2} \int w^{-1/2} dw - 4 \int \frac{du}{\sqrt{u^2+2^2}} \\ &= \frac{1}{2} (2\sqrt{w}) - 4 \ln(u + \sqrt{u^2+4}) + C \\ &= \sqrt{x^2-2x+5} - 4 \ln(x-1 + \sqrt{(x-1)^2+4}) + C \\ &= \boxed{\sqrt{x^2-2x+5} - 4 \ln(x-1 + \sqrt{x^2-2x+5}) + C}.\end{aligned}$$

18. We force the the derivative of the denominator to appear in the numerator. We then complete the square with $x^2 - 4x + 3 = (x - 2)^2 - 1$, and let $u = x - 2$, so $du = dx$. We obtain

$$\begin{aligned}\int \frac{x+1}{x^2-4x+3} dx &= \int \frac{\frac{1}{2}(2x-4)+3}{x^2-4x+3} dx \\ &= \frac{1}{2} \int \frac{(2x-4)}{x^2-4x+3} dx + 3 \int \frac{dx}{(x-2)^2-1} \\ &= \frac{1}{2} \ln|x^2-4x+3| + 3 \int \frac{du}{u^2-1} \\ &= \frac{1}{2} \ln|x^2-4x+3| + 3 \left(\frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| \right) + C \\ &= \frac{1}{2} \ln|x^2-4x+3| + 3 \left(\frac{1}{2} \ln|(x-2)-1| - \frac{1}{2} \ln|(x-2)+1| \right) + C \\ &= \boxed{\frac{1}{2} \ln|x^2-4x+3| + \frac{3}{2} \ln|x-3| - \frac{3}{2} \ln|x-1| + C}.\end{aligned}$$

19. We complete the square: $x^2 - 2x + 5 = (x - 1)^2 + 4$, and let $u = x - 1$ to obtain

$$\begin{aligned}\int_1^3 \frac{dx}{\sqrt{x^2-2x+5}} &= \int_1^3 \frac{dx}{\sqrt{(x-1)^2+4}} \\ &= \int_0^2 \frac{du}{\sqrt{u^2+2^2}} \\ &= \left[\ln(u + \sqrt{u^2+4}) \right]_{u=0}^{u=2} \\ &= \ln(2 + \sqrt{2^2+4}) - \ln(0 + \sqrt{0^2+4}) \\ &= \ln(2\sqrt{2}+2) - \ln 2 \\ &= \boxed{\ln(\sqrt{2}+1)}.\end{aligned}$$

20. We complete the square: $2x - x^2 = 1 - (x - 1)^2$. Then we let $u = x - 1$ to obtain

$$\begin{aligned}
 \int_{1/2}^1 \frac{x^2 dx}{\sqrt{2x - x^2}} &= \int_{1/2}^1 \frac{x^2 dx}{\sqrt{1 - (x - 1)^2}} \\
 &= \int_{-1/2}^0 \frac{(u + 1)^2}{\sqrt{1 - u^2}} du \\
 &= \int_{-1/2}^0 \frac{u^2 + 2u + 1}{\sqrt{1 - u^2}} du \\
 &= \int_{-1/2}^0 \frac{u^2}{\sqrt{1 - u^2}} du + \int_{-1/2}^0 \frac{2u}{\sqrt{1 - u^2}} du + \int_{-1/2}^0 \frac{1}{\sqrt{1 - u^2}} du \\
 &= \int_{-1/2}^0 \frac{u^2}{\sqrt{1 - u^2}} du + \int_{-1/2}^0 \frac{2u}{\sqrt{1 - u^2}} du + [\sin^{-1} u]_{-1/2}^0 \\
 &= \int_{-1/2}^0 \frac{u^2}{\sqrt{1 - u^2}} du + \int_{-1/2}^0 \frac{2u}{\sqrt{1 - u^2}} du + \frac{\pi}{6}.
 \end{aligned}$$

For the first integral we let $u = \sin \theta$, then we have

$$\begin{aligned}
 \int_{-1/2}^0 \frac{u^2}{\sqrt{1 - u^2}} du &= \int_{-\pi/6}^0 \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\
 &= \frac{1}{2} \int_{-\pi/6}^0 (1 - \cos(2\theta)) d\theta \\
 &= \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_{-\pi/6}^0 \\
 &= \frac{1}{12} \pi - \frac{1}{8} \sqrt{3}.
 \end{aligned}$$

And for the second integral we let $w = 1 - u^2$ to obtain

$$\begin{aligned}
 \int_{-1/2}^0 \frac{2u}{\sqrt{1 - u^2}} du &= \int_{3/4}^1 w^{-1/2} (-dw) \\
 &= [-2\sqrt{w}]_{3/4}^1 \\
 &= \sqrt{3} - 2.
 \end{aligned}$$

And so we obtain

$$\begin{aligned}
 \int_{1/2}^1 \frac{x^2 dx}{\sqrt{2x - x^2}} &= \left(\frac{1}{12} \pi - \frac{1}{8} \sqrt{3} \right) + (\sqrt{3} - 2) + \frac{\pi}{6} \\
 &= \boxed{\frac{1}{4} \pi + \frac{7}{8} \sqrt{3} - 2}.
 \end{aligned}$$

21. We complete the square: $e^{2x} + e^x + 1 = \left(e^x + \frac{1}{2}\right)^2 + \frac{3}{4}$. Then we let $u = e^x + \frac{1}{2}$ to obtain

$$\begin{aligned}
 \int \frac{e^x dx}{\sqrt{e^{2x} + e^x + 1}} &= \int \frac{e^x dx}{\sqrt{\left(e^x + \frac{1}{2}\right)^2 + \frac{3}{4}}} \\
 &= \int \frac{du}{\sqrt{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} \\
 &= \ln \left(u + \sqrt{u^2 + \frac{3}{4}} \right) + C \\
 &= \ln \left(e^x + \frac{1}{2} + \sqrt{\left(e^x + \frac{1}{2}\right)^2 + \frac{3}{4}} \right) + C \\
 &= \boxed{\ln \left(e^x + \frac{1}{2} + \sqrt{e^{2x} + e^x + 1} \right) + C}.
 \end{aligned}$$

22. We complete the square: $\sin^2 x + 4 \sin x + 3 = (\sin x + 2)^2 - 1$. Then we let $u = \sin x + 2$ to obtain

$$\begin{aligned}
 \int \frac{\cos x dx}{\sqrt{\sin^2 x + 4 \sin x + 3}} &= \int \frac{\cos x dx}{\sqrt{(\sin x + 2)^2 - 1}} \\
 &= \int \frac{du}{\sqrt{u^2 - 1}} \\
 &= \ln |u + \sqrt{u^2 - 1}| + C \\
 &= \ln \left(\sin x + 2 + \sqrt{(\sin x + 2)^2 - 1} \right) + C \\
 &= \boxed{\ln \left(\sin x + 2 + \sqrt{\sin^2 x + 4 \sin x + 3} \right) + C}.
 \end{aligned}$$

23. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}
 \int \frac{2x - 3}{\sqrt{4x - x^2 - 3}} dx &= \int \frac{-(4 - 2x) + 1}{\sqrt{4x - x^2 - 3}} dx \\
 &= - \int \frac{4 - 2x}{\sqrt{4x - x^2 - 3}} dx + \int \frac{1}{\sqrt{1 - (x - 2)^2}} dx.
 \end{aligned}$$

For the first integral let $w = 4x - x^2 - 3$ so $dw = (4 - 2x) dx$. In the second integral, complete the square with $4x - x^2 - 3 = 1 - (x - 2)^2$, and let $u = x - 2$, so $du = dx$. We obtain

$$\begin{aligned}
 \int \frac{2x - 3}{\sqrt{4x - x^2 - 3}} dx &= - \int w^{-1/2} dw + \int \frac{du}{\sqrt{1 - u^2}} \\
 &= -(2\sqrt{w}) + \sin^{-1} u + C \\
 &= \boxed{-2\sqrt{4x - x^2 - 3} + \sin^{-1} (x - 2) + C}.
 \end{aligned}$$

24. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}\int \frac{x+3}{\sqrt{x^2+2x+2}} dx &= \int \frac{\frac{1}{2}(2x+2)+2}{\sqrt{x^2+2x+2}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+2}} dx + 2 \int \frac{dx}{\sqrt{(x+1)^2+1}}.\end{aligned}$$

For the first integral we let $w = x^2 + 2x + 2$ so $dw = (2x + 2) dx$. In the second integral, we complete the square with $x^2 + 2x + 2 = (x + 1)^2 + 1$, and let $u = x + 1$, so $du = dx$. We obtain

$$\begin{aligned}\int \frac{x+3}{\sqrt{x^2+2x+2}} dx &= \frac{1}{2} \int w^{-1/2} dw + 2 \int \frac{du}{\sqrt{u^2+1}} \\ &= \frac{1}{2} (2\sqrt{w}) + 2 \ln(u + \sqrt{u^2+1}) + C \\ &= \sqrt{x^2+2x+2} + 2 \ln(x+1 + \sqrt{(x+1)^2+1}) + C \\ &= \boxed{\sqrt{x^2+2x+2} + 2 \ln(x+1 + \sqrt{x^2+2x+2}) + C}.\end{aligned}$$

25. We complete the square: $x^2 - 2x + 10 = (x - 1)^2 + 9$. Then we let $u = x - 1$, so $du = dx$, and obtain

$$\begin{aligned}\int \frac{dx}{(x^2 - 2x + 10)^{3/2}} &= \int \frac{dx}{((x - 1)^2 + 9)^{3/2}} \\ &= \int \frac{du}{(u^2 + 9)^{3/2}}.\end{aligned}$$

Let $u = 3 \tan \theta$, then $du = 3 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{dx}{(x^2 - 2x + 10)^{3/2}} &= \int \frac{3 \sec^2 \theta}{((3 \tan \theta)^2 + 9)^{3/2}} d\theta \\ &= 3 \int \frac{\sec^2 \theta}{(9 \tan^2 \theta + 9)^{3/2}} d\theta \\ &= \frac{1}{9} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\ &= \frac{1}{9} \int \cos \theta d\theta \\ &= \frac{1}{9} \sin \theta + C.\end{aligned}$$

We have $\tan \theta = \frac{u}{3}$, so $\cot \theta = \frac{3}{u}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{3}{u}\right)^2} = \frac{1}{u} \sqrt{u^2 + 9}$. So $\sin \theta = \frac{u}{\sqrt{u^2+9}}$ and we obtain

$$\begin{aligned}\int \frac{dx}{(x^2 - 2x + 10)^{3/2}} &= \frac{1}{9} \frac{u}{\sqrt{u^2+9}} + C \\ &= \frac{x-1}{9\sqrt{(x-1)^2+9}} + C \\ &= \boxed{\frac{x-1}{9\sqrt{x^2-2x+10}} + C}.\end{aligned}$$

26. We complete the square: $x^2 - 2x + 10 = (x - 1)^2 + 9$. Then we let $u = x - 1$ to obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 - 2x + 10}} &= \int \frac{dx}{\sqrt{(x - 1)^2 + 9}} \\
 &= \int \frac{du}{\sqrt{u^2 + 3^2}} \\
 &= \ln \left(u + \sqrt{u^2 + 9} \right) + C \\
 &= \ln \left(x - 1 + \sqrt{(x - 1)^2 + 9} \right) + C \\
 &= \boxed{\ln \left(x - 1 + \sqrt{x^2 - 2x + 10} \right) + C}.
 \end{aligned}$$

27. We complete the square: $x^2 + 2x - 3 = (x + 1)^2 - 4$. Then we let $u = x + 1$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + 2x - 3}} &= \int \frac{dx}{\sqrt{(x + 1)^2 - 4}} \\
 &= \int \frac{du}{\sqrt{u^2 - 4}} \\
 &= \ln \left| u + \sqrt{u^2 - 4} \right| + C \\
 &= \ln \left| x + 1 + \sqrt{(x + 1)^2 - 4} \right| + C \\
 &= \boxed{\ln \left| x + 1 + \sqrt{x^2 + 2x - 3} \right| + C}.
 \end{aligned}$$

28. We complete the square: $x^2 - 4x - 1 = (x - 2)^2 - 5$. Then we let $u = x - 2$, so $du = dx$, so we obtain

$$\begin{aligned}
 \int x \sqrt{x^2 - 4x - 1} \, dx &= \int x \sqrt{(x - 2)^2 - 5} \, dx \\
 &= \int (u + 2) \sqrt{u^2 - 5} \, du \\
 &= \int u \sqrt{u^2 - 5} \, du + 2 \int \sqrt{u^2 - 5} \, du.
 \end{aligned}$$

With the first integral, let $w = u^2 - 5$, so $dw = 2 \, du$, and we obtain

$$\begin{aligned}
 \int u \sqrt{u^2 - 5} \, du &= \frac{1}{2} \int \sqrt{w} \, dw \\
 &= \frac{1}{2} \left(\frac{2}{3} \right) w^{\frac{3}{2}} + C \\
 &= \frac{1}{3} (u^2 - 5)^{\frac{3}{2}} + C.
 \end{aligned}$$

And for the second integral, let $u = \sqrt{5} \sec \theta$, so $du = \sqrt{5} \sec \theta \tan \theta d\theta$, and we obtain

$$\begin{aligned}
 \int \sqrt{u^2 - 5} du &= \int \sqrt{5 \sec^2 \theta - 5} \sqrt{5} \sec \theta \tan \theta d\theta \\
 &= 5 \int \sec \theta \tan^2 \theta d\theta \\
 &= 5 \int (\sec^3 \theta - \sec \theta) d\theta \\
 &= 5 \left(\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) + C \\
 &= 5 \left(\frac{1}{2} \left(\frac{u}{\sqrt{5}} \right) \sqrt{\frac{u^2}{5} - 1} - \frac{1}{2} \ln \left| \frac{u}{\sqrt{5}} + \sqrt{\frac{u^2}{5} - 1} \right| \right) + C \\
 &= \frac{1}{2} u \sqrt{u^2 - 5} - \frac{5}{2} \ln |\sqrt{u^2 - 5} + u| + C.
 \end{aligned}$$

So we obtain

$$\begin{aligned}
 \int x \sqrt{x^2 - 4x - 1} dx &= \frac{1}{3} (u^2 - 5)^{\frac{3}{2}} + 2 \left(\frac{1}{2} u \sqrt{u^2 - 5} - \frac{5}{2} \ln |\sqrt{u^2 - 5} + u| \right) + C \\
 &= \frac{\left((x-2)^2 - 5 \right)^{\frac{3}{2}}}{3} + (x-2) \sqrt{(x-2)^2 - 5} \\
 &\quad - 5 \ln |x-2 + \sqrt{(x-2)^2 - 5}| + C \\
 &= \boxed{\frac{(x^2 - 4x - 1)^{\frac{3}{2}}}{3} + (x-2) \sqrt{x^2 - 4x - 1} - 5 \ln |x-2 + \sqrt{x^2 - 4x - 1}| + C}.
 \end{aligned}$$

29. We complete the square: $5 + 4x - x^2 = 9 - (x-2)^2$. Then we let $u = x-2$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{\sqrt{5 + 4x - x^2}}{x-2} dx &= \int \frac{\sqrt{9 - (x-2)^2}}{x-2} dx \\
 &= \int \frac{\sqrt{9 - u^2}}{u} du.
 \end{aligned}$$

Let $u = 3 \sin \theta$, so $du = 3 \cos \theta$, and we have

$$\begin{aligned}
\int \frac{\sqrt{5+4x-x^2}}{x-2} dx &= \int \frac{\sqrt{9-(3\sin\theta)^2}}{3\sin\theta} (3\cos\theta) d\theta \\
&= 3 \int \frac{\cos^2\theta}{\sin\theta} d\theta \\
&= 3 \int \frac{1-\sin^2\theta}{\sin\theta} d\theta \\
&= 3 \int (\csc\theta - \sin\theta) d\theta \\
&= 3(-\ln|\csc\theta + \cot\theta| + \cos\theta) + C \\
&= 3\left(-\ln\left|\frac{3}{u} + \frac{\sqrt{9-u^2}}{u}\right| + \frac{1}{3}\sqrt{9-u^2}\right) + C \\
&= -3\ln\left|\frac{3}{x-2} + \frac{\sqrt{9-(x-2)^2}}{x-2}\right| + \sqrt{9-(x-2)^2} + C \\
&= \boxed{3\ln|x-2| - 3\ln|3 + \sqrt{5+4x-x^2}| + \sqrt{5+4x-x^2} + C}.
\end{aligned}$$

30. We complete the square: $5+4x-x^2 = 9-(x-2)^2$. Then we let $u = x-2$, so $du = dx$, and we obtain

$$\begin{aligned}
\int \sqrt{5+4x-x^2} dx &= \int \sqrt{9-(x-2)^2} dx \\
&= \int \sqrt{9-u^2} du.
\end{aligned}$$

Let $u = 3\sin\theta$, so $du = 3\cos\theta$, and we obtain

$$\begin{aligned}
\int \sqrt{5+4x-x^2} dx &= \int \sqrt{9-(3\sin\theta)^2} (3\cos\theta) d\theta \\
&= 9 \int \cos^2\theta d\theta \\
&= \frac{9}{2} \int (1 + \cos(2\theta)) d\theta \\
&= \frac{9}{2} \left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{9}{2}(\theta + \sin\theta \cos\theta) + C \\
&= \frac{9}{2} \left(\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3} \left(\frac{\sqrt{9-u^2}}{3}\right)\right) + C \\
&= \frac{9}{2} \sin^{-1}\left(\frac{u}{3}\right) + \frac{1}{2}u\sqrt{9-u^2} + C \\
&= \frac{9}{2} \sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{9-(x-2)^2} + C \\
&= \boxed{\frac{9}{2} \sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C}.
\end{aligned}$$

31. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{x^2 + 2x - 3}} &= \int \frac{\frac{1}{2}(2x + 2) - 1}{\sqrt{x^2 + 2x - 3}} \, dx \\ &= \frac{1}{2} \int \frac{2x + 2}{\sqrt{x^2 + 2x - 3}} \, dx - \int \frac{dx}{\sqrt{(x + 1)^2 - 4}}.\end{aligned}$$

In the first integral, let $w = x^2 + 2x - 3$, so $dw = (2x + 2) \, dx$. In the second integral, complete the square with $x^2 + 2x - 3 = (x + 1)^2 - 4$, and let $u = x + 1$, so $du = dx$. We obtain

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{x^2 + 2x - 3}} &= \frac{1}{2} \int w^{-1/2} \, dw - \int \frac{du}{\sqrt{u^2 - 4}} \\ &= \frac{1}{2} (2\sqrt{w}) - \ln |u + \sqrt{u^2 - 4}| + C \\ &= \sqrt{x^2 + 2x - 3} - \ln |x + 1 + \sqrt{(x + 1)^2 - 4}| + C \\ &= \boxed{\sqrt{x^2 + 2x - 3} - \ln |x + 1 + \sqrt{x^2 + 2x - 3}| + C}.\end{aligned}$$

32. We force the the derivative of the quadratic to appear in the numerator.

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{x^2 - 4x + 3}} &= \int \frac{\frac{1}{2}(2x - 4) + 2}{\sqrt{x^2 - 4x + 3}} \, dx \\ &= \frac{1}{2} \int \frac{2x - 4}{\sqrt{x^2 - 4x + 3}} \, dx + 2 \int \frac{dx}{\sqrt{(x - 2)^2 - 1}}.\end{aligned}$$

In the first integral let $w = x^2 - 4x + 3$, so $dw = (2x - 4) \, dx$. In the second integral, complete the square with $x^2 - 4x + 3 = (x - 2)^2 - 1$, and let $u = x - 2$, so $du = dx$. We obtain

$$\begin{aligned}\int \frac{x \, dx}{\sqrt{x^2 - 4x + 3}} &= \frac{1}{2} \int w^{-1/2} \, dw + 2 \int \frac{du}{\sqrt{u^2 - 1}} \\ &= \frac{1}{2} (2\sqrt{w}) + 2 \ln |u + \sqrt{u^2 - 1}| + C \\ &= \sqrt{x^2 - 4x + 3} + 2 \ln |x - 2 + \sqrt{(x - 2)^2 - 1}| + C \\ &= \boxed{\sqrt{x^2 - 4x + 3} + 2 \ln |x - 2 + \sqrt{x^2 - 4x + 3}| + C}.\end{aligned}$$

Applications and Extensions

33. Let $u = x + h$, then

$$\int \frac{dx}{\sqrt{(x + h)^2 + k}} = \int \frac{du}{\sqrt{u^2 + k}}.$$

We then apply the result of exercise 97, section 7.3, to obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{(x + h)^2 + k}} &= \ln [u + \sqrt{u^2 + k}] + C \\ &= \boxed{\ln [\sqrt{(x + h)^2 + k} + x + h] + C}.\end{aligned}$$

34. We complete the square: $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$. Then let $u = x + \frac{b}{2a}$, so $du = dx$, and we obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \int \frac{dx}{\sqrt{a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}}} \\
 &= \int \frac{du}{\sqrt{au^2 - \frac{b^2 - 4ac}{4a}}} \\
 &= \int \frac{du}{\sqrt{a\left(u^2 - \frac{b^2 - 4ac}{4a^2}\right)}} \\
 &= \frac{1}{\sqrt{a}} \int \frac{du}{\sqrt{u^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2}} \quad (\text{using } a > 0 \text{ and } b^2 - 4ac > 0) \\
 &= \frac{1}{\sqrt{a}} \ln \left| u + \sqrt{u^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2} \right| + C \\
 &= \frac{1}{\sqrt{a}} \ln \left| x + \frac{b}{2a} + \sqrt{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}} \right| + C.
 \end{aligned}$$

And then we rewrite this and obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \frac{1}{\sqrt{a}} \ln \left| x + \frac{b}{2a} + \sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} \right| + C \\
 &= \frac{1}{\sqrt{a}} \ln \left| x + \frac{b}{2a} + \frac{1}{\sqrt{a}} \sqrt{ax^2 + bx + c} \right| + C \\
 &= \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{ax^2 + bx + c} + \frac{b}{2\sqrt{a}}}{\sqrt{a}} \right| + C \\
 &= \frac{1}{\sqrt{a}} \ln \left| \sqrt{ax^2 + bx + c} + \frac{b}{2\sqrt{a}} \right| - \frac{1}{\sqrt{a}} \ln \sqrt{a} + C \\
 &= \boxed{\frac{1}{\sqrt{a}} \ln \left| \sqrt{ax^2 + bx + c} + \sqrt{ax^2 + bx + c} + \frac{b}{2\sqrt{a}} \right| + C}.
 \end{aligned}$$

Challenge Problems

35. We rewrite

$$\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}} = \sqrt{\frac{(a+x)^2}{a^2 - x^2}} = \frac{\sqrt{(a+x)^2}}{\sqrt{a^2 - x^2}} = \frac{a+x}{\sqrt{a^2 - x^2}}.$$

Then we obtain

$$\begin{aligned}
 \int \sqrt{\frac{a+x}{a-x}} dx &= \int \frac{a+x}{\sqrt{a^2 - x^2}} dx \\
 &= \int \frac{a}{\sqrt{a^2 - x^2}} dx + \int \frac{x}{\sqrt{a^2 - x^2}} dx.
 \end{aligned}$$

Let $u = a^2 - x^2$, so $du = -2x \, dx$. Then we have

$$\begin{aligned}\int \sqrt{\frac{a+x}{a-x}} \, dx &= a \sin^{-1} \frac{x}{a} + \int \frac{-\frac{1}{2} du}{\sqrt{u}} \\ &= a \sin^{-1} \frac{x}{a} - \sqrt{u} + C \\ &= \boxed{a \sin^{-1} \frac{x}{a} - \sqrt{a^2 - x^2} + C}.\end{aligned}$$

7.5: Integration of Rational Functions Using Partial Fractions

Concepts and Vocabulary

1. (a) less than
2. True
3. True
4. False, the term $\frac{D}{(x+1)^4}$ is also needed

Skill Building

5. We use long division to obtain

$$\begin{aligned}\int \frac{x^2+1}{x+1} dx &= \int \left(x-1 + \frac{2}{x+1} \right) dx \\ &= \boxed{\frac{1}{2}x^2 - x + 2 \ln |x+1| + C}.\end{aligned}$$

6. We use long division to obtain

$$\begin{aligned}\int \frac{x^2+4}{x-2} dx &= \int \left(x+2 + \frac{8}{x-2} \right) dx \\ &= \boxed{\frac{1}{2}x^2 + 2x + 8 \ln |x-2| + C}.\end{aligned}$$

7. We use long division to obtain

$$\begin{aligned}\int \frac{x^3+3x-4}{x-2} dx &= \int \left(x^2+2x+7 + \frac{10}{x-2} \right) dx \\ &= \boxed{\frac{1}{3}x^3 + x^2 + 7x + 10 \ln |x-2| + C}.\end{aligned}$$

8. We use long division to obtain

$$\begin{aligned}\int \frac{x^3-3x^2+4}{x+3} dx &= \int \left(x^2-6x+18 - \frac{50}{x+3} \right) dx \\ &= \boxed{\frac{1}{3}x^3 - 3x^2 + 18x - 50 \ln |x+3| + C}.\end{aligned}$$

9. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(x-2)(x+1)} &= \frac{A}{x-2} + \frac{B}{x+1} \\ 1 &= A(x+1) + B(x-2)\end{aligned}$$

When $x = -1$ we obtain $B = -\frac{1}{3}$, and when $x = 2$ we have $A = \frac{1}{3}$. So we obtain

$$\begin{aligned}\int \frac{1}{(x-2)(x+1)} dx &= \int \left(\frac{1/3}{x-2} - \frac{1/3}{x+1} \right) dx \\ &= \boxed{\frac{1}{3} \ln |x-2| - \frac{1}{3} \ln |x+1| + C}.\end{aligned}$$

10. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(x+4)(x-1)} &= \frac{A}{x+4} + \frac{B}{x-1} \\ 1 &= A(x-1) + B(x+4)\end{aligned}$$

When $x = 1$ we obtain $B = \frac{1}{5}$, and when $x = -4$ we have $A = -\frac{1}{5}$. So we obtain

$$\begin{aligned}\int \frac{1}{(x+4)(x-1)} dx &= \int \left(\frac{-1/5}{x+4} + \frac{1/5}{x-1} \right) dx \\ &= \boxed{-\frac{1}{5} \ln |x+4| + \frac{1}{5} \ln |x-1| + C}.\end{aligned}$$

11. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\ x &= A(x-2) + B(x-1)\end{aligned}$$

When $x = 2$ we obtain $B = 2$, and when $x = 1$ we have $A = -1$. So we obtain

$$\begin{aligned}\int \frac{x dx}{(x-1)(x-2)} &= \int \left(-\frac{1}{x-1} + \frac{2}{x-2} \right) dx \\ &= \boxed{-\ln |x-1| + 2 \ln |x-2| + C}.\end{aligned}$$

12. We use partial fractions to obtain

$$\begin{aligned}\frac{3x}{(x+2)(x-4)} &= \frac{A}{x+2} + \frac{B}{x-4} \\ 3x &= A(x-4) + B(x+2)\end{aligned}$$

When $x = 4$ we obtain $B = 2$, and when $x = -2$ we have $A = 1$. So we obtain

$$\begin{aligned}\int \frac{3x dx}{(x+2)(x-4)} &= \int \left(\frac{1}{x+2} + \frac{2}{x-4} \right) dx \\ &= \boxed{\ln |x+2| + 2 \ln |x-4| + C}.\end{aligned}$$

13. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{(3x-2)(2x+1)} &= \frac{A}{3x-2} + \frac{B}{2x+1} \\ x &= A(2x+1) + B(3x-2)\end{aligned}$$

When $x = -\frac{1}{2}$ we obtain $B = \frac{1}{7}$, and when $x = \frac{2}{3}$ we have $A = \frac{2}{7}$. So we obtain

$$\begin{aligned}\int \frac{x dx}{(3x-2)(2x+1)} &= \int \left(\frac{2/7}{3x-2} + \frac{1/7}{2x+1} \right) dx \\ &= \boxed{\frac{2}{21} \ln |3x-2| + \frac{1}{14} \ln |2x+1| + C}.\end{aligned}$$

14. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(2x+3)(4x-1)} &= \frac{A}{2x+3} + \frac{B}{4x-1} \\ 1 &= A(4x-1) + B(2x+3)\end{aligned}$$

When $x = \frac{1}{4}$ we obtain $B = \frac{2}{7}$, and when $x = -\frac{3}{2}$ we have $A = -\frac{1}{7}$. So we obtain

$$\begin{aligned}\int \frac{dx}{(2x+3)(4x-1)} &= \int \left(\frac{-1/7}{2x+3} + \frac{2/7}{4x-1} \right) dx \\ &= \boxed{-\frac{1}{14} \ln |2x+3| + \frac{1}{14} \ln |4x-1| + C}.\end{aligned}$$

15. We use partial fractions to obtain

$$\begin{aligned}\frac{x-3}{(x+2)(x+1)^2} &= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ x-3 &= A(x+1)^2 + B(x+1)(x+2) + C(x+2)\end{aligned}$$

When $x = -1$ we obtain $C = -4$, and when $x = -2$ we have $A = -5$. When $x = 0$ we obtain $B = 5$. So we obtain

$$\begin{aligned}\int \frac{x-3}{(x+2)(x+1)^2} dx &= \int \left(\frac{-5}{x+2} + \frac{5}{x+1} + \frac{-4}{(x+1)^2} \right) dx \\ &= \int \frac{-5}{x+2} dx + \int \frac{5}{x+1} dx + \int \frac{-4}{(x+1)^2} dx \\ &= \boxed{-5 \ln |x+2| + 5 \ln |x+1| + \frac{4}{x+1} + C}.\end{aligned}$$

16. We use partial fractions to obtain

$$\begin{aligned}\frac{x+1}{x^2(x-2)} &= \frac{A}{x-2} + \frac{B}{x} + \frac{C}{x^2} \\ x+1 &= Ax^2 + Bx(x-2) + C(x-2)\end{aligned}$$

When $x = 0$ we obtain $C = -1/2$, and when $x = 2$ we have $A = 3/4$. When $x = 1$ we obtain $B = -3/4$. So we obtain

$$\begin{aligned}\int \frac{x+1}{x^2(x-2)} dx &= \int \left(\frac{3/4}{x-2} + \frac{-3/4}{x} + \frac{-1/2}{x^2} \right) dx \\ &= \int \frac{3/4}{x-2} dx + \int \frac{-3/4}{x} dx + \int \frac{-1/2}{x^2} dx \\ &= \boxed{\frac{3}{4} \ln |x-2| - \frac{3}{4} \ln |x| + \frac{1}{2x} + C}.\end{aligned}$$

17. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2}{(x-1)^2(x+1)} &= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ x^2 &= A(x-1)^2 + B(x-1)(x+1) + C(x+1)\end{aligned}$$

When $x = 1$ we obtain $C = 1/2$, and when $x = -1$ we have $A = 1/4$. When $x = 0$ we obtain $B = 3/4$. So we obtain

$$\begin{aligned}\int \frac{x^2}{(x-1)^2(x+1)} dx &= \int \left(\frac{1/4}{x+1} + \frac{3/4}{x-1} + \frac{1/2}{(x-1)^2} \right) dx \\ &= \int \frac{1/4}{x+1} dx + \int \frac{3/4}{x-1} dx + \int \frac{1/2}{(x-1)^2} dx \\ &= \boxed{\frac{1}{4} \ln |x+1| + \frac{3}{4} \ln |x-1| - \frac{1}{2(x-1)} + C}.\end{aligned}$$

18. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2 + x}{(x+2)(x-1)^2} &= \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ x^2 + x &= A(x-1)^2 + B(x-1)(x+2) + C(x+2)\end{aligned}$$

When $x = 1$ we obtain $C = 2/3$, and when $x = -2$ we have $A = 2/9$. When $x = 0$ we obtain $B = 7/9$. So we obtain

$$\begin{aligned}\int \frac{x^2 + x}{(x+2)(x-1)^2} dx &= \int \left(\frac{2/9}{x+2} + \frac{7/9}{x-1} + \frac{2/3}{(x-1)^2} \right) dx \\ &= \int \frac{2/9}{x+2} dx + \int \frac{7/9}{x-1} dx + \int \frac{2/3}{(x-1)^2} dx \\ &= \boxed{\frac{2}{9} \ln |x+2| + \frac{7}{9} \ln |x-1| - \frac{2}{3(x-1)} + C}.\end{aligned}$$

19. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ 1 &= A(x^2+1) + (Bx+C)x\end{aligned}$$

When $x = 0$ we obtain $A = 1$. So

$$\begin{aligned}1 &= (1)(x^2+1) + (Bx+C)x \\ 1 &= (B+1)x^2 + Cx + 1\end{aligned}$$

Equating coefficients, we obtain $B = -1$ and $C = 0$. We now have

$$\begin{aligned}\int \frac{dx}{x(x^2+1)} &= \int \left(\frac{1}{x} + \frac{-x}{x^2+1} \right) dx \\ &= \int \frac{1}{x} dx + \int \frac{-x}{x^2+1} dx \\ &= \boxed{\ln |x| - \frac{1}{2} \ln (x^2+1) + C}.\end{aligned}$$

20. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(x+1)(x^2+4)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+4} \\ 1 &= A(x^2+4) + (Bx+C)(x+1)\end{aligned}$$

When $x = -1$ we obtain $A = 1/5$. So

$$\begin{aligned}1 &= (1/5)(x^2+4) + (Bx+C)(x+1) \\ 1 &= \left(B + \frac{1}{5}\right)x^2 + (B+C)x + \left(C + \frac{4}{5}\right)\end{aligned}$$

Equating coefficients, we obtain $B = -1/5$ and $C = 1/5$. We now have

$$\begin{aligned}\int \frac{dx}{(x+1)(x^2+4)} &= \int \left(\frac{1/5}{x+1} + \frac{-(1/5)x + 1/5}{x^2+4} \right) dx \\ &= \int \frac{1/5}{x+1} dx + \int \frac{-(1/5)x}{x^2+4} dx + \int \frac{1/5}{x^2+4} dx \\ &= \boxed{\frac{1}{5} \ln |x+1| - \frac{1}{10} \ln (x^2+4) + \frac{1}{10} \tan^{-1} \left(\frac{x}{2} \right) + C}.\end{aligned}$$

21. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2 + 2x + 4} \\ x^2 + 2x + 3 &= A(x^2 + 2x + 4) + (Bx+C)(x+1)\end{aligned}$$

When $x = -1$ we obtain $A = 2/3$. So

$$\begin{aligned}x^2 + 2x + 3 &= A(x^2 + 2x + 4) + (Bx+C)(x+1) \\ x^2 + 2x + 3 &= \left(B + \frac{2}{3}\right)x^2 + \left(B + C + \frac{4}{3}\right)x + \left(C + \frac{8}{3}\right)\end{aligned}$$

Equating coefficients, we obtain $B = 1/3$ and $C = 1/3$. We now have

$$\begin{aligned}\int \frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)} dx &= \int \left(\frac{2/3}{x+1} + \frac{(1/3)x + 1/3}{x^2 + 2x + 4} \right) dx \\ &= \int \frac{2/3}{x+1} dx + \frac{1}{6} \int \frac{2x+2}{x^2 + 2x + 4} dx \\ &= \boxed{\frac{2}{3} \ln|x+1| + \frac{1}{6} \ln(x^2 + 2x + 4) + C}.\end{aligned}$$

22. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} &= \frac{A}{x} + \frac{Bx+C}{x^2 + 3x + 3} \\ x^2 - 11x - 18 &= A(x^2 + 3x + 3) + (Bx+C)x\end{aligned}$$

When $x = 0$ we obtain $A = -6$. So

$$\begin{aligned}x^2 - 11x - 18 &= A(x^2 + 3x + 3) + (Bx+C)x \\ x^2 - 11x - 18 &= (B-6)x^2 + (C-18)x - 18\end{aligned}$$

Equating coefficients, we obtain $B = 7$ and $C = 7$. We now have

$$\begin{aligned}\int \frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} dx &= \int \left(\frac{-6}{x} + \frac{7x+7}{x^2 + 3x + 3} \right) dx \\ &= \int \frac{-6}{x} dx + \int \frac{\frac{7}{2}(2x+3) - \frac{7}{2}}{x^2 + 3x + 3} dx \\ &= \int \frac{-6}{x} dx + \frac{7}{2} \int \frac{2x+3}{x^2 + 3x + 3} dx - \frac{7}{2} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\ &= -6 \ln|x| + \frac{7}{2} \ln(x^2 + 3x + 3) - \frac{7}{2} \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{x + \frac{3}{2}}{\frac{\sqrt{3}}{2}} \right) + C \\ &= \boxed{-6 \ln|x| + \frac{7}{2} \ln(x^2 + 3x + 3) - \frac{7\sqrt{3}}{3} \tan^{-1} \left(\frac{\sqrt{3}(2x+3)}{3} \right) + C}.\end{aligned}$$

23. We expand to obtain

$$\begin{aligned}\int \frac{2x+1}{(x^2+16)^2} dx &= \int \frac{2x}{(x^2+16)^2} dx + \int \frac{1}{(x^2+16)^2} dx \\ &= -\frac{1}{x^2+16} + \int \frac{1}{(x^2+16)^2} dx.\end{aligned}$$

Let $x = 4 \tan \theta$ to obtain

$$\begin{aligned}
 \int \frac{2x+1}{(x^2+16)^2} dx &= -\frac{1}{x^2+16} + \int \frac{4 \sec^2 \theta}{256 \sec^4 \theta} d\theta \\
 &= -\frac{1}{x^2+16} + \frac{1}{64} \int \cos^2 \theta d\theta \\
 &= -\frac{1}{x^2+16} + \frac{1}{128} \int (1 + \cos 2\theta) d\theta \\
 &= -\frac{1}{x^2+16} + \frac{1}{128} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= -\frac{1}{x^2+16} + \frac{1}{128} \theta + \frac{1}{128} \cos \theta \sin \theta + C \\
 &= -\frac{1}{x^2+16} + \frac{1}{128} \tan^{-1} \left(\frac{x}{4} \right) + \frac{1}{128} \frac{4}{\sqrt{x^2+16}} \frac{x}{\sqrt{x^2+16}} + C \\
 &= \boxed{-\frac{1}{x^2+16} + \frac{1}{128} \tan^{-1} \left(\frac{x}{4} \right) + \frac{x}{32(x^2+16)} + C}.
 \end{aligned}$$

24. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x^2+2x+3}{(x^2+4)^2} &= \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} \\
 x^2+2x+3 &= (Ax+B)(x^2+4) + (Cx+D) \\
 x^2+2x+3 &= Ax^3+Bx^2+(4A+C)x+(4B+D).
 \end{aligned}$$

Equating coefficients, we obtain $A = 0$, $B = 1$, $C = 2$, and $D = -1$, so

$$\begin{aligned}
 \int \frac{x^2+2x+3}{(x^2+4)^2} dx &= \int \left(\frac{1}{x^2+4} + \frac{2x-1}{(x^2+4)^2} \right) dx \\
 &= \int \frac{1}{x^2+4} dx + \int \frac{2x}{(x^2+4)^2} dx - \int \frac{1}{(x^2+4)^2} dx \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \int \frac{1}{(x^2+4)^2} dx.
 \end{aligned}$$

Let $x = 2 \tan \theta$ to obtain

$$\begin{aligned}
 \int \frac{x^2+2x+3}{(x^2+4)^2} dx &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \int \frac{2 \sec^2 \theta}{16 \sec^4 \theta} d\theta \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{1}{8} \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{1}{16} \int (1 + \cos 2\theta) d\theta \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{1}{16} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{1}{16} \theta - \frac{1}{16} \cos \theta \sin \theta + C \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{1}{16} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{16} \frac{2}{\sqrt{x^2+4}} \frac{x}{\sqrt{x^2+4}} + C \\
 &= \boxed{\frac{7}{16} \tan^{-1} \left(\frac{x}{2} \right) - \frac{1}{x^2+4} - \frac{x}{8(x^2+4)} + C}.
 \end{aligned}$$

25. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x^3}{(x^2 + 16)^3} &= \frac{Ax + B}{x^2 + 16} + \frac{Cx + D}{(x^2 + 16)^2} + \frac{Ex + F}{(x^2 + 16)^3} \\
 x^3 &= (Ax + B)(x^2 + 16)^2 + (Cx + D)(x^2 + 16) + Ex + F \\
 x^3 &= Ax^5 + Bx^4 + (32A + C)x^3 + (32B + D)x^2 \\
 &\quad + (256A + 16C + E)x + (256B + F + 16D).
 \end{aligned}$$

Equating coefficients, we obtain $A = 0$, $B = 0$, $C = 1$, $D = 0$, $E = -16$, and $F = 0$. So

$$\begin{aligned}
 \int \frac{x^3 dx}{(x^2 + 16)^3} &= \int \left(\frac{x}{(x^2 + 16)^2} + \frac{(-16)x}{(x^2 + 16)^3} \right) dx \\
 &= \int \frac{x}{(x^2 + 16)^2} dx - 16 \int \frac{x}{(x^2 + 16)^3} dx \\
 &= \boxed{-\frac{1}{2(x^2 + 16)} + \frac{4}{(x^2 + 16)^2} + C}.
 \end{aligned}$$

26. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x^2}{(x^2 + 4)^3} &= \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} + \frac{Ex + F}{(x^2 + 4)^3} \\
 x^2 &= (Ax + B)(x^2 + 4)^2 + (Cx + D)(x^2 + 4) + Ex + F \\
 x^2 &= Ax^5 + Bx^4 + (8A + C)x^3 + (8B + D)x^2 + (16A + 4C + E)x + (16B + F + 4D).
 \end{aligned}$$

Equating coefficients, we obtain $A = 0$, $B = 0$, $C = 0$, $D = 1$, $E = 0$, and $F = -4$. So

$$\int \frac{x^2 dx}{(x^2 + 4)^3} = \int \left(\frac{1}{(x^2 + 4)^2} + \frac{-4}{(x^2 + 4)^3} \right) dx.$$

Let $x = 2 \tan \theta$ to obtain

$$\begin{aligned}
 \int \frac{x^2 dx}{(x^2 + 4)^3} &= \int \left(\frac{1}{(4 \tan^2 \theta + 4)^2} + \frac{-4}{(4 \tan^2 \theta + 4)^3} \right) (2 \sec^2 \theta) d\theta \\
 &= \int \left(\frac{1}{16 \sec^4 \theta} + \frac{-4}{64 \sec^6 \theta} \right) (2 \sec^2 \theta) d\theta \\
 &= \frac{1}{8} \int \left(\frac{1}{\sec^4 \theta} - \frac{1}{\sec^6 \theta} \right) (\sec^2 \theta) d\theta \\
 &= \frac{1}{8} \int (\cos^2 \theta - \cos^4 \theta) d\theta \\
 &= \frac{1}{8} \int \left[\frac{1 + \cos(2\theta)}{2} - \left(\frac{1 + \cos(2\theta)}{2} \right)^2 \right] d\theta \\
 &= \frac{1}{8} \int \left(\frac{1 + \cos(2\theta)}{2} - \frac{1 + 2 \cos(2\theta) + \cos^2(2\theta)}{4} \right) d\theta \\
 &= \frac{1}{8} \int \left(\frac{1}{4} - \frac{\cos^2(2\theta)}{4} \right) d\theta \\
 &= \frac{1}{32} \int \left(1 - \frac{1 + \cos(4\theta)}{2} \right) d\theta \\
 &= \frac{1}{64} \int (1 - \cos(4\theta)) d\theta \\
 &= \frac{1}{64} \left(\theta - \frac{1}{4} \sin(4\theta) \right) + C \\
 &= \frac{1}{64} \left(\theta - \frac{1}{2} \sin(2\theta) \cos(2\theta) \right) + C \\
 &= \frac{1}{64} (\theta - \sin \theta \cos \theta (2 \cos^2 \theta - 1)) + C \\
 &= \frac{1}{64} \left(\tan^{-1} \left(\frac{x}{2} \right) - \frac{x}{\sqrt{x^2 + 4}} \frac{2}{\sqrt{x^2 + 4}} \left(2 \left(\frac{2}{\sqrt{x^2 + 4}} \right)^2 - 1 \right) \right) + C \\
 &= \boxed{\frac{1}{64} \tan^{-1} \left(\frac{x}{2} \right) + \frac{x(x^2 - 4)}{32(x^2 + 4)^2} + C}.
 \end{aligned}$$

27. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x}{x^2 + 2x - 3} &= \frac{x}{(x + 3)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x - 1} \\
 x &= A(x - 1) + B(x + 3)
 \end{aligned}$$

When $x = 1$ we obtain $B = 1/4$, and when $x = -3$ we have $A = 3/4$. So we obtain

$$\begin{aligned}
 \int \frac{x dx}{x^2 + 2x - 3} &= \int \left(\frac{3/4}{x + 3} + \frac{1/4}{x - 1} \right) dx \\
 &= \boxed{\frac{3}{4} \ln |x + 3| + \frac{1}{4} \ln |x - 1| + C}.
 \end{aligned}$$

28. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x^2 - x - 8}{(x + 1)(x^2 + 5x + 6)} &= \frac{x^2 - x - 8}{(x + 1)(x + 2)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x + 3} \\
 x^2 - x - 8 &= A(x + 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x + 2)
 \end{aligned}$$

When $x = -3$ we obtain $C = 2$, when $x = -2$ we have $B = 2$, and when $x = -1$ we have $A = -3$. So we obtain

$$\begin{aligned}\int \frac{x^2 - x - 8}{(x+1)(x^2+5x+6)} dx &= \int \left(\frac{-3}{x+1} + \frac{2}{x+2} + \frac{2}{x+3} \right) dx \\ &= \boxed{-3 \ln |x+1| + 2 \ln |x+2| + 2 \ln |x+3| + C}.\end{aligned}$$

29. We use partial fractions to obtain

$$\begin{aligned}\frac{10x^2 + 2x}{(x-1)^2(x^2+2)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2} \\ 10x^2 + 2x &= A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2\end{aligned}$$

When $x = 1$ we obtain $B = 4$. We expand and obtain

$$\begin{aligned}10x^2 + 2x &= A(x-1)(x^2+2) + 4(x^2+2) + (Cx+D)(x-1)^2 \\ 10x^2 + 2x &= (A+C)x^3 + (-A-2C+D+4)x^2 + (2A+C-2D)x + (-2A+D+8)\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}0 &= A+C \\ 10 &= -A-2C+D+4 \\ 2 &= 2A+C-2D \\ 0 &= -2A+D+8\end{aligned}$$

We obtain $A = -C$, so $10 = -(-C) - 2C + D + 4 = D - C + 4$ and $D = 6 + C$. Then $0 = -2A + D + 8 = -2(-C) + (6 + C) + 8 = 3C + 14$, and $C = -14/3$. We now obtain $A = 14/3$, and $D = 4/3$. So

$$\begin{aligned}\int \frac{10x^2 + 2x}{(x-1)^2(x^2+2)} dx &= \int \left(\frac{14/3}{x-1} + \frac{4}{(x-1)^2} + \frac{(-14/3)x + 4/3}{x^2+2} \right) dx \\ &= \int \frac{14/3}{x-1} dx + \int \frac{4}{(x-1)^2} dx - \frac{14}{3} \int \frac{x}{x^2+2} dx + \frac{4}{3} \int \frac{dx}{x^2+2} \\ &= \boxed{\frac{14}{3} \ln |x-1| - \frac{4}{x-1} - \frac{7}{3} \ln (x^2+2) + \frac{2\sqrt{2}}{3} \tan^{-1} \left(\frac{\sqrt{2}x}{2} \right) + C}.\end{aligned}$$

30. We use partial fractions to obtain

$$\begin{aligned}\frac{x+4}{x^2(x^2+4)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} \\ x+4 &= Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2\end{aligned}$$

When $x = 0$ we obtain $B = 1$. We expand and obtain

$$\begin{aligned}x+4 &= Ax(x^2+4) + 1(x^2+4) + (Cx+D)x^2 \\ x+4 &= (A+C)x^3 + (D+1)x^2 + 4Ax + 4\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}0 &= A+C \\ 0 &= D+1 \\ 1 &= 4A\end{aligned}$$

We obtain $A = 1/4$, $C = -1/4$, and $D = -1$. So

$$\begin{aligned}\int \frac{x+4}{x^2(x^2+4)} dx &= \int \left(\frac{1/4}{x} + \frac{1}{x^2} + \frac{(-1/4)x-1}{x^2+4} \right) dx \\ &= \int \frac{1/4}{x} dx + \int \frac{1}{x^2} dx - \frac{1}{4} \int \frac{x}{x^2+4} dx - \int \frac{dx}{x^2+4} \\ &= \boxed{\frac{1}{4} \ln |x| - \frac{1}{x} - \frac{1}{8} \ln (x^2+4) - \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C}.\end{aligned}$$

31. We use partial fractions to obtain

$$\begin{aligned}\frac{7x+3}{x^3-2x^2-3x} &= \frac{7x+3}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3} \\ 7x+3 &= A(x+1)(x-3) + Bx(x-3) + Cx(x+1)\end{aligned}$$

When $x = 3$ we obtain $C = 2$, when $x = -1$, we get $B = -1$, and when $x = 0$, we have $A = -1$. So

$$\begin{aligned}\int \frac{7x+3}{x^3-2x^2-3x} dx &= \int \left(\frac{-1}{x} + \frac{-1}{x+1} + \frac{2}{x-3} \right) dx \\ &= \int \frac{-1}{x} dx + \int \frac{-1}{x+1} dx + \int \frac{2}{x-3} dx \\ &= \boxed{-\ln |x| - \ln |x+1| + 2 \ln |x-3| + C}.\end{aligned}$$

32. We use partial fractions to obtain

$$\begin{aligned}\frac{x^5+1}{x^6-x^4} &= \frac{(x+1)(x^4-x^3+x^2-x+1)}{x^4(x-1)(x+1)} \\ \frac{x^4-x^3+x^2-x+1}{x^4(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x-1} \\ x^4-x^3+x^2-x+1 &= Ax^3(x-1) + Bx^2(x-1) + Cx(x-1) + D(x-1) + Ex^4\end{aligned}$$

When $x = 1$ we obtain $E = 1$, and when $x = 0$, we have $D = -1$. So

$$\begin{aligned}x^4-x^3+x^2-x+1 &= Ax^3(x-1) + Bx^2(x-1) + Cx(x-1) + (-1)(x-1) + (1)x^4 \\ x^4-x^3+x^2-x+1 &= (A+1)x^4 + (B-A)x^3 + (C-B)x^2 + (-C-1)x + 1.\end{aligned}$$

Equating coefficients, we obtain $A = 0$, $B = -1$, and $C = 0$. We now integrate

$$\begin{aligned}\int \frac{x^5+1}{x^6-x^4} dx &= \int \left(\frac{-1}{x^2} + \frac{-1}{x^4} + \frac{1}{x-1} \right) dx \\ &= \int \frac{-1}{x^2} dx + \int \frac{-1}{x^4} dx + \int \frac{1}{x-1} dx \\ &= \boxed{\frac{1}{x} + \frac{1}{3x^3} + \ln |x-1| + C}.\end{aligned}$$

33. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2}{(x-2)(x-1)^2} &= \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ x^2 &= A(x-1)^2 + B(x-2)(x-1) + C(x-2).\end{aligned}$$

When $x = 1$ we obtain $C = -1$, and when $x = 2$, we have $A = 4$. So

$$\begin{aligned}x^2 &= 4(x-1)^2 + B(x-2)(x-1) + (-1)(x-2) \\x^2 &= (B+4)x^2 + (-3B-9)x + (2B+6).\end{aligned}$$

Equating coefficients, we obtain $B = -3$. We now obtain

$$\begin{aligned}\int \frac{x^2}{(x-2)(x-1)^2} dx &= \int \left(\frac{4}{x-2} + \frac{-3}{x-1} + \frac{-1}{(x-1)^2} \right) dx \\&= \int \frac{4}{x-2} dx + \int \frac{-3}{x-1} dx + \int \frac{-1}{(x-1)^2} dx \\&= \boxed{4 \ln |x-2| - 3 \ln |x-1| + \frac{1}{x-1} + C}.\end{aligned}$$

34. We use partial fractions to obtain

$$\begin{aligned}\frac{x^2+1}{(x+3)(x-1)^2} &= \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\x^2+1 &= A(x-1)^2 + B(x+3)(x-1) + C(x+3).\end{aligned}$$

When $x = 1$ we obtain $C = 1/2$, and when $x = -3$, we have $A = 5/8$. So

$$\begin{aligned}x^2+1 &= (5/8)(x-1)^2 + B(x+3)(x-1) + (1/2)(x+3) \\x^2+1 &= \left(B + \frac{5}{8}\right)x^2 + \left(2B - \frac{3}{4}\right)x + \left(\frac{17}{8} - 3B\right).\end{aligned}$$

Equating coefficients, we obtain $B = 3/8$. We now obtain

$$\begin{aligned}\int \frac{x^2+1}{(x+3)(x-1)^2} dx &= \int \left(\frac{5/8}{x+3} + \frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} \right) dx \\&= \int \frac{5/8}{x+3} dx + \int \frac{3/8}{x-1} dx + \int \frac{1/2}{(x-1)^2} dx \\&= \boxed{\frac{5}{8} \ln |x+3| + \frac{3}{8} \ln |x-1| - \frac{1}{2(x-1)} + C}.\end{aligned}$$

35. We use partial fractions to obtain

$$\begin{aligned}\frac{2x+1}{x^3-1} &= \frac{2x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \\2x+1 &= A(x^2+x+1) + (Bx+C)(x-1).\end{aligned}$$

When $x = 1$ we obtain $A = 1$. So

$$\begin{aligned}2x+1 &= (1)(x^2+x+1) + (Bx+C)(x-1) \\2x+1 &= (B+1)x^2 + (C-B+1)x + (1-C).\end{aligned}$$

Equating coefficients, we obtain $B = -1$, and $C = 0$. We now obtain

$$\begin{aligned}
 \int \frac{2x+1}{x^3-1} dx &= \int \left(\frac{1}{x-1} + \frac{-x}{x^2+x+1} \right) dx \\
 &= \int \frac{1}{x-1} dx + \int \frac{-\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1} dx \\
 &= \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx + \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\
 &= \boxed{\ln|x-1| - \frac{1}{2} \ln(x^2+x+1) + \frac{\sqrt{3}}{3} \tan^{-1} \left(\frac{\sqrt{3}(2x+1)}{3} \right) + C}.
 \end{aligned}$$

36. We use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{x^3-8} &= \frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4} \\
 1 &= A(x^2+2x+4) + (Bx+C)(x-2).
 \end{aligned}$$

When $x = 2$ we obtain $A = \frac{1}{12}$. So

$$\begin{aligned}
 1 &= \left(\frac{1}{12} \right) (x^2+2x+4) + (Bx+C)(x-2) \\
 1 &= \left(B + \frac{1}{12} \right) x^2 + \left(C - 2B + \frac{1}{6} \right) x + \left(\frac{1}{3} - 2C \right).
 \end{aligned}$$

Equating coefficients, we obtain $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$. We now obtain

$$\begin{aligned}
 \int \frac{dx}{x^3-8} &= \int \left(\frac{\frac{1}{12}}{x-2} + \frac{-\frac{1}{12}x - \frac{1}{3}}{x^2+2x+4} \right) dx \\
 &= \int \frac{\frac{1}{12}}{x-2} dx + \int \frac{\frac{-1}{24}(2x+2) - \frac{1}{4}}{x^2+2x+4} dx \\
 &= \int \frac{\frac{1}{12}}{x-2} dx - \frac{1}{24} \int \frac{2x+2}{x^2+2x+4} dx - \frac{1}{4} \int \frac{1}{(x+1)^2 + (\sqrt{3})^2} dx \\
 &= \boxed{\frac{1}{12} \ln|x-2| - \frac{1}{24} \ln(x^2+2x+4) - \frac{\sqrt{3}}{12} \tan^{-1} \left(\frac{\sqrt{3}(x+1)}{3} \right) + C}.
 \end{aligned}$$

37. We use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{x^2-9} &= \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \\
 1 &= A(x+3) + B(x-3).
 \end{aligned}$$

When $x = -3$ we obtain $B = -\frac{1}{6}$, and when $x = 3$ we have $A = \frac{1}{6}$. So we obtain

$$\begin{aligned}
 \int_0^1 \frac{dx}{x^2-9} &= \int_0^1 \left(\frac{1/6}{x-3} + \frac{-1/6}{x+3} \right) dx \\
 &= \left[\frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| \right]_0^1 \\
 &= \frac{1}{6} \ln|1-3| - \frac{1}{6} \ln|1+3| - \left(\frac{1}{6} \ln|0-3| - \frac{1}{6} \ln|0+3| \right) \\
 &= \boxed{-\frac{1}{6} \ln 2}.
 \end{aligned}$$

38. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x^2 - 25} &= \frac{1}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5} \\ 1 &= A(x+5) + B(x-5).\end{aligned}$$

When $x = -5$ we obtain $B = \frac{-1}{10}$, and when $x = 5$ we have $A = \frac{1}{10}$. So we obtain

$$\begin{aligned}\int_2^4 \frac{dx}{x^2 - 25} &= \int_2^4 \left(\frac{\frac{1}{10}}{x-5} + \frac{\frac{-1}{10}}{x+5} \right) dx \\ &= \left[\frac{1}{10} \ln |x-5| - \frac{1}{10} \ln |x+5| \right]_2^4 \\ &= \frac{1}{10} \ln |4-5| - \frac{1}{10} \ln |4+5| - \left(\frac{1}{10} \ln |2-5| - \frac{1}{10} \ln |2+5| \right) \\ &= \boxed{\frac{1}{10} \ln 7 - \frac{3}{10} \ln 3}.\end{aligned}$$

39. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{16 - x^2} &= \frac{-1}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4} \\ -1 &= A(x+4) + B(x-4).\end{aligned}$$

When $x = -4$ we obtain $B = \frac{1}{8}$, and when $x = 4$ we have $A = \frac{-1}{8}$. So we obtain

$$\begin{aligned}\int_{-2}^3 \frac{dx}{16 - x^2} &= \int_{-2}^3 \left(\frac{-1/8}{x-4} + \frac{1/8}{x+4} \right) dx \\ &= \left[\frac{-1}{8} \ln |x-4| + \frac{1}{8} \ln |x+4| \right]_{-2}^3 \\ &= \frac{-1}{8} \ln |3-4| + \frac{1}{8} \ln |3+4| - \left(\frac{-1}{8} \ln |-2-4| + \frac{1}{8} \ln |-2+4| \right) \\ &= \frac{1}{8} \ln 7 - \frac{1}{8} \ln 2 + \frac{1}{8} \ln 6 = \boxed{\frac{1}{8} \ln 21}.\end{aligned}$$

40. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{9 - x^2} &= \frac{-1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \\ -1 &= A(x+3) + B(x-3).\end{aligned}$$

When $x = -3$ we obtain $B = \frac{1}{6}$, and when $x = 3$ we have $A = \frac{-1}{6}$. So we obtain

$$\begin{aligned}\int_1^2 \frac{dx}{9 - x^2} &= \int_1^2 \left(\frac{-1/6}{x-3} + \frac{1/6}{x+3} \right) dx \\ &= \left[\frac{-1}{6} \ln |x-3| + \frac{1}{6} \ln |x+3| \right]_1^2 \\ &= \frac{-1}{6} \ln |2-3| + \frac{1}{6} \ln |2+3| - \left(\frac{-1}{6} \ln |1-3| + \frac{1}{6} \ln |1+3| \right) \\ &= \boxed{\frac{1}{6} \ln 5 - \frac{1}{6} \ln 2}.\end{aligned}$$

41. Let $x = \sin \theta$, and substitute to obtain

$$\int \frac{\cos \theta}{\sin^2 \theta + \sin \theta - 6} d\theta = \int \frac{dx}{x^2 + x - 6} = \int \frac{dx}{(x-2)(x+3)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1}{(x-2)(x+3)} &= \frac{A}{x-2} + \frac{B}{x+3} \\ 1 &= A(x+3) + B(x-2) \end{aligned}$$

When $x = -3$ we obtain $B = -\frac{1}{5}$, and when $x = 2$ we have $A = \frac{1}{5}$. So we obtain

$$\begin{aligned} \int \frac{\cos \theta}{\sin^2 \theta + \sin \theta - 6} d\theta &= \int \left(\frac{1/5}{x-2} + \frac{-1/5}{x+3} \right) dx \\ &= \frac{1}{5} \ln |x-2| - \frac{1}{5} \ln |x+3| + C \\ &= \boxed{\frac{1}{5} \ln (2 - \sin \theta) - \frac{1}{5} \ln (\sin \theta + 3) + C}. \end{aligned}$$

42. Let $u = \cos x$, and substitute to obtain

$$\int \frac{\sin x}{\cos^2 x - 2 \cos x - 8} dx = \int \frac{-du}{u^2 - 2u - 8} = \int \frac{-du}{(u-4)(u+2)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{-1}{(u-4)(u+2)} &= \frac{A}{u-4} + \frac{B}{u+2} \\ -1 &= A(u+2) + B(u-4) \end{aligned}$$

When $u = -2$ we obtain $B = \frac{1}{6}$, and when $u = 4$ we have $A = -\frac{1}{6}$. So we obtain

$$\begin{aligned} \int \frac{\sin x}{\cos^2 x - 2 \cos x - 8} dx &= \int \left(\frac{-1/6}{u-4} + \frac{1/6}{u+2} \right) du \\ &= -\frac{1}{6} \ln |u-4| + \frac{1}{6} \ln |u+2| + C \\ &= \boxed{-\frac{1}{6} \ln (4 - \cos x) + \frac{1}{6} \ln (\cos x + 2) + C}. \end{aligned}$$

43. Let $x = \cos \theta$, and substitute to obtain

$$\int \frac{\sin \theta}{\cos^3 \theta + \cos \theta} d\theta = \int \frac{-dx}{x^3 + x} = \int \frac{-dx}{x(x^2 + 1)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{-1}{x(x^2 + 1)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ -1 &= A(x^2 + 1) + (Bx + C)x \end{aligned}$$

When $x = 0$ we obtain $A = -1$. So

$$\begin{aligned} -1 &= (-1)(x^2 + 1) + (Bx + C)x \\ -1 &= (B-1)x^2 + Cx - 1 \end{aligned}$$

Equating coefficients, we obtain $B = 1$ and $C = 0$. We now have

$$\begin{aligned}
 \int \frac{\sin \theta}{\cos^3 \theta + \cos \theta} d\theta &= \int \left(\frac{-1}{x} + \frac{x}{x^2 + 1} \right) dx \\
 &= \int \frac{-1}{x} dx + \int \frac{x}{x^2 + 1} dx \\
 &= -\ln |x| + \frac{1}{2} \ln (x^2 + 1) + C \\
 &= \boxed{-\ln |\cos \theta| + \frac{1}{2} \ln (\cos^2 \theta + 1) + C}.
 \end{aligned}$$

44. Let $x = \sin \theta$, and substitute to obtain

$$\int \frac{4 \cos \theta}{\sin^3 \theta + 2 \sin \theta} d\theta = \int \frac{4 dx}{x^3 + 2x} = \int \frac{4 dx}{x(x^2 + 2)}.$$

We use partial fractions to obtain

$$\begin{aligned}
 \frac{4}{x(x^2 + 2)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 2} \\
 4 &= A(x^2 + 2) + (Bx + C)x
 \end{aligned}$$

When $x = 0$ we obtain $A = 2$. So

$$\begin{aligned}
 4 &= 2(x^2 + 2) + (Bx + C)x \\
 4 &= (B + 2)x^2 + Cx + 4
 \end{aligned}$$

Equating coefficients, we obtain $B = -2$ and $C = 0$. We now have

$$\begin{aligned}
 \int \frac{4 \cos \theta}{\cos^3 \theta + \cos \theta} d\theta &= \int \left(\frac{2}{x} + \frac{-2x}{x^2 + 2} \right) dx \\
 &= 2 \int \frac{1}{x} dx - \int \frac{2x}{x^2 + 2} dx \\
 &= 2 \ln |x| - \ln (x^2 + 2) + C \\
 &= \boxed{2 \ln |\sin \theta| - \ln (\sin^2 \theta + 2) + C}.
 \end{aligned}$$

45. Let $x = e^t$, and substitute to obtain

$$\int \frac{e^t}{e^{2t} + e^t - 2} dt = \int \frac{dx}{x^2 + x - 2} = \int \frac{dx}{(x - 1)(x + 2)}.$$

We use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{(x - 1)(x + 2)} &= \frac{A}{x - 1} + \frac{B}{x + 2} \\
 1 &= A(x + 2) + B(x - 1)
 \end{aligned}$$

When $x = -2$ we obtain $B = -\frac{1}{3}$, and when $x = 1$ we have $A = \frac{1}{3}$. So we obtain

$$\begin{aligned}
 \int \frac{e^t}{e^{2t} + e^t - 2} dt &= \int \left(\frac{1/3}{x - 1} + \frac{-1/3}{x + 2} \right) dx \\
 &= \int \frac{1/3}{x - 1} dx + \int \frac{-1/3}{x + 2} dx \\
 &= \frac{1}{3} \ln |x - 1| - \frac{1}{3} \ln |x + 2| + C \\
 &= \boxed{\frac{1}{3} \ln |e^t - 1| - \frac{1}{3} \ln (e^t + 2) + C}.
 \end{aligned}$$

46. Let $y = e^x$, and substitute to obtain

$$\int \frac{e^x}{e^{2x} + e^x - 6} dx = \int \frac{dy}{y^2 + y - 6} = \int \frac{dy}{(y-2)(y+3)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1}{(y-2)(y+3)} &= \frac{A}{y-2} + \frac{B}{y+3} \\ 1 &= A(y+3) + B(y-2) \end{aligned}$$

When $y = -3$ we obtain $B = -\frac{1}{5}$, and when $y = 2$ we have $A = \frac{1}{5}$. So we obtain

$$\begin{aligned} \int \frac{e^x}{e^{2x} + e^x - 6} dx &= \int \left(\frac{1/5}{y-2} + \frac{-1/5}{y+3} \right) dy \\ &= \int \frac{1/5}{y-2} dy + \int \frac{-1/5}{y+3} dy \\ &= \frac{1}{5} \ln |y-2| - \frac{1}{5} \ln |y+3| + C \\ &= \boxed{\frac{1}{5} \ln |e^x - 2| - \frac{1}{5} \ln (e^x + 3) + C}. \end{aligned}$$

47. Let $y = e^x$, and substitute to obtain

$$\int \frac{e^x}{e^{2x} - 1} dx = \int \frac{dy}{y^2 - 1} = \int \frac{dy}{(y-1)(y+1)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1}{(y-1)(y+1)} &= \frac{A}{y-1} + \frac{B}{y+1} \\ 1 &= A(y+1) + B(y-1) \end{aligned}$$

When $y = -1$ we obtain $B = -\frac{1}{2}$, and when $y = 1$ we have $A = \frac{1}{2}$. So we obtain

$$\begin{aligned} \int \frac{e^x}{e^{2x} - 1} dx &= \int \left(\frac{1/2}{y-1} + \frac{-1/2}{y+1} \right) dy \\ &= \int \frac{1/2}{y-1} dy + \int \frac{-1/2}{y+1} dy \\ &= \frac{1}{2} \ln |y-1| - \frac{1}{2} \ln |y+1| + C \\ &= \boxed{\frac{1}{2} \ln |e^x - 1| - \frac{1}{2} \ln (e^x + 1) + C}. \end{aligned}$$

48. We rewrite the integral as

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x}{e^{2x} - 1} dx.$$

Let $y = e^x$, and substitute to obtain

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{dy}{y^2 - 1} = \int \frac{dy}{(y-1)(y+1)}.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{1}{(y-1)(y+1)} &= \frac{A}{y-1} + \frac{B}{y+1} \\ 1 &= A(y+1) + B(y-1)\end{aligned}$$

When $y = -1$ we obtain $B = -\frac{1}{2}$, and when $y = 1$ we have $A = \frac{1}{2}$. So we obtain

$$\begin{aligned}\int \frac{dx}{e^x - e^{-x}} &= \int \left(\frac{1/2}{y-1} + \frac{-1/2}{y+1} \right) dy \\ &= \int \frac{1/2}{y-1} dy + \int \frac{-1/2}{y+1} dy \\ &= \frac{1}{2} \ln |y-1| - \frac{1}{2} \ln |y+1| + C \\ &= \boxed{\frac{1}{2} \ln |e^x - 1| - \frac{1}{2} \ln (e^x + 1) + C}.\end{aligned}$$

49. We rewrite the integral as

$$\int \frac{dt}{e^{2t} + 1} = \int \frac{e^t}{e^t(e^{2t} + 1)} dt.$$

Let $x = e^t$, and substitute to obtain

$$\int \frac{dt}{e^{2t} + 1} = \int \frac{dx}{x(x^2 + 1)}.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x(x^2 + 1)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ 1 &= A(x^2 + 1) + (Bx + C)x\end{aligned}$$

When $x = 0$ we obtain $A = 1$. So

$$\begin{aligned}1 &= (1)(x^2 + 1) + (Bx + C)x \\ 1 &= (B + 1)x^2 + Cx + 1\end{aligned}$$

Equating coefficients, we obtain $B = -1$ and $C = 0$. We now have

$$\begin{aligned}\int \frac{dt}{e^{2t} + 1} &= \int \left(\frac{1}{x} + \frac{-x}{x^2 + 1} \right) dx \\ &= \int \frac{1}{x} dx + \int \frac{-x}{x^2 + 1} dx \\ &= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + C \\ &= \ln (e^t) - \frac{1}{2} \ln (e^{2t} + 1) + C \\ &= \boxed{t - \frac{1}{2} \ln (e^{2t} + 1) + C}.\end{aligned}$$

50. We rewrite the integral as

$$\int \frac{dt}{e^{3t} + e^t} = \int \frac{e^t}{e^{2t}(e^{2t} + 1)} dt.$$

Let $x = e^t$, and substitute to obtain

$$\int \frac{dt}{e^{3t} + e^t} = \int \frac{dx}{x^2(x^2 + 1)}.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1}{x^2(x^2 + 1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} \\ 1 &= Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2 \end{aligned}$$

When $x = 0$ we obtain $B = 1$. We expand and obtain

$$\begin{aligned} 1 &= Ax(x^2 + 1) + (1)(x^2 + 1) + (Cx + D)x^2 \\ 1 &= (A + C)x^3 + (D + 1)x^2 + Ax + 1 \end{aligned}$$

Equating coefficients, we obtain $A = 0$, $C = 0$, and $D = -1$. So

$$\begin{aligned} \int \frac{dt}{e^{3t} + e^t} &= \int \left(\frac{1}{x^2} + \frac{-1}{x^2 + 1} \right) dx \\ &= \int \frac{1}{x^2} dx - \int \frac{dx}{x^2 + 1} \\ &= -\frac{1}{x} - \tan^{-1} x + C \\ &= \boxed{-e^{-t} - \tan^{-1}(e^t) + C}. \end{aligned}$$

51. Let $u = \sin x - 1$, and substitute to obtain

$$\begin{aligned} \int \frac{\sin x \cos x}{(\sin x - 1)^2} dx &= \int \frac{u + 1}{u^2} du \\ &= \int \left(\frac{1}{u} + u^{-2} \right) du \\ &= \ln |u| - \frac{1}{u} + C \\ &= \boxed{\ln |\sin x - 1| - \frac{1}{\sin x - 1} + C}. \end{aligned}$$

52. Let $u = \cos x - 2$, and substitute to obtain

$$\begin{aligned} \int \frac{\cos x \sin x}{(\cos x - 2)^2} dx &= \int \frac{-(u + 2)}{u^2} du \\ &= \int \left(-\frac{1}{u} - 2u^{-2} \right) du \\ &= -\ln |u| + \frac{2}{u} + C \\ &= \boxed{-\ln (2 - \cos x) + \frac{2}{\cos x - 2} + C}. \end{aligned}$$

53. Let $u = \sin x$, and substitute to obtain

$$\int \frac{\cos x}{(\sin^2 x + 9)^2} dx = \int \frac{du}{(u^2 + 9)^2}.$$

Let $u = 3 \tan \theta$, and substitute to obtain

$$\begin{aligned}
 \int \frac{\cos x}{(\sin^2 x + 9)^2} dx &= \int \frac{3 \sec^2 \theta d\theta}{\left((3 \tan \theta)^2 + 9\right)^2} \\
 &= \frac{1}{27} \int \cos^2 \theta d\theta \\
 &= \frac{1}{27} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{1}{54} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{1}{54} \tan^{-1} \frac{u}{3} + \frac{1}{54} \sin \theta \cos \theta + C \\
 &= \frac{1}{54} \tan^{-1} \frac{u}{3} + \frac{1}{54} \frac{\tan \theta}{\sec^2 \theta} + C.
 \end{aligned}$$

Since $\tan \theta = \frac{u}{3}$, $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{u}{3}\right)^2 + 1} = \frac{1}{3} \sqrt{u^2 + 9}$, and we have

$$\begin{aligned}
 \int \frac{\cos x}{(\sin^2 x + 9)^2} dx &= \frac{1}{54} \tan^{-1} \frac{u}{3} + \frac{1}{54} \frac{\frac{u}{3}}{\left(\frac{1}{3} \sqrt{u^2 + 9}\right)^2} + C \\
 &= \frac{1}{54} \tan^{-1} \frac{u}{3} + \frac{1}{18} \frac{u}{u^2 + 9} + C \\
 &= \boxed{\frac{1}{54} \tan^{-1} \frac{\sin x}{3} + \frac{1}{18} \frac{\sin x}{\sin^2 x + 9} + C}.
 \end{aligned}$$

54. Let $u = \cos x$, and substitute to obtain

$$\int \frac{\sin x}{(\cos^2 x + 4)^2} dx = \int \frac{-du}{(u^2 + 4)^2}.$$

Let $u = 2 \tan \theta$, and substitute to obtain

$$\begin{aligned}
 \int \frac{\sin x}{(\cos^2 x + 4)^2} dx &= \int \frac{-2 \sec^2 \theta d\theta}{\left((2 \tan \theta)^2 + 4\right)^2} \\
 &= \frac{-1}{8} \int \cos^2 \theta d\theta \\
 &= \frac{-1}{8} \int \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{-1}{16} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\
 &= \frac{-1}{16} \tan^{-1} \frac{u}{2} + \frac{-1}{16} \sin \theta \cos \theta + C \\
 &= \frac{-1}{16} \tan^{-1} \frac{u}{2} + \frac{-1}{16} \frac{\tan \theta}{\sec^2 \theta} + C.
 \end{aligned}$$

Since $\tan \theta = \frac{u}{2}$, $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{\left(\frac{u}{2}\right)^2 + 1} = \frac{1}{2}\sqrt{u^2 + 4}$, and we have

$$\begin{aligned} \int \frac{\sin x}{(\cos^2 x + 4)^2} dx &= \frac{-1}{16} \tan^{-1} \frac{u}{2} + \frac{-1}{16} \frac{\frac{u}{2}}{\left(\frac{1}{2}\sqrt{u^2 + 4}\right)^2} + C \\ &= \frac{-1}{16} \tan^{-1} \frac{u}{2} - \frac{1}{8} \frac{u}{u^2 + 4} + C \\ &= \boxed{\frac{-1}{16} \tan^{-1} \frac{\cos x}{2} - \frac{1}{8} \frac{\cos x}{\cos^2 x + 4} + C}. \end{aligned}$$

55. The area under the graph is given by

$$A = \int_3^5 \frac{4}{x^2 - 4} dx = \int_3^5 \frac{4}{(x+2)(x-2)} dx.$$

We use partial fractions to obtain

$$\begin{aligned} \frac{4}{(x+2)(x-2)} &= \frac{A}{x+2} + \frac{B}{x-2} \\ 4 &= A(x-2) + B(x+2) \end{aligned}$$

When $x = 2$ we obtain $B = 1$, and when $x = -2$ we have $A = -1$. So we obtain

$$\begin{aligned} A &= \int_3^5 \left(\frac{-1}{x+2} + \frac{1}{x-2} \right) dx \\ &= [-\ln(x+2) + \ln(x-2)]_3^5 \\ &= -\ln(5+2) + \ln(5-2) - (-\ln(3+2) + \ln(3-2)) \\ &= \ln 3 + \ln 5 - \ln 7 = \boxed{\ln \frac{15}{7}}. \end{aligned}$$

56. The area under the graph is given by

$$A = \int_4^6 \frac{x-4}{(x+3)^2} dx.$$

Let $u = x + 3$ to obtain

$$\begin{aligned} A &= \int_7^9 \frac{(u-3)-4}{u^2} du \\ &= \int_7^9 \left(\frac{1}{u} - \frac{7}{u^2} \right) du \\ &= \left[\ln u + \frac{7}{u} \right]_7^9 \\ &= \ln 9 + \frac{7}{9} - \left(\ln 7 + \frac{7}{7} \right) \\ &= \boxed{\ln 9 - \ln 7 - \frac{2}{9}}. \end{aligned}$$

57. The area under the graph is given by

$$A = \int_0^2 \frac{8}{x^3 + 1} dx = \int_0^2 \frac{8}{(x+1)(x^2 - x + 1)} dx.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{8}{(x+1)(x^2-x+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\ 8 &= A(x^2-x+1) + (Bx+C)(x+1).\end{aligned}$$

When $x = -1$ we obtain $A = 8/3$. We expand and obtain

$$\begin{aligned}8 &= (8/3)(x^2-x+1) + (Bx+C)(x+1) \\ 8 &= \left(B + \frac{8}{3}\right)x^2 + \left(B + C - \frac{8}{3}\right)x + \left(C + \frac{8}{3}\right).\end{aligned}$$

Equating coefficients, we have $B = -8/3$ and $C = 16/3$. We now have

$$\begin{aligned}A &= \int_0^2 \left(\frac{8/3}{x+1} + \frac{(-8/3)x + 16/3}{x^2-x+1} \right) dx \\ &= \left[\frac{8}{3} \ln(x+1) \right]_0^2 + \int_0^2 \frac{(-4/3)(2x-1)x + 4}{x^2-x+1} dx \\ &= \frac{8}{3} \ln(2+1) - \frac{8}{3} \ln(0+1) - \frac{4}{3} \int_0^2 \frac{2x-1}{x^2-x+1} dx + 4 \int_0^2 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{8}{3} \ln 3 - \frac{4}{3} [\ln(x^2-x+1)]_0^2 + 4 \left[\frac{2\sqrt{3}}{3} \tan^{-1} \left(\frac{\sqrt{3}(2x-1)}{3} \right) \right]_0^2 \\ &= \frac{8}{3} \ln 3 - \frac{4}{3} (\ln(2^2-2+1) - \ln(0^2-0+1)) \\ &\quad + 4 \left[\frac{2\sqrt{3}}{3} \tan^{-1} \left(\frac{\sqrt{3}(2(2)-1)}{3} \right) - \frac{2\sqrt{3}}{3} \tan^{-1} \left(\frac{\sqrt{3}(2(0)-1)}{3} \right) \right] \\ &= \boxed{\frac{4}{3} \ln 3 + \frac{4\sqrt{3}}{3} \pi}.\end{aligned}$$

58. The volume is given by

$$V = \int_3^5 \pi \left(\frac{4}{x^2-4} \right)^2 dx = \pi \int_3^5 \frac{16}{(x-2)^2(x+2)^2} dx.$$

We use partial fractions to obtain

$$\begin{aligned}\frac{16}{(x-2)^2(x+2)^2} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} \\ 16 &= A(x-2)(x+2)^2 + B(x+2)^2 + C(x-2)^2(x+2) + D(x-2)^2.\end{aligned}$$

When $x = -2$ we obtain $D = 1$, and when $x = 2$ we have $B = 1$. We expand and obtain

$$\begin{aligned}16 &= A(x-2)(x+2)^2 + (1)(x+2)^2 + C(x-2)^2(x+2) + (1)(x-2)^2 \\ 16 &= (A+C)x^3 + (2A-2C+2)x^2 + (-4A-4C)x + (8C-8A+8).\end{aligned}$$

Equating coefficients, we have $A + C = 0$ and $2A - 2C + 2 = 0$. So $C = -A$, and

$2A - 2(-A) + 2 = 4A + 2 = 0$. We obtain $A = -1/2$ and $C = 1/2$. We now have

$$\begin{aligned}
 V &= \pi \int_3^5 \left(\frac{-1/2}{x-2} + \frac{1}{(x-2)^2} + \frac{1/2}{x+2} + \frac{1}{(x+2)^2} \right) dx \\
 &= \pi \left(\int_3^5 \frac{-1/2}{x-2} dx + \int_3^5 \frac{1}{(x-2)^2} dx + \int_3^5 \frac{1/2}{x+2} dx + \int_3^5 \frac{1}{(x+2)^2} dx \right) \\
 &= \pi \left(\left[-\frac{1}{2} \ln(x-2) \right]_3^5 + \left[-\frac{1}{x-2} \right]_3^5 + \left[\frac{1}{2} \ln(x+2) \right]_3^5 + \left[-\frac{1}{x+2} \right]_3^5 \right) \\
 &= \pi \left[-\frac{1}{2} \ln(5-2) - \left(-\frac{1}{2} \ln(3-2) \right) + \left(-\frac{1}{5-2} \right) - \left(-\frac{1}{3-2} \right) \right. \\
 &\quad \left. + \frac{1}{2} \ln(5+2) - \frac{1}{2} \ln(3+2) + \left(-\frac{1}{5+2} \right) - \left(-\frac{1}{3+2} \right) \right] \\
 &= \boxed{\pi \left(\frac{1}{2} \ln 7 - \frac{1}{2} \ln 5 - \frac{1}{2} \ln 3 + \frac{76}{105} \right)}.
 \end{aligned}$$

59. (a) We test possible rational roots, and obtain that the zeros of q are $\boxed{-4, -2, \text{ and } 3}$.

(b) From part (a), we obtain $q(x) = (x+4)(x+2)(x-3)$.

(c) We use partial fractions to obtain

$$\begin{aligned}
 \frac{3x-7}{x^3+3x^2-10x-24} &= \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+4} \\
 3x-7 &= A(x-3)(x+4) + B(x+2)(x+4) + C(x+2)(x-3).
 \end{aligned}$$

When $x = -4$ we obtain $C = -19/14$. When $x = 3$ we get $B = \frac{2}{35}$. And when $x = -2$, we have $A = \frac{13}{10}$. We now obtain

$$\begin{aligned}
 \int \frac{3x-7}{x^3+3x^2-10x-24} dx &= \int \left(\frac{13/10}{x+2} + \frac{2/35}{x-3} + \frac{-19/14}{x+4} \right) dx \\
 &= \int \frac{13/10}{x+2} dx + \int \frac{2/35}{x-3} dx + \int \frac{-19/14}{x+4} dx \\
 &= \boxed{\frac{13}{10} \ln|x+2| + \frac{2}{35} \ln|x-3| - \frac{19}{14} \ln|x+4| + C}.
 \end{aligned}$$

Challenge Problems

60. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, so $dx = 2u du$. We obtain

$$\begin{aligned}
 \int \frac{x dx}{3 + \sqrt{x}} &= \int \frac{u^2}{3 + u} (2u) du \\
 &= 2 \int \frac{u^3}{u+3} du.
 \end{aligned}$$

Let $y = u + 3$, then we have

$$\begin{aligned}
 \int \frac{x \, dx}{3 + \sqrt{x}} &= 2 \int \frac{(y-3)^3}{y} \, dy \\
 &= 2 \int \frac{y^3 - 9y^2 + 27y - 27}{y} \, dy \\
 &= 2 \int \left(y^2 - 9y + 27 - \frac{27}{y} \right) \, dy \\
 &= 2 \left(\frac{1}{3} y^3 - \frac{9}{2} y^2 + 27y - 27 \ln |y| \right) + C \\
 &= \frac{2}{3} (u+3)^3 - 9(u+3)^2 + 54(u+3) - 54 \ln |u+3| + C \\
 &= \boxed{\frac{2}{3} (\sqrt{x}+3)^3 - 9(\sqrt{x}+3)^2 + 54(\sqrt{x}+3) - 54 \ln (\sqrt{x}+3) + C}.
 \end{aligned}$$

61. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, so $dx = 2u \, du$. We obtain

$$\int \frac{dx}{\sqrt{x}+2} = \int \frac{2u}{u+2} \, du.$$

Let $y = u + 2$, then we have

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x}+2} &= 2 \int \frac{y-2}{y} \, du \\
 &= 2 \int \left(1 - \frac{2}{y} \right) \, dy \\
 &= 2(y - 2 \ln |y|) + C \\
 &= 2(u+2) - 4 \ln |u+2| + C \\
 &= \boxed{2\sqrt{x} - 4 \ln (\sqrt{x}+2) + C}.
 \end{aligned}$$

62. Let $u = \sqrt[3]{x}$, $du = \frac{1}{3x^{2/3}} dx$, so $dx = 3u^2 \, du$. We obtain

$$\begin{aligned}
 \int \frac{dx}{x - \sqrt[3]{x}} &= \int \frac{3u^2}{u^3 - u} \, du \\
 &= \int \frac{3u}{u^2 - 1} \, du \\
 &= \int \frac{3u}{(u-1)(u+1)} \, du.
 \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
 \frac{3u}{(u-1)(u+1)} &= \frac{A}{u-1} + \frac{B}{u+1} \\
 3u &= A(u+1) + B(u-1).
 \end{aligned}$$

When $u = -1$ we obtain $B = 3/2$, and when $u = 1$ we have $A = 3/2$. We now obtain

$$\begin{aligned}
 \int \frac{dx}{x - \sqrt[3]{x}} &= \int \left(\frac{3/2}{u-1} + \frac{3/2}{u+1} \right) \, du \\
 &= \frac{3}{2} \ln |u-1| + \frac{3}{2} \ln |u+1| + C \\
 &= \boxed{\frac{3}{2} \ln |\sqrt[3]{x}-1| + \frac{3}{2} \ln |\sqrt[3]{x}+1| + C}.
 \end{aligned}$$

63. Let $u = \sqrt[3]{x}$, $du = \frac{1}{3x^{2/3}} dx$, so $dx = 3u^2 du$. We obtain

$$\begin{aligned}\int \frac{x dx}{\sqrt[3]{x} - 1} &= \int \frac{u^3}{u - 1} (3u^2) du \\ &= 3 \int \frac{u^5}{u - 1} du.\end{aligned}$$

Let $y = u - 1$, then we have

$$\begin{aligned}\int \frac{x dx}{\sqrt[3]{x} - 1} &= 3 \int \frac{(y+1)^5}{y} dy \\ &= 3 \int \frac{y^5 + 5y^4 + 10y^3 + 10y^2 + 5y + 1}{y} dy \\ &= 3 \int \left(y^4 + 5y^3 + 10y^2 + 10y + 5 + \frac{1}{y} \right) dy \\ &= 3 \left(\frac{1}{5}y^5 + \frac{5}{4}y^4 + \frac{10}{3}y^3 + 5y^2 + 5y + \ln|y| \right) + C \\ &= \frac{3}{5}(u-1)^5 + \frac{15}{4}(u-1)^4 + 10(u-1)^3 + 15(u-1)^2 + 15(u-1) + 3\ln|u-1| + C \\ &= \frac{3}{5}u^5 + \frac{3}{4}u^4 + u^3 + \frac{3}{2}u^2 + 3u + 3\ln|u-1| + C \\ &= \boxed{\frac{3}{5}x^{5/3} + \frac{3}{4}x^{4/3} + x + \frac{3}{2}x^{2/3} + 3x^{1/3} + 3\ln|x^{1/3} - 1| + C}.\end{aligned}$$

64. Let $u = x^{1/6}$, $du = \frac{1}{6x^{5/6}} dx$, so $dx = 6u^5 du$. We obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= \int \frac{6u^5}{u^3 + u^2} du \\ &= \int \frac{6u^3}{u + 1} du.\end{aligned}$$

Let $y = u + 1$, and substitute to obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= \int \frac{6(y-1)^3}{y} dy \\ &= \int \frac{6y^3 - 18y^2 + 18y - 6}{y} dy \\ &= \int \left(6y^2 - 18y + 18 - \frac{6}{y} \right) dy \\ &= 2y^3 - 9y^2 + 18y - 6\ln|y| + C \\ &= 2(u+1)^3 - 9(u+1)^2 + 18(u+1) - 6\ln|u+1| + C \\ &= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + C \\ &= \boxed{2\sqrt{x} - 3\sqrt[3]{x} + 6x^{1/6} - 6\ln|x^{1/6} + 1| + C}.\end{aligned}$$

65. Let $u = x^{1/6}$, $du = \frac{1}{6x^{5/6}} dx$, so $dx = 6u^5 du$. We obtain

$$\begin{aligned}\int \frac{dx}{3\sqrt{x} - \sqrt[3]{x}} &= \int \frac{6u^5}{3u^3 - u^2} du \\ &= \int \frac{6u^3}{3u - 1} du.\end{aligned}$$

Let $y = 3u - 1$, and substitute to obtain

$$\begin{aligned}
 \int \frac{dx}{3\sqrt{x} - \sqrt[3]{x}} &= \int \frac{6\left(\frac{y+1}{3}\right)^3 \frac{1}{3} dy}{y} \\
 &= \int \frac{\frac{2}{9}y^3 + \frac{2}{3}y^2 + \frac{2}{3}y + \frac{2}{9}}{3y} dy \\
 &= \int \left(\frac{2}{27}y^2 + \frac{2}{9}y + \frac{2}{9} + \frac{2}{27y} \right) dy \\
 &= \frac{2}{81}y^3 + \frac{1}{9}y^2 + \frac{2}{9}y + \frac{2}{27} \ln|y| + C \\
 &= \frac{2}{81}(3u-1)^3 + \frac{1}{9}(3u-1)^2 + \frac{2}{9}(3u-1) + \frac{2}{27} \ln|3u-1| + C \\
 &= \frac{2}{3}u^3 + \frac{1}{3}u^2 + \frac{2}{9}u + \frac{2}{27} \ln|3u-1| + C \\
 &= \boxed{\frac{2}{3}x^{1/2} + \frac{1}{3}x^{1/3} + \frac{2}{9}x^{1/6} + \frac{2}{27} \ln|3x^{1/6} - 1| + C}.
 \end{aligned}$$

66. Let $u = 2 + 3x$, and substitute to obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt[3]{2+3x}} &= \int \frac{\frac{1}{3} du}{\sqrt[3]{u}} \\
 &= \frac{1}{3} \int u^{-1/3} du \\
 &= \frac{1}{3} \left(\frac{3}{2} u^{2/3} \right) + C \\
 &= \boxed{\frac{1}{2}(2+3x)^{2/3} + C}.
 \end{aligned}$$

67. Let $u = 1 + 2x$, and substitute to obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt[4]{1+2x}} &= \int \frac{\frac{1}{2} du}{\sqrt[4]{u}} \\
 &= \frac{1}{2} \int u^{-1/4} du \\
 &= \frac{1}{2} \left(\frac{4}{3} u^{3/4} \right) + C \\
 &= \boxed{\frac{2}{3}(1+2x)^{3/4} + C}.
 \end{aligned}$$

68. Let $u = 1 + x$, and substitute to obtain

$$\begin{aligned}
 \int \frac{x dx}{(1+x)^{3/4}} &= \int \frac{(u-1) du}{u^{3/4}} \\
 &= \int \left(u^{1/4} - u^{-3/4} \right) du \\
 &= \frac{4}{5} u^{5/4} - 4u^{1/4} + C \\
 &= \boxed{\frac{4}{5}(1+x)^{5/4} - 4(1+x)^{1/4} + C}.
 \end{aligned}$$

69. Let $u = 1 + x$, and substitute to obtain

$$\begin{aligned}\int \frac{dx}{(1+x)^{2/3}} &= \int \frac{du}{u^{2/3}} \\ &= \int u^{-2/3} du \\ &= 3u^{1/3} + C \\ &= \boxed{3(1+x)^{1/3} + C}.\end{aligned}$$

70. Let $u = \sqrt[3]{x}$, $du = \frac{1}{3x^{2/3}} dx$, so $dx = 3u^2 du$. We obtain

$$\int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1} dx = \int \frac{u + 1}{u - 1} 3u^2 du.$$

Let $y = u - 1$, then we have

$$\begin{aligned}\int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1} dx &= 3 \int \frac{(y+1)+1}{y} (y+1)^2 dy \\ &= 3 \int \frac{(y+2)(y+1)^2}{y} dy \\ &= 3 \int \frac{y^3 + 4y^2 + 5y + 2}{y} dy \\ &= 3 \int \left(y^2 + 4y + 5 + \frac{2}{y} \right) dy \\ &= 3 \left(\frac{1}{3} y^3 + 4 \left(\frac{1}{2} \right) y^2 + 5y + 2 \ln |y| \right) + C \\ &= 3 \left(\frac{1}{3} (u-1)^3 + 4 \left(\frac{1}{2} \right) (u-1)^2 + 5(u-1) + 2 \ln |u-1| \right) + C \\ &= u^3 + 3u^2 + 6u + 6 \ln |u-1| + C \\ &= \boxed{x + 3x^{2/3} + 6\sqrt[3]{x} + 6 \ln |\sqrt[3]{x} - 1| + C}.\end{aligned}$$

71. Let $u = x^{1/6}$, $du = \frac{1}{6x^{5/6}} dx$, so $dx = 6u^5 du$. We obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{x}(1+\sqrt[3]{x})^2} &= \int \frac{6u^5}{u^3(1+u^2)^2} du \\ &= 6 \int \frac{(1+u^2) - 1}{(1+u^2)^2} du \\ &= 6 \int \left(\frac{1}{1+u^2} - \frac{1}{(1+u^2)^2} \right) du \\ &= 6 \tan^{-1} u - 6 \int \frac{1}{(1+u^2)^2} du.\end{aligned}$$

In the last integral, let $u = \tan \theta$, and substitute to obtain

$$\begin{aligned}
 \int \frac{1}{(1+u^2)^2} du &= \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\
 &= \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C \\
 &= \frac{1}{2} \tan^{-1} u + \frac{1}{2} \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} + C \\
 &= \frac{1}{2} \tan^{-1} u + \frac{u}{2(u^2+1)} + C.
 \end{aligned}$$

We now obtain

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x}(1+\sqrt[3]{x})^2} &= 6 \tan^{-1} u - 6 \left(\frac{1}{2} \tan^{-1} u + \frac{u}{2(u^2+1)} \right) + C \\
 &= 3 \tan^{-1} u - \frac{3u}{u^2+1} + C \\
 &= \boxed{3 \tan^{-1} (x^{1/6}) - 3 \frac{x^{1/6}}{x^{1/3}+1} + C}.
 \end{aligned}$$

72. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
 \int \frac{dx}{1-\sin x} &= \int \frac{1}{1-\frac{2z}{1+z^2}} \frac{2dz}{1+z^2} \\
 &= 2 \int \frac{dz}{1-2z+z^2} \\
 &= 2 \int (1-z)^{-2} dz \\
 &= 2(1-z)^{-1} + C \\
 &= \boxed{\frac{2}{1-\tan \frac{x}{2}} + C}.
 \end{aligned}$$

73. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
 \int \frac{dx}{1+\sin x} &= \int \frac{1}{1+\frac{2z}{1+z^2}} \frac{2dz}{1+z^2} \\
 &= 2 \int \frac{dz}{1+2z+z^2} \\
 &= 2 \int (1+z)^{-2} dz \\
 &= -2(1+z)^{-1} + C \\
 &= \boxed{-\frac{2}{1+\tan \frac{x}{2}} + C}.
 \end{aligned}$$

74. With the substitution $z = \tan \frac{x}{2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{dx}{1 - \cos x} &= \int \frac{1}{1 - \frac{1-z^2}{1+z^2}} \frac{2dz}{1+z^2} \\ &= \int \frac{dz}{z^2} \\ &= -\frac{1}{z} + C \\ &= -\frac{1}{\tan \frac{x}{2}} + C \\ &= \boxed{-\cot \frac{x}{2} + C}. \end{aligned}$$

75. With the substitution $z = \tan \frac{x}{2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{dx}{3 + 2 \cos x} &= \int \frac{1}{3 + 2 \frac{1-z^2}{1+z^2}} \frac{2dz}{1+z^2} \\ &= 2 \int \frac{dz}{z^2 + (\sqrt{5})^2} \\ &= \frac{2\sqrt{5}}{5} \tan^{-1} \left(\frac{\sqrt{5}}{5} z \right) + C \\ &= \boxed{\frac{2\sqrt{5}}{5} \tan^{-1} \left(\frac{\sqrt{5}}{5} \tan \frac{x}{2} \right) + C}. \end{aligned}$$

76. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{dx}{\sin x + \cos x} &= \int \frac{1}{\frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} \frac{2dz}{1+z^2} \\ &= \int \frac{2dz}{-z^2 + 2z + 1} \\ &= \int \frac{-2dz}{z^2 - 2z - 1} \\ &= \int \frac{-2dz}{(z - (1 + \sqrt{2}))(z - (1 - \sqrt{2}))} \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned} \frac{-2}{(z - (1 + \sqrt{2}))(z - (1 - \sqrt{2}))} &= \frac{A}{z - (1 + \sqrt{2})} + \frac{B}{z - (1 - \sqrt{2})} \\ -2 &= A(z - (1 - \sqrt{2})) + B(z - (1 + \sqrt{2})) \end{aligned}$$

When $z = 1 - \sqrt{2}$ we obtain $B = \sqrt{2}/2$, and when $z = 1 + \sqrt{2}$ we have $A = -\sqrt{2}/2$. So we obtain

$$\begin{aligned} \int \frac{dx}{\sin x + \cos x} &= \int \left(\frac{-\sqrt{2}/2}{z - (1 + \sqrt{2})} + \frac{\sqrt{2}/2}{z - (1 - \sqrt{2})} \right) dx \\ &= -\frac{\sqrt{2}}{2} \ln |z - 1 - \sqrt{2}| + \frac{\sqrt{2}}{2} \ln |z - 1 + \sqrt{2}| + C \\ &= \boxed{-\frac{\sqrt{2}}{2} \ln |\tan \frac{x}{2} - 1 - \sqrt{2}| + \frac{\sqrt{2}}{2} \ln |\tan \frac{x}{2} - 1 + \sqrt{2}| + C}. \end{aligned}$$

77. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{dx}{1 - \sin x + \cos x} &= \int \frac{1}{1 - \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} \frac{2dz}{1+z^2} \\ &= \int \frac{dz}{1-z} \\ &= -\ln|1-z| \\ &= \boxed{-\ln\left|1 - \tan \frac{x}{2}\right| + C} \end{aligned}$$

78. Let $u = 3 + \cos x$, substitute to obtain

$$\begin{aligned} \int \frac{\sin x dx}{3 + \cos x} &= \int \frac{-du}{u} \\ &= -\ln|u| + C \\ &= -\ln(3 + \cos x) + C. \end{aligned}$$

79. With the substitution $z = \tan \frac{x}{2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{dx}{\tan x - 1} &= \int \frac{\frac{2}{1+z^2}}{\frac{2z}{1-z^2} - 1} dz \\ &= \int \frac{2(1-z^2)}{(1+z^2)(2z - (1-z^2))} dz \\ &= \int \frac{2(1-z^2)}{(z^2+1)(z^2+2z-1)} dz. \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned} \frac{2(1-z^2)}{(z^2+1)(z^2+2z-1)} &= \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+2z-1} \\ -2z^2+2 &= (Az+B)(z^2+2z-1) + (Cz+D)(z^2+1) \\ -2z^2+2 &= (A+C)z^3 + (2A+B+D)z^2 + (2B-A+C)z + (D-B). \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} A+C &= 0 \\ 2A+B+D &= -2 \\ 2B-A+C &= 0 \\ D-B &= 2 \end{aligned}$$

We solve this system and obtain $A = -1$, $B = -1$, $C = 1$, and $D = 1$. So

$$\begin{aligned} \int \frac{dx}{\tan x - 1} &= \int \left(\frac{z+1}{z^2+2z-1} - \frac{z+1}{z^2+1} \right) dz \\ &= \frac{1}{2} \int \frac{2(z+1)}{z^2+2z-1} - \frac{1}{2} \int \frac{2z}{z^2+1} dz - \int \frac{1}{z^2+1} dz \\ &= \frac{1}{2} \ln|z^2+2z-1| - \frac{1}{2} \ln(z^2+1) - \tan^{-1} z + C \\ &= \boxed{\frac{1}{2} \ln\left|\tan^2\left(\frac{x}{2}\right) + 2\tan\frac{x}{2} - 1\right| - \frac{1}{2} \ln\left(\tan^2\left(\frac{x}{2}\right) + 1\right) - \frac{x}{2} + C} \end{aligned}$$

80. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
 \int \frac{dx}{\tan x - \sin x} &= \int \frac{\frac{2}{1+z^2}}{\frac{2z}{1-z^2} - \frac{2z}{1+z^2}} dz \\
 &= \int \frac{2(1-z^2)}{2z((1+z^2) - (1-z^2))} dz \\
 &= \int \frac{2(1-z^2)}{4z^3} dz \\
 &= \frac{1}{2} \int \left(z^{-3} - \frac{1}{z} \right) dz \\
 &= \frac{1}{2} \left(-\frac{1}{2} z^{-2} - \ln |z| \right) + C \\
 &= \boxed{-\frac{1}{4} \cot^2 \left(\frac{x}{2} \right) - \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + C}.
 \end{aligned}$$

81. With the substitution $z = \tan \frac{x}{2}$, $\sec x = \frac{1+z^2}{1-z^2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
 \int \frac{\sec x dx}{\tan x - 2} &= \int \frac{\frac{1+z^2}{1-z^2} \frac{2}{1+z^2}}{\frac{2z}{1-z^2} - 2} dz \\
 &= \int \frac{1}{z - (1-z^2)} dz \\
 &= \int \frac{1}{z^2 + z - 1} dz \\
 &= \int \frac{1}{\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)} dz.
 \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right)} &= \frac{A}{z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} + \frac{B}{z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} \\
 1 &= A\left(z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right) + B\left(z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)\right).
 \end{aligned}$$

When $z = -\frac{1}{2}\sqrt{5} - \frac{1}{2}$ we obtain $B = -\sqrt{5}/5$, and when $z = \frac{1}{2}\sqrt{5} - \frac{1}{2}$ we have $A = \sqrt{5}/5$. So we obtain

$$\begin{aligned}
 \int \frac{\sec x dx}{\tan x - 2} &= \int \left(\frac{\sqrt{5}/5}{z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} + \frac{-\sqrt{5}/5}{z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)} \right) dz \\
 &= \frac{\sqrt{5}}{5} \ln \left| z - \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \right| - \frac{\sqrt{5}}{5} \ln \left| z - \left(-\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \right| + C \\
 &= \boxed{\frac{\sqrt{5}}{5} \ln \left| \tan \frac{x}{2} - \frac{1}{2}\sqrt{5} + \frac{1}{2} \right| - \frac{\sqrt{5}}{5} \ln \left| \tan \frac{x}{2} + \frac{1}{2}\sqrt{5} + \frac{1}{2} \right| + C}.
 \end{aligned}$$

82. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\cot x = \frac{1-z^2}{2z}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{\cot x \, dx}{1 + \sin x} &= \int \frac{\frac{1-z^2}{2z}}{1 + \frac{2z}{1+z^2}} \left(\frac{2}{1+z^2} \right) dz \\ &= \int \frac{\frac{1-z^2}{z}}{1 + z^2 + 2z} dz \\ &= \int \frac{(1-z)(1+z)}{z(1+z)^2} dz \\ &= \int \frac{1-z}{z(1+z)} dz. \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned} \frac{1-z}{z(1+z)} &= \frac{A}{z} + \frac{B}{1+z} \\ 1-z &= A(1+z) + Bz. \end{aligned}$$

When $z = -1$, we have $B = -2$, and when $z = 0$, $A = 1$. We now obtain

$$\begin{aligned} \int \frac{\cot x \, dx}{1 + \sin x} &= \int \left(\frac{1}{z} + \frac{-2}{1+z} \right) dz \\ &= \ln |z| - 2 \ln |1+z| + C \\ &= \boxed{\ln \left| \tan \frac{x}{2} \right| - 2 \ln \left| 1 + \tan \frac{x}{2} \right| + C}. \end{aligned}$$

83. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, $\sec x = \frac{1+z^2}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int \frac{\sec x \, dx}{1 + \sin x} &= \int \frac{\frac{1+z^2}{1-z^2} \frac{2}{1+z^2}}{1 + \frac{2z}{1+z^2}} dz \\ &= \int \frac{-2(1+z^2)}{(z-1)(z+1)^3} dz \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned} \frac{-2(1+z^2)}{(z-1)(z+1)^3} &= \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2} + \frac{D}{(z+1)^3} \\ -2(1+z^2) &= A(z+1)^3 + B(z-1)(z+1)^2 + C(z-1)(z+1) + D(z-1). \end{aligned}$$

When $z = -1$, we have $D = 2$, and when $z = 1$, $A = -1/2$. We now obtain

$$\begin{aligned} -2z^2 - 2 &= (-1/2)(z+1)^3 + B(z-1)(z+1)^2 + C(z-1)(z+1) + 2(z-1) \\ -2z^2 - 2 &= \left(B - \frac{1}{2} \right) z^3 + \left(B + C - \frac{3}{2} \right) z^2 + \left(\frac{1}{2} - B \right) z - \left(B + C + \frac{5}{2} \right). \end{aligned}$$

Equating coefficients, we obtain $B = 1/2$ and $C = -1$. We now have

$$\begin{aligned}
\int \frac{\sec x \, dx}{1 + \sin x} &= \int \left(\frac{-1/2}{z-1} + \frac{1/2}{z+1} + \frac{-1}{(z+1)^2} + \frac{2}{(z+1)^3} \right) dz \\
&= \int \frac{-1/2}{z-1} dz + \int \frac{1/2}{z+1} dz + \int \frac{-1}{(z+1)^2} dz + \int \frac{2}{(z+1)^3} dz \\
&= -\frac{1}{2} \ln |z-1| + \frac{1}{2} \ln |z+1| + \frac{1}{z+1} - \frac{1}{(z+1)^2} + C \\
&= \boxed{-\frac{1}{2} \ln \left| \tan \frac{x}{2} - 1 \right| + \frac{1}{2} \ln \left| \tan \frac{x}{2} + 1 \right| + \frac{1}{\tan \frac{x}{2} + 1} - \frac{1}{(\tan \frac{x}{2} + 1)^2} + C}.
\end{aligned}$$

84. With the substitution $z = \tan \frac{x}{2}$, $\sin x = \frac{2z}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sin x + 1} &= \int_0^1 \frac{1}{\frac{2z}{1+z^2} + 1} \frac{2dz}{1+z^2} \\
&= 2 \int_0^1 \frac{dz}{1+2z+z^2} \\
&= 2 \int_0^1 (1+z)^{-2} dz \\
&= \left[-2(1+z)^{-1} \right]_0^1 \\
&= -2(1+1)^{-1} - \left(-2(1+0)^{-1} \right) \\
&= \boxed{1}.
\end{aligned}$$

85. With the substitution $z = \tan \frac{x}{2}$, $\csc x = \frac{1+z^2}{2z}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$, we obtain

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{\csc x}{3+4\tan x} dx &= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{\frac{1+z^2}{2z} \frac{2}{1+z^2}}{3+4\frac{2z}{1-z^2}} dz \\
&= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{\frac{1}{z}(1-z^2)}{3(1-z^2)+8z} dz \\
&= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \frac{z^2-1}{z(3z+1)(z-3)} dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{z^2-1}{z(3z+1)(z-3)} &= \frac{A}{z} + \frac{B}{3z+1} + \frac{C}{z-3} \\
z^2-1 &= A(3z+1)(z-3) + Bz(z-3) + Cz(3z+1).
\end{aligned}$$

When $z = 0$, we have $A = 1/3$, when $z = -1/3$, $B = -4/5$, and when $z = 3$, $C = 4/15$.

We now obtain

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{\csc x}{3 + 4 \tan x} dx &= \int_{\tan(\pi/8)}^{\sqrt{3}/3} \left(\frac{1/3}{z} + \frac{-4/5}{3z+1} + \frac{4/15}{z-3} \right) dz \\
&= \left[\frac{1}{3} \ln z - \frac{4}{15} \ln(3z+1) + \frac{4}{15} \ln(3-z) \right]_{\tan(\pi/8)}^{\sqrt{3}/3} \\
&= \frac{1}{3} \ln \frac{\sqrt{3}}{3} - \frac{4}{15} \ln \left(3 \frac{\sqrt{3}}{3} + 1 \right) + \frac{4}{15} \ln \left(3 - \frac{\sqrt{3}}{3} \right) \\
&\quad - \left(\frac{1}{3} \ln \left(\tan \frac{\pi}{8} \right) - \frac{4}{15} \ln \left(3 \tan \frac{\pi}{8} + 1 \right) + \frac{4}{15} \ln \left(3 - \tan \frac{\pi}{8} \right) \right) \\
&= \boxed{\frac{1}{3} \ln \frac{\sqrt{3}}{3} - \frac{1}{3} \ln(\sqrt{2}-1) - \frac{4}{15} \ln(\sqrt{3}+1) - \frac{4}{15} \ln(4-\sqrt{2}) + \frac{4}{15} \ln(3\sqrt{2}-2) + \frac{4}{15} \ln \left(3 - \frac{\sqrt{3}}{3} \right)}.
\end{aligned}$$

86. With the substitution $z = \tan \frac{x}{2}$, $\cos x = \frac{1-z^2}{1+z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x dx}{2 - \cos x} &= \int_0^1 \frac{\frac{1-z^2}{1+z^2} \frac{2}{1+z^2}}{2 - \frac{1-z^2}{1+z^2}} dz \\
&= \int_0^1 \frac{2(1-z^2)}{(1+z^2)(2(1+z^2) - (1-z^2))} dz \\
&= \int_0^1 \frac{2(1-z^2)}{(z^2+1)(3z^2+1)} dz.
\end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned}
\frac{2(1-z^2)}{(z^2+1)(3z^2+1)} &= \frac{Az+B}{z^2+1} + \frac{Cz+D}{3z^2+1} \\
2(1-z^2) &= (Az+B)(3z^2+1) + (Cz+D)(z^2+1) \\
2-2z^2 &= (3A+C)z^3 + (3B+D)z^2 + (A+D)z + (B+D).
\end{aligned}$$

Equating coefficients, we obtain

$$\begin{aligned}
3A+C &= 0 \\
3B+D &= -2 \\
A+D &= 0 \\
B+D &= 2
\end{aligned}$$

We solve this system, and obtain $A=0$, $B=-2$, $C=0$, and $D=4$. So

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x dx}{2 - \cos x} &= \int_0^1 \left(-\frac{2}{z^2+1} + \frac{4}{3z^2+1} \right) dz \\
&= -2 \int_0^1 \frac{1}{z^2+1} dz + \frac{4}{3} \int_0^1 \frac{1}{z^2 + \left(\frac{\sqrt{3}}{3}\right)^2} dz \\
&= [-2 \tan^{-1} z]_0^1 + \left[\frac{4\sqrt{3}}{3} \tan^{-1} \left(\sqrt{3}z \right) \right]_0^1 \\
&= -2 \tan^{-1} 1 - (-2 \tan^{-1} 0) + \frac{4\sqrt{3}}{3} \tan^{-1} (\sqrt{3}(1)) - \frac{4\sqrt{3}}{3} \tan^{-1} (\sqrt{3}(0)) \\
&= \boxed{\frac{4\sqrt{3}}{9} \pi - \frac{1}{2} \pi}.
\end{aligned}$$

87. With the substitution $z = \tan \frac{x}{2}$, $\tan x = \frac{2z}{1-z^2}$, and $dx = \frac{2dz}{1+z^2}$ we obtain

$$\begin{aligned} \int_0^{\pi/4} \frac{4dx}{\tan x + 1} &= \int_0^{\tan(\pi/8)} \frac{4 \frac{2}{1+z^2}}{\frac{2z}{1-z^2} + 1} dz \\ &= \int_0^{\tan(\pi/8)} \frac{8(1-z^2)}{(1+z^2)(2z+(1-z^2))} dz \\ &= \int_0^{\tan(\pi/8)} \frac{8(z^2-1)}{(z^2+1)(z^2-2z-1)} dz. \end{aligned}$$

We use partial fractions to obtain

$$\begin{aligned} \frac{8(z^2-1)}{(z^2+1)(z^2-2z-1)} &= \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2-2z-1} \\ 8(z^2-1) &= (Az+B)(z^2-2z-1) + (Cz+D)(z^2+1) \\ 8z^2-8 &= (A+C)z^3 + (B-2A+D)z^2 + (C-2B-A)z + (D-B). \end{aligned}$$

Equating coefficients, we obtain

$$\begin{aligned} A+C &= 0 \\ B-2A+D &= 8 \\ -A-2B+C &= 0 \\ -B+D &= -8 \end{aligned}$$

We solve this system, and obtain $A = -4$, $B = 4$, $C = 4$, and $D = -4$. We now have

$$\begin{aligned} \int_0^{\pi/2} \frac{4dx}{\tan x + 1} &= \int_0^{\tan(\pi/8)} \left(\frac{-4z+4}{z^2+1} + \frac{4z-4}{z^2-2z-1} \right) dz \\ &= -2 \int_0^{\tan(\pi/8)} \frac{2z}{z^2+1} dz + 4 \int_0^{\tan(\pi/8)} \frac{1}{z^2+1} dz + 2 \int_0^{\tan(\pi/8)} \frac{2z-2}{z^2-2z-1} dz \\ &= -2 [\ln(z^2+1)]_0^{\tan(\pi/8)} + 4 [\tan^{-1} z]_0^{\tan(\pi/8)} + 2 [\ln(1+2z-z^2)]_0^{\tan(\pi/8)} \\ &= -2 \ln(\tan^2(\pi/8)+1) - (-2) \ln(0^2+1) + 4 \tan^{-1}(\tan(\pi/8)) - 4 \tan^{-1} 0 \\ &\quad + 2 \ln(1+2 \tan(\pi/8) - \tan^2(\pi/8)) - 2 \ln(1+2(0) - (0)^2) \\ &= -2 \ln\left(\left(\sqrt{2}-1\right)^2+1\right) + 4\left(\frac{1}{8}\pi\right) + 2 \ln\left(1+2\left(\sqrt{2}-1\right) - \left(\sqrt{2}-1\right)^2\right) \\ &= 2 \ln\left(\frac{2\sqrt{2}-2}{2-\sqrt{2}}\right) + \frac{1}{2}\pi \\ &= 2 \ln(\sqrt{2}) + \frac{1}{2}\pi \\ &= \boxed{\ln 2 + \frac{\pi}{2}}. \end{aligned}$$

88. With the substitution $z = \tan \frac{x}{2}$, $\csc x = \frac{1+z^2}{2z}$, and $dx = \frac{2dz}{1+z^2}$, we obtain

$$\begin{aligned}
 \int \csc x \, dx &= \int \frac{1+z^2}{2z} \frac{2}{1+z^2} dz \\
 &= \int \frac{1}{z} dz \\
 &= \ln |z| + C \\
 &= \ln \left| \tan \frac{x}{2} \right| + C \\
 &= \ln \frac{\left| \sin \frac{x}{2} \right|}{\left| \cos \frac{x}{2} \right|} + C \\
 &= \ln \frac{\sqrt{\frac{1-\cos x}{2}}}{\sqrt{\frac{1+\cos x}{2}}} + C \\
 &= \boxed{\ln \sqrt{\frac{1-\cos x}{1+\cos x}} + C}.
 \end{aligned}$$

89. We simplify as follows

$$\begin{aligned}
 \ln \sqrt{\frac{1-\cos x}{1+\cos x}} &= \ln \sqrt{\frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}} \\
 &= \ln \frac{1-\cos x}{\sqrt{1-\cos^2 x}} \\
 &= \ln \frac{1-\cos x}{\sqrt{\sin^2 x}} \\
 &= \ln \frac{1-\cos x}{|\sin x|} \\
 &= \ln \left| \frac{1-\cos x}{\sin x} \right| \\
 &= \ln \left| \frac{1}{\sin x} - \frac{\cos x}{\sin x} \right| \\
 &= \ln |\csc x - \cot x|.
 \end{aligned}$$

90. The domain of the function given by $\tanh^{-1}(x)$ is the interval $(-1, 1)$, so the function given by $\tanh^{-1}(x)$ is not an antiderivative of $\frac{1}{1-x^2}$ on the interval $[2, 3]$. We use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{1-x^2} &= \frac{-1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \\
 -1 &= A(x+1) + B(x-1).
 \end{aligned}$$

When $x = -1$ we obtain $B = \frac{1}{2}$, and when $x = 1$ we have $A = -\frac{1}{2}$. So we obtain

$$\begin{aligned}
 \int_2^3 \frac{1}{1-x^2} dx &= \int_2^3 \left(\frac{-1/2}{x-1} + \frac{1/2}{x+1} \right) dx \\
 &= \left[-\frac{1}{2} \ln |x-1| + \frac{1}{2} \ln |x+1| \right]_2^3 \\
 &= -\frac{1}{2} \ln |3-1| + \frac{1}{2} \ln |3+1| - \left(-\frac{1}{2} \ln |2-1| + \frac{1}{2} \ln |2+1| \right) \\
 &= \boxed{\frac{\ln 2 - \ln 3}{2}}.
 \end{aligned}$$

91. We simplify as follows

$$\begin{aligned}
 \ln \left| \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right| &= \ln \left| \frac{1 + \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}}{1 - \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}} \right| \\
 &= \ln \left| \frac{1 + \frac{\sin^2 \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}}}{1 - \frac{\sin^2 \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}}} \right| \\
 &= \ln \left| \frac{1 + \frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}}{1 - \frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}} \right| \\
 &= \ln \left| \frac{1 + \frac{1-\cos x}{\sin x}}{1 - \frac{1-\cos x}{\sin x}} \right| \\
 &= \ln \left| \frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} \right| \\
 &= \ln \left| \frac{\tan x + \sec x - 1}{\tan x - \sec x + 1} \frac{\sec x + \tan x}{\sec x + \tan x} \right| \\
 &= \ln \left| \frac{(\tan x + \sec x - 1)(\sec x + \tan x)}{(\tan x - \sec x + 1)(\sec x + \tan x)} \right| \\
 &= \ln \left| \frac{(\tan x + \sec x - 1)(\sec x + \tan x)}{\tan x \sec x + \tan^2 x - \sec^2 x - \sec x \tan x + \sec x + \tan x} \right| \\
 &= \ln \left| \frac{(\tan x + \sec x - 1)(\sec x + \tan x)}{-1 + \sec x + \tan x} \right| \\
 &= \ln |\sec x + \tan x|.
 \end{aligned}$$

92. We factor $1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$, and use partial fractions to obtain

$$\begin{aligned}
 \frac{1}{1+x^4} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\
 1 &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\
 1 &= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D).
 \end{aligned}$$

We solve the system

$$\begin{aligned} A + C &= 0 \\ B + D - \sqrt{2}A + \sqrt{2}C &= 0 \\ A + C - \sqrt{2}B + \sqrt{2}D &= 0 \\ B + D &= 1 \end{aligned}$$

and obtain $A = \sqrt{2}/4$, $B = 1/2$, $C = -\sqrt{2}/4$, and $D = 1/2$. We now obtain

$$\begin{aligned} \int \frac{1}{1+x^4} dx &= \int \left(\frac{(\sqrt{2}/4)x + 1/2}{x^2 + \sqrt{2}x + 1} + \frac{(-\sqrt{2}/4)x + 1/2}{x^2 - \sqrt{2}x + 1} \right) dx \\ &= \int \left(\frac{(\sqrt{2}/8)(2x + \sqrt{2}) + \frac{1}{4}}{x^2 + \sqrt{2}x + 1} + \frac{\left(-\frac{\sqrt{2}}{8}\right)(2x - \sqrt{2}) + \frac{1}{4}}{x^2 - \sqrt{2}x + 1} \right) dx \\ &= \frac{\sqrt{2}}{8} \int \frac{2x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} dx + \frac{1}{4} \int \frac{dx}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} \\ &\quad - \frac{\sqrt{2}}{8} \int \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} dx + \frac{1}{4} \int \frac{dx}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} \\ &= \boxed{\frac{\sqrt{2}}{8} \ln(x^2 + \sqrt{2}x + 1) + \frac{\sqrt{2}}{4} \tan^{-1}(\sqrt{2}x + 1) - \frac{\sqrt{2}}{8} \ln(x^2 - \sqrt{2}x + 1) + \frac{\sqrt{2}}{4} \tan^{-1}(\sqrt{2}x - 1) + C}. \end{aligned}$$

7.6: Integration Using Numerical Techniques

Concepts and Vocabulary

1. True

2. True

Skill Building

3. When $n = 3$ we have $\Delta x = \frac{6-0}{3} = 2$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x) dx &\approx \frac{\Delta x}{2} [f(0) + 2f(2) + 2f(4) + f(6)] \\ &= \frac{2}{2}(6 + 2(3) + 2(3) + 4) \\ &= \boxed{22}.\end{aligned}$$

When $n = 6$ we have $\Delta x = \frac{6-0}{6} = 1$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x) dx &\approx \frac{\Delta x}{2} [f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)] \\ &= \frac{1}{2}(6 + 2(3) + 2(3) + 2(4) + 2(3) + 2(2) + 4) \\ &= \boxed{20}.\end{aligned}$$

4. When $n = 2$ we have $\Delta x = \frac{5-1}{2} = 2$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_1^5 f(x) dx &\approx \frac{\Delta x}{2} [f(1) + 2f(3) + f(5)] \\ &= \frac{2}{2}(0 + 2(3) + 6) \\ &= \boxed{12}.\end{aligned}$$

When $n = 4$ we have $\Delta x = \frac{5-1}{4} = 1$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_1^5 f(x) dx &\approx \frac{\Delta x}{2} [f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] \\ &= \frac{1}{2}(0 + 2(2.5) + 2(3) + 2(6) + 6) \\ &= \boxed{\frac{29}{2} = 14.5}.\end{aligned}$$

5. When $n = 2$ we have $\Delta x = \frac{6-0}{2} = 3$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x) dx &\approx \frac{\Delta x}{3} [f(0) + 4f(3) + f(6)] \\ &= \frac{3}{3}(6 + 4(4) + 4) \\ &= \boxed{26}.\end{aligned}$$

When $n = 6$ we have $\Delta x = \frac{6-0}{6} = 1$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^6 f(x) dx &\approx \frac{\Delta x}{3} [f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)] \\ &= \frac{1}{3} (6 + 4(3) + 2(3) + 4(4) + 2(3) + 4(2) + 4) \\ &= \boxed{\frac{58}{3}}.\end{aligned}$$

6. When $n = 2$ we have $\Delta x = \frac{5-1}{2} = 2$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_1^5 f(x) dx &\approx \frac{\Delta x}{3} [f(1) + 4f(3) + f(5)] \\ &= \frac{2}{3} (0 + 4(3) + 6) \\ &= \boxed{12}.\end{aligned}$$

When $n = 4$ we have $\Delta x = \frac{5-1}{4} = 1$. So Simpson's Rule provides the approximation

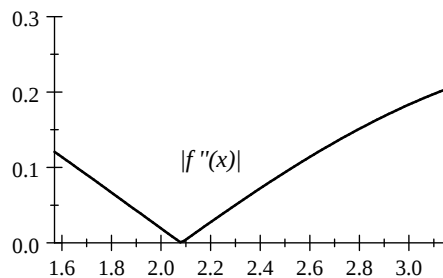
$$\begin{aligned}\int_1^5 f(x) dx &\approx \frac{\Delta x}{3} [f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] \\ &= \frac{1}{3} (0 + 4(2.5) + 2(3) + 4(6) + 6) \\ &= \boxed{\frac{46}{3}}.\end{aligned}$$

7. (a) With $n = 3$ we have $\Delta x = \frac{\pi - \pi/2}{3} = \pi/6$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx &\approx \frac{\Delta x}{2} \left[f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{5\pi}{6}\right) + f(\pi) \right] \\ &= \frac{\pi/6}{2} \left[\frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} + 2 \frac{\sin \frac{2\pi}{3}}{\frac{2\pi}{3}} + 2 \frac{\sin \frac{5\pi}{6}}{\frac{5\pi}{6}} + \frac{\sin \pi}{\pi} \right] \\ &= \frac{\pi}{12} \left[\frac{2}{\pi} + \frac{3}{2} \frac{\sqrt{3}}{\pi} + \frac{6}{5\pi} + 0 \right] \\ &\approx \boxed{0.483}.\end{aligned}$$

(b) We have $\frac{d}{dx} \left(\frac{\sin x}{x} \right) = -\frac{1}{x^2} (\sin x - x \cos x)$, and $\frac{d^2}{dx^2} \left(\frac{\sin x}{x} \right) = -\frac{1}{x^3} (x^2 \sin x - 2 \sin x + 2x \cos x)$. The graph of $|f''(x)| = \left| -\frac{1}{x^3} (x^2 \sin x - 2 \sin x + 2x \cos x) \right|$ shows that the maximum occurs at $x = \pi$, and so $M = \left| -\frac{1}{\pi^3} (\pi^2 \sin \pi - 2 \sin \pi + 2\pi \cos \pi) \right| = \frac{2}{\pi^2}$. We obtain

$$\text{Error} \leq \frac{(\pi - \pi/2)^3 \left(\frac{2}{\pi^2} \right)}{12(3)^2} \approx 0.007$$



(c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(\pi - \pi/2)^3 \left(\frac{2}{\pi^2}\right)}{12(n)^2} = \frac{\pi}{48n^2} < 0.0001 \\ \frac{\pi}{48(0.0001)} &< n^2 \\ \sqrt{\frac{\pi}{48(0.0001)}} &< n \\ 25.58 &< n \end{aligned}$$

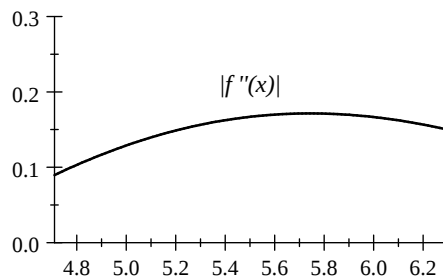
So we need $n = \boxed{26}$.

8. (a) With $n = 3$ we have $\Delta x = \frac{2\pi - 3\pi/2}{3} = \pi/6$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_{3\pi/2}^{2\pi} \frac{\cos x}{x} dx &\approx \frac{\Delta x}{2} \left[f\left(\frac{3\pi}{2}\right) + 2f\left(\frac{5\pi}{3}\right) + 2f\left(\frac{11\pi}{6}\right) + f(2\pi) \right] \\ &= \frac{\pi/6}{2} \left[\frac{\cos \frac{3\pi}{2}}{\frac{3\pi}{2}} + 2\frac{\cos \frac{5\pi}{3}}{\frac{5\pi}{3}} + 2\frac{\cos \frac{11\pi}{6}}{\frac{11\pi}{6}} + \frac{\cos(2\pi)}{2\pi} \right] \\ &= \frac{\pi}{12} \left[0 + \frac{3}{5\pi} + \frac{6}{11} \frac{\sqrt{3}}{\pi} + \frac{1}{2\pi} \right] \\ &\approx \boxed{0.1704}. \end{aligned}$$

- (b) We have $\frac{d}{dx}\left(\frac{\cos x}{x}\right) = -\frac{1}{x^2}(\cos x + x \sin x)$, and $\frac{d^2}{dx^2}\left(\frac{\cos x}{x}\right) = \frac{1}{x^3}(2 \cos x - x^2 \cos x + 2x \sin x)$. The graph of $|f''(x)| = \left|\frac{1}{x^3}(2 \cos x - x^2 \cos x + 2x \sin x)\right|$ shows that the maximum $M < 0.2$. We obtain

$$\text{Error} \leq \frac{(2\pi - 3\pi/2)^3 (0.2)}{12(3)^2} \approx 0.0072$$



(c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(2\pi - 3\pi/2)^3(0.2)}{12(n)^2} = \frac{\pi^3}{480n^2} < 0.0001 \\ \frac{\pi^3}{0.0001(480)} &< n^2 \\ \sqrt{\frac{\pi^3}{0.0001(480)}} &< n \\ 25.42 &< n\end{aligned}$$

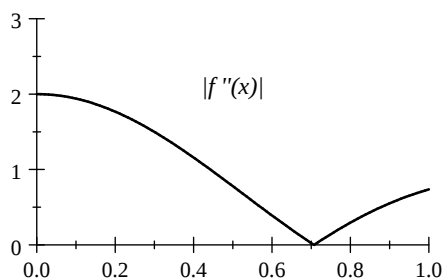
So we need $n = \boxed{26}$.

9. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^1 e^{-x^2} dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{2} \left[e^{-(0)^2} + 2e^{-(\frac{1}{4})^2} + 2e^{-(\frac{1}{2})^2} + 2e^{-(\frac{3}{4})^2} + e^{-(1)^2} \right] \\ &= \frac{1}{8} \left[1 + 2e^{-\frac{1}{16}} + 2e^{-\frac{1}{4}} + 2e^{-\frac{9}{16}} + e^{-1} \right] \\ &\approx \boxed{0.743}.\end{aligned}$$

- (b) We have $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$, and $\frac{d^2}{dx^2}(e^{-x^2}) = 4x^2e^{-x^2} - 2e^{-x^2}$. The graph of $|f''(x)| = |4x^2e^{-x^2} - 2e^{-x^2}|$ shows that the maximum $M = 2$. We obtain

$$\text{Error} \leq \frac{(1-0)^3(2)}{12(4)^2} = \frac{1}{96}$$



(c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(1-0)^3(2)}{12(n)^2} = \frac{1}{6n^2} < 0.0001 \\ \frac{1}{6(0.0001)} &< n^2 \\ \sqrt{\frac{1}{6(0.0001)}} &< n \\ 40.82 &< n\end{aligned}$$

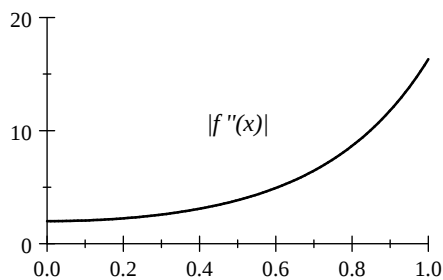
So we need $n = \boxed{41}$.

10. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^1 e^{x^2} dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{2} \left[e^{(0)^2} + 2e^{(\frac{1}{4})^2} + 2e^{(\frac{1}{2})^2} + 2e^{(\frac{3}{4})^2} + e^{(1)^2} \right] \\ &= \frac{1}{8} \left[1 + 2e^{\frac{1}{16}} + 2e^{\frac{1}{4}} + 2e^{\frac{9}{16}} + e \right] \\ &\approx \boxed{1.491}.\end{aligned}$$

- (b) We have $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$, and $\frac{d^2}{dx^2}(e^{x^2}) = 2e^{x^2} + 4x^2e^{x^2}$. The graph of $|f''(x)| = |2e^{x^2} + 4x^2e^{x^2}|$ shows that the maximum $M = 2e + 4(1)^2e^{1^2} \approx 16.31$. We obtain

$$\text{Error} \leq \frac{(1-0)^3(16.31)}{12(4)^2} \approx 0.085$$



- (c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(1-0)^3(16.31)}{12(n)^2} = \frac{1.359}{n^2} < 0.0001 \\ \frac{1.359}{0.0001} &< n^2 \\ \sqrt{\frac{1.359}{0.0001}} &< n \\ 116.6 &< n\end{aligned}$$

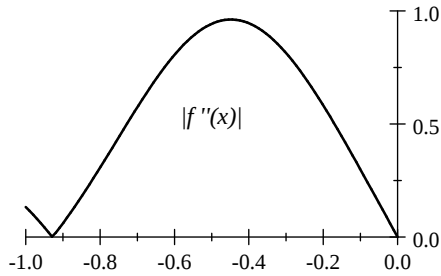
So we need $n = \boxed{117}$.

11. (a) With $n = 4$ we have $\Delta x = \frac{0-(-1)}{4} = \frac{1}{4}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_{-1}^0 \frac{dx}{\sqrt{1-x^3}} &\approx \frac{\Delta x}{2} \left[f(-1) + 2f\left(\frac{-3}{4}\right) + 2f\left(\frac{-1}{2}\right) + 2f\left(\frac{-1}{4}\right) + f(0) \right] \\ &= \frac{1/4}{2} \left[\frac{1}{\sqrt{1-(-1)^3}} + 2\frac{1}{\sqrt{1-(-3/4)^3}} + 2\frac{1}{\sqrt{1-(-1/2)^3}} \right. \\ &\quad \left. + 2\frac{1}{\sqrt{1-(-1/4)^3}} + \frac{1}{\sqrt{1-(0)^3}} \right] \\ &= \frac{1}{8} \left[\frac{1}{2}\sqrt{2} + \frac{16}{91}\sqrt{91} + \frac{4}{3}\sqrt{2} + \frac{16}{65}\sqrt{65} + 1 \right] \\ &\approx \boxed{0.907}. \end{aligned}$$

- (b) We have $\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^3}}\right) = \frac{3x^2}{2(1-x^3)^{\frac{3}{2}}}$, and $\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{1-x^3}}\right) = \frac{3x(5x^3+4)}{4(1-x^3)^{\frac{5}{2}}}$. The graph of $|f''(x)| = \left|\frac{3x(5x^3+4)}{4(1-x^3)^{\frac{5}{2}}}\right|$ shows that the maximum $M \leq 1$. We obtain

$$\text{Error} \leq \frac{(0-(-1))^3(1)}{12(4)^2} = \frac{1}{192}$$



- (c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(0-(-1))^3(1)}{12(n)^2} = \frac{1}{12n^2} < 0.0001 \\ \frac{1}{0.0001(12)} &< n^2 \\ \sqrt{\frac{1}{0.0001(12)}} &< n \\ 28.87 &< n \end{aligned}$$

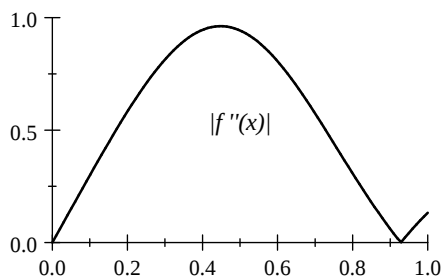
So we need $n = \boxed{29}$.

12. (a) With $n = 3$ we have $\Delta x = \frac{1-0}{3} = \frac{1}{3}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1+x^3}} &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1) \right] \\ &= \frac{1/3}{2} \left[\frac{1}{\sqrt{1+(0)^3}} + 2\frac{1}{\sqrt{1+(\frac{1}{3})^3}} + 2\frac{1}{\sqrt{1+(\frac{2}{3})^3}} + \frac{1}{\sqrt{1+(1)^3}} \right] \\ &= \frac{1}{6} \left[1 + \frac{3}{7}\sqrt{21} + \frac{6}{35}\sqrt{105} + \frac{1}{2}\sqrt{2} \right] \\ &\approx \boxed{0.9046}. \end{aligned}$$

- (b) We have $\frac{d}{dx} \left(\frac{1}{\sqrt{1+x^3}} \right) = \frac{-3x^2}{2(1+x^3)^{\frac{3}{2}}}$, and $\frac{d^2}{dx^2} \left(\frac{1}{\sqrt{1+x^3}} \right) = \frac{3x(5x^3-4)}{4(1+x^3)^{\frac{5}{2}}}$. The graph of $|f''(x)| = \left| \frac{3x(5x^3-4)}{4(1+x^3)^{\frac{5}{2}}} \right|$ shows that the maximum $M \leq 1$. We obtain

$$\text{Error} \leq \frac{(1-0)^3(1)}{12(4)^2} \approx 0.0052$$



- (c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(1-0)^3(1)}{12(n)^2} = \frac{1}{12n^2} < 0.0001 \\ \frac{1}{0.0001(12)} &< n^2 \\ \sqrt{\frac{1}{0.0001(12)}} &< n \\ 28.87 &< n \end{aligned}$$

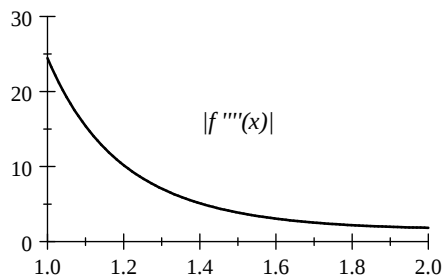
So we need $n = \boxed{29}$.

13. (a) With $n = 4$ we have $\Delta x = \frac{2-1}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_1^2 \frac{e^x}{x} dx &\approx \frac{\Delta x}{3} \left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2) \right] \\ &= \frac{1/4}{3} \left[\frac{e^1}{1} + 4\frac{e^{5/4}}{5/4} + 2\frac{e^{3/2}}{3/2} + 4\frac{e^{7/4}}{7/4} + \frac{e^2}{2} \right] \\ &= \frac{1}{12} \left[e + \frac{16}{5}e^{5/4} + \frac{4}{3}e^{3/2} + \frac{16}{7}e^{7/4} + \frac{1}{2}e^2 \right] \\ &\approx \boxed{3.059}. \end{aligned}$$

- (b) We have $\frac{d^4}{dx^4}\left(\frac{e^x}{x}\right) = \frac{1}{x^5}e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$. The graph of $|f^4(x)|$ shows $M \leq |f^{(4)}(1)| = 9e$. We obtain

$$\text{Error} \leq \frac{(2-1)^5(9e)}{180(4)^4} \approx 5.309 \times 10^{-4}$$



- (c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(2-1)^5(9e)}{180(n)^4} < 0.0001 \\ \frac{9e}{0.0001(180)} &< n^4 \\ \sqrt[4]{\frac{9e}{0.0001(180)}} &< n \\ 6.072 &< n \end{aligned}$$

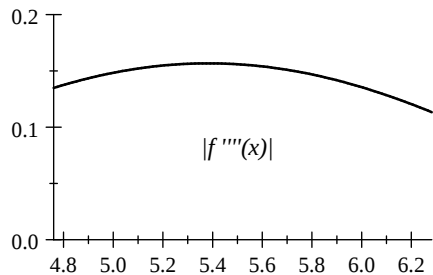
Since n must be even, we need $n = \boxed{8}$.

14. (a) With $n = 4$ we have $\Delta x = \frac{2\pi - 3\pi/2}{4} = \frac{\pi}{8}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_{3\pi/2}^{2\pi} \frac{\cos x}{x} dx &\approx \frac{\Delta x}{3} \left[f\left(\frac{3\pi}{2}\right) + 4f\left(\frac{13\pi}{8}\right) + 2f\left(\frac{7\pi}{4}\right) + 4f\left(\frac{15\pi}{8}\right) + f(2\pi) \right] \\ &= \frac{\pi/8}{3} \left[\frac{\cos(3\pi/2)}{3\pi/2} + 4\frac{\cos(13\pi/8)}{13\pi/8} + 2\frac{\cos(7\pi/4)}{7\pi/4} + 4\frac{\cos(15\pi/8)}{15\pi/8} + \frac{\cos(2\pi)}{2\pi} \right] \\ &\approx \boxed{0.1759}. \end{aligned}$$

- (b) We have $\frac{d^4}{dx^4}\left(\frac{\cos x}{x}\right) = \frac{1}{x^5}(24\cos x - 12x^2\cos x + x^4\cos x - 4x^3\sin x + 24x\sin x)$. The graph of $|f^4(x)|$ shows $M \leq 0.2$. We obtain

$$\text{Error} \leq \frac{(2\pi - 3\pi/2)^5(0.2)}{180(4)^4} \approx 4.151 \times 10^{-5}$$



(c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(2\pi - 3\pi/2)^5(0.2)}{180(n)^4} = \frac{1.063 \times 10^{-2}}{n^4} < 0.0001 \\ \frac{1.063 \times 10^{-2}}{0.0001} &< n^4 \\ \sqrt[4]{\frac{1.063 \times 10^{-2}}{0.0001}} &< n \\ 3.211 &< n\end{aligned}$$

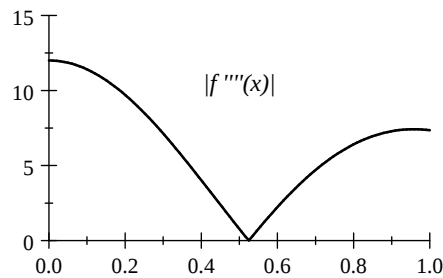
Since n must be even, we need $n = \boxed{4}$.

15. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^1 e^{-x^2} dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{3} \left[e^{-(0)^2} + 4e^{-(1/4)^2} + 2e^{-(1/2)^2} + 4e^{-(3/4)^2} + e^{-(1)^2} \right] \\ &= \frac{1}{12} \left[1 + 4e^{-\frac{1}{16}} + 2e^{-\frac{1}{4}} + 4e^{-\frac{9}{16}} + e^{-1} \right] \\ &\approx \boxed{0.747}.\end{aligned}$$

(b) We have $\frac{d^4}{dx^4}(e^{-x^2}) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$. The graph of $|f^4(x)|$ shows $M \leq 12$. We obtain

$$\text{Error} \leq \frac{(1-0)^5(12)}{180(4)^4} \approx 2.604 \times 10^{-4}$$



(c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(1-0)^5(12)}{180(n)^4} = \frac{1}{15n^4} < 0.0001 \\ \frac{1}{0.0001(15)} &< n^4 \\ \sqrt[4]{\frac{1}{0.0001(15)}} &< n \\ 5.081 &< n\end{aligned}$$

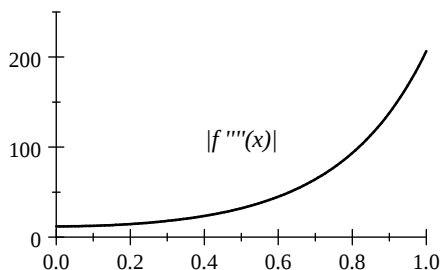
We need $n = \boxed{6}$.

16. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^1 e^{x^2} dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{3} \left[e^{(0)^2} + 4e^{(1/4)^2} + 2e^{(1/2)^2} + 4e^{(3/4)^2} + e^{(1)^2} \right] \\ &= \frac{1}{12} \left[1 + 4e^{\frac{1}{16}} + 2e^{\frac{1}{4}} + 4e^{\frac{9}{16}} + e \right] \\ &\approx \boxed{1.464}.\end{aligned}$$

- (b) We have $\frac{d^4}{dx^4}(e^{x^2}) = 4e^{x^2}(4x^4 + 12x^2 + 3)$. The graph of $|f''''(x)|$ shows $M = 76e \approx 206.6$. We obtain

$$\text{Error} \leq \frac{(1-0)^5(76e)}{180(4)^4} \approx 4.483 \times 10^{-3}$$



- (c) We require

$$\begin{aligned}\text{Error} &\leq \frac{(1-0)^5(76e)}{180(n)^4} \approx \frac{1.148}{(n)^4} < 0.0001 \\ \frac{1.148}{0.0001} &< n^4 \\ \sqrt[4]{\frac{1.148}{0.0001}} &< n \\ 10.35 &< n\end{aligned}$$

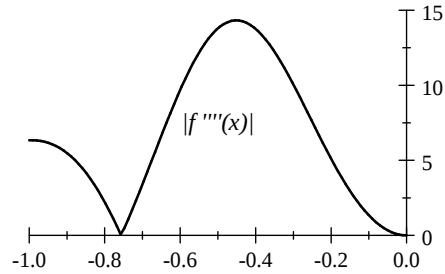
Since n must be even, we need $n = \boxed{12}$.

17. (a) With $n = 4$ we have $\Delta x = \frac{0-(-1)}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt{1-x^3}} &\approx \frac{\Delta x}{3} \left[f(-1) + 4f\left(\frac{-3}{4}\right) + 2f\left(\frac{-1}{2}\right) + 4f\left(\frac{-1}{4}\right) + f(0) \right] \\ &= \frac{1/4}{3} \left[\frac{1}{\sqrt{1-(-1)^3}} + 4\frac{1}{\sqrt{1-(-3/4)^3}} + 2\frac{1}{\sqrt{1-(-1/2)^3}} \right. \\ &\quad \left. + 4\frac{1}{\sqrt{1-(-1/4)^3}} + \frac{1}{\sqrt{1-(0)^3}} \right] \\ &= \frac{1}{12} \left[\frac{1}{2}\sqrt{2} + \frac{32}{91}\sqrt{91} + \frac{4}{3}\sqrt{2} + \frac{32}{65}\sqrt{65} + 1 \right] \\ &\approx \boxed{0.910}.\end{aligned}$$

(b) We have $\frac{d^4}{dx^4} \left(\frac{1}{\sqrt{1-x^3}} \right) = \frac{135x^2(7x^6+40x^3+16)}{16(1-x^3)^{\frac{9}{2}}}$. The graph of $|f^4(x)|$ shows $M \leq 15$. We obtain

$$\text{Error} \leq \frac{(0 - (-1))^5(15)}{180(4)^4} \approx 3.255 \times 10^{-4}$$



(c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(0 - (-1))^5(15)}{180(n)^4} = \frac{1}{12(n)^4} < 0.0001 \\ \frac{1}{0.0001(12)} &< n^4 \\ \sqrt[4]{\frac{1}{0.0001(12)}} &< n \\ 5.373 &< n \end{aligned}$$

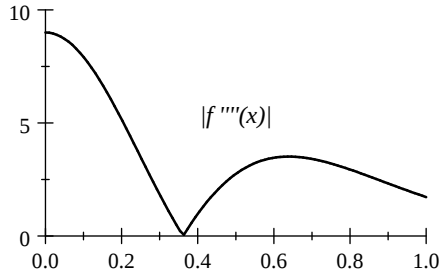
We need $n = \boxed{6}$.

18. (a) With $n = 4$ we have $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1+x^2}} &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{3} \left[\frac{1}{\sqrt{1+(0)^2}} + 4\frac{1}{\sqrt{1+(1/4)^2}} + 2\frac{1}{\sqrt{1+(1/2)^2}} \right. \\ &\quad \left. + 4\frac{1}{\sqrt{1+(3/4)^2}} + \frac{1}{\sqrt{1+(1)^2}} \right] \\ &= \frac{1}{12} \left[1 + \frac{16}{17}\sqrt{17} + \frac{4}{5}\sqrt{5} + \frac{16}{5} + \frac{1}{2}\sqrt{2} \right] \\ &\approx \boxed{0.8814}. \end{aligned}$$

(b) We have $\frac{d^4}{dx^4} \left(\frac{1}{\sqrt{1+x^2}} \right) = \frac{24x^4-72x^2+9}{(x^2+1)^{\frac{9}{2}}}$. The graph of $|f^4(x)|$ shows $M = 9$. We obtain

$$\text{Error} \leq \frac{(1-0)^5(9)}{180(4)^4} = \frac{1}{5120} \approx 1.953 \times 10^{-4}$$



(c) We require

$$\begin{aligned} \text{Error} &\leq \frac{(1-0)^5(9)}{180(n)^4} = \frac{1}{20(n)^4} < 0.0001 \\ \frac{1}{0.0001(20)} &< n^4 \\ \sqrt[4]{500} &< n \\ 4.729 &< n \end{aligned}$$

We need $n = \boxed{6}$.

19. (a) $\int_1^2 \frac{dx}{x} = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2$.

(b) With $n = 5$ we have $\Delta x = \frac{2-1}{5} = \frac{1}{5}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_1^2 \frac{dx}{x} &\approx \frac{\Delta x}{2} \left[f(1) + 2f\left(\frac{6}{5}\right) + 2f\left(\frac{7}{5}\right) + 2f\left(\frac{8}{5}\right) + 2f\left(\frac{9}{5}\right) + f(2) \right] \\ &= \frac{1/5}{2} \left[\frac{1}{1} + 2\frac{1}{6/5} + 2\frac{1}{7/5} + 2\frac{1}{8/5} + 2\frac{1}{9/5} + \frac{1}{2} \right] \\ &= \frac{1}{10} \left[1 + \frac{5}{3} + \frac{10}{7} + \frac{5}{4} + \frac{10}{9} + \frac{1}{2} \right] \\ &= \frac{1753}{2520} \\ &\approx \boxed{0.6956}. \end{aligned}$$

(c) With $n = 6$ we have $\Delta x = \frac{2-1}{6} = \frac{1}{6}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_1^2 \frac{dx}{x} &\approx \frac{\Delta x}{3} \left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{5}{3}\right) + 4f\left(\frac{11}{6}\right) + f(2) \right] \\ &= \frac{1/6}{3} \left[\frac{1}{1} + 4\frac{1}{7/6} + 2\frac{1}{4/3} + 4\frac{1}{3/2} + 2\frac{1}{5/3} + 4\frac{1}{11/6} + \frac{1}{2} \right] \\ &= \frac{1}{18} \left[\frac{1}{1} + \frac{24}{7} + \frac{3}{2} + \frac{8}{3} + \frac{6}{5} + \frac{24}{11} + \frac{1}{2} \right] \\ &= \frac{14411}{20790} \\ &\approx \boxed{0.6932}. \end{aligned}$$

20. The area is given by the integral $\int_2^{4.4} f(x) dx$, which we approximate by Simpson's Rule, with $\Delta x = 0.4$.

$$\begin{aligned}\int_2^{4.4} f(x) dx &\approx \frac{\Delta x}{3} [f(2.0) + 4f(2.4) + 2f(2.8) + 4f(3.2) + 2f(3.6) + 4f(4.0) + f(4.4)] \\ &= \frac{0.4}{3} [3.03 + 4(4.61) + 2(5.80) + 4(6.59) + 2(7.76) + 4(8.46) + 9.19] \\ &\approx \boxed{15.73}.\end{aligned}$$

21. The arc length is given by $\int_0^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{\pi/2} \sqrt{1 + (\cos x)^2} dx = \int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx$.

- (a) With $n = 4$ we have $\Delta x = \frac{(\pi/2)-0}{4} = \frac{\pi}{8}$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/8}{3} \left[\sqrt{1 + \cos^2 0} + 4\sqrt{1 + \cos^2 \frac{\pi}{8}} + 2\sqrt{1 + \cos^2 \frac{\pi}{4}} \right. \\ &\quad \left. + 4\sqrt{1 + \cos^2 \frac{3\pi}{8}} + \sqrt{1 + \cos^2 \frac{\pi}{2}} \right] \\ &\approx \boxed{1.910}.\end{aligned}$$

- (b) With $n = 3$ we have $\Delta x = \frac{(\pi/2)-0}{3} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/6}{2} \left[\sqrt{1 + \cos^2 0} + 2\sqrt{1 + \cos^2 \frac{\pi}{6}} + 2\sqrt{1 + \cos^2 \frac{\pi}{3}} + \sqrt{1 + \cos^2 \frac{\pi}{2}} \right] \\ &\approx \boxed{1.910}.\end{aligned}$$

22. The arc length is given by $\int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^4 \sqrt{1 + (e^x)^2} dx = \int_0^4 \sqrt{1 + e^{2x}} dx$.

- (a) With $n = 4$ we have $\Delta x = \frac{4-0}{4} = 1$. So Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^4 \sqrt{1 + e^{2x}} dx &\approx \frac{\Delta x}{3} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \\ &= \frac{1}{3} \left[\sqrt{1 + e^{2(0)}} + 4\sqrt{1 + e^{2(1)}} + 2\sqrt{1 + e^{2(2)}} + 4\sqrt{1 + e^{2(3)}} + \sqrt{1 + e^{2(4)}} \right] \\ &\approx \boxed{54.32}.\end{aligned}$$

- (b) With $n = 8$ we have $\Delta x = \frac{4-0}{8} = \frac{1}{2}$. So the Trapezoidal Rule provides the approxi-

mation

$$\begin{aligned}
 \int_0^4 \sqrt{1+e^{2x}} dx &\approx \frac{\Delta x}{2} [f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + 2f(2) \\
 &\quad + 2f(5/2) + 2f(3) + 2f(7/2) + f(4)] \\
 &= \frac{1}{2} \left[\sqrt{1+e^{2(0)}} + 2\sqrt{1+e^{2(\frac{1}{2})}} + 2\sqrt{1+e^{2(1)}} + 2\sqrt{1+e^{2(\frac{3}{2})}} \right. \\
 &\quad \left. + 2\sqrt{1+e^{2(2)}} + 2\sqrt{1+e^{2(\frac{5}{2})}} + 2\sqrt{1+e^{2(3)}} + 2\sqrt{1+e^{2(\frac{7}{2})}} + \sqrt{1+e^{2(4)}} \right] \\
 &\approx \boxed{55.17}.
 \end{aligned}$$

23. The work is given by the integral $\int_1^{2.5} p dV$, which we approximate by Simpson's Rule, with $\Delta V = 0.25$.

$$\begin{aligned}
 \int_1^{2.5} p dV &\approx \frac{\Delta V}{3} [f(1.0) + 4f(1.25) + 2f(1.5) + 4f(1.75) + 2f(2.0) + 4f(2.25) + f(2.5)] \\
 &= \frac{0.25}{3} [68.7 + 4(55.0) + 2(45.8) + 4(39.3) + 2(34.4) + 4(30.5) + 27.5] \\
 &\approx \boxed{62.983 \text{ inch-pounds}}.
 \end{aligned}$$

24. The work is given by the integral $\int_0^{50} F dx$, which we approximate by the Trapezoidal Rule, with $\Delta x = 5$.

$$\begin{aligned}
 \int_0^{50} F dx &\approx \frac{\Delta x}{2} [f(0) + 2f(5) + 2f(10) + 2f(15) + 2f(20) \\
 &\quad + 2f(25) + f(30) + 2f(35) + 2f(40) + 2f(45) + f(50)] \\
 &= \frac{5}{2} [100 + 2(80) + 2(66) + 2(56) + 2(50) + 2(45) + 2(40) + 2(36) + 2(33) + 2(30) + 28] \\
 &= \boxed{2500}.
 \end{aligned}$$

25. The volume is given by the integral $\int_0^{150} S dx$, which we approximate by the Trapezoidal Rule, with $\Delta x = 25$.

$$\begin{aligned}
 \int_0^{150} S dx &\approx \frac{\Delta x}{2} [f(0) + 2f(25) + 2f(50) + 2f(75) + 2f(100) + 2f(125) + f(150)] \\
 &= \frac{25}{2} [105 + 2(118) + 2(142) + 2(120) + 2(110) + 2(90) + 78] \\
 &\approx \boxed{16,787.5 \text{ m}^3}.
 \end{aligned}$$

26. Let y represent the vertical distance of the pond at position x . Then the area is given by the integral $\int_0^{20} y dx$, which we approximate by Simpson's Rule, with $\Delta x = 5$.

$$\begin{aligned}
 \int_0^{20} y dx &\approx \frac{\Delta x}{3} [f(0) + 4f(5) + 2f(10) + 4f(15) + f(20)] \\
 &= \frac{5}{3} [0 + 4(12) + 2(19) + 4(13) + 2(13) + 0] \\
 &\approx \boxed{273.3 \text{ ft}^2}.
 \end{aligned}$$

27. The volume is given by the integral $\int_0^{25} A \, dx$. We first approximate by the Trapezoidal Rule, with $\Delta x = 2.5$.

$$\begin{aligned}\int_0^{25} A \, dx &\approx \frac{\Delta x}{2} [f(0) + 2f(2.5) + 2f(5.0) + 2f(7.5) + 2f(10) + 2f(12.5) \\ &\quad + 2f(15.0) + 2f(17.5) + 2f(20) + 2f(22.5) + f(25)] \\ &= \frac{2.5}{2} [0 + 2(2510) + 2(3860) + 2(4870) + 2(5160) + 2(5590) \\ &\quad + 2(5810) + 2(6210) + 2(6890) + 2(7680) + 8270] \\ &\approx \boxed{131,787.5 \text{ m}^3}.\end{aligned}$$

We next approximate by Simpson's Rule.

$$\begin{aligned}\int_0^{25} A \, dx &\approx \frac{\Delta x}{3} [f(0) + 4f(2.5) + 2f(5.0) + 4f(7.5) + 2f(10) + 4f(12.5) \\ &\quad + 2f(15.0) + 4f(17.5) + 2f(20) + 4f(22.5) + f(25)] \\ &= \frac{2.5}{3} [0 + 4(2510) + 2(3860) + 4(4870) + 2(5160) + 4(5590) \\ &\quad + 2(5810) + 4(6210) + 2(6890) + 4(7680) + 8270] \\ &\approx \boxed{132,625 \text{ m}^3}.\end{aligned}$$

28. The area is given by the integral $\int_0^{80} y \, dx$. We approximate by the Trapezoidal Rule, with $\Delta x = 10$.

$$\begin{aligned}\int_0^{80} y \, dx &\approx \frac{\Delta x}{2} [f(0) + 2f(10) + 2f(20) + 2f(30) + 2f(40) + \\ &\quad + 2f(50) + 2f(60) + 2f(70) + f(80)] \\ &= \frac{10}{2} [5 + 2(10) + 2(13.2) + 2(15) \\ &\quad + 2(15.6) + 2(12) + 2(6) + 2(4) + 0] \\ &\approx \boxed{783.0 \text{ m}^2}.\end{aligned}$$

29. The volume is given by the integral $\int_2^8 \pi y^2 \, dx$. We approximate by the Trapezoidal Rule, with $\Delta x = 2$.

$$\begin{aligned}\int_2^8 \pi y^2 \, dx &\approx \frac{\Delta x}{2} [f(2) + 2f(4) + 2f(6) + f(8)] \\ &= \frac{2}{2} [\pi(1)^2 + 2\pi(3)^2 + 2\pi(3.5)^2 + \pi(3)^2] \\ &\approx \boxed{164.934}.\end{aligned}$$

30. The distance traveled is given by the integral $\int_0^3 v \, dt$.

(a) We approximate by the Trapezoidal Rule, with $\Delta t = 0.5$.

$$\begin{aligned}\int_0^3 v \, dt &\approx \frac{\Delta t}{2} [f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)] \\ &= \frac{0.5}{2} [5.1 + 2(5.3) + 2(5.6) + 2(6.1) + 2(6.8) + 2(6.7) + 6.5] \\ &= \boxed{18.15 \text{ m}}.\end{aligned}$$

(b) We approximate by Simpson's Rule, with $\Delta t = 0.5$.

$$\begin{aligned}\int_0^3 v \, dt &\approx \frac{\Delta t}{3} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] \\ &= \frac{0.5}{3} [5.1 + 4(5.3) + 2(5.6) + 4(6.1) + 2(6.8) + 4(6.7) + 6.5] \\ &= \boxed{18.13 \text{ m}}.\end{aligned}$$

31. The volume is given by $\int_0^1 \pi (\sin^{-1} y)^2 \, dy$.

(a) With $n = 4$ we have $\Delta y = \frac{1-0}{4} = \frac{1}{4}$. Using the disk method, Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^1 \pi (\sin^{-1} y)^2 \, dy &\approx \frac{\Delta y}{3} \left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1) \right] \\ &= \frac{1/4}{3} \left[\pi (\sin^{-1} 0)^2 + 4 \left(\pi \left(\sin^{-1} \frac{1}{4} \right)^2 \right) + 2 \left(\pi \left(\sin^{-1} \frac{1}{2} \right)^2 \right) \right. \\ &\quad \left. + 4 \left(\pi \left(\sin^{-1} \frac{3}{4} \right)^2 \right) + \pi (\sin^{-1} 1)^2 \right] \\ &\approx \boxed{1.6095}.\end{aligned}$$

Using the shell method, we have $\Delta x = \frac{\pi/2-0}{4} = \frac{\pi}{8}$. Simpson's Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} 2\pi x (1 - \sin x) \, dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/8}{3} \left[2\pi(0)(1 - \sin(0)) + 4 \left(2\pi \left(\frac{\pi}{8} \right) \left(1 - \sin \left(\frac{\pi}{8} \right) \right) \right) + 2 \left(2\pi \left(\frac{\pi}{4} \right) \left(1 - \sin \left(\frac{\pi}{4} \right) \right) \right) \right. \\ &\quad \left. + 4 \left(2\pi \left(\frac{3\pi}{8} \right) \left(1 - \sin \left(\frac{3\pi}{8} \right) \right) \right) + 2\pi \left(\frac{\pi}{2} \right) \left(1 - \sin \left(\frac{\pi}{2} \right) \right) \right] \\ &\approx \boxed{1.4709}.\end{aligned}$$

(b) With $n = 3$ we have $\Delta y = \frac{1-0}{3} = \frac{1}{3}$. Using the disk method, Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^1 \pi (\sin^{-1} y)^2 \, dy &\approx \frac{\Delta y}{2} \left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1) \right] \\ &= \frac{1/3}{2} \left[\pi (\sin^{-1} 0) + 2 \left(\pi \left(\sin^{-1} \frac{1}{3} \right)^2 \right) + 2 \left(\pi \left(\sin^{-1} \frac{2}{3} \right)^2 \right) + \pi (\sin^{-1} 1)^2 \right] \\ &\approx \boxed{1.9705}.\end{aligned}$$

Using the shell method, we have $\Delta x = \frac{\pi/2-0}{3} = \frac{\pi}{6}$. The Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_0^{\pi/2} 2\pi x (1 - \sin x) \, dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/6}{2} \left[2\pi(0)(1 - \sin(0)) + 2 \left(2\pi \left(\frac{\pi}{6} \right) \left(1 - \sin \left(\frac{\pi}{6} \right) \right) \right) \right. \\ &\quad \left. + 2 \left(2\pi \left(\frac{\pi}{3} \right) \left(1 - \sin \left(\frac{\pi}{3} \right) \right) \right) + 2\pi \left(\frac{\pi}{2} \right) \left(1 - \sin \left(\frac{\pi}{2} \right) \right) \right] \\ &\approx \boxed{1.3228}.\end{aligned}$$

32. The arc length is given by $\int_0^8 \sqrt{1 + (dy/dx)^2} dx$. We use implicit differentiation, and obtain

$$\begin{aligned} 9x^2 + 100y^2 &= 900 \\ 18x + 200y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{9x}{100y} \end{aligned}$$

So $\left(\frac{dy}{dx}\right)^2 = \left(-\frac{9x}{100y}\right)^2 = \frac{81x^2}{100(100y^2)} = \frac{81x^2}{100(900-9x^2)} = \frac{9x^2}{100(100-x^2)}$, and

$$\begin{aligned} \int_0^8 \sqrt{1 + (dy/dx)^2} dx &= \int_0^8 \sqrt{1 + \frac{9x^2}{100(100-x^2)}} dx \\ &= \int_0^8 \frac{1}{10} \sqrt{\frac{10000 - 91x^2}{100 - x^2}} dx \end{aligned}$$

The Trapezoidal Rule, with $n = 4$ and $\Delta x = \frac{8-0}{4} = 2$ is used.

$$\begin{aligned} \int_0^8 \sqrt{1 + (dy/dx)^2} dx &\approx \frac{\Delta x}{2} [f(0) + 2f(2) + 2f(4) + 2f(6) + f(8)] \\ &= \frac{2}{2} \left(\frac{1}{10}\right) \left[\sqrt{\frac{10000 - 91(0)^2}{100 - (0)^2}} + 2 \left(\sqrt{\frac{10000 - 91(2)^2}{100 - (2)^2}} \right) \right. \\ &\quad \left. + 2 \left(\sqrt{\frac{10000 - 91(4)^2}{100 - (4)^2}} \right) + 2 \left(\sqrt{\frac{10000 - 91(6)^2}{100 - (6)^2}} \right) + \sqrt{\frac{10000 - 91(8)^2}{100 - (8)^2}} \right] \\ &\approx \boxed{8.148}. \end{aligned}$$

33. (a) With $n = 6$ we have $\Delta x = \frac{\pi-0}{6} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_0^\pi f(x) dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{5\pi}{6}\right) + f(2\pi) \right] \\ &= \frac{\pi/6}{2} \left[1 + 2 \frac{\sin(\pi/6)}{\pi/6} + 2 \frac{\sin(\pi/3)}{\pi/3} + 2 \frac{\sin(\pi/2)}{\pi/2} \right. \\ &\quad \left. + 2 \frac{\sin(2\pi/3)}{2\pi/3} + 2 \frac{\sin(5\pi/6)}{5\pi/6} + \frac{\sin(2\pi)}{2\pi} \right] \\ &\approx \boxed{1.845}. \end{aligned}$$

(b) Simpson's Rule provides the approximation

$$\begin{aligned} \int_0^\pi f(x) dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + 4f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 4f\left(\frac{5\pi}{6}\right) + f(2\pi) \right] \\ &= \frac{\pi/6}{3} \left[1 + 4 \frac{\sin(\pi/6)}{\pi/6} + 2 \frac{\sin(\pi/3)}{\pi/3} + 4 \frac{\sin(\pi/2)}{\pi/2} \right. \\ &\quad \left. + 2 \frac{\sin(2\pi/3)}{2\pi/3} + 4 \frac{\sin(5\pi/6)}{5\pi/6} + \frac{\sin(2\pi)}{2\pi} \right] \\ &\approx \boxed{1.852}. \end{aligned}$$

34. (a) With $n = 20$ we have $\Delta x = \frac{1-(-1)}{20} = \frac{1}{10}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned}\int_{-1}^1 5e^{-x^2} dx &\approx \frac{\Delta x}{2} [f(-1) + 2f(-0.9) + \cdots + 2f(0.9) + f(1)] \\ &= \frac{1/10}{2} \left[5e^{-(-1)^2} + 2\left(5e^{-(-0.9)^2}\right) + \cdots + 2\left(5e^{-(0.9)^2}\right) + 5e^{-(1)^2} \right] \\ &\approx \boxed{7.462}.\end{aligned}$$

- (b) Simpson's Rule provides the approximation

$$\begin{aligned}\int_{-1}^1 5e^{-x^2} dx &\approx \frac{\Delta x}{3} [f(-1) + 4f(-0.9) + 2f(-0.8) + \cdots + 4f(0.9) + f(1)] \\ &= \frac{1/10}{3} \left[5e^{-(-1)^2} + 4\left(5e^{-(-0.9)^2}\right) + 2\left(5e^{-(-0.9)^2}\right) \right. \\ &\quad \left. + \cdots + 4\left(5e^{-(0.9)^2}\right) + 5e^{-(1)^2} \right] \\ &\approx \boxed{7.468}.\end{aligned}$$

- (c) Using a CAS, we obtain $\int_{-1}^1 5e^{-x^2} dx \approx \boxed{7.468}$.

Challenge Problems

35. Since $T_n = \frac{1}{2}(L_n + R_n)$ where L_n is the Riemann Sum using left endpoints, and R_n is the Riemann Sum using right endpoints, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} T_n &= \lim_{n \rightarrow \infty} \frac{1}{2}(L_n + R_n) \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} L_n + \frac{1}{2} \lim_{n \rightarrow \infty} R_n \\ &= \frac{1}{2} \int_a^b f(x) dx + \frac{1}{2} \int_a^b f(x) dx \\ &= \int_a^b f(x) dx.\end{aligned}$$

36. Since $f^{(4)}(x) = \frac{d^4}{dx^4}(Ax^3 + Bx^2 + Cx + D) = 0$, we have $M = 0$, and the error obtained using Simpson's Rule satisfies

$$\text{Error} \leq \frac{(b-a)^5(0)}{180n^4} = 0.$$

So an approximation using Simpson's Rule, with any n , gives the exact value of $\int_a^b f(x) dx$.

7.7: Integration Using Tables and Computer Algebra Systems

Skill Building

1. We use formula 126,

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C,$$

with $a = 2$ and $b = 1$. We obtain

$$\begin{aligned} \int e^{2x} \cos x \, dx &= \frac{e^{2x}}{2^2 + 1^2} (2 \cos x + \sin x) + C \\ &= \boxed{\frac{1}{5} e^{2x} (2 \cos x + \sin x) + C}. \end{aligned}$$

2. First let $u = 2x + 3$, so $x = \frac{u-3}{2}$ and $\frac{1}{2} du = dx$, and substitute to obtain

$$\begin{aligned} \int e^{5x+1} \sin(2x+3) \, dx &= \int e^{5(\frac{u-3}{2})+1} \sin u \, \frac{1}{2} du \\ &= \frac{1}{2} \int e^{\frac{5}{2}u - \frac{13}{2}} \sin u \, du \\ &= \frac{1}{2} e^{-\frac{13}{2}} \int e^{\frac{5}{2}u} \sin u \, du. \end{aligned}$$

We now use formula 125,

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C,$$

with $a = \frac{5}{2}$ and $b = 1$. We obtain

$$\begin{aligned} \frac{1}{2} e^{-\frac{13}{2}} \int e^{\frac{5}{2}u} \sin u \, du &= \frac{1}{2} e^{-\frac{13}{2}} \left(\frac{e^{\frac{5}{2}u}}{\left(\frac{5}{2}\right)^2 + 1^2} \left(\frac{5}{2} \sin u - \cos u \right) \right) + C \\ &= \frac{1}{2} e^{-\frac{13}{2}} \left(\frac{e^{\frac{5}{2}(2x+3)}}{\left(\frac{5}{2}\right)^2 + 1^2} \left(\frac{5}{2} \sin(2x+3) - \cos(2x+3) \right) \right) + C \\ &= \boxed{\frac{2}{29} e^{5x+1} \left(\frac{5}{2} \sin(2x+3) - \cos(2x+3) \right) + C}. \end{aligned}$$

3. We use formula 40,

$$\int x \sqrt{a+bx} \, dx = \frac{2}{15b^2} (3bx - 2a)(a+bx)^{3/2} + C,$$

with $b = 4$ and $a = 3$. We obtain

$$\begin{aligned} \int x \sqrt{4x+3} \, dx &= \int x \sqrt{3+4x} \, dx \\ &= \frac{2}{15(4)^2} (3(4)x - 2(3))(3+4x)^{3/2} + C \\ &= \frac{1}{120} (12x - 6)(3+4x)^{3/2} + C \\ &= \boxed{\frac{1}{20} (2x - 1)(3+4x)^{3/2} + C}. \end{aligned}$$

4. We use formula 61,

$$\int \frac{1}{(x^2 - a^2)^{3/2}} dx = -\frac{x}{\sqrt{x^2 - a^2}} + C,$$

with $a = 1$. We obtain

$$\begin{aligned} \int \frac{1}{(x^2 - 1)^{3/2}} dx &= \int \frac{1}{(x^2 - 1^2)^{3/2}} dx \\ &= -\frac{x}{\sqrt{x^2 - 1^2}} + C \\ &= \boxed{-\frac{x}{\sqrt{x^2 - 1}} + C} \end{aligned}$$

5. First let $u = x + 1$, so $x = u - 1$ and $du = dx$, and substitute to obtain

$$\begin{aligned} \int (x + 1)\sqrt{4x + 5} dx &= \int u\sqrt{4(u - 1) + 5} du \\ &= \int u\sqrt{1 + 4u} du. \end{aligned}$$

We use formula 40,

$$\int x\sqrt{a + bx} dx = \frac{2}{15b^2}(3bx - 2a)(a + bx)^{3/2} + C,$$

with $b = 4$ and $a = 1$. We obtain

$$\begin{aligned} \int (x + 1)\sqrt{4x + 5} dx &= \int u\sqrt{1 + 4u} du \\ &= \frac{2}{15(4)^2}(3(4)u - 2(1))(1 + 4u)^{3/2} + C \\ &= \frac{1}{60}(6u - 1)(1 + 4u)^{3/2} + C \\ &= \frac{1}{60}(6(x + 1) - 1)(1 + 4(x + 1))^{3/2} + C \\ &= \boxed{\frac{1}{60}(6x + 5)(4x + 5)^{3/2} + C}. \end{aligned}$$

6. We let $u = 2x + 3$, so $\frac{1}{2}du = dx$. We substitute to obtain

$$\int \frac{dx}{[(2x + 3)^2 - 1]^{3/2}} = \int \frac{1}{(u^2 - 1)^{3/2}} \frac{1}{2} du.$$

We use formula 61,

$$\int \frac{1}{(x^2 - a^2)^{3/2}} dx = -\frac{x}{\sqrt{x^2 - a^2}} + C,$$

with $a = 1$. We obtain

$$\begin{aligned}
\int \frac{dx}{\left[(2x+3)^2 - 1\right]^{3/2}} &= \frac{1}{2} \int \frac{1}{(u^2 - 1)^{3/2}} du \\
&= \frac{1}{2} \int \frac{1}{(u^2 - 1^2)^{3/2}} du \\
&= -\frac{1}{2} \frac{u}{\sqrt{u^2 - 1^2}} + C \\
&= -\frac{u}{2\sqrt{u^2 - 1}} + C \\
&= \boxed{-\frac{2x+3}{2\sqrt{(2x+3)^2 - 1}} + C}.
\end{aligned}$$

7. We use formula 45,

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right| + C, \quad a > 0,$$

with $a = 6 > 0$, and $b = 4$. We obtain

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4x+6}} &= \int \frac{dx}{x\sqrt{6+4x}} \\
&= \boxed{\frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{6+4x} - \sqrt{6}}{\sqrt{6+4x} + \sqrt{6}} \right| + C}.
\end{aligned}$$

8. We use formula 45,

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right| + C, \quad a > 0,$$

with $a = 8 > 0$, and $b = 1$. We obtain

$$\int \frac{dx}{x\sqrt{8+x}} = \boxed{\frac{1}{\sqrt{8}} \ln \left| \frac{\sqrt{8+x} - \sqrt{8}}{\sqrt{8+x} + \sqrt{8}} \right| + C}.$$

9. We use formula 47,

$$\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}},$$

with $a = 6$, and $b = 4$. We obtain

$$\begin{aligned}
\int \frac{\sqrt{4x+6}dx}{x} &= \int \frac{\sqrt{6+4x}dx}{x} \\
&= 2\sqrt{6+4x} + 6 \int \frac{dx}{x\sqrt{6+4x}}.
\end{aligned}$$

We now use formula 45,

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right| + C, \quad a > 0,$$

with $a = 6 > 0$, and $b = 4$. We obtain

$$\begin{aligned}\int \frac{\sqrt{4x+6}dx}{x} &= 2\sqrt{6+4x} + 6\left(\frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{6+4x}-\sqrt{6}}{\sqrt{6+4x}+\sqrt{6}} \right| \right) + C \\ &= \boxed{2\sqrt{6+4x} + \sqrt{6} \ln \left| \frac{\sqrt{6+4x}-\sqrt{6}}{\sqrt{6+4x}+\sqrt{6}} \right| + C}.\end{aligned}$$

10. We use formula 48,

$$\int \frac{\sqrt{a+bx}}{x^2} dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{a+bx}},$$

with $a = 8$, and $b = 1$. We obtain

$$\int \frac{\sqrt{8+x}}{x^2} dx = -\frac{\sqrt{8+x}}{x} + \frac{1}{2} \int \frac{dx}{x\sqrt{8+x}}.$$

We now use formula 45,

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} \right| + C, \quad a > 0,$$

with $a = 8 > 0$, and $b = 1$. We obtain

$$\begin{aligned}\int \frac{\sqrt{8+x}}{x^2} dx &= -\frac{\sqrt{8+x}}{x} + \frac{1}{2} \left(\frac{1}{\sqrt{8}} \ln \left| \frac{\sqrt{8+x}-\sqrt{8}}{\sqrt{8+x}+\sqrt{8}} \right| \right) + C \\ &= \boxed{-\frac{\sqrt{8+x}}{x} + \frac{\sqrt{2}}{8} \ln \left| \frac{\sqrt{8+x}-2\sqrt{2}}{\sqrt{8+x}+2\sqrt{2}} \right| + C}.\end{aligned}$$

11. We repeatedly use formula 123,

$$\int x^m (\ln x)^n dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

with $m = 3$ and $n = 2$ initially, and obtain

$$\begin{aligned}\int x^3 (\ln x)^2 dx &= \frac{x^{3+1} (\ln x)^2}{3+1} - \frac{2}{3+1} \int x^3 (\ln x)^{2-1} dx \\ &= \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{2} \int x^3 \ln x dx.\end{aligned}$$

And we use formula 123 again, with $m = 3$ and $n = 1$, to obtain

$$\begin{aligned}\int x^3 (\ln x)^2 dx &= \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{2} \left(\frac{x^{3+1} (\ln x)^1}{3+1} - \frac{1}{3+1} \int x^3 (\ln x)^{1-1} dx \right) \\ &= \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{2} \left(\frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx \right) \\ &= \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{8} x^4 \ln x + \frac{1}{8} \left(\frac{1}{4} x^4 \right) + C \\ &= \boxed{\frac{1}{4} x^4 (\ln x)^2 - \frac{1}{8} x^4 \ln x + \frac{1}{32} x^4 + C}.\end{aligned}$$

12. We use formula 123,

$$\int x^m (\ln x)^n dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

with $m = 3$ and $n = 2$, and obtain

$$\begin{aligned} \int x^3 (\ln x)^2 dx &= \frac{x^{3+1} (\ln x)^2}{3+1} - \frac{2}{3+1} \int x^3 (\ln x)^{2-1} dx \\ &= \frac{x^4 (\ln x)^2}{4} - \frac{1}{2} \int x^3 \ln x dx. \end{aligned}$$

We now use formula 120,

$$\int x^n \ln x dx = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$$

with $n = 3$ and obtain

$$\begin{aligned} \int x^3 (\ln x)^2 dx &= \frac{x^4 (\ln x)^2}{4} - \frac{1}{2} \left(\frac{x^{3+1}}{3+1} \left(\ln x - \frac{1}{3+1} \right) \right) + C \\ &= \boxed{\frac{1}{4} x^4 \ln^2 x - \frac{1}{8} x^4 \ln x + \frac{1}{32} x^4 + C}. \end{aligned}$$

13. Let $u = 2x$ to obtain

$$\int \sin^{-1}(2x) dx = \frac{1}{2} \int \sin^{-1} u du,$$

and then by formula 106,

$$\begin{aligned} &= \frac{1}{2} \left(u \sin^{-1} u + \sqrt{1-u^2} \right) + C \\ &= \frac{1}{2} \left((2x) \sin^{-1}(2x) + \sqrt{1-(2x)^2} \right) + C \\ &= \boxed{x \sin^{-1}(2x) + \frac{1}{2} \sqrt{1-4x^2} + C}. \end{aligned}$$

14. Let $u = -3x$ to obtain

$$\int \tan^{-1}(-3x) dx = -\frac{1}{3} \int \tan^{-1} u du,$$

and then by formula 108,

$$\begin{aligned} &= -\frac{1}{3} \left(u \tan^{-1} u - \frac{1}{2} \ln(1+u^2) \right) + C \\ &= -\frac{1}{3} \left((3x) \tan^{-1}(3x) - \frac{1}{2} \ln(1+(3x)^2) \right) + C \\ &= \boxed{-x \tan^{-1}(3x) + \frac{1}{6} \ln(1+9x^2) + C}, \end{aligned}$$

15. We complete the square, and write $3x - x^2 = \frac{9}{4} - (x - \frac{3}{2})^2$. Then substitute $u = x - \frac{3}{2}$, split the integral, and use symmetry to obtain

$$\begin{aligned}
 \int_1^2 \frac{x^3}{\sqrt{3x - x^2}} dx &= \int_1^2 \frac{x^3}{\sqrt{\frac{9}{4} - (x - \frac{3}{2})^2}} dx \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(u + \frac{3}{2})^3}{\sqrt{\frac{9}{4} - u^2}} du \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u^3 + \frac{9}{2}u^2 + \frac{27}{4}u + \frac{27}{8}}{\sqrt{\frac{9}{4} - u^2}} du \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u^3 + \frac{27}{4}u}{\sqrt{\frac{9}{4} - u^2}} du + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{9}{2}u^2 + \frac{27}{8}}{\sqrt{\frac{9}{4} - u^2}} du \\
 &= 0 + 2 \int_0^{\frac{1}{2}} \frac{(-\frac{9}{2})(\frac{9}{4} - u^2) + \frac{27}{2}}{\sqrt{\frac{9}{4} - u^2}} du \\
 &= -9 \int_0^{\frac{1}{2}} \sqrt{\frac{9}{4} - u^2} du + 27 \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{9}{4} - u^2}} du.
 \end{aligned}$$

Now use Formulas 62 and 17 to get

$$\begin{aligned}
 \int_1^2 \frac{x^3}{\sqrt{3x - x^2}} dx &= -9 \left[\frac{u}{2} \sqrt{\frac{9}{4} - u^2} + \frac{9}{2} \sin^{-1} \frac{u}{\frac{3}{2}} \right]_0^{\frac{1}{2}} + 27 \left[\sin^{-1} \frac{u}{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
 &= -9 \left(\frac{\frac{1}{2}}{2} \sqrt{\frac{9}{4} - \left(\frac{1}{2}\right)^2} + \frac{9}{8} \sin^{-1} \frac{2(\frac{1}{2})}{3} - \left(\frac{0}{2} \sqrt{\frac{9}{4} - 0^2} + \frac{9}{4} \sin^{-1} \frac{2(0)}{3} \right) \right) \\
 &\quad + 27 \left(\sin^{-1} \frac{2(\frac{1}{2})}{3} - \sin^{-1} \frac{2(0)}{3} \right) \\
 &= -9 \left(\frac{\sqrt{2}}{4} + \frac{9}{8} \sin^{-1} \frac{1}{3} - 0 \right) + 27 \left(\sin^{-1} \frac{1}{3} - 0 \right) \\
 &= \boxed{\frac{135}{8} \sin^{-1} \frac{1}{3} - \frac{9\sqrt{2}}{4}}.
 \end{aligned}$$

16. We use formula 59,

$$\int \frac{dx}{x^2 \sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C,$$

with $a = \sqrt{2}$. We obtain

$$\begin{aligned}
 \int_1^e \frac{dx}{x^2 \sqrt{x^2 + 2}} &= \int_1^e \frac{dx}{x^2 \sqrt{x^2 + (\sqrt{2})^2}} \\
 &= \left[-\frac{\sqrt{x^2 + (\sqrt{2})^2}}{(\sqrt{2})^2 x} \right]_1^e \\
 &= -\frac{\sqrt{e^2 + 2}}{2e} - \left(-\frac{\sqrt{1^2 + 2}}{2(1)} \right) \\
 &= \boxed{-\frac{\sqrt{e^2 + 2}}{2e} + \frac{\sqrt{3}}{2}}.
 \end{aligned}$$

17. (a) Using a CAS we obtain

$$\int e^{2x} \cos x \, dx = \boxed{\frac{e^{2x}}{5} (\sin x + 2 \cos x) + C}.$$

(b) We obtained in problem 1,

$$\int e^{2x} \sin x \, dx = \boxed{\frac{1}{5} e^{2x} (2 \cos x + \sin x) + C}.$$

(c) Since

$$\frac{e^{2x}}{5} (\sin x + 2 \cos x) = \frac{1}{5} e^{2x} (2 \cos x + \sin x)$$

we see that the results are equivalent.

18. (a) Using a CAS we obtain

$$\int e^{5x+1} \sin(2x+3) \, dx = \boxed{-\frac{1}{29} e^{5x+1} (2 \cos(2x+3) - 5 \sin(2x+3)) + C}.$$

(b) We obtained in problem 2,

$$\int e^{2x} \sin x \, dx = \boxed{\frac{2}{29} e^{5x+1} \left(\frac{5}{2} \sin(2x+3) - \cos(2x+3) \right) + C}.$$

(c) Since

$$\begin{aligned}
 -\frac{1}{29} e^{5x+1} (2 \cos(2x+3) - 5 \sin(2x+3)) &= -\frac{2}{29} e^{5x+1} \left(\cos(2x+3) - \frac{5}{2} \sin(2x+3) \right) \\
 &= \frac{2}{29} e^{5x+1} \left(\frac{5}{2} \sin(2x+3) - \cos(2x+3) \right)
 \end{aligned}$$

we see that the results are equivalent.

19. (a) Using a CAS we obtain

$$\int x \sqrt{4x+3} \, dx = \boxed{\frac{1}{20} (2x-1)(4x+3)^{\frac{3}{2}} + C}.$$

(b) We obtained in problem 3,

$$\int x\sqrt{4x+3} \, dx = \boxed{\frac{1}{20}(2x-1)(3+4x)^{3/2} + C}.$$

(c) Since

$$\frac{1}{20}(2x-1)(4x+3)^{\frac{3}{2}} = \frac{1}{20}(2x-1)(3+4x)^{3/2}$$

we see that the results are equivalent.

20. (a) Using a CAS we obtain

$$\int \frac{1}{(x^2-1)^{3/2}} \, dx = \boxed{-\frac{x}{\sqrt{x^2-1}} + C}.$$

(b) We obtained in problem 4,

$$\int \frac{1}{(x^2-1)^{3/2}} \, dx = \boxed{-\frac{x}{\sqrt{x^2-1}} + C}$$

(c) Since

$$-\frac{x}{\sqrt{x^2-1}} = -\frac{x}{\sqrt{x^2-1}}$$

we see that the results are equivalent.

21. (a) Using a CAS we obtain

$$\int (x+1)\sqrt{4x+5} \, dx = \boxed{\frac{1}{60}(4x+5)^{\frac{3}{2}}(6x+5) + C}.$$

(b) We obtained in problem 5,

$$\int (x+1)\sqrt{4x+5} \, dx = \boxed{\frac{1}{60}(6x+5)(4x+5)^{3/2} + C}.$$

(c) Since

$$\frac{1}{60}(4x+5)^{\frac{3}{2}}(6x+5) = \frac{1}{60}(6x+5)(4x+5)^{3/2}$$

we see that the results are equivalent.

22. (a) Using a CAS we obtain

$$\int \frac{dx}{\left[(2x+3)^2-1\right]^{3/2}} = \boxed{-\frac{1}{8} \frac{4x+6}{\sqrt{x^2+3x+2}} + C}.$$

(b) We obtained in problem 6,

$$\int \frac{dx}{\left[(2x+3)^2-1\right]^{3/2}} = \boxed{-\frac{2x+3}{2\sqrt{(2x+3)^2-1}} + C}.$$

(c) Since

$$\begin{aligned}
-\frac{1}{8} \frac{4x+6}{\sqrt{x^2+3x+2}} &= -\frac{1}{4} \frac{2x+3}{\sqrt{x^2+3x+2}} \\
&= -\frac{2x+3}{2\sqrt{4x^2+12x+8}} \\
&= -\frac{2x+3}{2\sqrt{4(x^2+3x+2)}} \\
&= -\frac{2x+3}{2\sqrt{4x^2+12x+8}} \\
&= -\frac{2x+3}{2\sqrt{4x^2+12x+9-1}} \\
&= -\frac{2x+3}{2\sqrt{(2x+3)^2-1}},
\end{aligned}$$

we see that the results are equivalent.

23. (a) Using a CAS we obtain

$$\int \frac{dx}{x\sqrt{4x+6}} = \boxed{-\frac{1}{3}\sqrt{6} \tanh^{-1}\left(\frac{\sqrt{4x+6}}{6}\sqrt{6}\right) + C}.$$

(b) We obtained in problem 7,

$$\int \frac{dx}{x\sqrt{4x+6}} = \boxed{\frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{6+4x}-\sqrt{6}}{\sqrt{6+4x}+\sqrt{6}} \right| + C}.$$

(c) Using the identity

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

we obtain

$$\begin{aligned}
-\frac{1}{3}\sqrt{6} \tanh^{-1}\left(\frac{\sqrt{4x+6}}{6}\sqrt{6}\right) &= -\frac{1}{3}\sqrt{6} \left(\frac{1}{2} \ln \left| \frac{1 + \frac{\sqrt{4x+6}}{6}\sqrt{6}}{1 - \frac{\sqrt{4x+6}}{6}\sqrt{6}} \right| \right) \\
&= \frac{1}{\sqrt{6}} \ln \left| \frac{1 - \frac{\sqrt{4x+6}}{\sqrt{6}}}{1 + \frac{\sqrt{4x+6}}{\sqrt{6}}} \right| \\
&= \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{6} - \sqrt{4x+6}}{\sqrt{6} + \sqrt{4x+6}} \right| \\
&= \frac{1}{\sqrt{6}} \ln \left| \frac{\sqrt{6+4x} - \sqrt{6}}{\sqrt{6+4x} + \sqrt{6}} \right|
\end{aligned}$$

and we see that the results are equivalent.

24. (a) Using a CAS we obtain

$$\int \frac{dx}{x\sqrt{8+x}} = \boxed{-\frac{1}{2}\sqrt{2} \tanh^{-1}\left(\frac{1}{4}\sqrt{x+8}\sqrt{2}\right) + C}.$$

(b) We obtained in problem 8,

$$\int \frac{dx}{x\sqrt{8+x}} = \boxed{\frac{1}{\sqrt{8}} \ln \left| \frac{\sqrt{8+x}-\sqrt{8}}{\sqrt{8+x}+\sqrt{8}} \right| + C}.$$

(c) Using the identity

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|,$$

we obtain

$$\begin{aligned} -\frac{1}{2}\sqrt{2} \tanh^{-1} \left(\frac{1}{4}\sqrt{x+8}\sqrt{2} \right) &= -\frac{1}{2}\sqrt{2} \left(\frac{1}{2} \ln \left| \frac{1+\frac{1}{4}\sqrt{x+8}\sqrt{2}}{1-\frac{1}{4}\sqrt{x+8}\sqrt{2}} \right| \right) \\ &= \frac{1}{2\sqrt{2}} \ln \left| \frac{1-\frac{1}{4}\sqrt{x+8}\sqrt{2}}{1+\frac{1}{4}\sqrt{x+8}\sqrt{2}} \right| \\ &= \frac{1}{\sqrt{8}} \ln \left| \frac{4/\sqrt{2}-\sqrt{x+8}}{4/\sqrt{2}+\sqrt{x+8}} \right| \\ &= \frac{1}{\sqrt{8}} \ln \left| \frac{\sqrt{8}-\sqrt{x+8}}{\sqrt{8}+\sqrt{x+8}} \right| \\ &= \frac{1}{\sqrt{8}} \ln \left| \frac{\sqrt{8+x}-\sqrt{8}}{\sqrt{8+x}+\sqrt{8}} \right|, \end{aligned}$$

and we see that the results are equivalent.

25. (a) Using a CAS we obtain

$$\int \frac{\sqrt{4x+6}dx}{x} = \boxed{2\sqrt{4x+6} - 2\sqrt{6} \tanh^{-1} \left(\frac{1}{6}\sqrt{4x+6}\sqrt{6} \right) + C}.$$

(b) We obtained in problem 9,

$$\int \frac{\sqrt{4x+6}dx}{x} = \boxed{2\sqrt{6+4x} + \sqrt{6} \ln \left| \frac{\sqrt{6+4x}-\sqrt{6}}{\sqrt{6+4x}+\sqrt{6}} \right| + C}.$$

(c) Using the identity

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|,$$

we obtain

$$\begin{aligned} 2\sqrt{4x+6} - 2\sqrt{6} \tanh^{-1} \left(\frac{1}{6}\sqrt{4x+6}\sqrt{6} \right) &= 2\sqrt{4x+6} - 2\sqrt{6} \left(\frac{1}{2} \ln \left| \frac{1+\frac{1}{6}\sqrt{4x+6}\sqrt{6}}{1-\frac{1}{6}\sqrt{4x+6}\sqrt{6}} \right| \right) \\ &= 2\sqrt{6+4x} + \sqrt{6} \ln \left| \frac{\sqrt{6}-\sqrt{4x+6}}{\sqrt{6}+\sqrt{4x+6}} \right| \\ &= 2\sqrt{6+4x} + \sqrt{6} \ln \left| \frac{\sqrt{6+4x}-\sqrt{6}}{\sqrt{6+4x}+\sqrt{6}} \right| \end{aligned}$$

and we see that the results are equivalent.

26. (a) Using a CAS we obtain

$$\int \frac{\sqrt{8+x}}{x^2} dx = \boxed{-\frac{\sqrt{x+8}}{x} - \frac{1}{4}\sqrt{2} \tanh^{-1}\left(\frac{1}{4}\sqrt{x+8}\sqrt{2}\right) + C}.$$

(b) We obtained in problem 10,

$$\int \frac{\sqrt{8+x}}{x^2} dx = \boxed{-\frac{\sqrt{8+x}}{x} + \frac{\sqrt{2}}{8} \ln \left| \frac{\sqrt{8+x}-2\sqrt{2}}{\sqrt{8+x}+2\sqrt{2}} \right| + C}.$$

(c) Using the identity

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|,$$

we obtain

$$\begin{aligned} -\frac{\sqrt{x+8}}{x} - \frac{1}{4}\sqrt{2} \tanh^{-1}\left(\frac{1}{4}\sqrt{x+8}\sqrt{2}\right) &= -\frac{\sqrt{8+x}}{x} - \frac{1}{4}\sqrt{2} \left(\frac{1}{2} \ln \left| \frac{1 + \frac{1}{4}\sqrt{x+8}\sqrt{2}}{1 - \frac{1}{4}\sqrt{x+8}\sqrt{2}} \right| \right) \\ &= -\frac{\sqrt{8+x}}{x} + \frac{\sqrt{2}}{8} \ln \left| \frac{2\sqrt{2} - \sqrt{x+8}}{2\sqrt{2} + \sqrt{x+8}} \right| \\ &= -\frac{\sqrt{8+x}}{x} + \frac{\sqrt{2}}{8} \ln \left| \frac{\sqrt{x+8} - 2\sqrt{2}}{\sqrt{x+8} + 2\sqrt{2}} \right| \end{aligned}$$

and we see that the results are equivalent.

27. (a) Using a CAS we obtain

$$\int x^3 (\ln x)^2 dx = \boxed{\frac{1}{32}x^4 (8 \ln^2 x - 4 \ln x + 1) + C}.$$

(b) We obtained in problem 11,

$$\int \frac{e^x}{x^3} dx = \boxed{\frac{1}{4}x^4 (\ln x)^2 - \frac{1}{8}x^4 \ln x + \frac{1}{32}x^4 + C}.$$

(c) Since

$$\frac{1}{32}x^4 (8 \ln^2 x - 4 \ln x + 1) = \frac{1}{4}x^4 (\ln x) - \frac{1}{8}x^4 \ln x + \frac{1}{32}x^4,$$

we see that the results are equivalent.

28. (a) Using a CAS we obtain

$$\int x^3 (\ln x)^2 dx = \boxed{\frac{1}{32}x^4 (8 \ln^2 x - 4 \ln x + 1) + C}.$$

(b) We obtained in problem 12,

$$\int x^3 (\ln x)^2 dx = \boxed{\frac{1}{4}x^4 \ln^2 x - \frac{1}{8}x^4 \ln x + \frac{1}{32}x^4 + C}.$$

(c) Since

$$\frac{1}{32}x^4 (8 \ln^2 x - 4 \ln x + 1) = \frac{1}{4}x^4 \ln^2 x - \frac{1}{8}x^4 \ln x + \frac{1}{32}x^4$$

we see that the results are equivalent.

29. (a) Using a CAS we obtain

$$\int \sin^{-1}(2x) dx = \boxed{\frac{1}{2}\sqrt{1-4x^2} + x \sin^{-1}(2x) + C}.$$

(b) We obtained in problem 13,

$$\int \sin^{-1}(2x) dx = \boxed{x \sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C}.$$

(c) Since

$$\frac{1}{2}\sqrt{1-4x^2} + x \sin^{-1}(2x) = x \sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2}$$

we see that the results are equivalent.

30. (a) Using a CAS we obtain

$$\int \tan^{-1}(-3x) dx = \boxed{-x \tan^{-1}(3x) + \frac{1}{6} \ln(9x^2 + 1) + C}.$$

(b) We obtained in problem 14,

$$\int \tan^{-1}(3x) dx = \boxed{-x \tan^{-1}(3x) + \frac{1}{6} \ln(1 + 9x^2) + C}.$$

(c) Since

$$-x \tan^{-1}(3x) + \frac{1}{6} \ln(9x^2 + 1) = -x \tan^{-1}(3x) + \frac{1}{6} \ln(1 + 9x^2)$$

we see that the results are equivalent.

31. (a) Using a CAS we obtain

$$\int_1^2 \frac{x^3}{\sqrt{3x-x^2}} dx = \boxed{\frac{135}{8} \sin^{-1} \frac{1}{3} - \frac{9}{4} \sqrt{2}}.$$

(b) We obtained in problem 15,

$$\int_1^e \frac{dx}{x^2 \sqrt{x^2 + 2}} = \boxed{\frac{135}{8} \sin^{-1} \frac{1}{3} - \frac{9}{4} \sqrt{2}}.$$

(c) Since

$$\frac{135}{8} \sin^{-1} \frac{1}{3} - \frac{9}{4} \sqrt{2} = \frac{135}{8} \sin^{-1} \frac{1}{3} - \frac{9}{4} \sqrt{2}$$

we see that the results are equivalent.

32. (a) Using a CAS we obtain

$$\int_1^e \frac{dx}{x^2 \sqrt{x^2 + 2}} = \boxed{\frac{1}{2} \sqrt{3} - \frac{1}{2} e^{-1} \sqrt{e^2 + 2}}.$$

(b) We obtained in problem 16,

$$\int_1^e \frac{dx}{x^2 \sqrt{x^2 + 2}} = \boxed{-\frac{\sqrt{e^2+2}}{2e} + \frac{\sqrt{3}}{2}}.$$

(c) Since

$$\begin{aligned}\frac{1}{2}\sqrt{3} - \frac{1}{2}e^{-1}\sqrt{e^2+2} &= -\frac{1}{2}e^{-1}\sqrt{e^2+2} + \frac{1}{2}\sqrt{3} \\ &= -\frac{\sqrt{e^2+2}}{2e} + \frac{\sqrt{3}}{2}\end{aligned}$$

we see that the results are equivalent.

33. Using a CAS we obtain

$$\int \sqrt{1+x^3} \, dx = \int \sqrt{x^3+1} \, dx$$

and we conclude that this integral cannot be expressed using elementary functions.

34. Using a CAS we obtain

$$\int \sqrt{1+\sin x} \, dx = \frac{2}{\cos x}(\sin x - 1)\sqrt{\sin x + 1} + C$$

and we conclude that this integral can be expressed using elementary functions.

35. Using a CAS we obtain

$$\int e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi} \operatorname{erf}(x) + C$$

and we conclude that this integral cannot be expressed using elementary functions.

36. Using a CAS we obtain

$$\int \frac{\cos x}{x} \, dx = \operatorname{Ci}(x) + C$$

and we conclude that this integral cannot be expressed using elementary functions.

37. Using a CAS we obtain

$$\int x \tan x \, dx = \frac{1}{2}ix^2 - i \operatorname{dilog}(ie^{ix}) - i\pi \tan^{-1}(e^{ix}) - i \operatorname{dilog}(-ie^{ix}) + C$$

and we conclude that this integral cannot be expressed using elementary functions.

38. Using a CAS we obtain

$$\int \sqrt{1+e^x} \, dx = \boxed{2\sqrt{e^x+1} + \ln(\sqrt{e^x+1}-1) - \ln(\sqrt{e^x+1}+1) + C}.$$

7.8: Improper Integrals

Concepts and Vocabulary

1. (c), an improper
2. (b) diverges
3. False
4. True
5. False
6. $\lim_{t \rightarrow b^-} \int_a^t f(x) dx$

Skill Building

7. The integral is improper, since upper limit of integration is ∞ .
8. The integral is not improper, since the function $f(x) = x^3$ is continuous on the closed interval $[0, 5]$.
9. The integral is not improper, since the function $f(x) = \frac{1}{x-1}$ is continuous on the closed interval $[2, 3]$.
10. The integral is improper, since the function $f(x) = \frac{1}{x-1}$ is undefined, at the endpoint $x = 1$.
11. The integral is improper, since the function $f(x) = \frac{1}{x}$ is undefined, at the endpoint $x = 0$.
12. The integral is not improper, since the function $f(x) = \frac{1}{x^2+1}$ is continuous on the closed interval $[-1, 1]$.
13. The integral is improper, since the function $f(x) = \frac{x}{x^2-1}$ is undefined, at the endpoint $x = 1$.
14. The integral is improper, since the upper limit of integration is ∞ .
15. We evaluate

$$\begin{aligned} \int_1^\infty \frac{dx}{x^3} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^3} \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2b^2} - \left(-\frac{1}{2(1)^2} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2b^2} \right] \\ &= \frac{1}{2}. \end{aligned}$$

The improper integral converges to $\frac{1}{2}$.

16. We evaluate

$$\begin{aligned}\int_{-\infty}^{-10} \frac{dx}{x^2} &= \lim_{a \rightarrow -\infty} \int_a^{-10} \frac{dx}{x^2} \\&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{x} \right]_a^{-10} \\&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{-10} - \left(-\frac{1}{a} \right) \right] \\&= \lim_{a \rightarrow -\infty} \left[\frac{1}{a} + \frac{1}{10} \right] \\&= \frac{1}{10}.\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{10}}$.

17. We evaluate

$$\begin{aligned}\int_0^{\infty} e^{2x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{2x} dx \\&= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{2x} \right]_0^b \\&= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{2(b)} - \frac{1}{2} e^{2(0)} \right] \\&= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{2b} - \frac{1}{2} \right] \\&= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

18. We evaluate

$$\begin{aligned}\int_0^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\&= \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\&= \lim_{b \rightarrow \infty} [-e^{-b} - (-e^{-0})] \\&= \lim_{b \rightarrow \infty} [1 - e^{-b}] \\&= 1.\end{aligned}$$

The improper integral converges to $\boxed{1}$.

19. We evaluate

$$\begin{aligned}\int_{-\infty}^{-1} \frac{4}{x} dx &= \lim_{a \rightarrow -\infty} \int_a^{-1} \frac{4}{x} dx \\&= \lim_{a \rightarrow -\infty} [4 \ln |x|]_a^{-1} \\&= \lim_{a \rightarrow -\infty} [4 \ln |-1| - 4 \ln |a|] \\&= \lim_{a \rightarrow -\infty} [-4 \ln |a|] \\&= -\infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

20. We evaluate

$$\begin{aligned}\int_1^\infty \frac{4}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{4}{x} dx \\&= \lim_{b \rightarrow \infty} [4 \ln |x|]_1^b \\&= \lim_{b \rightarrow \infty} [4 \ln |b| - 4 \ln |1|] \\&= \lim_{b \rightarrow \infty} [4 \ln |b|] \\&= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

21. We evaluate

$$\begin{aligned}\int_3^\infty \frac{dx}{(x-1)^4} &= \lim_{b \rightarrow \infty} \int_3^b \frac{dx}{(x-1)^4} \\&= \lim_{b \rightarrow \infty} \left[-\frac{1}{3(x-1)^3} \right]_3^b \\&= \lim_{b \rightarrow \infty} \left[-\frac{1}{3(b-1)^3} - \left(-\frac{1}{3(3-1)^3} \right) \right] \\&= \lim_{b \rightarrow \infty} \left[\frac{1}{24} - \frac{1}{3(b-1)^3} \right] \\&= \frac{1}{24}.\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{24}}$.

22. We evaluate

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{(x-1)^4} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{(x-1)^4} \\&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{3(x-1)^3} \right]_a^0 \\&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{3(0-1)^3} - \left(-\frac{1}{3(a-1)^3} \right) \right] \\&= \lim_{a \rightarrow -\infty} \left[\frac{1}{3(a-1)^3} + \frac{1}{3} \right] \\&= \frac{1}{3}.\end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{3}}$.

23. We evaluate

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{x^2 + 4} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + 4} \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_a^0 + \lim_{b \rightarrow \infty} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^b \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1} \frac{0}{2} - \frac{1}{2} \tan^{-1} \frac{a}{2} \right] + \lim_{b \rightarrow \infty} \left[\frac{1}{2} \tan^{-1} \frac{b}{2} - \frac{1}{2} \tan^{-1} \frac{0}{2} \right] \\
 &= \lim_{a \rightarrow -\infty} \left[-\frac{1}{2} \tan^{-1} \frac{a}{2} \right] + \lim_{b \rightarrow \infty} \left[\frac{1}{2} \tan^{-1} \frac{b}{2} \right] \\
 &= \frac{-1}{2} \left(-\frac{\pi}{2} \right) + \frac{1}{2} \left(\frac{\pi}{2} \right) \\
 &= \frac{\pi}{2}
 \end{aligned}$$

The improper integral converges to $\boxed{\frac{\pi}{2}}$.

24. We evaluate

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{x^2 + 1} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + 1} \\
 &= \lim_{a \rightarrow -\infty} [\tan^{-1} x]_a^0 + \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b \\
 &= \lim_{a \rightarrow -\infty} [\tan^{-1} 0 - \tan^{-1} a] + \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 0] \\
 &= \lim_{a \rightarrow -\infty} [-\tan^{-1} a] + \lim_{b \rightarrow \infty} [\tan^{-1} b] \\
 &= -\left(-\frac{\pi}{2} \right) + \left(\frac{\pi}{2} \right) \\
 &= \pi.
 \end{aligned}$$

The improper integral converges to $\boxed{\pi}$.

25. We evaluate

$$\begin{aligned}
 \int_0^1 \frac{dx}{x^2} &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x^2} \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{x} \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{a} \right) \right] \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{1}{a} - 1 \right] \\
 &= \infty.
 \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

26. We evaluate

$$\begin{aligned}\int_0^1 \frac{dx}{x^3} &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x^3} \\&= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2x^2} \right]_a^1 \\&= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2(1)^2} - \left(-\frac{1}{2a^2} \right) \right] \\&= \lim_{a \rightarrow 0^+} \left[\frac{1}{2a^2} - \frac{1}{2} \right] \\&= \infty.\end{aligned}$$

The improper integral diverges.

27. We evaluate

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x} \\&= \lim_{a \rightarrow 0^+} [\ln |x|]_a^1 \\&= \lim_{a \rightarrow 0^+} [\ln |1| - \ln |a|] \\&= \lim_{a \rightarrow 0^+} [-\ln |a|] \\&= \infty.\end{aligned}$$

The improper integral diverges.

28. We evaluate

$$\begin{aligned}\int_4^6 \frac{dx}{x-4} &= \lim_{a \rightarrow 4^+} \int_a^6 \frac{dx}{x-4} \\&= \lim_{a \rightarrow 4^+} [\ln |x-4|]_a^6 \\&= \lim_{a \rightarrow 4^+} [\ln |6-4| - \ln |a-4|] \\&= \lim_{a \rightarrow 4^+} [\ln 2 - \ln |a-4|] \\&= \infty.\end{aligned}$$

The improper integral diverges.

29. We evaluate

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{4-x}} &= \lim_{b \rightarrow 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} \\&= \lim_{b \rightarrow 4^-} [-2\sqrt{4-x}]_0^b \\&= \lim_{b \rightarrow 4^-} \left[-2\sqrt{4-b} - (-2\sqrt{4-0}) \right] \\&= \lim_{b \rightarrow 4^-} [4 - 2\sqrt{4-b}] \\&= 4.\end{aligned}$$

The improper integral converges to 4.

30. Let $u = 5 - x$ and substitute to obtain

$$\begin{aligned}
 \int_1^5 \frac{x \, dx}{\sqrt{5-x}} &= \int_4^0 \frac{(5-u)(-du)}{\sqrt{u}} \\
 &= \int_0^4 (5u^{-1/2} - u^{1/2}) \, du \\
 &= \lim_{a \rightarrow 0^+} \int_a^4 (5u^{-1/2} - u^{1/2}) \, du \\
 &= \lim_{a \rightarrow 0^+} \left[10\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}} \right]_a^4 \\
 &= \lim_{a \rightarrow 0^+} \left[10\sqrt{4} - \frac{2}{3}(4)^{\frac{3}{2}} - \left(10\sqrt{a} - \frac{2}{3}a^{\frac{3}{2}} \right) \right] \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{2}{3}a^{\frac{3}{2}} - 10\sqrt{a} + \frac{44}{3} \right] \\
 &= \frac{44}{3}.
 \end{aligned}$$

The improper integral converges to $\boxed{\frac{44}{3}}$.

31. We split the integral into two improper integrals:

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}}.$$

We consider

$$\begin{aligned}
 \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{dx}{\sqrt[3]{x}} \\
 &= \lim_{b \rightarrow 0^-} \left[\frac{3}{2}x^{\frac{2}{3}} \right]_{-1}^b \\
 &= \lim_{b \rightarrow 0^-} \left[\frac{3}{2}b^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}} \right] \\
 &= \lim_{b \rightarrow 0^-} \left[\frac{3}{2}b^{\frac{2}{3}} - \frac{3}{2} \right] \\
 &= -\frac{3}{2}.
 \end{aligned}$$

And

$$\begin{aligned}
 \int_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt[3]{x}} \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{3}{2}x^{\frac{2}{3}} \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}(a)^{\frac{2}{3}} \right] \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{3}{2} - \frac{3}{2}a^{\frac{2}{3}} \right] \\
 &= \frac{3}{2}.
 \end{aligned}$$

We conclude that the improper integral $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ converges to $-\frac{3}{2} + \frac{3}{2} = \boxed{0}$.

32. We split the integral into two improper integrals:

$$\int_0^3 \frac{dx}{(x-2)^2} = \int_0^2 \frac{dx}{(x-2)^2} + \int_2^3 \frac{dx}{(x-2)^2}.$$

We consider

$$\begin{aligned} \int_0^2 \frac{dx}{(x-2)^2} &= \lim_{b \rightarrow 2^-} \int_0^b \frac{dx}{(x-2)^2} \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{1}{x-2} \right]_0^b \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{1}{b-2} - \left(-\frac{1}{0-2} \right) \right] \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{1}{b-2} - \frac{1}{2} \right] \\ &= \infty. \end{aligned}$$

We conclude that the improper integral $\int_0^3 \frac{dx}{(x-2)^2}$ diverges.

33. We evaluate

$$\begin{aligned} \int_0^\infty \cos x \, dx &= \lim_{b \rightarrow \infty} \int_0^b \cos x \, dx \\ &= \lim_{b \rightarrow \infty} [\sin x]_0^b \\ &= \lim_{b \rightarrow \infty} [\sin b - \sin 0] \\ &= \lim_{b \rightarrow \infty} [\sin b]. \end{aligned}$$

Since this limit does not exist, the improper integral diverges.

34. We evaluate

$$\begin{aligned} \int_0^\infty \sin(\pi x) \, dx &= \lim_{b \rightarrow \infty} \int_0^b \sin(\pi x) \, dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{\pi} \cos(\pi b) - \left(-\frac{1}{\pi} \cos(\pi(0)) \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{\pi} - \frac{1}{\pi} \cos \pi b \right]. \end{aligned}$$

Since this limit does not exist, the improper integral diverges.

35. We evaluate

$$\begin{aligned} \int_{-\infty}^0 e^x \, dx &= \lim_{a \rightarrow -\infty} \int_a^0 e^x \, dx \\ &= \lim_{a \rightarrow -\infty} [e^x]_a^0 \\ &= \lim_{a \rightarrow -\infty} [e^0 - e^a] \\ &= \lim_{a \rightarrow -\infty} [1 - e^a] \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

The improper integral converges to $\boxed{1}$.

36. We evaluate

$$\begin{aligned}
 \int_{-\infty}^0 e^{-x} dx &= \lim_{a \rightarrow -\infty} \int_a^0 e^{-x} dx \\
 &= \lim_{a \rightarrow -\infty} [-e^{-x}]_a^0 \\
 &= \lim_{a \rightarrow -\infty} [-e^{-0} - (-e^{-a})] \\
 &= \lim_{a \rightarrow -\infty} [-1 + e^{-a}] \\
 &= \infty.
 \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

37. Let $u = x^2$ and substitute

$$\begin{aligned}
 \int_0^{\pi/2} \frac{x dx}{\sin x^2} &= \int_0^{\pi^2/4} \frac{\frac{1}{2} du}{\sin u} \\
 &= \frac{1}{2} \int_0^{\pi^2/4} \csc u du \\
 &= \lim_{a \rightarrow 0^+} \frac{1}{2} \int_a^{\pi^2/4} \csc u du \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2} \ln |\csc u + \cot u| \right]_a^{\pi^2/4} \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2} \ln \left| \csc \frac{\pi^2}{4} + \cot \frac{\pi^2}{4} \right| + \frac{1}{2} \ln |\csc a + \cot a| \right] \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2} \ln \left| \csc \frac{\pi^2}{4} + \cot \frac{\pi^2}{4} \right| + \frac{1}{2} \ln |\csc a + \cot a| \right] \\
 &= \infty
 \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

38. We evaluate

$$\begin{aligned}
 \int_0^1 \frac{\ln x dx}{x} &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{\ln x dx}{x} \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{1}{2} \ln^2 x \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left[\frac{1}{2} \ln^2 1 - \frac{1}{2} \ln^2 a \right] \\
 &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2} \ln^2 a \right] \\
 &= -\infty
 \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

39. We evaluate

$$\begin{aligned}
 \int_0^1 \frac{dx}{1-x^2} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(1-x)(1+x)} \\
 &= \lim_{b \rightarrow 1^-} \int_0^b \left(\frac{1}{2(x+1)} - \frac{1}{2(x-1)} \right) dx \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln(x+1) - \frac{1}{2} \ln|x-1| \right]_0^b \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln(b+1) - \frac{1}{2} \ln|b-1| - \left(\frac{1}{2} \ln(0+1) - \frac{1}{2} \ln|0-1| \right) \right] \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln(b+1) - \frac{1}{2} \ln|b-1| \right] \\
 &= \infty
 \end{aligned}$$

The improper integral diverges.

40. We evaluate

$$\begin{aligned}
 \int_1^2 \frac{dx}{\sqrt{x^2-1}} &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{\sqrt{x^2-1}} \\
 &= \lim_{a \rightarrow 1^+} \left[\ln \left| x + \sqrt{x^2-1} \right| \right]_a^2 \quad (\text{by Formula 55}) \\
 &= \lim_{a \rightarrow 1^+} \left[\ln \left| 2 + \sqrt{2^2-1} \right| - \ln \left| a + \sqrt{a^2-1} \right| \right] \\
 &= \lim_{a \rightarrow 1^+} \left[\ln(\sqrt{3}+2) - \ln \left| a + \sqrt{a^2-1} \right| \right] \\
 &= \ln(\sqrt{3}+2).
 \end{aligned}$$

The improper integral converges to $\ln(\sqrt{3}+2)$.

41. We evaluate the improper integral, and use a partial fraction decomposition.

$$\begin{aligned}
 \int_0^1 \frac{dx}{(1-x^2)^2} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(1-x^2)^2} \\
 &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(1-x)^2(1+x)^2} \\
 &= \lim_{b \rightarrow 1^-} \int_0^b \left(\frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2} \right) dx \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{1}{4} \ln|x+1| - \frac{1}{4} \ln|x-1| - \frac{1}{4(x-1)} - \frac{1}{4(x+1)} \right]_0^b \\
 &= \lim_{b \rightarrow 1^-} \left(\frac{1}{4} \ln|b+1| - \frac{1}{4} \ln|b-1| - \frac{1}{4b-4} - \frac{1}{4b+4} \right) \\
 &= \infty.
 \end{aligned}$$

The improper integral diverges.

42. We split the integral into two improper integrals:

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}.$$

We consider

$$\begin{aligned}
 \int_0^1 \frac{dx}{(x-1)^2} &= \lim_{t \rightarrow 1^-} \int_0^t \frac{dx}{(x-1)^2} \\
 &= \lim_{t \rightarrow 1^-} \left[\frac{1}{1-x} \right]_0^t \\
 &= \lim_{t \rightarrow 1^-} \left(\frac{1}{1-t} - \frac{1}{1} \right) \\
 &= \infty.
 \end{aligned}$$

We conclude that the improper integral $\int_0^2 \frac{dx}{(x-1)^2}$ diverges.

43. We evaluate

$$\begin{aligned}
 \int_0^{\pi/4} \tan(2x) dx &= \lim_{b \rightarrow \frac{\pi}{4}^-} \int_0^b \tan(2x) dx \\
 &= \lim_{b \rightarrow \frac{\pi}{4}^-} \left[\frac{1}{2} \ln |\sec 2x| \right]_0^b \\
 &= \lim_{b \rightarrow \frac{\pi}{4}^-} \left(\frac{1}{2} \ln |\sec 2b| \right) \\
 &= \infty
 \end{aligned}$$

The improper integral diverges.

44. We evaluate

$$\begin{aligned}
 \int_0^{\pi/2} \csc x dx &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \csc x dx \\
 &= \lim_{a \rightarrow 0^+} [-\ln |\csc x + \cot x|]_a^{\pi/2} \\
 &= \lim_{a \rightarrow 0^+} (\ln |\csc a + \cot a|) \\
 &= \infty
 \end{aligned}$$

The improper integral diverges.

45. We evaluate

$$\int_0^\infty \frac{x dx}{\sqrt{x+1}} = \lim_{b \rightarrow \infty} \int_0^b \frac{x dx}{\sqrt{x+1}}.$$

Let $u = x + 1$, and substitute to obtain

$$\begin{aligned}
 \int_0^\infty \frac{x dx}{\sqrt{x+1}} &= \lim_{b \rightarrow \infty} \int_1^{b+1} \frac{u-1}{\sqrt{u}} du \\
 &= \lim_{b \rightarrow \infty} \int_1^{b+1} (u^{1/2} - u^{-1/2}) du \\
 &= \lim_{b \rightarrow \infty} \left[\frac{2}{3} u^{3/2} - 2\sqrt{u} \right]_1^{b+1} \\
 &= \lim_{b \rightarrow \infty} \left[\frac{2}{3} (b+1)^{3/2} - 2\sqrt{b+1} - \left(\frac{2}{3} (1+1)^{3/2} - 2\sqrt{1+1} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\sqrt{b+1} \left(\frac{2}{3} (b+1) - 2 \right) + \frac{2}{3} \sqrt{2} \right] \\
 &= \infty.
 \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

46. We evaluate

$$\begin{aligned}\int_2^\infty \frac{dx}{x\sqrt{x^2-1}} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}} \\ &= \lim_{b \rightarrow \infty} [\sec^{-1} x]_2^b \\ &= \lim_{b \rightarrow \infty} (\sec^{-1} b - \sec^{-1} 2) \\ &= \frac{\pi}{2} - \frac{\pi}{3} \\ &= \frac{\pi}{6}.\end{aligned}$$

The improper integral converges to $\boxed{\frac{\pi}{6}}$.

47. We split the integral into two improper integrals:

$$\int_{-\infty}^\infty \frac{dx}{x^2+4x+5} = \int_{-\infty}^0 \frac{dx}{x^2+4x+5} + \int_0^\infty \frac{dx}{x^2+4x+5}.$$

We consider

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+4x+5} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2+4x+5} \\ &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x+2)^2+1} \\ &= \lim_{b \rightarrow \infty} [\tan^{-1}(x+2)]_0^b \\ &= \lim_{b \rightarrow \infty} [\tan^{-1}(b+2) - \tan^{-1}(0+2)] \\ &= \frac{\pi}{2} - \tan^{-1} 2.\end{aligned}$$

We now determine

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{x^2+4x+5} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{x^2+4x+5} \\ &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{(x+2)^2+1} \\ &= \lim_{a \rightarrow -\infty} [\tan^{-1}(x+2)]_a^0 \\ &= \lim_{a \rightarrow -\infty} [\tan^{-1}(0+2) - \tan^{-1}(a+2)] \\ &= \tan^{-1} 2 - \left(-\frac{\pi}{2}\right) \\ &= \tan^{-1} 2 + \frac{\pi}{2}.\end{aligned}$$

We conclude that the improper integral converges, with $\int_{-\infty}^\infty \frac{dx}{x^2+4x+5} = \left(\frac{\pi}{2} - \tan^{-1} 2\right) + \left(\tan^{-1} 2 + \frac{\pi}{2}\right) = \boxed{\pi}$.

48. We split the integral into two improper integrals:

$$\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^0 \frac{dx}{e^x + e^{-x}} + \int_0^\infty \frac{dx}{e^x + e^{-x}}.$$

We consider

$$\begin{aligned}\int_0^\infty \frac{dx}{e^x + e^{-x}} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{e^x + e^{-x}} \\ &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x dx}{e^{2x} + 1}\end{aligned}$$

Let $u = e^x$, and substitute to obtain

$$\begin{aligned}\int_0^\infty \frac{dx}{e^x + e^{-x}} &= \lim_{b \rightarrow \infty} \int_1^{e^b} \frac{du}{u^2 + 1} \\ &= \lim_{b \rightarrow \infty} [\tan^{-1} u]_1^{e^b} \\ &= \lim_{b \rightarrow \infty} [\tan^{-1} e^b - \tan^{-1} 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4}.\end{aligned}$$

We now determine

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{e^x + e^{-x}} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{e^x + e^{-x}} \\ &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{e^x dx}{e^{2x} + 1}\end{aligned}$$

Let $u = e^x$, and substitute to obtain

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{e^x + e^{-x}} &= \lim_{a \rightarrow -\infty} \int_{e^a}^1 \frac{du}{u^2 + 1} \\ &= \lim_{a \rightarrow -\infty} [\tan^{-1} u]_{e^a}^1 \\ &= \lim_{a \rightarrow -\infty} [\tan^{-1} 1 - \tan^{-1} e^a] \\ &= \frac{\pi}{4} - (0) \\ &= \frac{\pi}{4}.\end{aligned}$$

We conclude that the improper integral converges, with $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = \left(\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$.

49. We evaluate

$$\begin{aligned}\int_{-\infty}^2 \frac{dx}{\sqrt{4-x}} &= \lim_{a \rightarrow -\infty} \int_a^2 \frac{dx}{\sqrt{4-x}} \\ &= \lim_{a \rightarrow -\infty} [-2\sqrt{4-x}]_a^2 \\ &= \lim_{a \rightarrow -\infty} [-2\sqrt{4-2} - (-2\sqrt{4-a})] \\ &= \lim_{a \rightarrow -\infty} [2\sqrt{4-a} - 2\sqrt{2}] \\ &= \infty.\end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

50. We evaluate

$$\int_{-\infty}^1 \frac{x \, dx}{\sqrt{2-x}} = \lim_{a \rightarrow -\infty} \int_a^1 \frac{x \, dx}{\sqrt{2-x}}$$

Let $u = 2 - x$ and substitute to obtain

$$\begin{aligned} \int_{-\infty}^1 \frac{x \, dx}{\sqrt{2-x}} &= \lim_{a \rightarrow -\infty} \int_{2-a}^1 \frac{(2-u)}{\sqrt{u}} (-du) \\ &= - \lim_{a \rightarrow -\infty} \int_{2-a}^1 \left(2u^{-1/2} - u^{1/2} \right) du \\ &= - \lim_{a \rightarrow -\infty} \left[4\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}} \right]_{2-a}^1 \\ &= - \lim_{a \rightarrow -\infty} \left[\left(4\sqrt{1} - \frac{2}{3}(1)^{\frac{3}{2}} \right) - \left(4\sqrt{2-a} - \frac{2}{3}(2-a)^{\frac{3}{2}} \right) \right] \\ &= - \lim_{a \rightarrow -\infty} \left(\frac{2}{3}(2-a)^{\frac{3}{2}} - 4\sqrt{2-a} + \frac{10}{3} \right) \\ &= - \lim_{a \rightarrow -\infty} \left(\sqrt{2-a} \left(\frac{2}{3}(2-a) - 4 \right) + \frac{10}{3} \right) \\ &= -\infty. \end{aligned}$$

The improper integral diverges.

51. We evaluate

$$\begin{aligned} \int_2^4 \frac{2x \, dx}{\sqrt[3]{x^2-4}} &= \lim_{a \rightarrow 2^+} \int_a^4 \frac{2x \, dx}{\sqrt[3]{x^2-4}} \\ &= \lim_{a \rightarrow 2^+} \left[\frac{3}{2}(x^2-4)^{2/3} \right]_a^4 \\ &= \lim_{a \rightarrow 2^+} \left[\frac{3}{2}(4^2-4)^{2/3} - \frac{3}{2}(a^2-4)^{2/3} \right] \\ &= \lim_{a \rightarrow 2^+} \left[3(18)^{1/3} - \frac{3}{2}(a^2-4)^{2/3} \right] \\ &= \boxed{3\sqrt[3]{18}}. \end{aligned}$$

The improper integral converges to $3\sqrt[3]{18}$.

52. We evaluate

$$\begin{aligned} \int_0^\pi \frac{1}{1-\cos x} \, dx &= \lim_{a \rightarrow 0^+} \int_a^\pi \frac{1}{1-\cos x} \, dx \\ &= \lim_{a \rightarrow 0^+} \int_a^\pi \frac{1+\cos x}{\sin^2 x} \, dx \\ &= \lim_{a \rightarrow 0^+} \int_a^\pi (\csc^2 x + \csc x \cot x) \, dx \end{aligned}$$

$$\begin{aligned}
&= \lim_{a \rightarrow 0^+} [-\cot x - \csc x]_a^\pi \\
&= \lim_{a \rightarrow 0^+} \left[-\frac{\cos x + 1}{\sin x} \right]_a^\pi \\
&= \lim_{a \rightarrow 0^+} \left[-\frac{(\cos x + 1)(\cos x - 1)}{\sin x(\cos x - 1)} \right]_a^\pi \\
&= \lim_{a \rightarrow 0^+} \left[\frac{\sin^2 x}{\sin x(\cos x - 1)} \right]_a^\pi \\
&= \lim_{a \rightarrow 0^+} \left[\frac{\sin x}{\cos x - 1} \right]_a^\pi \\
&= \lim_{a \rightarrow 0^+} \left[\frac{\sin \pi}{\cos \pi - 1} - \left(\frac{\sin a}{\cos a - 1} \right) \right] \\
&= \lim_{a \rightarrow 0^+} \left[-\frac{\sin a}{\cos a - 1} \right] \\
&= \lim_{a \rightarrow 0^+} \left[\frac{\cos a + 1}{\sin a} \right] \\
&= \infty.
\end{aligned}$$

The improper integral diverges.

53. We split the integral into two improper integrals:

$$\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3}.$$

We consider

$$\begin{aligned}
\int_{-1}^0 \frac{dx}{x^3} &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{dx}{x^3} \\
&= \lim_{b \rightarrow 0^-} \left[-\frac{1}{2x^2} \right]_{-1}^b \\
&= \lim_{b \rightarrow 0^-} \left[-\frac{1}{2b^2} - \left(-\frac{1}{2(-1)^2} \right) \right] \\
&= \lim_{b \rightarrow 0^-} \left[\frac{1}{2} - \frac{1}{2b^2} \right] \\
&= -\infty.
\end{aligned}$$

We conclude that the improper integral $\int_{-1}^1 \frac{dx}{x^3}$ diverges.

54. We split the integral into two improper integrals:

$$\int_0^2 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^2 \frac{dx}{x-1}.$$

We consider

$$\begin{aligned}
\int_0^1 \frac{dx}{x-1} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{x-1} \\
&= \lim_{b \rightarrow 1^-} [\ln |x-1|]_0^b \\
&= \lim_{b \rightarrow 1^-} [\ln |b-1| - \ln |0-1|] \\
&= \lim_{b \rightarrow 1^-} [\ln |b-1|] \\
&= -\infty.
\end{aligned}$$

We conclude that the improper integral $\int_0^2 \frac{dx}{x-1}$ diverges.

55. We split the integral into two improper integrals:

$$\int_0^2 \frac{dx}{(x-1)^{1/3}} = \int_0^1 \frac{dx}{(x-1)^{1/3}} + \int_1^2 \frac{dx}{(x-1)^{1/3}}.$$

We consider

$$\begin{aligned}
\int_0^1 \frac{dx}{(x-1)^{1/3}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x-1)^{1/3}} \\
&= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (x-1)^{\frac{2}{3}} \right]_0^b \\
&= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (b-1)^{\frac{2}{3}} - \frac{3}{2} (0-1)^{\frac{2}{3}} \right] \\
&= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (b-1)^{\frac{2}{3}} - \frac{3}{2} \right] \\
&= -\frac{3}{2}.
\end{aligned}$$

And

$$\begin{aligned}
\int_1^2 \frac{dx}{(x-1)^{1/3}} &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{(x-1)^{1/3}} \\
&= \lim_{a \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{\frac{2}{3}} \right]_a^2 \\
&= \lim_{a \rightarrow 1^+} \left[\frac{3}{2} (2-1)^{\frac{2}{3}} - \frac{3}{2} (a-1)^{\frac{2}{3}} \right] \\
&= \lim_{a \rightarrow 1^+} \left[\frac{3}{2} - \frac{3}{2} (a-1)^{\frac{2}{3}} \right] \\
&= \frac{3}{2}.
\end{aligned}$$

We conclude that the improper integral $\int_0^2 \frac{dx}{(x-1)^{1/3}}$ converges to $-\frac{3}{2} + \frac{3}{2} = \boxed{0}$.

56. We split the integral into two improper integrals:

$$\int_{-1}^1 \frac{dx}{x^{5/3}} = \int_{-1}^0 \frac{dx}{x^{5/3}} + \int_0^1 \frac{dx}{x^{5/3}}.$$

We consider

$$\begin{aligned}
 \int_{-1}^0 \frac{dx}{x^{5/3}} &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{dx}{x^{5/3}} \\
 &= \lim_{b \rightarrow 0^-} \left[-\frac{3}{2x^{2/3}} \right]_{-1}^b \\
 &= \lim_{b \rightarrow 0^-} \left[-\frac{3}{2b^{2/3}} - \left(-\frac{3}{2(-1)^{2/3}} \right) \right] \\
 &= \lim_{b \rightarrow 0^-} \left[\frac{3}{2} - \frac{3}{2b^{2/3}} \right] \\
 &= -\infty.
 \end{aligned}$$

We conclude that the improper integral $\int_{-1}^1 \frac{dx}{x^{5/3}}$ diverges.

57. We evaluate

$$\begin{aligned}
 \int_1^2 \frac{dx}{(2-x)^{3/4}} &= \lim_{b \rightarrow 2^-} \int_1^b \frac{dx}{(2-x)^{3/4}} \\
 &= \lim_{b \rightarrow 2^-} \left[-4(2-x)^{1/4} \right]_1^b \\
 &= \lim_{b \rightarrow 2^-} \left[-4(2-b)^{1/4} - \left(-4(2-1)^{1/4} \right) \right] \\
 &= \lim_{b \rightarrow 2^-} \left[4 - 4\sqrt[4]{2-b} \right] \\
 &= 4.
 \end{aligned}$$

The improper integral converges to 4.

58. We evaluate

$$\begin{aligned}
 \int_0^4 \frac{dx}{\sqrt{8x-x^2}} &= \lim_{a \rightarrow 0^+} \int_a^4 \frac{dx}{\sqrt{8x-x^2}} \\
 &= \lim_{a \rightarrow 0^+} \int_a^4 \frac{dx}{\sqrt{16-(4-x)^2}} \\
 &= \lim_{a \rightarrow 0^+} \left[-\sin^{-1} \left(\frac{4-x}{4} \right) \right]_a^4 \\
 &= \lim_{a \rightarrow 0^+} \left[-\sin^{-1} \left(\frac{4-4}{4} \right) - \left(-\sin^{-1} \left(\frac{4-a}{4} \right) \right) \right] \\
 &= \lim_{a \rightarrow 0^+} \left[\sin^{-1} \left(\frac{4-a}{4} \right) \right] \\
 &= \frac{\pi}{2}.
 \end{aligned}$$

The improper integral converges to $\frac{\pi}{2}$.

59. We evaluate

$$\begin{aligned}
 \int_a^{3a} \frac{2x \, dx}{(x^2 - a^2)^{3/2}} &= \lim_{t \rightarrow a^+} \int_t^{3a} \frac{2x \, dx}{(x^2 - a^2)^{3/2}} \\
 &= \lim_{t \rightarrow a^+} \left[-2(x^2 - a^2)^{-1/2} \right]_t^{3a} \\
 &= \lim_{t \rightarrow a^+} \left[-2((3a)^2 - a^2)^{-1/2} - \left(-2(t^2 - a^2)^{-1/2} \right) \right] \\
 &= \lim_{t \rightarrow a^+} \left[-2(8a^2)^{-1/2} + 2(t^2 - a^2)^{-1/2} \right] \\
 &= \infty.
 \end{aligned}$$

The improper integral diverges.

60. We evaluate

$$\begin{aligned}
 \int_0^3 \frac{x \, dx}{(9 - x^2)^{3/2}} &= \lim_{b \rightarrow 3^-} \int_0^b \frac{x \, dx}{(9 - x^2)^{3/2}} \\
 &= \lim_{b \rightarrow 3^-} \left[(9 - x^2)^{-1/2} \right]_0^b \\
 &= \lim_{b \rightarrow 3^-} \left[(9 - b^2)^{-1/2} - (9 - 0^2)^{-1/2} \right] \\
 &= \lim_{b \rightarrow 3^-} \left[(9 - b^2)^{-1/2} - \frac{1}{3} \right] \\
 &= \infty.
 \end{aligned}$$

The improper integral diverges.

61. We evaluate

$$\begin{aligned}
 \int_0^\infty x e^{-x^2} \, dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2} \, dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-x^2} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-b^2} - \left(-\frac{1}{2} e^{-0^2} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2} e^{-b^2} \right] \\
 &= \frac{1}{2}.
 \end{aligned}$$

The improper integral converges to $\frac{1}{2}$.

62. We evaluate, using formula 125,

$$\begin{aligned}
 \int_0^\infty e^{-x} \sin x \, dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} \sin x \, dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{-x} (-\sin x - \cos x) \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} e^{-b} (-\sin b - \cos b) - \left(\frac{1}{2} e^{-0} (-\sin 0 - \cos 0) \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2} e^{-b} (\cos b + \sin b) \right] \\
 &= \frac{1}{2}.
 \end{aligned}$$

The improper integral converges to $\boxed{\frac{1}{2}}$.

63. (a) We have

$$\frac{1}{\sqrt{x^2 - 1}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x}.$$

By the comparison test, since $p = 1$ and $\int_1^\infty \frac{1}{x} \, dx$ diverges, we conclude that $\int_1^\infty \frac{1}{\sqrt{x^2 - 1}} \, dx$ diverges.

64. (a) We have

$$\frac{2}{\sqrt{x^2 - 4}} > \frac{2}{\sqrt{x^2}} = \frac{2}{x}.$$

By the comparison test, since $p = 1$ and $\int_2^\infty \frac{1}{x} \, dx$ diverges, we conclude that $\int_2^\infty \frac{2}{\sqrt{x^2 - 4}} \, dx$ diverges.

65. (a) We have

$$\frac{1 + e^{-x}}{x} \geq \frac{1}{x}.$$

By the comparison test, since $p = 1$ and $\int_1^\infty \frac{1}{x} \, dx$ diverges, we conclude that $\int_1^\infty \frac{1 + e^{-x}}{x} \, dx$ diverges.

66. (a) For $x \geq 1$, we have

$$e^{-x} \leq \frac{1}{x}$$

so

$$\frac{3e^{-x}}{x} \leq \frac{3}{x^2}.$$

By the comparison test, since $p = 2$ and $\int_1^\infty \frac{3}{x^2} \, dx$ converges, we conclude that $\int_1^\infty \frac{3e^{-x}}{x} \, dx$ converges.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{3e^{-x}}{x} \, dx \approx \boxed{0.6582}.$$

67. (a) We have

$$\sin^2 x \leq 1$$

so

$$\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$$

By the comparison test, since $p = 2$ and $\int_1^\infty \frac{1}{x^2} dx$ converges, we conclude that $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{\sin^2 x}{x^2} dx \approx \boxed{0.673}.$$

68. (a) We have

$$\cos^2 x \leq 1$$

so

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}.$$

By the comparison test, since $p = 2$ and $\int_1^\infty \frac{1}{x^2} dx$ converges, we conclude that $\int_1^\infty \frac{\cos^2 x}{x^2} dx$ converges.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{\cos^2 x}{x^2} dx \approx \boxed{0.3265}.$$

69. (a) We have

$$\frac{1}{(x+1)\sqrt{x}} \leq \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}.$$

By the comparison test, since $p = \frac{3}{2}$ and $\int_1^\infty \frac{1}{x^{3/2}} dx$ converges, we conclude that $\int_1^\infty \frac{1}{(x+1)\sqrt{x}} dx$ converges.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{1}{(x+1)\sqrt{x}} dx = \boxed{\frac{\pi}{2}}.$$

70. (a) We have

$$\frac{1}{x\sqrt{1+x^2}} \leq \frac{1}{x\sqrt{x^2}} = \frac{1}{x^2}.$$

By the comparison test, since $p = 2$ and $\int_1^\infty \frac{1}{x^2} dx$ converges, we conclude that $\int_1^\infty \frac{1}{x\sqrt{1+x^2}} dx$ converges.

(b) Using a CAS we obtain

$$\int_1^\infty \frac{1}{x\sqrt{1+x^2}} dx \approx \boxed{0.8814}.$$

Applications and Extensions

71. The area is given by the improper integral $\int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$. We evaluate

$$\begin{aligned}
 \int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx &= \lim_{b \rightarrow \infty} \int_0^b \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx \\
 &= \lim_{b \rightarrow \infty} [\ln|x+1| - \ln|x+2|]_0^b \\
 &= \lim_{b \rightarrow \infty} [\ln|b+1| - \ln|b+2| - (\ln|0+1| - \ln|0+2|)] \\
 &= \lim_{b \rightarrow \infty} [\ln 2 + \ln|b+1| - \ln|b+2|] \\
 &= \lim_{b \rightarrow \infty} \left[\ln 2 + \ln \left| \frac{b+1}{b+2} \right| \right] \\
 &= \boxed{\ln 2}.
 \end{aligned}$$

72. The area is given by the improper integral $\int_0^\infty \frac{1}{1+x^2} dx$. We evaluate

$$\begin{aligned}
 \int_0^\infty \frac{1}{1+x^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx \\
 &= \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b \\
 &= \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 0] \\
 &= \lim_{b \rightarrow \infty} [\tan^{-1} b] \\
 &= \boxed{\frac{\pi}{2}}.
 \end{aligned}$$

73. The volume is given by the improper integral $\int_0^\infty \pi(e^{-x})^2 dx$. We evaluate

$$\begin{aligned}
 \int_0^\infty \pi(e^{-x})^2 dx &= \lim_{b \rightarrow \infty} \int_0^b \pi e^{-2x} dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} \pi e^{-2x} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} \pi e^{-2b} - \left(-\frac{1}{2} \pi e^{-2(0)} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \pi - \frac{1}{2} \pi e^{-2b} \right] \\
 &= \boxed{\frac{\pi}{2}}.
 \end{aligned}$$

74. The volume is given by the improper integral $\int_1^\infty \pi \left(\frac{1}{\sqrt{x}} \right)^2 dx$. We evaluate

$$\begin{aligned}
 \int_1^\infty \pi \left(\frac{1}{\sqrt{x}} \right)^2 dx &= \lim_{b \rightarrow \infty} \int_1^b \pi \frac{1}{x} dx \\
 &= \lim_{b \rightarrow \infty} [\pi \ln|x|]_1^b \\
 &= \lim_{b \rightarrow \infty} [\pi \ln|b| - \pi \ln|1|] \\
 &= \lim_{b \rightarrow \infty} [\pi \ln b] \\
 &= \infty.
 \end{aligned}$$

So the volume of the solid of revolution is $\boxed{\text{not defined}}$.

75. The area is given by the improper integral $\int_{-\infty}^{\infty} \frac{8a^3}{x^2+4a^2} dx = \int_{-\infty}^0 \frac{8a^3}{x^2+4a^2} dx + \int_0^{\infty} \frac{8a^3}{x^2+4a^2} dx$.

We evaluate first

$$\begin{aligned} \int_{-\infty}^0 \frac{8a^3}{x^2+4a^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{8a^3}{x^2+4a^2} dx \\ &= \lim_{t \rightarrow -\infty} \left[(8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{x}{2a} \right) \right]_t^0 \\ &= \lim_{t \rightarrow -\infty} \left[(8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{0}{2a} \right) - (8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{t}{2a} \right) \right] \\ &= \lim_{t \rightarrow -\infty} \left[-4a^2 \tan^{-1} \left(\frac{t}{2a} \right) \right] \\ &= (-4a^2) \left(-\frac{\pi}{2} \right) \\ &= 2\pi a^2. \end{aligned}$$

We next evaluate

$$\begin{aligned} \int_0^{\infty} \frac{8a^3}{x^2+4a^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{8a^3}{x^2+4a^2} dx \\ &= \lim_{t \rightarrow \infty} \left[(8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{x}{2a} \right) \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[(8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{t}{2a} \right) - (8a^3) \frac{1}{2a} \tan^{-1} \left(\frac{0}{2a} \right) \right] \\ &= \lim_{t \rightarrow \infty} \left[4a^2 \tan^{-1} \left(\frac{t}{2a} \right) \right] \\ &= (4a^2) \left(\frac{\pi}{2} \right) \\ &= 2\pi a^2. \end{aligned}$$

So the area under the graph is $2\pi a^2 + 2\pi a^2 = \boxed{4\pi a^2}$.

76. (a) One can consider the total reaction as the rate of reaction times time. When the rate is not constant, the total reaction is given by the integral of the rate of reaction for $t \geq 0$. This integral is equal to the area under the graph of $y = r(t)$ on $[0, \infty)$.

(b) We evaluate the improper integral

$$\begin{aligned} \int_0^{\infty} r(t) dt &= \int_0^{\infty} te^{-t^2} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b te^{-t^2} dt \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-t^2} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-b^2} - \left(-\frac{1}{2} e^{-0^2} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2} e^{-b^2} \right] \\ &= \frac{1}{2}. \end{aligned}$$

So the total reaction is $\boxed{\frac{1}{2}}$.

77. (a) We have $R(t) = 100$, and $r = 0.08$. So the present value of the asset is given by the improper integral

$$\begin{aligned}
 \int_0^{\infty} R(t)e^{-rt} dt &= \lim_{b \rightarrow \infty} \int_0^b 100e^{-(0.08)t} dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{100}{-0.08} e^{-0.08t} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{100}{-0.08} e^{-0.08b} - \left(\frac{100}{-0.08} e^{-0.08(0)} \right) \right] \\
 &= \lim_{b \rightarrow \infty} [1250 - 1250e^{-0.08b}] \\
 &= 1250.
 \end{aligned}$$

The present value of the asset is \$1250.00.

- (b) Now with $R(t) = 1000 + 80t$, and $r = 0.07$, the present value of the asset is given by the improper integral

$$\begin{aligned}
 \int_0^{\infty} R(t)e^{-rt} dt &= \lim_{b \rightarrow \infty} \int_0^b (1000 + 80t)e^{-(0.07)t} dt \\
 &= \lim_{b \rightarrow \infty} \left[1000 \int_0^b e^{-(0.07)t} dt + 80 \int_0^b te^{-at} dt \right] \\
 &= \lim_{b \rightarrow \infty} \left[1000 \frac{1}{-0.07} e^{-0.07t} + 80 \left(-\frac{1}{(0.07)^2} e^{-(0.07)t} ((0.07)t + 1) \right) \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[1000 \frac{1}{-0.07} e^{-0.07b} + 80 \left(-\frac{1}{(0.07)^2} e^{-(0.07)b} ((0.07)b + 1) \right) \right. \\
 &\quad \left. - \left(1000 \frac{1}{-0.07} e^{-0.07(0)} + 80 \left(-\frac{1}{(0.07)^2} e^{-(0.07)(0)} ((0.07)(0) + 1) \right) \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1000}{-0.07} e^{-0.07b} + 80 \left(-\frac{1}{(0.07)^2} e^{-(0.07)b} ((0.07)b + 1) \right) \right. \\
 &\quad \left. - \left(\frac{1000}{-0.07} - \frac{80}{(0.07)^2} \right) \right] \\
 &= - \left(\frac{1000}{-0.07} - \frac{80}{(0.07)^2} \right) \\
 &\approx 30,612.
 \end{aligned}$$

The present value of the asset is approximately $\boxed{\$30,612}$.

78. We evaluate the improper integral

$$\begin{aligned}
\int_0^\infty Ri^2 dt &= \lim_{b \rightarrow \infty} \int_0^b R \left(I e^{-Rt/L} \right)^2 dt \\
&= \lim_{b \rightarrow \infty} \int_0^b RI^2 e^{-2Rt/L} dt \\
&= \lim_{b \rightarrow \infty} \left[\frac{RI^2}{-2R/L} e^{-2Rt/L} \right]_0^b \\
&= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} LI^2 e^{-2Rb/L} - \left(-\frac{1}{2} LI^2 e^{-2R(0)/L} \right) \right] \\
&= \lim_{b \rightarrow \infty} \left[\frac{1}{2} LI^2 - \frac{1}{2} LI^2 e^{-2Rb/L} \right] \\
&= \boxed{\frac{1}{2} LI^2}.
\end{aligned}$$

79. We evaluate the improper integral

$$\begin{aligned}
\frac{2\pi N I r}{10} \int_x^\infty \frac{dy}{(r^2 + y^2)^{3/2}} &= \frac{2\pi N I r}{10} \lim_{b \rightarrow \infty} \int_x^b \frac{dy}{(r^2 + y^2)^{3/2}} \\
&= \frac{2\pi N I r}{10} \lim_{b \rightarrow \infty} \left[\frac{1}{r^2} \frac{y}{\sqrt{r^2 + y^2}} \right]_x^b \quad (\text{by Formula 61}) \\
&= \frac{2\pi N I r}{10} \lim_{b \rightarrow \infty} \left[\frac{1}{r^2} \frac{b}{\sqrt{r^2 + b^2}} - \frac{1}{r^2} \frac{x}{\sqrt{r^2 + x^2}} \right] \\
&= \frac{2\pi N I r}{10} \lim_{b \rightarrow \infty} \left[\frac{1}{r^2} \frac{1}{\sqrt{(r/b)^2 + 1}} - \frac{1}{r^2} \frac{x}{\sqrt{r^2 + x^2}} \right] \\
&= \frac{2\pi N I r}{10} \frac{1}{r^2} \left(1 - \frac{x}{\sqrt{r^2 + x^2}} \right) \\
&= \boxed{\frac{\pi N I}{5r} \left(1 - \frac{x}{\sqrt{r^2 + x^2}} \right)}.
\end{aligned}$$

80. We evaluate the improper integral by splitting into two integrals.

$$\frac{r I m}{10} \int_{-\infty}^\infty \frac{dy}{(r^2 + y^2)^{3/2}} = \frac{r I m}{10} \left[\int_{-\infty}^0 \frac{dy}{(r^2 + y^2)^{3/2}} + \int_0^\infty \frac{dy}{(r^2 + y^2)^{3/2}} \right]$$

We first evaluate

$$\begin{aligned}
 \int_{-\infty}^0 \frac{dy}{(r^2 + y^2)^{3/2}} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dy}{(r^2 + y^2)^{3/2}} \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{r^2} \frac{y}{\sqrt{r^2 + y^2}} \right]_a^0 \quad (\text{by Formula 61}) \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{r^2} \frac{0}{\sqrt{r^2 + 0^2}} - \frac{1}{r^2} \frac{a}{\sqrt{r^2 + a^2}} \right] \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{r^2 \sqrt{1 + (r/a)^2}} \right] \\
 &= \frac{1}{r^2}.
 \end{aligned}$$

And we evaluate

$$\begin{aligned}
 \int_0^\infty \frac{dy}{(r^2 + y^2)^{3/2}} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dy}{(r^2 + y^2)^{3/2}} \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{r^2} \frac{y}{\sqrt{r^2 + y^2}} \right]_0^b \quad (\text{by Formula 61}) \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{r^2} \frac{b}{\sqrt{r^2 + b^2}} - \frac{1}{r^2} \frac{0}{\sqrt{r^2 + 0^2}} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{r^2 \sqrt{1 + (r/b)^2}} \right] \\
 &= \frac{1}{r^2}.
 \end{aligned}$$

So

$$\begin{aligned}
 \frac{rIm}{10} \int_{-\infty}^\infty \frac{dy}{(r^2 + y^2)^{3/2}} &= \frac{rIm}{10} \left[\frac{1}{r^2} + \frac{1}{r^2} \right] \\
 &= \boxed{\frac{mI}{5r}}.
 \end{aligned}$$

81. We have $W = \int_1^\infty F(r) dr$, where F is the force acting on an object. So we evaluate the improper integral

$$\begin{aligned}
 \int_1^\infty F(r) dr &= \lim_{b \rightarrow \infty} \int_1^b \frac{GmM}{r^2} dr \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{GmM}{r} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{GmM}{b} - \left(-\frac{GmM}{1} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[GmM - \frac{GmM}{b} \right] \\
 &= \boxed{GmM}.
 \end{aligned}$$

82. If $\alpha \geq 0$, then $\int_0^1 x^\alpha dx$ converges as an ordinary definite integral. If $-1 < \alpha < 0$, then

$$\begin{aligned}\int_0^1 x^\alpha dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^\alpha dx \\ &= \lim_{a \rightarrow 0^+} \left[\frac{x^{\alpha+1}}{\alpha+1} \right]_a^1 \\ &= \lim_{a \rightarrow 0^+} \left[\frac{1^{\alpha+1}}{\alpha+1} - \frac{a^{\alpha+1}}{\alpha+1} \right] \\ &= \frac{1}{\alpha+1},\end{aligned}$$

and the improper integral converges. If $\alpha = -1$, then

$$\begin{aligned}\int_0^1 x^\alpha dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^{-1} dx \\ &= \lim_{a \rightarrow 0^+} [\ln |x|]_a^1 \\ &= \lim_{a \rightarrow 0^+} [\ln |1| - \ln |a|] \\ &= \infty,\end{aligned}$$

and the improper integral diverges. And if $\alpha < -1$,

$$\begin{aligned}\int_0^1 x^\alpha dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^\alpha dx \\ &= \lim_{a \rightarrow 0^+} \left[\frac{x^{\alpha+1}}{\alpha+1} \right]_a^1 \\ &= \lim_{a \rightarrow 0^+} \left[\frac{1^{\alpha+1}}{\alpha+1} - \frac{a^{\alpha+1}}{\alpha+1} \right] \\ &= \infty,\end{aligned}$$

and the improper integral diverges. So $\int_0^1 x^\alpha dx$ converges for $\boxed{-1 < \alpha}$.

83. We evaluate the improper integral

$$\int_0^\infty x e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx.$$

Let $u = x$, $du = dx$, and $dv = e^{-x} dx$, $v = -e^{-x}$, and use integration by parts.

$$\begin{aligned}\int_0^\infty x e^{-x} dx &= \lim_{b \rightarrow \infty} \left[[x(-e^{-x})]_0^b - \int_0^b (-e^{-x}) dx \right] \\ &= \lim_{b \rightarrow \infty} \left[-be^{-b} + \int_0^b e^{-x} dx \right] \\ &= \lim_{b \rightarrow \infty} \left[-be^{-b} + [-e^{-x}]_0^b \right] \\ &= \lim_{b \rightarrow \infty} \left[-be^{-b} + [-e^{-b} - (-e^{-0})] \right] \\ &= \lim_{b \rightarrow \infty} \left[-be^{-b} + 1 - e^{-b} \right] \\ &= \boxed{1}.\end{aligned}$$

84. We evaluate the improper integral

$$\int_0^1 x \ln x \, dx = \lim_{a \rightarrow 0^+} \int_a^1 x \ln x \, dx.$$

Let $u = \ln x$, $du = \frac{1}{x} dx$, and $dv = x \, dx$, $v = \frac{1}{2}x^2$, and use integration by parts.

$$\begin{aligned} \int_0^1 x \ln x \, dx &= \lim_{a \rightarrow 0^+} \left[\left[(\ln x) \left(\frac{1}{2}x^2 \right) \right]_a^1 - \int_a^1 \left(\frac{1}{2}x^2 \right) \left(\frac{1}{x} \right) dx \right] \\ &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2}a^2 \ln a - \frac{1}{2} \int_a^1 x \, dx \right] \\ &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2}a^2 \ln a - \frac{1}{2} \left[\frac{1}{2}x^2 \right]_a^1 \right] \\ &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{2}a^2 \ln a - \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2}a^2 \right) \right] \\ &= \lim_{a \rightarrow 0^+} \left[\frac{1}{4}a^2 - \frac{1}{2}a^2 \ln a - \frac{1}{4} \right] \\ &= \boxed{-\frac{1}{4}}. \end{aligned}$$

85. We evaluate the improper integral

$$\int_0^\infty e^{-x} \cos x \, dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} \cos x \, dx.$$

We use integration by parts to obtain formula 126, and so

$$\begin{aligned} \int_0^\infty e^{-x} \cos x \, dx &= \lim_{b \rightarrow \infty} \left[\frac{1}{2}e^{-x}(-\cos x + \sin x) \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2}e^{-b}(-\cos b + \sin b) - \left(\frac{1}{2}e^{-0}(-\cos 0 + \sin 0) \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{2}e^{-b}(\cos b - \sin b) \right] \\ &= \boxed{\frac{1}{2}}. \end{aligned}$$

86. We evaluate the improper integral

$$\int_0^\infty \tan^{-1} x \, dx = \lim_{b \rightarrow \infty} \int_0^b \tan^{-1} x \, dx.$$

Let $u = \tan^{-1} x$, $du = \frac{1}{1+x^2} dx$, and $dv = dx$, $v = x$, and use integration by parts.

$$\begin{aligned} \int_0^\infty \tan^{-1} x \, dx &= \lim_{b \rightarrow \infty} \left[[x \tan^{-1} x]_0^b - \int_0^b x \left(\frac{1}{1+x^2} \right) dx \right] \\ &= \lim_{b \rightarrow \infty} \left[b \tan^{-1} b - \left[\frac{1}{2} \ln(x^2 + 1) \right]_0^b \right] \\ &= \lim_{b \rightarrow \infty} \left[b \tan^{-1} b - \frac{1}{2} \ln(b^2 + 1) \right] \\ &= \infty, \end{aligned}$$

and the improper integral $\boxed{\text{diverges}}$.

87. We evaluate the improper integral

$$\begin{aligned}
 \int_0^\infty \sin x \, dx &= \lim_{b \rightarrow \infty} \int_0^b \sin x \, dx \\
 &= \lim_{b \rightarrow \infty} [-\cos x]_0^b \\
 &= \lim_{b \rightarrow \infty} [-\cos b - (-\cos 0)] \\
 &= \lim_{b \rightarrow \infty} [1 - \cos b].
 \end{aligned}$$

Since this limit does not exist, the improper integral $\int_0^\infty \sin x \, dx$ diverges. We now evaluate the improper integral

$$\begin{aligned}
 \int_{-\infty}^0 \sin x \, dx &= \lim_{a \rightarrow -\infty} \int_a^0 \sin x \, dx \\
 &= \lim_{a \rightarrow -\infty} [-\cos x]_a^0 \\
 &= \lim_{a \rightarrow -\infty} [-\cos 0 - (-\cos a)] \\
 &= \lim_{a \rightarrow -\infty} [\cos a - 1].
 \end{aligned}$$

Since this limit does not exist, the improper integral $\int_{-\infty}^0 \sin x \, dx$ diverges. We now determine

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \int_{-t}^t \sin x \, dx &= \lim_{t \rightarrow \infty} [\cos x]_{-t}^t \\
 &= \lim_{t \rightarrow \infty} [\cos t - (\cos(-t))] \\
 &= \lim_{t \rightarrow \infty} [\cos t - \cos(t)] \\
 &= \lim_{t \rightarrow \infty} [0] \\
 &= \boxed{0}.
 \end{aligned}$$

88. If we let $f(x) = x + \frac{1/\pi}{1+x^2}$, then

$$\begin{aligned}
 \int_0^\infty \left(x + \frac{1/\pi}{1+x^2} \right) dx &= \lim_{b \rightarrow \infty} \int_0^b \left(x + \frac{1/\pi}{1+x^2} \right) dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2}x^2 + \frac{1}{\pi} \tan^{-1} x \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{2}b^2 + \frac{1}{\pi} \tan^{-1} b \right] \\
 &= \infty,
 \end{aligned}$$

so this integral diverges. And

$$\begin{aligned}
 \int_{-\infty}^0 x \, dx &= \lim_{a \rightarrow -\infty} \int_a^0 \left(x + \frac{1/\pi}{1+x^2} \right) dx \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{1}{2}x^2 + \frac{1}{\pi} \tan^{-1} x \right]_a^0 \\
 &= \lim_{a \rightarrow -\infty} \left[-\frac{1}{2}a^2 - \frac{1}{\pi} \tan^{-1} a \right] \\
 &= -\infty,
 \end{aligned}$$

so this integral also diverges. But

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \int_{-t}^t \left(x + \frac{1/\pi}{1+x^2} \right) dx &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} x^2 + \frac{1}{\pi} \tan^{-1} x \right]_{-t}^t \\
 &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} t^2 + \frac{1}{\pi} \tan^{-1} t - \left(\frac{1}{2} (-t)^2 + \frac{1}{\pi} \tan^{-1} (-t) \right) \right] \\
 &= \lim_{t \rightarrow \infty} \left[\frac{2}{\pi} \tan^{-1} t \right] \\
 &= 1.
 \end{aligned}$$

89. Since

$$\frac{1}{\sqrt{2+\sin x}} \geq \frac{1}{\sqrt{2+1}} = \frac{1}{\sqrt{3}}$$

for all x , and since $\int_0^\infty \frac{1}{3} dx$ diverges, by the Comparison Test, $\int_0^\infty \frac{1}{\sqrt{2+\sin x}} dx$ diverges.

90. Since for $x \geq 2$

$$\frac{\ln x}{\sqrt{x^2-1}} \geq \frac{\ln 2}{\sqrt{x^2}} = \frac{\ln 2}{x}$$

and $\int_2^\infty \frac{\ln 2}{x} dx$ diverges, by the Comparison Test, $\int_2^\infty \frac{\ln x}{\sqrt{x^2-1}} dx$ diverges.

91. (a) We evaluate the improper integral, and let $u = x^n$, $du = nx^{n-1} dx$, $dv = e^{-x} dx$, and $v = -e^{-x}$. We obtain

$$\begin{aligned}
 \int_0^\infty x^n e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x^n e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \left[[x^n (-e^{-x})]_0^b - \int_0^b nx^{n-1} (-e^{-x}) dx \right] \\
 &= \lim_{b \rightarrow \infty} \left[-b^n e^{-b} + n \int_0^b x^{n-1} e^{-x} dx \right] \\
 &= \lim_{b \rightarrow \infty} [-b^n e^{-b}] + \lim_{b \rightarrow \infty} \left[n \int_0^b x^{n-1} e^{-x} dx \right] \\
 &= 0 + n \lim_{b \rightarrow \infty} \left[\int_0^b x^{n-1} e^{-x} dx \right] \\
 &= n \int_0^\infty x^{n-1} e^{-x} dx
 \end{aligned}$$

(b) Using part (a) repeatedly, we obtain

$$\begin{aligned}
\int_0^\infty x^n e^{-x} dx &= n \int_0^\infty x^{n-1} e^{-x} dx \\
&= n(n-1) \int_0^\infty x^{n-2} e^{-x} dx \\
&= \dots \\
&= [n(n-1) \cdots 1] \int_0^\infty e^{-x} dx \\
&= (n!) \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\
&= (n!) \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\
&= (n!) \lim_{b \rightarrow \infty} [-e^{-b} - (-e^{-0})] \\
&= (n!) \lim_{b \rightarrow \infty} [1 - e^{-b}] \\
&= n!.
\end{aligned}$$

92. We evaluate the improper integral, and obtain for $p \neq 1$

$$\begin{aligned}
\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{b \rightarrow \infty} \int_e^b \frac{dx}{x(\ln x)^p} \\
&= \lim_{b \rightarrow \infty} \left[\frac{\ln^{1-p} x}{1-p} \right]_e^b \\
&= \lim_{b \rightarrow \infty} \left[\frac{\ln^{1-p} b}{1-p} - \frac{\ln^{1-p} e}{1-p} \right] \\
&= \lim_{b \rightarrow \infty} \left[\frac{\ln^{1-p} b}{1-p} - \frac{1}{1-p} \right].
\end{aligned}$$

If $p > 1$, then $1-p < 0$, and

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{b \rightarrow \infty} \left[\frac{\ln^{1-p} b}{1-p} - \frac{1}{1-p} \right] = \frac{1}{p-1}.$$

If $p < 1$, then $1-p > 0$, and

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{b \rightarrow \infty} \left[\frac{\ln^{1-p} b}{1-p} - \frac{1}{1-p} \right] = \infty.$$

And if $p = 1$, we have

$$\begin{aligned}
\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{b \rightarrow \infty} \int_e^b \frac{dx}{x \ln x} \\
&= \lim_{b \rightarrow \infty} [\ln(\ln x)]_e^b \\
&= \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln e)] \\
&= \lim_{b \rightarrow \infty} [\ln(\ln b) - 0] \\
&= \infty.
\end{aligned}$$

So $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

93. We evaluate the improper integral, and obtain for $p \neq 1$

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^p} &= \lim_{t \rightarrow a^+} \int_t^b \frac{dx}{(x-a)^p} \\ &= \lim_{t \rightarrow a^+} \left[\frac{(x-a)^{1-p}}{1-p} \right]_t^b \\ &= \lim_{t \rightarrow a^+} \left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p} \right].\end{aligned}$$

If $p > 1$, then $1-p < 0$, and

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{t \rightarrow a^+} \left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p} \right] = \infty.$$

If $0 < p < 1$, then $1-p > 0$, and

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{t \rightarrow a^+} \left[\frac{(b-a)^{1-p}}{1-p} - \frac{(t-a)^{1-p}}{1-p} \right] = \frac{(b-a)^{1-p}}{1-p}.$$

And if $p = 1$, we have

$$\begin{aligned}\int_a^b \frac{dx}{(x-a)^p} &= \lim_{t \rightarrow a^+} \int_t^b \frac{dx}{x-a} \\ &= \lim_{t \rightarrow a^+} [\ln |x-a|]_t^b \\ &= \lim_{t \rightarrow a^+} [\ln |b-a| - \ln |t-a|] \\ &= \infty.\end{aligned}$$

So $\int_a^b \frac{dx}{(x-a)^p}$ converges if $0 < p < 1$, and diverges if $p \geq 1$.

94. We evaluate the improper integral, and obtain for $p \neq 1$

$$\begin{aligned}\int_a^b \frac{dx}{(b-x)^p} &= \lim_{t \rightarrow b^-} \int_a^t \frac{dx}{(b-x)^p} \\ &= \lim_{t \rightarrow b^-} \left[\frac{-(b-x)^{1-p}}{1-p} \right]_a^t \\ &= \lim_{t \rightarrow b^-} \left[\frac{-(b-t)^{1-p}}{1-p} - \left(-\frac{(b-a)^{1-p}}{1-p} \right) \right].\end{aligned}$$

If $p > 1$, then $1-p < 0$, and

$$\int_a^b \frac{dx}{(b-x)^p} = \lim_{t \rightarrow b^-} \left[\frac{-(b-t)^{1-p}}{1-p} - \left(-\frac{(b-a)^{1-p}}{1-p} \right) \right] = \infty.$$

If $0 < p < 1$, then $1-p > 0$, and

$$\int_a^b \frac{dx}{(b-x)^p} = \lim_{t \rightarrow b^-} \left[\frac{-(b-t)^{1-p}}{1-p} - \left(-\frac{(b-a)^{1-p}}{1-p} \right) \right] = \frac{(b-a)^{1-p}}{1-p}.$$

And if $p = 1$, we have

$$\begin{aligned}
 \int_a^b \frac{dx}{(b-x)^p} &= \lim_{t \rightarrow b^-} \int_a^t \frac{dx}{b-x} \\
 &= \lim_{t \rightarrow b^-} [-\ln |b-x|]_a^t \\
 &= \lim_{t \rightarrow b^-} [-\ln |b-t| - (-\ln |b-a|)] \\
 &= \infty.
 \end{aligned}$$

So $\int_a^b \frac{dx}{(b-x)^p}$ converges if $0 < p < 1$, and diverges if $p \geq 1$.

95. If $p = 0$, then $\frac{1}{(x-a)^p} = 1$ for all x in $(a, b]$, and so $\int_a^b \frac{dx}{(x-a)^p} = b - a$. Likewise, $\frac{1}{(b-a)^p} = 1$ for all x in $[a, b)$, so $\int_a^b \frac{dx}{(b-x)^p} = 1$. And if $p < 0$, then $\frac{1}{(x-a)^p}$ and $\frac{1}{(b-a)^p}$ are continuous on $[a, b]$, so the integrals $\int_a^b \frac{dx}{(x-a)^p}$ and $\int_a^b \frac{dx}{(b-x)^p}$ are ordinary definite integrals, and hence converge. For example, $\int_1^2 \frac{dx}{(x-1)^{-1}} = \int_1^2 (x-1) dx = \left[\frac{1}{2}(x-1)^2 \right]_1^2 = \frac{1}{2}$.

96. Suppose $\int_a^\infty g(x) dx$ diverges, then since $g(x) \geq 0$, we know that $\int_a^\infty g(x) dx = \infty$. If M is any real number, then there exists b such that for every $b \leq t$,

$$M \leq \int_a^b g(x) dx \leq \int_a^b f(x) dx \leq \int_a^t f(x) dx.$$

So $\lim_{t \rightarrow \infty} \int_a^t f(x) dx = \infty$, by definition of an infinite limit. So $\int_a^\infty f(x) dx$ diverges.

Now suppose $\int_a^\infty f(x) dx$ converges. Then $\int_a^\infty g(x) dx$ must also converge, since from the above, we otherwise would conclude that $\int_a^\infty f(x) dx$ diverges.

97. We evaluate, for $s > 0$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^\infty e^{-sx} f(x) dx \\
 &= \int_0^\infty e^{-sx} x dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} x dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s^2} e^{-sx} (sx + 1) \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s^2} e^{-bs} (bs + 1) - \left(-\frac{1}{s^2} e^{-(0)s} ((0)s + 1) \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{s^2} - \frac{1}{s^2} e^{-bs} (bs + 1) \right] \\
 &= \boxed{\frac{1}{s^2}}.
 \end{aligned}$$

98. We evaluate, for $s > 0$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^{\infty} e^{-sx} f(x) dx \\
 &= \int_0^{\infty} e^{-sx} \cos x dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \cos x dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-xs}(\sin x - s \cos x)}{s^2 + 1} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-bs}(\sin b - s \cos b)}{s^2 + 1} - \frac{e^{-(0)s}(\sin(0) - s \cos(0))}{s^2 + 1} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{s}{s^2 + 1} + e^{-bs} \frac{\sin b - s \cos b}{s^2 + 1} \right] \\
 &= \boxed{\frac{s}{s^2 + 1}}.
 \end{aligned}$$

99. We evaluate, for $s > 0$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^{\infty} e^{-sx} f(x) dx \\
 &= \int_0^{\infty} e^{-sx} \sin x dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \sin x dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{-e^{-sx}(\cos x + s \sin x)}{s^2 + 1} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{-e^{-bs}(\cos b + s \sin b)}{s^2 + 1} - \frac{-e^{-(0)s}(\cos(0) + s \sin(0))}{s^2 + 1} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{s^2 + 1} - e^{-bs} \frac{\cos b + s \sin b}{s^2 + 1} \right] \\
 &= \boxed{\frac{1}{s^2 + 1}}.
 \end{aligned}$$

100. We evaluate, for $s > 1$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^\infty e^{-sx} f(x) dx \\
 &= \int_0^\infty e^{-sx} e^x dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{x(1-s)} dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{x(1-s)}}{1-s} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{b(1-s)}}{1-s} - \frac{e^{(0)(1-s)}}{1-s} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{s-1} - \frac{e^{-b(s-1)}}{s-1} \right] \\
 &= \boxed{\frac{1}{s-1}}.
 \end{aligned}$$

But for $s \leq 1$, $\int_0^\infty e^{-sx} e^x dx$ diverges.

101. We evaluate, for $s > a$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^\infty e^{-sx} f(x) dx \\
 &= \int_0^\infty e^{-sx} e^{ax} dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{x(a-s)} dx \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{x(a-s)}}{a-s} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{b(a-s)}}{a-s} - \frac{e^{0(a-s)}}{a-s} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-b(s-a)}}{a-s} - \frac{1}{a-s} \right] \\
 &= \boxed{\frac{1}{s-a}}.
 \end{aligned}$$

But for $s \leq a$, $\int_0^\infty e^{-sx} e^x dx$ diverges.

102. We evaluate, for $s > 0$,

$$\begin{aligned}
 L\{f(x)\} &= \int_0^\infty e^{-sx} f(x) dx \\
 &= \int_0^\infty e^{-sx} (1) dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-sx} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-bs} - \left(-\frac{1}{s} e^{-(0)s} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{s} - \frac{1}{s} e^{-bs} \right] \\
 &= \boxed{\frac{1}{s}}.
 \end{aligned}$$

Challenge Problems

103. The arc length is given by $\int_0^1 \sqrt{1 + (dy/dx)^2} dx$. We differentiate, and obtain

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{x - x^2} - \sin^{-1} \sqrt{x} \right) \\
 &= \frac{1}{2} (x - x^2)^{-1/2} (1 - 2x) - \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \frac{1}{2\sqrt{x}} \\
 &= \frac{1 - 2x}{2\sqrt{x - x^2}} - \frac{1}{2\sqrt{x - x^2}} \\
 &= \frac{-x}{\sqrt{x - x^2}}
 \end{aligned}$$

So

$$\begin{aligned}
 \sqrt{1 + (dy/dx)^2} &= \sqrt{1 + \left(\frac{-x}{\sqrt{x - x^2}} \right)^2} \\
 &= \sqrt{1 + \frac{x^2}{x - x^2}} \\
 &= \frac{1}{\sqrt{1 - x}}
 \end{aligned}$$

We now evaluate the improper integral.

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{1 - x}} dx &= \lim_{b \rightarrow 1^-} \int_0^b (1 - x)^{-1/2} dx \\
 &= \lim_{b \rightarrow 1^-} \left[-2\sqrt{1 - x} \right]_0^b \\
 &= \lim_{b \rightarrow 1^-} \left[-2\sqrt{1 - b} - (-2\sqrt{1 - 0}) \right] \\
 &= \lim_{b \rightarrow 1^-} \left[2 - 2\sqrt{1 - b} \right] \\
 &= \boxed{2}.
 \end{aligned}$$

104. We evaluate the improper integral

$$\int_{-\infty}^a e^{(x-e^x)} dx = \lim_{x \rightarrow -\infty} \int_x^a e^x e^{-e^x} dx.$$

Let $u = e^x$, then $du = e^x dx$, and we obtain

$$\begin{aligned} \int_{-\infty}^a e^{(x-e^x)} dx &= \lim_{x \rightarrow -\infty} \int_{e^x}^{e^a} e^{-u} du \\ &= \lim_{x \rightarrow -\infty} [-e^{-u}]_{e^x}^{e^a} \\ &= \lim_{x \rightarrow -\infty} [-e^{-e^a} - (-e^{-e^x})] \\ &= \lim_{x \rightarrow -\infty} [e^{-e^x} - e^{-e^a}] \\ &= \boxed{1 - e^{-e^a}}. \end{aligned}$$

105. We split the integral into two improper integrals:

$$\int_{-\infty}^{\infty} e^{(x-e^x)} dx = \lim_{x \rightarrow -\infty} \int_x^0 e^x e^{-e^x} dx + \lim_{x \rightarrow \infty} \int_0^x e^x e^{-e^x} dx.$$

For the first integral, let $u = e^x$, then $du = e^x dx$, and we obtain

$$\begin{aligned} \lim_{x \rightarrow -\infty} \int_x^0 e^x e^{-e^x} dx &= \lim_{x \rightarrow -\infty} \int_{e^x}^1 e^{-u} du \\ &= \lim_{x \rightarrow -\infty} [-e^{-u}]_{e^x}^1 \\ &= \lim_{x \rightarrow -\infty} [-e^{-1} - (-e^{-e^x})] \\ &= \lim_{x \rightarrow -\infty} [e^{-e^x} - e^{-1}] \\ &= 1 - \frac{1}{e}. \end{aligned}$$

For the second integral, let $u = e^x$, then $du = e^x dx$, and we obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_0^x e^x e^{-e^x} dx &= \lim_{x \rightarrow \infty} \int_1^{e^x} e^{-u} du \\ &= \lim_{x \rightarrow \infty} [-e^{-u}]_1^{e^x} \\ &= \lim_{x \rightarrow \infty} [-e^{-e^x} - (-e^{-1})] \\ &= \lim_{x \rightarrow \infty} [e^{-1} - e^{-e^x}] \\ &= \frac{1}{e}. \end{aligned}$$

So we have

$$\int_{-\infty}^{\infty} e^{(x-e^x)} dx = \left(1 - \frac{1}{e}\right) + \frac{1}{e} = \boxed{1}.$$

106. (a) We need to show that $\int_0^{\ln 2} e^{-x} dx = \int_{\ln 2}^{\infty} e^{-x} dx$. For the first integral we have

$$\begin{aligned}\int_0^{\ln 2} e^{-x} dx &= [-e^{-x}]_0^{\ln 2} \\ &= -e^{-\ln 2} - (-e^{-0}) \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2}.\end{aligned}$$

The improper integral is determined next,

$$\begin{aligned}\int_{\ln 2}^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_{\ln 2}^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} [-e^{-x}]_{\ln 2}^b \\ &= \lim_{b \rightarrow \infty} [-e^{-b} - (-e^{-\ln 2})] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} - e^{-b} \right] \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2}.\end{aligned}$$

So the areas are equal.

(b) The volume obtained by rotating the first region about the x-axis is

$$\begin{aligned}\int_0^{\ln 2} \pi (e^{-x})^2 dx &= \int_0^{\ln 2} \pi e^{-2x} dx \\ &= \left[-\frac{\pi}{2} e^{-2x} \right]_0^{\ln 2} \\ &= -\frac{\pi}{2} e^{-2(\ln 2)} - \left(-\frac{\pi}{2} e^{-2(0)} \right) \\ &= \frac{3\pi}{8}.\end{aligned}$$

The volume obtained by rotating the second region about the x-axis is given by the improper integral

$$\begin{aligned}\int_{\ln 2}^{\infty} \pi (e^{-x})^2 dx &= \lim_{b \rightarrow \infty} \int_{\ln 2}^b \pi e^{-2x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{\pi}{2} e^{-2x} \right]_{\ln 2}^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{\pi}{2} e^{-2b} - \left(-\frac{\pi}{2} e^{-2 \ln 2} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{8} \pi - \frac{1}{2} \pi e^{-2b} \right] \\ &= \frac{\pi}{8}.\end{aligned}$$

So the volume of the first solid is three times the volume of the second solid.

107. Since $a < b$, $f(x) \geq 0$ for all x . The integral of f is 0 outside of the interval $[a, b]$, and so we have

$$\int_{-\infty}^{\infty} f(x) dx = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a}(b-a) = 1.$$

We conclude that f is a probability density function.

108. We have $f(x) \geq 0$ for all x . The integral of f is 0 for $x < 0$, and so we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^{\infty} \frac{1}{a} e^{-x/a} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{a} e^{-x/a} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}b} - \left(-e^{-\frac{1}{a}(0)} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[1 - e^{-\frac{1}{a}b} \right] \\ &= 1. \end{aligned}$$

We conclude that f is a probability density function.

109. We need to evaluate

$$\begin{aligned} \int_{-\infty}^{\infty} xf(x) dx &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \int_a^b x dx \\ &= \frac{1}{b-a} \left[\frac{1}{2}b^2 - \frac{1}{2}a^2 \right] \\ &= \frac{(b-a)(b+a)}{2(b-a)} \\ &= \frac{a+b}{2}. \end{aligned}$$

The mean of f is $\mu = \boxed{\frac{a+b}{2}}$.

110. We need to evaluate

$$\begin{aligned} \int_{-\infty}^{\infty} xf(x) dx &= \int_0^{\infty} \frac{1}{a} xe^{-x/a} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{a} xe^{-x/a} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}x}(a+x) \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}b}(a+b) - \left(-e^{-\frac{1}{a}(0)}(a+0) \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[a - e^{-\frac{1}{a}b}(a+b) \right] \\ &= a. \end{aligned}$$

The mean of f is $\mu = \boxed{a}$.

111. We need to evaluate

$$\begin{aligned}
 \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx &= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx \\
 &= \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\
 &= \frac{1}{b-a} \left[\frac{1}{3} \left(x - \frac{a+b}{2}\right)^3 \right]_a^b \\
 &= \frac{1}{b-a} \left[\frac{1}{3} \left(b - \frac{a+b}{2}\right)^3 - \frac{1}{3} \left(a - \frac{a+b}{2}\right)^3 \right] \\
 &= \frac{1}{b-a} \left[\frac{1}{3} \left(\frac{1}{2}b - \frac{1}{2}a\right)^3 - \frac{1}{3} \left(\frac{1}{2}a - \frac{1}{2}b\right)^3 \right] \\
 &= \frac{1}{24(b-a)} [2(b-a)^3] \\
 &= \frac{(b-a)^2}{12}.
 \end{aligned}$$

The variance of f is $\sigma^2 = \boxed{\frac{(b-a)^2}{12}}$. The standard deviation is $\sigma = \frac{b-a}{2\sqrt{3}}$.

112. We need to evaluate

$$\begin{aligned}
 \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx &= \int_0^{\infty} \frac{1}{a} (x-a)^2 e^{-x/a} dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{a} (x-a)^2 e^{-x/a} dx \\
 &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}x} (a^2 + x^2) \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-e^{-\frac{1}{a}b} (a^2 + b^2) - \left(-e^{-\frac{1}{a}0} (a^2 + 0^2) \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left[a^2 - e^{-\frac{1}{a}b} (a^2 + b^2) \right] \\
 &= a^2.
 \end{aligned}$$

The variance of f is $\sigma^2 = \boxed{a^2}$. The standard deviation is $\sigma = a$.

Chapter 7 Review Exercises

1. We complete the square: $x^2 + 4x + 20 = (x + 2)^2 + 16$. Then we let $u = x + 2$ to obtain

$$\begin{aligned} \int \frac{dx}{x^2 + 4x + 20} &= \int \frac{dx}{(x + 2)^2 + 4^2} \\ &= \int \frac{du}{u^2 + 4^2} \\ &= \frac{1}{4} \tan^{-1} \frac{u}{4} + C \\ &= \boxed{\frac{1}{4} \tan^{-1} \left(\frac{x+2}{4} \right) + C}. \end{aligned}$$

2. We rewrite the integral to obtain the derivative of the denominator, complete the square with $y^2 + y + 1 = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4}$, and let $u = y + \frac{1}{2}$ to obtain

$$\begin{aligned} \int \frac{y + 1}{y^2 + y + 1} dy &= \int \frac{\frac{1}{2}(2y + 1) + \frac{1}{2}}{y^2 + y + 1} dy \\ &= \frac{1}{2} \int \frac{2y + 1}{y^2 + y + 1} dx + \frac{1}{2} \int \frac{dx}{\left(y + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{2} \ln(y^2 + y + 1) + \frac{1}{2} \int \frac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1}{2} \ln(y^2 + y + 1) + \frac{1}{2} \left(\frac{2}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}/2} \right) + C \\ &= \frac{1}{2} \ln(y^2 + y + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2y + 1}{\sqrt{3}} + C \\ &= \boxed{\frac{1}{2} \ln(y^2 + y + 1) + \frac{\sqrt{3}}{3} \tan^{-1} \frac{\sqrt{3}(2y+1)}{3} + C}. \end{aligned}$$

3. Factor out $\sec \phi \tan \phi$. Then let $u = \sec \phi$, so $du = \sec \phi \tan \phi d\phi$. We substitute and obtain

$$\begin{aligned} \int \sec^3 \phi \tan \phi d\phi &= \int \sec^2 \phi \sec \phi \tan \phi d\phi \\ &= \int u^2 du \\ &= \frac{1}{3} u^3 + C \\ &= \boxed{\frac{1}{3} \sec^3 \phi + C}. \end{aligned}$$

4. We use the identity $\cot^2 \theta = \csc^2 \theta - 1$.

$$\begin{aligned} \int \cot^2 \theta \csc \theta d\theta &= \int (\csc^2 \theta - 1) \csc \theta d\theta \\ &= \int \csc^3 \theta d\theta - \int \csc \theta d\theta \\ &= -\frac{1}{2} \csc \theta \cot \theta + \frac{1}{2} \ln |\csc \theta - \cot \theta| \\ &\quad - \ln |\csc \theta - \cot \theta| + C \quad (\text{using Formulas 89 and 14}) \\ &= \boxed{-\frac{1}{2} \csc \theta \cot \theta - \frac{1}{2} \ln |\csc \theta - \cot \theta| + C}. \end{aligned}$$

5. Factor out $\sin \phi$ and use the identity $\sin^2 \phi = 1 - \cos^2 \phi$.

$$\begin{aligned}\int \sin^3 \phi \, d\phi &= \int \sin^2 \phi \sin \phi \, d\phi \\ &= \int (1 - \cos^2 \phi) \sin \phi \, d\phi.\end{aligned}$$

Let $u = \cos \phi$, then $du = -\sin \phi \, d\phi$, so $\sin \phi \, d\phi = -du$. We substitute and obtain

$$\begin{aligned}\int \sin^3 \phi \, d\phi &= \int (1 - u^2)(-du) \\ &= \int (u^2 - 1) \, du \\ &= \frac{1}{3}u^3 - u + C \\ &= \boxed{\frac{1}{3} \cos^3 \phi - \cos \phi + C}.\end{aligned}$$

6. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned}\int \frac{x^2}{\sqrt{4 - x^2}} \, dx &= \int \frac{(2 \sin \theta)^2}{\sqrt{4 - (2 \sin \theta)^2}} (2 \cos \theta) \, d\theta \\ &= 8 \int \frac{\sin^2 \theta}{\sqrt{4 - 4 \sin^2 \theta}} \cos \theta \, d\theta \\ &= 4 \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta \\ &= 4 \int \sin^2 \theta \, d\theta \\ &= 4 \int \frac{1 - \cos(2\theta)}{2} \, d\theta \\ &= 2 \int (1 - \cos(2\theta)) \, d\theta \\ &= 2 \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\ &= 2\theta - 2 \sin \theta \cos \theta + C.\end{aligned}$$

We have $\theta = \sin^{-1}(\frac{x}{2})$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/2)^2} = \frac{1}{2}\sqrt{4 - x^2}$. We obtain

$$\begin{aligned}\int \frac{x^2}{\sqrt{4 - x^2}} \, dx &= 2 \sin^{-1} \left(\frac{x}{2} \right) - 2 \left(\frac{x}{2} \right) \left(\frac{1}{2} \sqrt{4 - x^2} \right) + C \\ &= \boxed{2 \sin^{-1} \left(\frac{x}{2} \right) - \frac{1}{2} x \sqrt{4 - x^2} + C}.\end{aligned}$$

7. Let $u = x + 2$, so $du = dx$, then substitute.

$$\int \frac{dx}{\sqrt{(x+2)^2 - 1}} = \int \frac{dx}{\sqrt{u^2 - 1}}$$

Let $u = \sec \theta$, then $du = \sec \theta \tan \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{(x+2)^2 - 1}} &= \int \frac{\sec \theta \tan \theta d\theta}{\sqrt{\sec^2 \theta - 1}} \\ &= \int \frac{\sec \theta \tan \theta d\theta}{\tan \theta} \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

We have $\sec \theta = u = x + 2$, so $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{u^2 - 1} = \sqrt{(x+2)^2 - 1}$. We obtain

$$\int \frac{dx}{\sqrt{(x+2)^2 - 1}} = \boxed{\ln \left| x + 2 + \sqrt{(x+2)^2 - 1} \right| + C}.$$

8. We use integration by parts with $u = x$ and $dv = \sin(2x) dx$. Then $du = dx$ and $v = \frac{1}{2}(-\cos(2x)) = -\frac{1}{2}\cos(2x)$. We obtain

$$\begin{aligned} \int_0^{\pi/4} x \sin(2x) dx &= \left[x \left(-\frac{1}{2} \cos(2x) \right) \right]_0^{\pi/4} - \int_0^{\pi/4} \left(-\frac{1}{2} \cos(2x) \right) dx \\ &= \left(-\frac{1}{2} \left(\frac{\pi}{4} \right) \cos \left(2 \left(\frac{\pi}{4} \right) \right) \right) - \left(-\frac{1}{2} (0) \cos(2(0)) \right) + \frac{1}{2} \int_0^{\pi/4} \cos(2x) dx \\ &= 0 + \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} \\ &= \frac{1}{2} \left(\frac{1}{2} \sin 2 \left(\frac{\pi}{4} \right) \right) - \frac{1}{2} \left(\frac{1}{2} \sin 2(0) \right) \\ &= \boxed{\frac{1}{4}}. \end{aligned}$$

9. We use integration by parts with $u = v$ and $dw = \csc^2 w dw$. Then $du = dv$ and $w = -\cot v$. We obtain

$$\begin{aligned} \int v \csc^2 v dv &= v(-\cot v) - \int (-\cot v) dv \\ &= -v \cot v + \int \cot v dv \\ &= \boxed{-v \cot v + \ln |\sin v| + C}. \end{aligned}$$

10. Factor out $\cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned} \int \sin^2 x \cos^3 x dx &= \int \sin^2 x \cos^2 x \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cos x dx. \end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. We substitute and obtain

$$\begin{aligned}
 \int \sin^2 x \cos^3 x \, dx &= \int u^2(1-u^2) \, du \\
 &= \int (u^2 - u^4) \, du \\
 &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\
 &= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}.
 \end{aligned}$$

11. Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta \, d\theta$. We substitute and obtain

$$\begin{aligned}
 \int (4-x^2)^{3/2} \, dx &= \int \left(4 - (2 \sin \theta)^2\right)^{3/2} (2 \cos \theta) \, d\theta \\
 &= 2 \int (2 \cos \theta)^3 \cos \theta \, d\theta \\
 &= 16 \int \cos^4 \theta \, d\theta \\
 &= 16 \int \left(\frac{1 + \cos(2\theta)}{2}\right)^2 \, d\theta \\
 &= 4 \int (1 + 2 \cos(2\theta) + \cos^2(2\theta)) \, d\theta \\
 &= 4(\theta + \sin(2\theta)) + 4 \int \frac{1 + \cos(4\theta)}{2} \, d\theta \\
 &= 4\theta + 4 \sin(2\theta) + 2\theta + \frac{1}{2} \sin(4\theta) + C \\
 &= 6\theta + 8 \sin \theta \cos \theta + \sin(2\theta) \cos(2\theta) + C \\
 &= 6\theta + 8 \sin \theta \cos \theta + 2 \sin \theta \cos \theta (1 - 2 \sin^2 \theta) + C
 \end{aligned}$$

We have $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, and $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/2)^2} = \frac{1}{2}\sqrt{4 - x^2}$. We obtain

$$\begin{aligned}
 \int (4-x^2)^{3/2} \, dx &= 6 \sin^{-1}\left(\frac{x}{2}\right) + 8\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right) \\
 &\quad + 2\left(\frac{x}{2}\right)\left(\frac{1}{2}\sqrt{4-x^2}\right)\left(1 - 2\left(\frac{x}{2}\right)^2\right) + C \\
 &= \boxed{6 \sin^{-1}\left(\frac{x}{2}\right) + 2x\sqrt{4-x^2} + \frac{1}{4}x\sqrt{4-x^2}(2-x^2) + C}.
 \end{aligned}$$

12. We use partial fractions to obtain

$$\begin{aligned}
 \frac{3x^2 + 1}{x^3 + 2x^2 - 3x} &= \frac{3x^2 + 1}{x(x+3)(x-1)} \\
 &= \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} \\
 3x^2 + 1 &= A(x+3)(x-1) + Bx(x-1) + Cx(x+3).
 \end{aligned}$$

When $x = 1$ we obtain $C = 1$, when $x = -3$, we have $B = 7/3$, and when $x = 0$, we obtain

$A = -1/3$. So

$$\begin{aligned}\int \frac{3x^2 + 1}{x^3 + 2x^2 - 3x} dx &= \int \left(\frac{-1/3}{x} + \frac{7/3}{x+3} + \frac{1}{x-1} \right) dx \\ &= \int \frac{-1/3}{x} dx + \int \frac{7/3}{x+3} dx + \int \frac{1}{x-1} dx \\ &= \boxed{-\frac{1}{3} \ln |x| + \frac{7}{3} \ln |x+3| + \ln |x-1| + C}.\end{aligned}$$

13. Let $u = e^t$, so $du = e^t dt$. We substitute to obtain

$$\begin{aligned}\int \frac{e^{2t}}{e^t - 2} dt &= \int \frac{e^t}{e^t - 2} e^t dt \\ &= \int \frac{u}{u - 2} du \\ &= \int \frac{u - 2 + 2}{u - 2} du \\ &= \int \left(1 + \frac{2}{u - 2} \right) du \\ &= u + 2 \ln |u - 2| + C \\ &= \boxed{e^t + 2 \ln |e^t - 2| + C}.\end{aligned}$$

14. We rewrite the integral, and obtain

$$\int \frac{dy}{5 + 4y + 4y^2} = \int \frac{dy}{4 + (1 + 2y)^2}.$$

Let $u = 1 + 2y$, so $du = 2 dy$, and substitute:

$$\begin{aligned}\int \frac{dy}{5 + 4y + 4y^2} &= \int \frac{\frac{1}{2} du}{2^2 + u^2} \\ &= \frac{1}{2} \left(\frac{1}{2} \right) \tan^{-1} \frac{u}{2} + C \\ &= \boxed{\frac{1}{4} \tan^{-1} \frac{1+2y}{2} + C}.\end{aligned}$$

15. We use partial fractions to obtain

$$\begin{aligned}\frac{x}{x^4 - 16} &= \frac{x}{(x^2 - 4)(x^2 + 4)} \\ &= \frac{x}{(x - 2)(x + 2)(x^2 + 4)} \\ &= \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 4}\end{aligned}$$

so

$$x = A(x + 2)(x^2 + 4) + B(x - 2)(x^2 + 4) + (Cx + D)(x - 2)(x + 2).$$

Let $x = 2$ to obtain $A = 1/16$, and $x = -2$ to get $B = 1/16$. We now have

$$\begin{aligned}x &= \frac{1}{16}(x + 2)(x^2 + 4) + \frac{1}{16}(x - 2)(x^2 + 4) + (Cx + D)(x - 2)(x + 2) \\ x &= \left(C + \frac{1}{8} \right) x^3 + Dx^2 + \left(\frac{1}{2} - 4C \right) x - 4D.\end{aligned}$$

Equating coefficients, we obtain $C = -1/8$ and $D = 0$. So

$$\begin{aligned}
 \int \frac{x \, dx}{x^4 - 16} &= \int \left(\frac{1/16}{x-2} + \frac{1/16}{x+2} + \frac{(-1/8)x}{x^2+4} \right) dx \\
 &= \frac{1}{16} \ln |x-2| + \frac{1}{16} \ln |x+2| - \frac{1}{16} \ln (x^2+4) + C \\
 &= \frac{1}{16} \ln |x^2-4| - \frac{1}{16} \ln (x^2+4) + C \\
 &= \boxed{\frac{1}{16} \ln \left| \frac{x^2-4}{x^2+4} \right| + C}.
 \end{aligned}$$

16. Let $y = x^2$, so $dy = 2x \, dx$ and $x \, dx = \frac{1}{2} dy$. We substitute and obtain

$$\begin{aligned}
 \int x^3 e^{x^2} \, dx &= \int x^2 e^{x^2} x \, dx \\
 &= \int y e^y \frac{1}{2} dy \\
 &= \frac{1}{2} \int y e^y \, dy.
 \end{aligned}$$

We use integration by parts with $u = y$, $du = dy$, $dv = e^y \, dy$, and $v = e^y$. We obtain

$$\begin{aligned}
 \int x^3 e^{x^2} \, dx &= \frac{1}{2} \left(y e^y - \int e^y \, dy \right) \\
 &= \frac{1}{2} (y e^y - e^y) + C \\
 &= \boxed{\frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C}.
 \end{aligned}$$

17. Let $u = y + 1$, so $du = dy$ and $y = u - 1$. We substitute and obtain

$$\begin{aligned}
 \int \frac{y^2 \, dy}{(y+1)^3} &= \int \frac{(u-1)^2 \, du}{u^3} \\
 &= \int \frac{u^2 - 2u + 1}{u^3} \, du \\
 &= \int \left(\frac{1}{u} - 2u^{-2} + u^{-3} \right) \, du \\
 &= \ln |u| + \frac{2}{u} - \frac{1}{2u^2} + C \\
 &= \boxed{\ln |y+1| + \frac{2}{y+1} - \frac{1}{2(y+1)^2} + C}.
 \end{aligned}$$

18. Let $x = 5 \tan \theta$, then $dx = 5 \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 25}} &= \int \frac{5 \sec^2 \theta}{(5 \tan \theta)^2 \sqrt{(5 \tan \theta)^2 + 25}} d\theta \\ &= \frac{1}{5} \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{25 \tan^2 \theta + 25}} d\theta \\ &= \frac{1}{25} \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta \\ &= \frac{1}{25} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \frac{1}{25} \int \frac{\cos \theta}{\sin^2 \theta} d\theta. \end{aligned}$$

Let $u = \sin \theta$, then $du = \cos \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 25}} &= \frac{1}{25} \int \frac{1}{u^2} du \\ &= \frac{1}{25} \left(-\frac{1}{u} \right) + C \\ &= -\frac{1}{25} \csc \theta + C. \end{aligned}$$

We have $\tan \theta = \frac{x}{5}$, so $\cot \theta = \frac{5}{x}$, and $\csc \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + \left(\frac{5}{x}\right)^2} = \frac{1}{x} \sqrt{x^2 + 25}$. We obtain

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 + 25}} &= -\frac{1}{25} \frac{1}{x} \sqrt{x^2 + 25} + C \\ &= \boxed{-\frac{\sqrt{x^2 + 25}}{25x} + C}. \end{aligned}$$

19. We use integration by parts with $u = x$ and $dv = \sec^2 x dx$. Then $du = dx$ and $v = \tan x$. We obtain

$$\begin{aligned} \int x \sec^2 x dx &= x \tan x - \int \tan x dx \\ &= \boxed{x \tan x - \ln |\sec x| + C}. \end{aligned}$$

20. We complete the square: $16 + 4x - 2x^2 = 2(8 + 2x - x^2) = 2(9 - (x - 1)^2)$. Then we let $u = (x - 1)$ to obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{16 + 4x - 2x^2}} &= \int \frac{dx}{\sqrt{2(9 - (x - 1)^2)}} \\ &= \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{3^2 - u^2}} \\ &= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{u}{3} \right) + C \\ &= \boxed{\frac{\sqrt{2}}{2} \sin^{-1} \left(\frac{x-1}{3} \right) + C}. \end{aligned}$$

21. We use integration by parts with $u = \ln(1 - y)$, $du = \frac{-1}{1-y} dy$, $dv = dy$, and $v = y$. We obtain

$$\begin{aligned}
 \int \ln(1 - y) dy &= y \ln(1 - y) - \int y \left(\frac{-1}{1-y} \right) dy \\
 &= y \ln(1 - y) - \int \frac{-y}{1-y} dy \\
 &= y \ln(1 - y) - \int \frac{(1-y) - 1}{1-y} dy \\
 &= y \ln(1 - y) - \int \left(1 - \frac{1}{1-y} \right) dy \\
 &= \boxed{y \ln(1 - y) - y - \ln(1 - y) + C}.
 \end{aligned}$$

22. We use partial fractions to obtain

$$\begin{aligned}
 \frac{x^3 - 2x - 1}{(x^2 + 1)^2} &= \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \\
 x^3 - 2x - 1 &= (Ax + B)(x^2 + 1) + Cx + D \\
 x^3 - 2x - 1 &= Ax^3 + Bx^2 + (A + C)x + (B + D).
 \end{aligned}$$

We equate coefficients, and obtain $A = 1$, $B = 0$, $C = -3$, and $D = -1$. We now have

$$\begin{aligned}
 \int \frac{x^3 - 2x - 1}{(x^2 + 1)^2} dx &= \int \left(\frac{x}{x^2 + 1} - \frac{3x + 1}{(x^2 + 1)^2} \right) dx \\
 &= \frac{1}{2} \ln(x^2 + 1) - 3 \int \frac{x}{(x^2 + 1)^2} dx - \int \frac{1}{(x^2 + 1)^2} dx \\
 &= \frac{1}{2} \ln(x^2 + 1) + \frac{3}{2(x^2 + 1)} - \int \frac{1}{(x^2 + 1)^2} dx.
 \end{aligned}$$

For the remaining integral, we let $x = \tan \theta$, and obtain

$$\begin{aligned}
 \int \frac{1}{(x^2 + 1)^2} dx &= \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\
 &= \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 + \cos(2\theta)) d\theta \\
 &= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\
 &= \frac{1}{2} \left(\theta + \frac{\tan \theta}{\sec^2 \theta} \right) + C.
 \end{aligned}$$

Since $x = \tan \theta$, $\sec \theta = \sqrt{\tan^2 \theta + 1} = \sqrt{x^2 + 1}$, and we have

$$\begin{aligned}
 \int \frac{1}{(x^2 + 1)^2} dx &= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{(\sqrt{x^2 + 1})^2} + C \\
 &= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{x^2 + 1} + C.
 \end{aligned}$$

And so we obtain

$$\int \frac{x^3 - 2x - 1}{(x^2 + 1)^2} dx = \boxed{\frac{1}{2} \ln(x^2 + 1) + \frac{3}{2(x^2 + 1)} - \frac{1}{2} \tan^{-1} x - \frac{1}{2} \frac{x}{x^2 + 1} + C}.$$

23. We use partial fractions to obtain

$$\begin{aligned} \frac{3x^2 + 2}{x^3 - x^2} &= \frac{3x^2 + 2}{x^2(x - 1)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} \\ 3x^2 + 2 &= Ax(x - 1) + B(x - 1) + Cx^2. \end{aligned}$$

Let $x = 1$ to obtain $C = 5$. Let $x = 0$ to get $B = -2$. Then

$$\begin{aligned} 3x^2 + 2 &= Ax(x - 1) + (-2)(x - 1) + 5x^2 \\ 3x^2 + 2 &= (A + 5)x^2 + (-A - 2)x + 2. \end{aligned}$$

We equate coefficients and determine $A = -2$. We now have

$$\begin{aligned} \int \frac{3x^2 + 2}{x^3 - x^2} dx &= \int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{5}{x - 1} \right) dx \\ &= \boxed{-2 \ln |x| + \frac{2}{x} + 5 \ln |x - 1| + C}. \end{aligned}$$

24. Let $y = \sqrt{2/3} \tan \theta$, then $dy = \sqrt{2/3} \sec^2 \theta d\theta$. We substitute and obtain

$$\begin{aligned} \int \frac{dy}{\sqrt{2 + 3y^2}} &= \int \frac{\sqrt{2/3} \sec^2 \theta d\theta}{\sqrt{2 + 3(\sqrt{2/3} \tan \theta)^2}} \\ &= \int \frac{\sqrt{2/3} \sec^2 \theta d\theta}{\sqrt{2 + 2 \tan^2 \theta}} \\ &= \frac{1}{\sqrt{3}} \int \frac{1}{\sec \theta} (\sec^2 \theta) d\theta \\ &= \frac{1}{\sqrt{3}} \int \sec \theta d\theta \\ &= \frac{1}{\sqrt{3}} \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

We have $\tan \theta = \sqrt{3/2}y$, and $\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + (3/2)y^2} = \frac{\sqrt{2}}{2} \sqrt{2 + 3y^2}$. We obtain

$$\begin{aligned} \int \frac{dy}{\sqrt{2 + 3y^2}} &= \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{2}}{2} \sqrt{2 + 3y^2} + \sqrt{3/2}y \right| + C \\ &= \frac{1}{\sqrt{3}} \ln \left| \frac{\sqrt{2}}{2} \sqrt{2 + 3y^2} + \frac{\sqrt{6}}{2}y \right| + C \\ &= \boxed{\frac{\sqrt{3}}{3} \ln \left| \sqrt{4 + 6y^2} + \sqrt{6}y \right| + C}. \end{aligned}$$

25. We use integration by parts with $u = \sin^{-1} x$, $du = \frac{1}{\sqrt{1-x^2}} dx$, $dv = x^2 dx$, and $v = \frac{1}{3}x^3$. We obtain

$$\begin{aligned}\int x^2 \sin^{-1} x \, dx &= \frac{1}{3}x^3 \sin^{-1} x - \int \left(\frac{1}{3}x^3\right) \frac{1}{\sqrt{1-x^2}} dx \\ &= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3} \int \frac{x^2}{\sqrt{1-x^2}} x \, dx.\end{aligned}$$

Now let $u = 1 - x^2$, so $du = -2x \, dx$, $x \, dx = -\frac{1}{2} du$, and $x^2 = 1 - u$. We substitute and obtain

$$\begin{aligned}\int x^2 \sin^{-1} x \, dx &= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3} \int \frac{1-u}{\sqrt{u}} \left(-\frac{1}{2}\right) du \\ &= \frac{1}{3}x^3 \sin^{-1} x + \frac{1}{6} \int (u^{-1/2} - u^{1/2}) du \\ &= \frac{1}{3}x^3 \sin^{-1} x + \frac{1}{6} \left(2\sqrt{u} - \frac{2}{3}u^{3/2}\right) + C \\ &= \boxed{\frac{1}{3}x^3 \sin^{-1} x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C}.\end{aligned}$$

26. Let $x = \frac{4}{3} \tan \theta$, then $dx = \frac{4}{3} \sec^2 \theta \, d\theta$. We substitute and obtain

$$\begin{aligned}\int \sqrt{16+9x^2} \, dx &= \int \sqrt{16+9\left(\frac{4}{3}\tan\theta\right)^2} \left(\frac{4}{3}\sec^2\theta\right) d\theta \\ &= \frac{4}{3} \int \sqrt{16+16\tan^2\theta} \sec^2\theta \, d\theta \\ &= \frac{4}{3} \int (4\sec\theta) \sec^2\theta \, d\theta \\ &= \frac{16}{3} \int \sec^3\theta \, d\theta \\ &= \frac{16}{3} \left[\frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln |\sec\theta + \tan\theta| \right] + C \\ &= \frac{8}{3} \sec\theta \tan\theta + \frac{8}{3} \ln |\sec\theta + \tan\theta| + C.\end{aligned}$$

We have $\tan \theta = \frac{3x}{4}$, and $\sec \theta = \sqrt{1+\tan^2 \theta} = \sqrt{1+\left(\frac{3x}{4}\right)^2} = \frac{1}{4}\sqrt{16+9x^2}$. We obtain

$$\begin{aligned}\int \sqrt{16+9x^2} \, dx &= \frac{8}{3} \left(\frac{1}{4} \sqrt{16+9x^2} \right) \left(\frac{3x}{4} \right) + \frac{8}{3} \ln \left| \frac{1}{4} \sqrt{16+9x^2} + \frac{3x}{4} \right| + C \\ &= \frac{1}{2} x \sqrt{16+9x^2} + \frac{8}{3} \ln \left| \sqrt{16+9x^2} + 3x \right| - \frac{8}{3} \ln 4 + C \\ &= \boxed{\frac{1}{2} x \sqrt{16+9x^2} + \frac{8}{3} \ln \left| \sqrt{16+9x^2} + 3x \right| + C}.\end{aligned}$$

27. We use partial fractions to obtain

$$\begin{aligned}\frac{1}{x^2+2x} &= \frac{1}{2(x+2)} \\ &= \frac{A}{x} + \frac{B}{x+2} \\ 1 &= A(x+2) + Bx.\end{aligned}$$

When $x = -2$ we obtain $B = -1/2$, and when $x = 0$, we have $A = 1/2$. So

$$\begin{aligned}\int \frac{dx}{x^2 + 2x} &= \int \left(\frac{1/2}{x} + \frac{-1/2}{x+2} \right) dx \\ &= \frac{1}{2} \ln |x| - \frac{1}{2} \ln |x+2| + C \\ &= \boxed{\frac{1}{2} \ln \left| \frac{x}{x+2} \right| + C}.\end{aligned}$$

28. We use formula 105, with $m = 4$, $n = 4$, and we have

$$\begin{aligned}\int \sin^4 y \cos^4 y dy &= -\frac{\sin^3 y \cos^5 y}{4+4} + \frac{4-1}{4+4} \int \sin^{4-2} y \cos^4 y dy \\ &= -\frac{\sin^3 y \cos^5 y}{8} + \frac{3}{8} \int \sin^2 y \cos^4 y dy.\end{aligned}$$

With $m = 2$, $n = 4$ we have

$$\begin{aligned}\int \sin^4 y \cos^4 y dy &= -\frac{\sin^3 y \cos^5 y}{8} + \frac{3}{8} \left(-\frac{\sin y \cos^5 y}{2+4} + \frac{2-1}{2+4} \int \sin^{2-2} y \cos^4 y dy \right) \\ &= -\frac{\sin^3 y \cos^5 y}{8} - \frac{\sin y \cos^5 y}{16} + \frac{1}{16} \int \cos^4 y dy.\end{aligned}$$

We now can use formula 91, and obtain

$$\begin{aligned}\int \sin^4 y \cos^4 y dy &= -\frac{\sin^3 y \cos^5 y}{8} - \frac{\sin y \cos^5 y}{16} + \frac{1}{16} \left(\frac{1}{4} \cos^3 y \sin y + \frac{4-1}{4} \int \cos^2 y dy \right) \\ &= -\frac{\sin^3 y \cos^5 y}{8} - \frac{\sin y \cos^5 y}{16} + \frac{\cos^3 y \sin y}{64} + \frac{3}{64} \int \cos^2 y dy.\end{aligned}$$

Finally, from formula 85, we have

$$\begin{aligned}\int \sin^4 y \cos^4 y dy &= -\frac{\sin^3 y \cos^5 y}{8} - \frac{\sin y \cos^5 y}{16} + \frac{\cos^3 y \sin y}{64} + \frac{3}{64} \left(\frac{y}{2} + \frac{\sin 2y}{4} \right) + C \\ &= \boxed{-\frac{\sin^3 y \cos^5 y}{8} - \frac{\sin y \cos^5 y}{16} + \frac{\cos^3 y \sin y}{64} + \frac{3y}{128} + \frac{3 \sin 2y}{256} + C}.\end{aligned}$$

29. We use partial fractions to obtain

$$\begin{aligned}\frac{w-2}{1-w^2} &= \frac{w-2}{(1-w)(1+w)} \\ &= \frac{A}{1-w} + \frac{B}{1+w} \\ w-2 &= A(1+w) + B(1-w).\end{aligned}$$

Let $w = -1$ to obtain $B = -3/2$, and let $w = 1$ to obtain $A = -1/2$. We now have

$$\begin{aligned}\int \frac{w-2}{1-w^2} dw &= \int \left(\frac{-1/2}{1-w} + \frac{-3/2}{1+w} \right) dw \\ &= \boxed{\frac{1}{2} \ln |1-w| - \frac{3}{2} \ln |1+w| + C}.\end{aligned}$$

30. Let $u = x^2 - 4$, so $du = 2x dx$, and $x dx = \frac{1}{2} du$. We substitute and obtain

$$\begin{aligned}\int \frac{x}{\sqrt{x^2 - 4}} dx &= \int \frac{1}{\sqrt{u}} \left(\frac{1}{2}\right) du \\ &= \frac{1}{2} \int u^{-1/2} du \\ &= \frac{1}{2} (2\sqrt{u}) + C \\ &= \boxed{\sqrt{x^2 - 4} + C}.\end{aligned}$$

31. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}} dx$ and $\frac{1}{\sqrt{x}} dx = 2 du$. We substitute and obtain

$$\begin{aligned}\int \frac{1}{\sqrt{x}} \cos^2 \sqrt{x} dx &= \int (\cos^2 u)(2) du \\ &= 2 \int \frac{1 + \cos(2u)}{2} du \\ &= \int (1 + \cos(2u)) du \\ &= u + \frac{1}{2} \sin(2u) + C \\ &= u + \sin u \cos u + C \\ &= \boxed{\sqrt{x} + \sin \sqrt{x} \cos \sqrt{x} + C}.\end{aligned}$$

32. We use formula 96, and obtain

$$\begin{aligned}\int \sin\left(\frac{\pi}{2}x\right) \sin(\pi x) dx &= -\frac{\sin\left(\frac{\pi}{2} + \pi\right)x}{2\left(\frac{\pi}{2} + \pi\right)} + \frac{\sin\left(\frac{\pi}{2} - \pi\right)x}{2\left(\frac{\pi}{2} - \pi\right)} + C \\ &= -\frac{\sin\left(\frac{3\pi}{2}x\right)}{3\pi} + \frac{\sin\left(-\frac{\pi}{2}x\right)}{-\pi} + C \\ &= \boxed{-\frac{\sin\left(\frac{3\pi}{2}x\right)}{3\pi} + \frac{\sin\left(\frac{\pi}{2}x\right)}{\pi} + C}.\end{aligned}$$

33. We use formula 98, and obtain

$$\begin{aligned}\int \sin x \cos(2x) dx &= -\frac{\cos(1+2)x}{2(1+2)} - \frac{\cos(1-2)x}{2(1-2)} + C \\ &= \boxed{\frac{1}{2} \cos x - \frac{1}{6} \cos(3x) + C}.\end{aligned}$$

34. Let $x = 2 \cos \theta$, so $dx = -2 \sin \theta d\theta$, and the limits of integration are $\theta = \cos^{-1} \frac{0}{2} = \frac{\pi}{2}$ and $\theta = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$. We substitute and obtain

$$\begin{aligned}\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx &= \int_{\pi/2}^{\pi/3} \frac{(2 \cos \theta)^2}{\sqrt{4-(2 \cos \theta)^2}} (-2 \sin \theta) d\theta \\ &= -8 \int_{\pi/2}^{\pi/3} \frac{\cos^2 \theta \sin \theta}{2 \sin \theta} d\theta \\ &= 4 \int_{\pi/3}^{\pi/2} \cos^2 \theta d\theta\end{aligned}$$

$$\begin{aligned}
&= 4 \int_{\pi/3}^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta \\
&= 2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\
&= 2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{\pi/2} \\
&= 2 \left[\frac{\pi}{2} + \frac{1}{2} \sin \left(2 \left(\frac{\pi}{2} \right) \right) - \left(\frac{\pi}{3} + \frac{1}{2} \sin \left(2 \left(\frac{\pi}{3} \right) \right) \right) \right] \\
&= 2 \left[\frac{1}{2} \pi - \left(\frac{1}{3} \pi + \frac{1}{4} \sqrt{3} \right) \right] \\
&= \boxed{\frac{1}{3} \pi - \frac{1}{2} \sqrt{3}}.
\end{aligned}$$

35. Let $u = 1 + x^2$, then $du = 2x dx$, so $x dx = \frac{1}{2} du$. The lower limit of integration is $u = 1 + 0^2 = 1$, and the upper limit becomes $u = 1 + (\sqrt{3})^2 = 4$. We substitute to obtain

$$\begin{aligned}
\int_0^{\sqrt{3}} \frac{x dx}{\sqrt{1+x^2}} &= \int_1^4 \frac{\frac{1}{2} du}{\sqrt{u}} \\
&= \int_1^4 \frac{1}{2} u^{-1/2} du \\
&= \left[u^{1/2} \right]_1^4 \\
&= 4^{1/2} - 1^{1/2} \\
&= \boxed{1}.
\end{aligned}$$

36. (a) Let $u = 2x$, so $du = 2 dx$. We substitute and obtain

$$\int \frac{\cos^2 2x}{\sin^3 2x} dx = \frac{1}{2} \int \cos^2 u (\sin u)^{-3} du.$$

We use Formula 105 with $m = -3$ and $n = 2$, and obtain

$$\begin{aligned}
\int \frac{\cos^2 2x}{\sin^3 2x} dx &= \frac{1}{2} \left(\frac{(\sin u)^{-3+1} \cos^{2-1} u}{-3+2} + \frac{2-1}{-3+2} \int \cos^{2-2} u (\sin u)^{-3} du \right) \\
&= \frac{-\cos u}{2 \sin^2 u} - \frac{1}{2} \int \csc^3 u du.
\end{aligned}$$

Next use Formula 89, and we have

$$\begin{aligned}
\int \frac{\cos^2 2x}{\sin^3 2x} dx &= \frac{-\cos u}{2 \sin^2 u} - \frac{1}{2} \left(-\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln |\csc u - \cot u| \right) + C \\
&= -\frac{1}{2} \cos 2x \csc^2 2x + \frac{1}{4} \csc 2x \cot 2x - \frac{1}{4} \ln |\csc 2x - \cot 2x| + C.
\end{aligned}$$

(b) Using a CAS we obtain

$$\int \frac{\cos^2 2x}{\sin^3 2x} dx = \boxed{-\frac{1}{4} \frac{\cos^3(2x)}{\sin^2(2x)} - \frac{1}{4} \cos(2x) - \frac{1}{4} \ln |\csc(2x) - \cot(2x)| + C}.$$

(c) We simplify the first two terms of the antiderivative in part (a),

$$\begin{aligned}
-\frac{1}{2} \cos 2x \csc^2 2x + \frac{1}{4} \csc 2x \cot 2x &= -\frac{1}{2} \cos 2x \csc^2 2x + \frac{1}{4} \csc 2x (\cos 2x \csc 2x) \\
&= -\frac{1}{2} \cos 2x \csc^2 2x + \frac{1}{4} \csc^2 2x \cos 2x \\
&= -\frac{1}{4} \cos (2x) \csc^2 (2x).
\end{aligned}$$

And likewise, in part (b), we obtain

$$\begin{aligned}
-\frac{1}{4} \frac{\cos^3 (2x)}{\sin^2 (2x)} - \frac{1}{4} \cos (2x) &= -\frac{1}{4} \frac{\cos (2x) (1 - \sin^2 (2x))}{\sin^2 (2x)} - \frac{1}{4} \cos (2x) \\
&= -\frac{1}{4} \cos (2x) \csc^2 (2x) + \frac{1}{4} \cos (2x) - \frac{1}{4} \cos (2x) \\
&= -\frac{1}{4} \cos (2x) \csc^2 (2x).
\end{aligned}$$

And so adding the same logarithm terms, we see that our results above are equal.

37. We integrate by parts, with $u = \tan^{-1} x$, $du = \frac{1}{1+x^2} dx$, $dv = x^n dx$, and $v = \frac{1}{n+1} x^{n+1}$. We obtain

$$\begin{aligned}
\int x^n \tan^{-1} x dx &= \left(\frac{1}{n+1} x^{n+1} \right) \tan^{-1} x - \int \left(\frac{1}{n+1} x^{n+1} \right) \frac{1}{1+x^2} dx \\
&= \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2} dx.
\end{aligned}$$

38. We integrate by parts, with $u = x^n$, $du = nx^{n-1} dx$, $dv = (ax+b)^{1/2} dx$, and $v = \frac{2}{3a}(ax+b)^{3/2}$. We obtain

$$\begin{aligned}
\int x^n (ax+b)^{1/2} dx &= (x^n) \left(\frac{2}{3a} (ax+b)^{3/2} \right) - \int \left(\frac{2}{3a} (ax+b)^{3/2} \right) (nx^{n-1}) dx \\
&= \frac{2x^n}{3a} (ax+b)^{3/2} - \frac{2n}{3a} \int (ax+b)(ax+b)^{1/2} (x^{n-1}) dx \\
&= \frac{2x^n}{3a} (ax+b)^{3/2} \\
&\quad - \frac{2n}{3a} \left(\int (ax)(ax+b)^{1/2} (x^{n-1}) dx + \int (b)(ax+b)^{1/2} (x^{n-1}) dx \right) \\
&= \frac{2x^n}{3a} (ax+b)^{3/2} - \frac{2n}{3} \int x^n (ax+b)^{1/2} dx - \frac{2bn}{3a} \int x^{n-1} (ax+b)^{1/2} dx.
\end{aligned}$$

We add $\frac{2n}{3} \int x^n (ax+b)^{1/2} dx$ to each side and obtain

$$\begin{aligned}
\left(1 + \frac{2n}{3} \right) \int x^n (ax+b)^{1/2} dx &= \frac{2x^n}{3a} (ax+b)^{3/2} - \frac{2bn}{3a} \int x^{n-1} (ax+b)^{1/2} dx \\
\left(\frac{2n+3}{3} \right) \int x^n (ax+b)^{1/2} dx &= \frac{2x^n}{3a} (ax+b)^{3/2} - \frac{2bn}{3a} \int x^{n-1} (ax+b)^{1/2} dx \\
\int x^n (ax+b)^{1/2} dx &= \frac{3}{2n+3} \frac{2x^n}{3a} (ax+b)^{3/2} - \frac{3}{2n+3} \frac{2bn}{3a} \int x^{n-1} (ax+b)^{1/2} dx \\
&= \frac{2x^n}{(2n+3)a} (ax+b)^{3/2} - \frac{2bn}{(2n+3)a} \int x^{n-1} (ax+b)^{1/2} dx
\end{aligned}$$

39. We evaluate

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-1/2} e^{-x^{1/2}} dx.$$

Let $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2} dx$. We substitute to obtain

$$\begin{aligned} \int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= \lim_{b \rightarrow \infty} \int_1^{b^{1/2}} 2e^{-u} du \\ &= \lim_{b \rightarrow \infty} [-2e^{-u}]_1^{b^{1/2}} \\ &= \lim_{b \rightarrow \infty} [-2e^{-b^{1/2}} - (-2e^{-1})] \\ &= \lim_{b \rightarrow \infty} [-2e^{-b^{1/2}} + 2e^{-1}] \\ &= \frac{2}{e}. \end{aligned}$$

The improper integral converges to $\boxed{\frac{2}{e}}$.

40. We evaluate

$$\int_0^1 \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} \sin(x^{1/2}) dx.$$

Let $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2} dx$. We substitute to obtain

$$\begin{aligned} \int_0^1 \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= \lim_{a \rightarrow 0^+} \int_{\sqrt{a}}^1 2 \sin u du \\ &= \lim_{a \rightarrow 0^+} [-2 \cos u]_{\sqrt{a}}^1 \\ &= \lim_{a \rightarrow 0^+} [-2 \cos 1 - (-2 \cos \sqrt{a})] \\ &= \lim_{a \rightarrow 0^+} [-2 \cos 1 + 2 \cos \sqrt{a}] \\ &= -2 \cos 1 + 2(1) \\ &= 2 - 2 \cos 1. \end{aligned}$$

The improper integral converges to $\boxed{2 - 2 \cos 1}$.

41. We evaluate

$$\int_0^1 \frac{x dx}{\sqrt{1-x^2}} = \lim_{b \rightarrow 1^-} \int_0^b \frac{x dx}{\sqrt{1-x^2}}.$$

Let $u = 1 - x^2$ and substitute to obtain

$$\begin{aligned} \int_0^1 \frac{x dx}{\sqrt{1-x^2}} &= \lim_{b \rightarrow 1^-} \int_1^{1-b^2} \frac{-\frac{1}{2} du}{\sqrt{u}} \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{2}\right) \int_1^{1-b^2} u^{-1/2} du \\ &= \lim_{b \rightarrow 1^-} \left[\left(-\frac{1}{2}\right) 2\sqrt{u}\right]_1^{1-b^2} \\ &= \lim_{b \rightarrow 1^-} [-\sqrt{u}]_1^{1-b^2} \\ &= \lim_{b \rightarrow 1^-} [-\sqrt{1-b^2} - (-\sqrt{1})] \\ &= 1. \end{aligned}$$

The improper integral converges to $\boxed{1}$.

42. We evaluate

$$\int_{-\infty}^0 xe^x dx = \lim_{a \rightarrow -\infty} \int_a^0 xe^x dx.$$

We let $u = x$, $du = dx$, $dv = e^x dx$, and $v = e^x$ and integrate by parts to obtain

$$\begin{aligned} \int_{-\infty}^0 xe^x dx &= \lim_{a \rightarrow -\infty} \left([xe^x]_a^0 - \int_a^0 e^x dx \right) \\ &= \lim_{a \rightarrow -\infty} \left(-ae^a - [e^x]_a^0 \right) \\ &= \lim_{a \rightarrow -\infty} \left(-ae^a - (e^0 - e^a) \right) \\ &= -1. \end{aligned}$$

The improper integral converges to $\boxed{-1}$.

43. We evaluate

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin x}{\cos x} dx &= \lim_{b \rightarrow (\pi/2)^-} \int_0^b \tan x dx \\ &= \lim_{b \rightarrow (\pi/2)^-} [\ln |\sec x|]_0^b \\ &= \lim_{b \rightarrow (\pi/2)^-} [\ln |\sec b| - \ln |\sec 0|] \\ &= \lim_{b \rightarrow (\pi/2)^-} [\ln |\sec b|] \\ &= \infty \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

44. We evaluate

$$\int_1^\infty \frac{\sqrt{1+x^{1/8}}}{x^{3/4}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\sqrt{1+x^{1/8}}}{x^{3/4}} dx.$$

Let $u = 1 + x^{1/8}$, then $du = \frac{1}{8}x^{-7/8} dx$, so $dx = 8(u-1)^7 du$. We substitute and obtain

$$\begin{aligned} \int_1^\infty \frac{\sqrt{1+x^{1/8}}}{x^{3/4}} dx &= \lim_{b \rightarrow \infty} \int_2^{1+b^{1/8}} \frac{\sqrt{u}}{(u-1)^6} 8(u-1)^7 du \\ &= \lim_{b \rightarrow \infty} 8 \int_2^{1+b^{1/8}} \sqrt{u} (u-1) du \\ &= \lim_{b \rightarrow \infty} 8 \int_2^{1+b^{1/8}} (u^{3/2} - u^{1/2}) du \\ &= \lim_{b \rightarrow \infty} \left[\frac{16}{15} \left(3u^{\frac{5}{2}} - 5u^{\frac{3}{2}} \right) \right]_2^{1+b^{1/8}} \\ &= \lim_{b \rightarrow \infty} \left[\frac{16}{15} \left(3 \left(1 + b^{1/8} \right)^{\frac{5}{2}} - 5 \left(1 + b^{1/8} \right)^{\frac{3}{2}} \right) - \frac{16}{15} \left(3(2)^{\frac{5}{2}} - 5(2)^{\frac{3}{2}} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{16}{15} \left(1 + b^{1/8} \right)^{\frac{3}{2}} \left(3 \left(1 + b^{1/8} \right) - 5 \right) - \frac{16}{15} \left(3(2)^{\frac{5}{2}} - 5(2)^{\frac{3}{2}} \right) \right] \\ &= \infty. \end{aligned}$$

The improper integral $\boxed{\text{diverges}}$.

45. Since

$$\frac{1 + e^{-x}}{x} \geq \frac{1}{x}$$

for all x , and since $\int_1^\infty \frac{1}{x} dx$ diverges, by the Comparison Test, $\int_1^\infty \frac{1+e^{-x}}{x} dx$ diverges.

46. Since

$$\frac{x}{(1+x)^3} \leq \frac{x}{x^3} = \frac{1}{x^2}$$

for all $x > 0$, and since $\int_0^\infty \frac{1}{x^2} dx$ converges, by the Comparison Test, $\int_0^\infty \frac{x}{(1+x)^3} dx$ converges.

47. We use integration by parts with $u = x^2$ and $dv = \cos x dx$. Then

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx$$

Then $f(x) = \boxed{x^2 \sin x}$.

48. (a) The volume is given by the integral $\int_1^e \pi(\ln x)^2 dx$. We use integration by parts with $u = (\ln x)^2$ and $dv = \pi dx$. Then $du = \frac{2 \ln x}{x} dx$ and $v = \pi x$. We obtain

$$\begin{aligned} \int_1^e \pi(\ln x)^2 dx &= \left[(\ln x)^2 (\pi x) \right]_1^e - \int_1^e (\pi x) \left(\frac{2 \ln x}{x} \right) dx \\ &= \left[(\ln e)^2 (\pi e) - (\ln 1)^2 (\pi(1)) \right] - 2\pi \int_1^e \ln x dx \\ &= \pi e - 2\pi \int_1^e \ln x dx \end{aligned}$$

We use integration by parts again with $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. We obtain

$$\begin{aligned} \int_1^e \pi(\ln x)^2 dx &= \pi e - 2\pi \left([(\ln x)x]_1^e - \int_1^e x \left(\frac{1}{x} \right) dx \right) \\ &= \pi e - 2\pi \left([(\ln e)e - (\ln 1)1] - \int_1^e dx \right) \\ &= \pi e - 2\pi(e - (e - 1)) \\ &= \boxed{\pi e - 2\pi}. \end{aligned}$$

(b) The volume is given by the integral $\int_1^e 2\pi x \ln x dx$. We use integration by parts with $u = \ln x$ and $dv = 2\pi x dx$. Then $du = \frac{1}{x} dx$ and $v = \pi x^2$. We obtain

$$\begin{aligned} \int_1^e 2\pi x \ln x dx &= [(\ln x)(\pi x^2)]_1^e - \int_1^e (\pi x^2) \left(\frac{1}{x} \right) dx \\ &= \left[(\ln e)(\pi e^2) - (\ln 1)(\pi(1)^2) \right] - \pi \int_1^e x dx \\ &= \pi e^2 - \pi \left[\frac{1}{2} x^2 \right]_1^e \\ &= \pi e^2 - \pi \left(\frac{1}{2} e^2 - \frac{1}{2} (1)^2 \right) \\ &= \boxed{\frac{1}{2} \pi e^2 + \frac{1}{2} \pi}. \end{aligned}$$

49. The arc length is given by $\int_0^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{\pi/2} \sqrt{1 + (\cos x)^2} dx = \int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx$.

(a) With $n = 3$ we have $\Delta x = \frac{(\pi/2)-0}{3} = \frac{\pi}{6}$. So the Trapezoidal Rule provides the approximation

$$\begin{aligned} \int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx &\approx \frac{\Delta x}{2} \left[f(0) + 2f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/6}{2} \left[\sqrt{1 + \cos^2 0} + 2\sqrt{1 + \cos^2 \frac{\pi}{6}} + 2\sqrt{1 + \cos^2 \frac{\pi}{3}} + \sqrt{1 + \cos^2 \frac{\pi}{2}} \right] \\ &\approx \boxed{1.910}. \end{aligned}$$

(b) With $n = 4$ we have $\Delta x = \frac{(\pi/2)-0}{4} = \frac{\pi}{8}$. So Simpson's Rule provides the approximation

$$\begin{aligned} \int_0^{\pi/2} \sqrt{1 + \cos^2 x} dx &\approx \frac{\Delta x}{3} \left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{4}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi/8}{3} \left[\sqrt{1 + \cos^2 0} + 4\sqrt{1 + \cos^2 \frac{\pi}{8}} + 2\sqrt{1 + \cos^2 \frac{\pi}{4}} + 4\sqrt{1 + \cos^2 \frac{3\pi}{8}} + \sqrt{1 + \cos^2 \frac{\pi}{2}} \right] \\ &\approx \boxed{1.910}. \end{aligned}$$

50. The distance traveled is given by the integral $\int_1^4 v dt$. We approximate by the Trapezoidal Rule, with $\Delta t = 0.5$.

$$\begin{aligned} \int_1^4 v dx &\approx \frac{\Delta t}{2} [f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)] \\ &= \frac{0.5}{2} [3 + 2(4.3) + 2(4.6) + 2(5.1) + 2(5.8) + 2(6.2) + 6.6] \\ &= \boxed{15.4 \text{ meters}}. \end{aligned}$$

51. The area is given by the improper integral

$$\begin{aligned} \int_0^1 x^{-2/3} dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^{-2/3} dx \\ &= \lim_{a \rightarrow 0^+} \left[3x^{1/3} \right]_a^1 \\ &= \lim_{a \rightarrow 0^+} \left[3(1)^{1/3} - 3(a)^{1/3} \right] \\ &= 3. \end{aligned}$$

The area of the region is $\boxed{3}$.

52. The volume is given by the improper integral

$$\begin{aligned} \int_0^1 \pi \left(x^{-2/3} \right)^2 dx &= \lim_{a \rightarrow 0^+} \int_a^1 \pi x^{-4/3} dx \\ &= \lim_{a \rightarrow 0^+} \left[-3\pi x^{-1/3} \right]_a^1 \\ &= \lim_{a \rightarrow 0^+} \left[-3\pi(1)^{-1/3} - \left(-3\pi(a)^{-1/3} \right) \right] \\ &= \infty. \end{aligned}$$

The improper integral diverges, so the volume does not exist.

Chapter 8: Infinite Series

8.1 Sequences

Concepts and Vocabulary

1. **False:** A sequence is a function whose domain is the set of positive *integers* and not the set of positive *real numbers*.
2. **False:** A sequence $\{s_n\}$ is convergent to a limit L if $\lim_{n \rightarrow \infty} s_n = L$. ($L = 0$ is a special case, but in general a sequence that is convergent will converge to a real number L .)
3. **True:** This is because if $f(x)$ is a related function of the sequence $\{s_n\}$ then if $\lim_{x \rightarrow \infty} f(x) = L$ it follows that $\lim_{n \rightarrow \infty} s_n = L$ (theorem, p.544).
4. **(b) :** Bounded, because no term of $\{s_n\}$ will ever exceed in absolute value, K .
5. **False:** Consider the sequence $\{s_n\} = \{\sin(\frac{\pi}{2}n)\}$ (see p.549, fig.13). This sequence is bounded by ± 1 , but it never converges as it oscillates between -1 and $+1$.
6. **True:** A sequence whose n^{th} term cannot be bounded in absolute value by any finite real number cannot converge to a finite real number.
7. **False:** A sequence is decreasing if and only if $s_n > s_{n+1}$ for $n \geq 1$.
8. **False:** As an example, consider the sequence $\{t_n\} = \left\{1 + \frac{(-1)^n}{n^2}\right\}$ (see fig.14, p.549). This sequence is not monotonic, but it converges to the limit 1.
9. **False:** If the sequence $\{s_n\}$ is increasing then the algebraic ratio $\frac{s_{n+1}}{s_n} > 1$ for all $n \geq 1$.
10. **(b) :** This follows from the properties of a monotonic function. A function whose derivative is < 0 at $x = x_0$ is decreasing in an open interval about $x = x_0$. Since the function is a related function of the sequence $\{s_n\}$ it follows that the sequence is also decreasing.
11. **True:** It is only the eventual (large n) terms of the sequence which determine its convergence or divergence properties.
12. **False:** A bounded, monotonic sequence will be a convergent sequence (see theorem, p.549).

Skill Building

13. $s_n = \frac{n+1}{n}$. Put $n = 1, 2, 3, 4$. Then $s_1 = 2$; $s_2 = \frac{3}{2}$; $s_3 = \frac{4}{3}$; $s_4 = \frac{5}{4}$. The first four terms of the sequence are $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$.
14. $s_n = \frac{2}{n^2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = 2$; $s_2 = \frac{2}{4} = \frac{1}{2}$; $s_3 = \frac{2}{9}$; $s_4 = \frac{2}{16} = \frac{1}{8}$. The first four terms of the sequence are $2, \frac{1}{2}, \frac{2}{9}, \frac{1}{8}$.
15. $s_n = \ln n$. Put $n = 1, 2, 3, 4$. Then $s_1 = \ln 1 = 0$; $s_2 = \ln 2$; $s_3 = \ln 3$; $s_4 = \ln 4$. The first four terms of the sequence are $0, \ln 2, \ln 3, \ln 4$.

16. $s_n = \frac{n}{\ln(n+1)}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1}{\ln 2}$; $s_2 = \frac{2}{\ln 3}$; $s_3 = \frac{3}{\ln 4}$; $s_4 = \frac{4}{\ln 5}$. The first four terms of the sequence are $\boxed{\frac{1}{\ln 2}, \frac{2}{\ln 3}, \frac{3}{\ln 4}, \frac{4}{\ln 5}}$.

17. $s_n = \frac{(-1)^{n+1}}{2n+1}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{(-1)^2}{2+1} = \frac{1}{3}$; $s_2 = \frac{(-1)^3}{4+1} = -\frac{1}{5}$; $s_3 = \frac{(-1)^4}{6+1} = \frac{1}{7}$; $s_4 = \frac{(-1)^5}{8+1} = -\frac{1}{9}$. The first four terms of the sequence are $\boxed{\frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{9}}$.

18. $s_n = \frac{1-(-1)^n}{2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1-(-1)^1}{2} = \frac{1+1}{2} = 1$; $s_2 = \frac{1-(-1)^2}{2} = \frac{1-1}{2} = 0$; $s_3 = \frac{1-(-1)^3}{2} = \frac{1+1}{2} = 1$; $s_4 = \frac{1-(-1)^4}{2} = \frac{1-1}{2} = 0$. The first four terms of the sequence are $\boxed{1, 0, 1, 0}$.

19. $s_n = \begin{cases} (-1)^{n+1} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$ Put $n = 1, 2, 3, 4$. Then $s_1 = 1$; $s_2 = (-1)^{2+1} = (-1)^3 = -1$; $s_3 = 1$; $s_4 = (-1)^{4+1} = (-1)^5 = -1$. The first four terms of the sequence are $\boxed{1, -1, 1, -1}$.

20. $s_n = \begin{cases} n^2 + n & \text{if } n \text{ is even} \\ 4n + 1 & \text{if } n \text{ is odd.} \end{cases}$ Put $n = 1, 2, 3, 4$. Then $s_1 = 4(1) + 1 = 5$; $s_2 = 2^2 + 2 = 6$; $s_3 = 4(3) + 1 = 13$; $s_4 = 4^2 + 4 = 20$. The first four terms of the sequence are $\boxed{5, 6, 13, 20}$.

21. $s_n = \frac{n!}{2^n}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1!}{2^1} = \frac{1}{2}$; $s_2 = \frac{2!}{2^2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{1}{2}$; $s_3 = \frac{3!}{2^3} = \frac{3 \cdot 2 \cdot 1}{8} = \frac{3}{4}$; $s_4 = \frac{4!}{2^4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{16} = \frac{3}{2}$. The first four terms of the sequence are $\boxed{\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2}}$.

22. $s_n = \frac{n!}{n^2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1!}{1^2} = 1$; $s_2 = \frac{2!}{2^2} = \frac{2 \cdot 1}{4} = \frac{1}{2}$; $s_3 = \frac{3!}{3^2} = \frac{3 \cdot 2 \cdot 1}{9} = \frac{2}{3}$; $s_4 = \frac{4!}{4^2} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{16} = \frac{3}{2}$. The first four terms of the sequence are $\boxed{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}}$.

23. $2, 4, 6, 8, 10, \dots$ Observe that this is the set of even numbers. The n^{th} of term of the sequence is $\boxed{2n}$.

24. $1, 3, 5, 7, 9, \dots$ Observe that this is the set of odd numbers. The n^{th} term of the sequence is $\boxed{2n - 1}$. (We don't write $2n + 1$ because the first term, for $n = 1$, needs to be 1.)

25. $2, 4, 8, 16, 32, \dots$ Notice that each term is two times the preceding term. So the n^{th} term $= 2 \times (n-1)^{\text{st}}$ term $= 2 \times 2 \times (n-2)^{\text{nd}}$ term $= \dots$ etc. The n^{th} term of the sequence is $\boxed{2^n}$.

26. $1, 8, 27, 64, 125, \dots$ Here, each term is a perfect cube. The n^{th} term of the sequence is $\boxed{n^3}$.

27. $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots$ Notice that every even term appears with a minus sign. So we need a factor of $(-1)^{n+1}$ to appear in the n^{th} term of the sequence, which is $\boxed{(-1)^{n+1} \frac{1}{n+1}}$.

28. $1, -2, 3, -4, 5, \dots$ Notice that every even term appears with a minus sign. So we need a factor of $(-1)^{n+1}$ to appear in the n^{th} term of the sequence, which is $\boxed{(-1)^{n+1} n}$.

29. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ Each term has both numerator and denominator increasing, with denominator being 1 more than the numerator. So the n^{th} term of the sequence is $\boxed{\frac{n}{n+1}}$.

30. $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \dots$ The numerators are all perfect squares, while the denominators are just the consecutive positive integers plus 1. So the n^{th} term of the sequence is $\boxed{\frac{n^2}{n+1}}$.

31. $1, 1, 2, 6, 24, 120, 720, \dots$ The rapidly increasing sequence indicates that the factorial could be involved. The n^{th} term of the sequence $\boxed{(n-1)!}$ generates this sequence because $(1-1)! = 0! = 1$; $(2-1)! = 1! = 1$; $(3-1)! = 2! = 2$, and so on.

32. $1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots$ The rapidly increasing denominators could mean the factorial appears there. The n^{th} term of the sequence $\boxed{\frac{1}{(n-1)!}}$ generates this sequence because $\frac{1}{(1-1)!} = \frac{1}{0!} = \frac{1}{1} = 1$; $\frac{1}{(2-1)!} = \frac{1}{1!} = 1$; $\frac{1}{(3-1)!} = \frac{1}{2!} = \frac{1}{2}$, and so on.

33. The n^{th} term of the sequence is $s_n = \frac{3}{n}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{3}{n} = 3 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 3 \cdot 0 = 0,$$

where the constant multiple property of convergent sequences was used. The limit of the sequence is $\boxed{0}$.

34. The n^{th} term of the sequence is $s_n = -\frac{2}{n}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} -\frac{2}{n} = (-2) \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = (-2) \cdot 0 = 0,$$

where the constant multiple property of convergent sequences was used. The limit of the sequence is $\boxed{0}$.

35. The n^{th} term of the sequence is $s_n = 1 - \frac{1}{n}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{1}$.

36. The n^{th} term of the sequence is $s_n = \frac{1}{n} + 4$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + 4\right) = \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} 4 = 0 + 4 = 4,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{4}$.

37. The n^{th} term of the sequence is $s_n = \frac{4n+2}{n} = 4 + \frac{2}{n}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(4 + \frac{2}{n}\right) = \lim_{n \rightarrow \infty} 4 + \lim_{n \rightarrow \infty} \frac{2}{n} = \lim_{n \rightarrow \infty} 4 + 2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 4 + 2 \cdot 0 = 4,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{4}$.

38. The n^{th} term of the sequence is $s_n = \frac{2n+1}{n} = 2 + \frac{1}{n}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(2 + \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n} = 2 + 0 = 2,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{2}$.

39. The n^{th} term of the sequence is $s_n = \left(\frac{2-n}{n^2}\right)^4 = \left(\frac{2}{n^2} - \frac{1}{n}\right)^4$. We have $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{2}{n^2} - \frac{1}{n}\right)^4 = \left[\lim_{n \rightarrow \infty} \left(\frac{2}{n^2} - \frac{1}{n}\right)\right]^4 = \left[\lim_{n \rightarrow \infty} \frac{2}{n^2} - \lim_{n \rightarrow \infty} \frac{1}{n}\right]^4 = \left[2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} - \lim_{n \rightarrow \infty} \frac{1}{n}\right]^4 = [2 \cdot 0 - 0]^4 = 0$, where the power, sum and difference, and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{0}$.

40. The n^{th} term of the sequence is $s_n = \left(\frac{n^3-2n}{n^3}\right)^2 = \left(1 - \frac{2}{n^2}\right)^2$. We have $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{2}{n^2}\right)^2 = \left[\lim_{n \rightarrow \infty} \left(1 - \frac{2}{n^2}\right)\right]^2 = \left[\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{2}{n^2}\right]^2 = \left[\lim_{n \rightarrow \infty} 1 - 2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2}\right]^2 = [1 - 2 \cdot 0]^2 = 1^2 = 1$, where the power, sum and difference, and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{1}$.

41. The n^{th} term of the sequence is $s_n = \sqrt{\frac{n+1}{n^2}} = \sqrt{\frac{1}{n} + \frac{1}{n^2}}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{n} + \frac{1}{n^2}} = \left[\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)\right]^{\frac{1}{2}} = \left[\lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n^2}\right]^{\frac{1}{2}} = [0 + 0]^{\frac{1}{2}} = 0,$$

where the power and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{0}$.

42. The n^{th} term of the sequence is $s_n = \sqrt[3]{8 - \frac{1}{n}} = \left(8 - \frac{1}{n}\right)^{\frac{1}{3}}$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(8 - \frac{1}{n}\right)^{\frac{1}{3}} = \left[\lim_{n \rightarrow \infty} \left(8 - \frac{1}{n}\right)\right]^{\frac{1}{3}} = \left[\lim_{n \rightarrow \infty} 8 - \lim_{n \rightarrow \infty} \frac{1}{n}\right]^{\frac{1}{3}} = [8 - 0]^{\frac{1}{3}} = 2,$$

where the power and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{2}$.

43. The n^{th} term of the sequence is $s_n = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right)$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right) &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right) \\ &= \left(1 - \lim_{n \rightarrow \infty} \frac{1}{n}\right) \cdot \left(1 - \lim_{n \rightarrow \infty} \frac{1}{n^2}\right) \\ &= (1 - 0)(1 - 0) = 1, \end{aligned}$$

where the product and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{1}$.

44. The n^{th} term of the sequence is $s_n = \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{n^3}\right)$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{n^3}\right) &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right) \cdot \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^3}\right) \\ &= \left(1 - \lim_{n \rightarrow \infty} \frac{1}{n}\right) \cdot \left(1 - \lim_{n \rightarrow \infty} \frac{1}{n^2}\right) \cdot \left(1 - \lim_{n \rightarrow \infty} \frac{1}{n^3}\right) \\ &= (1 - 0)(1 - 0)(1 - 0) = 1, \end{aligned}$$

where the product and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{1}$.

45. Consider a sequence $\{s_n\}$ with $s_n = \frac{n+1}{3n} = \frac{1}{3} + \frac{1}{3n}$. This sequence converges because

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{3n}\right) = \lim_{n \rightarrow \infty} \frac{1}{3} + \lim_{n \rightarrow \infty} \frac{1}{3n} = \lim_{n \rightarrow \infty} \frac{1}{3} + \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{3} + 0 = \frac{1}{3},$$

where the sum and difference and constant multiple properties of convergent sequences were used. Then by the theorem on p.543, since the function $f(x) = \ln x$ is continuous at $x = \frac{1}{3}$, the sequence $\{f(s_n)\}$

converges, and limit of the sequence $\{f(s_n)\} = \{\ln(\frac{n+1}{3n})\}$ is $\boxed{\ln\left(\frac{1}{3}\right)}$.

46. Consider a sequence $\{s_n\}$ with $s_n = \frac{n^2+2}{2n^2+3}$. This sequence converges because

$$\begin{aligned}\lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{n^2+2}{2n^2+3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n^2}}{2 + \frac{3}{n^2}} = \frac{\lim_{n \rightarrow \infty} (1 + \frac{2}{n^2})}{\lim_{n \rightarrow \infty} (2 + \frac{3}{n^2})} \\ &= \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{2}{n^2}}{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{3}{n^2}} = \frac{\lim_{n \rightarrow \infty} 1 + 2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2}}{\lim_{n \rightarrow \infty} 2 + 3 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2}} \\ &= \frac{1 + 2 \cdot 0}{2 + 3 \cdot 0} = \frac{1}{2},\end{aligned}$$

where the quotient, sum and difference, and constant multiple properties of convergent sequences were used. The function $f(x) = \ln x$ is continuous at $x = \frac{1}{2}$. By the theorem on p. 543, the sequence $\{f(s_n)\}$

converges, and the limit of the sequence $\{f(s_n)\} = \left\{\ln\left(\frac{n^2+2}{2n^2+3}\right)\right\}$ is $\boxed{\ln\left(\frac{1}{2}\right)}$.

47. Consider the sequence $\{s_n\}$ with $s_n = \frac{4}{n} - 2$. This sequence converges because

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{4}{n} - 2\right) = \lim_{n \rightarrow \infty} \frac{4}{n} - \lim_{n \rightarrow \infty} 2 = 4 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} - \lim_{n \rightarrow \infty} 2 = 4 \cdot 0 - 2 = -2,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The function $f(x) = e^x$ is continuous at $x = -2$. By the theorem on p. 543, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \{e^{(4/n)-2}\}$ is $\boxed{e^{-2}}$.

48. Consider the sequence $\{s_n\}$ with $s_n = 3 + \frac{6}{n}$. This sequence converges because

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(3 + \frac{6}{n}\right) = \lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} \frac{6}{n} = \lim_{n \rightarrow \infty} 3 + 6 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 3 + 6 \cdot 0 = 3,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The function $f(x) = e^x$ is continuous at $x = 3$. By the theorem on p.543, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \{e^{3+(6/n)}\}$ is $\boxed{e^3}$.

49. Consider the sequence $\{s_n\}$ with $s_n = \frac{1}{n}$. This sequence converges because $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$. The function $f(x) = \sin x$ is continuous at $x = 0$. By the theorem on p.543, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{\sin\left(\frac{1}{n}\right)\right\}$ will be $\sin(0) = \boxed{0}$.

50. Consider the sequence $\{s_n\}$ with $s_n = \frac{1}{n}$. This sequence converges because $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$. The function $f(x) = \cos x$ is continuous at $x = 0$. By the theorem on p.543, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{\cos\left(\frac{1}{n}\right)\right\}$ will be $\cos(0) = \boxed{1}$.

51. The function $f(x) = \frac{x^2-4}{x^2+x-2}$ is a related function of the sequence $\left\{\frac{n^2-4}{n^2+n-2}\right\}$. Since

$$\begin{aligned}
\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^2 - 4}{x^2 + x - 2} = \lim_{x \rightarrow \infty} \frac{1 - \frac{4}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} \\
&= \frac{\lim_{x \rightarrow \infty} (1 - \frac{4}{x^2})}{\lim_{x \rightarrow \infty} (1 + \frac{1}{x} - \frac{2}{x^2})} = \frac{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} \frac{4}{x^2}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{1}{x} - \lim_{x \rightarrow \infty} \frac{2}{x^2}} \\
&= \frac{1 - 0}{1 + 0 - 0} = 1,
\end{aligned}$$

the sequence $\left\{ \frac{n^2-4}{n^2+n-2} \right\}$ also converges and its limit is $\boxed{1}$.

52. The function $f(x) = \frac{x+2}{x^2+6x+8}$ is a related function of the sequence $\left\{ \frac{n+2}{n^2+6n+8} \right\}$. Since

$$\begin{aligned}
\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x+2}{x^2+6x+8} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{2}{x^2}}{1 + \frac{6}{x} + \frac{8}{x^2}} \\
&= \frac{\lim_{x \rightarrow \infty} (\frac{1}{x} + \frac{2}{x^2})}{\lim_{x \rightarrow \infty} (1 + \frac{6}{x} + \frac{8}{x^2})} = \frac{\lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{2}{x^2}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{6}{x} + \lim_{x \rightarrow \infty} \frac{8}{x^2}} \\
&= \frac{0+0}{1+0+0} = 0,
\end{aligned}$$

the sequence $\left\{ \frac{n+2}{n^2+6n+8} \right\}$ also converges and its limit is $\boxed{0}$.

53. We write the n^{th} term of the sequence $\{s_n\} = \left\{ \frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right\}$ as

$$s_n = \frac{n^2}{2n+1} - \frac{n^2}{2n-1} = \frac{n^2[(2n-1) - (2n+1)]}{(2n)^2 - 1^2} = \frac{n^2(-2)}{4n^2 - 1} = -\frac{2n^2}{4n^2 - 1}.$$

The function $f(x) = -\frac{2x^2}{4x^2-1}$ is a related function of the sequence $\{s_n\}$. Since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} -\frac{2x^2}{4x^2-1} = \lim_{x \rightarrow \infty} \frac{-2}{4 - \frac{1}{x^2}} = \frac{\lim_{x \rightarrow \infty} (-2)}{\lim_{x \rightarrow \infty} (4 - \frac{1}{x^2})} = \frac{\lim_{x \rightarrow \infty} (-2)}{\lim_{x \rightarrow \infty} 4 - \lim_{x \rightarrow \infty} \frac{1}{x^2}} = \frac{-2}{4-0} = -\frac{1}{2},$$

the sequence $\{s_n\} = \left\{ \frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right\}$ also converges and its limit is $\boxed{-\frac{1}{2}}$.

54. The function $f(x) = \frac{6x^4-5}{7x^4+3}$ is a related function of the sequence $\left\{ \frac{6n^4-5}{7n^4+3} \right\}$. Since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{6x^4-5}{7x^4+3} = \lim_{x \rightarrow \infty} \frac{6 - \frac{5}{x^4}}{7 + \frac{3}{x^4}} = \frac{\lim_{x \rightarrow \infty} (6 - \frac{5}{x^4})}{\lim_{x \rightarrow \infty} (7 + \frac{3}{x^4})} = \frac{\lim_{x \rightarrow \infty} 6 - \lim_{x \rightarrow \infty} \frac{5}{x^4}}{\lim_{x \rightarrow \infty} 7 + \lim_{x \rightarrow \infty} \frac{3}{x^4}} = \frac{6-0}{7+0} = \frac{6}{7},$$

the sequence $\left\{ \frac{6n^4-5}{7n^4+3} \right\}$ also converges and its limit is $\boxed{\frac{6}{7}}$.

55. The function $f(x) = \frac{\sqrt{x}+2}{\sqrt{x}+5}$ is a related function of the sequence $\left\{ \frac{\sqrt{n}+2}{\sqrt{n}+5} \right\}$. Since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\sqrt{x}+2}{\sqrt{x}+5} = \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{\sqrt{x}}}{1 + \frac{5}{\sqrt{x}}} = \frac{\lim_{x \rightarrow \infty} (1 + \frac{2}{\sqrt{x}})}{\lim_{x \rightarrow \infty} (1 + \frac{5}{\sqrt{x}})} = \frac{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{5}{\sqrt{x}}} = \frac{1+0}{1+0} = 1,$$

the sequence $\left\{\frac{\sqrt{n+2}}{\sqrt{n+5}}\right\}$ also converges and its limit is $\boxed{1}$.

56. The function $f(x) = \frac{\sqrt{x}}{e^x}$ is a related function of the sequence $\left\{\frac{\sqrt{n}}{e^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}e^x} = 0,$$

the sequence $\left\{\frac{\sqrt{n}}{e^n}\right\}$ also converges and its limit is $\boxed{0}$.

57. The function $f(x) = \frac{x^2}{3^x}$ is a related function of the sequence $\left\{\frac{n^2}{3^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{3^x} = \lim_{x \rightarrow \infty} \frac{2x}{3^x \ln 3} = \lim_{x \rightarrow \infty} \frac{2}{3^x (\ln 3)^2} = 0,$$

the sequence $\left\{\frac{n^2}{3^n}\right\}$ also converges and its limit is $\boxed{0}$.

58. The function $f(x) = \frac{(x-1)^2}{e^x}$ is a related function of the sequence $\left\{\frac{(n-1)^2}{e^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{(x-1)^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2(x-1)}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0,$$

the sequence $\left\{\frac{(n-1)^2}{e^n}\right\}$ also converges and its limit is $\boxed{0}$.

59. To see if the sequence $\{s_n\} = \left\{\frac{(-1)^n}{3n^2}\right\}$ converges, consider the absolute value of the n^{th} term:

$$|s_n| = \left|\frac{(-1)^n}{3n^2}\right| \leq \frac{1}{3n^2}.$$

That means $-\frac{1}{3n^2} \leq \frac{(-1)^n}{3n^2} \leq \frac{1}{3n^2}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{3n^2}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{3n^2}\right\}$. Since $a_n \leq s_n \leq b_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{-1}{3n^2} = (-\frac{1}{3}) \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} = -\frac{1}{3} \cdot 0 = 0$ and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{3n^2} = \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} = \frac{1}{3} \cdot 0 = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{(-1)^n}{3n^2}\right\}$ converges, and its limit is $\boxed{0}$.

60. To see if the sequence $\{s_n\} = \left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges, consider the absolute value of the n^{th} term:

$$|s_n| = \left|\frac{(-1)^n}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}}.$$

That means $-\frac{1}{\sqrt{n}} \leq \frac{(-1)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{\sqrt{n}}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{\sqrt{n}}\right\}$. Since $a_n \leq s_n \leq b_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{-1}{\sqrt{n}} = 0$ and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges, and its limit is $\boxed{0}$.

61. To see if the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ converges, consider the absolute value of the n^{th} term:

$$|s_n| = \left|\frac{\sin n}{n}\right| \leq \frac{1}{n}.$$

That is, $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. Since $a_n \leq s_n \leq b_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} -\frac{1}{n} = 0$ and

$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ converges, and its limit is $\boxed{0}$.

62. To see if the sequence $\{s_n\} = \left\{\frac{\cos n}{n}\right\}$ converges, consider the absolute value of the n^{th} term:

$$|s_n| = \left|\frac{\cos n}{n}\right| \leq \frac{1}{n}.$$

That is, $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. Since $a_n \leq s_n \leq b_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{-1}{n} = 0$ and

$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{\cos n}{n}\right\}$ converges, and its limit is $\boxed{0}$.

63. The sequence $\{s_n\} = \{\cos(\pi n)\}$ oscillates between -1 and $+1$ for odd n or even n respectively. There is no single number that the terms of the sequence approach as $n \rightarrow \infty$. So the sequence $\{s_n\} = \{\cos(\pi n)\}$ diverges.

64. The sequence $\{s_n\} = \left\{\cos\left(\frac{\pi}{2}n\right)\right\}$ oscillates between $0, -1$ or $+1$ depending on whether n is odd, a multiple of 2 , or a multiple of 4 , respectively. There is no single number that the terms of the sequence approach as $n \rightarrow \infty$. So the sequence $\{s_n\} = \left\{\cos\left(\frac{\pi}{2}n\right)\right\}$ diverges.

65. The sequence is $\{s_n\} = \{\sqrt{n}\}$. The terms of the sequence are $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots$. Given any positive number $M > 1$, we choose a positive integer $N > M^2 > 1$. Then whenever $n > N > 1$, we have

$$s_n = \sqrt{n} > \sqrt{N} > M.$$

That is, the sequence $\{s_n\} = \{\sqrt{n}\}$ diverges to infinity.

66. The sequence is $\{s_n\} = \{n^2\}$. The terms of the sequence are $1, 4, 9, 16, \dots$. Given any positive number $M > 1$, we choose a positive integer $N > \sqrt{M} > 1$. Then whenever $n > N > 1$, we have

$$s_n = n^2 > N^2 > M.$$

That is, the sequence $\{s_n\} = \{n^2\}$ diverges to infinity.

67. The sequence $\{s_n\} = \left\{\left(-\frac{1}{3}\right)^n\right\}$ converges to the limit of $\boxed{0}$ because $-1 < -\frac{1}{3} < 1$.

68. The sequence $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$ converges to the limit of $\boxed{0}$ because $-1 < \frac{1}{3} < 1$.

69. The sequence $\{s_n\} = \left\{\left(\frac{5}{4}\right)^n\right\}$ diverges to infinity because $\frac{5}{4} > 1$.

70. The sequence $\{s_n\} = \left\{\left(\frac{\pi}{2}\right)^n\right\}$ diverges to infinity because $\frac{\pi}{2} > 1$.

71. The sequence is $\{s_n\} = \left\{\frac{n+(-1)^n}{n}\right\}$. The terms of the sequence are $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \dots$. The sequence appears to approach the limit $L = 1$ by oscillating on either side of 1 . Let $\epsilon > 0$. Then

$$|s_n - L| = \left|\frac{n+(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}.$$

We have $|s_n - L| < \epsilon$ if $n > \lceil \frac{1}{\epsilon} \rceil = N$. This establishes that the sequence converges to the limit $\boxed{1}$.
(Note: $\lceil x \rceil$ means the integer part of x .)

72. The sequence is $\{s_n\} = \left\{\frac{1}{n} + (-1)^n\right\}$. The terms of the sequence are $0, \frac{1}{2} + 1 = \frac{3}{2}, \frac{1}{3} - 1 = -\frac{2}{3}, \frac{1}{4} + 1 = \frac{5}{4}, \frac{1}{5} - 1 = -\frac{4}{5}, \dots$. The terms of the sequence oscillate between positive and negative numbers, with the positive terms tending to the limit of 1 from above, and the negative terms tending to a limit of -1 from above. Since there is no single number that the terms of the sequence approach as $n \rightarrow \infty$, the sequence $\{s_n\} = \left\{\frac{1}{n} + (-1)^n\right\}$ diverges.

73. The sequence $\{s_n\} = \left\{\frac{\ln n}{n}\right\}$. The sequence has limit

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0,$$

using L'Hôpital rule. This proves the sequence is convergent. By the theorem in the middle of p.547, the sequence being convergent is bounded, that is, it is bounded both from above and from below.

74. The sequence is $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$. We have the following bound

$$|s_n| = \left|\frac{\sin n}{n}\right| \leq \frac{1}{n}$$

which means $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\left\{\frac{1}{n}\right\}$. As $n \rightarrow \infty$, both these bounding sequences converge to 0. This means the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ is convergent to the same limit by the Squeeze Theorem. By the theorem in the middle of p.547, the sequence being convergent is bounded, that is, it is bounded both from above and from below.

75. The sequence is $\{s_n\} = \left\{n + \frac{1}{n}\right\}$. As $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$, but n grows without limit. So there is no upper bound. However, a lower bound for the sequence is 2 (the value of s_n for $n = 1$), in that all terms of the sequence exceed 2. So the sequence $\{s_n\} = \left\{n + \frac{1}{n}\right\}$ is bounded only from below.

76. The sequence $\{s_n\} = \left\{\frac{3}{n+1}\right\}$ is bounded from below by 0 (the limit of s_n as $n \rightarrow \infty$), and bounded from above by $\frac{3}{2}$ (the value of s_n for $n = 1$, since the sequence is decreasing). So the sequence $\{s_n\} = \left\{\frac{3}{n+1}\right\}$ is bounded both from above and from below.

77. The sequence $\{s_n\} = \left\{\frac{n^2}{n+1}\right\}$ is bounded from below by $\frac{1}{2}$ (the value of s_n for $n = 1$), but it is not bounded from above since

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n^2}{n+1} = \lim_{x \rightarrow \infty} \frac{x^2}{x+1} = \lim_{x \rightarrow \infty} \frac{2x}{1} = \infty,$$

by using L'Hôpital's rule on a related function of the sequence. So the sequence $\{s_n\} = \left\{\frac{n^2}{n+1}\right\}$ is bounded only from below.

78. The sequence $\{s_n\} = \left\{\frac{2^n}{n^2}\right\}$ is bounded from below. The terms of the sequence are $\frac{2}{1}, \frac{4}{4} = 1, \frac{8}{9}, \frac{16}{16} = 1, \frac{32}{25}, \dots$. Note that the lower bound of the sequence is 1. It cannot be less than 1, since $2^n \geq n^2$ for all values of n . That this sequence has no upper bound is seen by examining

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \lim_{x \rightarrow \infty} \frac{2^x}{x^2} = \lim_{x \rightarrow \infty} \frac{2^x(\ln 2)}{2x} = \lim_{x \rightarrow \infty} \frac{2^x(\ln 2)^2}{2} = \infty,$$

by using L'Hôpital's rule on a related function of the sequence. So the sequence $\{s_n\} = \left\{\frac{2^n}{n^2}\right\}$ is bounded only from below.

79. The terms of the sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ are $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots$. The sequence is bounded below by $-\frac{1}{2}$ and is bounded above by 0, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n = 0$, because $-1 \leq -\frac{1}{2} \leq 1$. So the sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ is bounded both from above and from below.

80. The terms of sequence $\{s_n\} = \{\sqrt{n}\}$ are $1, \sqrt{2}, \sqrt{3}, \dots$. The sequence is bounded from below by 1 but grows without limit as $n \rightarrow \infty$ (as was shown in Problem **65**), and is not bounded from above. So the sequence $\{s_n\} = \{\sqrt{n}\}$ is bounded only from below.

81. The first three terms of the sequence $\{s_n\} = \left\{\frac{3^n}{(n+1)^3}\right\}$ are $s_1 = 0.375$, $s_2 \approx 0.333$, $s_3 \approx 0.422$. Since $s_1 > s_2$, but $s_2 < s_3$, the sequence is not monotonic. To see if it eventually increases, use the Algebraic Ratio Test on the sequence. We have

$$\frac{s_{n+1}}{s_n} = \frac{3^{n+1}/(n+2)^3}{3^n/(n+1)^3} = 3 \cdot \left(\frac{n+1}{n+2}\right)^3 > 1$$

when $\left(\frac{n+1}{n+2}\right)^3 > \frac{1}{3}$ or $\frac{n+1}{n+2} > \frac{1}{\sqrt[3]{3}}$. That is

$$\begin{aligned}\sqrt[3]{3}n + \sqrt[3]{3} &> n + 2 \\ (\sqrt[3]{3} - 1)n &> 2 - \sqrt[3]{3} \\ n &> \frac{2 - \sqrt[3]{3}}{\sqrt[3]{3} - 1} = \frac{0.5577\dots}{0.4422\dots} > 1.\end{aligned}$$

So for $n \geq 2$ the sequence $\{s_n\} = \left\{\frac{3^n}{(n+1)^3}\right\}$ will be a nonmonotonic and an eventually increasing one.

82. We use the Algebraic Difference Test on the sequence $\{s_n\} = \left\{\frac{2n+1}{n}\right\}$. We have

$$s_{n+1} - s_n = \frac{2n+3}{n+1} - \frac{2n+1}{n} = \frac{2n^2 + 3n - 2n^2 - 3n - 1}{n(n+1)} = -\frac{1}{n(n+1)} < 0,$$

whenever $n \geq 1$. This means the sequence $\{s_n\} = \left\{\frac{2n+1}{n}\right\}$ is a decreasing one for $n \geq 1$.

83. The sequence is $\left\{\frac{\ln n}{\sqrt{n}}\right\}$. A related function of the sequence is $f(x) = \frac{\ln x}{\sqrt{x}}$. Compute the derivative:

$$f'(x) = \frac{\sqrt{x} \cdot (\ln x)' - \ln x \cdot (\sqrt{x})'}{(\sqrt{x})^2} = \frac{\sqrt{x} \cdot \frac{1}{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{1}{x^{3/2}} \left[1 - \frac{\ln x}{2}\right] < 0$$

when $1 - \frac{\ln x}{2} < 0$, or $\ln x > 2$ or $x > e^2 = 7.389\dots$. So the sequence $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ is an eventually decreasing one for $n \geq 8$. This means that for $n < 8$, the sequence is increasing, so it is nonmonotonic.

84. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{\sqrt{n+1}}{n}\right\}$. We have

$$\frac{s_{n+1}}{s_n} = \frac{\sqrt{n+2}/(n+1)}{\sqrt{n+1}/n} = \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{n}{(n+1)} < 1$$

whenever

$$\begin{aligned}\sqrt{n+2}(n) &< \sqrt{n+1}(n+1) \\ (n+2)n^2 &< (n+1)(n+1)^2 \\ n^3 + 2n^2 &< n^3 + 3n^2 + 3n + 1 \\ 0 &< n^2 + 3n + 1.\end{aligned}$$

This happens whenever $n \geq 1$. So the sequence $\{s_n\} = \left\{\frac{\sqrt{n+1}}{n}\right\}$ is a decreasing one for $n \geq 1$, and is monotonic as well.

85. The sequence is $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$. We use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{(1/3)^{n+1}}{(1/3)^n} = \frac{1}{3} < 1.$$

So the sequence $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$ is a decreasing one for $n \geq 1$, and is monotonic as well.

86. The sequence is $\{s_n\} = \left\{\frac{n^2}{5^n}\right\}$. We use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)^2/5^{n+1}}{n^2/5^n} = \frac{(n+1)^2 5^n}{5^{n+1} n^2} = \frac{1}{5} \left(1 + \frac{1}{n}\right)^2 < 1$$

whenever

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^2 &< 5 \\ 1 + \frac{1}{n} &< \sqrt{5} \\ \frac{1}{n} &< \sqrt{5} - 1 \\ n &> \frac{1}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{4} = 0.809... \end{aligned}$$

So the sequence $\{s_n\} = \left\{\frac{n^2}{5^n}\right\}$ is a decreasing one for $n \geq 1$, and it is also monotonic.

87. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{n!}{3^n}\right\}$. We have:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{n!} \cdot \frac{3^n}{3^{n+1}} = \frac{1}{3} \cdot (n+1) \geq 1$$

whenever $n+1 \geq 3$, or $n \geq 2$. So the sequence $\{s_n\} = \left\{\frac{n!}{3^n}\right\}$ is an eventually nondecreasing one for $n \geq 2$. Since the sequence increases for $n < 2$, it is nonmonotonic.

88. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{n!}{n^2}\right\}$. We have:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{n!} \cdot \frac{n^2}{(n+1)^2} = n+1 \cdot \frac{n^2}{(n+1)^2} = \frac{n^2}{n+1} > 1$$

if $n^2 > n+1$. This happens whenever $n^2 - n - 1 > 0$ or $n > \frac{1+\sqrt{5}}{2} = 1.618...$. So the sequence $\{s_n\} = \left\{\frac{n!}{n^2}\right\}$ is an increasing one for $n \geq 2$. Since the sequence is decreasing for $n < 2$, it is nonmonotonic.

89. The sequence $\{s_n\} = \{ne^{-n}\}$ is bounded from below by zero, since its terms are positive for all n . Examine the derivative of a related function $f(x) = xe^{-x}$ of the sequence :

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$$

if $x > 1$. This shows the sequence is a decreasing one for all $n \geq 1$. Since the sequence is bounded from below and is decreasing, it will converge.

90. The sequence is $\{s_n\} = \{\tan^{-1}n\}$. Since $\tan^{-1}n < \frac{\pi}{2}$ for any positive integer n , the sequence is bounded from above by $\frac{\pi}{2}$. Consider a related function of the sequence $f(x) = \tan^{-1}x$. Since $f'(x) = \frac{1}{1+x^2} > 0$ when $x > 0$, this shows the sequence is increasing for $n \geq 1$. Since the sequence is increasing and is bounded from above, it will converge.

91. The sequence is $\{s_n\} = \left\{\frac{n}{n+1}\right\}$. Since $n < n+1$ for all $n \geq 1$ the sequence is bounded from above by 1. Using the Algebraic Ratio Test,

$$\frac{s_{n+1}}{s_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} > 1$$

provided

$$\begin{aligned}(n+1)^2 &> n(n+2) \\ n^2 + 2n + 1 &> n^2 + 2n \\ 1 &> 0.\end{aligned}$$

So the sequence is always increasing for any $n \geq 1$. Since the sequence is bounded from above and is increasing, it will converge.

92. The sequence is $\{s_n\} = \left\{\frac{n}{n^2+1}\right\}$. The sequence is bounded from below by 0 since the terms of the sequence are positive for all n . Consider a related function of the sequence, $f(x) = \frac{x}{x^2+1}$. Now its derivative

$$f'(x) = \frac{(x^2+1)x' - x(x^2+1)'}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$$

for $x \geq 1$. Since a related function of the sequence is nonincreasing for $x \geq 1$ this shows the sequence is nonincreasing for $n \geq 1$. Since the sequence is bounded from below and is nonincreasing, it will converge.

93. The sequence $\{s_n\} = \left\{2 - \frac{1}{n}\right\}$ is bounded from above by 2 since $\frac{1}{n} < 1$ for all values of n . Also, this sequence is increasing since the Algebraic Difference Test gives

$$s_{n+1} - s_n = 2 - \frac{1}{n+1} - 2 + \frac{1}{n} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0$$

for $n \geq 1$. Since the sequence is bounded from above and is increasing, it will converge.

94. The sequence $\{s_n\} = \left\{\frac{n}{2^n}\right\}$ is bounded from below by 0 since its terms are positive for all n . To show the sequence is monotonic, we use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} \left(1 + \frac{1}{n}\right) \leq 1$$

whenever $1 + \frac{1}{n} \leq 2$, or $\frac{1}{n} \leq 1$, or $n \geq 1$. This shows the sequence is nonincreasing for $n \geq 1$. Since the sequence is bounded from below and is nonincreasing, it will converge.

95. The sequence $\{s_n\} = \left\{\frac{3}{n} + 6\right\}$ is bounded from below by 6 since $\frac{3}{n} > 0$ for all n . To check the monotonicity of the sequence, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = \frac{3}{n+1} + 6 - \frac{3}{n} - 6 = -\frac{3}{n(n+1)} < 0$$

whenever $n \geq 1$. This shows the sequence is decreasing. Since the sequence is decreasing and is bounded from below, it will converge. Next, compute

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{3}{n} + 6\right) = \lim_{n \rightarrow \infty} \frac{3}{n} + \lim_{n \rightarrow \infty} 6 = 0 + 6 = 6,$$

where the sum and difference property of convergent sequences was used. So the limit to which the sequence will converge is $\boxed{6}$.

96. The sequence $\{s_n\} = \left\{2 - \frac{4}{n}\right\}$ is bounded from above by 2 since $\frac{4}{n} > 0$ for all n . To check the monotonicity of the sequence, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = 2 - \frac{4}{n+1} - 2 + \frac{4}{n} = \frac{4}{n(n+1)} > 0$$

for $n \geq 1$. This shows the sequence is an increasing one. Since the sequence is bounded from above by 2 and increasing, it will $\boxed{\text{converge}}$. Next, compute

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(2 - \frac{4}{n}\right) = \lim_{n \rightarrow \infty} 2 - 4 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 2 - 4 \cdot 0 = 2,$$

where the sum and difference and the constant multiple properties of convergent sequences have been used. So the limit to which the sequence will converge is $\boxed{2}$.

97. The sequence is $\{s_n\} = \left\{\ln\left(\frac{n+1}{3n}\right)\right\}$. The sequence $\{a_n\} = \left\{\frac{n+1}{3n}\right\} = \left\{\frac{1}{3} + \frac{1}{3n}\right\}$ is bounded from below by $\frac{1}{3}$ since $\frac{1}{3n} > 0$ for all n . Also, the sequence $\{a_n\}$ is decreasing as can be seen by the Algebraic Difference Test:

$$a_{n+1} - a_n = \frac{1}{3} + \frac{1}{3(n+1)} - \frac{1}{3} - \frac{1}{3n} = -\frac{1}{3n(n+1)} < 0$$

for $n \geq 1$. So the sequence $\{a_n\}$, being bounded from below and decreasing, converges. To find the limit of the sequence $\{a_n\}$, we compute:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{3n}\right) = \lim_{n \rightarrow \infty} \frac{1}{3} + \lim_{n \rightarrow \infty} \frac{1}{3n} = \frac{1}{3} + 0 = \frac{1}{3},$$

where the sum and difference property of convergent sequences has been used. Since $f(x) = \ln x$ is continuous at $x = \frac{1}{3}$, then by the theorem on p.543, the sequence $\{f(a_n)\} = \{s_n\}$ will also $\boxed{\text{converge}}$, and to the limit of $\ln\left(\frac{1}{3}\right) = \boxed{-\ln 3}$.

98. The sequence is $\{s_n\} = \left\{\cos\left(n\pi + \frac{\pi}{2}\right)\right\}$. For any value of n , the value of $s_n = 0$. This means the sequence is bounded from below, is nonincreasing, and so it $\boxed{\text{converges}}$ to the limit of $\boxed{0}$.

99. The sequence $\{s_n\} = \{(-1)^n \sqrt{n}\}$ has terms $-1, \sqrt{2}, -\sqrt{3}, \sqrt{4}, -\sqrt{5}, \dots$. As $n \rightarrow \infty$, the sequence oscillates between increasing positive and negative terms without bound. So the sequence $\boxed{\text{diverges}}$.

100. The sequence is $\{s_n\} = \left\{\frac{(-1)^n}{2n}\right\}$. Consider

$$|s_n| = \left|\frac{(-1)^n}{2n}\right| \leq \frac{1}{2n}$$

that is, $-\frac{1}{2n} \leq \frac{(-1)^n}{2n} \leq \frac{1}{2n}$.

If $\{a_n\} = \left\{-\frac{1}{2n}\right\}$ and $\{b_n\} = \left\{\frac{1}{2n}\right\}$, we have $a_n \leq s_n \leq b_n$. Also, $\{a_n\}$ and $\{b_n\}$ are convergent sequences because 0 is an upper bound for the increasing sequence $\{a_n\}$ and 0 is a lower bound for the decreasing sequence $\{b_n\}$. By the Squeeze Theorem, this shows that the sequence $\{s_n\} = \left\{\frac{(-1)^n}{2n}\right\}$ also $\boxed{\text{converges}}$ and to the same limit of $\boxed{0}$.

101. The sequence $\{s_n\} = \left\{\frac{3^{n+1}}{4^{n+1}}\right\} = \left\{\left(\frac{3}{4}\right)^n + \frac{1}{4^n}\right\}$ is bounded from below by 0 since the n^{th} term is positive for all n . To check monotonicity, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = \frac{3^{n+1} + 1}{4^{n+1}} - \frac{3^n + 1}{4^n} = \frac{3 \cdot 3^n + 1 - 4 \cdot 3^n - 4}{4^{n+1}} = \frac{-3^n - 3}{4^{n+1}} < 0$$

for $n \geq 1$. This means the sequence is decreasing. Since the sequence $\{s_n\} = \{\frac{3^n+1}{4^n}\}$ is bounded from below and decreasing, it converges. To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n + \lim_{n \rightarrow \infty} \frac{1}{4^n} = 0 + 0 = 0,$$

since $-1 \leq \frac{3}{4} \leq 1$. So the sequence converges to a limit of 0.

102. The sequence is $\{s_n\} = \{n + \sin \frac{1}{n}\}$. As $\lim_{n \rightarrow \infty} n = \infty$, the sequence will diverge.

103. The sequence $\{s_n\} = \left\{\frac{\ln(n+1)}{n+1}\right\}$ is bounded from below by 0 since $\ln(n+1) > 0$ for all $n \geq 1$. A related function of the sequence is $f(x) = \frac{\ln(x+1)}{x+1}$. Now since

$$f'(x) = \frac{(x+1)\frac{1}{x+1} - \ln(x+1) \cdot 1}{(x+1)^2} = \frac{1 - \ln(x+1)}{(x+1)^2} < 0$$

for $x > e - 1$, it means the function is decreasing, and so the sequence is decreasing as well for $n \geq 2$. Since the sequence is bounded from below and decreasing, it converges. To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n+1} = \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{x+1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{1} = 0$$

using L'Hôpital's rule. So the sequence converges to a limit of 0.

104. A related function of the sequence $\{s_n\} = \left\{\frac{\ln(n+1)}{\sqrt{n}}\right\}$ is $f(x) = \frac{\ln(x+1)}{\sqrt{x}}$. By the Theorem on p.544, we have

$$\lim_{n \rightarrow \infty} s_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x+1} = \lim_{x \rightarrow \infty} \frac{2 \cdot \frac{1}{\sqrt{x}}}{1} = 0,$$

using L'Hôpital's rule. So the sequence converges to a limit of 0.

105. The sequence $\{s_n\} = \{0.5^n\}$ converges to a limit of 0, since $-1 < 0.5 < 1$.

106. The sequence $\{s_n\} = \{(-2)^n\}$ diverges, since $-2 < -1$.

107. The sequence $\{s_n\} = \left\{\cos \frac{\pi}{n}\right\}$ is bounded from above by 1 since $|\cos \frac{\pi}{n}| \leq 1$. A related function of the sequence is $f(x) = \cos \frac{\pi}{x}$. Since

$$f'(x) = -\sin \frac{\pi}{x} \cdot \left(-\frac{\pi}{x^2}\right) = \frac{\pi}{x^2} \sin \frac{\pi}{x} > 0$$

for $x > 1$, the function is increasing, which means the sequence is increasing for $n \geq 1$. Since the sequence is bounded above and increasing, it converges. To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \cos \frac{\pi}{n} = \cos 0 = 1.$$

So the sequence converges to a limit of 1.

108. The sequence $\{s_n\} = \left\{\sin \frac{\pi}{n}\right\}$ is bounded from below by 0 since $\sin \frac{\pi}{n} > 0$ for all $n \geq 1$. A related function of the sequence is $f(x) = \sin \frac{\pi}{x}$. Since

$$f'(x) = \cos \frac{\pi}{x} \cdot \left(-\frac{\pi}{x^2}\right) = -\frac{\pi}{x^2} \cos \frac{\pi}{x} < 0$$

for $x > 2$, the function is decreasing, which means the sequence is decreasing for $n \geq 2$. Since the sequence $\{s_n\} = \{\sin \frac{\pi}{n}\}$ is bounded below and decreasing, it converges. To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = \sin 0 = 0.$$

So the sequence converges to a limit of 0.

109. The sequence is $\{s_n\} = \{\cos(\frac{n}{e^n})\}$. Consider the sequence $\{b_n\} = \{\frac{n}{e^n}\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. A related function of the sequence $\{b_n\}$ is $f(x) = \frac{x}{e^x}$. We have

$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} < 0$$

if $x > 1$, which shows the function $f(x)$ is decreasing. So the sequence $\{b_n\}$ is decreasing for $n > 1$. This establishes that the sequence $\{b_n\}$ converges since it is bounded from below and is decreasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0,$$

using L'Hôpital's rule. Since $g(x) = \cos x$ is continuous at $x = 0$, from the theorem on p.543, it follows that the sequence $\{g(b_n)\} = \{s_n\} = \{\cos(\frac{n}{e^n})\}$ also converges, to the value $\cos 0 = \span style="border: 1px solid black; padding: 0 2px;">1.$

110. The sequence is $\{s_n\} = \{\sin(\frac{(n+1)^3}{e^n})\}$. Consider the sequence $\{b_n\} = \{\frac{(n+1)^3}{e^n}\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. A related function of the sequence $\{b_n\}$ is $f(x) = \frac{(x+1)^3}{e^x}$. We have

$$f'(x) = \frac{e^x \cdot 3(x+1)^2 - (x+1)^3 \cdot e^x}{e^{2x}} = \frac{e^x(x+1)^2[3 - (x+1)]}{e^{2x}} \leq 0$$

whenever $x \geq 2$, which shows the function $f(x)$ is a nonincreasing function. So $\{b_n\}$ is a nonincreasing sequence for $n \geq 2$. This establishes that the sequence $\{b_n\}$ converges, since it is bounded from below and nonincreasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{(x+1)^3}{e^x} = \lim_{x \rightarrow \infty} \frac{3(x+1)^2}{e^x} = \lim_{x \rightarrow \infty} \frac{3 \cdot 2(x+1)}{e^x} = \lim_{x \rightarrow \infty} \frac{3 \cdot 2 \cdot 1}{e^x} = 0,$$

using L'Hôpital's rule. Since $g(x) = \sin x$ is continuous at $x = 0$, from the theorem on p.543, the sequence $\{g(b_n)\} = \{s_n\} = \{\sin(\frac{(n+1)^3}{e^n})\}$ also converges, with a limit $\sin 0 = \span style="border: 1px solid black; padding: 0 2px;">0.$

111. The sequence is $\{s_n\} = \{e^{1/n}\}$. Consider the sequence $\{b_n\} = \{\frac{1}{n}\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. Since

$$\frac{b_{n+1}}{b_n} = \frac{n}{n+1} < 1$$

when $n \geq 1$, the sequence $\{b_n\}$ is decreasing according to the Algebraic Ratio Test. Because the sequence $\{b_n\}$ is bounded from below and decreasing, it converges. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Now the function $f(x) = e^x$ is continuous at $x = 0$. So by the theorem on p.543, this means the sequence $\{f(b_n)\} = \{s_n\} = \{e^{1/n}\}$ also converges, to a limit $e^0 = \span style="border: 1px solid black; padding: 0 2px;">1.$

112. The sequence is $\{s_n\} = \{\frac{e^n}{n}\}$. Consider a related function of the sequence, $f(x) = \frac{e^x}{x}$. We have using L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} s_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x}{x} = \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty.$$

This means the sequence $\{s_n\} = \{\frac{e^n}{n}\}$ *diverges.*

113. The sequence $\{s_n\} = \{1 + (\frac{1}{2})^n\}$ is bounded from below by 1, since $(\frac{1}{2})^n > 0$ for all $n \geq 1$. Using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 + \frac{1}{2^{n+1}} - 1 - \frac{1}{2^n} = \frac{1}{2 \cdot 2^n} - \frac{1}{2^n} = -\frac{1}{2 \cdot 2^n} = -\frac{1}{2^{n+1}} < 0$$

for $n \geq 1$. This shows the sequence is a decreasing one. Since the sequence $\{s_n\} = \{1 + (\frac{1}{2})^n\}$ is bounded from below and decreasing, it *converges.* To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[1 + \left(\frac{1}{2} \right)^n \right] = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n = 1 + 0 = 1,$$

since $-1 < \frac{1}{2} < 1$. So the sequence converges to a limit of 1.

114. The sequence $\{s_n\} = \{1 - (\frac{1}{2})^n\}$ is bounded from above by 1, since $(\frac{1}{2})^n > 0$ for all $n \geq 1$. Using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 - \frac{1}{2^{n+1}} - 1 + \frac{1}{2^n} = -\frac{1}{2 \cdot 2^n} + \frac{1}{2^n} = \frac{1}{2 \cdot 2^n} = \frac{1}{2^{n+1}} > 0$$

for $n \geq 1$. This shows the sequence is an increasing one. Since the sequence $\{s_n\} = \{1 - (\frac{1}{2})^n\}$ is bounded from above and increasing, it *converges.* To find the limit, compute:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[1 - \left(\frac{1}{2} \right)^n \right] = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n = 1 - 0 = 1,$$

since $-1 < \frac{1}{2} < 1$. So the sequence converges to a limit of 1.

Applications and Extensions

115. The sequence $\{s_n\} = \left\{ \frac{n^2 \tan^{-1} n}{n^2 + 1} \right\}$ is bounded from above since

$$\frac{n^2 \tan^{-1} n}{n^2 + 1} < \tan^{-1} n < \frac{\pi}{2}$$

for all $n \geq 1$. A related function of the sequence is $f(x) = \frac{x^2 \tan^{-1} x}{x^2 + 1}$. Since

$$f'(x) = \frac{(x^2 + 1)[2x \tan^{-1} x + \frac{x^2}{1+x^2}] - x^2 \tan^{-1} x \cdot 2x}{(x^2 + 1)^2} = \frac{2x \tan^{-1} x + x^2}{(x^2 + 1)^2} > 0$$

for $x > 1$, the function is increasing, which means the sequence $\{s_n\}$ is increasing for $n \geq 1$. Since the sequence is bounded from above and increasing, it *converges.*

116. The sequence $\{s_n\} = \{n \sin \frac{1}{n}\}$ has an upper bound of 1 since $|\frac{\sin n}{n}| \leq \frac{1}{n}$. A related function of the sequence is $f(x) = x \sin \frac{1}{x}$. We have

$$f'(x) = \sin \frac{1}{x} + x \cdot \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} > 0$$

whenever $\tan \frac{1}{x} > \frac{1}{x}$. This inequality is seen to be satisfied for all $x \geq 1$ as follows: Let $g(y) = \tan y - y$. Then $g'(y) = \sec^2 y - 1 \geq 0$ when $\frac{1}{\cos^2 y} \geq 1$ or $\cos^2 y \leq 1$, which is true for all $y \geq 0$. This shows that $g(y)$ is an increasing function for all $y \in [0, \frac{\pi}{2})$. Since $g(0) = 0$, this means that $g(y) \geq 0$ for all $y \in [0, \frac{\pi}{2})$, that is, $\tan y \geq y$. If $y = \frac{1}{x}$, then this means that $g(x) = \tan \frac{1}{x} - \frac{1}{x}$ is increasing (at least) in the interval $x \in [1, \infty)$, and since $g(1) = \tan 1 - 1 \approx 0.557 > 0$, it follows that $g(x) > 0$, or $\tan \frac{1}{x} > \frac{1}{x}$ for $x \in [1, \infty)$. That is, the function $f(x)$ is increasing, and the sequence in turn is increasing for $n \geq 1$. Since the sequence $\{s_n\} = \{n \sin \frac{1}{n}\}$ is bounded from above and increasing, it converges.

Alternative Solution: We invoke the Theorem on p.544, and compute the limit on a related function $f(x) = x \sin \frac{1}{x}$ as follows:

$$\lim_{n \rightarrow \infty} s_n = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{y \rightarrow 0} \frac{1}{y} \sin y = 1,$$

where $y = \frac{1}{x}$, and using the standard limit result that $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$. Since a related function of the sequence converges to a finite limit, it means the sequence also converges (and converges to the same limit).

117. The sequence is $\{s_n\} = \left\{ \frac{n + \sin n}{n + \cos(4n)} \right\}$. Consider $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$. Since $\left| \frac{\sin n}{n} \right| \leq \frac{1}{n}$, we have $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} \pm \frac{1}{n} = 0$, by the Squeeze theorem, $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$. Similarly, since $\left| \frac{\cos(4n)}{n} \right| \leq \frac{1}{n}$, we have $-\frac{1}{n} \leq \frac{\cos(4n)}{n} \leq \frac{1}{n}$, and since $\lim_{n \rightarrow \infty} \pm \frac{1}{n} = 0$, by the Squeeze theorem, $\lim_{n \rightarrow \infty} \frac{\cos(4n)}{n} = 0$. We have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n + \sin n}{n + \cos(4n)} = \lim_{n \rightarrow \infty} \frac{1 + \frac{\sin n}{n}}{1 + \frac{\cos(4n)}{n}} = \frac{1 + \lim_{n \rightarrow \infty} \frac{\sin n}{n}}{1 + \lim_{n \rightarrow \infty} \frac{\cos(4n)}{n}} = \frac{1 + 0}{1 + 0} = 1.$$

The terms of the sequence oscillate on either side of 1, while this calculation shows that the sequence converges to a limit of 1.

118. The sequence is $\{s_n\} = \left\{ \frac{n^2}{2n+1} \sin \frac{1}{n} \right\}$. A related function of the sequence is $f(x) = \frac{x^2}{2x+1} \sin \frac{1}{x}$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{2x+1} \sin \frac{1}{x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{2x+1} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{y \rightarrow 0} \frac{1}{2+y} \frac{\sin y}{y} \quad (\text{Use } y = \frac{1}{x}) \\ &= \lim_{y \rightarrow 0} \frac{1}{2+y} \cdot \lim_{y \rightarrow 0} \frac{\sin y}{y} = \frac{1}{2} \cdot 1 = \frac{1}{2}. \end{aligned}$$

So the sequence $\{s_n\} = \left\{ \frac{n^2}{2n+1} \sin \frac{1}{n} \right\}$ converges to a limit of $\frac{1}{2}$.

119. The sequence is $\{s_n\} = \{\ln n - \ln(n+1)\} = \left\{ \ln \frac{n}{n+1} \right\}$. The sequence $\{b_n\} = \left\{ \frac{n}{n+1} \right\}$ is bounded from above by 1 since $\frac{n}{n+1} < 1$ for all $n \geq 1$. The sequence $\{b_n\}$ is increasing by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n+2} \frac{n+1}{n} = \frac{n^2 + 2n + 1}{n^2 + 2n} > 1$$

for $n \geq 1$. So the sequence $\{b_n\}$, being both bounded from above and increasing, is convergent. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{1}{1+0} = 1.$$

Since $f(x) = \ln x$ is a continuous function at $x = 1$, by the theorem on p.543, the sequence

$\{f(b_n)\} = \{s_n\} = \{\ln n - \ln(n+1)\} = \left\{\ln \frac{n}{n+1}\right\}$ converges to the limit $\ln 1 = 0$.

120. The sequence is $\{s_n\} = \left\{\ln n^2 + \ln \frac{1}{n^2+1}\right\} = \left\{\ln \left(\frac{n^2}{n^2+1}\right)\right\}$. The sequence $\{b_n\} = \left\{\frac{n^2}{n^2+1}\right\}$ is bounded from above by 1 since $\frac{n^2}{n^2+1} < 1$ for all $n \geq 1$. The sequence $\{b_n\}$ is increasing by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2}{(n+1)^2+1} \cdot \frac{(n^2+1)}{n^2} = \frac{n^2(n+1)^2 + (n+1)^2}{n^2[(n+1)^2+1]} = \frac{n^2(n+1)^2 + n^2 + 2n + 1}{n^2(n+1)^2 + n^2} > 1$$

for $n \geq 1$. So the sequence $\{b_n\}$, being both bounded from above and increasing, is convergent. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = \frac{1}{1+0} = 1.$$

Since $f(x) = \ln x$ is continuous at $x = 1$, by the theorem on p.543, the sequence

$\{f(b_n)\} = \{s_n\} = \left\{\ln n^2 + \ln \frac{1}{n^2+1}\right\}$ converges to the limit $\ln 1 = 0$.

121. The sequence is $\{s_n\} = \left\{\frac{n^2}{\sqrt{n^2+1}}\right\}$. Since

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{1 + \frac{1}{n^2}}} = \infty,$$

this sequence diverges.

122. The sequence is $\{s_n\} = \left\{\frac{5^n}{(n+1)^2}\right\}$. Using the L'Hôpital rule on a related function of the sequence we have:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{5^n}{(n+1)^2} = \lim_{x \rightarrow \infty} \frac{5^x}{(x+1)^2} = \lim_{x \rightarrow \infty} \frac{5^x \ln 5}{2(x+1)} = \lim_{x \rightarrow \infty} \frac{5^x (\ln 5)^2}{2} = \infty.$$

This sequence diverges.

123. The sequence is $\{s_n\} = \left\{\frac{2^n}{(2)(4)(6)\cdots(2n)}\right\} = \left\{\frac{2^n}{2^n(1 \cdot 2 \cdot 3 \cdots n)}\right\} = \left\{\frac{1}{n!}\right\}$. It is bounded from below by 0 since $\frac{1}{n!} > 0$ for all $n \geq 1$. Using the Algebraic Ratio Test, we have

$$\frac{s_{n+1}}{s_n} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} < 1$$

for $n \geq 1$. This shows the sequence is decreasing. Since the sequence $\{s_n\} = \left\{\frac{2^n}{(2)(4)(6)\cdots(2n)}\right\}$ is bounded from below and decreasing, it will converge.

124. The sequence is $\{s_n\} = \left\{\frac{3^{n+1}}{(3)(6)(9)\cdots(3n)}\right\} = \left\{\frac{3^{n+1}}{3^n \cdot n!}\right\} = \left\{\frac{3}{n!}\right\}$. It is bounded from below by 0 since $\frac{3}{n!} > 0$ for all $n \geq 1$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} < 1$$

for $n \geq 1$. This shows the sequence is decreasing. Since the sequence $\{s_n\} = \left\{\frac{3^{n+1}}{(3)(6)(9)\cdots(3n)}\right\}$ is bounded from below and decreasing, it will converge.

125. Let $f(x) = x^2 + x \cos x + 1$. Completing the square yields

$$f(x) = \left(x + \frac{\cos x}{2}\right)^2 + 1 - \frac{\cos^2 x}{4}$$

so that

$$f(x) \geq 1 - \frac{\cos^2 x}{4} \geq \frac{3}{4} > 0$$

since $\left(x + \frac{\cos x}{2}\right)^2 \geq 0$ and $\cos^2 x \leq 1$ for all x . Moreover,

$$\begin{aligned} f'(x) &= 2x - x \sin x + \cos x \\ &= x(2 - \sin x) + \cos x \\ &\geq x + \cos x && \text{since } 2 - \sin x \geq 1 \text{ because } |\sin x| \leq 1; \\ &\geq x - 1 && \text{since } -1 \leq \cos x \leq 1; \\ &> 0 \end{aligned}$$

for $x > 1$. This shows that $f(x)$ is a positive and increasing function of x for $x > 1$. It follows that

$$g(x) = \frac{1}{f(x)} = \frac{1}{x^2 + x \cos x + 1}$$

is a positive and decreasing function (because $g'(x) = -\frac{1}{f(x)^2} \cdot f'(x) < 0$ since $f'(x) > 0$) for $x > 1$. As $g(x)$ is a related function of the sequence $\{s_n\} = \left\{\frac{1}{n^2 + n \cos n + 1}\right\}$, the sequence $\{s_n\}$ is decreasing and bounded from below by 0, so it converges.

126 (a) The first eight terms of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21$.

(b) We verify that the n^{th} term given satisfies the recursive definition of the Fibonacci sequence:

1.

$$u_1 = \frac{(1 + \sqrt{5})^1 - (1 - \sqrt{5})^1}{2^1 \sqrt{5}} = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$$

2.

$$u_2 = \frac{(1 + \sqrt{5})^2 - (1 - \sqrt{5})^2}{2^2 \sqrt{5}} = \frac{(1 + 2\sqrt{5} + 5) - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

3.

$$\begin{aligned} u_{n+2} - u_n &= \frac{(1 + \sqrt{5})^{n+2} - (1 - \sqrt{5})^{n+2}}{2^{n+2} \sqrt{5}} - \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \\ &= \frac{(1 + \sqrt{5})^2 (1 + \sqrt{5})^n - (1 - \sqrt{5})^2 (1 - \sqrt{5})^n}{4(2^n \sqrt{5})} - \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \\ &= \frac{(1 + 5 + 2\sqrt{5})(1 + \sqrt{5})^n - (1 + 5 - 2\sqrt{5})(1 - \sqrt{5})^n}{4(2^n \sqrt{5})} - \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \\ &= \frac{(3 + \sqrt{5})(1 + \sqrt{5})^n - (3 - \sqrt{5})(1 - \sqrt{5})^n}{2(2^n \sqrt{5})} - \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \\ &= \frac{(3 + \sqrt{5})(1 + \sqrt{5})^n - (3 - \sqrt{5})(1 - \sqrt{5})^n - 2(1 + \sqrt{5})^n + 2(1 - \sqrt{5})^n}{2(2^n \sqrt{5})} \\ &= \frac{(3 - 2 + \sqrt{5})(1 + \sqrt{5})^n - (3 - 2 - \sqrt{5})(1 - \sqrt{5})^n}{2^{n+1} \sqrt{5}} \\ &= \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1} \sqrt{5}} = u_{n+1}. \end{aligned}$$

That is, $u_{n+2} = u_{n+1} + u_n$. This means that the n^{th} term of the Fibonacci sequence is given by $u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$, since we have verified that it satisfies the recursive definition of the Fibonacci sequence.

127. The fish population in Mirror Lake is $p_n = rp_{n-1} + h$. We have

$$\begin{aligned} p_1 &= rp_0 + h; \\ p_2 &= rp_1 + h = r(rp_0 + h) + h \\ &= r^2p_0 + rh + h; \\ p_3 &= rp_2 + h = r(r^2p_0 + rh + h) + h \\ &= r^3p_0 + (r^2 + r + 1)h; \dots \text{ and so on.} \end{aligned}$$

The pattern is such that

$$p_n = r^n p_0 + (1 + r + r^2 + \dots + r^{n-1})h.$$

Let $S_n = 1 + r + r^2 + \dots + r^{n-1}$. Then $rS_n = r + r^2 + r^3 + \dots + r^n$. Subtracting, we have $rS_n - S_n = r^n - 1$, or $S_n = \frac{r^n - 1}{r - 1}$. So, the fish population for $p_0 = 3000$ is

$$p_n = 3000r^n + \left(\frac{r^n - 1}{r - 1} \right) h.$$

As $n \rightarrow \infty$, $r^n \rightarrow 0$ since $0 < r < 1$. We have

$$\lim_{n \rightarrow \infty} p_n = p_0 \lim_{n \rightarrow \infty} r^n + \left(\frac{1 - \lim_{n \rightarrow \infty} r^n}{1 - r} \right) h = 0 + \left(\frac{1 - 0}{1 - r} \right) h = \frac{h}{1 - r}.$$

So the sequence *converges.*

128. (a) After one time constant, the charge remaining $Q_1 = \frac{Q_0}{e} = Q_0 e^{-1}$. After two time constants, the charge remaining $Q_2 = \frac{Q_0}{e^2} = Q_0 e^{-2}$. So after n time constants, the charge remaining is $Q_n = Q_0 e^{-n}$.

(b) The sequence $\{Q_n\}$ is bounded from below by 0 since the charge Q_n is positive for all $n \geq 1$, and the sequence is decreasing as seen by the Algebraic Ratio Test:

$$\frac{Q_{n+1}}{Q_n} = \frac{Q_0 e^{-n-1}}{Q_0 e^{-n}} = \frac{1}{e} < 1.$$

Since the sequence $\{Q_n\}$ is bounded from below and is decreasing, it *converges.* Since the sequence

converges, we compute the limit: $\lim_{n \rightarrow \infty} Q_n = \lim_{n \rightarrow \infty} Q_0 e^{-n} =$ $0.$

129. (a) Intensity after the first reflection $I_1 = (0.95)I_0$. Intensity after the second reflection $I_2 = (0.95)I_1 = (0.95)(0.95)I_0 = (0.95)^2 I_0$. So, proceeding this way, the intensity after n reflections will be $I_n = (0.95)^n I_0$.

(b) We require $I_n \leq (0.02)I_0$ for the intensity after n reflections to have decreased by at least 98%. So

$$\begin{aligned} I_n &\leq (0.02)I_0 \\ n \log(0.95) &\leq \log(0.02) \\ n &\geq \frac{\log(0.02)}{\log(0.95)} \approx 76.3 \end{aligned}$$

(Note: The last inequality was reversed because for numbers less than 1, the log function is negative, so we are dividing by a negative number, which reverses inequalities.) We want at least 77 reflections.

130. (a) Let r be the average number of neutrons released per fission event. Each of these neutrons causes another fission event, each releasing on average r neutrons, for a total of r^2 neutrons on average after the second fission event. So after n fission events, r^n neutrons on average are released. For $r = 2.5$, the sequence that describes the average number of neutrons after n fission events is $\{r^n\} = \boxed{\{(2.5)^n\}}$.

(b) Since $r = 2.5 > 1$, $\lim_{n \rightarrow \infty} r^n = \infty$, so the sequence $\boxed{\text{diverges}}$.

(c) The number of fission events in the fission chain reaction of Uranium-235 increases without bound, leading to a nuclear reactor meltdown or an explosion in a nuclear bomb.

Challenge Problems

131. (a) The sequence is $\{s_n\} = \{e^{n/(n+2)}\}$. Consider the sequence $\{b_n\} = \left\{\frac{n}{n+2}\right\}$. This sequence is bounded from above by 1 since $\frac{n}{n+2} < 1$ for all $n \geq 1$. Also, the sequence $\{b_n\}$ is increasing since by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)(n+2)}{(n+3)n} = \frac{n^2 + 3n + 2}{n^2 + 3n} > 1$$

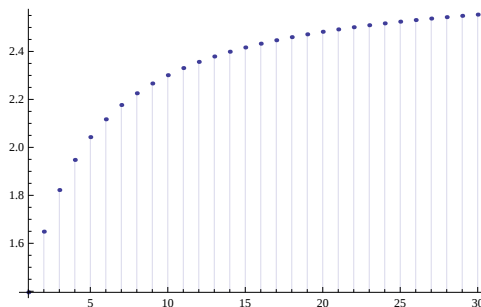
for all $n \geq 1$. So $\{b_n\}$ is a convergent sequence since it is bounded from above and increasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{n+2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = \frac{1}{1+0} = 1.$$

Since $f(x) = e^x$ is continuous at $x = 1$, by the theorem on p.543, the sequence $\{f(b_n)\} = \{s_n\} = \{e^{n/(n+2)}\}$ also $\boxed{\text{converges}}$.

(b) $\lim_{n \rightarrow \infty} e^{n/(n+2)} = \lim_{n \rightarrow \infty} f(b_n) = f(\lim_{n \rightarrow \infty} b_n) = f(1) = e^1 = \boxed{e}$.

(c) As seen by plotting the sequence $\{s_n\} = \{e^{n/(n+2)}\}$, the points seem to reach the asymptotic value of e . See figure below:



132. We need to show that when $0 < r < 1$, $\lim_{n \rightarrow \infty} r^n = 0$. Let $r = \frac{1}{1+p}$, where $p > 0$. Expanding by the Binomial Theorem,

$$|r|^n = \frac{1}{|(1+p)^n|} = \frac{1}{\left|1 + np + \frac{n(n-1)}{2}p^2 + \dots + p^n\right|} < \frac{1}{np}.$$

This means $-\frac{1}{np} < r^n < \frac{1}{np}$. Since $\lim_{n \rightarrow \infty} \left(-\frac{1}{np}\right) = 0 = \lim_{n \rightarrow \infty} \left(\frac{1}{np}\right)$, by the Squeeze Theorem, we have $\lim_{n \rightarrow \infty} r^n = 0$, which was to be proved.

133. Suppose $-1 < r < 0$, and let $s = -r$. Then $0 < s < 1$. Moreover

$$r^n = (-1)^n s^n$$

so

$$-s^n \leq r^n \leq s^n$$

for all n . From Problem 132, $\lim_{n \rightarrow \infty} s^n = 0$, therefore $\lim_{n \rightarrow \infty} r^n = 0$ by the Squeeze Theorem.

134. We need to show that if $r > 1$, then $\lim_{n \rightarrow \infty} r^n = \infty$. Let $r = 1 + p$, where $p > 0$. Then, by the Binomial Theorem, we have

$$r^n = (1 + p)^n = 1 + np + \frac{n(n-1)}{2}p^2 + \cdots + p^n > np,$$

or $r^n > np$. Since $\lim_{n \rightarrow \infty} np = \infty$ if $p > 0$, it follows that $\lim_{n \rightarrow \infty} r^n = \infty$, which was to be proved.

135. If $r < -1$, then $r = 1 + p$ means that $p < -2$. If n is even, then $r^n > 0$, and $r^n > n|p|$, from the calculation of the previous problem 134. If n is odd, then $r^n < 0$ and $r^n < -n|p|$. As $n \rightarrow \infty$, r^n oscillates without bound between large positive and negative values, so $\lim_{n \rightarrow \infty} r^n$ does not exist for $r < -1$, as was to be shown.

136. Recall that a function f is continuous at $x_0 = L$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - L| < \delta$ implies $|f(x) - f(L)| < \epsilon$. Let $\{s_n\}$ be a sequence with a limit L . Then $\lim_{n \rightarrow \infty} s_n = L$ means that for every $\epsilon' > 0$, there exists an $n > N$ such that $|s_n - L| < \epsilon'$. Choose x to be s_n . Set $\delta = \epsilon'$. Then for $0 < |s_n - L| < \epsilon'$, continuity of f means for $n > N$, $|f(s_n) - f(L)| < \epsilon$. But this means that $\lim_{n \rightarrow \infty} f(s_n) = f(L)$, which was to be shown.

137. We have $|s_n| = |s_n - L + L| \leq |s_n - L| + |L|$, using the Triangle Inequality. Similarly, $|L| = |L - s_n + s_n| \leq |L - s_n| + |s_n| = |s_n - L| + |s_n|$. Together these can be written as $||s_n| - |L|| \leq |s_n - L|$. If $|s_n - L| < \epsilon$ for $n > N$, then $||s_n| - |L|| \leq |s_n - L| < \epsilon$. This means that $\lim_{n \rightarrow \infty} |s_n| = |L|$.

The converse is not true. For example, the sequence $\{(-1)^n\}$ converges to 1, while the sequence $\{(-1)^n\}$ is nonconvergent as its terms oscillate between -1 and $+1$.

138. Suppose the limit of a sequence is *not* unique. Then we would have $\lim_{n \rightarrow \infty} s_n = L_1$, and $\lim_{n \rightarrow \infty} s_n = L_2$. Then we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} (s_n - s_n) &= \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_n \\ \lim_{n \rightarrow \infty} 0 &= 0 = L_1 - L_2 \end{aligned}$$

$$\text{or,} \quad L_1 = L_2.$$

This proves that the limit of a convergent sequence is indeed unique, since our calculation shows $L_1 = L_2$.

139. The definition of a limit at infinity of a function $f(x)$ is: $\lim_{x \rightarrow \infty} f(x) = L$ if for every $\epsilon > 0$, there is an N such that $|f(x) - L| < \epsilon$ for $x > N$.

The definition of a limit of a sequence $\{s_n\}$ is: $\lim_{n \rightarrow \infty} s_n = L$ if for every $\epsilon > 0$ there is an N such that $|s_n - L| < \epsilon$ for $n > N$.

The definitions appear identical, even though in the case of functions, the choice of N is not necessarily an integer as it is in the case of sequences.

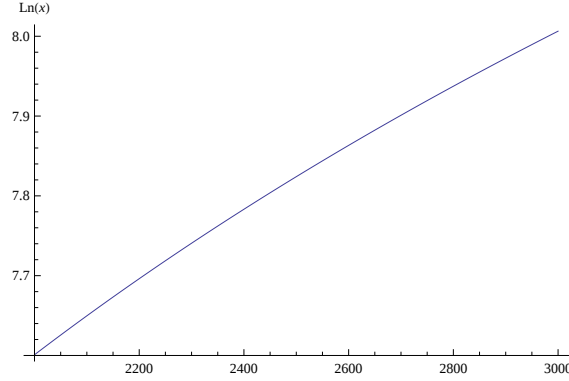
140. (a) A related function of the sequence $\{\ln n\}$ is $f(x) = \ln x$. Since $f'(x) = \frac{1}{x} > 0$ for $x > 0$, the related function of the sequence is increasing, which means the sequence $\{\ln n\}$ is increasing for $n \geq 1$.

(b) Using the algebraic ratio test, we have $\frac{\ln(n+1)}{\ln n} > 1$, since $\ln n$ is an increasing function of n . This shows that the sequence $\{\ln n\}$ is unbounded from above.

(c) The sequence $\{\ln n\}$ is an increasing sequence that is unbounded from above, and so it will diverge.

(d) We need an N such that $N > e^{20} \approx 4.852 \times 10^8$. So the smallest N so that $\ln N > 20$ is $N = e^{20} + 1$.

(e) The zoomed-in graph of $\ln x$ for large values of x appears below:



(f) The graph suggests that the function $y = \ln x$ increases as x increases, hence the sequence $\{\ln n\}$ will diverge without bound as well, confirming the result in (c).

141. (a) For $a, b > 0$,

$$\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0.$$

So $\frac{a+b}{2} > \sqrt{ab}$. If $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$, then using the result we have proved, namely $\frac{a_n+b_n}{2} > \sqrt{a_n b_n}$ for all n , we have $a_{n+1} > b_{n+1}$ for all n , or, equivalently, $a_n > b_n$ for all n . Because of this, we can write $a_{n+1} = \frac{a_n+b_n}{2} < \frac{a_n+a_n}{2} = a_n$, i.e., $a_{n+1} < a_n$. So by the Algebraic Difference Test, the sequence $\{a_n\}$ decreases.

Now $a_n > b_n$ also implies $b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n$, i.e., $b_{n+1} > b_n$. So by the algebraic ratio test, the sequence $\{b_n\}$ increases. We have, since $\{a_n\}$ is decreasing $a_n < a_{n-1} < a_{n-2} < \dots < a_1$ and since $\{b_n\}$ is increasing, $b_n < b_{n+1} < b_{n+2} < \dots$. Combining this with $b_n < a_n$, or equivalently, $b_{n+1} < a_{n+1}$, we have:

$$b_1 < b_2 < \dots < b_n < b_{n+1} < a_{n+1} < a_n < \dots < a_1$$

for all n , and, in particular, $b_n < b_{n+1} < a_1$ for all n .

(b) From the chain of inequalities that appear above, it follows that $b_1 < a_{n+1} < a_n$ for all n .

(c) Let's first prove $\sqrt{a} - \sqrt{b} < a - b$ when $a > b > 0$:

$$\sqrt{a} - \sqrt{b} < a - b$$

$$\text{is the same as } \sqrt{a} - a < \sqrt{b} - b$$

$$\sqrt{a}(1 - \sqrt{a}) < \sqrt{b}(1 - \sqrt{b}).$$

$$\text{Squaring, } a(1 + a - 2\sqrt{a}) < b(1 + b - 2\sqrt{b})$$

$$< a(1 + a - 2\sqrt{b}), \quad \text{since } b < a$$

$$a(1 + a) - 2a\sqrt{a} < a(1 + a) - 2a\sqrt{b}$$

$$-2a\sqrt{a} < -2a\sqrt{b}$$

$$a\sqrt{a} > a\sqrt{b}$$

$$\sqrt{a} > \sqrt{b}$$

$$a > b$$

which is the case, ending the proof. Consider

$$0 < a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})^2 < \frac{1}{2}(a_n - b_n) < \dots < \frac{a_1 - b_1}{2^n}$$

by repeating the process n times, and using the result proved above in the form $\sqrt{a_n} - \sqrt{b_n} < a_n - b_n$ since $b_n < a_n$. This shows that $0 < a_{n+1} - b_{n+1} < \frac{a_1 - b_1}{2^n}$.

(d) The result of part (c) shows that as $n \rightarrow \infty$, the difference between a_{n+1} and b_{n+1} , gets arbitrarily small. Since $\lim_{n \rightarrow \infty} \frac{a_1 - b_1}{2^n} = (a_1 - b_1) \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$, since $-1 < \frac{1}{2} < 1$, and $\lim_{n \rightarrow \infty} 0 = 0$, by the Squeeze Theorem, we have $\lim_{n \rightarrow \infty} (a_{n+1} - b_{n+1}) = 0$, which means that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$, as required to be proved. (Note that each limit separately exists because the sequence $\{a_n\}$ is bounded from below by 0 and is decreasing, so it converges to some limit; the sequence $\{b_n\}$ is bounded from above by a_1 —see the string of inequalities in part (a)—and is increasing, so it also converges to some limit.)

142. $s_n = \frac{2^{n-1}4^n}{n!}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{2^n 4^{n+1}}{(n+1)!} \frac{n!}{2^{n-1}4^n} = \frac{2 \cdot 4}{n+1} = \frac{8}{n+1} \leq 1$$

for $n \geq 7$. This shows the sequence $\{s_n\}$ is nonincreasing. Since each term of the sequence is positive, $s_n > 0$ for $n \geq 1$, the sequence $\{s_n\}$ is bounded from below. Since the sequence $\{s_n\}$ is nonincreasing and bounded from below, it *converges*.

143. $s_n = \frac{n!}{3^n \cdot 4^n}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{3^{n+1} \cdot 4^{n+1}} \frac{3^n \cdot 4^n}{n!} = \frac{n+1}{12} \geq 1$$

for $n \geq 11$. The sequence is nondecreasing, so there is no upper bound. So the sequence *diverges*.

144. To see if the sequence $\{s_n\}$ is monotonic, we find the ratio $\frac{s_{n+1}}{s_n}$.

$$\frac{s_{n+1}}{s_n} = \frac{\frac{(n+1)!}{3^{n+1} + 8(n+1)}}{\frac{n!}{3^n + 8n}} = \frac{(n+1)!}{n!} \frac{3^n + 8n}{3^{n+1} + 8(n+1)} = (n+1) \frac{3^n + 8n}{3^{n+1} + 8(n+1)}.$$

Using L'Hôpital's rule on a related function of the fractional term,

$$\lim_{n \rightarrow \infty} \frac{3^n + 8n}{3^{n+1} + 8(n+1)} = \lim_{x \rightarrow \infty} \frac{3^x + 8x}{3^{x+1} + 8(x+1)} = \lim_{x \rightarrow \infty} \frac{(\ln 3)3^x + 8}{(\ln 3)3^{x+1} + 8} = \lim_{x \rightarrow \infty} \frac{(\ln 3)2^x}{(\ln 3)^2 3^{x+1}} = \lim_{x \rightarrow \infty} \frac{3^x}{3^{x+1}} = \frac{1}{3}.$$

For large n , we have $\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = \lim_{n \rightarrow \infty} \frac{(n+1)}{3} = \infty$. The increasing sequence $\{s_n\}$ is not bounded from above, so it will diverge.

145. The sequence is $\{s_n\} = \{(3^n + 5^n)^{1/n}\} = \left\{5 \left(1 + \left(\frac{3}{5}\right)^n\right)^{1/n}\right\}$. The sequence is bounded from below by 5 since $\left(\frac{3}{5}\right)^n > 0$ and $\left(1 + \left(\frac{3}{5}\right)^n\right)^{1/n} > 1$. (The n^{th} root of a number greater than 1 is greater than 1.) A related function of the sequence is $f(x) = 5 \left(1 + \left(\frac{3}{5}\right)^x\right)^{1/x}$. Note that $f(x) > 0$ when $x > 0$. Taking the logarithm of both sides and differentiating with respect to x , we get

$$\ln f = \ln 5 + \frac{1}{x} \ln \left(1 + \left(\frac{3}{5}\right)^x\right);$$

$$\frac{f'}{f} = 0 - \frac{1}{x^2} \ln \left(1 + \left(\frac{3}{5}\right)^x\right) + \frac{1}{x} \cdot \frac{1}{\left(1 + \left(\frac{3}{5}\right)^x\right)} \cdot \left(\frac{3}{5}\right)^x \cdot \ln \left(\frac{3}{5}\right) < 0$$

whenever $x > 0$ since $\ln \left(\frac{3}{5}\right) < 0$. Since $f(x) > 0$, we have shown that $f'(x) < 0$, or the function is decreasing for $x > 0$. This means the sequence is decreasing for $n \geq 1$. Since the sequence is bounded from below and is decreasing, it converges, as was to be shown.

146. (a) Let a_1 and N be positive numbers. Then

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{N}{a_n}\right) = \frac{a_n^2 + N}{2a_n} > 0$$

for all $n \geq 1$. Since

$$a_n^2 - 2a_n\sqrt{n} + N = (a_n - \sqrt{N})^2 \geq 0,$$

it follows that

$$a_n^2 + N \geq 2a_n\sqrt{n}, \text{ or } \frac{a_n^2 + N}{2a_n} \geq \sqrt{N}.$$

So we have

$$a_{n+1} = \frac{a_n^2 + N}{2a_n} \geq \sqrt{N} \text{ for } n \geq 1,$$

that is, $a_n \geq \sqrt{N}$ for $n \geq 2$. By the Algebraic Ratio Test, we get

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} \left(1 + \frac{N}{a_n^2} \right) \leq \frac{1}{2} \left(1 + \frac{N}{N} \right) = 1$$

for $n \geq 2$ since $a_n \geq \sqrt{N}$ for $n \geq 2$. We have shown that the sequence $\{a_n\}$ is a nonincreasing sequence that is bounded from below, so it converges. Suppose the limit of the sequence is L . Taking the limit as $n \rightarrow \infty$ of both sides of the equation

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{N}{a_n} \right)$$

yields $L = \frac{1}{2} \left(L + \frac{N}{L} \right)$ or $2L^2 = L^2 + N$, or $L = \sqrt{N}$.

(b) We want to approximate $\sqrt{28}$, so $N = 28$. Let $a_1 = 1$ (Any other value may be chosen, in which case the answers will vary for a_3 , but for a_6 , the answers will be much closer if not identical to three decimal places.) Then $a_2 = \frac{1}{2} \left[a_1 + \frac{28}{a_1} \right] = \frac{1}{2} [1 + 28] = \frac{29}{2}$. Then

$$a_3 = \frac{1}{2} \left[\frac{29}{2} + \frac{28}{(29/2)} \right] = \frac{1}{2} \left[\frac{29}{2} + \frac{56}{29} \right] = \frac{29}{4} + \frac{28}{29} \approx 8.216. \text{ Next, } a_3 = \frac{1}{2} \left[8.216 + \frac{28}{8.216} \right] \approx 5.812;$$

$a_4 = \frac{1}{2} \left[5.812 + \frac{28}{5.812} \right] \approx 5.315$; $a_5 = \frac{1}{2} \left[5.315 + \frac{28}{5.315} \right] \approx 5.292$; $a_6 = \frac{1}{2} \left[5.292 + \frac{28}{5.292} \right] \approx 5.292$. To three decimal places, we have using a calculator, $\sqrt{28} \approx 5.292$. We see that a_6 is 100% accurate and a_3 is about 91% accurate.

147. The given sequence can be rewritten as $\{s_n\} = \left\{ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \right\} = \left\{ \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-2) \cdot (2n-1)}{(2 \cdot 4 \cdot 6 \cdots (2n-2))^2 (2n)} \right\} = \left\{ \frac{(2n-1)!}{(2^{n-1} (1 \cdot 2 \cdot 3 \cdots (n-1)))^2 \cdot 2n} \right\} = \left\{ \frac{(2n-1)!}{2^{2n-2} \cdot (n-1)! (n-1)! \cdot 2n} \right\} = \left\{ \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \right\}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{(2n+1)!}{2^{2n+1} (n+1)! n!} \cdot \frac{2^{2n-1} n! (n-1)!}{(2n-1)!} = \frac{(2n+1)(2n)(2n-1)!(n-1)! 2^{2n-1}}{(2n-1)! (n+1)n(n-1)! 2^2 2^{2n-1}} = \frac{2n+1}{2n+2} < 1$$

for $n \geq 1$ which means the sequence $\{s_n\}$ is decreasing. Since the sequence is decreasing, the upper bound on the terms of the sequence is $s_1 = \frac{1}{2}$. So the sequence is bounded below by 0 and above by $\frac{1}{2}$. Hence the sequence is both monotonic and bounded as needed to be shown.

148. Expanding $\left(1 + \frac{1}{n}\right)^n$ using the Binomial Theorem, we obtain

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= \binom{n}{0} + \binom{n}{1} \left(\frac{1}{n}\right)^1 + \binom{n}{2} \left(\frac{1}{n}\right)^2 + \cdots + \binom{n}{k} \left(\frac{1}{n}\right)^k + \cdots + \binom{n}{n} \left(\frac{1}{n}\right)^n \\ &= 1 + n \left(\frac{1}{n}\right) + \underbrace{\frac{n(n-1)}{2!} \left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \left(\frac{1}{n}\right)^k + \cdots + \left(\frac{1}{n}\right)^n}_{n+1 \text{ terms}}. \end{aligned}$$

Simplifying, we have

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{1 \left(1 - \frac{1}{n}\right)}{2!} + \cdots + \frac{1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)}{k!} + \\ &\quad \cdots + \frac{1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)}{n!}. \end{aligned} \tag{1}$$

Next, we expand $\left(1 + \frac{1}{n+1}\right)^{n+1}$:

$$\begin{aligned} \left(1 + \frac{1}{n+1}\right)^{n+1} &= \binom{n+1}{0} + \binom{n+1}{1} \left(\frac{1}{n+1}\right)^1 + \binom{n+1}{2} \left(\frac{1}{n+1}\right)^2 + \cdots + \binom{n+1}{n+1} \left(\frac{1}{n+1}\right)^{n+1} \\ &= 1 + 1 + \underbrace{\frac{1 \left(1 - \frac{1}{n+1}\right)}{2!} + \frac{1 \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right)}{3!} + \cdots + \frac{1 \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n-1}{n+1}\right)}{n!}}_{n+1 \text{ terms}} \\ &\quad + \frac{1 \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n}{n+1}\right)}{(n+1)!}. \end{aligned} \quad (2)$$

We make two observations from equations (1) and (2):

- After the first two terms, each term of (2) is greater than the corresponding term of (1).
- The expansion of (2) has one more positive term than the expansion of (1).

So this demonstrates that

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

which means the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is increasing.

To show that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is bounded from above, we first note that

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{1 \left(1 - \frac{1}{n}\right)}{2!} + \cdots + \frac{1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)}{n!} \\ &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}. \end{aligned} \quad (3)$$

Now, for $n \geq 2$, we have

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n(n-1)(n-2) \cdots 3 \cdot 2 \geq \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n-1 \text{ terms}} = 2^{n-1},$$

that is,

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

for $n \geq 2$. Going back to inequality (3), we have

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &< 1 + \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \\ &\leq 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right). \end{aligned} \quad (4)$$

Let

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}. \quad (5)$$

Then, multiplying through by $\frac{1}{2}$, we get

$$\frac{1}{2}S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}. \quad (6)$$

Subtracting (6) from (5) gives

$$S_n - \frac{1}{2}S_n = \frac{1}{2}S_n = 1 - \frac{1}{2^n},$$

so that

$$S_n = \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} = 2 \left[1 - \left(\frac{1}{2}\right)^n \right] < 2.$$

We continue the estimation of the inequality (4):

$$\left(1 + \frac{1}{n}\right)^n < 1 + S_n < 1 + 2 = 3,$$

which proves that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is bounded from above.

149. Let the limit of the convergent sequence $\{s_n\}$ be S . The n^{th} term of the arithmetic mean sequence $\{a_n\}$ is $a_n = \frac{1}{n}[s_1 + s_2 + \cdots + s_n]$. Let $\epsilon > 0$. Since s_n is convergent to the limit S , then there exists an N such that for all $n \geq N$ $|s_n - S| < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned} |a_n - S| &= \left| \frac{1}{n} \sum_{k=1}^n (s_k) - S \right| = \frac{1}{n} \left| \sum_{k=1}^n (s_k) - ns \right| = \frac{1}{n} \left| \sum_{k=1}^n (s_k - S) \right| \\ &\leq \frac{1}{n} \sum_{k=1}^N |s_k - S| + \frac{1}{n} \sum_{k=N+1}^n |s_k - S| \end{aligned}$$

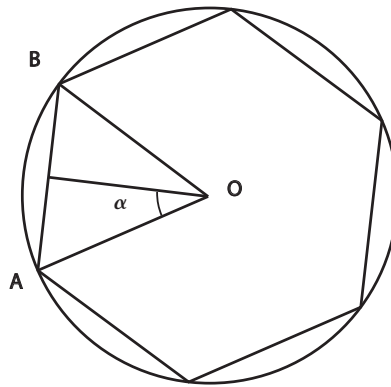
Since for $n \geq N$, $|s_n - S| < \frac{\epsilon}{2}$, the sum in the second term satisfies $\sum_{k=N+1}^n |s_k - S| < n \frac{\epsilon}{2}$. Let

$\sum_{k=1}^N |s_k - S| = M$. Choose $n \geq \max\{N, \frac{2}{\epsilon}M\}$. Then we have $|a_n - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, proving the result that the sequence of arithmetic means $\{a_n\}$ converges, and converges to the same limit as the original sequence $\{s_n\}$.

150. (a) From the figure below, which shows a circle of radius R circumscribing a regular polygon of n sides, we see that area of the triangle with the angle $\alpha = \frac{\pi}{n}$ indicated is

$A_{\triangle} = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} \cdot R \sin \alpha \cdot R \cos \alpha$. Area of the polygon,

$$A_n = 2n \cdot A_{\triangle} = 2n \cdot \frac{1}{2} R^2 \sin \alpha \cos \alpha = \frac{n}{2} R^2 \sin 2\alpha, \text{ or } \boxed{A_n = \frac{n}{2} R^2 \sin \left(\frac{2\pi}{n} \right)}.$$

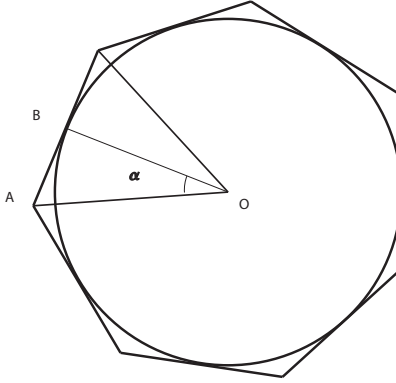


(b) We have, as the number of sides of the polygon grows without limit,

$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{n}{2} R^2 \sin \left(\frac{2\pi}{n} \right) = \lim_{n \rightarrow \infty} \pi R^2 \frac{\sin \left(\frac{2\pi}{n} \right)}{\frac{2\pi}{n}} = \pi R^2 \lim_{y \rightarrow 0} \frac{\sin y}{y} = \pi R^2,$$

where the substitution of $y = \frac{2\pi}{n}$ was made, and the standard result $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$ was used. This shows that as the polygon's sides increase, its area approaches that of the circumscribing circle.

151. (a) From the figure below, which shows a circle of radius r inscribed in a polygon of n sides, we have $\alpha = \frac{\pi}{n}$. Now, the area of the $\triangle ABO$, $A_{\triangle} = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}r \cdot r \tan \alpha = \frac{1}{2}r^2 \tan \left(\frac{\pi}{n}\right)$. The area of the polygon, $A_n = 2n \times A_{\triangle} = \boxed{nr^2 \tan \left(\frac{\pi}{n}\right)}$.

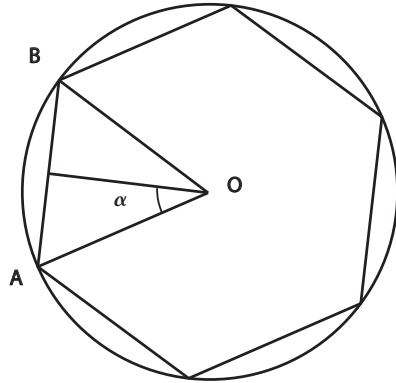


(b) As the number of sides of the polygon increases without limit, we have:

$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} nr^2 \tan \frac{\pi}{n} = \pi r^2 \lim_{n \rightarrow \infty} \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} = \pi r^2 \lim_{y \rightarrow 0} \frac{\tan y}{y} = \pi r^2,$$

where the substitution of $y = \frac{\pi}{n}$ was made, and the standard result $\lim_{y \rightarrow 0} \frac{\tan y}{y} = 1$ was used. This shows that as the polygon's sides increase, its area approaches that of the inscribed circle.

152. (a) From the figure below, which shows a circle of radius R circumscribing a regular polygon of n sides, we see that the length of the segment AB is $2R \sin \alpha = 2 \left(R \sin \left(\frac{\pi}{n}\right)\right)$. The polygon has n sides, so its perimeter will be n times the length of the segment AB, or $\boxed{P_n = 2nR \sin \left(\frac{\pi}{n}\right)}$.



(b) As the number of sides is increased without limit, we expect the perimeter of inscribed polygon to approach the circumference of the circle.

$$\lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} 2nR \sin \left(\frac{\pi}{n}\right) = 2\pi R \lim_{n \rightarrow \infty} \frac{\sin \left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 2\pi R \lim_{y \rightarrow 0} \frac{\sin y}{y} = 2\pi R,$$

where the substitution of $y = \frac{\pi}{n}$ was made, and the standard result $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$ was used.

153. According to the definition of a convergent sequence (p.541) a sequence $\{s_l\}$ converges to a real number L if, for any number $\epsilon > 0$, there is a positive integer N so that $|s_l - L| < \epsilon$ for all integers $l > N$. Consider the absolute value of the difference of s_n and s_m :

$$|s_n - s_m| = |s_n - L + L - s_m| \leq |s_n - L| + |L - s_m|,$$

where we have used the triangle inequality. Since both sequences $\{s_n\}$ and $\{s_m\}$ converge to the same limit L , applying the definition, we can find positive integers N_1 and N_2 such that for any number $\epsilon > 0$, $|s_n - L| < \epsilon/2$ for all $n > N_1$ and $|s_m - L| < \epsilon/2$ for all $m > N_2$. . Then, since $|L - s_m| = |s_m - L|$, we have

$$|s_n - s_m| \leq |s_n - L| + |s_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

i.e., $|s_n - s_m| < \epsilon$, for $n, m > N$ where N is the larger of $\{N_1, N_2\}$. This shows that every convergent sequence must be a Cauchy sequence.

8.2 Infinite Series

Concepts and Vocabulary

1. **(b)**: $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$ is called an infinite series.
2. **(d)**: $S_n = \sum_{k=1}^n a_k$ is the n^{th} partial sum, and the sequence $\{S_n\}$ is called the sequence of partial sums.
3. **True**: If a series converges then the sequence of partial sums converges. Conversely, if the sequence of partial sums converges, then the series also converges.
4. **False**: The geometric series $\sum_{k=1}^{\infty} ar^{k-1}$, $a \neq 0$, converges if $|r| < 1$, but diverges when $|r| = 1$.
5. $S = \boxed{\frac{a}{1-r}}$ is the sum of the convergent geometric series $\sum_{k=1}^{\infty} ar^{k-1}$, $a \neq 0$.
6. **False**: The limit of the n^{th} term of a series as $n \rightarrow \infty$ does not tell us if the series itself will converge. In this instance, the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges even though $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Skill Building

7. $S_4 = \sum_{k=1}^4 \left(\frac{3}{4}\right)^{k-1} = \left(\frac{3}{4}\right)^0 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} = \boxed{\frac{175}{64}}$.
8. $S_4 = \sum_{k=1}^4 \frac{(-1)^{k+1}}{3^{k-1}} = \frac{(-1)^2}{3^0} + \frac{(-1)^3}{3^1} + \frac{(-1)^4}{3^2} + \frac{(-1)^5}{3^3} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} = \boxed{\frac{20}{27}}$.
9. $S_4 = \sum_{k=1}^4 k = 1 + 2 + 3 + 4 = \boxed{10}$.
10. $S_4 = \sum_{k=1}^4 \ln k = \ln 1 + \ln 2 + \ln 3 + \ln 4 = \ln(1 \cdot 2 \cdot 3 \cdot 4) = \boxed{\ln 24}$.
11. $S_n = \sum_{k=1}^n \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} = \frac{1}{n+3}$.
We have
$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{n+3}\right) = \lim_{n \rightarrow \infty} \frac{1}{3} - \lim_{n \rightarrow \infty} \frac{1}{n+3} = \frac{1}{3} - 0 = \frac{1}{3}.$$

Since the sequence of partial sums converges to $\frac{1}{3}$, this means the sum of the telescoping series is $\boxed{\frac{1}{3}}$.

12. $S_n = \sum_{k=1}^n \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{(n+1)^2}$.
We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)^2}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} = 1 - 0 = 1.$$

Since the sequence of partial sums converges to 1, this means the sum of the telescoping series is $\boxed{1}$.

$$13. S_n = \sum_{k=1}^n \left(\frac{1}{3^{k+1}} - \frac{1}{3^k} \right) = \left(\frac{1}{3^2} - \frac{1}{3} \right) + \left(\frac{1}{3^3} - \frac{1}{3^2} \right) + \cdots + \left(\frac{1}{3^n} - \frac{1}{3^{n-1}} \right) + \left(\frac{1}{3^{n+1}} - \frac{1}{3^n} \right) = \frac{1}{3^{n+1}} - \frac{1}{3}.$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{3^{n+1}} - \frac{1}{3} \right) = \lim_{n \rightarrow \infty} \frac{1}{3^{n+1}} - \lim_{n \rightarrow \infty} \frac{1}{3} = 0 - \frac{1}{3} = -\frac{1}{3}.$$

Since the sequence of partial sums converges to $-\frac{1}{3}$, this means the sum of the telescoping series is $\boxed{-\frac{1}{3}}$.

$$14. S_n = \sum_{k=1}^n \left(\frac{1}{4^{k+1}} - \frac{1}{4^k} \right) = \left(\frac{1}{4^2} - \frac{1}{4} \right) + \left(\frac{1}{4^3} - \frac{1}{4^2} \right) + \cdots + \left(\frac{1}{4^n} - \frac{1}{4^{n-1}} \right) + \left(\frac{1}{4^{n+1}} - \frac{1}{4^n} \right) = \frac{1}{4^{n+1}} - \frac{1}{4}.$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{4^{n+1}} - \frac{1}{4} \right) = \lim_{n \rightarrow \infty} \frac{1}{4^{n+1}} - \lim_{n \rightarrow \infty} \frac{1}{4} = 0 - \frac{1}{4} = -\frac{1}{4}.$$

Since the sequence of partial sums converges to $-\frac{1}{4}$, this means the sum of the telescoping series is $\boxed{-\frac{1}{4}}$.

$$15. \text{ We write the general term as a difference of terms: } S_n = \sum_{k=1}^n \frac{1}{4k^2-1} = \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \frac{1}{2} \left(\frac{1}{2n-3} - \frac{1}{2n-1} \right) + \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{2} - \frac{1}{2} \left(\frac{1}{2n+1} \right).$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2} \left(\frac{1}{2n+1} \right) \right) = \lim_{n \rightarrow \infty} \frac{1}{2} - \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{2n+1} \right) = \frac{1}{2} - 0 = \frac{1}{2}.$$

Since the sequence of partial sums converges to $\frac{1}{2}$, this means the sum of the telescoping series is $\boxed{\frac{1}{2}}$.

16. We write the general term as a difference of terms:

$$S_n = \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right) = \frac{1}{2} \left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \frac{1}{2} \left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \cdots + \frac{1}{2} \left(\frac{1}{(n-1)n} - \frac{1}{n(n+1)} \right) + \frac{1}{2} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right).$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right) = \lim_{n \rightarrow \infty} \frac{1}{4} - \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{(n+1)(n+2)} \right) = \frac{1}{4} - 0 = \frac{1}{4}.$$

Since the sequence of partial sums converges to $\frac{1}{4}$, this means the sum of the telescoping series is $\boxed{\frac{1}{4}}$.

17. $\sum_{k=1}^{\infty} (\sqrt{2})^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sqrt{2}$. The series will *diverge* since $|r| > 1$.

18. $\sum_{k=1}^{\infty} (0.33)^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = 0.33$. Since $|r| < 1$, this series will *converge* to a sum of $\frac{a}{1-r} = \frac{1}{1-0.33} = \frac{1}{0.67} = \frac{100}{67}$.

19. $\sum_{k=1}^{\infty} 5 \left(\frac{1}{6} \right)^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 5$ and $r = \frac{1}{6}$. Since $|r| < 1$, this series will *converge* to a sum of $\frac{1}{1-r} = \frac{5}{1-\frac{1}{6}} = \frac{5}{\frac{5}{6}} = \boxed{6}$.

20. $\sum_{k=1}^{\infty} 4(1.1)^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 4$ and $r = 1.1$. The series will diverge since $|r| > 1$.

21. $\sum_{k=0}^{\infty} 7\left(\frac{1}{3}\right)^k = 7\left(\frac{1}{3}\right)^0 + \sum_{k=1}^{\infty} 7\left(\frac{1}{3}\right)^k = 7 + \sum_{k=1}^{\infty} \frac{7}{3}\left(\frac{1}{3}\right)^{k-1}$. The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{7}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{\frac{7}{3}}{1-\frac{1}{3}} = \frac{7/3}{2/3} = \frac{7}{2}$. The original series will converge to a sum of $7 + \frac{7}{2} = \boxed{\frac{21}{2}}$.

22. $\sum_{k=0}^{\infty} \left(\frac{7}{4}\right)^k = \left(\frac{7}{4}\right)^0 + \sum_{k=1}^{\infty} \left(\frac{7}{4}\right)^k = 1 + \sum_{k=1}^{\infty} \frac{7}{4}\left(\frac{7}{4}\right)^{k-1}$. The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{7}{4}$ and $r = \frac{7}{4}$. Since $|r| > 1$, this series will diverge, which means the original series will diverge as well.

23. $\sum_{k=1}^{\infty} (-0.38)^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -0.38$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1}{1-(-0.38)} = \frac{1}{1.38} = \frac{100}{138} = \boxed{\frac{50}{69}}$.

24. $\sum_{k=1}^{\infty} (-0.38)^k = \sum_{k=1}^{\infty} (-0.38)(-0.38)^{k-1}$. This has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = -0.38$ and $r = -0.38$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{-0.38}{1-(-0.38)} = -\frac{0.38}{1.38} = -\frac{38}{138} = \boxed{-\frac{19}{69}}$.

25. $\sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k} = \frac{2^{0+1}}{3^0} + \sum_{k=1}^{\infty} 2 \cdot \left(\frac{2}{3}\right)^k = 2 + \sum_{k=1}^{\infty} 2 \cdot \left(\frac{2}{3}\right) \left(\frac{2}{3}\right)^{k-1} = 2 + \sum_{k=1}^{\infty} \frac{4}{3} \left(\frac{2}{3}\right)^{k-1}$.

The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{3}$ and $r = \frac{2}{3}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{4/3}{1-\frac{2}{3}} = \frac{4/3}{1/3} = 4$. The original series will converge to a sum of $2 + 4 = \boxed{6}$.

26. $\sum_{k=0}^{\infty} \frac{5^k}{6^{k+1}} = \frac{5^0}{6^{0+1}} + \sum_{k=1}^{\infty} \frac{5^k}{6^{k+1}} = \frac{1}{6} + \sum_{k=1}^{\infty} \frac{5}{6^2} \cdot \frac{5^{k-1}}{6^{k-1}} = \frac{1}{6} + \sum_{k=1}^{\infty} \frac{5}{36} \cdot \left(\frac{5}{6}\right)^{k-1}$.

The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{5}{36}$ and $r = \frac{5}{6}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{5/36}{1-\frac{5}{6}} = \frac{5/36}{1/6} = \frac{5}{6}$. The original series converges to a sum of $\frac{1}{6} + \frac{5}{6} = \boxed{1}$.

27. $\sum_{k=0}^{\infty} \frac{1}{4^{k+1}} = \frac{1}{4^{0+1}} + \sum_{k=1}^{\infty} \frac{1}{4^{k+1}} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4^2} \cdot \left(\frac{1}{4}\right)^{k-1}$.

The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{4^2}$ and $r = \frac{1}{4}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1/4^2}{1-\frac{1}{4}} = \frac{1/16}{3/4} = \frac{1}{12}$. The original series converges to a sum of $\frac{1}{4} + \frac{1}{12} = \boxed{\frac{1}{3}}$.

28. $\sum_{k=0}^{\infty} \frac{4^{k+1}}{3^k} = \frac{4^{0+1}}{3^0} + \sum_{k=1}^{\infty} \frac{4^{k+1}}{3^k} = 4 + \sum_{k=1}^{\infty} \frac{4^2}{3} \left(\frac{4}{3}\right)^{k-1}.$

The second term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4^2}{3}$ and $r = \frac{4}{3}$. Since $|r| > 1$, this series *diverges*, and so does the original series.

29. $\sum_{k=1}^{\infty} \sin^{k-1} \left(\frac{\pi}{2}\right)$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sin \frac{\pi}{2} = 1$. Since $|r| = 1$, this series *diverges*.

30. $\sum_{k=1}^{\infty} \tan^{k-1} \left(\frac{\pi}{4}\right)$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \tan \frac{\pi}{4} = 1$. Since $|r| = 1$, this series *diverges*.

31. $\sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| > 1$, this series *diverges*.

32. $\sum_{k=1}^{\infty} \left(-\frac{2}{3}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -\frac{2}{3}$. Since $|r| < 1$, this series will *converge* to a sum of $\frac{a}{1-r} = \frac{1}{1-(-\frac{2}{3})} = \frac{1}{5/3} = \boxed{\frac{3}{5}}.$

33. $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \left(\frac{1}{3}\right)^n + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series *converges* to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{1}{2/3} = \boxed{\frac{3}{2}}.$

34. $1 + \frac{1}{4} + \frac{1}{16} + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{1}{4}$. Since $|r| < 1$, this series *converges* to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \boxed{\frac{4}{3}}.$

35. $1 + 2 + 4 + \cdots + 2^n + \cdots = \sum_{k=1}^{\infty} 2^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = 2$. Since $|r| > 1$, this series will *diverge*.

36. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^{n-1}}{2^{n-1}} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{k-1}} = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -\frac{1}{2}$. Since $|r| < 1$, this series *converges* to a sum of $\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{3/2} = \boxed{\frac{2}{3}}.$

37. $\left(\frac{1}{7}\right)^2 + \left(\frac{1}{7}\right)^3 + \cdots + \left(\frac{1}{7}\right)^n + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^{k-1} - 1 - \frac{1}{7}.$

The first term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{1}{7}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{7}} = \frac{1}{6/7} = \frac{7}{6}$. The original series *converges* to a sum of

$$\frac{7}{6} - 1 - \frac{1}{7} = \boxed{\frac{1}{42}}.$$

$$38. \left(\frac{3}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \cdots + \left(\frac{3}{4}\right)^n + \cdots = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1} - 1 - \frac{3}{4} - \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4.$$

The first term has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{3}{4}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{3}{4}} = \frac{1}{1/4} = 4$. The original series converges to a sum of

$$4 - 1 - \frac{3}{4} - \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4 = 3 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} - \frac{81}{256} = \boxed{\frac{243}{256}}.$$

39. The series can be written $\sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k'=1}^{\infty} \frac{1}{k'}$, where $k' = k + 1$. This is the harmonic series, which diverges.

40. $\sum_{k=4}^{\infty} k^{-1} = \sum_{k=4}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k} - 1 - \frac{1}{2} - \frac{1}{3}$. The first term is the harmonic series. It diverges, and so does the original series.

41. $\sum_{k=1}^{\infty} \frac{1}{100^k} = \sum_{k=1}^{\infty} \frac{1}{100} \cdot \left(\frac{1}{100}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{100}$ and $r = \frac{1}{100}$.

Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{1/100}{99/100} = \boxed{\frac{1}{99}}$.

42. $\sum_{k=1}^{\infty} e^{-k} = \sum_{k=1}^{\infty} \frac{1}{e^k} = \sum_{k=1}^{\infty} \frac{1}{e} \cdot \left(\frac{1}{e}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{e}$ and $r = \frac{1}{e}$.

Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \boxed{\frac{1}{e-1}}$.

43. $\sum_{k=1}^{\infty} (-10k)$. The n^{th} partial sum is $S_n = \sum_{k=1}^n (-10k) = (-10) \sum_{k=1}^n k = (-10) \cdot \frac{n(n+1)}{2}$, using the standard formula for the sum of the first n integers. We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-10) \frac{n(n+1)}{2} = -\infty.$$

Since the sequence of partial sums $\{S_n\}$ diverges, the series diverges as well.

44. $\sum_{k=1}^{\infty} \frac{3k}{5}$. The n^{th} partial sum is $S_n = \sum_{k=1}^n \frac{3k}{5} = \frac{3}{5} \sum_{k=1}^n k = \frac{3}{5} \cdot \frac{n(n+1)}{2}$, using the standard formula for the sum of the first n integers. We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{3}{5} \frac{n(n+1)}{2} = \infty.$$

Since the sequence of partial sums $\{S_n\}$ diverges, the series diverges as well.

45. $\sum_{k=1}^{\infty} \cos^{k-1} \left(\frac{2\pi}{3}\right)$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \cos \frac{2\pi}{3} = -\frac{1}{2}$. Since

$|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$.

46. $\sum_{k=1}^{\infty} \sin^{k-1} \left(\frac{\pi}{6}\right)$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sin \frac{\pi}{6} = \frac{1}{2}$. Since

$|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$.

47. $\sum_{k=1}^{\infty} \frac{\tan^k(\frac{\pi}{4})}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$, since $\tan \frac{\pi}{4} = 1$. This is a harmonic series and so it *diverges*.

48. $\sum_{k=1}^{\infty} \frac{\sin^k(\frac{\pi}{2})}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$, since $\sin \frac{\pi}{2} = 1$. This is a harmonic series and so it *diverges*.

49. $\sum_{k=1}^{\infty} \cos(\pi k) = \cos \pi + \cos 2\pi + \cos 3\pi + \cdots + \cos n\pi + \cdots$. Now, $\cos \pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc. In general, $\cos n\pi = (-1)^n$. So the series can be written $\sum_{k=1}^{\infty} (-1)^k = \sum_{k=1}^{\infty} (-1)(-1)^{k-1}$, which has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = -1$ and $r = -1$. Since $|r| = 1$, this series *diverges*.

50. $\sum_{k=1}^{\infty} \sin(\frac{\pi k}{2}) = \sin \frac{\pi}{2} + \sin \pi + \sin \frac{3\pi}{2} + \sin 2\pi + \cdots$. Now, $\sin \frac{\pi}{2} = 1$, $\sin \pi = 0$, $\sin \frac{3\pi}{2} = -1$, $\sin 2\pi = 0$, etc. We see that the partial sums $S_n = \sum_{k=1}^n \sin(\frac{\pi k}{2})$ will alternate between the values of $+1$ and 0 , depending on the value of n . The sequence of partial sums is $\{S_n\} = \{1, 0, 1, 0, 1, 0, \dots\}$. Since the limit of the sequence of partial sums does not exist, the series *diverges*.

51. $\sum_{k=1}^{\infty} 2^{-k} 3^{k+1} = \sum_{k=1}^{\infty} \frac{3^{k+1}}{2^k} = \sum_{k=1}^{\infty} \frac{3^2 \cdot 3^{k-1}}{2 \cdot 2^{k-1}} = \sum_{k=1}^{\infty} \frac{9}{2} \left(\frac{3}{2}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{9}{2}$ and $r = \frac{3}{2}$. Since $|r| > 1$, this series *diverges*.

52. $\sum_{k=1}^{\infty} 3^{1-k} 2^{1+k} = \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^{k-1}} = \sum_{k=1}^{\infty} 2^2 \cdot \left(\frac{2}{3}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = 2^2 = 4$ and $r = \frac{2}{3}$. Since $|r| < 1$, this series will *converge* to a sum of $\frac{a}{1-r} = \frac{4}{1-\frac{2}{3}} = \frac{4}{\frac{1}{3}} = \frac{4}{1} = \boxed{12}$.

53. $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k = \sum_{k=1}^{\infty} \left(-\frac{1}{3}\right) \left(-\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = -\frac{1}{3}$ and $r = -\frac{1}{3}$. Since $|r| < 1$, this series *converges* to a sum of $\frac{a}{1-r} = \frac{-\frac{1}{3}}{1-(-\frac{1}{3})} = \frac{-1/3}{4/3} = \boxed{-\frac{1}{4}}$.

54. $\sum_{k=1}^{\infty} \frac{\pi}{3^k} = \sum_{k=1}^{\infty} \frac{\pi}{3} \cdot \left(\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{\pi}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series *converges* to a sum of $\frac{a}{1-r} = \frac{\frac{\pi}{3}}{1-\frac{1}{3}} = \frac{\pi/3}{2/3} = \boxed{\frac{\pi}{2}}$.

55. $\sum_{k=1}^{\infty} \ln \frac{k}{k+1} = \sum_{k=1}^{\infty} (\ln k - \ln(k+1))$. The n^{th} partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^n (\ln k - \ln(k+1)) = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) \\ &= \ln 1 - \ln(n+1) = -\ln(n+1). \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} [-\ln(n+1)] = -\infty.$$

Since the limit of the partial sums diverges, the sequence $\{S_n\}$ is a divergent sequence. This in turn means that the original series is *divergent*.

56. $\sum_{k=1}^{\infty} [e^{2k^{-1}} - e^{2(k+1)^{-1}}]$. The n^{th} partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^n (e^{2k^{-1}} - e^{2(k+1)^{-1}}) = (e^{2/1} - e^{2/2}) + (e^{2/2} - e^{2/3}) + \cdots + (e^{2/(n-1)} - e^{2/n}) + (e^{2/n} - e^{2/(n+1)}) \\ &= e^2 - e^{2/(n+1)}. \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n (e^{2k^{-1}} - e^{2(k+1)^{-1}}) = \lim_{n \rightarrow \infty} e^2 - \lim_{n \rightarrow \infty} e^{2/(n+1)} = e^2 - e^{\lim_{n \rightarrow \infty} 2/(n+1)} = e^2 - e^0 = e^2 - 1.$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series converges to the same limit $e^2 - 1$.

57. $\sum_{k=1}^{\infty} \left(\sin \frac{1}{k} - \sin \frac{1}{k+1} \right)$. The n^{th} partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\sin \frac{1}{k} - \sin \frac{1}{k+1} \right) \\ &= \left(\sin \frac{1}{1} - \sin \frac{1}{2} \right) + \left(\sin \frac{1}{2} - \sin \frac{1}{3} \right) + \cdots + \left(\sin \frac{1}{n-1} - \sin \frac{1}{n} \right) + \left(\sin \frac{1}{n} - \sin \frac{1}{n+1} \right) \\ &= \sin 1 - \sin \frac{1}{n+1}. \end{aligned}$$

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(\sin 1 - \sin \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} \sin 1 - \lim_{n \rightarrow \infty} \sin \frac{1}{n+1} = \sin 1 - \sin \left(\lim_{n \rightarrow \infty} \frac{1}{n+1} \right) \\ &= \sin 1 - \sin 0 = \sin 1. \end{aligned}$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series converges to the same limit $\sin 1$.

58. $\sum_{k=1}^{\infty} \left(\tan \frac{1}{k} - \tan \frac{1}{k+1} \right)$. The n^{th} partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\tan \frac{1}{k} - \tan \frac{1}{k+1} \right) \\ &= \left(\tan \frac{1}{1} - \tan \frac{1}{2} \right) + \left(\tan \frac{1}{2} - \tan \frac{1}{3} \right) + \cdots + \left(\tan \frac{1}{n-1} - \tan \frac{1}{n} \right) + \left(\tan \frac{1}{n} - \tan \frac{1}{n+1} \right) \\ &= \tan 1 - \tan \frac{1}{n+1}. \end{aligned}$$

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(\tan 1 - \tan \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} \tan 1 - \lim_{n \rightarrow \infty} \tan \frac{1}{n+1} = \tan 1 - \tan \left(\lim_{n \rightarrow \infty} \frac{1}{n+1} \right) \\ &= \tan 1 - \tan 0 = \tan 1. \end{aligned}$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series converges to the same limit $\tan 1$.

59. Write the given decimal as follows:

$$\begin{aligned} 0.5555 \dots &= 0.5 + 0.05 + 0.005 + \dots = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \dots \\ &= \frac{5}{10} \left(1 + \frac{1}{10} + \frac{1}{100} + \dots \right) \\ &= \sum_{k=1}^{\infty} \frac{5}{10} \left(\frac{1}{10} \right)^{k-1}. \end{aligned}$$

This is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{5}{10}$ and $r = \frac{1}{10}$. Since $|r| < 1$, this series converges to a sum of

$$\frac{a}{1-r} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{5/10}{9/10} = \frac{5}{9}. \text{ So the decimal written as a rational number is } 0.5555 \dots = \boxed{\frac{5}{9}}.$$

60. Write the given decimal as follows:

$$\begin{aligned} 0.727272 \dots &= 0.72 + 0.0072 + 0.000072 + \dots = \frac{72}{100} + \frac{72}{10000} + \frac{72}{1000000} + \dots \\ &= \frac{72}{100} \left(1 + \frac{1}{100} + \frac{1}{10000} + \dots \right) \\ &= \sum_{k=1}^{\infty} \frac{72}{100} \left(\frac{1}{100} \right)^{k-1}. \end{aligned}$$

This is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{72}{100}$ and $r = \frac{1}{100}$. Since $|r| < 1$, this series converges to a sum

$$\text{of } \frac{a}{1-r} = \frac{\frac{72}{100}}{1-\frac{1}{100}} = \frac{72/100}{99/100} = \frac{72}{99} = \frac{8}{11}. \text{ So the decimal written as a rational number is } 0.727272 \dots = \boxed{\frac{8}{11}}.$$

61. Write the given decimal as follows:

$$\begin{aligned} 4.28555 \dots &= 4.28 + 0.00555 \dots = 4.28 + 0.005 + 0.0005 + 0.00005 + \dots \\ &= 4.28 + \frac{5}{1000} + \frac{5}{10000} + \frac{5}{100000} + \dots = 4.28 + \frac{5}{1000} \left(1 + \frac{1}{10} + \frac{1}{100} + \dots \right) \\ &= 4.28 + \sum_{k=1}^{\infty} \frac{5}{1000} \left(\frac{1}{10} \right)^{k-1}. \end{aligned}$$

The second term is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{5}{1000}$ and $r = \frac{1}{10}$. Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{5}{1000}}{1-\frac{1}{10}} = \frac{5/1000}{9/10} = \frac{5}{900}$. So the original series sums to the value of the decimal in rational form, which is $4.28 + \frac{5}{900} = \frac{428}{100} + \frac{5}{900} = \boxed{\frac{3857}{900}}.$

62. Write the given decimal as follows:

$$\begin{aligned} 7.162162162 \dots &= 7 + 0.162162162 \dots = 7 + 0.162 + 0.000162 + 0.000000162 + \dots \\ &= 7 + \frac{162}{1,000} + \frac{162}{1,000,000} + \frac{162}{1,000,000,000} + \dots \\ &= 7 + \frac{162}{1000} \left(1 + \frac{1}{1,000} + \frac{1}{1,000,000} + \dots \right) \\ &= 7 + \sum_{k=1}^{\infty} \frac{162}{1000} \left(\frac{1}{1000} \right)^{k-1}. \end{aligned}$$

The second term is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{162}{1000}$ and $r = \frac{1}{1000}$. Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{162}{1000}}{1-\frac{1}{1000}} = \frac{162/1000}{999/1000} = \frac{162}{999}$. So the original series sums to the value of the decimal in rational form, which is $7 + \frac{162}{999} = \boxed{\frac{265}{37}}$.

Applications and Extensions

63. Let the original height from which the ball is dropped be h_0 . The height attained after the first bounce $h_1 = h_0 \cdot r$, where r = fraction of the original height. The height attained after the second bounce $h_2 = h_1 \cdot r = (h_0 r)r = h_0 r^2$. Proceeding in this fashion, we can say that the height attained after the n^{th} bounce will be $h_n = h_0 r^n$.

The total vertical distance traveled by the ball, remembering that each bounce has an upward vertical travel followed by a downward vertical travel, is given by

$$\begin{aligned} H &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_n + \cdots \\ &= h_0 + 2(h_0 r + h_0 r^2 + h_0 r^3 + \cdots + h_0 r^n + \cdots) \\ &= h_0 + 2h_0 r(1 + r + r^2 + \cdots + r^{n-1} + \cdots) \\ &= h_0 + 2h_0 r \cdot \sum_{k=1}^{\infty} r^{k-1}. \end{aligned}$$

Since the process of bouncing is not an elastic collision for the ball in question, we have $|r| < 1$, so that the second term is a convergent geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ which sums to $\frac{a}{1-r}$, with $a = 1$, so the sum will be $\frac{1}{1-r}$. So the total vertical distance traversed by the bouncing ball is

$$H = h_0 + 2h_0 r \cdot \frac{1}{1-r} = h_0 \left[1 + \frac{2r}{1-r} \right] = h_0 \left(\frac{1+r}{1-r} \right) = 18\text{ft} \left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}} \right) = \frac{18 \cdot (5/3)}{1/3} \text{ft} = \boxed{90\text{ft.}}$$

64. (a) Let the initial dollar amount on January 1, 2015 be $A_1 = A_0$. The amount on January 2 will be $A_2 = rA_1 = rA_0$, where r = the fraction of the original amount given on January 1. On January 3, the amount received will be $r(A_2) = r^2 A_0$, and so forth. The total amount received

$A = A_0 + rA_0 + r^2 A_0 + \cdots = A_0(1 + r + r^2 + \cdots) = \sum_{k=1}^{\infty} A_0 r^{k-1}$. If $|r| < 1$, this is a convergent geometric series with sum $\frac{A_0}{1-r}$. So the total amount received $A = \frac{A_0}{1-r} = \frac{\$1000}{1-\frac{1}{9}} = \frac{\$1000}{1/10} = \boxed{\$10,000}$.

(b) We want the amount received on the n^{th} day to be less than one cent = \$0.01. So $A_n = r^{n-1} A_0 < \$0.01$ gives $r^{n-1} < \frac{\$0.01}{\$1000} = 10^{-5}$. Taking the $\log_{10}$ of both sides, remembering that $\log_{10} r$ is an increasing function of r , we have

$$\begin{aligned} (n-1) \log_{10} r &< -5 \log_{10} 10 \\ n-1 &> -\frac{5}{\log_{10} r} \\ n-1 &> -\frac{5}{\log_{10} \frac{9}{10}} \\ &= \frac{-5}{-0.04576} \approx 109.27, \end{aligned}$$

(where the inequality has been reversed since $|r| < 1$ gives negative values for $\log_{10} r$), or $n-1 = 110$ days, so $n = 111$ days. Now, 2015 is not a leap year, so February 2015 has 28 days. January has 31 days, March has 31, for a total of $28 + 31 + 31 = 90$ days. So a total time period of 111 days after (and including) January 1 would fall on April 21, 2015. On, and after this day, the amount paid out will be less than 1 cent.

65. (a) The population of rainbow trout in Mirror Lake in year n of the restocking program is

$p_n = 3000r^n + h \sum_{k=1}^n r^{k-1}$. As $n \rightarrow \infty$, the manager can expect the steady rainbow trout population to be

$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} 3000r^n + h \lim_{n \rightarrow \infty} \sum_{k=1}^n r^{k-1} = 3000 \lim_{n \rightarrow \infty} r^n + h \sum_{k=1}^{\infty} r^{k-1}.$$

Since $|r| < 1$, we have $\lim_{n \rightarrow \infty} r^n = 0$. Also the second term is a convergent geometric series with sum $\frac{h}{1-r}$.

So the steady population of rainbow trout is $p = 3000 \cdot 0 + \frac{h}{1-r} = \boxed{\frac{h}{1-r}}$.

(b) We require $p = 4000$ with $r = 0.5$. This means $h = p(1-r) = 4000(1-0.5) = \boxed{2000}$ is the number of fish to be added annually to obtain the steady population.

66. The multiplier for a marginal propensity to consume of 90% is $\sum_{k=1}^{\infty} (0.90)^{k-1} = \frac{1}{1-0.9} = \frac{1}{0.1} = \boxed{10}$.

67. (a) The expression for stock pricing can be written as

Price = $P + P \frac{1+i}{1+r} + P \left(\frac{1+i}{1+r} \right)^2 + P \left(\frac{1+i}{1+r} \right)^3 + \cdots = \sum_{k=1}^{\infty} P \left(\frac{1+i}{1+r} \right)^{k-1} = \frac{P}{1 - \left(\frac{1+i}{1+r} \right)} = \boxed{\frac{P(1+r)}{r-i}}$. (Here, we have used the fact that in most normal circumstances, the annual rate of return $r\%$ will always be greater than the annual dividend increase $i\%$, which means $\left| \frac{1+i}{1+r} \right| < 1$, and so the geometric series will converge to the sum we found rather than diverge.)

(b) If $r = 9\%$ the annual rate of return on a stock, $P = \$4.00$ what the stock currently pays in annual dividends, and $i = 3\%$ the annual increase in dividend, then the highest stock price the investor should pay will be about $\frac{P(1+r)}{r-i} = \frac{\$4(1+9)}{9-3} = \frac{\$4(10)}{6} = \boxed{\$6.67}$.

68. Area of the Koch snowflake is

$$\begin{aligned} A &= 1 + 3 \left(\frac{1}{9} \right) + 12 \left(\frac{1}{9} \right)^2 + 48 \left(\frac{1}{9} \right)^3 + 192 \left(\frac{1}{9} \right)^4 + \cdots \\ &= 1 + \frac{3}{9} \left[1 + 4 \cdot \frac{1}{9} + 16 \cdot \left(\frac{1}{9} \right)^2 + 64 \cdot \left(\frac{1}{9} \right)^3 + \cdots \right] \\ &= 1 + \frac{1}{3} \left[1 + \frac{4}{9} + \left(\frac{4}{9} \right)^2 + \left(\frac{4}{9} \right)^3 + \cdots \right] \\ &= 1 + \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{4}{9} \right)^{k-1} \\ &= 1 + \frac{1}{3} \cdot \frac{1}{1 - \frac{4}{9}} = 1 + \frac{1}{3} \cdot \frac{9}{9-4} = 1 + \frac{3}{5} \\ &= \boxed{\frac{8}{5} \text{ square units.}} \end{aligned}$$

69. Let the length of the race track be L . Let the tortoise be given a lead of $d < L$. If Achilles' speed is v_A and the tortoise's speed is v_T , then we are given that $v_A - v_T = v$, a (positive) constant. If the tortoise starts off at $t = 0$, then the time elapsed before Achilles starts is $\Delta t = \frac{d}{v_T}$. The time taken by Achilles to cover half the distance separating them would be $t_1 = \frac{d}{2v_A}$. By this time, the tortoise has moved out further by a distance $= v_T t_1 = \frac{v_T}{2v_A} d$. The new remaining distance between them $= \frac{d}{2} + \frac{v_T}{2v_A} d = \frac{d}{2} + \frac{v_A - v}{v_A} \frac{d}{2} = d - \frac{v}{v_A} \frac{d}{2} = d_1$, say.

The time taken by Achilles to cover half this new distance is $t_2 = \frac{d_1}{2v_A}$. The tortoise will by then have moved out further by a distance $= \frac{v_T}{2v_A}d_1$. The new remaining distance separating them,

$$\begin{aligned} d_2 &= d_1 - \frac{v}{v_A} \frac{d_1}{2} \\ &= d_1 \left(1 - \frac{v}{2v_A}\right) \\ &= d \left(1 - \frac{v}{2v_A}\right) \left(1 - \frac{v}{2v_A}\right) \\ &= d \left(1 - \frac{v}{2v_A}\right)^2. \end{aligned}$$

After n such halvings, the distance separating them $d_n = d \left(1 - \frac{v}{2v_A}\right)^n$. Since $\boxed{r = 1 - \frac{v}{2v_A}} < 1$, we have

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} dr^n = d \lim_{n \rightarrow \infty} r^n = 0.$$

This demonstrates that Achilles will eventually catch up with the tortoise, since the distance separating them tends to 0. In fact, we need to next show that this process takes a *finite* amount of time.

The total time for catching up after a time Δt has elapsed (this is the lead time given to the tortoise) is

$$\begin{aligned} T &= t_1 + t_2 + \cdots + t_n + \cdots \\ &= \frac{d}{2v_A} + \frac{d_1}{2v_A} + \frac{d_2}{2v_A} + \cdots + \frac{d_{n-1}}{2v_A} + \cdots \\ &= \frac{d}{2v_A} + r \frac{d}{2v_A} + r^2 \frac{d}{2v_A} + \cdots \\ &= \frac{d}{2v_A} (1 + r + r^2 + \cdots) \\ &= \frac{d}{2v_A} \sum_{k=1}^{\infty} r^{k-1} = \frac{d}{2v_A} \left(\frac{1}{1-r} \right) \\ &= \frac{d}{2v_A} \left(\frac{1}{1 - \left(1 - \frac{v}{2v_A}\right)} \right) = \frac{d}{2v_A} \left(\frac{1}{\frac{v}{2v_A}} \right) \\ &= \frac{d}{v}. \end{aligned}$$

This is exactly as we would expect, the time taken by Achilles to overtake the tortoise is just the original distance separating them divided by the relative difference in speeds. So we see by using a series argument that Achilles will catch up with the tortoise in a finite time.

Next, we show using a series argument that the total distance covered by Achilles in catching up with the tortoise is also finite. The total distance is

$$\begin{aligned} D &= \frac{d}{2} + \frac{d_1}{2} + \frac{d_2}{2} + \cdots \\ &= \frac{d}{2} + \frac{rd}{2} + \frac{r^2d}{2} + \cdots \\ &= \frac{1}{2}d(1 + r + r^2 + \cdots) \\ &= \frac{1}{2}d \sum_{k=1}^{\infty} r^{k-1} = \frac{1}{2}d \cdot \frac{1}{1-r} = \frac{1}{2}d \cdot \left(\frac{1}{1 - \left(1 - \frac{v}{2v_A}\right)} \right) = \frac{v_A}{v} d \\ &= \frac{v_A}{v_A - v_T} d. \end{aligned}$$

Since $v_T > 0$, this distance is bigger than d , as we would expect. If the speeds are such that $D < L$, then Achilles may even overtake the tortoise and win the race!

70. Probability of winning $= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots + \frac{1}{2^{2n-1}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \sum_{k=1}^{\infty} \frac{1}{2} \frac{1}{2^{2k-2}} = \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{4}\right)^{k-1}$. This

is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{2}$ and $r = \frac{1}{4}$, which, since $|r| < 1$, will converge to a sum of

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1/2}{3/4} = \boxed{\frac{2}{3}}.$$

71. (a) Amount of pig and cattle feces produced prior to cleaning on the first day $T(1) = p$. After the cleaning at the end of the first day, amount of feces left over $= (1-e)T(1) = (1-e)p$. Amount of feces produced on the second day, prior to cleaning, $T(2) = p + (1-e)p$. After the end of the second day's cleaning, amount of feces left over $= (1-e)T(2) = (1-e)[p + (1-e)p] = (1-e)p + (1-e)^2p$. Then, amount of feces produced on the third day, prior to cleaning, $T(3) = p + (1-e)p + (1-e)^2p$. Continuing this way, the total amount of accumulated fecal matter on the n^{th} day prior to cleaning will be

$$T(n) = p + (1-e)p + (1-e)^2p + \cdots + (1-e)^{n-1}p = \boxed{p \sum_{k=1}^n (1-e)^{k-1}}.$$

(b) $T = \lim_{n \rightarrow \infty} T(n) = \lim_{n \rightarrow \infty} p \sum_{k=1}^n (1-e)^{k-1} = \sum_{k=1}^{\infty} (1-e)^{k-1} = p \cdot \frac{1}{1-(1-e)} = \boxed{\frac{p}{e}}$, since the series is a

converging geometric series of the type $\sum_{k=1}^{\infty} r^{k-1}$ with $|r| = |1-e| < 1$.

(c) From part (b) we see that $T = \frac{p}{e}$. If we require that $T \leq L$, then we would have $T(n) \leq L$ for all n , since $n < \infty$ and $T(n)$ is a monotonic increasing function of n . Now the condition $T \leq L$ leads to $\frac{p}{e} \leq L$, or $e \geq \frac{p}{L}$. This shows that the minimum efficiency of cleaning is $e_{\min} = \boxed{\frac{p}{L}}$.

(d) We have $e_{\min} = \frac{p}{L} = \frac{120 \text{ kg}}{180 \text{ kg}} = \boxed{\frac{2}{3}}$. At $e = \frac{4}{5}$, we have $T(365) \approx T(\infty) = \frac{p}{e} = \frac{120 \text{ kg}}{4/5} = \boxed{150 \text{ kg}}$.

72. Write

$$\begin{aligned} 0.9999 \cdots &= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots \\ &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots \\ &= \frac{9}{10} \left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots \right) \\ &= \frac{9}{10} \sum_{k=1}^{\infty} \left(\frac{1}{10} \right)^{k-1} = \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = 1, \end{aligned}$$

that is $0.9999 \cdots = 1$, as was needed to be demonstrated.

73. The right hand side is a geometric series: $\sum_{k=1}^{\infty} \frac{1}{x^{k-1}} = \sum_{k=1}^{\infty} \left(\frac{1}{x}\right)^{k-1}$. If $|x| > 1$, then $\left|\frac{1}{x}\right| < 1$, so this series will converge to a sum of $\frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}$. So we have shown that $\frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{x^{k-1}}$ for $|x| > 1$.

74. The right hand side is a geometric series: $\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1} = \sum_{k=1}^{\infty} (-x)^{k-1}$. Since $|r| = |-x| = |x| < 1$, this geometric series will converge to a sum of $\frac{1}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$. So we have shown that $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$.

75. Since there is no closed form expression for the partial sums of the harmonic series, we have to proceed by trial and error calculating the partial sums. The partial sum $S_n = \sum_{k=1}^n \frac{1}{k}$ for $n = 10$ is 2.929, and for $n = 11$ it is 3.020. So $\boxed{n = 11}$ is the smallest number for which $\sum_{k=1}^n \frac{1}{k} \geq 3$.

76. Since there is no closed form expression for the partial sums of the harmonic series, we have to proceed by trial and error calculating the partial sums. The partial sum $S_n = \sum_{k=1}^n \frac{1}{k}$ for $n = 30$ is 3.995, and for $n = 31$ it is 4.027. So $\boxed{n = 31}$ is the smallest number for which $\sum_{k=1}^n \frac{1}{k} \geq 4$.

77. The given series can be rewritten as follows: $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$. Consider the n^{th} partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) \\ &= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \cdots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \\ &= 1 - \frac{1}{\sqrt{n+1}}. \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 1 - 0 = 1.$$

Since the sequence of partial sums converges, the series $\boxed{\text{converges}}$ as well, and has the sum $\boxed{1}$.

78. The given series can be rewritten as follows: $\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2(k+2)} \right)$, using partial fractions. Consider the n^{th} partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\frac{1}{2k} - \frac{1}{2(k+2)} \right) = \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+2} \right) \\ &= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \\ &\quad + \frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n} \right) + \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right). \end{aligned}$$

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{4} - \lim_{n \rightarrow \infty} \frac{1}{2(n+1)} - \lim_{n \rightarrow \infty} \frac{1}{2(n+2)} = \frac{3}{4} - 0 - 0 = \frac{3}{4}. \end{aligned}$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{3}{4}}$.

79. The given series can be rewritten using the techniques of partial fraction decomposition as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{\infty} \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} = \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right).$$

Consider the n^{th} partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right) = \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\ &= \left\{ \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n} \right) + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) \right\} \\ &\quad - \left\{ \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right\} \\ &= \left\{ \frac{1}{2} \left(\frac{1}{1} \right) - \frac{1}{2} \left(\frac{1}{n+1} \right) \right\} - \left\{ \frac{1}{2} \left(\frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{n+2} \right) \right\} \\ &= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}. \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \right) = \lim_{n \rightarrow \infty} \frac{1}{4} - \lim_{n \rightarrow \infty} \frac{1}{2(n+1)} + \lim_{n \rightarrow \infty} \frac{1}{2(n+2)} = \frac{1}{4} - 0 + 0 = \frac{1}{4}.$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{1}{4}}.$

80. The given series can be rewritten using the techniques of partial fraction decomposition as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \sum_{k=1}^{\infty} \frac{1}{6k} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)} - \frac{1}{6(k+3)} = \sum_{k=1}^{\infty} \frac{1}{6k} - \frac{3}{6(k+1)} + \frac{3}{6(k+2)} - \frac{1}{6(k+3)}.$$

Consider the n^{th} partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{1}{6} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \frac{1}{3} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) + \frac{1}{6} \left(\frac{1}{k+2} - \frac{1}{k+3} \right) \\ &= \left\{ \frac{1}{6} \left(\frac{1}{1} - \frac{1}{2} \right) + \frac{1}{6} \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \frac{1}{6} \left(\frac{1}{n-1} - \frac{1}{n} \right) + \frac{1}{6} \left(\frac{1}{n} - \frac{1}{n+1} \right) \right\} \\ &\quad - \left\{ \frac{1}{3} \left(\frac{1}{2} - \frac{1}{3} \right) + \frac{1}{3} \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \frac{1}{3} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{1}{3} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right\} \\ &\quad + \left\{ \frac{1}{6} \left(\frac{1}{3} - \frac{1}{4} \right) + \frac{1}{6} \left(\frac{1}{4} - \frac{1}{5} \right) + \cdots + \frac{1}{6} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \frac{1}{6} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \right\} \\ &= \left\{ \frac{1}{6} \left(\frac{1}{1} \right) - \frac{1}{6} \left(\frac{1}{n+1} \right) \right\} - \left\{ \frac{1}{3} \left(\frac{1}{2} \right) - \frac{1}{3} \left(\frac{1}{n+2} \right) \right\} + \left\{ \frac{1}{6} \left(\frac{1}{3} \right) - \frac{1}{6} \left(\frac{1}{n+3} \right) \right\} \\ &= \frac{1}{6} - \frac{1}{6} + \frac{1}{18} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)} \\ &= \frac{1}{18} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)}. \end{aligned}$$

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(\frac{1}{18} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{18} - \lim_{n \rightarrow \infty} \frac{1}{6(n+1)} + \lim_{n \rightarrow \infty} \frac{1}{3(n+2)} - \lim_{n \rightarrow \infty} \frac{1}{6(n+3)} \\ &= \frac{1}{18} - 0 + 0 - 0 = \frac{1}{18}. \end{aligned}$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{1}{18}}$.

81. We use the following result from the theory of partial fractions: $F(k) = \frac{1}{\prod_{i=1}^m (k+a_i)} = \sum_{i=1}^m \frac{A_i}{k+a_i}$, where $A_i = (k+a_i)F(k)|_{k=-a_i}$. To prove this, multiply both sides by $\prod_{i=1}^m (k+a_i)$. Then we get

$$\begin{aligned} 1 &= \sum_{i=1}^m \frac{A_i}{(k+a_i)} \prod_{i=1}^m (k+a_i) \\ &= A_1(k+a_2)(k+a_3)\cdots(k+a_m) + A_2(k+a_1)(k+a_3)\cdots(k+a_m) + A_3(k+a_1)(k+a_2)(k+a_4)\cdots(k+a_m) \\ &\quad + \cdots + A_{m-1}(k+a_1)\cdots(k+a_{m-2})(k+a_m) + A_m(k+a_1)\cdots(k+a_{m-1}). \end{aligned}$$

Note that each set of factors multiplying the coefficient A_i is missing the term $(k+a_i)$. So for instance, when we set $k = -a_1$, all terms vanish *except* the term multiplying A_1 :

$$1 = A_1(-a_1+a_2)(-a_1+a_3)\cdots(-a_1+a_m)$$

or,

$$A_1 = \frac{1}{(-a_1+a_2)(-a_1+a_3)\cdots(-a_1+a_m)} = (k+a_1)F(k)|_{k=-a_1}$$

where $F(k) = \frac{1}{\prod_{i=1}^m (k+a_i)}$. Similarly, for $k = -a_2$, we obtain $A_2 = (k+a_2)F(k)|_{k=-a_2}$, etc, and in general, for $k = -a_i$, we have $A_i = (k+a_i)F(k)|_{k=-a_i}$, which proves the stated result.

Let $F(k) = \frac{1}{k(k+1)(k+2)\cdots(k+a)}$. Since $a_1 = 0, a_2 = 1, a_3 = 2, \dots, a_i = i-1, \dots, a_m = a$, we get $A_1 = \frac{1}{a!}$, $A_2 = -\frac{a}{a!}$, $A_3 = (-1)^2 \frac{a(a-1)}{2a!}$, and so on. In general, for $1 \leq i \leq m$, $A_i = \frac{(-1)^{i-1}}{(i-1)!} \frac{a(a-1)\cdots(a-i+2)}{a!}$. For example, in Problem 79, we had the case $a = 2$. We have $A_1 = \frac{1}{2!} = \frac{1}{2}$, $A_2 = -\frac{2}{2!} = -1$, and $A_3 = (-1)^2 \frac{2(2-1)}{2 \cdot 2!} = \frac{1}{2}$. The partial fraction decomposition and the partial sum were:

$$\begin{aligned} \frac{1}{k(k+1)(k+2)} &= \frac{A_1}{k} + \frac{A_2}{k+1} + \frac{A_3}{k+2} \\ &= \frac{1}{2} \cdot \frac{1}{k} - \frac{1}{k+1} + \frac{1}{2} \cdot \frac{1}{k+2} \\ &= \frac{1}{2!} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right) \\ &= \frac{1}{2!} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \frac{1}{2!} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\ S_n &= \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{1}{2 \cdot 2!} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}. \end{aligned}$$

In Problem 80, we had the case $a = 3$, so $A_1 = \frac{1}{3!} = \frac{1}{6}$, $A_2 = -\frac{3}{3!} = -\frac{1}{2}$, $A_3 = (-1)^2 \frac{3(3-1)}{2 \cdot 3!} = \frac{1}{2}$, and

$A_4 = (-1)^3 \frac{3(3-1)(3-2)}{3! \cdot 3!} = -\frac{1}{6}$. The partial fraction decomposition and the partial sum were

$$\begin{aligned} \frac{1}{k(k+1)(k+2)(k+3)} &= \frac{A_1}{k} + \frac{A_2}{k+1} + \frac{A_3}{k+2} + \frac{A_4}{k+3} \\ &= \frac{1}{6} \cdot \frac{1}{k} - \frac{1}{2} \cdot \frac{1}{k+1} + \frac{1}{2} \cdot \frac{1}{k+2} - \frac{1}{6} \cdot \frac{1}{k+3} \\ &= \frac{1}{3!} \left(\frac{1}{k} - \frac{3}{k+1} + \frac{3}{k+2} - \frac{1}{k+3} \right) \\ &= \frac{1}{3!} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \frac{1}{3} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) + \frac{1}{6} \left(\frac{1}{k+2} - \frac{1}{k+3} \right) \\ S_n &= \sum_{k=1}^n \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3 \cdot 3!} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)}. \end{aligned}$$

For the general case of a , we get, as in the previous two problems, a series of terms multiplying the coefficients A_i . When they are added as telescoping series of the partial sum S_n , the result will have a term $\frac{1}{a \cdot a!}$ without any n -dependence, while all the surviving terms will have the form $\frac{\alpha_i A_i}{n+a_i}$ where α_i are whole numbers coming from the distribution of terms in the telescoping series. That is, we will obtain:

$$\begin{aligned} \frac{1}{k(k+1)(k+2) \cdots (k+a)} &= \frac{A_1}{k} + \frac{A_2}{k+1} + \frac{A_3}{k+2} + \frac{A_4}{k+3} + \cdots + \frac{A_i}{k+i-1} + \cdots + \frac{A_m}{k+a} \\ &= \frac{1}{a!} \cdot \frac{1}{k} - \frac{a}{a!} \cdot \frac{1}{k+1} + (-1)^2 \frac{a(a-1)}{2!a!} \cdot \frac{1}{k+2} + (-1)^3 \frac{a(a-1)(a-2)}{3!a!} \cdot \frac{1}{k+3} \\ &\quad + \cdots + (-1)^{i-1} \frac{a(a-1) \cdots (a-i)}{(i-1)!a!} \cdot \frac{1}{k+i-1} + \cdots + (-1)^{a-1} \frac{1}{a!} \cdot \frac{1}{k+a} \\ &= \frac{1}{a!} \left(\frac{1}{k} - \frac{a}{k+1} + \frac{a(a-1)}{2!(k+2)} - \frac{a(a-1)(a-2)}{3!(k+3)} + \cdots \right) \\ &\quad + \frac{1}{a!} \left(\cdots + (-1)^{i-1} \frac{a(a-1) \cdots (a-i)}{(i-1)!(k+i-1)} + \cdots + (-1)^{a-1} \frac{1}{k+a} \right) \\ &= \frac{1}{a!} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \frac{a-1}{a!} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) + \cdots + \frac{(-1)^{a-1}}{a!} \left(\frac{1}{k+a-1} - \frac{1}{k+a} \right) \\ S_n &= \sum_{k=1}^n \frac{1}{k(k+1)(k+2) \cdots (k+a)} = \frac{1}{a \cdot a!} + \sum_{k=1}^a \frac{\alpha_i A_i}{n+k}. \end{aligned}$$

In the limit, as $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} S_n = \boxed{\frac{1}{a \cdot a!}}$, which is also the sum of the series.

82. The right hand side of $\frac{x}{2+2x} = x + x^2 + x^3 + \cdots$ can be written as $\sum_{k=1}^{\infty} x \cdot x^{k-1}$. This is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = x$ and $r = x$. The series has a convergent sum, $\frac{a}{1-r} = \frac{x}{1-x}$ for $|r| = |x| < 1$. So the equation becomes:

$$\begin{aligned} \frac{x}{2(1+x)} &= \frac{x}{1-x} \\ 1-x &= 2+2x \\ 3x &= -1, \end{aligned}$$

or $\boxed{x = -\frac{1}{3}}.$

83. A convergent geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$, $|r| < 1$, has the sum $\frac{a}{1-r}$. If a is rational, then $a = \frac{p}{q}$, where p and q are integers that have no common factors, and $q \neq 0$; similarly, if r is rational, then

$r = \frac{s}{t} < 1$, where s and t are integers with no common factors, and $t \neq 0$. Then, we have

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{p}{q}}{1-\frac{s}{t}} = \frac{pt}{q(t-s)}.$$

Since pt and $q(t-s)$ are integers, the ratio is a rational number as well, with $q(t-s) \neq 0$, since $\frac{s}{t} < 1$ (possibly with common factors, which can be canceled out to simplify the result).

84. $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. When $r = 1$, $S_n = a + a + \cdots + a$ (n times) $= na$. On the other hand, since for $r \neq 1$, $S_n = \frac{a(r^n - 1)}{r - 1}$, we have

$$\lim_{r \rightarrow 1} S_n = \lim_{r \rightarrow 1} \frac{a(r^n - 1)}{r - 1} = a \lim_{r \rightarrow 1} \frac{nr^{n-1}}{1} = an = na,$$

by using L'Hôpital's rule. So the results of using either the series directly for $r = 1$, or applying a limiting procedure to the partial sum formula, agree with each other.

Challenge Problems

85. Since $\gamma = \lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n)$ and the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges very slowly, we would expect the limit above to converge very slowly as well. Using $S_n = \sum_{k=1}^n \frac{1}{k}$ gives $\gamma_n \approx S_n - \ln n$ as an approximation to γ . Using the given values of the partial sums, we have:

$$n = 10 : S_{10} = 2.92897; \gamma_{10} \approx S_{10} - \ln 10 \approx \boxed{0.62638}.$$

$$n = 20 : S_{20} = 3.59744; \gamma_{20} \approx S_{20} - \ln 20 \approx \boxed{0.60171}.$$

$$n = 50 : S_{50} = 4.49921; \gamma_{50} \approx S_{50} - \ln 50 \approx \boxed{0.58719}.$$

$$n = 100 : S_{100} = 5.18738; \gamma_{100} \approx S_{100} - \ln 100 \approx \boxed{0.58221}.$$

As expected, the rate of convergence of the limit defining γ is very slow as well.

86. Using (see solution to Problem 85) $\gamma_n \approx \sum_{k=1}^n \frac{1}{k} - \ln n$, we have $\gamma_{10^9} \approx \sum_{k=1}^{10^9} \frac{1}{k} - \ln 10^9$, or

$$\sum_{k=1}^{10^9} \frac{1}{k} \approx \gamma_{10^9} + \ln 10^9. \text{ Using } \gamma_{10^9} \approx \gamma \approx 0.5772, \text{ we have, to four decimal places, } \sum_{k=1}^{10^9} \frac{1}{k} \approx \boxed{21.3005}.$$

87. If a real number has a repeating decimal, the repeating part can be expressed as a geometric series. Such a real number may be represented as

$$R = x.yz \cdots abc \cdots abc \cdots abc \cdots$$

(x is the whole number part) where, if the repeated units $abc \cdots$ start in (say) the n^{th} decimal place, and have a string length r , we can write:

$$\begin{aligned} R &= x.yz \cdots + abc \cdots \times 10^{-n+1-r} + abc \cdots \times 10^{-n+1-2r} + abc \cdots \times 10^{-n+1-3r} + \cdots \\ &= \frac{xyz \cdots}{10^{n-1}} + abc \cdots \times 10^{-n+1-r} (1 + 10^{-r} + 10^{-2r} + \cdots) \\ &= \frac{xyz \cdots}{10^{n-1}} + \frac{abc \cdots}{10^{n-1+r}} \sum_{k=1}^{\infty} (10^{-r})^{k-1} \\ &= \frac{xyz \cdots}{10^{n-1}} + \frac{abc \cdots}{10^{n-1+r}} \left(\frac{1}{1 - 10^{-r}} \right) \\ &= \frac{xyz \cdots}{10^{n-1}} + \frac{abc \cdots}{10^{n-1}(10^r - 1)}, \end{aligned}$$

which, being a sum of rational numbers, is a rational number.

Conversely, any rational number is of the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$ with no common factors. Then performing long division produces a remainder of with either 0 which will terminate the long division, or result a repeating block of numbers because there can only be at most $q - 1$ different remainders. The moment you get a remainder that is non-zero, the rest of the long division will proceed as it had when the same remainder had earlier occurred, causing a repeating pattern to emerge in the quotient. So we have shown that a real number has a repeating decimal if and only if it is a rational number.

88. (a) Since $s_n \geq 0$ for all $n \geq 1$, the sequence of partial sums $\{S_n\}$ is a nondecreasing sequence, and so if $\{S_n\}$ is bounded, then it converges, because a bounded monotonic sequence converges. This in turn means that the series $\sum_{k=1}^{\infty} s_k$ converges. Conversely, if the original series converges, then the partial sums also converge, so by the theorem on p.547, the sequence $\{S_n\}$ must also be bounded.

(b) Write the harmonic series as $1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots$. Each of the bracketed terms is $> \frac{1}{2}$. The partial sums form an unbounded sequence since, for instance, $S_1 = 1$, $S_2 = S_{2^0} = 1 + \frac{1}{2}$; $S_4 = S_{2^2} > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} \cdot 2$; $S_8 = S_{2^3} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} \cdot 3$; $S_{16} = S_{2^4} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} \cdot 4$; and in general, $S_{2^n} \geq 1 + \sum_{k=0}^n \frac{1}{2} = 1 + \frac{1}{2} \cdot n$. This shows that the partial sums S_{2^n} are unbounded, since $\lim_{n \rightarrow \infty} S_{2^n} \geq \lim_{n \rightarrow \infty} (1 + \frac{1}{2} \cdot n) = \infty$, which means the sequence $\{S_{2^n}\}$ will diverge, so the series will diverge as well.

8.3 Properties of Series; Integral Test

Concepts and Vocabulary

1. **(a)** If the series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{n \rightarrow \infty} a_n = 0$.
2. **False:** For example, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ but $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (Harmonic series).
3. **True:** Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^3 = \infty$. The series cannot converge.
4. **True:** Any finite number of terms in a series do not determine convergence.
5. **False:** For example, $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$ converges as it is a telescoping series (the sum is 1), whereas $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ both separately diverge, being the harmonic series and the harmonic series from the second term forward, respectively.
6. **True:** If a series is convergent to S then a scalar multiple of each term yields a convergent series of sum cS .
7. **True:** This is the basis of the integral test of convergence.
8. **False:** If the terms are all positive, then the sequence of partial sums will be an increasing sequence, and if this sequence is not bounded, the series will be divergent.
9. The p -series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $\underline{p > 1}$ and diverges if $\underline{0 < p \leq 1}$.
10. **Converges.** $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a p -series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p > 1$, which converges.
11. **Diverges.** $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{-1/2}} = \sum_{k=1}^{\infty} k^{1/2}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^{1/2} = \infty \neq 0$. So by the Divergence Test, the series diverges.
12. **False:** The upper bound must be $1 + \frac{1}{p-1}$; the lower bound is correct.

Skill Building

13. The series is $\sum_{k=1}^{\infty} 16$. We have $\lim_{n \rightarrow \infty} a_n = 16 \neq 0$. So by the Divergence Test, the series *diverges.*
14. The series is $\sum_{k=1}^{\infty} \frac{k+9}{k}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{9}{n} \right) = 1 \neq 0$. So by the Divergence Test, the series *diverges.*
15. The series is $\sum_{k=1}^{\infty} \ln k$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln n = \infty \neq 0$. So by the Divergence Test, the series *diverges.*
16. The series is $\sum_{k=1}^{\infty} e^k$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} e^n = \infty \neq 0$. So by the Divergence Test, the series *diverges.*

17. The series is $\sum_{k=1}^{\infty} \frac{k^2}{k^2+4}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+4} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{4}{n^2}} = \frac{1}{1+0} = 1 \neq 0$. So by the Divergence Test, the series *diverges*.

18. The series is $\sum_{k=1}^{\infty} \frac{k^2+3}{\sqrt{k}}$. We have
 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2+3}{\sqrt{n}} = \lim_{n \rightarrow \infty} n^{3/2} + 3 \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = \lim_{n \rightarrow \infty} n^{3/2} + 0 = \infty \neq 0$. So by the Divergence Test, the series *diverges*.

19. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{1.01}}$. The function $f(x) = \frac{1}{x^{1.01}}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$\begin{aligned} I &= \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^{1.01}} = \lim_{b \rightarrow \infty} \left[\frac{x^{-1.01+1}}{-1.01+1} \right]_1^b \\ &= -\frac{1}{0.01} \lim_{b \rightarrow \infty} [b^{-0.01} - 1^{-0.01}] = -\frac{1}{0.01} [0 - 1] \\ &= 100. \end{aligned}$$

Since the improper integral I converges, by the integral test the series *converges* as well.

20. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{0.9}}$. The function $f(x) = \frac{1}{x^{0.9}}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^{0.9}} = \lim_{b \rightarrow \infty} \left[\frac{x^{-0.9+1}}{-0.9+1} \right]_1^b = \frac{1}{0.1} \lim_{b \rightarrow \infty} [b^{0.1} - 1^{0.1}] = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges* as well.

21. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k}$. The function $f(x) = \frac{\ln x}{x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} \leq 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx = \lim_{b \rightarrow \infty} \frac{1}{2} (\ln x)^2 \Big|_1^b = \lim_{b \rightarrow \infty} \frac{1}{2} [(\ln b)^2 - 0] = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges* as well.

22. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$. The function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{\ln x}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{\ln x} + x \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \sqrt{\ln x} + \frac{1}{\sqrt{\ln x}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$I = \int_2^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x\sqrt{\ln x}}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \int_2^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}} = \lim_{b \rightarrow \infty} \left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_{\ln 2}^{\ln b} = 2 \lim_{b \rightarrow \infty} \sqrt{u} \Big|_{\ln 2}^{\ln b} = 2 \lim_{b \rightarrow \infty} (\sqrt{\ln b} - \sqrt{\ln 2}) = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges* as well.

23. The series is $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty k e^{-k^2}$. The function $f(x) = x e^{-x^2}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$ since

$$f'(x) = 1 \cdot e^{-x^2} + x e^{-x^2} (-2x) = e^{-x^2} (1 - 2x^2) < 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$I = \int_1^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x^2} dx$. Let $u = e^{-x^2}$. Then $du = -2x e^{-x^2} dx$. We have

$$I = \int_1^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_{e^{-1}}^{e^{-b^2}} -\frac{1}{2} du = -\frac{1}{2} \lim_{b \rightarrow \infty} u \Big|_{e^{-1}}^{e^{-b^2}} = -\frac{1}{2} \lim_{b \rightarrow \infty} [e^{-b^2} - e^{-1}] = -\frac{1}{2} [0 - e^{-1}] = \frac{e^{-1}}{2}.$$

Since the improper integral I converges, by the integral test the series *converges* as well.

24. The series is $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty k e^{-k}$. The function $f(x) = x e^{-x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since

$$f'(x) = 1 \cdot e^{-x} + x(-e^{-x}) = e^{-x}(1 - x) \leq 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$I = \int_1^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x} dx$. Let $u = x$, $dv = e^{-x} dx$. Then, $du = dx$, $v = -e^{-x}$. Integrating by parts, we have

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left(-x e^{-x} \Big|_1^b + \int_1^b e^{-x} dx \right) = \lim_{b \rightarrow \infty} (-b e^{-b} + e^{-1}) + \lim_{b \rightarrow \infty} [-e^{-x}]_1^b \\ &= -\lim_{b \rightarrow \infty} \frac{b}{e^b} + \frac{1}{e} - \lim_{b \rightarrow \infty} (e^{-b} - e^{-1}) = -\lim_{b \rightarrow \infty} \frac{1}{e^b} + \frac{1}{e} - 0 + \frac{1}{e} \\ &= 0 + \frac{1}{e} - 0 + \frac{1}{e} = \frac{2}{e}, \end{aligned}$$

using L'Hôpital's rule on the first limit in the second line. Since the improper integral I converges, by the integral test the series *converges* as well.

25. The series is $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty \frac{1}{k^2+1}$. The function $f(x) = \frac{1}{x^2+1}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since $f'(x) = -\frac{1}{(x^2+1)^2} \cdot (2x) < 0$ for $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_1^b = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since the improper integral I converges, by the integral test the series *converges* as well.

26. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$. The function $f(x) = \frac{1}{x\sqrt{x^2-1}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{x^2-1}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{x^2-1} + x \cdot \frac{1}{2\sqrt{x^2-1}} \cdot (2x) = \frac{x^2-1+x^2}{\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$. Let $x = \sec u$. Then $dx = \sec u \tan u du$. We have

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \int_{\sec^{-1} 2}^{\sec^{-1} b} \frac{\sec u \tan u du}{\sec u \sqrt{\sec^2 u - 1}} = \lim_{b \rightarrow \infty} \int_{\sec^{-1} 2}^{\sec^{-1} b} \frac{\sec u \tan u du}{\sec u \tan u} = \lim_{b \rightarrow \infty} u \Big|_{\sec^{-1} 2}^{\sec^{-1} b} = \lim_{b \rightarrow \infty} (\sec^{-1} b - \sec^{-1} 2) \\ &= \frac{\pi}{2} - \sec^{-1} 2. \end{aligned}$$

Since the improper integral I converges, by the integral test the series *converges* as well.

27. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k \ln k}$. The function $f(x) = \frac{1}{x \ln x}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x \ln x$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} = \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges* as well.

28. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$. The function $f(x) = \frac{1}{x(\ln x)^3}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^3$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot (\ln x)^3 + x \cdot 3(\ln x)^2 \cdot \frac{1}{x} = (\ln x)^3 + 3(\ln x)^2 > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^3}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^3} = \lim_{b \rightarrow \infty} \left[\frac{u^{-3+1}}{-3+1} \right]_{\ln 2}^{\ln b} = -\frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2} \right] = \frac{1}{2} \frac{1}{(\ln 2)^2}.$$

Since the improper integral I converges, by the integral test the series *converges* as well.

29. The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p -series with $p = 2 > 1$. So it *converges*.

30. The series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is a p -series with $p = 4 > 1$. So it *converges*.

- 31.** The series $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p -series with $0 < p = \frac{1}{3} < 1$. So it *diverges.*
- 32.** The series $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is a p -series with $0 < p = \frac{2}{3} < 1$. So it *diverges.*
- 33.** The series $\sum_{k=1}^{\infty} \frac{1}{k^e}$ is a p -series with $p = e > 1$. So it *converges.*
- 34.** The series $\sum_{k=1}^{\infty} \frac{1}{k^\pi}$ is a p -series with $p = \pi > 1$. So it *converges.*
- 35.** $1 + \frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a p -series with $p = \frac{3}{2} > 1$. So it *converges.*
- 36.** $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p -series with $0 < p = \frac{1}{3} < 1$ and so *diverges.*
- 37.** $1 + \frac{1}{4\sqrt{2}} + \frac{1}{9\sqrt{3}} + \frac{1}{16\sqrt{4}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ is a p -series with $p = \frac{5}{2} > 1$ and so *converges.*
- 38.** $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p -series with $p = 3 > 1$ and so *converges.*
- 39.** The series is $\sum_{k=1}^{\infty} \frac{10}{k}$. Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=1}^{\infty} \frac{10}{k}$ *diverges* as well since it is a nonzero constant multiple of a divergent series.
- 40.** The series is $\sum_{k=1}^{\infty} \frac{2}{1+k}$. Since $\sum_{k=1}^{\infty} \frac{1}{1+k}$ is just the harmonic series $\sum_{k=2}^{\infty} \frac{1}{k}$, which diverges, the series $\sum_{k=1}^{\infty} \frac{2}{1+k}$ which is a nonzero constant multiple of a divergent series also *diverges*.
- 41.** The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2+1}{4k+1}$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2+1}{4n+1} = \lim_{n \rightarrow \infty} \frac{n+\frac{1}{n}}{4+\frac{1}{n}} = \infty \neq 0$, by the Divergence Test the series *diverges.*
- 42.** The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^3}{k^3+3}$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+3} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{3}{n^3}} = 1 \neq 0$, by the Divergence Test, the series *diverges.*
- 43.** The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (k + \frac{1}{k})$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (n + \frac{1}{n}) = \lim_{n \rightarrow \infty} n + 0 = \infty \neq 0$, by the Divergence Test, the series *diverges.*
- 44.** The series is $\sum_{k=1}^{\infty} (\frac{1}{3^k} - \frac{1}{4^k}) = \sum_{k=1}^{\infty} \frac{1}{3} (\frac{1}{3})^{k-1} - \sum_{k=1}^{\infty} \frac{1}{4} (\frac{1}{4})^{k-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1/3}{2/3} - \frac{1/4}{3/4} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Here, we used the fact that each term was a convergent geometric series, and the sum and difference property of convergent series to simplify the result. So, the series *converges.*
- 45.** The series is $\sum_{k=1}^{\infty} (\frac{1}{3k} - \frac{1}{4k}) = \sum_{k=1}^{\infty} \frac{1}{12k}$ is a constant multiple of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ which diverges. So the given series also *diverges.*

46. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(k - \frac{10}{k}\right)$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(n - \frac{10}{n}\right) = \lim_{n \rightarrow \infty} n - 0 = \infty \neq 0$. So by the Divergence Test the series *diverges*.

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2}k\right)$. Since $\sin\left(\frac{\pi}{2}k\right)$ is $+1$ for $k = 1, 5, 9, \dots$; 0 for $k = 2, 4, 6, \dots$; and -1 for $k = 3, 7, 11, \dots$, the partial sums $S_n = \sum_{k=1}^n a_k$ will oscillate between 0 and 1 for all n . The sequence of partial sums $\{S_n\} = \{1, 1, 0, 0, 1, 1, 0, 0, \dots\}$ does not converge to a limit, so the series will *diverge*.

48. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sec \pi k$. Since $\sec \pi k$ is -1 for $k = 1, 3, 5, \dots$, and $+1$ for $k = 2, 4, 6, \dots$, the partial sums $S_n = \sum_{k=1}^n a_k$ will oscillate between -1 and 0 . The series will *diverge* because the sequence of partial sums $\{S_n\} = \{-1, 0, -1, 0, \dots\}$ does not converge to a limit.

49. The series is $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{k+1}{k-2}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n-2} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{1-\frac{2}{n}} = \frac{1+0}{1-0} = 1 \neq 0$. So the series *diverges*.

50. The series is $\sum_{k=5}^{\infty} a_k = \sum_{k=5}^{\infty} \frac{2k^5+3}{k^5-4k^4}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n^5+3}{n^5-4n^4} = \lim_{n \rightarrow \infty} \frac{2+\frac{3}{n^5}}{1-\frac{4}{n}} = \frac{2+0}{1-0} = 2 \neq 0$. So the series *diverges*.

51. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^{1/2}}$. The function $f(x) = \frac{1}{x(\ln x)^{1/2}} = \frac{1}{x\sqrt{\ln x}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{\ln x}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{\ln x} + x \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x\sqrt{\ln x}}. \text{ Let } u = \ln x. \text{ Then } du = \frac{dx}{x}. \text{ We have}$$

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}} du = \lim_{b \rightarrow \infty} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{\ln 2}^{\ln b} = 2 \lim_{b \rightarrow \infty} \left(\sqrt{\ln b} - \sqrt{\ln 2} \right) = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges* as well.

52. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$. The function $f(x) = \frac{1}{x(\ln x)^2}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^2$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot (\ln x)^2 + x \cdot (2 \ln x) \cdot \frac{1}{x} = (\ln x)^2 + 2 \ln x > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find

$$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2}. \text{ Let } u = \ln x. \text{ Then } du = \frac{dx}{x}. \text{ We have}$$

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-2} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-1}}{-1} \right]_{\ln 2}^{\ln b} = - \lim_{b \rightarrow \infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2} \right) = - \left(0 - \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}.$$

Since the improper integral I converges, by the integral test the series *converges* as well.

53. The series is $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{2k}{k^2-4}$. The function $f(x) = \frac{2x}{x^2-4}$ is defined on $[3, \infty)$ and is continuous, positive, and decreasing for all $x \geq 3$, since

$$f'(x) = \frac{(x^2-4) \cdot 2 - 2x \cdot (2x)}{(x^2-4)^2} = -\frac{3x^2+8}{(x^2-4)^2} < 0 \text{ for } x \geq 3.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 3$. Using the integral test, we find

$I = \int_3^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_3^b \frac{2x dx}{x^2-4}$. Let $u = x^2 - 4$. Then $du = 2x dx$. We have

$$I = \lim_{b \rightarrow \infty} \int_5^{b^2-4} \frac{du}{u} = \lim_{b \rightarrow \infty} [\ln u]_5^{b^2-4} = \lim_{b \rightarrow \infty} (\ln(b^2-4) - \ln 5) = \infty.$$

Since the improper integral I diverges, by the integral test the series *diverges*.

54. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k)}$. The function $f(x) = \frac{1}{(2x-1)(2x)} = \frac{1}{2x-1} - \frac{1}{2x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$\begin{aligned} I &= \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \left(\int_1^b \frac{dx}{2x-1} - \frac{dx}{2x} \right) = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln |2x-1| - \frac{1}{2} \ln |x| \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln |2b-1| - \frac{1}{2} \ln |b| \right] = \frac{1}{2} \lim_{b \rightarrow \infty} \ln \left| \frac{2b-1}{b} \right| = \frac{1}{2} \lim_{b \rightarrow \infty} \ln \left| \frac{2-\frac{1}{b}}{1} \right| \\ &= \frac{1}{2} \ln \lim_{b \rightarrow \infty} \left| 2 - \frac{1}{b} \right| = \frac{1}{2} \ln 2. \end{aligned}$$

Since the improper integral I converges, by the integral test the series *converges* as well.

55. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$. The function $f(x) = \frac{1}{x(\ln x)^p}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To check whether $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^p$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot (\ln x)^p + x \cdot p(\ln x)^{p-1} \cdot \frac{1}{x} = (\ln x)^p + p(\ln x)^{p-1} > 0.$$

This final condition is satisfied for all $p > 0$ and for x in the interval $[2, \infty)$, so we conclude that the function $f(x)$ must be decreasing on $[2, \infty)$. Also we have $a_k = f(k)$ for all positive integers $k \geq 2$. To use the Integral Test evaluate $I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^p}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have, for $p \neq 1$,

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^p} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-p} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_{\ln 2}^{\ln b} = \frac{1}{1-p} \lim_{b \rightarrow \infty} \left(\frac{1}{(\ln b)^{p-1}} - \frac{1}{(\ln 2)^{p-1}} \right).$$

This improper integral converges if and only if $p > 1$, and diverges for $p < 1$, so the series converges, by the Integral Test, for $p > 1$, and diverges for $p < 1$.

Next, consider the case $p = 1$. We have

$$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} = \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series diverges for $p = 1$.

In conclusion, using the integral test, we have shown that the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges if and only if $p > 1$.

56. The series is $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$. The function $f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^p}$ is defined on $[3, \infty)$ and is continuous, positive, and decreasing for all $x \geq 3$. To check whether $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)[\ln(\ln x)]^p$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[3, \infty)$. We have

$$\begin{aligned} g'(x) &= \ln x [\ln(\ln x)]^p + x \cdot \frac{1}{x} [\ln(\ln x)]^p + x \ln x \cdot p [\ln(\ln x)]^{p-1} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \ln x [\ln(\ln x)]^p + [\ln(\ln x)]^p + p [\ln(\ln x)]^{p-1} \\ &= [\ln(\ln x)]^{p-1} [(\ln x + 1)(\ln(\ln x) + p)] \\ &> 0. \end{aligned}$$

This final condition is satisfied for all $p > 0$ and for x in the interval $[3, \infty)$, so we can conclude that the function $f(x)$ must be decreasing on $[3, \infty)$. Also we have $a_k = f(k)$ for all positive integers $k \geq 3$. Using the integral test, we find $I = \int_3^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_3^b \frac{dx}{x(\ln x)[\ln(\ln x)]^p}$. Let $u = \ln(\ln x)$. Then $du = \frac{dx}{x \ln x}$. We have for $p \neq 1$,

$$I = \lim_{b \rightarrow \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{du}{u^p} = \lim_{b \rightarrow \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} u^{-p} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_{\ln(\ln 3)}^{\ln(\ln b)} = \frac{1}{1-p} \lim_{b \rightarrow \infty} \left(\frac{1}{[\ln(\ln b)]^{p-1}} - \frac{1}{\ln[(\ln 3)]^{p-1}} \right).$$

This improper integral converges if and only if $p > 1$, and diverges if $p < 1$ so the series converges, by the Integral Test, for $p > 1$ and diverges for $p < 1$.

Next, consider the case $p = 1$. We have

$$I = \lim_{b \rightarrow \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{du}{u} = \lim_{b \rightarrow \infty} [\ln u]_{\ln(\ln 3)}^{\ln(\ln b)} = \lim_{b \rightarrow \infty} [\ln(\ln(\ln b)) - \ln(\ln(\ln 3))] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series diverges for $p = 1$. In conclusion, using the Integral Test, we have shown that the series $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$ converges if and only if $p > 1$.

57. $S = 1 + 2 + 4 + 8 + \cdots$ is a divergent geometric series, because the sequence of partial sums $S_n = \sum_{k=1}^n 2^{k-1}$ diverges since $|r| = 2 > 1$. Multiplying the series S by a constant nonzero multiple such as 2 again gives a divergent series. The equation $2S = -1 + S$ can be solved for S by the usual operations of arithmetic only if S and $2S$ are finite, which they are not. (Note that since both S and $2S$ are infinite, the equation $2S = -1 + S$ is satisfied identically, that is in the sense of $\infty = \infty$.) This is the reason for the paradox.

58. Let $a_k = 1$ and $b_k = -1$ for all k . Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge, but $\sum_{k=1}^{\infty} (a_k + b_k)$ converges. Let $a_k = b_k = 1$ for all k . Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge, but $\sum_{k=1}^{\infty} (a_k - b_k)$ converges.

For another example, let $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k+1}$ for all k . Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ being harmonic series diverge, but $\sum_{k=1}^{\infty} (a_k - b_k)$ will be a telescoping series and converge. Let $a_k = \frac{1}{k}$ and $b_k = -\frac{1}{k+1}$ for all k .

Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ being harmonic series diverge, but $\sum_{k=1}^{\infty} (a_k + b_k)$ will be a telescoping series and converge.

59. Using a CAS, we find that

(a) $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$

(b) We have $\pi^2 \approx 6 \sum_{k=1}^{100} \frac{1}{k^2} = 6(1.63498 \dots) \approx \boxed{9.8099}.$

60. (a) Since $\frac{1}{p-1} < \sum_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$, an interval of width 1 for the case of $p = 3$ is

$$\left(\frac{1}{p-1}, 1 + \frac{1}{p-1}\right) = \left(\frac{1}{3-1}, 1 + \frac{1}{3-1}\right) = \left(\frac{1}{2}, \frac{3}{2}\right).$$

(b) Using a CAS, $\sum_{k=1}^{100} \frac{1}{k^3} \approx \boxed{1.2020}.$

61. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k^6 e^{-k}$. The function $f(x) = x^6 e^{-x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 6$, since

$$f'(x) = 6x^5 e^{-x} - x^6 e^{-x} = x^5(6 - x)e^{-x} \leq 0 \text{ for } x \geq 6.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 6$. Using the integral test, we find

$$I = \int_6^{\infty} f(x) dx = \int_6^{\infty} x^6 e^{-x} dx = \frac{176112}{e},$$

using a CAS. Since the improper integral I converges, by the integral test the series $\sum_{k=6}^{\infty} a_k = \sum_{k=6}^{\infty} k^6 e^{-k}$

converges as well. The original series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k^6 e^{-k}$ differs from the convergent series

$\sum_{k=6}^{\infty} a_k = \sum_{k=6}^{\infty} k^6 e^{-k}$ by the five terms: $\sum_{k=1}^5 a_k = \sum_{k=1}^5 k^6 e^{-k} \approx 225.625$ using a CAS, so since the two series differ by a finite value, it means that the original series *converges* as well.

62. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+3}{k^2+6k+7}$. The function $f(x) = \frac{x+3}{x^2+6x+7}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since

$$f'(x) = \frac{(x^2 + 6x + 7) \cdot 1 - (x + 3) \cdot (2x + 6)}{(x^2 + 6x + 7)^2} = -\frac{x^2 + 6x + 11}{(x^2 + 6x + 7)^2} < 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x+3}{x^2+6x+7} dx = \frac{1}{2} \lim_{b \rightarrow \infty} (\ln(b^2 + 6b + 7) - \ln 14) = \infty,$$

using a CAS. Since the improper integral I diverges, by the integral test the series *diverges* as well.

63. The series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{5k+6}{k^3-1}$. The function $f(x) = \frac{5x+6}{x^3-1}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$, since

$$f'(x) = \frac{(x^3 - 1) \cdot 5 - (5x + 6) \cdot (3x^2)}{(x^3 - 1)^2} = -\frac{10x^3 + 18x^2 + 5}{(x^3 - 1)^2} < 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find using a CAS,

$$I = \int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{5x+6}{x^3-1} dx = \frac{1}{6} \left(-\sqrt{3}\pi + 11 \ln 7 + 2\sqrt{3} \tan^{-1} \left(\frac{5}{\sqrt{3}} \right) \right) \approx 3.375.$$

Since the improper integral I converges, by the integral test the series *converges* as well.

64. (a) For the series $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$: $S_{10} \approx 2.9561$ $S_{1000} \approx 7.7290$ $S_{100,000} \approx 12.7783$.

For the series $\sum_{k=1}^{\infty} \frac{1}{k^{1.01}}$: $S_{10} \approx 2.9023$ $S_{1000} \approx 7.2530$ $S_{100,000} \approx 11.4529$.

(b) The partial sums of the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^{1.01}}$ grow slower compared to the partial sums of the divergent series $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$, as one might have expected.

65. Since $\sum_{k=1}^{\infty} a_k$ is convergent, let the sequence of partial sums $\{S_n\}$ for $\sum_{k=1}^{\infty} a_k$ converge to L , that is, $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = L$. Similarly, since $\sum_{k=1}^{\infty} b_k$ is convergent, let the sequence of partial sums $\{S'_n\}$ for $\sum_{k=1}^{\infty} b_k$ converge to M , which means $\lim_{n \rightarrow \infty} S'_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n b_k = M$. The sequence of partial sums of $\sum_{k=1}^{\infty} (a_k + b_k)$ is $\{S_n + S'_n\}$, and we have

$$\lim_{n \rightarrow \infty} (S_n + S'_n) = \lim_{n \rightarrow \infty} S_n + \lim_{n \rightarrow \infty} S'_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k + \lim_{n \rightarrow \infty} \sum_{k=1}^n b_k = L + M,$$

using the additive property of limits, showing that the series $\sum_{k=1}^{\infty} (a_k + b_k)$ converges.

66. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series which converges to S ; this means that the sequence $\{S_n\}$ of partial sums $S_n = \sum_{k=1}^n a_k$ converges to S , or $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = S$. Consider the series $\sum_{k=1}^{\infty} ca_k$. The sequence of partial sums $\{S'_n\}$ of this series is $S'_n = \sum_{k=1}^n ca_k$. We have

$$\lim_{n \rightarrow \infty} S'_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n ca_k = c \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = cS,$$

using the scalar multiple property of limits. Since the sequence of partial sums $\{S'_n\}$ converges to cS , this means that the sum of the series $\sum_{k=1}^{\infty} ca_k = cS$.

67. We have $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^N a_k + \sum_{k=N+1}^{\infty} a_k = K + S = S + K$. This proves that $\sum_{k=1}^{\infty} a_k$ converges with sum $S + K$.

68. Let the series $\sum_{k=1}^{\infty} a_k$ converge to a sum L . This means the limit of the sequence of partial sums $\{S_n\}$ is L , that is $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = L$. If the series $\sum_{k=1}^{\infty} b_k$ diverges, then the limit of the sequence of partial

sums $\{S'_n\}$ is either nonexistent or is infinite, in other words, $\lim_{n \rightarrow \infty} S'_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n b_k$ does not exist or is infinite. The sequence of partial sums $\{S''_n\}$ of the series $\sum_{k=1}^{\infty} (a_k + b_k)$ is given by

$$S''_n = \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k = S_n + S'_n.$$

Taking the limits on both sides, we have

$$\lim_{n \rightarrow \infty} S''_n = \lim_{n \rightarrow \infty} S_n + \lim_{n \rightarrow \infty} S'_n = L + \lim_{n \rightarrow \infty} S'_n,$$

which does not exist or is infinite because $\lim_{n \rightarrow \infty} S'_n$ does not exist or is infinite. This shows that the sequence of partial sums $\{S''_n\}$ of the series $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges, so the series itself diverges.

69. Since the series $\sum_{k=1}^{\infty} a_k$ with $a_k > 0$ for all k converges, the limit of the sequence $\{S_n\}$ of partial sums with n^{th} term given by $S_n = \sum_{k=1}^n a_k$ is finite, say L . That is, $L = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$. The n^{th} term of the sequence $\{S'_n\}$ of partial sums for the series $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ is given by $S'_n = \sum_{k=1}^n \frac{a_k}{1+a_k}$. Since the terms $a_k > 0$ for all k , we have

$$0 < S'_n = \sum_{k=1}^n \frac{a_k}{a_k + 1} < \sum_{k=1}^n a_k < L,$$

which means that the partial sum is bounded from below by 0 and above by L . Also, the terms of the sequence of partial sums is an increasing sequence which can be seen using the Algebraic Difference Test:

$$S'_{n+1} - S'_n = \sum_{k=1}^{n+1} \frac{a_k}{a_k + 1} - \sum_{k=1}^n \frac{a_k}{a_k + 1} = \frac{a_{n+1}}{a_{n+1} + 1} > 0,$$

since $a_{n+1} > 0$ for all n . Since the sequence $\{S'_n\}$ of partial sums is bounded from above and increasing, it converges.

Challenge Problems

70. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k \ln(1 + \frac{1}{k})}$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{1}{n \ln(1 + \frac{1}{n})} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\ln(1 + \frac{1}{n})} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\ln(1 + \frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2}}{\frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) = 1 \neq 0, \end{aligned}$$

using L'Hôpital's rule on a related function of the sequence $\{a_k\}$. So the series *diverges*.

71. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{(\ln k)^p}$. The function $f(x) = \frac{1}{(\ln x)^p}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. Also, $a_k = f(k)$ for all positive integers k . Using the integral test, we find $I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{(\ln x)^p}$. Let $u = \frac{1}{(\ln x)^p}$, $dv = dx$. Then, $du = -p(\ln x)^{-p-1} \cdot \frac{1}{x} dx$, and $v = x$. Integrating by parts, we have

$$I = \lim_{b \rightarrow \infty} \left(\left[\frac{x}{(\ln x)^p} \right]_2^b + \int_2^b \frac{dx}{(\ln x)^{p+1}} \right).$$

The first term yields

$$\begin{aligned} \lim_{b \rightarrow \infty} \frac{b}{(\ln b)^p} - \frac{2}{(\ln 2)^p} &= \lim_{b \rightarrow \infty} \frac{1}{p(\ln b)^{p-1} \cdot \frac{1}{b}} - \frac{2}{(\ln 2)^p} = \lim_{b \rightarrow \infty} \frac{b}{p(\ln b)^{p-1}} - \frac{2}{(\ln 2)^p} \\ &= \cdots = \lim_{b \rightarrow \infty} \frac{b}{p! \ln b} - \frac{2}{(\ln 2)^p} = \frac{1}{p!} \lim_{b \rightarrow \infty} \frac{1}{\frac{1}{b}} - \frac{2}{(\ln 2)^p} = \infty, \end{aligned}$$

via a repeated application of L'Hôpital's rule to the first term. This means that the improper integral I diverges, so by the integral test the series *diverges* as well.

72. The series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{(\ln k)^q}{x^p}$. We assume in what follows that $p > 0$ and $q > 0$ are integers. The function $f(x) = \frac{(\ln x)^q}{x^p}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. Also, $a_k = f(k)$ for all positive integers k . To use the integral test, we evaluate

$$I(p, q) = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b x^{-p} (\ln x)^q dx.$$

Let $u = (\ln x)^q$, $dv = x^{-p} dx$. Then $du = \frac{(\ln x)^{q-1}}{x} dx$, $v = \frac{x^{-p+1}}{-p+1}$, assuming $p \neq 1$ (we will consider the case $p = 1$ separately below). Integrating by parts, we have

$$\begin{aligned} I(p, q) &= \lim_{b \rightarrow \infty} \left[\frac{(\ln x)^q \cdot x^{-p+1}}{-p+1} \Big|_2^b - \int_2^b \frac{x^{-p+1}}{-p+1} \cdot \frac{(\ln x)^{q-1}}{x} dx \right] \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left[(\ln x)^q x^{-p+1} \Big|_2^b - \int_2^b \frac{(\ln x)^{q-1}}{x^p} dx \right] \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^q}{b^{p-1}} - \frac{(\ln 2)^q}{2^{p-1}} - I(p, q-1) \right] \end{aligned}$$

The first term limit is of an indeterminate form, and can be found by repeated application of L'Hôpital's rule,

$$\lim_{b \rightarrow \infty} \frac{(\ln b)^q}{b^{p-1}} = \lim_{b \rightarrow \infty} \frac{q(\ln b)^{q-1} \cdot \frac{1}{b}}{(p-1)b^{p-2}} = \lim_{b \rightarrow \infty} \frac{q(\ln b)^{q-1}}{(p-1)b^{p-1}} = \cdots = \lim_{b \rightarrow \infty} \frac{q!}{(p-1)!b^{p-1}}.$$

This limit exists if $p > 1$. The other term obtained while integrating $I(p, q)$ by parts, namely $I(p, q-1)$ can again be integrated by parts, and will produce similar limits that are defined only if $p > 1$ —to see this, just replace q by $q-k$ everywhere in the above calculation, to see that

$$I(p, q-k) = \frac{1}{1-p} \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^{q-k}}{b^{p-1}} - \frac{(\ln 2)^{q-k}}{2^{p-1}} - I(p, q-k-1) \right],$$

where $k = 0, 1, 2, \dots, q-1$. In the final step, for $k = q$, we would obtain

$$I(p, 0) = \int_2^b \frac{dx}{x^p} = \frac{1}{1-p} \frac{1}{b^{p-1}},$$

and once again, this term has convergent limit as $b \rightarrow \infty$ only if $p > 1$. We have shown that the series converges for any $q > 0$ if $p > 1$.

If $p = 1$, then we have $dv = \frac{dx}{x}$ so $v = \ln x$. Proceeding with the integration of parts as before, we have

$$\begin{aligned}
 I(p = 1, q) &= \lim_{b \rightarrow \infty} \left[(\ln x)^q \cdot (\ln x)^1 \Big|_2^b - \int_2^b (\ln x)^1 \cdot \frac{(\ln x)^{q-1}}{x} dx \right] \\
 I(1, q) &= \lim_{b \rightarrow \infty} \left[(\ln x)^{q+1} \Big|_2^b \right] - \lim_{b \rightarrow \infty} \int_2^b \frac{(\ln x)^q}{x} dx \\
 &= \lim_{b \rightarrow \infty} [(\ln b)^{q+1} - (\ln 2)^{q+1}] - I(1, q) \\
 \text{or,} \quad 2I(1, q) &= \lim_{b \rightarrow \infty} [(\ln b)^{q+1} - (\ln 2)^{q+1}] \\
 I(1, q) &= \frac{1}{2} \lim_{b \rightarrow \infty} [(\ln b)^{q+1} - (\ln 2)^{q+1}],
 \end{aligned}$$

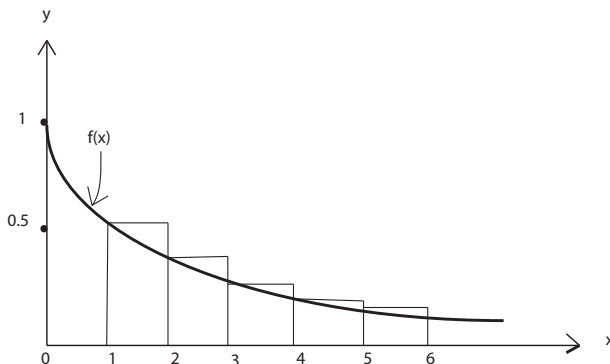
which exists only if $q + 1 \leq 0$, or $q \leq -1$. Since we assumed that $q > 0$, we conclude that the series does not converge for $p = 1$. So we have shown using the Integral Test that the original series converges for the integers $p > 1$ for any $q > 0$.

73. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k^x$. The function $f(y) = y^x$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $y \geq 1$, provided $x < 0$, since $f'(x) = xy^{x-1} < 0$ if $x < 0$ for $y \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the Integral Test, we find

$$I = \int_1^{\infty} f(y) dy = \lim_{b \rightarrow \infty} \int_1^b y^x dy = \lim_{b \rightarrow \infty} \frac{y^{x+1}}{x+1} \Big|_1^b = \lim_{b \rightarrow \infty} \frac{1}{(x+1)b^{-1-x}} - \frac{1}{(x+1)} = -\frac{1}{x+1}$$

that is, the limit exists and is finite only if $-1 - x > 0$, or $x < -1$. So for $x \leq 0$ the range of x values that lead to a convergent series is $(-\infty, -1)$. For $x > 0$, the function $f(y) = y^x$ is an increasing function of x , and the Integral Test is not applicable. But for this case, we have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^x = \infty \neq 0$, so the series diverges by the Divergence Test.

74. (a) Consider the finite sum $S_n = \sum_{k=1}^n \frac{1}{1+k^2}$. The function $f(x) = \frac{1}{1+x^2}$ is defined on $[0, \infty)$ and is continuous, positive, and decreasing for all $x \geq 0$. Also, $a_k = f(k)$ for all positive integers k . So we have $\int_1^n \frac{1}{1+x^2} dx \leq \sum_{k=1}^n \frac{1}{1+k^2} \leq \int_0^n \frac{dx}{1+x^2}$ or $\tan^{-1} n - \frac{\pi}{4} \leq S_n \leq \tan^{-1} n$. So, $S_n \leq \tan^{-1} n$. See figure below for the case $n = 6$:



(b) Since $S_n < \tan^{-1} n < \frac{\pi}{2}$, the sequence of partial sums $\{S_n\}$ is bounded from above. The terms of the sum S_n are positive, so the sequence $\{S_n\}$ is also increasing, as seen by the Algebraic Difference Test:

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} \frac{1}{1+k^2} - \sum_{k=1}^n \frac{1}{1+k^2} = \frac{1}{1+(n+1)^2} > 0.$$

Since the sequence of partial sums $\{S_n\}$ is bounded from above and is increasing, it converges to a finite limit, which means that the series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ also converges.

(c) From part (a), we have since $\tan^{-1} n - \frac{\pi}{4} \leq S_n \leq \tan^{-1} n$. Taking limits on every term of the inequality, we get $\lim_{n \rightarrow \infty} (\tan^{-1} n - \frac{\pi}{4}) \leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{1+k^2} \right) \leq \lim_{n \rightarrow \infty} (\tan^{-1} n)$. So, $\frac{\pi}{2} - \frac{\pi}{4} \leq \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leq \frac{\pi}{2}$, or $\frac{\pi}{4} \leq \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leq \frac{\pi}{2}$, as was required to be proved.

75. (a) $\sum_{k=1}^{10} \frac{1}{k^3} \approx \boxed{1.1975}.$

(b) We have since $\frac{1}{p-1} < \sum_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$, for the case of $p = 3$, $\frac{1}{3-1} < \sum_{k=1}^{\infty} \frac{1}{k^3} < 1 + \frac{1}{3-1}$, or

$$\boxed{\frac{1}{2} < \sum_{k=1}^{\infty} \frac{1}{k^3} < \frac{3}{2}}.$$

(c) We conclude that $\zeta(3)$ converges to a value between $\frac{1}{2}$ and $\frac{3}{2}$.

76. (a) For the rectangles under the curve $y = f(x)$, as shown in the left figure of the problem, the height of the tallest rectangle of unit width is $f(n+1)$, while the height of the smallest rectangle of unit width is $f(m)$. The area under the curve is larger than the areas of the rectangles of unit width under the curve, so we have $f(n+1) + \cdots + f(m) \leq \int_n^m f(x) dx$.

For the rectangles over the curve $y = f(x)$, as shown in the right figure of the problem, the height of the tallest rectangle of unit width is $f(n)$ while the height of the smallest rectangle of unit width is $f(m-1)$. The area under the curve is now smaller than the areas of all the rectangles of unit width over the curve, so we have $\int_n^m f(x) dx \leq f(n) + \cdots + f(m-1)$. Putting these two geometric results together gives us the desired inequalities,

$$f(n+1) + \cdots + f(m) \leq \int_n^m f(x) dx \leq f(n) + \cdots + f(m-1).$$

(b) We have, since $\sum_{k=1}^{\infty} f(k)$ converges, then $\sum_{k=n+1}^{\infty} f(k)$ and $\sum_{k=n}^{\infty} f(k)$ also converge, since either of these series differs from the convergent series $\sum_{k=1}^{\infty} f(k)$ only by a finite sum of positive terms. Since from part (a), we had $f(n+1) + \cdots + f(m) \leq \int_n^m f(x) dx \leq f(n) + \cdots + f(m-1)$, taking the limits on every term of the inequality as $m \rightarrow \infty$ gives us

$$\lim_{m \rightarrow \infty} \sum_{k=n+1}^m f(k) \leq \lim_{m \rightarrow \infty} \int_n^m f(x) dx \leq \lim_{m \rightarrow \infty} \sum_{k=n}^m f(k)$$

or $\sum_{k=n+1}^{\infty} f(k) \leq \int_n^{\infty} f(x) dx \leq \sum_{k=n+1}^{\infty} f(k)$, as needed to be shown, since the improper integral exists by the Squeeze Theorem.

(c) From the left hand inequality of part (b), we have $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \int_n^{\infty} \frac{1}{x^2} dx$, or

$$\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^n \frac{1}{k^2} \leq \left[-\frac{1}{x} \right]_n^{\infty}.$$

Now, we use $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (see Problem 75). The Error, which refers to the difference between the

full value of the convergent series and its n^{th} partial sum, is given by

$$\text{Error} = \left| \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2} \right| \leq \frac{1}{n}.$$

So for an error $\frac{1}{n} < \left(\frac{1}{2}\right) 10^{-2}$, we have $n > 200$. For an error $\frac{1}{n} < \left(\frac{1}{2}\right) 10^{-10}$, we have $n > 2 \times 10^{10}$.

8.4 Comparison Tests

Concepts and Vocabulary

1. (b) The series $\sum_{k=1}^{\infty} a_k$ is divergent by the Comparison Test.
2. **False:** We require $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0$ in order to be able to conclude that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both convergent or both divergent.
3. **False:** If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$, then all that can be concluded is either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or they both diverge.
4. **False:** The limit comparison test merely decides if the unknown series converges or diverges by comparing it with a known convergent or divergent series. It does not tell us what the sum of the series is.

Skill Building

5. Comparing the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ with the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ we have

$$\frac{1}{n(n+1)} = \frac{1}{n^2 + n} < \frac{1}{n^2}$$

whenever $n \geq 1$. Since the p -series $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges for $p = 2 > 1$, by the Comparison Test,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \text{ also } \boxed{\text{converges.}}$$

6. Comparing the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$ with the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ we have

$$\frac{1}{(n+1)^2} = \frac{1}{n^2 + 2n + 1} < \frac{1}{n^2}$$

for $n \geq 1$. By the Comparison Test, since the p -series $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges for $p = 2 > 1$, the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \text{ also } \boxed{\text{converges.}}$$

7. Comparing the n^{th} term of $\sum_{k=2}^{\infty} \frac{4^k}{7^{k+1}}$ with the n^{th} term of $\sum_{k=2}^{\infty} \left(\frac{4}{7}\right)^k$, we have

$$\frac{4^n}{7^{n+1}} < \frac{4^n}{7^n}$$

for $n \geq 1$. Since the geometric series $\sum_{k=2}^{\infty} \left(\frac{4}{7}\right)^k$ converges since $|r| = \frac{4}{7} < 1$, by the Comparison Test, the

$$\text{series } \sum_{k=2}^{\infty} \frac{4^k}{7^{k+1}} \text{ also } \boxed{\text{converges.}}$$

8. Comparing the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2^k)}$ with the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{2^k}$ we wish to have

$$\frac{1}{(2n-1)(2^n)} < \frac{1}{(2^n)}$$

or,

$$\frac{1}{2n-1} < 1$$

or,

$$2n-1 > 1,$$

or $n > 1$. So for $n > 1$, by the Comparison test, since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is a convergent geometric series, since

$|r| = \frac{1}{2} < 1$, the series $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2^k)}$ also *converges.*

9. Comparing the n^{th} term of $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ with the n^{th} term of $\sum_{k=2}^{\infty} \frac{1}{k}$ we have

$$\frac{1}{\sqrt{n(n-1)}} = \frac{1}{n\sqrt{1-\frac{1}{n}}} > \frac{1}{n}$$

for $n > 1$. Since $\sum_{k=2}^{\infty} \frac{1}{k}$ is the divergent harmonic series, by the Comparison Test, the series $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ also *diverges.*

10. Comparing the n^{th} term of $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{k-1}$ with the n^{th} term of $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$, we wish to have

$$\frac{\sqrt{n}}{n-1} > \frac{1}{\sqrt{n}}$$

or, cross-multiplying,

$$n > n-1,$$

which is true for all $n \geq 1$. Since the p -series $\sum_{k=2}^{\infty} \frac{1}{k^p} = \sum_{k=2}^{\infty} \frac{1}{k^{1/2}}$ diverges since $0 < p < 1$, by the Comparison Test, the series $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{k-1}$ also *diverges.*

11. Comparing the n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$ with the n^{th} term of the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ (convergent since $p = 3 > 1$) we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{n^3}$$

for $n \geq 1$. By the Comparison Test, the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$ also *converges.*

12. We compare the n^{th} term of $\sum_{k=1}^{\infty} \frac{6}{5k-2}$ with the n^{th} term of the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Since

$$\frac{6}{5n-2} > \frac{1}{n}$$

or

$$6n > 5n-2,$$

for $n > -2$, by the Comparison Test, the series $\sum_{k=1}^{\infty} \frac{6}{5k-2}$ also *diverges.*

13. We compare $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^\pi}$ with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^\pi}$, which converges since $p = \pi > 1$. We have

$$\frac{\sin^2 n}{n^\pi} \leq \frac{1}{n^\pi}$$

since $\sin^2 n \leq 1$ for all $n \geq 1$, so by the Comparison Test, the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^\pi}$ also *converges.*

14. We compare $\sum_{k=1}^{\infty} \frac{\cos^2 k}{k^2+1}$ with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

since $\cos^2 n \leq 1$ for all $n \geq 1$, so by the Comparison Test, the series $\sum_{k=1}^{\infty} \frac{\cos^2 k}{k^2+1}$ also *converges.*

15. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ behaves like

$$a_n = \frac{1}{(n+1)(n+2)} = \frac{1}{n^2 \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)} \approx \frac{1}{n^2}$$

for large n . So we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p -series since $p = 2 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)(n+2)} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 3n + 2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{3}{n} + \frac{2}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ also *converges.*

16. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ behaves like

$$a_n = \frac{1}{n^2+1} = \frac{1}{n^2 \left(1 + \frac{1}{n^2}\right)} \approx \frac{1}{n^2}$$

for large n . So we compare with the convergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ converges, then by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ also *converges.*

17. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ behaves like

$$a_n = \frac{1}{\sqrt{n^2+1}} = \frac{1}{n\sqrt{1 + \frac{1}{n^2}}} \approx \frac{1}{n}$$

for large n . So we compare with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ also *diverges.*

18. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+4}$ behaves like

$$a_n = \frac{\sqrt{n}}{n+4} = \frac{\sqrt{n}}{n(1+\frac{4}{n})} = \frac{1}{\sqrt{n}(1+\frac{4}{n})} \approx \frac{1}{\sqrt{n}}$$

for large n . So we compare with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which is divergent since $0 < p < 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{n+4}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n}{n+4} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{4}{n}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+4}$ also *diverges.*

19. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k^2+5}$ behaves like

$$a_n = \frac{3\sqrt{n}+2}{2n^2+5} = \frac{\sqrt{n}(3+\frac{2}{\sqrt{n}})}{n^2(2+\frac{5}{n^2})} \approx \frac{1}{n^{3/2}}$$

for large n . So we compare with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ which is convergent since $p = \frac{3}{2} > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{3\sqrt{n}+2}{2n^2+5}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{n\sqrt{n}(3\sqrt{n}+2)}{2n^2+5} = \lim_{n \rightarrow \infty} \frac{3n^2+2n\sqrt{n}}{2n^2+5} = \lim_{n \rightarrow \infty} \frac{3+\frac{2}{\sqrt{n}}}{2+\frac{5}{n^2}} = \frac{3}{2}.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k^2+5}$ also *converges.*

20. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k-3}$ behaves like

$$a_n = \frac{3\sqrt{n}+2}{2n-3} = \frac{\sqrt{n}(3+\frac{2}{\sqrt{n}})}{n(2-\frac{3}{n})} \approx \frac{1}{\sqrt{n}}$$

for large n . So we compare with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which is divergent since $0 < p = \frac{1}{2} < 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{3\sqrt{n}+2}{2n-3}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{3n+2\sqrt{n}}{2n-3} = \lim_{n \rightarrow \infty} \frac{3+\frac{2}{\sqrt{n}}}{2-\frac{3}{n}} = \frac{3}{2}.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k-3} \text{ also } \boxed{\text{diverges.}}$$

21. The n^{th} term of the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$ behaves like

$$a_n = \frac{1}{n\sqrt{n^2-1}} = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}} \approx \frac{1}{n^2}$$

for large n . So we compare with the p -series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the

$$\text{series } \sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}} \text{ also } \boxed{\text{converges.}}$$

22. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{(2k-1)^2}$ behaves like

$$a_n = \frac{n}{(2n-1)^2} = \frac{n}{n^2\left(2-\frac{1}{n}\right)^2} \approx \frac{1}{n}$$

for large n . So we compare with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n}{(2n-1)^2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{(2n-1)^2} = \lim_{n \rightarrow \infty} \frac{1}{\left(2-\frac{1}{n}\right)^2} = \frac{1}{4}.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{(2k-1)^2} \text{ also } \boxed{\text{diverges.}}$$

23. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k+4}{k2^k}$ behaves like

$$a_n = \frac{3n+4}{n2^n} = \left(3+\frac{4}{n}\right) \frac{1}{2^n} \approx \frac{1}{2^n}$$

for large n . So we compare with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{3n+4}{n2^n}}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \left(3+\frac{4}{n}\right) = 3.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k+4}{k2^k}$ also *converges.*

24. The n^{th} term of the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{k-1}{k2^k}$ behaves like

$$a_n = \frac{n-1}{n2^n} = \frac{1}{2^n} \left(1 - \frac{1}{n}\right) \approx \frac{1}{2^n}$$

for large n . So we compare with the convergent geometric series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{2^k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n-1}{n2^n}}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \frac{n-1}{n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{k-1}{k2^k}$ also *converges.*

25. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}$ behaves like $a_n = \frac{1}{2^{n+1}} \approx \frac{1}{2^n}$ for large n . So we compare with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^n}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{2^n}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}$ also *converges.*

26. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{3^{k+2}}$ behaves like $a_n = \frac{5}{3^{n+2}} \approx \frac{5}{3^n}$ for large n . So we compare with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^n}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{5}{3^{n+2}}}{\frac{1}{3^n}} = \lim_{n \rightarrow \infty} \frac{5 \cdot 3^n}{3^{n+2}} = \lim_{n \rightarrow \infty} \frac{5}{1 + \frac{2}{3^n}} = 5.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{3^{k+2}}$ also *converges.*

27. The n^{th} term of $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+5}{k^{k+1}}$ behaves like

$$a_n = \frac{n+5}{n^{n+1}} = \frac{n \left(1 + \frac{5}{n}\right)}{n^{n+1}} \approx \frac{1}{n^n}$$

for large n . So we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^k}$ which was shown (see Example 1, p. 576 of text) to be convergent. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+5}{n^{n+1}}}{\frac{1}{n^n}} = \lim_{n \rightarrow \infty} \frac{(n+5)n^n}{n^{n+1}} = \lim_{n \rightarrow \infty} \frac{n+5}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{5}{n}\right) = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+5}{k^{k+1}}$ also converges.

28. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{k^{k+1}}$ behaves like $a_n = \frac{5}{n^{n+1}} \approx \frac{5}{n^n}$ for large n . So we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^k}$ which was shown (see Example 1, p. 576 of text) to be convergent. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^{n+1}}}{\frac{1}{n^n}} = \lim_{n \rightarrow \infty} \frac{5n^n}{n^{n+1}} = \lim_{n \rightarrow \infty} \frac{5}{1 + \frac{1}{n^n}} = \frac{5}{1+0} = 5.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{k^{k+1}}$ also converges.

29. Comparing the n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{6k}{5k^2+2}$ with the n^{th} term of the harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ gives $\frac{6n}{5n^2+2} > \frac{1}{n}$ since $6n^2 > 5n^2 + 2$ for $n > \sqrt{2}$. This means for $n \geq 2$, the n^{th} terms satisfy $a_n > b_n$. Since the harmonic series $\sum_{k=1}^{\infty} b_k$ diverges, by the Comparison test, the given series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{6k}{5k^2+2}$ also diverges.

30. The n^{th} term of the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{6k+3}{2k^3-2}$ behaves like

$$a_n = \frac{6n+3}{2n^3-2} = \frac{n(6 + \frac{3}{n})}{n^3(2 - \frac{2}{n^3})} \approx \frac{1}{n^2}$$

for large n . So we compare with the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$ which is a convergent p -series (since $p = 2 > 1$). We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{6n+3}{2n^3-2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{6n^3+3n^2}{2n^3-2} = \lim_{n \rightarrow \infty} \frac{6 + \frac{3}{n}}{2 - \frac{2}{n}} = 3.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{6k+3}{2k^3-2}$ also converges.

31. The n^{th} term of the series $\sum_{k=1}^{\infty} \frac{7+k}{(1+k^2)^4}$ behaves like

$$a_n = \frac{7+n}{(1+n^2)^4} = \frac{n(\frac{7}{n}+1)}{n^8(\frac{1}{n^2}+1)^4} \approx \frac{1}{n^7}$$

for large n . So we compare with the series $\sum_{k=1}^{\infty} b_k = \frac{1}{k^7}$ which is a convergent p -series (since $p = 7 > 1$). We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{7+n}{(1+n^2)^4}}{\frac{1}{n^7}} = \lim_{n \rightarrow \infty} \frac{7n^7 + n^8}{(1+n^2)^4} = \lim_{n \rightarrow \infty} \frac{\frac{7}{n} + 1}{\left(\frac{1}{n^2} + 1\right)^4} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{7+k}{(1+k^2)^4}$ also converges.

32. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{7+k}{1+k^2}\right)^4$ behaves like

$$a_n = \left(\frac{7+n}{1+n^2}\right)^4 = \frac{n^4}{n^8} \left(\frac{\frac{7}{n} + 1}{\frac{1}{n^2} + 1}\right)^4 \approx \frac{1}{n^4}$$

for large n . So we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$ which is a convergent p -series, since $p = 4 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{7+n}{1+n^2}\right)^4}{\frac{1}{n^4}} = \lim_{n \rightarrow \infty} \left(\frac{7n+n^2}{1+n^2}\right)^4 = \lim_{n \rightarrow \infty} \left(\frac{\frac{7}{n} + 1}{\frac{1}{n^2} + 1}\right)^4 = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{7+k}{1+k^2}\right)^4$ also converges.

33. We compare the series $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k}$ with the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Comparing the n^{th} term of each series we have $\frac{e^{1/n}}{n} > \frac{1}{n}$ since $e^{1/n} > 1$ for any $n \geq 1$ (since the n^{th} root of a number bigger than 1 is also bigger than 1; here $e > 1$). By the Comparison Test, the original series also diverges.

34. We compare the series $\sum_{k=1}^{\infty} \frac{1}{1+e^k}$ with the convergent geometric series $\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$, which converges since $|r| = \frac{1}{e} < 1$. Comparing the n^{th} term of each series we have $\frac{1}{1+e^n} < \frac{1}{e^n}$ for all $n \geq 1$, so by the Comparison Test, the original series also converges.

35. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1+\frac{1}{k})^2}{e^k}$ behaves like $\frac{(1+\frac{1}{n})^2}{e^n} \approx \frac{1}{e^n}$ for large n . So we compare with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ which converges since $|r| = \frac{1}{e} < 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{(1+\frac{1}{n})^2}{e^n}}{\frac{1}{e^n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1+\frac{1}{k})^2}{e^k}$ also converges.

36. We compare the series $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ which converges since $|r| = \frac{1}{2} < 1$. Comparing the n^{th} term of each series we have $\frac{1}{n2^n} < \frac{1}{2^n}$ for $n > 1$. By the Comparison Test, we see that the original series also converges.

37. We compare the series $\sum_{k=1}^{\infty} \frac{1+\sqrt{k}}{k}$ with the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Since the n^{th} terms of the series satisfy $\frac{1+\sqrt{n}}{n} > \frac{1}{n}$ for $n \geq 1$, we see by the Comparison Test that the original series *diverges.*

38. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1+3\sqrt{k}}{k^2}$ behaves like

$$a_n = \frac{1+3\sqrt{n}}{n^2} = \frac{\sqrt{n} \left(\frac{1}{\sqrt{n}} + 3 \right)}{n^2} \approx \frac{1}{n^{3/2}}$$

for large n . So we compare with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, which converges since $p = \frac{3}{2} > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1+3\sqrt{n}}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{n^{3/2} + 3n^2}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}} + 3 \right) = 3.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1+3\sqrt{k}}{k^2}$ also *converges.*

39. We compare the series $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \sin^2 k$ with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ which converges since $|r| = \frac{1}{2} < 1$. Comparing the n^{th} term of each series we have $\left(\frac{1}{2}\right)^n \sin^2 n \leq \frac{1}{2^n}$ since $\sin^2 n \leq 1$ for all $n \geq 1$. By the Comparison test, we see that the original series also *converges.*

40. We compare the series $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^3}$ with the p -series $\sum_{k=1}^{\infty} \frac{\pi}{2k^3}$, convergent since $p = 3 > 1$. Since the n^{th} terms of the series satisfy $\frac{\tan^{-1} n}{n^3} < \frac{\pi}{2n^3}$ because $\tan^{-1} n < \frac{\pi}{2}$ for any finite and positive $n \geq 1$, we see that, by the Comparison Test, the original series also *converges.*

Applications and Extensions

41. We compare the series $\sum_{k=2}^{\infty} \frac{2}{k^3 \ln k}$ with the series $\sum_{k=2}^{\infty} \frac{2}{k^3}$, which is a convergent p -series since $p = 3 > 1$. Since the n^{th} terms of the series satisfy $\frac{2}{n^3 \ln n} < \frac{2}{n^3}$ since $\ln n > 1$ when $n \geq 3$ (recall $\ln e = 1$) we see that by the Comparison Test, the original series also *converges.*

42. (Using the result of Problem 51 below) we compare the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(\ln k)^4}$ with the divergent harmonic series $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{1}{k}$. Applying L'Hôpital's rule repeatedly ($\star$) and arranging the fractions after

each application, we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{a_n}{d_n} &= \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{\sqrt{n}(\ln n)^4}}{\frac{1}{n}} \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{(\ln n)^4} \\
 &\stackrel{*}{=} \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{2\sqrt{n}}}{\frac{4(\ln n)^3}{n}} \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{8(\ln n)^3} \\
 &\stackrel{*}{=} \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{2\sqrt{n}}}{\frac{24(\ln n)^2}{n}} \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{48(\ln n)^2} \\
 &\stackrel{*}{=} \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{2\sqrt{n}}}{\frac{96 \ln n}{n}} \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{192 \ln n} \\
 &\stackrel{*}{=} \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{2\sqrt{n}}}{\frac{192}{n}} \right] = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{384} \\
 &= \infty.
 \end{aligned}$$

So, by the result of Problem 51, since the series $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(\ln k)^4}$ also *diverges*.

43. We compare the series $\sum_{k=2}^{\infty} \frac{\ln k}{k+3}$ with the series $\sum_{k=2}^{\infty} \frac{1}{k+3}$, which is a divergent harmonic series. The n^{th} terms of the series satisfy $\frac{\ln n}{n+3} > \frac{1}{n+3}$ since $\ln n > 1$ if $n \geq 3$ (recall that $\ln e = 1$). By the Comparison test, we see that the original series also *diverges*.

44. We compare the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{(\ln k)^2}{k^{5/2}}$ with the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$, which is a convergent p -series since $p = \frac{3}{2} > 1$. The n^{th} term satisfies

$$a_n = \frac{(\ln n)^2}{n^{5/2}} < \frac{1}{n^{3/2}} = b_n$$

if $\ln n < n^{1/2}$. To find under what conditions $\ln n < n^{1/2}$, let $f(x) = \ln x - x^{1/2}$. We wish to find when this function is decreasing. Since $f'(x) = \frac{1}{x} - \frac{1}{2}x^{-1/2} < 0$ whenever $\sqrt{x} > 2$ or $x > 4$, and $f(4) = \ln 4 - \sqrt{2} < 0$, and we have shown that $f(x)$ is decreasing for $x > 4$, and is negative at $x = 4$, this proves that $f(x) < 0$ for $x > 4$. That is, $\ln n < n^{1/2}$ for $n \geq 4$. Note that the Comparison Test does not care about the relative behavior of series for any initial finite number of terms, but only about the *large* n behavior; due to this, by the Comparison Test, we see that the original series also *converges*.

45. We compare the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin \frac{1}{k}$ with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ which is a divergent harmonic series. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1,$$

where the substitution $y = \frac{1}{n}$ was made and the standard limit result $\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$ was used. Since the

limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ also *diverges*.

46. We compare the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \tan \frac{1}{k}$ with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ which is a divergent harmonic series. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim_{y \rightarrow 0} \frac{\tan y}{y} = \lim_{y \rightarrow 0} \frac{\sec^2 y}{1} = 1,$$

where the substitution $y = \frac{1}{n}$ was made and L'Hôpital's rule was used. Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \tan \frac{1}{k}$ also *diverges.*

47. Compare the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k!}$ with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$. Since

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \geq 2 \cdot 2 \cdot 2 \cdots 2 \quad (n \text{ factors of } 2),$$

we have $n! \geq 2^n$ or

$$a_n = \frac{1}{n!} \leq \frac{1}{2^n} = b_n$$

for all $n \geq 2$. So by the Comparison Test that the original series also *converges.*

48. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^k}$. We have

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \leq n \cdot n \cdot n \cdots n \cdot 2 \cdot 1 = 2n^{n-2} = 2 \frac{n^n}{n^2}$$

so that

$$a_n = \frac{n!}{n^n} \leq \frac{2}{n^2} = b_n.$$

Comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{2}{k^2}$, which is a constant multiple of a convergent p -series (since

$p = 2 > 1$), we conclude based on the Comparison Test that the original series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^k}$ *converges.*

49. (a) The n^{th} term of $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ satisfies $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$. So by the Comparison Test, since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (being a p -series for $p = 2 > 1$), the original series also *converges.*

(b) We have the n^{th} term of $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ satisfy $\frac{1}{n^2-1} > \frac{1}{n^2}$. This however provides no information about the convergence or divergence of the original series, since the n^{th} term of the original series is *greater* and not less than the n^{th} term of a convergent p -series.

(c) Let $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k^2-1}$ and $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2-1} = \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{n^2}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the

series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k^2-1}$ also *converges.* So, *yes*, the Limit Comparison Test can be used to decide the convergence of the series.

50. Compare the series $\sum_{k=1}^{\infty} \frac{d}{10^k}$ with the geometric series $\sum_{k=1}^{\infty} \frac{1}{10^{k-1}}$, which is convergent since $|r| = \frac{1}{10} < 1$. Since the n^{th} terms of the series satisfy $\frac{d_n}{10^n} < \frac{1}{10^{n-1}}$, since $d_n < 10$, by the Comparison Test, the original series also *converges.*

51. If $\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \infty$ it follows that we can choose an $M > 0$ such that there exists an $n > N$, such that for $n > N$, we have $\frac{a_n}{d_n} > M$, that is, $a_n > Md_n$. By the Comparison Test, since $\sum_{k=1}^{\infty} d_k$ diverges, so does $\sum_{k=1}^{\infty} a_k$.

52. If $\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = 0$ it follows that we can choose an $\epsilon > 0$ such that there exists $n > N$, such that for $n > N$, we have $\left| \frac{a_n}{d_n} - 0 \right| < \epsilon$, or $a_n < \epsilon d_n$. By the Comparison Test, since $\sum_{k=1}^{\infty} d_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

53. Let $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$ and $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{1}{k}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{x \rightarrow \infty} \frac{x}{\ln x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} x = \infty,$$

by using L'Hôpital's rule on a related function of the ratio of the n^{th} terms. So by the result of Problem 51, since the harmonic series $\sum_{k=2}^{\infty} d_k$ diverges, the original series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$ also *diverges.*

54. Let $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^2$ and $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{1}{k}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{d_n} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{\ln n}\right)^2}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{(\ln n)^2} = \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^2} \\ &= \lim_{x \rightarrow \infty} \frac{1}{2 \ln x \cdot \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{x}{2 \ln x} = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} = \frac{1}{2} \lim_{x \rightarrow \infty} x = \infty, \end{aligned}$$

by using L'Hôpital's rule on a related function of the ratio of the n^{th} terms. So by the result of Problem 51, since the harmonic series $\sum_{k=2}^{\infty} d_k$ diverges, the original series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^2$ also *diverges.*

55. Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ and $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = 2 \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0,$$

by using L'Hôpital's rule on a related function of the ratio of the n^{th} terms. So by the result of Problem 52, since the p -series $\sum_{k=2}^{\infty} d_k$ converges (since $p = \frac{3}{2} > 1$), the original series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ also *converges.*

56. Let $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$ and $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2(\ln n)^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{(\ln n)^2} = 0.$$

So by the result of Problem 52, since the p -series $\sum_{k=2}^{\infty} d_k$ converges (since $p = 2 > 1$), the original series

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2} \text{ also } \boxed{\text{converges.}}$$

57. If $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$ and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{e^k}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\ln n}}{\frac{1}{e^n}} = \lim_{n \rightarrow \infty} \frac{e^n}{\ln n} = \lim_{x \rightarrow \infty} \frac{e^x}{\ln x} = \lim_{x \rightarrow \infty} \frac{e^x}{\frac{1}{x}} = \lim_{x \rightarrow \infty} x e^x = \infty,$$

by using L'Hôpital's rule on a related function of the ratio of the n^{th} terms. Because the limit in question is not a finite positive number, but infinity, the Limit Comparison test is inconclusive in regards to whether the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$ converges or diverges, even though the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{e^k}$ converges, as it is a geometric series with $|r| = \frac{1}{e} < 1$.

58. (a) Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^p}$ and $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^q}$, where $1 < q < 2$ and $p > 1$. Then we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^p}}{\frac{1}{n^q}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{p-q}} = \lim_{x \rightarrow \infty} \frac{\ln x}{x^{p-q}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{(p-q)x^{p-q-1}} = \frac{1}{p-q} \lim_{x \rightarrow \infty} \frac{1}{x^{p-q}} = 0,$$

by L'Hôpital's rule, and since from $p > 1$, and $1 < q < 2$, we have $p - q > 0$, since $p > 1$ implies $p \geq 2 > q$. So by the result of Problem 52, since the p -series $\sum_{k=1}^{\infty} d_k$ converges (for $q > 1$), the original series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^p} \text{ also } \boxed{\text{converges.}}$$

(b) Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ and $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^q}$, where $1 < q < 2$, $p > 1$, and $r > 0$. Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{d_n} &= \lim_{n \rightarrow \infty} \frac{\frac{(\ln n)^r}{n^p}}{\frac{1}{n^q}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^r}{n^{p-q}} = \lim_{x \rightarrow \infty} \frac{(\ln x)^r}{x^{p-q}} \\ &= \lim_{x \rightarrow \infty} \frac{r(\ln x)^{r-1} \cdot \frac{1}{x}}{(p-q)x^{p-q-1}} = \frac{r}{p-q} \lim_{x \rightarrow \infty} \frac{(\ln x)^{r-1}}{x^{p-q}} = \dots \\ &= \frac{r!}{(p-q)^r} \lim_{x \rightarrow \infty} \frac{1}{x^{p-q}} = 0 \end{aligned}$$

by repeatedly applying L'Hôpital's rule, and since from $p > 1$, and $1 < q < 2$, we have $p - q > 0$, since $p > 1$ implies $p \geq 2 > q$. So by the result of Problem 52, since the p -series $\sum_{k=1}^{\infty} d_k$ converges (for $q > 1$), the

$$\text{original series } \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p} \text{ also } \boxed{\text{converges.}}$$

59. (a) Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^p}$ and $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$, where $0 < p \leq 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^p}}{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} \ln n = \infty.$$

By the results of exercise 51, since $\sum_{k=1}^{\infty} d_k$ is a divergent p -series (since $0 < p \leq 1$), it follows that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^p} \text{ also diverges, as was needed to be shown.}$$

(b) Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ and $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$, where $0 < p \leq 1$ and $r > 0$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{(\ln n)^r}{n^p}}{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} (\ln n)^r = \infty,$$

since r is given to be a positive real number. By the results of exercise 51, since $\sum_{k=1}^{\infty} d_k$ is a divergent p -series (since $0 < p \leq 1$), it follows that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ also diverges, as was needed to be shown.

60. The series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ diverges since it is of the form $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$, $p = 1$, which diverges by the results of Problem 59, part (a).

61. The series $\sum_{k=1}^{\infty} \frac{\sqrt{\ln k}}{\sqrt{k}}$ diverges since it is of the form $\sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$, with $0 < p = \frac{1}{2} \leq 1$, and $r = \frac{1}{2} > 0$, which converges by the results of Problem 59, part (b).

62. The series $\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$ converges since it is of the form $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$, where $p = 3 > 1$, which converges by the results of Problem 58, part (a).

63. The series $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^3 \ln k}}$ can be compared with the series $\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$, which is a convergent p -series, since $p = \frac{3}{2} > 1$. The n^{th} terms of the series satisfy $\frac{1}{\sqrt{n^3 \ln n}} < \frac{1}{n^{3/2}}$ if $\ln n > 1$, which is the case if $n \geq 3$ (recall that $\ln e = 1$). By the Comparison Test, the original series also converges. (Note: You cannot apply the results of Problem 58 or 59 to analyze this series as $r < 0$ here.)

64. (a) We first show that the p -series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p = 1$ using the Integral Test. Let $f(x) = \frac{1}{x^p}$ be a related function of the n^{th} term of the series. $f(x)$ is continuous and decreasing on $[1, \infty)$, and $f(k) = a_k$ for all $k \geq 1$. To apply the Integral Test, we evaluate

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b = -\frac{1}{p+1} \lim_{b \rightarrow \infty} [b^{-p-1} - 1^{-p-1}] = -\frac{1}{p+1} [0-1] = \frac{1}{p+1}.$$

Since the limit is a finite real number, the improper integral converges, so by the Integral Test, the series converges for $p > 1$. At $p = 1$, the integral in question evaluates to

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln x]_1^b = \lim_{b \rightarrow \infty} [\ln b - \ln 1] = \infty,$$

so by the Integral Test, since the improper integral diverges, so does the series for $p = 1$.

(b) For $0 < p < 1$, $n^p < n$ for $n \geq 1$, or

$$a_n = \frac{1}{n^p} < \frac{1}{n} = b_n$$

for $n \geq 1$. Since we have shown in part (a) that the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, by the Comparison Test,

so does the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ for $0 < p < 1$.

If $p = 0$, the series becomes $\sum_{k=1}^{\infty} 1$. Since $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$, the series diverges by the Divergence Test.

Finally, let $p < 0$. Then $p = -q$, with $q > 0$. Since

$$a_n = \frac{1}{n^p} = n^q \geq 1$$

for $n \geq 1$, and the series $\sum_{k=1}^{\infty} 1$ diverges (see above), by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ is divergent for $p < 0$.

65. If the series $\sum_{k=1}^{\infty} a_k$ converges, then by the Comparison Test the series $\sum_{k=1}^{\infty} \frac{a_k}{k}$ also converges since the n^{th} terms satisfy $\frac{a_n}{n} < a_n$ for $n > 1$, provided that a_n are all positive (and they are given to be positive).

66. The series $\sum_{k=1}^{\infty} \frac{1}{1+2^k}$ is convergent as can be seen by comparing it with the (convergent) geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. The n^{th} terms satisfy $\frac{1}{1+2^n} < \frac{1}{2^n}$ for all $n \geq 1$, so by the Comparison Test, the original series also converges.

67. We are given that the harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Choose $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p -series (since $p = 2 > 1$). Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

On the other hand, choose $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)}$. Since $\frac{1}{n \ln(n+1)} > \frac{1}{(n+1) \ln(n+1)}$, it suffices to show that

$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{1}{(k+1) \ln(k+1)}$ is divergent, for then the divergence of $\sum_{k=1}^{\infty} a_k$ would follow from the Comparison Test. That $\sum_{k=1}^{\infty} c_k$ is a divergent series is seen by using Integral Test. In fact, this was done in Problem 27,

of Section 8.3, where it is shown that $\sum_{k=2}^{\infty} \frac{1}{k \ln k} = \sum_{k=1}^{\infty} \frac{1}{(k+1) \ln(k+1)}$ is divergent. Now we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n \ln(n+1)}}{\frac{1}{n}} = \frac{1}{\ln(n+1)} = 0.$$

This shows that the Limit Comparison test gives no guidance as to the convergence or divergence of a given $\sum_{k=1}^{\infty} a_k$ series if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$.

Challenge Problems

68. The n^{th} term of the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{\ln(2k+1)}{\sqrt{k^2-2}\sqrt{k^3-2k-3}}$ behaves like

$$a_n = \frac{\ln(2n+1)}{\sqrt{n^2-2}\sqrt{n^3-2n-3}} = \frac{\ln(2n+1)}{n \cdot n^{3/2} \sqrt{1-\frac{2}{n^2}} \sqrt{1-\frac{2}{n^2}-\frac{3}{n^3}}} \approx \frac{\ln(2n+1)}{n^{5/2}}$$

for large n . We compare with the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{\ln(2k+1)}{k^{5/2}}$. To see that this series converges, examine the n^{th} term

$$b_n = \frac{\ln(2n+1)}{n^{5/2}} \leq \frac{\ln(2n+n)}{n^{5/2}} = \frac{\ln(3n)}{n^{5/2}} = \frac{\ln 3 + \ln n}{n^{5/2}} = \frac{\ln 3}{n^{5/2}} + \frac{\ln n}{n^{5/2}} = c_n + d_n$$

for $n \geq 2$. The series $\sum_{k=2}^{\infty} c_k = \sum_{k=2}^{\infty} \frac{\ln 3}{k^{5/2}}$ is a constant multiple of a convergent p -series (since $p = \frac{5}{2} > 1$), while the series $\sum_{k=2}^{\infty} d_k = \sum_{k=2}^{\infty} \frac{\ln k}{k^{5/2}}$ converges by the result of Problem 58. (In the terminology of that problem, $p = \frac{5}{2} > 1$.) So we have shown that the series $\sum_{k=2}^{\infty} b_k$ converges. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\ln(2n+1)}{\sqrt{n^2-2\sqrt{n^3-2n-3}}}}{\frac{\ln(2n+1)}{n^{5/2}}} = \lim_{n \rightarrow \infty} \frac{n^{5/2}}{\sqrt{n^2-2\sqrt{n^3-2n-3}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1-\frac{2}{n^2}}\sqrt{1-\frac{2}{n^2}-\frac{3}{n^3}}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{\ln(2k+1)}{\sqrt{k^2-2\sqrt{k^3-2k-3}}}$ also *converges.*

69. The n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{(k^3-k+1)\ln(2k+1)}}$ behaves like

$$a_n = \frac{\sqrt{n}}{\sqrt{(n^3-n+1)\ln(2n+1)}} = \frac{\sqrt{n}}{n^{3/2}\sqrt{1-\frac{1}{n^2}+\frac{1}{n^3}}} \cdot \frac{1}{\sqrt{\ln(2n+1)}} \approx \frac{1}{n} \cdot \frac{1}{\sqrt{\ln(2n+1)}}$$

for large n . We compare with $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{\ln(2k+1)}}$. This series diverges, and this is seen as follows: Let

$\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k\ln(2k+1)}$. This series diverges by comparing with the series $\sum_{k=2}^{\infty} c_k = \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ (which itself diverges by the Integral Test since $\lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b \rightarrow \infty} \ln(\ln b) - \ln(\ln 2) = \infty$) and applying the Limit Comparison Test: $\lim_{n \rightarrow \infty} \frac{d_n}{c_n} = \lim_{n \rightarrow \infty} \frac{\frac{n \ln n}{n \ln(2n+1)}}{\frac{1}{n \ln(2n+1)}} = \lim_{x \rightarrow \infty} \frac{\ln x}{\ln(2x+1)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{2}{2x+1}} = \lim_{x \rightarrow \infty} \left(2 + \frac{1}{x}\right) = 2$. Compare now the n^{th} terms of the series $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} d_k$:

$$\lim_{n \rightarrow \infty} \frac{b_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \frac{1}{\sqrt{\ln(2n+1)}}}{\frac{1}{n \ln(2n+1)}} = \lim_{n \rightarrow \infty} \sqrt{\ln(2n+1)} = \infty.$$

So, by the result of Problem 51, the series $\sum_{k=1}^{\infty} b_k$ indeed diverges. Now, finally, we compare the series

$\sum_{k=1}^{\infty} a_k$ with the series $\sum_{k=1}^{\infty} b_k$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n\sqrt{1-\frac{1}{n^2}+\frac{1}{n^3}}} \cdot \frac{1}{\sqrt{\ln(2n+1)}}}{\frac{1}{n} \cdot \frac{1}{\sqrt{\ln(2n+1)}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{n^2}+\frac{1}{n^3}}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{(k^3-k+1)\ln(2k+1)}}$ also *diverges.*

70. The n^{th} term of the series $\sum_{k=1}^{\infty} \frac{1+\sin k}{4^k}$ satisfies $0 < \frac{1+\sin n}{4^n} \leq \frac{2}{4^n} < \frac{3}{4^n}$, since $\sin n > 0$ for $n \geq 1$. So,

comparing with the convergent geometric series $\sum_{k=1}^{\infty} \frac{3}{4} \left(\frac{1}{4}\right)^{k-1}$ we see by the Comparison Test that the given series also *converges.*

71. We compare the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n^{1+1/n}} = \lim_{n \rightarrow \infty} \frac{n}{n \cdot n^{1/n}} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = \frac{1}{\lim_{n \rightarrow \infty} n^{1/n}}.$$

To evaluate $\lim_{n \rightarrow \infty} n^{1/n}$, let $y = n^{1/n}$. Then $\ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$. We have

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} e^{\ln y} = \lim_{n \rightarrow \infty} e^{\frac{\ln n}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\ln n}{n}} = e^{\lim_{x \rightarrow \infty} \frac{\ln x}{x}} = e^{\lim_{x \rightarrow \infty} \frac{1/x}{1}} = e^{\lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1,$$

using L'Hôpital's rule on a related function of $\frac{\ln n}{n}$. Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ also diverges.

72. (a) For large n , the n^{th} term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2-3k-2}{k^2(k+1)^2}$ behaves like

$$a_n = \frac{n^2 - 3n - 2}{n^2(n+1)^2} = \frac{n^2(1 - \frac{3}{n} - \frac{2}{n^2})}{n^2 \cdot n^2(1 + \frac{1}{n})^2} \approx \frac{1}{n^2}.$$

So we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p -series (since $p = 2 > 1$). We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{(1 - \frac{3}{n} - \frac{2}{n^2})}{n^2(1 + \frac{1}{n})^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{(1 - \frac{3}{n} - \frac{2}{n^2})}{(1 + \frac{1}{n})^2} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2-3k-2}{k^2(k+1)^2}$ also converges.

(b) We use a partial fraction decomposition on the k^{th} term of the series $\sum_{k=1}^{\infty} a_k$ as follows:

$$\frac{k^2 - 3k - 2}{k^2(k+1)^2} = \frac{A}{k^2} + \frac{B}{(k+1)^2} + \frac{C}{k} + \frac{D}{(k+1)}.$$

Comparing coefficients after taking the common denominator on the right hand side, we obtain $A = -2$; $B = 2$; $C = 1$; $D = -1$. So,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k^2 - 3k - 2}{k^2(k+1)^2} &= \sum_{k=1}^{\infty} -\frac{2}{k^2} + \sum_{k=1}^{\infty} \frac{2}{(k+1)^2} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{(k+1)} \right) \\ &= -2 \cdot \frac{\pi^2}{6} + 2 \left(\frac{\pi^2}{6} - 1 \right) + \lim_{n \rightarrow \infty} \left[\left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right] \\ &= -2 + \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = -2 + 1 = -1, \end{aligned}$$

where we have used the result $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (see p.572), and the fact that

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2} - 1. \text{ So the required sum of the series is } \boxed{-1}.$$

8.5 Alternating Series; Absolute Convergence

Concepts and Vocabulary

- 1. False:** For odd k , $(-1)^k = -1$ and $\cos k\pi = -1$. So $(-1)^k \cos(k\pi) = (-1)(-1) = +1$. For even k , $(-1)^k = +1$ and $\cos k\pi = +1$. So $(-1)^k \cos(k\pi) = (+1)(+1) = +1$. This means that the terms of the series are all positive for every k and so the series is not an alternating series.
- 2. False:** For odd k , $(-1)^k = -1$, so $[1 + (-1)^k] = 1 - 1 = 0$. For even k , $(-1)^k = +1$, so $[1 + (-1)^k] = 1 + 1 = 2$. So the terms of the series are alternately either 0 or positive, but that does not make it an alternating series, which by definition alternates *positive and negative* terms.
- 3. False:** It is not sufficient that $\lim_{n \rightarrow \infty} a_n = 0$ for the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ to be convergent. We also need to require that the a_k are nonincreasing, that is $a_k \geq a_{k+1}$ for all k .
- 4. False:** The correct conclusion under the stated conditions is that the error in using the sum of n terms S_n in order to approximate the sum S of the series is given by $|E_n| \leq a_{n+1}$ and not $|E_n| \leq a_n$.
- 5. False:** It is possible for an alternating series to be convergent even if it is not absolutely convergent. For example, the alternating harmonic series is convergent but not absolutely convergent (as it would then become a divergent harmonic series).
- 6. True:** This follows from the statement of the theorem on p. 586.

Skill Building

- 7.** The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series *converges*.

- 8.** The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2\sqrt{k}}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{n+1}} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series *converges*.

- 9.** The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k+1}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$. Since the limit is nonzero, by the Divergence Test, the series will *diverge*.

- 10.** The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}$. We have $\lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$. Since the limit is nonzero, by the Divergence Test, the series *diverges*.

- 11.** The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{5k^2+2}$. We have $\lim_{n \rightarrow \infty} \frac{n^2}{5n^2+2} = \lim_{n \rightarrow \infty} \frac{1}{5+\frac{2}{n^2}} = \frac{1}{5} \neq 0$. Since the limit is nonzero, by the Divergence Test, the series will *diverge*.

12. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k^2}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0 + 0 = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{(n+1)^2}}{\frac{n+1}{n^2}} = \lim_{n \rightarrow \infty} \frac{(n+2)n^2}{(n+1)^3} = \frac{n^3 + 2n^2}{n^3 + 3n^2 + 3n + 1} < 1 \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series *converges.*

13. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)2^k}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{(n+1)2^n} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+2)2^{n+1}}}{\frac{1}{(n+1)2^n}} = \frac{1}{2} \frac{n+1}{n+2} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series *converges.*

14. The series is $\sum_{k=2}^{\infty} (-1)^{k+1} a_k = \sum_{k=2}^{\infty} (-1)^{k+1} \frac{1}{k \ln k}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1) \ln(n+1)}}{\frac{1}{n \ln n}} = \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} < 1, \text{ for } n \geq 2,$$

since both n and $\ln n$ are monotonically increasing functions of n for $n \geq 2$. So the terms of the series are nonincreasing. By the Alternating Series Test, the series *converges.*

15. The series is $\sum_{k=2}^{\infty} (-1)^{k+1} a_k = \sum_{k=2}^{\infty} (-1)^{k+1} \frac{1}{1+2^{-k}}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1+2^{-n}} = \frac{1}{1+0} = 1 \neq 0$. Since the limit is nonzero, by the Divergence Test, the series *diverges.*

16. The series is $\sum_{k=0}^{\infty} (-1)^k a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \leq 1, \text{ for } n \geq 0.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series *converges.*

17. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k}{k+1} \right)^k$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n} = \frac{1}{e} \neq 0,$$

using the standard limit definition of e . Since the limit is nonzero, by the Divergence Test, the series *diverges.*

18. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{(k+1)^3}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{n^2}{n^3 \left(1 + \frac{1}{n} \right)^3} = \lim_{n \rightarrow \infty} \frac{1}{n \left(1 + \frac{1}{n} \right)^3} = 0.$$

Let $f(x) = \frac{x^2}{(x+1)^3}$ be a related function of the n^{th} term of the series. Examine the derivative:

$$f'(x) = \frac{(x+1)^3 \cdot 2x - x^2 \cdot 3(x+1)^2}{(x+1)^6} = \frac{(x+1)^2 x}{(x+1)^6} [2(x+1) - 3x] = \frac{(x+1)^2 x(2-x)}{(x+1)^6}.$$

So for $x > 2$, $f'(x) < 0$, which means that for $n \geq 2$, the a_n are nonincreasing. By the Alternating Series Test, the series converges.

19. We have $S_3 = \sum_{k=1}^3 (-1)^{k+1} \frac{1}{k^2} = (-1)^2 \frac{1}{1^2} + (-1)^3 \frac{1}{2^2} + (-1)^4 \frac{1}{3^2} = 1 - \frac{1}{4} + \frac{1}{9} \approx \text{0.8611.}$ The upper estimate to the error is $|E_3| \leq a_4 = \frac{1}{4^2} = \frac{1}{16} = \text{0.0625.}$

20. We have $S_2 = \sum_{k=0}^2 (-1)^k \frac{1}{k!} = (-1)^0 \frac{1}{0!} + (-1)^1 \frac{1}{1!} + (-1)^2 \frac{1}{2!} = 1 - 1 + \frac{1}{2} = \text{0.5000.}$ The upper estimate to the error is $|E_2| \leq a_3 = \frac{1}{3!} = \frac{1}{6} \approx \text{0.1667.}$

21. We have $S_3 = \sum_{k=1}^3 (-1)^{k+1} \frac{1}{k^4} = (-1)^2 \frac{1}{1^4} + (-1)^3 \frac{1}{2^4} + (-1)^4 \frac{1}{3^4} = 1 - \frac{1}{16} + \frac{1}{81} \approx \text{0.9498.}$ The upper estimate to the error is $|E_3| \leq a_4 = \frac{1}{4^4} = \frac{1}{256} \approx \text{0.0039.}$

22. We have $S_3 = \sum_{k=1}^3 (-1)^{k+1} \left(\frac{1}{\sqrt{k}}\right)^k = (-1)^2 \left(\frac{1}{\sqrt{1}}\right)^1 + (-1)^3 \left(\frac{1}{\sqrt{2}}\right)^2 + (-1)^4 \left(\frac{1}{\sqrt{3}}\right)^3 = 1 - \frac{1}{2} + \frac{1}{3\sqrt{3}} \approx \text{0.6925.}$ The upper estimate to the error is $|E_3| \leq a_4 = \left(\frac{1}{\sqrt{4}}\right)^4 = \text{0.0625.}$

23. We have $S_2 = \sum_{k=0}^2 (-1)^k \frac{1}{k!} \left(\frac{1}{3}\right)^k = (-1)^0 \frac{1}{0!} \left(\frac{1}{3}\right)^0 + (-1)^1 \frac{1}{1!} \left(\frac{1}{3}\right)^1 + (-1)^2 \frac{1}{2!} \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{3} + \frac{1}{18} \approx \text{0.7222.}$ The upper estimate to the error is $|E_2| \leq a_3 = \frac{1}{3!} \left(\frac{1}{3}\right)^3 = \frac{1}{6} \cdot \frac{1}{27} \approx \text{0.00617.}$

24. We have $S_2 = \sum_{k=0}^2 (-1)^k \frac{1}{k!} \left(\frac{1}{2}\right)^k = (-1)^0 \frac{1}{0!} \left(\frac{1}{2}\right)^0 + (-1)^1 \frac{1}{1!} \left(\frac{1}{2}\right)^1 + (-1)^2 \frac{1}{2!} \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} + \frac{1}{8} = \text{0.6250.}$ The upper estimate to the error is $|E_2| \leq a_3 = \frac{1}{3!} \left(\frac{1}{2}\right)^3 = \frac{1}{6} \cdot \frac{1}{8} \approx \text{0.0208.}$

25. We have

$$\begin{aligned} S_2 &= \sum_{k=0}^2 (-1)^k \frac{1}{2k+1} \left(\frac{1}{3}\right)^{2k+1} = (-1)^0 \frac{1}{2 \cdot 0 + 1} \left(\frac{1}{3}\right)^{2 \cdot 0 + 1} + (-1)^1 \frac{1}{2 \cdot 1 + 1} \left(\frac{1}{3}\right)^{2 \cdot 1 + 1} + (-1)^2 \frac{1}{2 \cdot 2 + 1} \left(\frac{1}{3}\right)^{2 \cdot 2 + 1} \\ &= \frac{1}{3} - \frac{1}{3} \left(\frac{1}{3}\right)^3 + \frac{1}{5} \left(\frac{1}{3}\right)^5 = \frac{1}{3} - \frac{1}{81} + \frac{1}{1215} \approx \text{0.3218.} \end{aligned}$$

The upper estimate to the error is $|E_2| \leq a_3 = \frac{1}{2 \cdot 3 + 1} \left(\frac{1}{3}\right)^{2 \cdot 3 + 1} = \frac{1}{7} \left(\frac{1}{3}\right)^7 = \frac{1}{15309} \approx \text{6.53} \times 10^{-5}.$

26. We have $S_3 = \sum_{k=1}^3 (-1)^{k+1} \frac{1}{k^k} = (-1)^2 \frac{1}{1^1} + (-1)^3 \frac{1}{2^2} + (-1)^4 \frac{1}{3^3} = 1 - \frac{1}{4} + \frac{1}{27} \approx \text{0.7870.}$ The upper estimate to the error is $|E_3| \leq a_4 = \frac{1}{4^4} \approx \text{0.0039.}$

27. Since $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ is a constant multiple of an alternating harmonic series which converges, the given series also converges. Further $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$, the series of absolute values, is a

constant multiple of the harmonic series which diverges, so it diverges too. Since the original series converges, but does not converge absolutely, it is *conditionally convergent*.

28. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-4}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{3n-4} = 0$. Further, by the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{3(n+1)-4}}{\frac{1}{3n-4}} = \frac{3n-4}{3n-1} < 1 \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the original series converges. Consider now the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3k-4}$. We compare the n^{th} term of this series with that of the harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{3n-4}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{3n-4} = \lim_{n \rightarrow \infty} \frac{1}{3 - \frac{4}{n}} = \frac{1}{3}.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ also diverges by the Limit Comparison Test. This means that the original series is convergent but not absolutely convergent. So in other words it is *conditionally convergent*.

29. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin k}{k^2+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2+1}$. We have $\left| \frac{\sin n}{n^2+1} \right| \leq \frac{1}{n^2+1} < \frac{1}{n^2}$. So, by the Comparison Test, comparing with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which (since $p = 2 > 1$) is convergent, we see that the series of absolute values $\sum_{k=1}^{\infty} a_k$ is convergent. This means that the original series is *absolutely convergent*.

30. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos k}{k^2}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\cos k}{k^2}$. We have $\left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2}$. So, by the Comparison Test, comparing with the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which (since $p = 2 > 1$) is convergent, we see that the series of absolute values $\sum_{k=1}^{\infty} a_k$ is convergent. This means that the original series is *absolutely convergent*.

31. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{5}\right)^k$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^k$. It converges since it is a geometric series with $|r| = \frac{1}{5} < 1$. This means that the original series is *absolutely convergent*.

32. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{5^k}{k!}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5^k}{k!}$.

Consider the following estimate, for $n \geq 6$:

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &\geq 6 \cdot 6 \cdot 6 \cdots 6 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 1 \\ &= 6^{n-5} \cdot 5! \\ &= 6^n \cdot \left(\frac{5!}{6^5}\right) \\ &= \frac{5}{324} \cdot 6^n. \end{aligned}$$

That is,

$$\frac{n!}{6^n} \geq \frac{5}{324} \quad \text{if } n \geq 6.$$

The n^{th} term of the series of absolute values satisfies

$$a_n = \frac{5^n}{n!} = \frac{6^n}{n!} \cdot \frac{5^n}{6^n} \leq \left(\frac{324}{5}\right) \left(\frac{5}{6}\right)^n = \left(\frac{324}{5}\right) b_n$$

for $n \geq 6$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^k$ is a convergent geometric series with $|r| = \frac{5}{6} < 1$, and a constant multiple of this series is also convergent, and also since the omission of the first 5 terms (since $n \geq 6$) will not affect a series' convergence or divergence properties, this means that the series of absolute values converges by the Comparison Test, so the original series is *absolutely convergent*.

33. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{e^k}{k}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{e^n}{n} = \lim_{x \rightarrow \infty} \frac{e^x}{x} = \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty \neq 0,$$

using L'Hôpital's rule on a related function of a_n . Since the limit is nonzero, by the Divergence Test, the series *diverges*.

34. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k^2}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \lim_{x \rightarrow \infty} \frac{2^x}{x^2} = \lim_{x \rightarrow \infty} \frac{2^x \ln 2}{2x} = \lim_{x \rightarrow \infty} \frac{2^x (\ln 2)^2}{2} = \infty \neq 0,$$

using L'Hôpital's rule on a related function of a_n . Since the limit is nonzero, by the Divergence Test, the series *diverges*.

35. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

The n^{th} term of this series satisfies $a_n = \frac{1}{n(n+1)} < \frac{1}{n^2}$ for all $n \geq 1$. Now the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges being a convergent p -series (since $p = 2 > 1$), so by the Comparison Test, the series of absolute values converges. This means that the original series is *absolutely convergent*.

36. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\sqrt{k+3}}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+3}}$.

The n^{th} term of this series satisfies $a_n = \frac{1}{n\sqrt{n+3}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ for all $n \geq 1$. Now the series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges being a convergent p -series (since $p = \frac{3}{2} > 1$), so by the Comparison Test, the series of absolute values converges. This means that the original series is *absolutely convergent*.

37. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sqrt{k}}{k^2+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$. The n^{th} term of this series behaves like $a_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2(1+\frac{1}{n^2})} \approx \frac{1}{n^{3/2}}$ for large n . We compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ is convergent (it is a convergent p -series with $p = \frac{3}{2} > 1$), by the Limit Comparison Test, the series of absolute values $\sum_{k=1}^{\infty} a_k$ is also convergent. This means that the original series is *absolutely convergent.*

38. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sqrt{k}}{k+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+1}$. The n^{th} term of this series behaves like $a_n = \frac{\sqrt{n}}{n+1} = \frac{n^{1/2}}{n(1+\frac{1}{n})} \approx \frac{1}{n^{1/2}}$ for large n . We compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ is a divergent p -series (since $0 < p = \frac{1}{2} < 1$), by the Limit Comparison Test, the series of absolute values is divergent as well. Next we need to examine the original series for convergence. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0$. Consider a related function of a_n , $f(x) = \frac{\sqrt{x}}{x+1}$. We examine the derivative:

$$f'(x) = \frac{(x+1) \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot 1}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0, \text{ for } x \geq 1.$$

The function is a nonincreasing one for $x \geq 1$, which means that the terms of the original series are nonincreasing for $n \geq 1$. By the Alternating Series Test, we see that the original series must be convergent. However, since the series of absolute values is divergent, this means that the original series must be *conditionally convergent.*

39. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln k}{k}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k}$. This series diverges by the Integral Test. Using $f(x) = \frac{\ln x}{x}$ which is a continuous, positive and decreasing function (since $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} \leq 0$) on $[e, \infty)$, and for which $a_k = f(k)$ for all values of $k \geq 3$, we have

$$I = \int_3^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_3^b \frac{\ln x dx}{x} = \lim_{b \rightarrow \infty} \frac{1}{2} [\ln x^2] \Big|_3^b = \lim_{b \rightarrow \infty} \frac{1}{2} (\ln b^2 - \ln 3^2) = \infty.$$

So the improper integral I diverges, showing that the series of absolute values also diverges. Since the series of absolute values diverges, we must continue to see if the original series is convergent. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0,$$

using L'Hôpital's rule on a related function of a_n . Next we examine the derivative of the related function:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0, \text{ for } x \geq 3,$$

recalling that $\ln e = 1$. This means that for $n \geq 3$, the a_n are nonincreasing. So by the Alternating Series Test, the original series converges. However, since the series of absolute values is divergent, this means that the original series is *conditionally convergent*.

40. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k^3}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^3}$.

The series $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p -series since $p = \frac{3}{2} > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{d_n} = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{3/2}} = \lim_{x \rightarrow \infty} \frac{\ln x}{x^{3/2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{3}{2}x^{1/2}} = \lim_{x \rightarrow \infty} \frac{2}{3x^{3/2}} = 0,$$

using L'Hôpital's rule on a related function of the ratio of n^{th} terms of the two series. By the result of Problem 52, in end of chapter exercises of Section 4, we conclude that the series of absolute values is convergent. But this means that the original series is *absolutely convergent*.

41. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{ke^k}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{ke^k}$.

We compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p -series (since $p = 2 > 1$). We have

$a_n = \frac{1}{ne^n} < \frac{1}{n^2}$ if $e^n > n$. (To see that this is true for all n , let $f(x) = e^x - x$. Then $f'(x) = e^x - 1 > 0$ for $x > 0$. This means the function $f(x)$ is an increasing one for $x > 0$, and in turn this shows $e^x > x$ for $x > 0$.) We see that the series of absolute values is convergent by the Comparison Test, so the original series must be *absolutely convergent*.

42. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{e^k}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{e^k}$. This is a geometric series that converges because $|r| = \frac{1}{e} < 1$, since $e > 1$. Since the series of absolute values converges, the original series must be *absolutely convergent*.

Applications and Extensions

43. The series is $\sum_{k=2}^{\infty} (-1)^k a_k = \sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$. The series of absolute values is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k \ln k}$. Consider the function $f(x) = \frac{1}{x \ln x}$, which is defined, positive, and decreasing on $[2, \infty)$, and such that $a_k = f(k)$ for all $k \geq 2$. We have $I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x}$. Let $u = \ln x$, then $du = \frac{dx}{x}$. Continuing,

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} = \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series of absolute values also diverges. Next, we examine the original series for convergence. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$. Also, consider the derivative of the related function of a_n :

$$f'(x) = -\frac{1}{x^2 \ln x} - \frac{1}{x^2 (\ln x)^2} < 0 \text{ for } x \geq 2.$$

So the a_n are nonincreasing. By the Alternating Series Test, the original series converges. But since it is not absolutely convergent, it is *conditionally convergent*.

44. The series is $\sum_{k=2}^{\infty} (-1)^k a_k = \sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^2}$. The series of absolute values is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$. Consider the function $f(x) = \frac{1}{x(\ln x)^2}$, which is defined, positive, and decreasing on $[2, \infty)$, and such that $a_k = f(k)$ for all $k \geq 2$. We have $I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2}$. Let $u = \ln x$, then $du = \frac{dx}{x}$. Continuing,

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-2} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-2+1}}{-2+1} \right]_{\ln 2}^{\ln b} = - \lim_{b \rightarrow \infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2} \right) = 0.$$

Since the improper integral I converges, by the Integral Test, the series of absolute values also converges. But this means that the original series is *absolutely convergent*.

45. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2}$. The series of absolute values is $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$. We have $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$, for all $n \geq 1$, since $|\sin n| \leq 1$ for all $n \geq 1$. Since the p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent (since $p = 2 > 1$), by the Comparison Test, we see that the series of absolute values is convergent. This means that the original series $\sum_{k=1}^{\infty} a_k$ is *absolutely convergent*.

46. The series is $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \tan^{-1} k}{k}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k}$. Its terms are all nonzero since $\tan^{-1} n > 0$ for $n \geq 1$. Comparing this series with $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\tan^{-1} n}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \tan^{-1} n = \frac{\pi}{2}.$$

Since the limit is a positive real number and the series $\sum_{k=1}^{\infty} b_k$ diverges because it is a harmonic series, we conclude that the series of absolute values $\sum_{k=1}^{\infty} a_k$ also diverges by the Limit Comparison Test. Next, we check to see if the original series converges or not. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{n} = \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x^2}}{1} = 0,$$

by using L'Hôpital's rule on a related function of a_n , $f(x) = \frac{\tan^{-1} x}{x}$. We have, examining its derivative:

$$f'(x) = \frac{x \cdot \frac{1}{1+x^2} - \tan^{-1} x \cdot 1}{x^2} = \frac{x - (1+x^2) \tan^{-1} x}{x^2(1+x^2)} \leq 0$$

whenever $x \leq (1+x^2) \tan^{-1} x$, or equivalently, whenever $\tan^{-1} x \geq \frac{x}{1+x^2}$. We prove that this is true for all $x \geq 0$. Let $x_0 \geq 0$. The value of the function $\tan^{-1} x$ at $x = x_0$ can be represented as a definite integral:

$$\tan^{-1} x_0 = \int_0^{x_0} \frac{dx}{1+x^2}$$

since $\tan^{-1} 0 = 0$. The function $g(x) = \frac{1}{1+x^2}$ is a nonincreasing function of x for $x \geq 0$ since

$$g'(x) = -\frac{2x}{(1+x^2)^2} \leq 0$$

for $x \geq 0$. So the smallest value of $g(x) = \frac{1}{1+x^2}$ on the interval $[0, x_0]$ is $g(x_0) = \frac{1}{1+x_0^2}$. So we have

$$\tan^{-1} x_0 = \int_0^{x_0} \frac{dx}{1+x^2} \geq \int_0^{x_0} \frac{dx}{1+x_0^2} = \frac{1}{1+x_0^2} \int_0^{x_0} dx = \frac{x_0}{1+x_0^2}.$$

That is, $\frac{x_0}{1+x_0^2} \leq \tan^{-1} x_0$, or $x_0 \leq (1+x_0^2) \tan^{-1} x_0$. Since $x_0 \geq 0$ was an arbitrary point, it follows that $x \leq (1+x^2) \tan^{-1} x$ for all $x \geq 0$, i.e., $f'(x) \leq 0$ for all $x \geq 0$. This means in the original series the a_n are nonincreasing for $n \geq 1$ and so the original series is convergent by the Alternating Series Test. However, the series of absolute values has been shown above to be divergent, so the original series is

conditionally convergent.

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{1/k}}$. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = \frac{1}{\lim_{n \rightarrow \infty} n^{1/n}}$. To evaluate $\lim_{n \rightarrow \infty} n^{1/n}$, set $y = n^{1/n}$. Then $\ln y = \frac{1}{n} \ln n$. We have

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

using L'Hôpital's rule on a related function of the ratio. So $\lim_{n \rightarrow \infty} \ln y = \ln \lim_{n \rightarrow \infty} y = 0$ gives

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} n^{1/n} = 1.$$

Since $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$, by the Divergence Test, the series will be *divergent*.

48. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k}{k+1} \right)^k$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} \neq 0,$$

using the limit definition of e . Since the limit is nonzero, by the Divergence Test, the series will be

divergent.

49. The series is $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k!}$. the n^{th} term of this series satisfies $a_n = \frac{1}{n!} \leq \frac{1}{n^2}$ when $n! \geq n^2$. Let $n \geq 4$. We estimate as follows:

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2)(n-3) \cdots 3 \cdot 2 \cdot 1 \\ &\geq n \cdot (n-1) \cdot (n-2) \cdot 1 \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot 1 \\ &= n(n-1)(n-2). \end{aligned}$$

Next, we demonstrate (for $n \geq 4$) that $n(n-1)(n-2) \geq n^2$:

$$\begin{aligned} n(n-1)(n-2) &\geq n^2 \\ n(n^2 - 3n + 2) &\geq n^2 \\ n^2 - 3n + 2 &\geq n \\ n^2 - 4n + 2 &\geq 0 \\ n^2 - 4n + 4 - 2 &\geq 0 \\ (n-2)^2 - 2 &\geq 0 \\ (n-2)^2 &\geq 2. \end{aligned}$$

That is, $n-2 \leq -\sqrt{2}$ or $n-2 \geq \sqrt{2}$. Keeping the positive root solution, we have $n \geq 2 + \sqrt{2} \approx 3.414$ which concurs with our initial assumption that $n \geq 4$. So we have shown the desired inequality, that is $n! > n^2$

for $n \geq 4$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series (since $p = 2 > 1$), by the Comparison

Test, the series of absolute values converges, which means the original series is *absolutely convergent*.

50. The series is $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^2}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$. The n^{th} term of this series satisfies $a_n = \frac{1}{(2n-1)^2} < \frac{1}{n^2}$, since $2n-1 > n$ for $n > 1$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series (since $p = 2 > 1$), by the Comparison Test, the series of absolute values converges, which means the original series is *absolutely convergent*.

51. The series expands as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. So positive terms are of the form $\sum_{k=1}^{\infty} \frac{1}{2k-1}$. Since the n^{th} term of this series satisfies $\frac{1}{2n-1} > \frac{1}{2n}$ for $n \geq 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{2k}$, being a constant multiple of the divergent harmonic series is divergent, the series of positive terms also diverges by the Comparison Test, as was needed to be proved.

52. The series expands as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. So negative terms are of the form $-\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$, which, being a constant multiple of the divergent harmonic series is divergent. So the series of negative terms diverges as was needed to be proved.

53. We group terms of the alternating harmonic series as follows: $S_1 = 1$;
 $S_2 = 1 + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8}\right) \approx -0.0416667$; $S_3 = S_1 + S_2 + \left(\frac{1}{3}\right) \approx +0.2916663$;
 $S_4 = S_1 + S_2 + S_3 + \left(-\frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}\right) \approx 0.02559557$, and so on. That is, for odd and even indices,

$$S_{2n-1} = S_1 + S_2 + \cdots + S_{2n-2} + \left(\frac{1}{2n-1}\right)$$

and

$$S_{2n} = S_1 + S_2 + \cdots + S_{2n-1} + \left(-\frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n}\right).$$

We show that the terms of the sequence $\{S_n\}$ are decreasing and bounded from below by 0. Using the Algebraic Difference Test,

$$S_{2n} - S_{2n-1} = S_{2n-1} - \frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n} - \frac{1}{2n-1}. \quad (1)$$

Examining equation (1), that the quantity on the right side will be negative for $n \geq 1$, proving that the sequence is a decreasing one. By its construction, the sequence is bounded from below by 0, so it converges. The sum to which the series $\sum_{k=1}^{\infty} S_k$ it converges is 0 since the limit of the sequence $\{S_n\}$ of partial sums is 0. So we have managed to find a way to group the terms of the alternating harmonic series to produce a series whose sum is 0.

54. By trial and error, we find that $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 2 < 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15}$. This means we can come close to 2 using the first 7 terms of the positive odd denominator numbers. Subsequently, we form sum groupings as in Problem 53, starting from $n = 15$. This way, we will be able to form groupings whose eventual sum will be 2.

55. We create blocks of numbers that sum to a little above 1, and then blocks that sum to a little under 1, and so on; adding these blocks up, one gets the sum to be infinite. This can be done because the sum of the positive terms of the conditionally convergent alternating harmonic series diverges, and the sum of the negative terms of the conditionally convergent alternating harmonic series diverges as well.

56. We have the series of absolute values of $e^{-x} \cos x + e^{-2x} \cos 2x + e^{-3x} \cos 3x + \cdots$ satisfy (since $|\cos \theta| \leq 1$),

$|e^{-x} \cos x| + |e^{-2x} \cos 2x| + |e^{-3x} \cos 3x| + \cdots \leq e^{-x} + e^{-2x} + e^{-3x} + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^{kx} = \sum_{k=1}^{\infty} \left[\left(\frac{1}{e}\right)^x\right]^k$. This last is a geometric series that converges since $0 < |r| = \left(\frac{1}{e}\right)^x < 1$ for all $x > 0$ (since $e^x > 1$ for $x > 0$). By the Comparison Test, the series of absolute values converges, so the original series converges absolutely, as was needed to be shown.

57. The n^{th} term of the series of absolute values of the series $1 + r \cos \theta + r^2 \cos(2\theta) + r^3 \cos(3\theta) + \cdots$ satisfies $|r^n \cos n\theta| \leq |r|^n$. Since $\sum_{k=1}^{\infty} |r|^k$ is a geometric series that is convergent when $0 < |r| < 1$, or $-1 < r < 1$, we find that the series of absolute values converges, and the original series *absolutely converges* for $-1 < r < 1$. Also, since the geometric series diverges when $|r| \geq 1$, we conclude that the given series *diverges* for $|r| \geq 1$.

58. The alternating harmonic series is conditionally convergent, so its terms can be rearranged to obtain different kinds of sums, as illustrated in the statement of the problem.

59. $N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, and $N - 1$ are both divergent harmonic series. They can be added and the sums rearranged to produce any number one wishes. (See for example the Problems 53-55 where we rearranged the alternating harmonic series, whose series of absolute values is the harmonic series. We were able to rearrange terms to obtain any sum we wanted.) Only series which are absolutely convergent can be added together and rearranged at will to produce a convergent series with a unique sum. (See p.588 for the list of properties of absolutely convergent series.)

60. The series will either converge or diverge. For instance consider the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the divergent (constant term) series $\sum_{k=1}^{\infty} 1$. The sum will diverge, since the partial sum $S_n = n + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and $\lim_{n \rightarrow \infty} S_n = \infty$. Their difference will diverge as well, since the partial sum will be $S_n = (1 - 1) + (1 - \frac{1}{2}) + (1 - \frac{1}{3}) + (1 - \frac{1}{4}) + \cdots + (1 - \frac{1}{n}) = 0 + \frac{1}{2} + \frac{3}{4} + \cdots + \frac{n-1}{n} = \sum_{k=1}^n \frac{k}{k+1}$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$, by the Divergence Test, the difference series also diverges.

Now consider the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k+1}$ which, being the harmonic series from the second term forward, also diverges. Then their difference will converge, and this is seen as follows. The n^{th} partial sum of the difference of these series will be

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

which converges to 1, since

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

So the series which is the term by term difference of the two series also converges.

These examples demonstrate that nothing conclusive can be said about the convergence or divergence when two divergent series are added or subtracted term by term from each other.

61. The summand in the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ can be split up into partial fractions as $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. The n^{th} partial sum has the form $S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n}$, due to

the telescoping. We have $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (1 - \frac{1}{n}) = 1$. Since the limit of the sequence $\{S_n\}$ of partial sums is 1, it means the series sums to 1.

62. To show the series $\sum_{k=1}^{\infty} (a_k + b_k)$ converges absolutely, consider the absolute value of the n^{th} term. We have $|a_n + b_n| \leq |a_n| + |b_n|$, by the Triangle Inequality. Let the sequence of partial sums of $\sum_{k=1}^{\infty} |a_k|$ be $\{S_n\}$ and the sequence of partial sums of $\sum_{k=1}^{\infty} |b_k|$ be $\{S'_n\}$, and the sequence of partial sums of $\sum_{k=1}^{\infty} |a_k + b_k|$ be $\{S''_n\}$. Then, since we have

$$0 \leq S''_n = \sum_{k=1}^n |a_k + b_k| \leq \sum_{k=1}^n |a_k| + |b_k| = \sum_{k=1}^n |a_k| + \sum_{k=1}^n |b_k| = S_n + S'_n$$

and (by assumption) the sequences $\{S_n\}$ and $\{S'_n\}$ converge, it follows that the sequence $\{S''_n\}$ is also convergent. Since absolutely convergent series can be rearranged and regrouped at will without changing the sum, it follows that $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

63. An absolutely convergent series can be rearranged at will (see Problem 66 below), so one may group all positive terms together, and all the negative terms together, and the sum of the series will not be affected. Since the entire series has a finite sum, it means that each of the component series will also have a finite sum, and so will converge separately.

64. Let the conditionally convergent series be $\sum_{k=1}^{\infty} a_k$. We assume the terms are nonzero; none of the arguments below will be affected by a few zero terms. Note that $b_n = \frac{a_n + |a_n|}{2}$ are all the positive n^{th} terms, and $c_n = \frac{a_n - |a_n|}{2}$ are all the negative n^{th} terms. Also note that $a_n = b_n + c_n$. Suppose that the series of positive terms is convergent. Then, because its terms are all positive, it is also absolutely convergent. Since $\sum_{k=1}^{\infty} a_k$ converges conditionally and $\sum_{k=1}^{\infty} b_k$ converges absolutely, then $\sum_{k=1}^{\infty} c_k$ converges absolutely, because $|c_n| = \frac{1}{2}(|a_n - |a_n||) \leq \frac{1}{2}(a_n + |a_n|) = b_n$ so by the Comparison Test, if $\sum_{k=1}^{\infty} b_k$ converges absolutely, then so must $\sum_{k=1}^{\infty} c_k$. But since $a_n = b_n + c_n$ this must mean that $\sum_{k=1}^{\infty} a_k$ converges absolutely (by the result of Problem 62) which contradicts its conditional convergence. This contradiction was a result of assuming that $\sum_{k=1}^{\infty} b_k$ was convergent, so that must be false. This means that the series of positive terms diverges. But since $c_n = a_n - b_n$ it means that $\sum_{k=1}^{\infty} c_k$ must also be divergent. So, finally, we have shown that the series of positive terms and the series of negative terms both diverge separately.

65. The n^{th} term of the series $\sum_{k=1}^{\infty} c_k$ is

$$|c_n| = \begin{cases} \frac{1}{a^n}, & n \text{ even} \\ \frac{1}{b^n}, & n \text{ odd} \end{cases}$$

Let $d = \min\{a, b\}$. Since $a > 1$ and $b > 1$, we have $\frac{1}{d} < 1$ and thus $|c_n| \leq \frac{1}{d^n}$. Since $\frac{1}{d} < 1$, the series $\sum_{k=1}^{\infty} \frac{1}{d^k}$ is a convergent geometric series, so $\sum_{k=1}^{\infty} |c_k|$ converges. This means that $\sum_{k=1}^{\infty} c_k$ *converges absolutely.*

66. If the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then the series of absolute values $\sum_{k=1}^{\infty} |a_k| = L$ must be convergent to a finite number (say) L . Since the sequence of partial sums $\{S_n\}$ is also convergent to the

limit L , this means for every $\epsilon > 0$, there must be an N_1 such that for $n > N_1$ we have

$$|S_n - L| = \left| S_n - \sum_{k=1}^{\infty} |a_k| \right| = \left| \sum_{k \geq N_1+1}^{\infty} |a_k| \right| < \frac{\epsilon}{2}.$$

That is, for any $\epsilon > 0$, there exists an N_1 such that $\sum_{k=N_1+1}^{\infty} |a_k| < \frac{\epsilon}{2}$. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a permutation on the indices of the series, that is a one to one and onto map defined on the positive integers. This achieves the rearrangement of the series from $\sum_{k=1}^{\infty} a_k$ to $\sum_{k=1}^{\infty} a_{f(k)}$. Since the series of absolute values is convergent, it means that the sequence of partial sums $S_n = |a_1| + |a_2| + \cdots + |a_n|$ is bounded from above by some real number, say K . Then the sequence of partial sums of the rearrangements $S'_n = |a_{f(1)}| + |a_{f(2)}| + \cdots + |a_{f(n)}|$ is also bounded from above by the same number K' . Since $\{S'_n\}$ is a bounded from above and monotonically increasing sequence (since all its terms are positive), it will converge.

To show that the limits are the same, we use the fact that there exists some $N_2 \geq N_1$ such that among the terms $a_{f(1)}, a_{f(2)}, \dots, a_{f(N_2)}$ we shall find all of the terms $a_1, a_2, \dots, a_{N_1}$. For $n > N_2$, let $\mathcal{D}$ be the (finite) set of subscripts $k \in \{f(1), f(2), f(3), \dots, f(n)\}$ that do not match up with any of the $\{1, 2, \dots, N_1\}$. Then for $n > N_2 \geq N_1$, we have

$$\left| \sum_{k=1}^n |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k| \right| = \left| \sum_{\mathcal{D}} |a_{f(k)}| \right| \leq \sum_{\mathcal{D}} |a_{f(k)}| \leq \sum_{k=N_1+1}^{\infty} |a_k| < \frac{\epsilon}{2},$$

where the final inequality follows from the fact that all the terms $|a_{f(k)}|$ with $k \in \mathcal{D}$ are to be found among the set $\{|a_{N_1+1}|, |a_{N_1+2}|, \dots\}$, since $N_1 \leq N_2 < n$ and $\mathcal{D}$, which is a finite set, only contains elements not in the set $\{1, 2, 3, \dots, N_1\}$. Now, we compute:

$$\left| \sum_{k=1}^n |a_{f(k)}| - L \right| = \left| \sum_{k=1}^n |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k| + \sum_{k=1}^{N_1} |a_k| - L \right| \leq \left| \sum_{k=1}^n |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k| \right| + \left| \sum_{k=1}^{N_1} |a_k| - L \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that the series of absolute values obtained by a rearrangement of the original series converges to L . So we have shown that a rearrangement of the original absolutely convergent series will result in a series which itself is absolutely convergent to the same limit as the original series.

67. The proof of this result is identical in every respect to the proof carried out in the textbook, pages 582-583 (Alternating Series Test); the only change is that instead of writing the alternating series as $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ with $a_k > 0$ in the textbook, it is written as $\sum_{k=1}^{\infty} a_k$ in this problem. (That is, the minus signs are now included in the terms a_k themselves.) So, we have to be careful about putting absolute value signs around the terms while importing the proof from the book to this problem. For instance, in the text, the hypothesis for nondecreasing terms is written as $a_{k+1} \leq a_k$ while in the problem the hypothesis for nondecreasing terms applies to the absolute values: $|a_{k+1}| \leq |a_k|$ for all $k \geq 1$. As in the book, form the series $\{S_{2n}\}$ of partial sums and show that it is nondecreasing; form the series $\{S_{2n+1}\}$ of partial sums and show that it is nonincreasing; finally show that both these series tend to the same limit as $n \rightarrow \infty$.

Challenge Problems

68. The series can be written as $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} \sin\left(\frac{(-1)^k}{k}\right) = \sum_{k=1}^{\infty} (-1)^k \sin\left(\frac{1}{k}\right)$, since $\sin(-\theta) = -\sin \theta$.

The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$. Note that the terms of the series are nonzero because

$\sin\left(\frac{1}{n}\right) > 0$ for $n \geq 1$. Comparing this series to the divergent harmonic series, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$, we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

employing the substitution $x = \frac{1}{n}$, and using the standard limit result at the end. Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ is divergent, by the Limit Comparison Test, we see that the series $\sum_{k=1}^{\infty} a_k$ must be divergent. So the series of absolute values diverges. Next, we need to see if the original series itself converges. The n^{th} term satisfies $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sin \frac{1}{n} = 0$. Also, if $f(x) = \sin \frac{1}{x}$ is a related function of the n^{th} term of the series, examining its derivative yields $f'(x) = -\frac{1}{x^2} \cos \frac{1}{x} < 0$ for $x > 1$ (recall that for θ between 0 and 1 radians, $\cos \theta$ is a positive function). This shows that the related function, and the sequence it is related to, are nonincreasing. So by the Alternating Series Test, the original series converges. But since the series of absolute values has been shown above to be divergent, this means that the original series is *conditionally convergent*.

69. The series is $\sum_{k=2}^{\infty} (-1)^k a_k = \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt[p]{k^3+1}}$. The absolute value series is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{(n^3+1)^{1/p}}$. The n^{th} term of the series behaves like

$$a_n = \frac{1}{(n^3+1)^{1/p}} = \frac{1}{n^{3/p} (1 + \frac{1}{n^3})^{1/p}} \approx \frac{1}{n^{3/p}}$$

for large n . So we compare with the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^{3/p}}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n^3+1)^{3/p}}}{\frac{1}{n^{3/p}}} = \lim_{n \rightarrow \infty} \left(\frac{n^3}{n^3+1} \right)^{1/p} = \lim_{n \rightarrow \infty} \frac{1}{(1 + \frac{1}{n^3})^{1/p}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^q}$ is a convergent “ q -series” since for $2 < p < 3$, we have $q = \frac{3}{p} > 1$, it means that the absolute value series is also convergent by the Limit Comparison Test. This in turn implies that the original series is *absolutely convergent* if $2 < p < 3$.

If $p \geq 3$, then the large n behavior of the n^{th} term is $\frac{1}{n^{3/p}} = \frac{1}{n^q}$ where $q \leq 1$. We compare with the divergent series $\sum_{k=2}^{\infty} b_k = \frac{1}{k^q}$ for $q \leq 1$; the computation is identical to the one shown above, except the conclusion is that the original series *diverges* if $p \geq 3$.

70. The series is $\sum_{k=2}^{\infty} (-1)^k a_k = \sum_{k=2}^{\infty} (-1)^k k^{(1-k)/k}$. The series of absolute values is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} k^{(1-k)/k} = \sum_{k=2}^{\infty} \frac{k^{1/k}}{k}$. Comparing with the divergent harmonic series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k}$ we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^{1/n}}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n^{1/n} = 1,$$

where the reader is referred to Problem 47 for the evaluation of the final limit. Since the limit is a positive number, and the series $\sum_{k=2}^{\infty} b_k$ diverges, by the comparison test, the series of absolute values diverges as well. Next, we check to see if the original series is convergent. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{n} = 0$. Let $f(x) = \frac{x^{1/x}}{x}$ be a related function of the a_n . Let $y = x^{1/x}$. Then $\ln y = \frac{1}{x} \ln x$. Differentiating, we have

$$\frac{y'}{y} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}.$$

Next, examine the derivative of $f(x) = \frac{x^{1/x}}{x} = \frac{y}{x}$:

$$f'(x) = \frac{y'}{x} - \frac{y}{x^2} = \frac{x^{1/x}}{x^2}(1 - \ln x) - \frac{x^{1/x}}{x^2} = \frac{x^{1/x}}{x^2}(-\ln x) < 0,$$

whenever $x > 1$. Since the related function is decreasing the series is nonincreasing for $n \geq 1$. By the Alternating Series Test, the original series converges. However, since the series of absolute values diverges as seen above, the original series is *conditionally convergent*.

71. The series is $\sum_{k=2}^{\infty} \frac{(-1)^k}{(\ln k)^{\ln k}}$. The series of absolute values is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}}$. We have

$$(\ln n)^{\ln n} = \left(e^{\ln(\ln n)}\right)^{\ln n} = (e^{\ln n})^{\ln(\ln n)} = n^{\ln(\ln n)},$$

since $(e^a)^b = (e^b)^a$. Since $\ln n$ is an increasing function, so is $\ln(\ln n)$. So there will exist some $n > 2$ for which $\ln(\ln n) \geq 2$. Solving, we find that $n \geq e^{e^2} \approx 1618.18$. So we see that the n^{th} term of this series satisfies $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$ for $n \geq 1619$. Since the p -series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k^2}$ converges ($p = 2 > 1$), we see by the Comparison Test that the series of absolute values also converges. (We note that convergence does not depend on the omission of a finite number of terms from the comparison, in this instance the first 1618 terms, recalling the series starts with $n = 2$.) But this means that the original series is *absolutely convergent*.

72. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$. We have $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0$. Let $f(x) = \frac{\sqrt{x}}{x+1}$ be a related function of the absolute value of the n^{th} term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{(x+1) \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot 1}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing, or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 67, we see that the series *converges*.

73. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2}$. We have $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{1}{\left(\sqrt{n} + \frac{1}{\sqrt{n}}\right)^2} = 0$. Let $f(x) = \frac{x}{(x+1)^2}$ be a related function of the absolute value of the n^{th} term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{(x+1)^2 \cdot 1 - x \cdot 2(x+1) \cdot 1}{(x+1)^4} = \frac{(x+1)}{(x+1)^4} [x+1-2x] = \frac{1-x}{(1+x)^3} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 67, we see that the series *converges*.

74. The series is $\sum_{k=2}^{\infty} (-1)^k a_k = \sum_{k=2}^{\infty} (-1)^k \ln \frac{k+1}{k}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln \frac{n+1}{n} = \ln \lim_{n \rightarrow \infty} \frac{n+1}{n} = \ln \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = \ln(1+0) = 0.$$

Let $f(x) = \ln \frac{x+1}{x}$ be a related function of the absolute value of the n^{th} term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{x}{x+1} \cdot \frac{x \cdot 1 - (x+1) \cdot 1}{x^2} = -\frac{1}{x(x+1)} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 67, we see that the series *converges.*

75. (a) Since $\lim_{n \rightarrow \infty} a_n = 0$, we have $\lim_{n \rightarrow \infty} \Delta a_n = \lim_{n \rightarrow \infty} (a_n - a_{n+1}) = 0$. We are given that the sequence $\{\Delta a_n\}$ decreases. Therefore, by the Alternating Series Test, $\sum_{k=1}^{\infty} (-1)^{k+1} \Delta a_k$ is a convergent alternating series.

(b) Let $R_n = \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k$. R_n is a well defined value since $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is a convergent alternating series. Expanding R_n , we have

$$\begin{aligned} R_n &= (-1)^{n+2} a_{n+1} + (-1)^{n+3} a_{n+2} + (-1)^{n+4} a_{n+3} + (-1)^{n+5} a_{n+4} + \cdots \\ &= (-1)^{n+2} [a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots] \\ &= (-1)^{n+2} [(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots]. \end{aligned}$$

Since the sequence $\{a_n\}$ decreases, each of the pairs grouped in this last expression is positive, so the absolute value of R_n is given by

$$|R_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots.$$

Rewriting this produces

$$|R_n| = \frac{a_n}{2} + \frac{1}{2} [-a_n + 2a_{n+1} - 2a_{n+2} + 2a_{n+3} - 2a_{n+4} + \cdots] \quad (1)$$

$$\begin{aligned} &= \frac{a_n}{2} + \frac{1}{2} [(-a_n + a_{n+1}) + (a_{n+1} - a_{n+2}) + (-a_{n+2} + a_{n+3}) + \cdots] \\ &= \frac{a_n}{2} + \frac{1}{2} [-(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) - (a_{n+2} - a_{n+3}) + \cdots] \\ &= \frac{a_n}{2} + \frac{1}{2} [-\Delta a_n + \Delta a_{n+1} - \Delta a_{n+2} + \Delta a_{n+3} - \cdots] \\ &= \frac{a_n}{2} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \Delta a_{k+n-1}, \end{aligned} \quad (2)$$

as was required to be shown. Rewriting equation (2) produces

$$|R_n| = \frac{a_n}{2} + \frac{1}{2} [(-\Delta a_n + \Delta a_{n+1}) + (-\Delta a_{n+2} + \Delta a_{n+3}) + \cdots].$$

Since $\{\Delta a_n\}$ is a decreasing sequence, each of the pairs grouped in this last expression is negative, so we have

$$|R_n| < \frac{a_n}{2}.$$

(c) Use equation (1) from part (b) to rewrite the absolute value of R_n as follows:

$$\begin{aligned} |R_n| &= \frac{a_{n+1}}{2} + \frac{1}{2} [a_{n+1} - 2a_{n+2} + 2a_{n+3} - 2a_{n+4} + \cdots] \\ &= \frac{a_{n+1}}{2} + \frac{1}{2} [(a_{n+1} - a_{n+2}) + (-a_{n+2} + a_{n+3}) + (a_{n+3} - a_{n+4}) + \cdots] \\ &= \frac{a_{n+1}}{2} + \frac{1}{2} [\Delta a_{n+1} - \Delta a_{n+2} + \Delta a_{n+3} - \cdots] \\ &= \frac{a_{n+1}}{2} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \Delta a_{k+n}, \end{aligned} \quad (3)$$

as was required to be shown. Rewriting equation (3) produces

$$|R_n| = \frac{a_{n+1}}{2} + \frac{1}{2} [(\Delta a_{n+1} - \Delta a_{n+2}) + (\Delta a_{n+3} - \Delta a_{n+4}) + \cdots].$$

Since $\{\Delta a_n\}$ is a decreasing sequence, each of the pairs grouped in this last expression is positive, so we have

$$|R_n| > \frac{a_{n+1}}{2}.$$

8.6 Ratio Test; Root Test

Concepts and Vocabulary

1. False: We have $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cos(n+1)\pi}{\cos n\pi} \right| = 1$ for all n , so $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$. This means the Ratio Test is inconclusive, and you cannot conclude divergence as claimed.

2. False: For a counterexample, consider the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right|^2 = 1,$$

whereas (see p.572) $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. So the value of $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ has no relation to the actual value of the sum

$$\sum_{k=1}^{\infty} a_k.$$

3. False: If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Ratio Test is inconclusive. For a counterexample, consider the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = 1.$$

However, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ being the harmonic series, diverges.

4. False: Since the Root Test works by extracting the n^{th} root, a series that has an n^{th} power in its n^{th} term (as opposed to one that has an n^{th} root in its n^{th} term) works well with the Root Test (providing, of course, the Root Test is applicable in the first place).

Skill Building

5. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{4k^2-1}{2^k}$ is a series of nonzero terms. The ratio of the $n+1^{\text{st}}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4(n+1)^2-1}{2^{n+1}}}{\frac{4n^2-1}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \left[\frac{4(n^2+2n+1)-1}{4n^2-1} \right] = \frac{1}{2} \left(\frac{4n^2+8n+3}{4n^2-1} \right).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left| \frac{4n^2+8n+3}{4n^2-1} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{4 + \frac{8}{n} + \frac{3}{n^2}}{4 - \frac{1}{n^2}} \right| = \frac{1}{2} \cdot \left| \frac{4+0+0}{4-0} \right| = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series converges by the Ratio Test.

6. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(2k+1)2^k}$ is a series of nonzero terms. The ratio of the $n+1^{\text{st}}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(2(n+1)+1)2^{n+1}}}{\frac{1}{(2n+1)2^n}} = \frac{(2n+1)2^n}{(2n+3)2^{n+1}} = \frac{1}{2} \left(\frac{2n+1}{2n+3} \right).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left| \frac{2n+1}{2n+3} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \right| = \frac{1}{2} \cdot \left| \frac{2+0}{2+0} \right| = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series *converges* by the Ratio Test.

7. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^k$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1) \left(\frac{2}{3}\right)^{n+1}}{n \left(\frac{2}{3}\right)^n} = \left(\frac{2}{3}\right) \frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right) \left| \frac{n+1}{n} \right| = \lim_{n \rightarrow \infty} \frac{2}{3} \cdot \left| 1 + \frac{1}{n} \right| = \frac{2}{3} \cdot |1+0| = \frac{2}{3} < 1.$$

Since the limit is less than 1, the series *converges* by the Ratio Test.

8. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5^k}{k^2}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{n+1}}{(n+1)^2}}{\frac{5^n}{n^2}} = \frac{n^2}{(n+1)^2} \cdot \frac{5^{n+1}}{5^n} = 5 \frac{n^2}{(n+1)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} 5 \left| \frac{n^2}{(n+1)^2} \right| = 5 \lim_{n \rightarrow \infty} \left| \frac{1}{\left(1 + \frac{1}{n}\right)^2} \right| = 5 \cdot \left| \frac{1}{(1+0)^2} \right| = 5 > 1.$$

By the Ratio Test, since the limit is greater than 1, the series *diverges*.

9. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{10^k}{(2k)!}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(2(n+1))!}}{\frac{10^n}{(2n)!}} = \frac{10^{n+1}}{10^n} \cdot \frac{(2n)!}{(2n+2)!} = 10 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{10}{(2n+2)(2n+1)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{10}{(2n+2)(2n+1)} \right| = 0.$$

Since the limit is less than 1, the series *converges* by the Ratio Test.

10. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(2k)!}{5^k 3^{k-1}}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1))!}{5^{n+1} 3^{n+1-1}}}{\frac{(2n)!}{5^n 3^{n-1}}} = \frac{5^n}{5^{n+1}} \cdot \frac{3^{n-1}}{3^n} \cdot \frac{(2n+2)!}{(2n)!} = \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \frac{1}{15} (2n+2)(2n+1).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{15} (2n+2)(2n+1) \right| = \infty.$$

Since the limit is ∞ , the series *diverges* by the Ratio Test.

11. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{(2k-2)!}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{(2(n+1)-2)!}}{\frac{n}{(2n-2)!}} = \frac{n+1}{n} \cdot \frac{(2n-2)!}{(2n)!} = \frac{n+1}{n} \cdot \frac{(2n-2)!}{2n(2n-1)(2n-2)!} = \frac{n+1}{n} \cdot \frac{1}{2n(2n-1)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \cdot \frac{1}{2n(2n-1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{2n(2n-1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| 1 + \frac{1}{n} \right| \cdot 0 = |1+0| \cdot 0 = 1 \cdot 0 = 0. \end{aligned}$$

Since the limit is less than 1, the series *converges* by the Ratio Test.

12. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(k+1)!}{3^k}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)!}{3^{n+1}}}{\frac{(n+1)!}{3^n}} = \frac{3^n}{3^{n+1}} \cdot \frac{(n+2)!}{(n+1)!} = \frac{1}{3} \frac{(n+2)(n+1)!}{(n+1)!} = \frac{1}{3}(n+2).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{3}(n+2) \right| = \infty.$$

By the Ratio Test, since the limit is ∞ , the series *diverges*.

13. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{k(k+1)}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)(n+2)}}{\frac{2^n}{n(n+1)}} = \frac{2^{n+1}}{2^n} \cdot \frac{n(n+1)}{(n+1)(n+2)} = 2 \cdot \frac{n}{n+2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2n}{n+2} \right| = \lim_{n \rightarrow \infty} \left| \frac{2}{1 + \frac{2}{n}} \right| = 2 > 1.$$

Since the limit is greater than 1, the series *diverges* by the Ratio Test.

14. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^2(k+1)^2}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^2(n+2)^2}}{\frac{n!}{n^2(n+1)^2}} = \frac{(n+1)!}{n!} \cdot \frac{n^2(n+1)^2}{(n+1)^2(n+2)^2} = \frac{(n+1)n!}{n!} \cdot \frac{n^2}{(n+2)^2} = \frac{(n+1)n^2}{(n+2)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)n^2}{(n+2)^2} \right| = \lim_{n \rightarrow \infty} \left| (n+1) \cdot \frac{1}{\left(1 + \frac{2}{n}\right)^2} \right| \\ &= \infty. \end{aligned}$$

By the Ratio Test, since the limit is ∞ , the series *diverges.*

15. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^3}{k!}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{(n+1)^3}{n^3} = \frac{n!}{(n+1)n!} \cdot \frac{(n+1)^3}{n^3} = \frac{1}{n+1} \cdot \frac{(n+1)^3}{n^3}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \cdot \frac{(n+1)^3}{n^3} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| \cdot \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^3 \right| = |0| \cdot |1| = 0.$$

By the Ratio Test, since the limit is less than 1, the series *converges.*

16. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^{k+1}}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}} = \frac{(n+1)!}{n!} \cdot \frac{n^{n+1}}{(n+1)^{n+2}} = \frac{(n+1)n!}{n!} \cdot \frac{n^{n+1}}{(n+1)^{n+2}} = \left(\frac{n}{n+1} \right)^{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \cdot \left(\frac{n}{n+1} \right)^n \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right| = |1| \cdot \left| \frac{1}{e} \right| = \frac{1}{e} < 1.$$

By the Ratio Test, since the limit is less than 1, the series *converges.*

17. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3^{k-1}}{k \cdot 2^k}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^n}{(n+1) \cdot 2^{n+1}}}{\frac{3^{n-1}}{n \cdot 2^n}} = \frac{3^n}{3^{n-1}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n}{n+1} = 3 \cdot \frac{1}{2} \cdot \frac{n}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3}{2} \cdot \frac{n}{n+1} \right| = \frac{3}{2} \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = \frac{3}{2} \cdot |1| = \frac{3}{2} > 1.$$

By the Ratio Test, since the limit is greater than 1, the series *diverges.*

18. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k(k+2)}{3^k}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)(n+3)}{3^{n+1}}}{\frac{n(n+2)}{3^n}} = \frac{3^n}{3^{n+1}} \cdot \frac{(n+1)(n+3)}{n(n+2)} = \frac{1}{3} \cdot \frac{(n+1)(n+3)}{n(n+2)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{3} \cdot \frac{(n+1)(n+3)}{n(n+2)} \right| = \frac{1}{3} \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right) \left(1 + \frac{3}{n}\right)}{\left(1 + \frac{2}{n}\right)} \right| = \frac{1}{3} \cdot |1| = \frac{1}{3} < 1.$$

By the Ratio Test, since the limit is less than 1, the series *converges.*

19. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{e^k}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{e^n}{e^{n+1}} \cdot \frac{n+1}{n} = \frac{1}{e} \cdot \frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{e} \cdot \frac{n+1}{n} \right| = \frac{1}{e} \lim_{n \rightarrow \infty} \left| 1 + \frac{1}{n} \right| = \frac{1}{e} \cdot |1| = \frac{1}{e} < 1.$$

By the Ratio Test, since the limit is less than 1, the series *converges.*

20. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{e^k}{k^3}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{e^{n+1}}{(n+1)^3}}{\frac{e^n}{n^3}} = \frac{e^{n+1}}{e^n} \cdot \frac{n^3}{(n+1)^3} = e \cdot \frac{n^3}{(n+1)^3}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| e \cdot \frac{n^3}{(n+1)^3} \right| = e \lim_{n \rightarrow \infty} \left| \frac{1}{\left(1 + \frac{1}{n}\right)^3} \right| = e \cdot |1| = e > 1.$$

By the Ratio Test, since the limit is greater than 1, the series *diverges.*

21. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k \cdot 2^k$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1) \cdot 2^{n+1}}{n \cdot 2^n} = 2 \cdot \frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| 2 \cdot \frac{n+1}{n} \right| = 2 \lim_{n \rightarrow \infty} \left| 1 + \frac{1}{n} \right| = 2 \cdot |1| = 2 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series *diverges.*

22. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{4^k}{k}$ is a series of nonzero terms. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{n+1}}{\frac{4^n}{n}} = \frac{4^{n+1}}{4^n} \cdot \frac{n}{n+1} = 4 \cdot \frac{n}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| 4 \cdot \frac{n}{n+1} \right| = 4 \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = 4 \cdot |1| = 4 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series *diverges.*

23. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{2k+1}{5k+1} \right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{2n+1}{5n+1} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{2n+1}{5n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 + \frac{1}{n}}{5 + \frac{1}{n}} \right| = \left| \frac{2}{5} \right| = \frac{2}{5} < 1.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

24. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{3k-1}{2k+1}\right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{3n-1}{2n+1}\right)^n\right|} = \lim_{n \rightarrow \infty} \left|\frac{3n-1}{2n+1}\right| = \lim_{n \rightarrow \infty} \left|\frac{3 - \frac{1}{n}}{2 + \frac{1}{n}}\right| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1.$$

Since the limit is greater than 1, by the Root Test, the series *diverges.*

25. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{k}{5}\right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{n}{5}\right)^n\right|} = \lim_{n \rightarrow \infty} \left|\frac{n}{5}\right| = \infty.$$

Since the limit is greater than 1, by the Root Test, the series *diverges.*

26. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\pi^{2k}}{k^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{\pi^{2n}}{n^n}\right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{\pi^2}{n}\right|^n} = \lim_{n \rightarrow \infty} \left|\frac{\pi^2}{n}\right| = 0.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

27. $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \left(\frac{\ln k}{k}\right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|} = \lim_{n \rightarrow \infty} \left|\frac{\ln n}{n}\right| = \lim_{x \rightarrow \infty} \left|\frac{\ln x}{x}\right| = \lim_{x \rightarrow \infty} \left|\frac{\frac{1}{x}}{1}\right| = \lim_{x \rightarrow \infty} \left|\frac{1}{x}\right| = 0,$$

using L'Hôpital's rule on a related function of the n^{th} root of the n^{th} term of the series. Since the limit is less than 1, by the Root Test, the series *converges.*

28. $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{1}{\ln n}\right)^n\right|} = \lim_{n \rightarrow \infty} \left|\frac{1}{\ln n}\right| = 0.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

29. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{\sqrt{k^2+1}}{3k}\right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{\sqrt{n^2+1}}{3n}\right)^n\right|} = \lim_{n \rightarrow \infty} \left|\frac{\sqrt{n^2+1}}{3n}\right| = \lim_{n \rightarrow \infty} \left|\sqrt{\frac{1}{9} + \frac{1}{9n^2}}\right| = \left|\sqrt{\frac{1}{9}}\right| = \frac{1}{\sqrt{9}} = \frac{1}{3} < 1.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

30. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{\sqrt{4k^2+1}}{k} \right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{\sqrt{4n^2+1}}{n} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{4n^2+1}}{n} \right| = \lim_{n \rightarrow \infty} \left| \sqrt{4 + \frac{1}{n^2}} \right| = |\sqrt{4}| = 2 > 1.$$

Since the limit is greater than 1, by the Root Test, the series *diverges.*

31. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{2^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^2}{2^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{2/n}}{2} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left(n^{1/n} \right)^2 = \frac{1}{2} \left(\lim_{n \rightarrow \infty} n^{1/n} \right)^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2} < 1.$$

(Here, and in the next three problems numbered 32, 33, and 34, and elsewhere below, we use the following result: $\lim_{n \rightarrow \infty} n^{1/n} = 1$. We can derive this as follows. Let $y = n^{1/n}$. Then $\ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$. We have

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} e^{\ln y} = \lim_{n \rightarrow \infty} e^{\frac{\ln n}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\ln n}{n}} = e^{\lim_{x \rightarrow \infty} \frac{\ln x}{x}} = e^{\lim_{x \rightarrow \infty} \frac{1/x}{1}} = e^{\lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1,$$

using L'Hôpital's rule on a related function of $\frac{\ln n}{n}$. This proves the claimed result.)

Since the limit is less than 1, by the Root Test, the series *converges.*

32. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^3}{3^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^3}{3^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{3/n}}{3} \right| = \frac{1}{3} \lim_{n \rightarrow \infty} \left(n^{1/n} \right)^3 = \frac{1}{3} \left(\lim_{n \rightarrow \infty} n^{1/n} \right)^3 = \frac{1}{3} \cdot 1^3 = \frac{1}{3} < 1,$$

where we used the result (see Problem 31) $\lim_{n \rightarrow \infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series *converges.*

33. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^4}{5^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^4}{5^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{4/n}}{5} \right| = \frac{1}{5} \lim_{n \rightarrow \infty} \left(n^{1/n} \right)^4 = \frac{1}{5} \left(\lim_{n \rightarrow \infty} n^{1/n} \right)^4 = \frac{1}{5} \cdot 1^4 = \frac{1}{5} < 1,$$

where we used the result (see Problem 31) $\lim_{n \rightarrow \infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series *converges.*

34. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{3^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n}{3^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{1/n}}{3} \right| = \frac{1}{3} \lim_{n \rightarrow \infty} n^{1/n} = \frac{1}{3} \cdot 1 = \frac{1}{3} < 1,$$

where we used the result (see Problem 31) $\lim_{n \rightarrow \infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series *converges.*

35. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{10}{(3k+1)^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{10}{(3n+1)^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{10^{1/n}}{3n+1} \right| = \lim_{n \rightarrow \infty} 10^{1/n} \cdot \lim_{n \rightarrow \infty} \frac{1}{3n+1} = 10^0 \cdot 0 = 0.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

36. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{k^2}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(1 + \frac{1}{n}\right)^{n^2} \right|} = \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^n \right| = e > 1.$$

Since the limit is greater than 1, by the Root Test, the series *diverges.*

37. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(k+1)(k+2)}{k!}$ is a series of nonzero terms. Since the n^{th} term involves the factorial, we could try the Ratio Test. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)(n+3)}{(n+1)!}}{\frac{(n+1)(n+2)}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{(n+3)}{(n+1)} = \frac{n!}{(n+1)n!} \cdot \frac{(n+3)}{(n+1)} = \frac{(n+3)}{(n+1)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+3)}{(n+1)^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{n \left(1 + \frac{3}{n}\right)}{n^2 \left(1 + \frac{1}{n}\right)^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{3}{n}\right)}{\left(1 + \frac{1}{n}\right)^2} \right| = 0 \cdot 1 = 0.$$

Since the limit is less than 1, by the Ratio Test, the series *converges.*

38. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{(3k+1)!}$ is a series of nonzero terms. Since the n^{th} term involves the factorial, we could try the Ratio Test. The ratio of the $n+1^{st}$ and the n^{th} term is

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)!}{(3(n+1)+1)!}}{\frac{n!}{(3n+1)!}} = \frac{(3n+1)!}{(3n+4)!} \cdot \frac{(n+1)!}{n!} \\ &= \frac{(3n+1)!}{(3n+4)(3n+3)(3n+2)(3n+1)!} \cdot \frac{(n+1)n!}{n!} \\ &= \frac{(n+1)}{(3n+4)(3n+3)(3n+2)}. \end{aligned}$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{(3n+4)(3n+3)(3n+2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{n \left(1 + \frac{1}{n}\right)}{n^3 \left(3 + \frac{4}{n}\right) \left(3 + \frac{3}{n}\right) \left(3 + \frac{2}{n}\right)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{n^2} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right)}{\left(3 + \frac{4}{n}\right) \left(3 + \frac{3}{n}\right) \left(3 + \frac{2}{n}\right)} \right| \\ &= 0 \cdot \frac{1}{27} = 0. \end{aligned}$$

Since the limit is less than 1, by the Ratio Test, the series *converges.*

39. The given series can be written as $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k \ln k}{2^k} = 0 + \sum_{k=2}^{\infty} \frac{k \ln k}{2^k} = \sum_{k=2}^{\infty} \frac{k \ln k}{2^k}$ which is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we could apply the Root Test. Let us first establish a limit result: $\lim_{n \rightarrow \infty} (\ln n)^{1/n} = 1$. To prove this, set $y = (\ln n)^{1/n}$. Then, $\ln y = \frac{1}{n} \ln(\ln n) = \frac{\ln(\ln n)}{n}$. We have

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} e^{\ln y} = \lim_{n \rightarrow \infty} e^{\frac{\ln(\ln n)}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\ln(\ln n)}{n}} = e^{\lim_{x \rightarrow \infty} \frac{\ln(\ln x)}{x}} = e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1}} = e^{\lim_{x \rightarrow \infty} \frac{1}{x \ln x}} = e^0 = 1,$$

using L'Hôpital's rule on a related function of $\frac{\ln(\ln n)}{n}$. This proves the required result. Applying the Root Test, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n \ln n}{2^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{1/n} (\ln n)^{1/n}}{2} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} n^{1/n} \cdot \lim_{n \rightarrow \infty} (\ln n)^{1/n} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} < 1.$$

(Here, we also used the result, proved in Problem 31, that $\lim_{n \rightarrow \infty} n^{1/n} = 1$, in addition to the limit result proved above.) Since the limit is less than 1, by the Root Test, the series converges.

The series $\sum_{k=2}^{\infty} \frac{k \ln k}{2^k}$ which is a series of nonzero terms, can also be seen to converge using the Ratio Test.

The ratio of the $n+1^{\text{st}}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1) \ln(n+1)}{2^{n+1}}}{\frac{n \ln n}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n} = \frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left| 1 + \frac{1}{n} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)}{\ln n} \right| \\ &= \frac{1}{2} \cdot 1 \cdot \lim_{x \rightarrow \infty} \left| \frac{\ln(x+1)}{\ln x} \right| = \frac{1}{2} \lim_{x \rightarrow \infty} \left| \frac{\frac{1}{x+1}}{1} \right| \\ &= \frac{1}{2} \lim_{x \rightarrow \infty} \left| \frac{1}{x+1} \right| = \frac{1}{2} \cdot 0 = 0, \end{aligned}$$

using L'Hôpital's rule on a related function of the last ratio. Since the limit is less than 1, by the Ratio Test, the series converges, in agreement with the Root Test above.

40. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left[\ln \left(e^3 + \frac{1}{k} \right) \right]^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left[\ln \left(e^3 + \frac{1}{n} \right) \right]^n \right|} = \lim_{n \rightarrow \infty} \left| \ln \left(e^3 + \frac{1}{n} \right) \right| \\ &= \ln \left[\lim_{n \rightarrow \infty} e^3 + \lim_{n \rightarrow \infty} \frac{1}{n} \right] \\ &= \ln [e^3 + 0] = 3 > 1. \end{aligned}$$

Since the limit is greater than 1, by the Root Test, the series diverges.

41. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin^k \left(\frac{1}{k} \right)$ is a series of nonzero terms (since $\sin \theta$ is nonvanishing on the interval $0 < \theta \leq 1$ rad). Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \sin^n \left(\frac{1}{n} \right) \right|} = \lim_{n \rightarrow \infty} \left| \sin \left(\frac{1}{n} \right) \right| = \left| \sin \left\{ \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \right\} \right| = |\sin 0| = 0.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

42. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k}{2^{k^2}}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^n}{2^{n^2}} \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^{n/n}}{2^{n^2/n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{2^n} \right| = \lim_{x \rightarrow \infty} \left| \frac{x}{2^x} \right| = \lim_{x \rightarrow \infty} \left| \frac{1}{2^x(\ln 2)} \right| = 0,$$

applying L'Hôpital's rule to a related function of the n^{th} root of the n^{th} term of the series. Since the limit is less than 1, by the Root Test, the series *converges.*

43. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1+\frac{1}{k})^{2k}}{e^k}$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we apply the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{\left(1 + \frac{1}{n}\right)^{2n}}{e^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right)^2}{e} \right| = \frac{1}{e} \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \right]^2 = \frac{1}{e} \cdot 1^2 = \frac{1}{e} < 1.$$

Since the limit is less than 1, by the Root Test, the series *converges.*

44. $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{2^k(k+1)}{k^2(k+2)}$ is a series of nonzero terms. It is not clear which test is the better one. The presence of an n^{th} power in the n^{th} term indicates the Root Test; however the linear factors might be easier to handle with the Ratio Test. Let us try the problem both ways.

Ratio Test: The ratio of the $n+1^{\text{st}}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}(n+2)}{(n+1)^2(n+3)}}{\frac{2^n(n+1)}{n^2(n+2)}} = \frac{2^{n+1}}{2^n} \cdot \frac{n^2(n+2)^2}{(n+1)^3(n+3)} = 2 \cdot \frac{n^2(n+2)^2}{(n+1)^3(n+3)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| 2 \cdot \frac{n^2(n+2)^2}{(n+1)^3(n+3)} \right| = 2 \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{2}{n}\right)^2}{\left(1 + \frac{1}{n}\right)^3 \left(1 + \frac{3}{n}\right)} \right| = 2 \cdot 1 = 2 > 1.$$

Since the limit is greater than 1, the series *diverges* by the Ratio Test.

Root Test: Examine the limit of the absolute value of the n^{th} root of the n^{th} term:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{2^n(n+1)}{n^2(n+2)} \right|} = \lim_{n \rightarrow \infty} \left| \frac{2}{n^{2/n}} \cdot \frac{(n+1)^{1/n}}{(n+2)^{1/n}} \right| \\ &= 2 \lim_{n \rightarrow \infty} \left(\frac{1}{n^{1/n}} \right)^2 \cdot \lim_{n \rightarrow \infty} \left| \frac{n^{1/n} \cdot \left(1 + \frac{1}{n}\right)^{1/n}}{n^{1/n} \cdot \left(1 + \frac{2}{n}\right)^{1/n}} \right| \\ &= 2 \cdot 1^2 \cdot \left| \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{1/n}}{\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^{1/n}} \right| \\ &= 2 \cdot 1^2 \cdot \frac{1}{1} = 2 > 1, \end{aligned} \tag{8.1}$$

where the limit result from Problem 31 has been used to address the first factor's limit (in equation 8.1). Since the limit is greater than 1, the series *diverges* by the Root test, in agreement with the analysis by the Ratio Test.

Applications and Extensions

45. For $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$, the divergent harmonic series (whose divergence can be established using the Integral Test), we have applying the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = \left| \frac{1}{1} \right| = 1.$$

So the Ratio Test is inconclusive, even though we know by other means that the series diverges.

46. For $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, a convergent p -series (since $p = 2 > 1$), we have applying the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(1 + \frac{1}{n})^2} \right| = \left| \frac{1}{1^2} \right| = 1.$$

So the Ratio Test is inconclusive, even though we know by other means that the series converges.

47. Consider the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{|\sin k|}{2^k}$. This series converges, because the n^{th} term satisfies

$$a_n = \frac{|\sin n|}{2^n} \leq \frac{1}{2^n}.$$

Comparing with the convergent geometric series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$ (convergent since $|r| = \frac{1}{2} < 1$), we see by the Comparison Test that the series $\sum_{k=1}^{\infty} a_k$ indeed converges. However, computing the limit of the ratio of the absolute value of the $n+1^{\text{st}}$ and the n^{th} term gives:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{|\sin(n+1)|}{2^{n+1}}}{\frac{|\sin n|}{2^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{\sin(n+1)}{\sin n} \right| \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{\sin n \cos 1 + \cos n \sin 1}{\sin n} \right| \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} |\cos 1 + \cot n \sin 1|, \end{aligned}$$

where the addition formula $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ has been used to simplify the limit. As $n \rightarrow \infty$, the limit of $\cot n$ does not exist, as on any interval $[n, n + \pi]$, $\cot n$ takes values ranging from $-\infty$ to $+\infty$. So, we have an example of a series $\sum_{k=1}^{\infty} a_k$ where $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \neq \infty$ does not exist despite its being a convergent series.

48. A slight modification of the convergent series in Problem 47 produces an example of a divergent series without a well-defined limit. Consider the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} |\sin k|$. That this series is divergent is seen examining the limit of the n^{th} term: $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} |\sin n|$, which does not exist as on any interval $[n, n + 2\pi]$, the function $\sin n$ takes all values from -1 to $+1$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series $\sum_{k=1}^{\infty} a_k$ is divergent as claimed, using the Divergence Test. Computing the ratio of the absolute value of the $n+1^{\text{st}}$ and the n^{th} term gives:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sin(n+1)}{\sin n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sin n \cos 1 + \cos n \sin 1}{\sin n} \right| = \lim_{n \rightarrow \infty} |\cos 1 + \cot n \sin 1|,$$

where the addition formula $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ has been used to simplify the limit. As $n \rightarrow \infty$, the limit of $\cot n$ does not exist, as on any interval $[n, n + \pi]$, $\cot n$ takes values ranging from $-\infty$ to $+\infty$. So, we have an example of a series $\sum_{k=1}^{\infty} a_k$ where $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \neq \infty$ does not exist despite its being a divergent series.

49. (a) To apply the Ratio Test to the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!}$ (note that the series has nonzero terms) we find the ratio of the $n+1^{st}$ and the n^{th} term to be

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-1)^{n+1} 3^{n+1}}{(n+1)!}}{\frac{(-1)^n 3^n}{n!}} = \frac{(-1)^{n+1}(-1)^n}{(-1)^n} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)n!} = (-1) \cdot 3 \cdot \frac{1}{n+1} = -\frac{3}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| -\frac{3}{n+1} \right| = |-3| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 3 \cdot 0 = 0.$$

Since the limit is less than 1, the series converges by the Ratio Test.

(b) Using a CAS, we directly evaluate the sum to be $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!} = \boxed{\frac{1 - e^3}{e^3}}.$

50. The n^{th} term of the series $\frac{1}{3} - \frac{2^3}{3^2} + \frac{3^3}{3^3} - \frac{4^3}{3^4} + \cdots + \frac{(-1)^{n-1} n^3}{3^n} + \cdots$ is $a_n = \frac{(-1)^{n-1} n^3}{3^n}$. The ratio of the $n+1^{st}$ and the n^{th} term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-1)^{n+1-1} (n+1)^3}{3^{n+1}}}{\frac{(-1)^{n-1} n^3}{3^n}} = \frac{(-1)^n}{(-1)^{n-1}} \cdot \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} = (-1) \cdot \frac{1}{3} \cdot \left(\frac{n+1}{n} \right)^3.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)}{3} \left(\frac{n+1}{n} \right)^3 \right| = \frac{1}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^3 = \frac{1}{3} \cdot 1 = \frac{1}{3} < 1.$$

Since the limit is less than 1, by the Ratio Test, we see that the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^3}{3^k}$ converges.

51. Observe that for a fixed positive integer n , $a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \cdots \left(\frac{n}{n} \right) \leq \left(\frac{1}{n} \right) \cdot 1 \cdot 1 \cdots 1 = \frac{1}{n}$, which is convergent since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Also, $0 < \frac{n!}{n^n}$, for all $n \geq 1$. By the Squeeze Theorem, since $\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{n!}{n^n} \leq \lim_{n \rightarrow \infty} \frac{1}{n}$, we have $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$.

52. For $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$, since the terms are nonzero, the Root Test gives

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{1}{n^{1/n}} \right| = 1,$$

using the limit result from Problem 31. Likewise, for $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, since the terms are nonzero, the Root Test gives

$$\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{n^2} \right|} = \lim_{n \rightarrow \infty} \left| \frac{1}{n^{2/n}} \right| = \lim_{n \rightarrow \infty} \left(\frac{1}{n^{1/n}} \right)^2 = \left(\lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} \right)^2 = 1^2 = 1,$$

using, again, the limit result from Problem 31. So we find that the Root Test is inconclusive for the limit value of 1, because it cannot distinguish between the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the convergent p -series (convergent since $p = 2 > 1$) $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

53. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is a series of nonzero terms if $x \neq 0$ (If $x = 0$, then the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} 0 = 0$, so the series converges for $x = 0$.) The ratio of the $n+1^{\text{st}}$ and the n^{th} term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} = \frac{n^2}{(n+1)^2} \cdot \frac{x^{n+1}}{x^n} = \frac{n^2}{(n+1)^2} \cdot x.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \cdot x \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{(1 + \frac{1}{n})^2} \right| = |x| \cdot |1| = |x|.$$

If $|x| < 1$, then the series converges by the Ratio Test, if $|x| > 1$, the series diverges by the Ratio Test, while when $|x| = 1$, the Ratio Test is inconclusive. However, when $|x| = 1$, that is, $x = \pm 1$, the series becomes either the p -series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is known to converge since $p = 2 > 1$, or the alternating series $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ which converges by the Alternating Series Test (since we have the n^{th} term satisfy $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$, and also $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} < 1$ for $n \geq 1$). So finally, the range of values of x for which there will be convergence is $\boxed{|x| \leq 1}$, or, equivalently, $\boxed{-1 \leq x \leq 1}$.

54. For the series $\sum_{k=1}^{\infty} a_k$, with the property that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, let $\left| \frac{a_{n+1}}{a_n} \right| = f(n)$. Then we have

$$|a_{n+1}| = \left| \frac{a_{n+1}}{a_n} \cdot \frac{a_n}{a_{n-1}} \cdots \frac{a_2}{a_1} \cdot a_1 \right| = f(n)f(n-1) \cdots f(1)|a_1|.$$

We examine the limit of the sequence of partial sums $\{S_{n+1}\} = \sum_{k=1}^{n+1} a_k$ of the series $\sum_{k=1}^{\infty} a_k$. We have

$$\lim_{n \rightarrow \infty} |S_{n+1}| = \lim_{n \rightarrow \infty} \left| \sum_{k=1}^{n+1} a_k \right| > \lim_{n \rightarrow \infty} |a_{n+1}| = \lim_{n \rightarrow \infty} f(n)f(n-1) \cdots f(1)|a_1| = \infty,$$

since by hypothesis $\lim_{n \rightarrow \infty} f(n) = \infty$. This implies the sequence of partial sums $\{S_{n+1}\}$ diverges, so the series itself will diverge.

55. To prove the root test, let $\sum_{k=1}^{\infty} a_k$ be a series with nonzero terms. Let $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L \neq \infty$ exist. We consider three cases: $0 \leq L < 1$, $L > 1$ and $L = 1$.

$0 \leq L < 1$: Let $0 \leq L < r < 1$. Let $\epsilon = r - L > 0$. Since $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{1/n} = L < \infty$, there exists some $N > 0$, such that if $n > N$, we have

$$\begin{aligned} \left| |a_n|^{1/n} - L \right| &< \epsilon = r - L \\ L - r &< |a_n|^{1/n} - L < r - L \\ 2L - r &< |a_n|^{1/n} < r, \end{aligned}$$

or $|a_n| < r^n$. Now, $\sum_{k=0}^{\infty} r^k$ is a geometric series and since $0 < r < 1$, it is convergent to a sum of $\frac{1}{1-r}$. Also, since $|a_n| < r^n$ for $n > N$, by the Comparison Test, the series $\sum_{k=N}^{\infty} |a_n|$ is convergent as well. The series $\sum_{k=1}^{\infty} |a_n|$ differs from the series $\sum_{k=N}^{\infty} |a_n|$ by a finite number of terms, namely $\sum_{k=1}^{N-1} |a_n|$. Also, $\sum_{k=1}^{N-1} |a_n| < \infty$, since the a_n are finite for $n < N$. So, since the convergence or divergence property of a series is not affected by the omission or addition of a finite number of terms whose sum is finite, we conclude that the series $\sum_{k=1}^{\infty} |a_k|$ converges absolutely, and so converges.

$L > 1$: Since $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$, we can find some $n \geq N$ such that $|a_n|^{1/n} > 1$. This means $|a_n| > 1^n = 1$ for $n \geq N$, and $\lim_{n \rightarrow \infty} |a_n| \geq 1 \neq 0$, so the series diverges by the Divergence Test. This shows that when $L > 1$, the Root Test indicates that the series for which $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ is a divergent series.

$L = 1$: To show that the Root Test for this case is inconclusive, we exhibit a convergent, divergent, and a conditionally convergent series, all of which produce the limit 1. From Problem 52, we saw that the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the convergent p -series (with $p = 2 > 1$) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ both yielded $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, so the Root Test cannot distinguish between them as far as convergence or divergence is concerned. Consider the conditionally convergent alternating harmonic series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. The terms are nonzero, so the Root Test can be applied. We have $|a_n| = \left| \frac{(-1)^{n+1}}{n} \right| = \frac{1}{n}$ for $n \geq 1$, which brings us back to the same calculation as for the harmonic series in Problem 52. So the Root Test gives the limit of 1 for this conditionally convergent series as well.

56. (a) Let $n = 2m$ be even, where $m \geq 1$ is any positive integer. Then $n + 1 = 2m + 1$ is odd. Since $a_{2k} = \frac{1}{2^k}$, we have $a_n = a_{2m} = \frac{1}{2^m}$. Since $a_{2k-1} = \frac{1}{2^{k+1}}$, we have

$$a_{n+1} = a_{2m+1} = a_{2(m+1)-1} = \frac{1}{2^{(m+1)+1}} = \frac{1}{2^{m+2}}.$$

By the Ratio Test, which is applicable since the terms of the series are nonzero, we have for even n ,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{2m \rightarrow \infty} \left| \frac{a_{2m+1}}{a_{2m}} \right| = \lim_{m \rightarrow \infty} \left| \frac{\frac{1}{2^{m+2}}}{\frac{1}{2^m}} \right| = \lim_{m \rightarrow \infty} \left| \frac{2^m}{2^{m+2}} \right| = \lim_{m \rightarrow \infty} \left| \frac{2^m}{2^m \cdot 2^2} \right| = \frac{1}{4}.$$

Let $n = 2m + 1$ be odd, where $m \geq 0$ is zero or any positive integer. Then $n + 1 = 2m + 2 = 2(m + 1)$ is even. Since $a_{2k-1} = \frac{1}{2^{k+1}}$, we have $a_n = a_{2m+1} = a_{2(m+1)-1} = \frac{1}{2^{(m+1)+1}} = \frac{1}{2^{m+2}}$. Since $a_{2k} = \frac{1}{2^k}$, we have $a_{n+1} = a_{2(m+1)} = \frac{1}{2^{m+1}}$. By the Ratio Test, which is applicable since the terms of the series are nonzero, we have for odd n ,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{2m+1 \rightarrow \infty} \left| \frac{a_{2(m+1)}}{a_{2m+1}} \right| = \lim_{m \rightarrow \infty} \left| \frac{\frac{1}{2^{m+1}}}{\frac{1}{2^{m+2}}} \right| = \lim_{m \rightarrow \infty} \left| \frac{2^{m+2}}{2^{m+1}} \right| = \lim_{m \rightarrow \infty} \left| \frac{2^m \cdot 2^2}{2^m \cdot 2} \right| = 2.$$

We see that the Ratio Test produces two different limits for the case of n even or n odd. In other words, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \neq \infty$ does not exist, so the Ratio Test is not conclusive as to whether the series converges or diverges.

(b) We consider the cases of n even and n odd with the root test. If n is even, then $n = 2m$ for any positive integer m , and as in part (a), $a_n = a_{2m} = \frac{1}{2^m}$. Applying the Root Test, which is applicable since the terms of the series are nonzero, we have for even n

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{2m \rightarrow \infty} \sqrt[2m]{|a_{2m}|} = \lim_{m \rightarrow \infty} \left(\frac{1}{2^m} \right)^{\frac{1}{2m}} = \left(\frac{1}{2} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

If n is odd, then $n = 2m + 1$ for any positive integer (or zero) m , and as in part (a), $a_n = a_{2m+1} = \frac{1}{2^{m+2}}$. Applying the Root Test, which is applicable since the terms of the series are nonzero, we have for odd n ,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{2m+1 \rightarrow \infty} \sqrt[2m+1]{|a_{2m+1}|} = \lim_{m \rightarrow \infty} \left(\frac{1}{2^{m+2}} \right)^{\frac{1}{(2m+1)}} = \lim_{m \rightarrow \infty} \frac{1}{2^{\frac{m+2}{2m+1}}} = \lim_{m \rightarrow \infty} 2^{-\frac{1+\frac{2}{m}}{2+\frac{1}{m}}} = 2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Since the limits are the same for both even and odd n , and since the even and the odd cases together comprise the entire series, we see that the Root Test gives us a limit which exists.

(c) Since the limit found by applying the Root Test in part (b) is less than 1, by the Root Test, the series in question converges.

Challenge Problems

57. We have

$$\sum_{k=1}^{\infty} a_k = 1 + \frac{2}{2^2} + \frac{3}{3^3} + \frac{1}{4^4} + \frac{2}{5^5} + \frac{3}{6^6} + \cdots < \sum_{k=1}^{\infty} b_k = 1 + \frac{2}{2^2} + \frac{3}{3^3} + \frac{4}{4^4} + \frac{5}{5^5} + \frac{6}{6^6} + \cdots + \frac{n}{n^n} + \cdots.$$

The n^{th} term of the series to the right of the inequality is $b_n = \frac{n}{n^n}$. By the Root Test, which applies since the terms of that series are nonzero, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n}{n^n} \right|} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{n} = \lim_{n \rightarrow \infty} n^{1/n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 1 \cdot 0 = 0,$$

where we have used the limit result established in Problem 31. Since the limit is less than 1, by the Root Test, the series $\sum_{k=1}^{\infty} b_k$ converges, and then by the Comparison Test, we see that the original series $\sum_{k=1}^{\infty} a_k$ converges as well.

58. We have

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k+2)!} = \sum_{k=1}^{\infty} \frac{(k+1)(k+1)}{(k+2)(k+1)(k!)} = \sum_{k=1}^{\infty} \left(\frac{k+1}{k+2} \right) \left(\frac{1}{k!} \right) < \sum_{k=1}^{\infty} \frac{1}{k!},$$

since $\frac{k+1}{k+2} < 1$ for $k \geq 1$. Now, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k!}$ is convergent since by the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Since the limit is less than 1, by the Ratio Test, the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k!}$ converges, and by the

Comparison Test, so does the series $\sum_{k=1}^{\infty} a_k$.

Alternately, using the suggested hint, the Limit Comparison test may be used. The n^{th} term of the original series behaves like

$$a_n = \frac{(n+1)^2}{(n+2)!} = \frac{(n+1)(n+1)}{(n+2)(n+1)(n!)} = \frac{n+1}{n+2} \cdot \frac{1}{n!} \approx \frac{1}{n!} = b_n$$

for large values of n . Comparing the series of positive terms $\sum_{k=1}^{\infty} a_k$ with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k!}$, which is convergent as noted earlier, and also has positive terms, we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{n+2} \cdot \frac{1}{n!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = \frac{1+0}{1+0} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the original series $\sum_{k=1}^{\infty} a_k$ converges.

59. The n^{th} term of the series $\sum_{k=1}^{\infty} c_k = a + b + a^2 + b^2 + a^3 + b^3 + \cdots$ is $c_n = a^n + b^n$. Since $0 < a < b$, the terms of the series are all nonzero, and the Root Test may be applied. We have

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} &= \sqrt[n]{a^n + b^n} = b \sqrt[n]{1 + \left(\frac{a^n}{b^n}\right)} = b \lim_{n \rightarrow \infty} \left[1 + \left(\frac{a}{b}\right)^n\right]^{1/n} \\ &= b \left[1 + \lim_{n \rightarrow \infty} \left(\frac{a}{b}\right)^n\right]^{\lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= b(1 + 0)^0 = b < 1,\end{aligned}$$

where, in order to obtain $\lim_{n \rightarrow \infty} \left(\frac{a}{b}\right)^n = 0$, we used the fact that $a < b$, so that $\frac{a}{b} < 1$, and to obtain the final inequality we used $b < 1$. Since the limit is less than 1, by the Root Test, the series converges.

60. Since the Ratio Test indicates the series $\sum_{k=1}^{\infty} a_k$ converges, it means $\lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. This means for any $\epsilon > 0$ there exists an N such that if $n \geq N$, then $\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon$, that is, $L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon$. We have

$$|a_n| = \frac{|a_n|}{|a_{n-1}|} \cdot \frac{|a_{n-1}|}{|a_{n-2}|} \cdots \frac{|a_{N+1}|}{|a_N|} \cdot |a_N|.$$

Applying the inequality above to each of the fractional terms, we get

$$\begin{aligned}(L - \epsilon)^{n-N} \cdot |a_N| &< |a_n| < (L + \epsilon)^{n-N} \cdot |a_N| \\ \text{or, taking the } n^{\text{th}} \text{ root, } & (L - \epsilon)^{1 - \frac{N}{n}} \cdot |a_N|^{\frac{1}{n}} < \sqrt[n]{|a_n|} < (L + \epsilon)^{1 - \frac{N}{n}} |a_N|^{\frac{1}{n}}.\end{aligned}$$

Taking the limit as $n \rightarrow \infty$ on this string of inequalities, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} (L - \epsilon)^{1 - \frac{N}{n}} \cdot \lim_{n \rightarrow \infty} |a_N|^{\frac{1}{n}} &\leq \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \lim_{n \rightarrow \infty} (L + \epsilon)^{1 - \frac{N}{n}} \cdot \lim_{n \rightarrow \infty} |a_N|^{\frac{1}{n}} \\ (L - \epsilon)^{\lim_{n \rightarrow \infty} (1 - \frac{N}{n})} \cdot |a_N|^{\lim_{n \rightarrow \infty} \frac{1}{n}} &\leq \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq (L + \epsilon)^{\lim_{n \rightarrow \infty} (1 - \frac{N}{n})} \cdot |a_N|^{\lim_{n \rightarrow \infty} \frac{1}{n}} \\ (L - \epsilon)^1 |a_N|^0 &\leq \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq (L + \epsilon)^1 |a_N|^0 \\ L - \epsilon &\leq \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq L + \epsilon.\end{aligned}$$

Since this is valid for any $\epsilon > 0$, in the limit, as $\epsilon \rightarrow 0^+$, since $\lim_{\epsilon \rightarrow 0^+} (L - \epsilon) = L$ and $\lim_{\epsilon \rightarrow 0^+} (L + \epsilon) = L$, by the Squeeze Theorem, we have $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ since $L < 1$ by assumption. So by the Root Test, the series converges.

We have shown that if the Ratio Test indicates a series converges, then so will the Root Test. That the converse is not true can be seen from Problem 56, where a series is seen to converge by the Root Test, but the Ratio Test is inconclusive.

8.7 Summary of Tests

Concepts and Vocabulary

- False:** The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ but diverges if $p = 1$ (it then becomes the divergent harmonic series).
- False:** According to the Test for Divergence, $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n \rightarrow \infty} a_n \neq 0$. The test says nothing if $\lim_{n \rightarrow \infty} a_n = 0$. For instance, the convergent p -series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ (convergent because $p = 2 > 1$), and the divergent harmonic series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ both satisfy $\lim_{n \rightarrow \infty} a_n = 0$.
- True:** This is the statement of the theorem on Absolute Convergence Test, see Chapter 8, Section 8.5, p.586.
- False:** For a counterexample, consider the alternating harmonic series, $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. Then the absolute value series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent as it is the harmonic series. However, since $\lim_{n \rightarrow \infty} a_n = 0$ and by the Algebraic Ratio test $\frac{a_{n+1}}{a_n} < 1$ for $n \geq 1$ means the terms of the series are nonincreasing, by the Alternating Series Test, we see that the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges. (This is called conditional convergence.)
- False:** For deciding convergence via the Ratio Test, it is not important that $\left| \frac{a_{n+1}}{a_n} \right| < 1$; we need to require that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. For example the divergent harmonic series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ satisfies $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n}{n+1} \right| < 1$ for $n \geq 1$, while $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{1+\frac{1}{n}} \right| = 1$ means that the Ratio Test is in fact inconclusive in regards to the harmonic series' convergence or divergence (rather the Integral Test can be used to show the harmonic series diverges).
- $0 < a_k \leq b_k$: This follows from the statement of the Comparison Test for Convergence, see Section 8.4, p. 576.

Skill Building

- The series is $\sum_{k=1}^{\infty} \frac{9k^3 + 5k^2}{k^{5/2} + 4}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{9n^3 + 5n^2}{n^{5/2} + 4} = \lim_{n \rightarrow \infty} \frac{n^3 \left(9 + \frac{5}{n}\right)}{n^{5/2} \left(1 + \frac{4}{n^{5/2}}\right)} = \lim_{n \rightarrow \infty} \left[n^{1/2} \cdot \frac{9 + \frac{5}{n}}{1 + \frac{4}{n^{5/2}}} \right] = \infty.$$

Since the limit of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

- The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+1}}$. The n^{th} term behaves like

$$a_n = \frac{1}{\sqrt{2n+1}} = \frac{1}{n^{1/2} \sqrt{2 + \frac{1}{n^{1/2}}}} \approx \frac{1}{n^{1/2}}$$

for large values of n . So we compare with the divergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$, which is divergent since $0 < p = \frac{1}{2} < 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^{1/2} \sqrt{2 + \frac{1}{n^{1/2}}}}}{\frac{1}{n^{1/2}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 + \frac{1}{n^{1/2}}}} = \frac{1}{\sqrt{2}}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series of absolute values $\sum_{k=1}^{\infty} a_k$ diverges.

Let's investigate whether the original series converges. We have $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n+1}} = 0$. Also by the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{2(n+1)+1}}}{\frac{1}{\sqrt{2n+1}}} = \frac{\sqrt{2n+1}}{\sqrt{2n+3}} < 1$$

for all $n \geq 1$. So the terms of the series are nonincreasing as well. By the Alternating Series Test, the original series converges. But since the series of absolute values diverges, the original series is

conditionally convergent.

9. The series is $6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = 6 + 2 + \sum_{k=1}^{\infty} \frac{2}{3^k} = 8 + \sum_{k=1}^{\infty} \frac{2}{3} \cdot \left(\frac{1}{3}\right)^{k-1}$. The latter term is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{2}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, it converges to a sum of $\frac{a}{1-r} = \frac{2/3}{1-1/3} = \frac{2/3}{2/3} = 1$. The original series converges to a sum of $8 + 1 = 9$, which is finite, so the series is convergent. Since all terms of the original series are positive, the series is also

absolutely convergent.

10. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{\pi}{k}$. The series of absolute values is $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sin \frac{\pi}{k} \right|$. We have

$$|a_n| = \frac{1}{n^2} \left| \sin \frac{\pi}{n} \right| \leq \frac{1}{n^2} = b_n$$

since $|\sin \theta| \leq 1$ for $0 < \theta = \frac{\pi}{n} \leq \pi$. By comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p -series (since $p = 2 > 1$), the series of absolute values converges by the Comparison Test, which means the original series

converges absolutely.

11. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k+1}{k^3+1}$. The n^{th} term behaves like

$$a_n = \frac{3n+2}{n^3+1} = \frac{n(3 + \frac{2}{n})}{n^3(1 + \frac{1}{n^3})} \approx \frac{1}{n^2}$$

for large values of n . We compare with the convergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent since $p = 2 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{3n+2}{n^3+1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{3n^3 + 2n^2}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{1 + \frac{1}{n^3}} = 3.$$

Since the limit is a positive real number, the series is convergent by the Limit Comparison Test. And since all the terms of the series are positive, it is also

absolutely convergent.

12. The series is $1 + \frac{2^2+1}{2^3+1} + \frac{3^2+1}{3^3+1} + \frac{4^2+1}{4^3+1} + \cdots = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2+1}{k^3+1}$. The n^{th} term behaves like

$$a_n = \frac{n^2+1}{n^3+1} = \frac{n^2 \left(1 + \frac{1}{n^2}\right)}{n^3 \left(1 + \frac{1}{n^3}\right)} \approx \frac{1}{n}$$

for large values of n . We compare with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^3+n}{n^3+1} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n^2}}{1 + \frac{1}{n^3}} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series *diverges.*

13. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+4}{k\sqrt{3k-2}}$. The n^{th} term behaves like

$$a_n = \frac{n+4}{n\sqrt{3n-2}} = \frac{n \left(1 + \frac{4}{n}\right)}{n^{3/2} \sqrt{3 - \frac{2}{n}}} \approx \frac{1}{n^{1/2}}$$

for large values of n . We compare with the divergent p -series (divergent since $0 < p = \frac{1}{2} < 1$)

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+4}{n\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n+4}{\sqrt{n}\sqrt{3n-2}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{4}{n}}{\sqrt{3 - \frac{2}{n}}} = \frac{1}{\sqrt{3}}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series *diverges.*

14. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^3}$. The series of absolute values is $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{|\sin k|}{k^3}$. We have

$$|a_n| = \frac{|\sin n|}{n^3} \leq \frac{1}{n^3} = b_n,$$

since $|\sin n| \leq 1$ for $n \geq 1$. Comparing with the convergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$, which is convergent since $p = 3 > 1$, we see by the Comparison Test that the series of absolute values converges. This means the original series is *absolutely convergent.*

15. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3^{2k-1}}{k^2+2k}$ is a series of nonzero terms. We apply the Ratio Test. The ratio of the $n+1^{\text{st}}$ and the n^{th} term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{2(n+1)-1}}{(n+1)^2+2(n+1)}}{\frac{3^{2n-1}}{n^2+2n}} = \frac{3^{2n+1}}{3^{2n-1}} \cdot \frac{n^2+2n}{(n^2+2n+1)+2n+1} = \frac{9n^2+18n}{n^2+4n+2}.$$

The limit of the absolute value of the ratio found above is

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{9n^2+18n}{n^2+4n+2} = \lim_{n \rightarrow \infty} \frac{9 + \frac{18}{n}}{1 + \frac{4}{n} + \frac{2}{n^2}} = 9 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series *diverges.*

16. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5^k}{k!}$ is a series of nonzero terms. By the Ratio Test, we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!} \right| = 5 \lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)n!} \right| = 0.$$

Since the limit is less than 1, by the Ratio Test, the series converges. Since all the terms are positive, the series is also *absolutely convergent*.

17. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(1 + \frac{2}{k}\right)^k$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = \lim_{n' \rightarrow \infty} \left(1 + \frac{1}{n'}\right)^{2n'} = \lim_{n' \rightarrow \infty} \left[\left(1 + \frac{1}{n'}\right)^{n'} \right]^2 = \left[\lim_{n' \rightarrow \infty} \left(1 + \frac{1}{n'}\right)^{n'} \right]^2 = e^2 \neq 0$$

where $n' = \frac{n}{2}$. Since the limit of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

18. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2+4}{e^k}$ is a series of nonzero terms. By the Ratio Test, we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2+4}{e^{n+1}}}{\frac{n^2+4}{e^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^n}{e^{n+1}} \cdot \frac{(n+1)^2+4}{n^2+4} \right| = \frac{1}{e} \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right)^2 + \frac{4}{n^2}}{1 + \frac{4}{n^2}} \right| = \frac{1}{e} \cdot |1| < 1.$$

Since the limit is less than 1, by the Ratio Test, the series converges. Also, since the terms of the series are all positive, this means the series is *absolutely convergent*.

19. The series is $\frac{2}{3} - \frac{3}{4} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{3} - \frac{5}{6} \cdot \frac{1}{4} + \dots = \sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k+2} \cdot \frac{1}{k}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+1}{k+2} \cdot \frac{1}{k}$. Comparing with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$, we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{n+2} \cdot \frac{1}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the absolute value series diverges.

Next, we examine the original series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ for convergence. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 1 \cdot 0 = 0.$$

By the Algebraic Ratio Test, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{n+3} \cdot \frac{1}{n+1}}{\frac{n+1}{n+2} \cdot \frac{1}{n}} = \frac{n(n+2)^2}{(n+1)^2(n+3)} < 1$$

provided the following holds:

$$\begin{aligned} n(n+2)^2 &< (n+1)^2(n+3) \\ n(n^2+4n+4) &< (n^2+2n+1)(n+3) \\ n^3+4n^2+4n &< n^3+2n^2+n+3n^2+6n+3 \\ 4n^2+4n &< 5n^2+7n+3 \end{aligned}$$

$$\text{that is, } n^2+3n+3 > 0,$$

which is the case for $n \geq 1$. So the terms of the series are nonincreasing. By the Alternating series Test, the original series converges. Since the series of absolute values diverges, the original series is

conditionally convergent.

20. The series is $2 + \frac{3}{2} \cdot \frac{1}{4} + \frac{4}{3} \cdot \frac{1}{4^2} + \frac{5}{4} \cdot \frac{1}{4^3} + \cdots = \sum_{k=1}^{\infty} \frac{k+1}{k} \cdot \frac{1}{4^{k-1}} = \sum_{k=1}^{\infty} \frac{k+1}{k 4^{k-1}} = \sum_{k=1}^{\infty} a_k$. Since the n^{th} term behaves like

$$a_n = \frac{n+1}{n 4^{n-1}} = \frac{1}{4^{n-1}} \left(1 + \frac{1}{n}\right) \approx \frac{1}{4^{n-1}}$$

for large n , we compare with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{3}{4^{k-1}} = \sum_{k=1}^{\infty} 3 \left(\frac{1}{4}\right)^{k-1}$ which is a convergent geometric series since $|r| = \frac{1}{4} < 1$. Now, we have

$$a_n = \frac{n+1}{n} \left(\frac{1}{4^{n-1}}\right) = \frac{n+1}{3n} \left(\frac{3}{4^{n-1}}\right) = \frac{n+1}{3n} b_n < \frac{n+n}{3n} b_n = \frac{2}{3} b_n < b_n$$

for all $n \geq 1$. So by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ converges. The terms of the series are positive, so this means that the series also *converges absolutely.*

21. The series is $1 + \frac{1 \cdot 3}{2!} + \frac{1 \cdot 3 \cdot 5}{3!} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} + \cdots$. Note that

$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{(2 \cdot 4 \cdot 6 \cdot 8)4!} = \frac{8!}{2^4 [1 \cdot 2 \cdot 3 \cdot 4]4!} = \frac{8!}{2^4 \cdot (4!)(4!)},$$

so the n^{th} term of the series can be written as $a_n = \frac{(2n)!}{2^n \cdot (n!)(n!)}$. By the Ratio Test, which is indicated since factorials are involved, and the terms of the series are nonzero, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(2(n+1))!}{2^{n+1} \cdot (n+1)!(n+1)!}}{\frac{(2n)!}{2^n \cdot (n!)(n!)}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n}{2^{n+1}} \cdot \frac{n!}{(n+1)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{(2(n+1))!}{(2n)!} \right| \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)n!} \cdot \frac{n!}{(n+1)n!} \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \right| \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{\left(2 + \frac{2}{n}\right) \left(2 + \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n}\right)} \right| \\ &= \frac{1}{2} \cdot 4 = 2 > 1. \end{aligned}$$

Since the limit is greater than 1, by the Ratio Test, the series $\sum_{k=1}^{\infty} a_k$ *diverges.*

22. The series is $\frac{1}{\sqrt{1 \cdot 2 \cdot 3}} + \frac{1}{\sqrt{2 \cdot 3 \cdot 4}} + \frac{1}{\sqrt{3 \cdot 4 \cdot 5}} + \cdots = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)(k+2)}}$. The n^{th} term satisfies

$$a_n = \frac{1}{\sqrt{n(n+1)(n+2)}} = \frac{1}{n^{3/2} \sqrt{\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}} < \frac{1}{n^{3/2}} = b_n$$

for $n \geq 1$. By comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, which is a convergent p -series (since $p = \frac{3}{2} > 1$),

by the Comparison Test, we see that the original series $\sum_{k=1}^{\infty} a_k$ converges. Since the terms of the series

$\sum_{k=1}^{\infty} a_k$ are all positive, this series is also *absolutely convergent.*

23. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{(2k)!}$ is a series of nonzero terms and has factorials, so the use of the Ratio Test is indicated. The ratio of the $n+1^{st}$ and n^{th} term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(2(n+1))!}}{\frac{n!}{(2n)!}} = \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2(n+1))!} = \frac{(n+1)n!}{n!} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{(n+1)}{(2n+2)(2n+1)}.$$

Taking the limit of the absolute value of the ratio found above gives

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{(2n+2)(2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{1}{2n+2} \cdot \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2 + \frac{1}{n}} = 0 \cdot \frac{1}{2} = 0.$$

Since the limit is less than 1, by the Ratio Test, the series is convergent. Additionally, the terms of the series are all positive, so this means that the series is also *absolutely convergent*.

24. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k^3 e^{-k^4}$. Now the function $f(x) = x^3 e^{-x^4}$ is defined on $[1, \infty)$, is continuous, and decreasing for all $x \geq 1$, since

$$f'(x) = 3x^2 e^{-x^4} + x^3(-4x^3)e^{-x^4} = (3x^2 - 4x^6)e^{-x^4} < 0$$

when $3x^2 - 4x^6 < 0$, or $3 < 4x^4$, or $x > \sqrt[4]{\frac{3}{4}}$, but $\sqrt[4]{\frac{3}{4}} < 1$. So, for $x \geq 1$, the series decreases, as claimed. Also, $f(k) = a_k$ for all integers $k \geq 1$. By the Integral Test, we have

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b x^3 e^{-x^4} dx.$$

Let $u = e^{-x^4}$. Then $du = -4x^3 e^{-x^4} dx$. Continuing, we have

$$I = -\frac{1}{4} \lim_{b \rightarrow \infty} \int_{e^{-1}}^{e^{-b^4}} du = -\frac{1}{4} \lim_{b \rightarrow \infty} [u]_{e^{-1}}^{e^{-b^4}} = -\frac{1}{4} \lim_{b \rightarrow \infty} (e^{-b^4} - e^{-1}) = -\frac{1}{4} (0 - e^{-1}) = \frac{1}{4e}.$$

Since the improper integral I converges, by the Integral Test, the series converges as well. Since the terms of the series are all positive, the series is *absolutely convergent*.

25. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+100}}$. Comparing with the divergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which diverges since $0 < p = \frac{1}{2} < 1$, we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n+100}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+100}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{100}{\sqrt{n}}} = 1.$$

Since the limit is a positive real number, the series *diverges* by the Limit Comparison Test.

26. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2+5k}{3+5k^2}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2+5n}{3+5n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n}}{\frac{3}{n^2} + 5} = \frac{1}{5} \neq 0.$$

Since the limit of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

27. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^4+4}}$. We have

$$a_n = \frac{1}{\sqrt[3]{n^4+4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{4/3}} = b_n.$$

Comparing the series with the convergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$, which is convergent since $p = \frac{4}{3} > 1$, we see that the original series $\sum_{k=1}^{\infty} a_k$ is convergent by the Comparison Test. Further, since all the terms of $\sum_{k=1}^{\infty} a_k$ are positive, the series is *absolutely convergent*.

28. The series can be written as $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{11} \left(-\frac{3}{2}\right)^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{11} \left(\frac{3}{2}\right)^k$. We have

$$\lim_{n \rightarrow \infty} |a_n| = \frac{1}{11} \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n = \infty \neq 0,$$

using the result stated in Section 8.1, p. 546, and proved in Section 8.1, Problem 134, that $\lim_{n \rightarrow \infty} r^n = \infty$ if $|r| > 1$. Since the limit of the absolute value of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

29. The series is $\frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k+2}$. We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{n+2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = 1 \neq 0.$$

Since the limit of the absolute value of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

30. The series can be written

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k(-4)^{3k}}{5^k} = \sum_{k=1}^{\infty} (-1)^{3k} \frac{k(4)^{3k}}{5^k} = \sum_{k=1}^{\infty} [(-1)^3]^k \frac{k(4^3)^k}{5^k} = \sum_{k=1}^{\infty} (-1)^k k \left(\frac{64}{5}\right)^k.$$

We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n \cdot \left(\frac{64}{5}\right)^n = \lim_{n \rightarrow \infty} n \cdot \lim_{n \rightarrow \infty} \left(\frac{64}{5}\right)^n = \infty,$$

since both limits are separately ∞ (for the second limit, we use the result quoted in Problem 28 above). Since the limit of the absolute value of the n^{th} term is nonzero, the series *diverges* by the Divergence Test.

31. The series is $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} \left(-\frac{1}{k}\right)^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^k}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^k}$, the “ k to the k series”. You could use the Table 6, p.598 to assert convergence. However, that this is convergent can be directly deduced using the root test (note that the terms of the series of absolute values are nonzero, and the n^{th} term has an n^{th} power, so the Root Test is applicable) :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n \rightarrow \infty} \left|\frac{1}{n}\right| = 0 < 1.$$

Since the limit is less than 1, by the Root Test, the series of absolute values converges, which means the original series is *absolutely convergent*.

32. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{2^k+1}$. We have

$$a_n = \frac{5}{2^n+1} < \frac{5}{2^n} = b_n.$$

Comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{5}{2^k} = \sum_{k=1}^{\infty} \frac{5}{2} \left(\frac{1}{2}\right)^{k-1}$, which is a convergent geometric series (since $|r| = \frac{1}{2} < 1$), we see that the original series converges by the Comparison Test. Since the terms of the original series are all positive, we conclude that series must be *absolutely convergent.*

33. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} e^{-k^2}$ is a series of nonzero terms. Since the n^{th} term has an n^{th} power, the Root Test would be ideal to use:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(|e^{-n^2}| \right)^{1/n} = \lim_{n \rightarrow \infty} |e^{-n}| = 0 < 1.$$

Since the limit is less than 1, by the Root Test, the series converges. Since its terms are all positive, the series also *converges absolutely.*

34. The series is $\frac{\sin \sqrt{1}}{1^{3/2}} + \frac{\sin \sqrt{2}}{2^{3/2}} + \frac{\sin \sqrt{3}}{3^{3/2}} + \cdots = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin \sqrt{k}}{k^{3/2}}$. The series of absolute values is

$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{|\sin \sqrt{k}|}{k^{3/2}}$. We have

$$|a_n| = \frac{|\sin \sqrt{n}|}{n^{3/2}} \leq \frac{1}{n^{3/2}} = b_n,$$

since $|\sin \sqrt{n}| \leq 1$ for all $n \geq 1$. Comparing with the convergent (since $p = \frac{3}{2} > 1$) p -series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, by the Comparison Test, we see the series of absolute values is convergent, which means that the original series must be *absolutely convergent.*

35. The series is $\sum_{k=2}^{\infty} (-1)^{k+1} a_k = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k(\ln k)^3}$. The series of absolute values is $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$.

Consider the function $f(x) = \frac{1}{x(\ln x)^3}$ which is defined, decreasing and continuous on $[2, \infty)$, and for which $a_k = f(k)$ for all $k \geq 2$. Applying the Integral Test, we have

$$I = \int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^3}.$$

Let $u = \ln x$. Then $du = \frac{dx}{x}$. Continuing,

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^3} = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} u^{-3} du = \lim_{b \rightarrow \infty} \left[\frac{u^{-3+1}}{-3+1} \right]_{\ln 2}^{\ln b} \\ &= -\frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{u^2} \right]_{\ln 2}^{\ln b} = -\frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2} \right] \\ &= -\frac{1}{2} \left[0 - \frac{1}{(\ln 2)^2} \right] = \frac{1}{2(\ln 2)^2} \end{aligned}$$

Since the improper integral I converges, by the Integral Test, the series also converges. Moreover, the terms of the series are all positive, so that the original series is *absolutely convergent.*

36. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{(2k)^k}$ is a series of nonzero terms. Since its n^{th} term involves an n^{th} power, the Root Test is the best choice:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{(2n)^n} \right|} = \lim_{n \rightarrow \infty} \left| \frac{1}{2n} \right| = 0.$$

Since the limit is less than 1, by the Root Test, the series converges. The terms of the series are all positive, so this series also is *absolutely convergent*.

37. The series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \left(\frac{\ln k}{1000} \right)^k$ is a series of nonzero terms. Since the n^{th} term involves an n^{th} power, we use the Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{\ln n}{1000} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{\ln n}{1000} \right| = \infty.$$

By the Root Test, since the limit is greater than 1, the series *diverges*.

38. The series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\cosh^2 k} = \sum_{k=1}^{\infty} \frac{1}{\left(\frac{e^k + e^{-k}}{2} \right)^2} = \sum_{k=1}^{\infty} \frac{4}{e^{2k} + e^{-2k} + 2}.$$

Here the standard definition $\cosh k = \frac{e^k + e^{-k}}{2}$ was used. We have

$$a_n = \frac{4}{e^{2n} + e^{-2n} + 2} < \frac{4}{e^{2n}} = b_n$$

since $e^{-2n} + 2 > 0$ for $n \geq 1$. Now, we can write the series $\sum_{k=1}^{\infty} b_k$ as follows:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{4}{e^{2k}} = \sum_{k=1}^{\infty} \frac{4}{(e^2)^k} = \sum_{k=1}^{\infty} \frac{4}{e^2} \cdot \left(\frac{1}{e^2} \right)^{k-1}.$$

We see that it is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{e^2}$ and $|r| = \frac{1}{e^2} < 1$ so it converges. By the Comparison Test, the original series also converges. Since the terms of the original series are positive, it is *absolutely convergent*.

39. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$. The n^{th} term satisfies

$$a_n = \frac{\tan^{-1} n}{n^2} < \frac{\pi}{2n^2} = b_n$$

since $\tan^{-1} n < \frac{\pi}{2}$ for any $n \geq 1$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{\pi}{2k^2}$ is a constant multiple of a convergent p -series (convergent since $p = 2 > 1$), the original series also converges by the Comparison Test. Since $\tan^{-1} n > 0$ for all $n \geq 1$, the terms of the original series are all positive, which in turn means that the original series is also *absolutely convergent*.

40. The n^{th} partial sum of the series $\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ is

$$\begin{aligned} S_n &= \sum_{k=1}^n (\sqrt{k+1} - \sqrt{k}) \\ &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n} - \sqrt{n-1}) + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - \sqrt{1}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} S_n = \infty$, the sequence $\{S_n\}$ of partial sums diverges, which means that the series diverges as well.

41. The n^{th} partial sum of the series $\sum_{k=4}^{\infty} \left(\frac{1}{k-3} - \frac{1}{k} \right)$ is

$$\begin{aligned} S_n &= \sum_{k=4}^n \left(\frac{1}{k-3} - \frac{1}{k} \right) \\ &= \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \\ &\quad + \cdots + \left(\frac{1}{n-7} - \frac{1}{n-4} \right) + \left(\frac{1}{n-6} - \frac{1}{n-3} \right) + \left(\frac{1}{n-5} - \frac{1}{n-2} \right) + \\ &\quad + \left(\frac{1}{n-4} - \frac{1}{n-1} \right) + \left(\frac{1}{n-3} - \frac{1}{n} \right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-2} - \frac{1}{n-1} - \frac{1}{n}. \end{aligned}$$

The limit of the sequence $\{S_n\}$ is:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-2} - \frac{1}{n-1} - \frac{1}{n} \right) = 1 + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}.$$

Since the sequence of partial sums $\{S_n\}$ converges, the series converges as well, and the limit of the sequence of partial sums is also the sum of the series, which is $\frac{11}{6}$.

42. The series is $\sum_{k=2}^{\infty} \ln \left(\frac{k}{k+1} \right) = \sum_{k=2}^{\infty} (\ln k - \ln(k+1))$. The n^{th} partial sum of the series is

$$\begin{aligned} S_n &= \sum_{k=2}^n (\ln k - \ln(k+1)) \\ &= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) \\ &= \ln 1 - \ln(n+1) = -\ln(n+1). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} S_n = -\infty$, the sequence of partial sums $\{S_n\}$ diverges, which in turn means that the series diverges.

43. The series is

$$1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots = \sum_{k=1}^{\infty} \frac{2^k \cdot (k!)(k!)}{(2k)!},$$

where we use the fact that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1)} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2k)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k)} = \frac{2^k \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots k]}{(2k)!} = \frac{2^k \cdot (k!)}{(2k)!}.$$

Since the terms of the series are nonzero, and factorials are involved in the n^{th} term, the Ratio Test suggests itself to be used. The ratio of the $n+1^{\text{st}}$ and the n^{th} term of the series is

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{2^{n+1} \cdot (n+1)!(n+1)!}{(2(n+1))!}}{\frac{2^n \cdot (n!)(n!)}{(2n)!}} = \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2(n+1))!} \\ &= 2 \cdot \frac{(n+1)n!}{n!} \cdot \frac{(n+1)n!}{n!} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \\ &= 2 \cdot \frac{(n+1)(n+1)}{(2n+2)(2n+1)}. \end{aligned}$$

The limit of the absolute value of the ratio found above is

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2 \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \right| = 2 \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n}\right)}{\left(2 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)} \right| = 2 \left| \frac{1 \cdot 1}{2 \cdot 2} \right| = 2 \cdot \frac{1}{4} = \frac{1}{2} < 1.$$

Since the limit is less than 1, by the Ratio Test, the series *converges.*

44. (a) The series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left[\left(\frac{2}{3} \right)^k - \frac{2}{k^2 + 2k} \right] = \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k - \sum_{k=1}^{\infty} \frac{2}{k^2 + 2k}$$

is the difference between two series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k$ and $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{2}{k^2 + 2k}$. We show below that each of these series converges, so the original series must converge, which in turn justifies our splitting the series into two parts (which is allowed only for absolutely convergent series). The series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k = \sum_{k=1}^{\infty} \left(\frac{2}{3} \right) \left(\frac{2}{3} \right)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$, with $a = \frac{2}{3}$, and $r = \frac{2}{3}$.

Since $|r| < 1$, this series converges to a sum of

$$\frac{a}{1-r} = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = 2.$$

The n^{th} term of the series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{2}{k^2 + 2k}$ satisfies

$$c_n = \frac{2}{n^2 + 2n} < \frac{2}{n^2} = d_n$$

for $n \geq 1$. Since the series $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{2}{k^2}$ is a constant multiple of a convergent p -series (since $p = 2 > 1$),

by the Comparison Test, we see that the series $\sum_{k=1}^{\infty} c_k$ converges. These calculations show that the original

series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} c_k$ converges.

(b) We have already summed the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k = 2$ above. To sum the series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{2}{k^2 + 2k}$, consider its n^{th} partial sum, which can be simplified using partial fractions and the telescoping nature of the terms as follows:

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{2}{k^2 + 2k} = \sum_{k=1}^n \frac{2}{k(k+2)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+2} \right) \\ &= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \\ &\quad + \left(\frac{1}{n-2} - \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}. \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{2}.$$

Since the sequence $\{S_n\}$ of partial sums converges to a sum, the series converges to the same sum, that is,

$\sum_{k=1}^{\infty} c_k = \frac{3}{2}$. Finally, the required sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} c_k = 2 - \frac{3}{2} = \boxed{\frac{1}{2}}.$$

45. (a) The series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left[\left(-\frac{1}{4}\right)^k + \frac{3}{k(k+1)} \right] = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k + \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$$

is the sum of two series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k$ and $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$. We show below that each of these series converges, so the original series converges as well, which justifies our splitting the series into two parts (which is allowed only for absolutely convergent series). The series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right) \left(-\frac{1}{4}\right)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$, with $a = -\frac{1}{4}$ and $r = -\frac{1}{4}$. Since $|r| < 1$, the series converges to a sum of

$$\frac{a}{1-r} = \frac{-\frac{1}{4}}{1 - \left(-\frac{1}{4}\right)} = \frac{-\frac{1}{4}}{1 + \frac{1}{4}} = \frac{-\frac{1}{4}}{\frac{5}{4}} = -\frac{1}{5}.$$

The n^{th} term of the series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$ satisfies

$$c_n = \frac{3}{n(n+1)} < \frac{3}{n^2} = d_n$$

for $n \geq 1$. Since the series $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty} \frac{3}{k^2}$ is a constant multiple of a convergent p -series (since $p = 2 > 1$),

by the Comparison Test, we see that the series $\sum_{k=1}^{\infty} c_k$ converges. These calculations demonstrate that the

original series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k$ converges.

(b) We have already summed the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k = -\frac{1}{5}$ above. To sum the series

$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$ consider its n^{th} partial sum, which can be simplified using partial fractions and the telescoping nature of the terms as follows:

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{3}{k(k+1)} = \sum_{k=1}^n 3 \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 3 \left(\frac{1}{1} - \frac{1}{2} \right) + 3 \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + 3 \left(\frac{1}{n-1} - \frac{1}{n} \right) + 3 \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 3 - \frac{3}{n+1}. \end{aligned}$$

We have

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(3 - \frac{3}{n+1} \right) = 3.$$

Since the sequence $\{S_n\}$ of partial sums converges to a sum, the series converges to the same sum, that is

$\sum_{k=1}^{\infty} c_k = 3$. Finally, the required sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = -\frac{1}{5} + 3 = \boxed{\frac{14}{5}}.$$

Challenge Problems

46. (a) The series is

$$1 - 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{9} - \frac{1}{4} + \frac{1}{27} + \frac{1}{5} - \frac{1}{81} - \cdots.$$

The n^{th} term of this series is

$$a_n = \left[(-1)^{n+1} \frac{1}{n} + (-1)^n \frac{1}{3^{n-1}} \right] = (-1)^{n-1} \left[\frac{1}{n} - \frac{1}{3^n} \right].$$

To show the series converges, we apply the Alternating Series Test. We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{1}{3^n} \right) = 0 - 0 = 0.$$

Also, by the Algebraic Difference Test,

$$|a_{n+1}| - |a_n| = \frac{1}{n+1} - \frac{1}{3^{n+1}} - \frac{1}{n} + \frac{1}{3^n} = \frac{1}{n+1} - \frac{1}{n} - \frac{1}{3} \cdot \frac{1}{3^n} + \frac{1}{3^n} = \frac{n-n-1}{n(n+1)} + \frac{2}{3} \cdot \frac{1}{3^n} = -\frac{1}{n(n+1)} + \frac{2}{3^{n+1}} < 0$$

provided

$$\frac{2}{3^{n+1}} < \frac{1}{n(n+1)}$$

or, $2n(n+1) < 3^{n+1}.$

The left side of the inequality is a polynomial whose terms are $\{4, 12, 24, \dots\}$ for $n = 1, 2, 3, \dots$, while the right side of the inequality is an exponential series with terms $\{9, 27, 81, \dots\}$ for $n = 1, 2, 3, \dots$. So we see that the inequality is satisfied for $n \geq 1$, so by the Alternating Series Test, the series converges.

(b) Since the series converges to a sum, we can split up the n^{th} term as two terms and sum them separately, so that the sum of the series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k$. Here, the first sum

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is the alternating harmonic series, which we know to be a convergent series (using, for example, the Alternating Series Test, or consulting the Table 6 on p.598). The alternating harmonic series has a sum

$$\sum_{k=1}^{\infty} b_k = \ln 2 \text{ (see the note on p.583).}$$

The second sum is

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{3^{k-1}} = \frac{-1}{3^0} + \frac{1}{3^1} + \frac{-1}{3^2} + \frac{1}{3^3} + \frac{-1}{3^4} \cdots = -1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots,$$

is a geometric series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{3^{k-1}} = \sum_{k=1}^{\infty} \left(-\frac{(-1)^{k-1}}{3^{k-1}} \right) = \sum_{k=1}^{\infty} -\left(-\frac{1}{3}\right)^{k-1}$ of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = -1$ and $r = -\frac{1}{3}$. Since $|r| < 1$, the geometric series also converges to a sum

$$\frac{a}{1-r} = \frac{-1}{1 - \left(-\frac{1}{3}\right)} = \frac{-1}{1 + 1/3} = -\frac{3}{4}.$$

So the sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = \boxed{\ln 2 - \frac{3}{4}}.$$

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{2k^3-1}$. The n^{th} term of this series satisfies

$$a_n = \frac{\ln n}{2n^3-1} < \frac{n}{2n^3-1} = b_n$$

since $\ln n < n$ for $n \geq 1$. (To see this, let $f(x) = \ln x - x$ be a related function of $\ln n - n$. Then $f'(x) = \frac{1}{x} - 1 \leq 0$ when $x \geq 1$. So the function is decreasing for $x \geq 1$. Since $f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$, and the function is decreasing for $x \geq 1$, it means that $f(x) < 0$ for all $x \geq 1$, that is, $\ln x < x$ for all $x \geq 1$. So, we have $\ln n < n$ for $n \geq 1$ as well.) Now the series $\sum_{k=1}^{\infty} b_k$ converges by comparing it with the convergent p -series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent since $p = 2 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{b_n}{c_n} = \lim_{n \rightarrow \infty} \frac{\frac{n}{2n^3-1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^3}{2n^3-1} = \lim_{n \rightarrow \infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} b_k$ converges; and this in turn, via the Comparison Test, means that the original series $\sum_{k=1}^{\infty} a_k$ converges as well.

48. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin^3\left(\frac{1}{k}\right)$. Note that the terms of the series are positive, since $0 < \frac{1}{n} \leq 1$ for $1 \leq n < \infty$, and $\sin \theta$ is positive for $\theta = \frac{1}{n}$ that satisfies $0 < \theta \leq 1$ (radians). We compare the series to $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$, which is a convergent p -series since $p = 3 > 1$. We have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin^3\left(\frac{1}{n}\right)}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \right]^3 = \left[\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \right]^3 = \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} \right]^3 = 1^3 = 1,$$

where we made the substitution $x = \frac{1}{n}$ and used the standard limits result $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Since the limit is a positive real number, by the Limit Comparison Test, the original series $\sum_{k=1}^{\infty} a_k$ is seen to converge.

Note: We could instead have used the Comparison Test for convergence using the series $\sum_{k=1}^{\infty} b_k$, since the n^{th} term of the absolute value series $\sum_{k=1}^{\infty} |a_k|$ satisfies

$$|a_n| = \left| \sin\left(\frac{1}{n}\right) \right|^3 \leq \frac{1}{n^3}$$

for all $n \geq 1$, since $|\sin \theta| < \theta$ for $\theta = \frac{1}{n}$ in the interval $(0, 1]$. Since the absolute value series converges by the Comparison Test, the original series converges absolutely, and therefore converges.

8.8 Power Series

Concepts and Vocabulary

1. **True:** The theorem regarding a power series in the form $\sum_{k=0}^{\infty} a_k(x-c)^k$ states that the series converges either at $x=c$, or converges absolutely for all x , or converges absolutely on $|x-c| < R$, where R is a positive number. (For a full statement, see p.602.)

2. **True:** This follows from the Ratio Test for convergence of a series.

3. **True:** This follows from the theorem concerning a power series in the form $\sum_{k=0}^{\infty} a_k(x-c)^k$ with $c=0$; see Problem 1 for the reasoning.

4. **False:** There is no reason that this happen. For example, the power series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges absolutely if $|x| < 1$ since

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = |x| < 1$$

for convergence by the Ratio Test, as claimed. This means the interval of convergence is $(-1, 1)$.

At $x=1$, the series becomes $\sum_{k=1}^{\infty} \frac{1}{k}$ which, being the harmonic series, diverges.

At $x=-1$, the series becomes $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ which, being a constant multiple of the convergent alternating harmonic series also converges. So the power series is convergent on the interval $[-1, 1)$: that is, it is convergent at the one endpoint of its interval of convergence but not at the other endpoint.

5. **True:** This follows from the theorem on the radius of convergence of a power series, which says the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely for all $|x-c| < R$. The radius R here is a property of the series and not of the value of c , which means that the power series will have the same radius of convergence for any value of c .

6. **False:** If R is the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, then $\sum_{k=0}^{\infty} a_k x^k$ converges in the interval $-R < x < R$, while $\sum_{k=0}^{\infty} a_k(x-3)^k$ converges in the interval $-R < x-3 < R$, or simplifying, $-R+3 < x < R+3$. This shows that the two series have different intervals of convergence.

7. **False:** This follows from the theorem on convergence and divergence of a power series, which says that if a power series $\sum_{k=0}^{\infty} a_k x^k$ converges for $x_0 \neq 0$, then it converges absolutely for all numbers x for which $|x| < |x_0|$, or $-|x_0| < x < |x_0|$. The endpoints $-|x_0|$ and $|x_0|$ have to be examined separately, and if the series converges at $|x_0| = 8$, then it does not follow that the series also converges at $-|x_0| = -8$.

8. **True:** From the theorem quoted in Problem 7, if $|x_0|=3$, then the power series converges in the interval $-3 < x < 3$, so it also converges for $x=1$.

9. **True:** As in Problems 7 and 8, if the series converges for $x=-4$, this means $|x_0|=4$, so since the series converges in the interval $-4 < x < 4$, it converges for $x=3$ as well.

10. False: If the series converges for $x = 3$, then it may converge or diverge at $x = 5$ because the radius of convergence of the series $\sum_{k=0}^{\infty} a_k x^k$ is not given or known. For instance, if the radius of convergence is 4, then the series will diverge at $x = 5$, but if the radius of convergence is 7, then the series will converge at $x = 5$.

11. False: Intervals of convergence for the power series $\sum_{k=0}^{\infty} a_k x^k$ could be of the form $|x| \leq R$, or $[-R, R]$, if convergence at the endpoints can be demonstrated. So we can have intervals of the form $[-2, 2]$ or $[-4, 4]$, but not $[-2, 4]$. This is because if the interval of convergence were $[-2, 4]$ it would mean the power series diverges for $x < -2$, but by the theorem on convergence and divergence of a power series, it would then mean that the series also diverges for $x > 2$, which would contradict the statement that it converges in the interval $[-2, 4]$.

12. False: All that can be said is that if a power series diverges for $x = x_1$, then it diverges for all $x > x_1$. It may or may not converge for $x < x_1$. As an illustration, the power series considered in Problem 4 above, $\sum_{k=1}^{\infty} \frac{x^k}{k}$ diverges for $x_1 = 3$, but it also diverges for $x = 2$, and $2 < 3$. On the other hand, it converges at $x = \frac{1}{2}$, (since the radius of convergence of the power series is 1), and $\frac{1}{2} < 3$ as well. So this shows that a series can either converge or diverge for x values less than a certain value x_1 that leads to divergence of the series.

Skill Building

13. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} kx^k$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = |x| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) = |x| \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$ or $-1 < x < 1$. For $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} kx^k = \sum_{k=0}^{\infty} (-1)^k k = 0 + -1 + 2 - 3 + 4 - \dots$$

which diverges since the sequence of partial sums is $\{S_n\} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$, and it oscillates between growing positive and negative numbers as n grows large. Since the sequence of partial sums is not convergent, the series is not convergent.

For $x = 1$, the series becomes

$$\sum_{k=0}^{\infty} kx^k = \sum_{k=0}^{\infty} k(1)^k = \sum_{k=0}^{\infty} k = \infty,$$

so it diverges.

Concluding, the power series $\sum_{k=0}^{\infty} kx^k$ converges for $\boxed{-1 < x < 1}$.

14. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)x^{n+1}}{3^{n+1}}}{\frac{nx^n}{3^n}} \right| = \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{|x|}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{|x|}{3} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 3$, or $-3 < x < 3$.

For $x = -3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{kx^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-k)(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^{k+1} k = 0 + 1 - 2 + 3 - 4 + \cdots$$

The sequence of partial sums of the series is $\{S_n\} = \{0, 1, -1, 2, -2, \dots\}$ which oscillates between growing positive and negative values as n grows large. Since the sequence of partial sums does not converge, neither does the series.

For $x = 3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{kx^k}{3^k} = \sum_{k=0}^{\infty} \frac{k3^k}{3^k} = \sum_{k=0}^{\infty} k = \infty,$$

so this series diverges.

Concluding, the power series $\sum_{k=0}^{\infty} \frac{kx^k}{3^k}$ converges for $\boxed{-3 < x < 3}$.

15. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = 1 + \sum_{k=1}^{\infty} \frac{(x+1)^k}{3^k}$ is a series of nonzero terms if $x \neq -1$. (At $x = -1$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x+1)^{n+1}}{3^{n+1}}}{\frac{(x+1)^n}{3^n}} \right| = \frac{1}{3} |x+1| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x+1| < 3$, or $-3 < x+1 < 3$, or $-4 < x < 2$.

At $x = -4$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-4+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots$$

The sequence of partial sums of the series is $\{S_n\} = \{1, 0, 1, 0, \dots\}$ which oscillates between 1 and 0 for all n . Since the sequence of partial sums does not converge, neither does the series.

At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(2+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{3^k}{3^k} = \sum_{k=0}^{\infty} 1 = \infty,$$

so this series diverges.

Concluding, the power series $\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k}$ converges for $\boxed{-4 < x < 2}$.

16. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}} \right| = |x-2| \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = |x-2| \lim_{n \rightarrow \infty} \frac{1}{(1 + \frac{1}{n})^2} = |x-2| \frac{1}{(1+0)^2} = |x-2| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x-2 < 1$, or $1 < x < 3$.

At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(1-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which is an alternating series whose n^{th} term goes to zero as $n \rightarrow \infty$; also the terms of the series are nonincreasing, because by the Algebraic Ratio Test $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2} < 1$ for $n \geq 1$. By the Alternating Series Test, the power series converges for $x = 1$.

At $x = 3$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(3-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p -series (since $p = 2 > 1$), so the power series converges for $x = 3$.

Concluding, the power series $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$ converges for $\boxed{1 \leq x \leq 3}$.

17. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{2^k(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}(n+2)}}{\frac{x^n}{2^n(n+1)}} \right| = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = \frac{|x|}{2} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 2$, which means the radius of convergence $\boxed{R = 2}$.

At $x = -2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-2)^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'-1}}{k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{(-1)^2 k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{k'}$$

where $k' = k + 1$; the series is the alternating harmonic series which converges. So the power series converges for $x = -2$.

At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{2^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k'=1}^{\infty} \frac{1}{k'}$$

where $k' = k + 1$; the series is the harmonic series which diverges. So the power series diverges for $x = 2$.

Concluding, the interval of convergence of the power series is $\boxed{-2 \leq x < 2}$.

(b) Before we do the root test, it would be useful to prove the following result which shall be used in several problems below: $\lim_{n \rightarrow \infty} (n+a)^{1/n} = 1$. To prove this, set $y = (n+a)^{1/n}$. Then

$\ln y = \frac{1}{n} \ln(n+a) = \frac{\ln(n+a)}{n}$. We have, using L'Hôpital's rule on a related function of the ratio:

$$\lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} e^{\ln y} = \lim_{n \rightarrow \infty} e^{\ln(n+a)/n} = \lim_{x \rightarrow \infty} e^{\ln(x+a)/x} = e^{\lim_{x \rightarrow \infty} \frac{\ln(x+a)}{x}} = e^{\lim_{x \rightarrow \infty} \frac{1/x}{1} / 1} = e^{\lim_{x \rightarrow \infty} \frac{1}{x+a}} = e^0 = 1.$$

This proves the claimed result. (Note that this formula holds for any value of a , as the proof does not restrict the value of a . For $a < 0$, we require $a > -n$, which for any fixed a occurs as $n \rightarrow \infty$.)

By the Root Test, using the limit result above for $a = 1$,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{2^n(n+1)} \right|} = \lim_{n \rightarrow \infty} \frac{|x|}{2(n+1)^{1/n}} = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{1}{(n+1)^{1/n}} = \frac{|x|}{2} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 \leq x < 2}$.

(c) Answers will vary.

18. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k(k+1)} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{2^k(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{2^{n+1}(n+2)}}{\frac{(-1)^n x^n}{2^n(n+1)}} \right| = \frac{|x|}{2} |(-1)| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = \frac{|x|}{2} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 2$, which means the radius of convergence is $\boxed{R = 2}$.

At $x = -2$, the series becomes

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k (-2)^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^k 2^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k'=1}^{\infty} \frac{1}{k'}$$

using $k' = k + 1$; the series is the harmonic series which diverges, so the power series diverges for $x = -2$.

At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'-1}}{k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{(-1)^2 k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{k'}$$

where $k' = k + 1$; the series is the alternative harmonic series, which converges. So the power series converges for $x = 2$. Concluding, the interval of convergence of the power series is $\boxed{-2 < x \leq 2}$.

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 1$, that is,

$\lim_{n \rightarrow \infty} (n+1)^{1/n} = 1$, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(-1)^n \frac{x^n}{2^n(n+1)}|} = \frac{|x|}{2} \lim_{n \rightarrow \infty} \frac{1}{(n+1)^{1/n}} = \frac{|x|}{2} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 < x \leq 2}$.

(c) Answers will vary.

19. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{k+5} = \frac{1}{5} + \sum_{k=1}^{\infty} \frac{x^k}{k+5}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to $\frac{1}{5}$.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+6}}{\frac{x^n}{n+5}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+5}{n+6} = |x| \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n}}{1 + \frac{6}{n}} = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1}$.

At $x = -1$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{k+5} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+5} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'-5}}{k'} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'+1}}{(-1)^6 k'} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'+1}}{k'}$$

where $k' = k + 5$; this is an alternating harmonic series from the fifth term onward. Since the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, and its convergence property is unaffected by the removal of a finite number of terms, the power series converges for $x = -1$.

At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{k+5} = \sum_{k=0}^{\infty} \frac{(1)^k}{k+5} = \sum_{k=0}^{\infty} \frac{1}{k+5} = \sum_{k'=5}^{\infty} \frac{1}{k'}$$

using $k' = k + 5$; this is the harmonic series from the fifth term forward. Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, and the divergence property remains unaffected by the removal of a finite number of terms, the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 5$, that is,

$\lim_{n \rightarrow \infty} (n+5)^{1/n} = 1$, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{n+5} \right|} = |x| \lim_{n \rightarrow \infty} \frac{1}{(n+5)^{1/n}} = |x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 \leq x < 1}$.

(c) Answers will vary.

20. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{1+k^2} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{1+k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{1+(n+1)^2}}{\frac{x^n}{1+n^2}} \right| = |x| \lim_{n \rightarrow \infty} \frac{1+n^2}{1+(n+1)^2} \\ &= |x| \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + 1}{\frac{1}{n^2} + \left(1 + \frac{1}{n}\right)^2} = |x| \frac{0+1}{0+(1+0)^2} \\ &= |x| < 1 \end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1}$.

At $x = -1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{1+k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{1+k^2}.$$

This is an alternating series. The n^{th} term satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{1+n^2} = 0.$$

Also, by the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{1+(n+1)^2}}{\frac{1}{1+n^2}} = \frac{1+n^2}{1+(n+1)^2} < 1 \text{ for } n \geq 0.$$

So the terms are nonincreasing. By the Alternating Series Test, the power series converges for $x = -1$.

At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{1+k^2} = \sum_{k=0}^{\infty} \frac{1}{1+k^2}.$$

The n^{th} term of the series satisfies

$$a_n = \frac{1}{1+n^2} < \frac{1}{n^2} = b_n \text{ for } n \geq 1.$$

Comparing with the convergent p -series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent since $p = 2 > 1$, the series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges. This means the series

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{1+k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{1+k^2}$$

also converges, which means the power series converges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1}$.

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim_{n \rightarrow \infty} n^{1/n} = 1$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{1+n^2} \right|} = |x| \lim_{n \rightarrow \infty} \frac{1}{(1+n^2)^{1/n}} = |x| \lim_{n \rightarrow \infty} \frac{1}{(n^2)^{1/n} \left(\frac{1}{n^2} + 1 \right)^{1/n}} \\ &= |x| \lim_{n \rightarrow \infty} \left(\frac{1}{n^{1/n}} \right)^2 \cdot \frac{1}{\left[\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + 1 \right) \right]^{\lim_{n \rightarrow \infty} \frac{1}{n}}} \\ &= |x| \cdot 1^2 \cdot \frac{1}{[0+1]^0} = |x| < 1 \end{aligned}$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 \leq x \leq 1}$.

(c) Answers will vary.

21. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k^2 x^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2 x^{n+1}}{3^{n+1}}}{\frac{n^2 x^n}{3^n}} \right| = \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} = \frac{|x|}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{|x|}{3} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 3$, which means the radius of convergence is $\boxed{R = 3}$.

At $x = -3$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k^2 x^k}{3^k} = \sum_{k=0}^{\infty} \frac{k^2 (-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k k^2.$$

We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n^2 = \infty,$$

so the series diverges by the Divergence Test. This means the power series diverges for $x = -3$.

At $x = 3$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k^2 3^k}{3^k} = \sum_{k=0}^{\infty} k^2.$$

Since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^2 = \infty,$$

the series diverges by the Divergence Test. So the power series diverges at $x = 3$.

Concluding, the interval of convergence of the power series is $\boxed{-3 < x < 3}$.

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim_{n \rightarrow \infty} n^{1/n} = 1$, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^2 x^n}{3^n} \right|} = \frac{|x|}{3} \lim_{n \rightarrow \infty} \left(n^{1/n} \right)^2 = \frac{|x|}{3} \left(\lim_{n \rightarrow \infty} n^{1/n} \right)^2 = \frac{|x|}{3} \cdot 1^2 = \frac{|x|}{3} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 3}$ and the interval of convergence is $\boxed{-3 < x < 3}$.

(c) Answers will vary.

22. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = 1 + \sum_{k=1}^{\infty} \frac{2^k x^k}{3^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{3^{n+1}}}{\frac{2^n x^n}{3^n}} \right| = \frac{2}{3} |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{3}{2}$, which means the radius of convergence is $\boxed{R = \frac{3}{2}}$.

At $x = -\frac{3}{2}$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = \sum_{k=0}^{\infty} \frac{2^k \left(-\frac{3}{2}\right)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums $\{S_n\} = \{1, 0, 1, 0, \dots\}$ has no limit, because its terms oscillate between 1 and 0 for all n . So the power series diverges for $x = -\frac{3}{2}$.

At $x = \frac{3}{2}$, the series becomes

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = \sum_{k=0}^{\infty} \frac{2^k \left(\frac{3}{2}\right)^k}{3^k} = \sum_{k=0}^{\infty} 1 = \infty.$$

So the power series diverges at $x = \frac{3}{2}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

(b) By the Root Test, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{2^n x^n}{3^n} \right|} = \frac{2}{3} |x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint

computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{3}{2}}$ and the interval

of convergence is $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

(c) Answers will vary.

23. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{2k+1}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)x^{n+1}}{2(n+1)+1}}{\frac{nx^n}{2n+1}} \right| = |x| \cdot \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \lim_{n \rightarrow \infty} \frac{2n+1}{2n+3} \\ &= |x| \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \right) \\ &= |x| \cdot 1 \cdot 1 < 1 \end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1}$.

At $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k(-1)^k}{2k+1}.$$

We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2} \neq 0.$$

By the Divergence Test, the series diverges. So the power series diverges for $x = -1$.

At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k(1)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k}{2k+1}.$$

We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$$

as above. By the Divergence Test, the series diverges. So the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1}$.

(b) By the Root Test, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{nx^n}{2n+1} \right|} = |x| \cdot \lim_{n \rightarrow \infty} \frac{1}{\left(2 + \frac{1}{n} \right)^{1/n}} = |x| \cdot \frac{1}{\lim_{n \rightarrow \infty} \left(2 + \frac{1}{n} \right)^{\lim_{n \rightarrow \infty} \frac{1}{n}}} = |x| \cdot \frac{1}{2^0} = |x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1}$.

(c) Answers will vary.

24. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (6x)^k = 1 + \sum_{k=1}^{\infty} (6x)^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(6x)^{n+1}}{(6x)^n} \right| = |6x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{1}{6}$, which means the radius

of convergence is $\boxed{R = \frac{1}{6}}$.

At $x = -\frac{1}{6}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (6x)^k = \sum_{k=0}^{\infty} \left(6 \cdot \left[-\frac{1}{6}\right]\right)^k = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums of the series, $\{S_n\} = \{1, 0, 1, 0, \dots\}$ diverges because the terms oscillate between 1 and 0 for all n . So the power series diverges when $x = -\frac{1}{6}$.

At $x = \frac{1}{6}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (6x)^k = \sum_{k=0}^{\infty} \left(6 \cdot \left[\frac{1}{6}\right]\right)^k = \sum_{k=0}^{\infty} (1)^k = \infty.$$

So the series diverges when $x = \frac{1}{6}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{1}{6} < x < \frac{1}{6}}$.

(b) By the Root Test, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(6x)^n|} = |6x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint

computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{1}{6}}$ and the interval

of convergence is $\boxed{-\frac{1}{6} < x < \frac{1}{6}}$.

(c) Answers will vary.

25. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (x-3)^k = 1 + \sum_{k=1}^{\infty} (x-3)^k$ is a series of nonzero terms for $x \neq 3$. (At $x = 3$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| = |x-3| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x-3 < 1$, or $2 < x < 4$.

So the radius of convergence of the series is $\boxed{R = 1}$.

At $x = 2$, the power series becomes

$$\sum_{k=0}^{\infty} (x-3)^k = \sum_{k=0}^{\infty} (2-3)^k = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums of the series, $\{S_n\} = \{1, 0, 1, 0, \dots\}$ diverges because the terms oscillate between 1 and 0 for all n . So the power series diverges when $x = 2$.

At $x = 4$, the power series becomes

$$\sum_{k=0}^{\infty} (x-3)^k = \sum_{k=0}^{\infty} (4-3)^k = \sum_{k=0}^{\infty} (1)^k = \sum_{k=0}^{\infty} 1 = \infty.$$

So the series diverges when $x = 4$.

Concluding, the interval of convergence of the power series is $\boxed{2 < x < 4}$.

(b) By the Root Test, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(x-3)^n|} = |x-3| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{2 < x < 4}$.

(c) Answers will vary.

26. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(2x)^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)(2x)^{n+1}}{3^{n+1}}}{\frac{n(2x)^n}{3^n}} \right| = \frac{|2x|}{3} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{|2x|}{3} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{|2x|}{3} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{3}{2}$ which means that the radius of convergence of the series is $\boxed{R = \frac{3}{2}}$.

At $x = -\frac{3}{2}$, the series reduces to

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(2x)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k \left(2 \left[-\frac{3}{2} \right] \right)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k k.$$

We have

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n = \infty.$$

By the Divergence Test, the series diverges. So the power series diverges at $x = -\frac{3}{2}$. At $x = \frac{3}{2}$, the series reduces to

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(2x)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k \left(2 \cdot \frac{3}{2} \right)^k}{3^k} = \sum_{k=0}^{\infty} k = \infty.$$

This means the power series diverges at $x = \frac{3}{2}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim_{n \rightarrow \infty} n^{1/n} = 1$, we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n(2x)^n}{3^n} \right|} = \left| \frac{2x}{3} \right| \cdot \lim_{n \rightarrow \infty} n^{1/n} = \left| \frac{2x}{3} \right| \cdot 1 < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{3}{2}}$ and the interval of convergence is $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

(c) Answers will vary.

27. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{k^3}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)^3}}{\frac{x^n}{n^3}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} = |x| \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n} \right)} = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, and the radius of convergence is $\boxed{R = 1}$.

At $x = -1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$. The n^{th} term of this series satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{|(-1)^n|}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n^3} = 0.$$

By the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \frac{n^3}{(n+1)^3} < 1$$

for $n \geq 1$. Since the terms of the series are nonincreasing, and the n^{th} term has 0 limit, the series converges by the Alternating Series Test. So the power series converges for $x = -1$.

At $x = 1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$. This is a convergent p -series which converges since $p = 3 > 1$, so the power series converges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1}$.

28. The power series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{x^k}{\ln k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln n}} \right| = |x| \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \\ &= |x| \lim_{y \rightarrow \infty} \frac{\ln y}{\ln(y+1)} = |x| \lim_{y \rightarrow \infty} \frac{\frac{1}{y}}{\frac{1}{y+1}} = |x| \lim_{y \rightarrow \infty} \frac{y+1}{y} \\ &= |x| \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y} \right) = |x| < 1, \end{aligned}$$

for convergence of the series by the Ratio Test, where we used a related function of the ratio $\frac{\ln n}{\ln(n+1)}$ in order to apply L'Hôpital's rule. So the series converges for $|x| < 1$, and the radius of convergence is $\boxed{R = 1}$.

At $x = -1$, the series becomes $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$. The n^{th} term of this series satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{|(-1)^n|}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0.$$

By the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln n}} = \frac{\ln n}{\ln(n+1)} < 1$$

if $n \geq 2$, since the natural logarithm function is an increasing function. Since the terms of the series are nonincreasing, and the n^{th} term has 0 limit, the series converges by the Alternating Series Test. So the power series converges for $x = -1$.

At $x = 1$, the series becomes $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$. The n^{th} term of this series satisfies

$$a_n = \frac{1}{\ln n} > \frac{1}{n} = b_n$$

for $n \geq 2$ since $\ln n < n$ for $n \geq 2$. (This is proved as follows: $f(x) = \ln x - x$ is a decreasing function of x for $x \geq 1$, since $f'(x) = \frac{1}{x} - 1 \leq 0$ for $x \geq 1$, and since $f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$, we have, because $f(x)$ is decreasing, $f(x) < 0$ for all $x \geq 1$, that is, $\ln x < x$ for all $x \geq 1$. This in turn means $\ln n < n$ for

$n \geq 1$. We restrict to $n \geq 2$ to ensure the denominator of the n^{th} term stays nonzero.) So, comparing with the series $\sum_{k=2}^{\infty} b_k = \sum_{k=2}^{\infty} \frac{1}{k}$, which diverges being the harmonic series from the second term onward, we see by the Comparison Test that the original series diverges. So the power series is divergent for $x = 1$. Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

29. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^3}}{\frac{(x-2)^n}{n^3}} \right| = |x-2| \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} = |x-2| \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^3} = |x-2| \frac{1}{(1+0)^3} = |x-2| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x - 2 < 1$, or $1 < x < 3$.

The radius of convergence of the series is $\boxed{R = 1}$.

At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(1-2)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}.$$

This series is convergent because it is absolutely convergent: the series of absolute values is

$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^3}$, which converges as it is a convergent p -series with $p = 3 > 1$. So the power series converges for $x = 1$.

At $x = 3$, the power series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(3-2)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(1)^k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

which converges as it is a convergent p -series with $p = 3 > 1$. So the power series converges for $x = 3$.

Concluding, the interval of convergence of the power series is $\boxed{1 \leq x \leq 3}$.

30. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(x-2)^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n(x-2)^n}{3^n} \right|} = \left| \frac{x-2}{3} \right| \lim_{n \rightarrow \infty} n^{1/n} = \frac{|x-2|}{3} \cdot 1 < 1$$

for convergence of the series by the Root Test (where we have used the result proved in problem 17(b), that $\lim_{n \rightarrow \infty} n^{1/n} = 1$). So the series converges for $|x-2| < 3$, or $-3 < x-2 < 3$, or $-1 < x < 5$. The radius

of convergence of the series is $\boxed{R = 3}$.

At $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(x-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(-1-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k k$$

which diverges by the Divergence Test since $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n = \infty$. So the power series diverges for $x = -1$.

At $x = 5$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(x-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(5-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(3^k)}{3^k} = \sum_{k=0}^{\infty} k = \infty$$

since the sum of all the nonnegative integers is infinite. This means the power series diverges for $x = 5$. Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 5}$.

31. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n \rightarrow \infty} \left| (-1) \cdot \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!} \right| \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} = |x|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} \\ &= |x|^2 \cdot 0 = 0 \end{aligned}$$

which means the series converges by the Ratio Test regardless of the value of x , that is, for all values of x . So the radius of convergence $\boxed{R = \infty}$, and the interval of convergence is $\boxed{-\infty < x < \infty}$.

32. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (kx)^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(nx)^n|} = |x| \lim_{n \rightarrow \infty} n = \infty$$

if $x \neq 0$, so the series diverges by the Root Test if $x \neq 0$. When $x = 0$, the series becomes

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} 0 = 0$, so the series converges for $x = 0$. Concluding, the radius of convergence of the series $\boxed{R = 0}$ and the series converges for $\boxed{x = 0}$.

33. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{kx^k}{\ln(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)x^{n+1}}{\ln(n+2)}}{\frac{nx^n}{\ln(n+1)}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)} \\ &= |x| \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \cdot \lim_{y \rightarrow \infty} \frac{\ln(y+1)}{\ln(y+2)} = |x| \cdot 1 \cdot \lim_{y \rightarrow \infty} \frac{\frac{1}{y+1}}{\frac{1}{y+2}} \\ &= |x| \lim_{y \rightarrow \infty} \frac{y+2}{y+1} = |x| \lim_{y \rightarrow \infty} \frac{1 + \frac{2}{y}}{1 + \frac{1}{y}} = |x| \cdot \frac{1+0}{1+0} = |x| \cdot 1 \\ &= |x| < 1 \end{aligned}$$

for convergence of the series by the Ratio Test, where we have used a related function of the ratio in the last factor of the limit, in order to apply L'Hôpital's rule to it. So the series converges for $-1 < x < 1$, or the radius of convergence $\boxed{R = 1}$.

At $x = -1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k(-1)^k}{\ln(k+1)}$. The absolute value of the n^{th} term satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{\ln(n+1)} = \lim_{z \rightarrow \infty} \frac{z}{\ln(z+1)} = \lim_{z \rightarrow \infty} \frac{1}{\frac{1}{z+1}} = \lim_{z \rightarrow \infty} (z+1) = \infty$$

where we apply L'Hôpital's rule to a related function of the absolute value of the n^{th} term of the series. By the Divergence Test, the power series diverges for $x = -1$.

At $x = 1$, the power series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} |a_n| = \infty$, we see that it diverges too.

Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1}$.

34. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{\ln(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{\ln(n+2)}}{\frac{x^n}{\ln(n+1)}} \right| = |x| \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)} = |x| \lim_{y \rightarrow \infty} \frac{\ln(y+1)}{\ln(y+2)} \\ &= |x| \lim_{y \rightarrow \infty} \frac{\frac{1}{y+1}}{\frac{1}{y+2}} = |x| \lim_{y \rightarrow \infty} \frac{y+2}{y+1} \\ &= |x| \lim_{y \rightarrow \infty} \frac{1 + \frac{2}{y}}{1 + \frac{1}{y}} = |x| \cdot \frac{1+0}{1+0} = |x| \cdot 1 < 1 \end{aligned}$$

for convergence of the series by the Ratio Test, where we have used a related function of the ratio in the limit in order to apply L'Hôpital's rule to it. So the series converges for $-1 < x < 1$, or the radius of convergence $\boxed{R = 1}$.

At $x = -1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+1)}$. The n^{th} term of this series satisfies

$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$. Also by the Algebraic Ratio Test, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln n}} \right| = \left| \frac{\ln n}{\ln(n+1)} \right| < 1$$

for $n \geq 1$, since the natural logarithm function is an increasing function. So the series is nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -1$.

At $x = 1$, the series reduces to $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\ln(n+1)}$. The n^{th} term satisfies

$$a_n = \frac{1}{\ln(n+1)} > \frac{1}{n+1} = b_n$$

for $n \geq 1$, where we have invoked the fact that $\ln(n+1) < n+1$ for $n \geq 1$ (see Problem 28 for a proof of a similar fact, $\ln n < n$ for $n \geq 2$; the proof here is analogous). Comparing with the series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is the divergent harmonic series from the second term onward,

by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ diverges. So the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

35. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(k+1)x^k}{4^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)(n+2)x^{n+1}}{4^{n+1}}}{\frac{n(n+1)x^n}{4^n}} \right| = \frac{|x|}{4} \lim_{n \rightarrow \infty} \left(\frac{n+2}{n} \right) = \frac{|x|}{4} \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n} \right) = \frac{|x|}{4} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-4 < x < 4$, or the radius of convergence is $\boxed{R = 4}$.

At $x = -4$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(k+1)(-4)^k}{4^k} = \sum_{k=0}^{\infty} (-1)^k k(k+1)$$

which diverges by the Divergence Test since $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} |n(n+1)| = \infty$, that is, a nonzero limit. So the power series diverges for $x = -4$.

At $x = 4$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(k+1)4^k}{4^k} = \sum_{k=0}^{\infty} k(k+1) = \infty$$

because the sum of product of consecutive nonnegative integers is infinite. (Alternately, we could use $\lim_{n \rightarrow \infty} a_n \neq 0$ and conclude divergence based on the Divergence Test.) So the power series diverges for $x = 4$.

Concluding, the interval of convergence of the power series is $\boxed{-4 < x < 4}$.

36. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k (x-5)^k}{k(k+1)}$ is a series of nonzero terms for $x \neq 5$. (At $x = 5$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (x-5)^{n+1}}{(n+1)(n+2)}}{\frac{(-1)^n (x-5)^n}{n(n+1)}} \right| = |x-5| \lim_{n \rightarrow \infty} \frac{n}{n+2} = |x-5| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = |x-5| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges if $-1 < x-5 < 1$, or $4 < x < 6$.

The radius of convergence is $\boxed{R = 1}$.

At $x = 4$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k (4-5)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

The n^{th} term of this series satisfies

$$a_n = \frac{1}{n(n+1)} < \frac{1}{n^2} = b_n$$

for $n \geq 1$. By the Comparison Test, since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, being a p -series for $p = 2 > 1$, the

series $\sum_{k=1}^{\infty} a_k$ converges as well. So the power series converges for $x = 4$.

At $x = 6$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k (6-5)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k (1)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)}.$$

We could deduce convergence by noting that the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is just the series at $x = 4$, which converges from the calculation above; since the series absolutely converges, it converges. Alternately, the Alternating Series Test may be invoked: the n^{th} term of the series satisfies $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0$; by the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} \right| = \left| \frac{n}{n+2} \right| < 1$$

for $n \geq 1$, which means the absolute values of the terms of the series are nonincreasing. So it converges by the Alternating Series Test, and the power series converges for $x = 6$.

Concluding, the interval of convergence of the power series is $\boxed{4 \leq x \leq 6}$.

37. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(x-3)^{2k}}{9^k} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(x-3)^{2k}}{9^k}$ is a series of nonzero terms if $x \neq 3$. (At $x = 3$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}(x-3)^{2(n+1)}}{9^{n+1}}}{\frac{(-1)^n(x-3)^{2n}}{9^n}} \right| = \frac{|x-3|^2}{9} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x-3| < 3$, or $-3 < x-3 < 3$, or $0 < x < 6$. The radius of convergence is $\boxed{R = 3}$.

At $x = 0$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(0-3)^{2k}}{9^k} = \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^{2k}(3^2)^k}{9^k} = \sum_{k=0}^{\infty} [(-1)^3]^k = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums is $\{S_n\} = \{1, 0, 1, 0, \dots\}$. Since the sequence of partial sums oscillates between 0 and 1 for large n , it does not attain a limit, that is, it diverges, which in turn means the series diverges. So the power series diverges for $x = 0$.

At $x = 6$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(6-3)^{2k}}{9^k} = \sum_{k=0}^{\infty} (-1)^k \frac{(3^2)^k}{9^k} = \sum_{k=0}^{\infty} (-1)^k.$$

This is identical to the series at $x = 0$, and since we showed that series diverges, this one does too. So the power series diverges for $x = 6$.

Concluding, the interval of convergence of the power series is $\boxed{0 < x < 6}$.

38. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{e^k} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{e^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{e^n} \right|} = \frac{|x|}{e} < 1$$

for convergence of the series by the Root Test. So the series converges for $|x| < e$, or $-e < x < e$, and the radius of convergence is $\boxed{R = e}$.

At $x = -e$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-e)^k}{e^k} = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums of this series has the form $\{S_n\} = \{1, 0, 1, 0, \dots\}$, which is nonconvergent since the terms of the series oscillate between 0 and 1 for large n . Since the sequence of partial sums diverges, the series diverges. So the power series diverges for $x = -e$.

At $x = e$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{e^k}{e^k} = \sum_{k=0}^{\infty} 1 = \infty$$

since the sum of all the nonnegative integers is infinite. So the power series diverges for $x = e$.

Concluding, the interval of convergence of the power series is $\boxed{-e < x < e}$.

39. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(2x)^k}{k!}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{(2x)^{n+1}}{(n+1)!}}{(-1)^n \frac{(2x)^n}{n!}} \right| \\ &= \lim_{n \rightarrow \infty} \left| (-1) \cdot 2x \cdot \frac{n!}{(n+1)!} \right| = |2x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} \\ &= |2x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \end{aligned}$$

for any value of x . Since the limit is less than 1, by the Ratio Test, the series converges for any value of x . This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

40. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(x+1)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(x+1)^k}{k!}$ is a series of nonzero terms for $x \neq -1$. (At $x = -1$, the series converges to 1.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1)!}}{\frac{(x+1)^n}{n!}} \right| = |x+1| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = |x+1| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = |x+1| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

for any value of x . Since the limit is less than 1, by the Ratio Test, the series converges for any value of x . This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

41. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{4k}}{k!} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^{4k}}{k!}$ is a series of nonzero terms for $x \neq 1$. (At $x = 1$, the series converges to 1.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{4(n+1)}}{(n+1)!}}{\frac{(x-1)^{4n}}{n!}} \right| = |(x-1)^4| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \\ &= |x-1|^4 \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = |x-1|^4 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \end{aligned}$$

for any value of x . Since the limit is less than 1, by the Ratio Test, the series converges for any value of x . This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

42. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x+1)^k}{k(k+1)(k+2)}$ is a series of nonzero terms for $x \neq -1$ (At $x = -1$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1)(n+2)(n+3)}}{\frac{(x+1)^n}{n(n+1)(n+2)}} \right| = |x+1| \lim_{n \rightarrow \infty} \frac{n}{n+3} = |x+1| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{3}{n}} = |x+1| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x+1 < 1$ or $-2 < x < 0$, and the radius of convergence is $\boxed{R = 1}$.

At $x = -2$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)}$. The n^{th} term of this series satisfies $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n(n+1)(n+2)} = 0$. By the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)(n+2)(n+3)}}{\frac{1}{n(n+1)(n+2)}} \right| = \left| \frac{n}{n+3} \right| < 1$$

for $n \geq 1$. This means the absolute value of the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -2$.

At $x = 0$, the power series reduces to $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$. The n^{th} term of this series satisfies

$$a_n = \frac{1}{n(n+1)(n+2)} < \frac{1}{n^3} = b_n$$

for all $n \geq 1$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p -series because $p = 3 > 1$, by the Comparison

Test, the series $\sum_{k=1}^{\infty} a_k$ converges. So the power series converges for $x = 0$.

Concluding, the interval of convergence of the power series is $\boxed{-2 \leq x \leq 0}$.

43. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k x^k}{k!}$ is a series of nonzero terms for $x \neq 0$ (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}}{\frac{n^n x^n}{n!}} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{n!}{(n+1)!} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| (n+1) \cdot \left(1 + \frac{1}{n}\right)^n \cdot \frac{n!}{(n+1)n!} \right| = |x| \lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^n \right| \\ &= |x|e < 1 \end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges if $|x| < \frac{1}{e}$, or $-\frac{1}{e} < x < \frac{1}{e}$, and the

radius of convergence is $\boxed{R = \frac{1}{e}}$. In what follows below, we shall use a useful approximation called

Stirling's approximation to help us analyze the various limits, and to put bounds on the n^{th} terms. This states that for large n , the value of $n!$ is given by

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n}\right) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

A proof of this approximation is found in most books on analysis, and shall not be given here, as it is not within the scope of the textbook. Without this result, one cannot perform the analyses to follow.

At $x = -\frac{1}{e}$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k \left(-\frac{1}{e}\right)^k}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{k^k}{k! e^k}.$$

The n^{th} term of this series satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n^n}{n! e^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{n}{e}\right)^n}{n!} = \lim_{n \rightarrow \infty} \frac{\left(\frac{n}{e}\right)^n}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi n}} = 0$$

using Stirling's approximation quoted above. By the Algebraic Ratio Test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{n^{n+1}}{(n+1)! e^{n+1}}}{\frac{n^n}{n! e^n}} \right| = \frac{n^n \cdot n}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{1}{e} = \frac{n}{n+1} \cdot \frac{1}{e} < 1$$

if $(n+1)e > n$, which is satisfied for all $n \geq 1$, since $e > 1$. This means the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -\frac{1}{e}$.

At $x = \frac{1}{e}$, the power series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k \left(\frac{1}{e}\right)^k}{k!} = \sum_{k=1}^{\infty} \frac{k^k}{k! e^k}.$$

The n^{th} term of this series satisfies

$$a_n = \frac{n^n}{n! e^n} = \frac{\left(\frac{n}{e}\right)^n}{n!} = \frac{\left(\frac{n}{e}\right)^n}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = \frac{1}{\sqrt{2\pi n}} > \frac{1}{\sqrt{3\pi n}} = b_n$$

for large $n > N$, where Stirling's approximation was again used. Since the series

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{3\pi k}} = \frac{1}{\sqrt{3\pi}} \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ is a constant multiple of a divergent p -series (divergent since

$0 < p = \frac{1}{2} < 1$), by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ also diverges. (Note that convergence or divergence of a series is not governed by the first N terms which are finite in number, and the sum of whose terms are finite in value.) So the power series diverges for $x = \frac{1}{e}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{1}{e} \leq x < \frac{1}{e}}$.

44. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k (x-2)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{3^k (x-2)^k}{k!}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 1.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{n+1} (x-2)^{n+1}}{(n+1)!}}{\frac{3^n (x-2)^n}{n!}} \right| = 3|x-2| \lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)!} \right| \\ &= 3|x-2| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = 3|x-2| \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0 \end{aligned}$$

for all values of x . Since the limit is less than 1, by the Ratio Test, the series converges for any value of x . This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is

$$\boxed{-\infty < x < \infty}.$$

45. [In the problems 45-48, we shall extensively use the following result: If $|r| < 1$, then the geometric series $\sum_{k'=0}^{\infty} ar^{k'} = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$, where $k' = k-1$. This allows us to start a geometric series at $k' = 0$ and use the standard results of Section 8.2.]

(a) The domain of $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k$ is the interval of convergence of the power series that defines

f . The series is a geometric series of the form $\sum_{k=0}^{\infty} r^k$, so it converges if $|r| = \left|\frac{x}{3}\right| < 1$, or $|x| < 3$, or

$-3 < x < 3$. So the domain of f is $\boxed{-3 < x < 3}$.

(b) $x = 2$ and $x = -1$ are in the domain $-3 < x < 3$ of f . So we evaluate:

$$f(2) = \sum_{k=0}^{\infty} \frac{2^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1 - \frac{2}{3}} = \frac{1}{\frac{1}{3}} = \boxed{3}.$$

$$f(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k} = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k = \frac{1}{1 - (-\frac{1}{3})} = \frac{1}{1 + \frac{1}{3}} = \frac{1}{\frac{4}{3}} = \boxed{\frac{3}{4}}.$$

(c) Since f is defined by a geometric series, we can find its sum in its domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = \frac{1}{1 - \frac{x}{3}} = \boxed{\frac{3}{3-x} \text{ for } -3 < x < 3}.$$

46. (a) The domain of $f(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^k = \sum_{k=0}^{\infty} \left(-\frac{x}{2}\right)^k = 1 + \sum_{k=1}^{\infty} \left(-\frac{x}{2}\right)^k$ is the interval of convergence of the power series that defines f . The series is geometric of the form $\sum_{k=0}^{\infty} r^k$, so it converges if

$|r| = \left|-\frac{x}{2}\right| = \left|\frac{x}{2}\right| < 1$, or $|x| < 2$, or $-2 < x < 2$. So the domain of f is $\boxed{-2 < x < 2}$.

(b) $x = 0$ and $x = 1$ are in the domain $-2 < x < 2$ of f . So we evaluate:

$$f(0) = 1 + \sum_{k=1}^{\infty} \left(-\frac{0}{2}\right)^k = 1 + 0 = \boxed{1}.$$

$$f(1) = \sum_{k=0}^{\infty} (-1)^k \frac{1^k}{2^k} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}.$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k} = \sum_{k=0}^{\infty} \left(-\frac{x}{2}\right)^k = \frac{1}{1 - (-\frac{x}{2})} = \frac{1}{1 + \frac{x}{2}} = \boxed{\frac{2}{2+x} \text{ for } -2 < x < 2}.$$

47. (a) The domain of $f(x) = \sum_{k=0}^{\infty} \frac{(x-2)^k}{2^k} = 1 + \sum_{k=1}^{\infty} \frac{(x-2)^k}{2^k}$ is the interval of convergence of the power series that defines f . The series is geometric of the form $\sum_{k=0}^{\infty} r^k$, so it converges if $|r| = \left|\frac{x-2}{2}\right| < 1$, or $|x-2| < 2$, or $-2 < x-2 < 2$, or $0 < x < 4$. So the domain of f is $\boxed{0 < x < 4}$.

(b) $x = 1$ and $x = 2$ are in the domain $0 < x < 4$ of f . So we evaluate:

$$f(1) = \sum_{k=0}^{\infty} \frac{(1-2)^k}{2^k} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}.$$

$$f(2) = 1 + \sum_{k=1}^{\infty} \frac{(2-2)^k}{2^k} = 1 + 0 = \boxed{1}.$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x-2}{2}\right)^k = \frac{1}{1 - \left(\frac{x-2}{2}\right)} = \frac{2}{2-x+2} = \boxed{\frac{2}{4-x} \text{ for } 0 < x < 4}.$$

48. (a) The domain of $f = \sum_{k=0}^{\infty} (-1)^k (x+3)^k = \sum_{k=0}^{\infty} [-(x+3)]^k = 1 + \sum_{k=1}^{\infty} [-(x+3)]^k$ is the interval of convergence of the power series that defines f . The series is geometric of the form $\sum_{k=0}^{\infty} r^k$, so it converges if $|r| = |-(x+3)| < 1$, or $-1 < x+3 < 1$, or $-4 < x < -2$. So the domain of f is $\boxed{-4 < x < -2}$.

(b) $x = -3$ and $x = -2.5 = -\frac{5}{2}$ are in the domain $-4 < x < 2$ of f . So we evaluate:

$$f(-3) = 1 + \sum_{k=1}^{\infty} [-(-3+3)]^k = \boxed{1}.$$

$$f(-2.5) = f\left(-\frac{5}{2}\right) = \sum_{k=0}^{\infty} \left[-\left(-\frac{5}{2} + 3\right) \right]^k = \sum_{k=0}^{\infty} \left[-\left(\frac{1}{2}\right) \right]^k = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}.$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} [-(x+3)]^k = \frac{1}{1 - [-(x+3)]} = \frac{1}{1+x+3} = \boxed{\frac{1}{x+4} \text{ for } -4 < x < -2}.$$

49. The series $\sum_{k=0}^{\infty} a_k x^k$ is a power series centered at $x = 0$. If the series is convergent for $x = 3$, then by the theorem on the convergence and divergence of power series, it is convergent for any x in a radius of convergence $R = 3 - 0 = 3$ about the point $x = 0$, that is, in the interval $-3 < x < 3$. Since convergence at $x = 3$ is given, we may extend the interval of convergence to $-3 < x \leq 3$. This means it is convergent at $x=2$, since $x = 2$ is found in the interval $-3 < x \leq 3$. However, the series may or may not converge at $x = 5$. So no, nothing can be said about the convergence at $x=5$.

50. The series $\sum_{k=0}^{\infty} a_k (x-2)^k$ is a power series centered at $x = 2$. If the series converges for $x = 6$, then it will necessarily converge for all other x in a radius of $R = 6 - 2 = 4$ about the point $x = 2$, that is, $-4 < x - 2 < 4$, or for all other x such that $-2 < x < 6$.

51. The series $\sum_{k=0}^{\infty} a_k x^k$ is a power series centered at $x = 0$. If it converges for $x = 6$, then the theorem on convergence and divergence of power series ensures that it will converge absolutely for all x in a radius of $R = 6 - 0 = 6$ about the point $x = 0$, that is, $-6 < x < 6$. Since convergence at $x = 6$ is given, we can extend the range of convergence to at least $-6 < x \leq 6$. If the series diverges for $x = -8$, then it diverges for all $x \leq -8$ and also for all $x > 8$. Based on these observations:

(a) TRUE: The series will converge at $x = 2$ since $-6 < 2 \leq 6$.

(b) FALSE: The series is only guaranteed to converge from (and not necessarily including) $x = -6$, up to (and including) $x = 6$. It may or may not converge in the interval $6 < x \leq 8$, so it may or may not converge for $x = 7$.

(c) FALSE: It follows from the statement of the theorem on convergence/divergence of a power series, see p.602, that convergence of the series at $x = 6$ implies absolute convergence for $|x| < 6$. Nothing is known about absolute convergence at $x = 6$.

(d) FALSE: The theorem only guarantees convergence on an open interval. So the series is guaranteed to converge on $-6 < x < 6$. (Convergence at $x = 6$ is given to us, so this range may be extended to $-6 < x \leq 6$.) The series may or may not converge at $x = -6$.

(e) TRUE: The series diverges at $x = 10$ because it will diverge for all $x > 8$.

(f) TRUE: The theorem guarantees absolute convergence at $x = 4$ since it is in the interval $-6 < x \leq 6$.

52. If the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(x-3)^k$ is 5, it means the series is absolutely convergent for $-5 < x-3 < 5$, or $-2 < x < 8$, and divergent for $x < -2$ and $x > 8$. Based on this:

- (a) *TRUE*: Since $x = 2$ falls in the interval of convergence $-2 < x < 8$.
 (b) *FALSE*: Since $x = 7$ falls within the interval of convergence $-2 < x < 8$, the series will not diverge at $x = 7$.
 (c) *NOT NECESSARILY TRUE*: At $x = 8$, the series may converge or it may diverge.
 (d) *FALSE*: $x = -6$ is in the range $x < -2$, so it will diverge.
 (e) *NOT NECESSARILY TRUE*: At $x = -2$, the series may converge or it may diverge.

53. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{k=0}^{\infty} (-x^3)^k.$$

(b) The geometric series converges if $|r| = |-x^3| < 1$, or $|x| < 1$. So the radius of convergence is $R = 1$ and the interval of convergence is $-1 < x < 1$.

54. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k.$$

(b) The geometric series converges if $|r| = |x^2| < 1$, or $|x| < 1$. So the radius of convergence is $R = 1$ and the interval of convergence is $-1 < x < 1$.

55. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{6-2x} = \left(\frac{1}{6}\right) \frac{1}{1-\frac{2x}{6}} = \left(\frac{1}{6}\right) \frac{1}{1-\frac{x}{3}} = \sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{x}{3}\right)^k.$$

(b) The geometric series converges if $|r| = \left|\frac{x}{3}\right| < 1$, or $|x| < 3$. So the radius of convergence is $R = 3$ and the interval of convergence is $-3 < x < 3$.

56. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{4}{x+2} = \frac{4}{2-(-x)} = \frac{2}{1-\left(-\frac{x}{2}\right)} = \sum_{k=0}^{\infty} 2 \left(-\frac{x}{2}\right)^k.$$

(b) The geometric series converges if $|r| = \left|-\frac{x}{2}\right| < 1$, or $|x| < 2$. So the radius of convergence is $R = 2$ and the interval of convergence is $-2 < x < 2$.

57. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{x}{1+x^3} = \frac{x}{1-(-x^3)} = \sum_{k=0}^{\infty} x \cdot (-x^3)^k = \sum_{k=0}^{\infty} (-1)^{3k} x^{3k+1} = \sum_{k=0}^{\infty} [(-1)^3]^k x^{3k+1} = \sum_{k=0}^{\infty} (-1)^k x^{3k+1}.$$

(b) The geometric series converges if $|r| = |(-x)^3| < 1$. So the radius of convergence is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1}$.

58. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{4x^2}{x+2} = \frac{2x^2}{1 - (-\frac{x}{2})} = \sum_{k=0}^{\infty} 2x^2 \left(-\frac{x}{2}\right)^k = \sum_{k=0}^{\infty} 8 \left(-\frac{x}{2}\right)^2 \left(-\frac{x}{2}\right)^k = \boxed{\sum_{k=0}^{\infty} 8 \left(-\frac{x}{2}\right)^{k+2}}.$$

(b) The geometric series converges if $|r| = \left|-\frac{x}{2}\right| < 1$, or $|x| < 2$. So the radius of convergence is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 < x < 2}$.

59. First, we check to see if the power series representing the function $f(x) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ has a nonzero radius of convergence:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = |x|^2 \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)!} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} = |x|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} \\ &= 0 \text{ for any value of } x. \end{aligned}$$

So the radius of convergence is $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned} f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\ f'(x) &= 1 - \frac{3x^2}{3 \cdot 2!} + \frac{5x^4}{5 \cdot 4!} - \cdots + \frac{(-1)^k (2k+1)x^{2k+1-1}}{(2k+1)(2k)!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}}. \end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned} f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\ \int_0^x f(x) dx &= \frac{x^2}{2} - \frac{x^4}{4 \cdot 3!} + \frac{x^6}{6 \cdot 5!} - \cdots + \frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+2)!}}. \end{aligned}$$

60. First, we check to see if the power series representing the function $f(x) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ has a

nonzero radius of convergence:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}} \right| = |x|^2 \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)!} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = |x|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} \\ &= 0, \text{ for any value of } x.\end{aligned}$$

So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned}f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\ f'(x) &= 0 - \frac{2x}{2 \cdot 1!} + \frac{4x^3}{4 \cdot 3!} - \cdots + \frac{(-1)^k (2k) x^{2k-1}}{(2k)(2k-1)!} + \cdots \\ &= -x + \frac{x^3}{3!} - \cdots + \frac{(-1)^k x^{2k-1}}{(2k-1)!} + \cdots \\ &= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)!}}.\end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned}f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\ \int_0^x f(x) dx &= x - \frac{x^3}{3 \cdot 2!} + \frac{x^5}{5 \cdot 4!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)(2k)!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}.\end{aligned}$$

61. Let's check to see if the power series representing the function $f(x) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ has a nonzero radius of convergence:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = |x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

for all values of x . So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned}f(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \\ f'(x) &= 0 + 1 + \frac{2x}{2 \cdot 1!} + \frac{3x^2}{3 \cdot 2!} + \cdots + \frac{kx^{k-1}}{k(k-1)!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!} + \cdots \\ &= \boxed{\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}}.\end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned}
 f(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \\
 \int_0^x f(x) dx &= x + \frac{x^2}{2 \cdot 1!} + \frac{x^3}{3 \cdot 2!} + \frac{x^4}{4 \cdot 3!} + \cdots + \frac{x^{k+1}}{(k+1)k!} + \cdots \\
 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{k+1}}{(k+1)!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}}.
 \end{aligned}$$

62. Let's check to see if the power series representing the function $f(x) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$ has a nonzero radius of convergence:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{(n+1)!}}{\frac{(-1)^n x^n}{n!}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = |x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

for all values of x . So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned}
 f(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^k x^k}{k!} + \cdots \\
 f'(x) &= 0 - 1 + \frac{2x}{2 \cdot 1!} - \frac{3x^2}{3 \cdot 2!} + \cdots + \frac{(-1)^k k x^{k-1}}{k(k-1)!} + \cdots \\
 &= -1 + x - \frac{x^2}{2!} + \cdots + \frac{(-1)^k x^{k-1}}{(k-1)!} + \cdots \\
 &= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k!}}.
 \end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned}
 f(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^k x^k}{k!} + \cdots \\
 \int_0^x f(x) dx &= x - \frac{x^2}{2 \cdot 1!} + \frac{x^3}{3 \cdot 2!} - \frac{x^4}{4 \cdot 3!} + \cdots + \frac{(-1)^k x^{k+1}}{(k+1)k!} + \cdots \\
 &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{k+1}}{(k+1)!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)!}}.
 \end{aligned}$$

63. We wish to find a power series representation of $f(x) = \frac{1}{(1+x)^2}$. The function

$g(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k$ has a geometric power series representation which converges for

$|r| = |-x| = |x| < 1$. Using the differentiation property of power series, we get

$$\begin{aligned}
 f(x) &= -\frac{dg}{dx} = \frac{1}{(1+x)^2} \\
 &= -\frac{d}{dx} \left[\sum_{k=0}^{\infty} (-x)^k \right] \\
 &= -\frac{d}{dx} [(-x)^0 + (-x)^1 + (-x)^2 + (-x)^3 + \cdots + (-x)^k + \cdots] \\
 &= -\frac{d}{dx} [1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots] \\
 &= -[0 - 1 + 2x - 3x^2 + \cdots + (-1)^k kx^{k-1} + \cdots] \\
 &= 1 - 2x + 3x^2 - \cdots + (-1)^{k+1} kx^{k-1} + \cdots \\
 &= \sum_{k=1}^{\infty} (-1)^{k+1} kx^{k-1} = \sum_{k=0}^{\infty} (-1)^k (k+1)x^k.
 \end{aligned}$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. Note that at the left endpoint $x = -1$ the series representation reduces to $\sum_{k=0}^{\infty} (-1)^{2k} (k+1)$, which is divergent by the

Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\sum_{k=0}^{\infty} (-1)^k (k+1)$, which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} (-1)^k (k+1)x^k, \quad -1 < x < 1.}$$

64. We wish to find a power series representation of $f(x) = \frac{1}{(1-x)^3}$. The function

$$h(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$$

has a geometric power series representation which converges for $|r| = |x| < 1$. Using the differentiation property of power series, we have

$$h'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots + kx^{k-1} + \cdots = \sum_{k=1}^{\infty} kx^{k-1}.$$

To see that this power series has a nonzero radius of convergence, with $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} kx^{k-1}$ being a series of nonzero terms, we compute

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x^n}{nx^{n-1}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+1}{n} = |x| \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = |x| \cdot 1 < 1$$

for convergence by the Ratio Test. Since the condition $|x| < 1$ was already satisfied for the series representation of the function $h(x)$ to be convergent, we see that the series representation of $h'(x)$ is convergent as well. So using the differentiation property of power series again, we have

$$h''(x) = \frac{2}{(1-x)^3} = 2 + 6x + \cdots + k(k-1)x^{k-2} + \cdots = \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \sum_{k'=0}^{\infty} (k'+2)(k'+1)x^{k'},$$

where we set $k' = k - 2$ or $k = k' + 2$. So finally

$$f(x) = \frac{1}{(1-x)^3} = \frac{1}{2} h''(x) = \frac{1}{2} \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{1}{2} \sum_{k=0}^{\infty} (k+2)(k+1)x^k.$$

The interval of convergence is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{1}{2} \sum_{k=0}^{\infty} (-1)^k (k+2)(k+1)$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{1}{2} \sum_{k=0}^{\infty} (k+2)(k+1)$, which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{1}{2} \sum_{k=0}^{\infty} (k+2)(k+1)x^k, \quad -1 < x < 1.}$$

65. We wish to find a power series representation of $f(x) = \frac{2}{3(1-x)^2}$. From the result of Problem 64 above, we have

$$f(x) = \frac{2}{3(1-x)^2} = \frac{2}{3} h'(x) = \frac{2}{3} \sum_{k=1}^{\infty} kx^{k-1}.$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{2}{3} \sum_{k=1}^{\infty} (-1)^{k-1} k$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{2}{3} \sum_{k=1}^{\infty} k$ which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{2}{3} \sum_{k=1}^{\infty} kx^{k-1}, \quad -1 < x < 1.}$$

66. We wish to find a power series representation of $f(x) = \frac{1}{(1-x)^4}$. From the result of Problem 64 above, we have that $h''(x) = \frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} k(k-1)x^{k-2}$. Let us compute the radius of convergence of this series:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)n x^{n-1}}{n(n-1)x^{n-2}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+1}{n-1} = |x| \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} = |x| \cdot 1 < 1$$

for convergence by the Ratio Test. Since the radius of convergence is nonzero, we apply the differentiation property of power series to this function to obtain

$$\begin{aligned} h'''(x) &= \frac{2 \cdot 3}{(1-x)^4} = \frac{d}{dx} [2 + 6x + 12x^2 + \cdots + k(k-1)x^{k-2} + \cdots] \\ &= 0 + 6 + 24x + \cdots + k(k-1)(k-2)x^{k-3} + \cdots \\ &= \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)x^k. \end{aligned}$$

Finally, we obtain

$$f(x) = \frac{1}{(1-x)^4} = \frac{1}{6} h'''(x) = \frac{1}{6} \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \frac{1}{6} \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)x^k.$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{1}{6} \sum_{k=0}^{\infty} (-1)^k (k+3)(k+2)(k+1)$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{1}{6} \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)$, which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{1}{6} \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)x^k, \quad -1 < x < 1.}$$

67. We wish to find a power series representation of $f(x) = \ln\left(\frac{1}{1+x}\right)$. Since

$$g(x) = \frac{1}{1+x} - \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

is a geometric series that converges if $|r| = |-x| = |x| < 1$, we can use the integration property of power series and integrate term by term:

$$\begin{aligned} \int_0^x g(x) dx &= \int_0^x t \frac{dx}{1+x} = \ln(1+x) \\ &= \int_0^x [1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots] dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^k \frac{x^{k+1}}{k+1} + \cdots \\ \text{or,} \quad \ln(1+x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)}. \end{aligned}$$

The function f can be written:

$$f(x) = \ln\left(\frac{1}{1+x}\right) = \ln 1 - \ln(1+x) = 0 - \ln(1+x).$$

So we have

$$f(x) = - \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1}$$

The interval of convergence is at least $|x| < 1$, or $-1 < x < 1$. At the left endpoint $x = -1$, the series representation reduces to $\sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which is the divergent harmonic series. At the right

endpoint $x = 1$, the series representation reduces to $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ which is a constant multiple of the convergent alternating harmonic series.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1}, \quad -1 < x \leq 1.}$$

68. We wish to find a power series representation of $f(x) = \ln(1-2x)$. Consider the function

$$g(x) = \frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k = 1 + 2x + 4x^2 + \cdots + 2^k x^k + \cdots$$

which is a function with a geometric power series representation that converges if $|r| = |2x| < 1$, or $|x| < \frac{1}{2}$. Since the interval of convergence is nonzero, we use the integration property of power series to integrate

term by term:

$$\begin{aligned}
 \int_0^x g(x) dx &= \int_0^x \frac{dx}{1-2x} = \frac{\ln|1-2x|}{-2} \\
 &= \int_0^x [1 + 2x + 4x^2 + \cdots + 2^k x^k + \cdots] dx \\
 &= x + x^2 + \frac{4}{3}x^3 + \cdots + 2^k \frac{x^{k+1}}{k+1} + \cdots \\
 \text{or, } \quad \ln|1-2x| &= -2 \left[x + x^2 + \frac{4}{3}x^3 + \cdots + \frac{2^k x^{k+1}}{k+1} + \cdots \right] = \sum_{k=0}^{\infty} (-1) \frac{2^{k+1} x^{k+1}}{k+1} \\
 &= \sum_{k=0}^{\infty} (-1) \frac{(2x)^{k+1}}{k+1}.
 \end{aligned}$$

The interval of convergence is at least $|x| < \frac{1}{2}$. At the left endpoint $x = -\frac{1}{2}$, the series representation reduces to $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ which is the convergent alternating harmonic series. At the right endpoint, $x = \frac{1}{2}$, the series representation reduces to $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1) \frac{1}{k+1}$ which is a constant multiple of the divergent harmonic series.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} (-1) \frac{(2x)^{k+1}}{k+1}, \quad -\frac{1}{2} \leq x < \frac{1}{2}.}$$

69. We wish to find a power series representation of $f(x) = \ln(1-x^2)$. Consider the function

$$g(x) = \frac{-2x}{1-x^2} = (-2x) \sum_{k=0}^{\infty} (x^2)^k = (-2x) \sum_{k=0}^{\infty} x^{2k} = \sum_{k=0}^{\infty} (-2)x^{2k+1} = -2x - 2x^3 - \cdots - 2x^{2k+1} - \cdots.$$

This is a geometric series that converges if $|r| = |x^2| < 1$, or $|x| < 1$, so the power series representation of the function g has a nonzero radius of convergence. Using the integration property of power series, we get

$$\begin{aligned}
 \int_0^x g(x) dx &= \int_0^x \frac{-2x dx}{1-x^2} = \ln|1-x^2| \\
 &= \int_0^x [-2x - 2x^3 - \cdots - 2x^{2k+1} - \cdots] dx \\
 &= -x^2 - \frac{2}{4}x^4 - \cdots - \frac{2}{2k+2}x^{2k+2} - \cdots \\
 &= -x^2 - \frac{x^4}{2} - \cdots - \frac{x^{2k+2}}{k+1} - \cdots \\
 \text{or, } \quad \ln|1-x^2| &= -\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k+1}.
 \end{aligned}$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$, the series representation reduces to $-\sum_{k=0}^{\infty} \frac{1}{k+1}$, which, being a constant multiple of the harmonic series, diverges. At the right endpoint $x = 1$, the series representation reduces to the same expression as it did at $x = -1$, so once again it diverges.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{-\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k+1}, \quad -1 < x < 1.}$$

70. We wish to find a power series representation of $f(x) = \ln(1 + x^2)$. Consider the function

$$g(x) = \frac{2x}{1+x^2} = \frac{2x}{1-(-x^2)} = 2x \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1} = 2x - 2x^3 + 2x^5 - \cdots + 2(-1)^k x^{2k+1} + \cdots$$

This is a geometric series that converges if $|r| = |x^2| < 1$, or $|x| < 1$, so the power series representation of the function g has a nonzero radius of convergence. Using the integration property of power series, we get

$$\begin{aligned} \int_0^x g(x) dx &= \int_0^x \frac{2x dx}{1+x^2} = \ln|1+x^2| \\ &= \int_0^x [2x - 2x^3 + 2x^5 - \cdots + 2(-1)^k x^{2k+1} + \cdots] dx \\ &= x^2 - \frac{2x^4}{4} + \frac{2x^6}{6} - \cdots + (-1)^k \frac{2x^{2k+2}}{2k+2} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots + (-1)^k \frac{x^{2k+2}}{k+1} + \cdots \end{aligned}$$

or, $\ln(1+x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1}$.

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$, the power series representation reduces to $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ which is the convergent alternating harmonic series. At the right endpoint $x = 1$, we obtain the same series as we had for $x = -1$, so once again the series converges.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1}, \quad -1 \leq x \leq 1.}$$

Applications and Extensions

71. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to zero.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$, or $-1 < x < 1$.

At $x = -1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which is the convergent alternating harmonic series. So the power series converges for $x = -1$.

At $x = 1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ which is the divergent harmonic series. So the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

72. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-4)^k}{k}$ is a series of nonzero terms for $x \neq 4$. (At $x = 4$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-4)^{n+1}}{n+1}}{\frac{(x-4)^n}{n}} \right| = |x-4| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x-4| \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = |x-4| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x - 4 < 1$, or $3 < x < 5$.

At $x = 3$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(3-4)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which is the convergent alternating harmonic series. So the power series converges for $x = 3$.

At $x = 5$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(5-4)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ which is the divergent harmonic series. So the power series diverges for $x = 5$.

Concluding, the interval of convergence of the power series is $\boxed{3 \leq x < 5}$.

73. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{2(n+1)+1}}{\frac{x^n}{2n+1}} \right| = |x| \lim_{n \rightarrow \infty} \frac{2n+1}{2n+3} = |x| \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$, or $-1 < x < 1$.

At $x = -1$, the series reduces to $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}$. The n^{th} term of this series satisfies

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0$. By the Algebraic Ratio Test, we see

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{2(n+1)+1}}{\frac{1}{2n+1}} \right| = \frac{2n+1}{2n+3} < 1$$

for $n \geq 1$. So the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -1$.

At $x = 1$, the series reduces to $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2k+1}$. The n^{th} term of this series satisfies

$$a_n = \frac{1}{2n+1} > \frac{1}{2n+2} = b_n.$$

Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2(k+1)}$ is a constant multiple of the series $\sum_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is

the divergent harmonic series from the second term forward, by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ diverges. So the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

74. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} \right| = |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = |x| \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^2 = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$ or $-1 < x < 1$.

At $x = -1$, the series reduces to $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$. The series of absolute values is $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p -series, since $p = 2 > 1$. Since the original series converges absolutely, it converges. So the power series is convergent for $x = -1$.

At $x = 1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is again a convergent p -series for $p = 2 > 1$. So the power series converges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1}$.

75. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} x^{k^2} = 1 + \sum_{k=1}^{\infty} x^{k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have, applying the Root Test,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|x^{n^2}|} = \lim_{n \rightarrow \infty} |x^{n^2/n}| = \lim_{n \rightarrow \infty} |x^n|.$$

Now, since $\lim_{n \rightarrow \infty} r^n = \infty$ if $|r| > 1$ and $\lim_{n \rightarrow \infty} r^n = 0$ if $|r| < 1$, we require for convergence by the Root Test, $|x| < 1$, or $-1 < x < 1$.

At $x = -1$, the series becomes $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^{k^2} = 1 - 1 + 1 - 1 + \cdots$. Since the sequence of partial sums $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ is not convergent, the power series diverges for $x = -1$.

At $x = 1$, the series becomes $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} 1 = \infty$, so the power series diverges for $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1}$.

76. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^a}{a^k} (x-a)^k$. This is a series of nonzero terms when $x \neq a$. (At $x = a$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^a (x-a)^{n+1}}{a^{n+1}}}{\frac{n^a (x-a)^n}{a^n}} \right| = \left| \frac{x-a}{a} \right| \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^a = \left| \frac{x-a}{a} \right| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x-a| < |a|$, or $-|a| < x-a < |a|$, or $-|a|+a < x < |a|+a$. If we assume $a > 0$, then $|a| = a$, and the power series converges for $0 < x < 2a$. Since $a \neq 0$ has been given, we have an interval of convergence of nonzero length. If we assume $a < 0$, then $|a| = -a$, and the power series converges for $2a < x < 0$.

At $x = 0$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^a}{a^k} (-a)^k = \sum_{k=1}^{\infty} (-1)^k k^a.$$

If $a = -1$, the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which, being an alternating harmonic series, converges. For $-1 < a < 0$, the series becomes an alternating p -series with $0 < p < 1$, which converges by the alternating series test; with $p = -a$, we have: $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{k^p} = 0$; also the terms of the series are nonincreasing, as can be seen by the Algebraic Ratio Test, since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{n^{p+1}}}{\frac{1}{n^p}} = \frac{n^p}{n^{p+1}} < 1$$

for $n \geq 1$. So by the Alternating Series Test, the series converges for $-1 \leq a < 0$ at $x = 0$. For $a > 0$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^k k^a$. By the Divergence Test, $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n^a = \infty$, so the series

diverges. Finally, if $a < -1$, then $p = -a > 1$, so the series becomes $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^p}$, which is a convergent p -series since $p > 1$. Summarizing, at $x = 0$, the power series converges for $a < -1$, $-1 \leq a < 0$, and diverges for $a > 0$.

At $x = 2a$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^a}{a^k} (2a-a)^k = \sum_{k=1}^{\infty} k^a.$$

If $a < -1$, then with $p = -a > 1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$, which is a convergent p -series since

$p > 1$. If $-1 \leq a < 0$, then using $p = -a$, we have $0 < p \leq 1$, which makes the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ a

divergent p -series. For $a > 0$, we use the Divergence Test to see $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^a = \infty$, so the series diverges. Summarizing, for $x = 2a$, the series converges when $a < -1$, and diverges for $-1 \leq a < 0$ and $a > 0$.

Concluding, we see the power series converges for the following ranges of the parameter a :

$$\boxed{a < -1 : 2a \leq x \leq 0; \quad -1 \leq a < 0 : 2a < x \leq 0; \quad a > 0 : 0 < x < 2a.}$$

77. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} (x-1)^k = 1 + \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} (x-1)^k$ is a series of nonzero terms if $x \neq 1$. (At $x = 1$, the series converges to 1.) We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{((n+1)!)^2 (x-1)^{n+1}}{(2(n+1))!}}{\frac{(n!)^2 (x-1)^n}{(2n)!}} \right| = |x-1| \lim_{n \rightarrow \infty} \left\{ \left[\frac{(n+1)!}{n!} \right]^2 \cdot \frac{(2n)!}{(2n+2)!} \right\} \\ &= |x-1| \lim_{n \rightarrow \infty} \left\{ \left[\frac{(n+1)n!}{n!} \right]^2 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right\} \\ &= |x-1| \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = |x-1| \lim_{n \rightarrow \infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n})}{(2+\frac{2}{n})(2+\frac{1}{n})} \\ &= |x-1| \cdot \frac{1}{4} < 1 \end{aligned}$$

for convergence by the Ratio Test if $|x-1| < 4$, or $-4 < x-1 < 4$, or $-3 < x < 5$.

At $x = -3$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} (-3-1)^k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} (-4)^k = \sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{(2k)!} 2^{2k}.$$

As in Problem 43, we have to use Stirling's Approximation,

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

We see the absolute value of the n^{th} term of the series is

$$\begin{aligned} |a_n| &= \frac{n! n! 2^{2n}}{(2n)!} \approx \frac{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right) \left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right) 2^{2n}}{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi \cdot 2n}} \\ &= \frac{n^{2n} \cdot 2^{2n} (2\pi n)}{2^{2n} n^{2n} \sqrt{4\pi n}} = \sqrt{n} \sqrt{\pi} \rightarrow \infty, \text{ as } n \rightarrow \infty \end{aligned}$$

which means the series diverges by the Divergence Test at $x = -3$.

At $x = 5$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} (5-1)^k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} 4^k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} 2^{2k}$$

which is identical to the series of absolute values of the series at $x = -3$: as such, it too diverges by the same calculation used to establish the divergence of the absolute value of the n^{th} term above. So the series diverges by the Divergence Test at $x = 5$.

Concluding, the interval of convergence of the power series is $\boxed{-3 < x < 5.}$

78. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{\sqrt{k!}}{(2k)!} x^k = 1 + \sum_{k=1}^{\infty} \frac{\sqrt{k!}}{(2k)!} x^k$ is a series of nonzero terms for $x \neq 0$. (At

$x = 0$, the series converges to 1.) We have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{\sqrt{(n+1)!} x^{n+1}}{(2(n+1))!}}{\frac{\sqrt{n!} x^n}{(2n)!}} \right| = |x| \lim_{n \rightarrow \infty} \left\{ \sqrt{\frac{(n+1)!}{n!}} \cdot \frac{(2n)!}{(2n+2)!} \right\} \\
 &= |x| \lim_{n \rightarrow \infty} \left\{ \sqrt{\frac{(n+1)n!}{n!}} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right\} = |x| \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{(2n+2)(2n+1)} \\
 &= |x| \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{n}}}{\left(2\sqrt{n} + \frac{2}{\sqrt{n}}\right) \left(2\sqrt{n} + \frac{1}{\sqrt{n}}\right)} \\
 &= 0
 \end{aligned}$$

for any value of x . So the interval of convergence is $\boxed{-\infty < x < \infty}$.

79. (a) We are given

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1.$$

Replacing x with x^2 , we have

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \boxed{\sum_{k=0}^{\infty} x^{2k}}.$$

(b) The interval of convergence is $|x^2| < 1$, as it is a geometric series, so it is $\boxed{-1 < x < 1}$.

80. (a) Using partial fractions, we can write

$$\begin{aligned}
 \frac{1}{1-x^2} &= \frac{1}{2} \left[\frac{1}{1-x} + \frac{1}{1+x} \right] \\
 &= \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + \cdots + x^{2k} + \cdots
 \end{aligned}$$

Since the radius of convergence of the geometric series is $|x| < 1$, it is nonzero, so we may integrate the series term by term to obtain:

$$\begin{aligned}
 \int_0^x \frac{dx}{1-x^2} &= \frac{1}{2} \left[\int_0^x \frac{dx}{1-x} + \int_0^x \frac{dx}{1+x} \right] = \int_0^x [1 + x^2 + x^4 + \cdots + x^{2n} + \cdots] dx \\
 \text{or,} \quad \frac{1}{2} [-\ln|1-x| + \ln|1+x|] &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k+1}}{2k+1} + \cdots \\
 \text{or,} \quad \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| &= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}}.
 \end{aligned}$$

(b) Since the series summed is geometric, it converges at least for $|x| < 1$, or $-1 < x < 1$. At $x = -1$, the series becomes $\sum_{k=0}^{\infty} a_n = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty} \frac{1}{2k+1}$. The absolute value of the n^{th} term of the series satisfies

$$|a_n| = \frac{1}{2n+1} > \frac{1}{2n+2} = \frac{1}{2} \cdot \frac{1}{n+1} = b_n.$$

Since $\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \frac{1}{2} \cdot \frac{1}{k+1}$ is a constant multiple of the divergent harmonic series, by the Comparison Test,

the series $\sum_{k=0}^{\infty} a_n$ diverges as well. At $x = 1$, the series becomes $\sum_{k=0}^{\infty} a_n = \sum_{k=0}^{\infty} \frac{1}{2k+1}$. We can use the same Comparison Test as above to conclude that the series diverges.

So the interval of convergence of the power series is $\boxed{-1 < x < 1.}$

81. We require $\ln \left| \frac{1+x}{1-x} \right| = \ln 2$, which happens when $\frac{1+x}{1-x} = 2$, or $1+x = 2(1-x)$, or $1+x = 2-2x$, or $3x = 1$, or $x = \frac{1}{3}$. So using the results of Problem 80, we have

$$\begin{aligned} \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \\ \frac{1}{2} \ln \left| \frac{1+\frac{1}{3}}{1-\frac{1}{3}} \right| &= \frac{1}{2} \ln \left| \frac{\frac{4}{3}}{\frac{2}{3}} \right| = \frac{1}{2} \ln 2 = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \cdots \\ \text{or, } \ln 2 &= \frac{2}{3} \left[1 + \frac{1}{3^3} + \frac{1}{5 \cdot 3^5} + \cdots \right] \\ &= \frac{2}{3} \left[1 + \frac{1}{27} + \frac{1}{405} + \cdots \right] \\ &\approx \frac{2}{3} [1 + 0.037 + 0.002 + \cdots] \\ &\approx \frac{2}{3} [1.039] \approx 0.6927 \approx \boxed{0.693.} \end{aligned}$$

82. Gregory's series is (see p.608)

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \\ \text{So, } \tan^{-1} 1 &= \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1} + \cdots \\ &\approx \sum_{k=0}^{1000} \frac{(-1)^k}{2k+1} \\ &\approx \boxed{0.7856479} \end{aligned}$$

using a CAS. This means

$$\pi \approx 4(0.7856479) \approx \boxed{3.142592.}$$

83. If R is the radius of convergence of the power series $\sum_{k=1}^{\infty} a_k x^k$, then the power series converges for $|x| < R$. By the Ratio Test, which is applicable if the terms of the series are nonzero (it suffices to assume $a_n \neq 0$ as $n \rightarrow \infty$) for $x \neq 0$ (at $x = 0$, the series converges to 0), we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < R \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

for convergence by the Ratio Test. So we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$, which was to be shown.

84. If the power series $\sum_{k=1}^{\infty} a_k x^k$ has a radius of convergence R , then the power series

$\sum_{k=1}^{\infty} a_k x^{2k} = \sum_{k=1}^{\infty} a_k (x^2)^k = \sum_{k=1}^{\infty} a_k y^k$ also has a radius of convergence R , where $y = x^2$. That is, the series converges for $|y| < R$, or $|x^2| < R$, or for $|x| < \sqrt{R}$. This means the radius of convergence of the power series $\sum_{k=1}^{\infty} a_k x^{2k}$ is $\sqrt{R}$.

85. Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series centered at $x = 0$, with a radius of convergence R and an interval of convergence $-R < x < R$. If the power series is absolutely convergent at $x = R$, then the series of absolute values $\sum_{k=0}^{\infty} |a_k| |x|^k$ is convergent for all $0 \leq |x| \leq R$, by the theorem on convergence and divergence of a power series. This means it is absolutely convergent for $-R \leq x \leq R$, or, it is also absolutely convergent at $x = -R$, which is the other endpoint of the interval of convergence. Similarly, consider a power series $\sum_{k=0}^{\infty} a_k (x - c)^k$ centered at $x = c$ with a radius of convergence R , and an interval of convergence $-R + c < x < R + c$. If this series is absolutely convergent at, say, the right endpoint $x = R + c$, then it is convergent for all x such that $0 \leq |x - c| \leq R$, or $-R \leq x - c \leq R$, or $-R + c \leq x \leq R + c$. That is, it is convergent at $x = -R + c$, which is the other endpoint of the interval of convergence. Conversely, suppose the series is absolutely convergent at the left endpoint, $x = -R + c$, then it is convergent for all x such that $0 \leq |x - c| \leq R$, or $-R \leq x - c \leq R$, or $-R + c \leq x \leq R + c$. In other words, it is also convergent at $x = R + c$. So we have shown that if a power series is absolutely convergent at one of its endpoints, it is absolutely convergent at the other endpoint.

86. From the result of Problem 83, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$ if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists for a convergent power series $\sum_{k=0}^{\infty} a_k x^k$, which converges for $|x| < R$. We will use this result in the calculations to follow.

Let series $\sum_{k=1}^{\infty} k a_k x^{k-1}$ be a series of nonzero terms for $x \neq 0$ for at least large values of k . (That is, we assume that $a_n \neq 0$ as $n \rightarrow \infty$.) At $x = 0$, note that the series converges to 0. Taking the limit of the absolute value of the ratio of the $n + 1^{\text{st}}$ and the n^{th} term of the series, we have

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)a_{n+1}x^n}{n a_n x^{n-1}} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) = |x| \cdot \frac{1}{R} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{|x|}{R} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. That is, the series converges for $|x| < R$, as was to be proved.

The series $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to a_0 .) Taking the limit of the absolute value of the ratio of the $n + 1^{\text{st}}$ and the n^{th} term of the series, we have

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{a_{n+1}x^{n+2}}{n+2}}{\frac{a_n x^{n+1}}{n+1}} \right| = |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = |x| \cdot \frac{1}{R} \cdot \lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \right) = \frac{|x|}{R} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. That is, the series converges for $|x| < R$, as was to be proved.

Challenge Problems

87. The differential equation is $(1 + x^2)y'' - 4xy' + 6y = 0$. Let the solution be represented by a power series of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$, with $a_0 \neq 0$ and $a_1 \neq 0$. We have then:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\ y'' &= 2a_2 + 6a_3 x + \cdots + n(n-1)a_n x^{n-2} + (n+1)n a_{n+1} x^{n-1} + (n+2)(n+1)a_{n+2} x^n + \cdots \\ x^2 y'' &= 2a_2 x^2 + 6a_3 x^3 + \cdots + n(n-1)a_n x^n + \cdots \\ x y' &= a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots + n a_n x^n + \cdots \end{aligned}$$

The coefficient of the x^n term of the series $(1+x^2)y'' - 4xy' + 6y = y'' + x^2y'' - 4xy' + 6y$ is given by

$$[(n+2)(n+1)a_{n+2} + n(n-1)a_n] - 4na_n + 6a_n = 0$$

in order that the differential equation $(1+x^2)y'' - 4xy' + 6y = 0$ be satisfied. That is,

$$\begin{aligned}(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n &= 0 \\(n+2)(n+1)a_{n+2} + (n^2 - n - 4n + 6)a_n &= 0 \\(n+2)(n+1)a_{n+2} + (n^2 - 5n + 6)a_n &= 0 \\(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n &= 0.\end{aligned}$$

Solving for a_{n+2} , we get

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}a_n.$$

We have assumed $a_0 \neq 0$ and $a_1 \neq 0$. Substituting values for n in the recursion relation above, we can determine the coefficients a_n in terms of a_0 and a_1 . So, setting $n = 0, 1, 2, \dots$, we have

$$\begin{aligned}a_2 = a_{0+2} &= -\frac{(0-2)(0-3)}{(0+2)(0+1)}a_0 = -\frac{(-2)(-3)}{2 \cdot 1}a_0 = -\frac{6}{2}a_0 = \boxed{-3a_0} \\a_3 = a_{1+2} &= -\frac{(1-2)(1-3)}{(1+2)(1+1)}a_1 = -\frac{(-1)(-2)}{3 \cdot 2}a_1 = -\frac{2}{6}a_1 = \boxed{-\frac{1}{3}a_1} \\a_4 = a_{2+2} &= -\frac{(2-2)(2-3)}{(2+2)(2+1)}a_2 = 0 \\a_5 = a_{3+2} &= -\frac{(3-2)(3-3)}{(3+2)(3+1)}a_2 = 0.\end{aligned}$$

All the higher even coefficients are 0 because a_{2n} for $n \geq 2$ will be related via the recursion relation to a_{2n-2} , and this to a_{2n-4} , and so forth until $a_4 = 0$ is reached. Similarly, all the higher odd coefficients will be 0 because a_{2n+1} for $n \geq 2$ will be related via the recursion relation to a_{2n-1} and this to a_{2n-3} , and so forth until $a_5 = 0$ is reached.

So the only nonzero coefficients of the series $y(x) = \sum_{k=0}^{\infty} a_k x^k$ are $a_0 \neq 0$, $a_1 \neq 0$ (by assumption) and

$$a_2 = -3a_0 \text{ and } a_3 = -\frac{1}{3}a_1.$$

So finally the solution to the differential equation is

$$\begin{aligned}y(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\&= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3\end{aligned}$$

or, $\boxed{y(x) = a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right)}.$

88. If the power series $\sum_{k=0}^{\infty} a_k 3^k$ converges, we have (if the terms of the series $a_n \neq 0$ as $n \rightarrow \infty$) applying the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} 3^{n+1}}{a_n 3^n} \right| = 3 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

for convergence of the series. This means

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < \frac{1}{3}.$$

To check the convergence of $\sum_{k=1}^{\infty} k a_k 2^k$ (which is also a series of nonzero terms), use the Ratio Test again:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(n+1)a_{n+1}2^{n+1}}{n a_n 2^n} \right| &= 2 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) = 1 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \\ &= 2 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot 1 < 2 \cdot \frac{1}{3} = \frac{2}{3} < 1 \end{aligned}$$

so this series will be convergent.

89. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-2)^k}{k 3^k}$ is a series of nonzero terms at $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1) 3^{n+1}}}{\frac{(x-2)^n}{n 3^n}} \right| = \frac{1}{3} |x-2| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = \frac{|x-2|}{3} \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{|x-2|}{3} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges if $|x-2| < 3$, or $-3 < x-2 < 3$, or $-1 < x < 5$.

At $x = -1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1-2)^k}{k 3^k} = \sum_{k=1}^{\infty} \frac{(-3)^k}{k 3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

which, being the alternating harmonic series, is convergent. So the power series converges for $x = -1$.

At $x = 5$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(5-2)^k}{k 3^k} = \sum_{k=1}^{\infty} \frac{3^k}{k 3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which, being the harmonic series, is divergent. So the power series diverges for $x = 5$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 5}$.

90. Let $-R < x_0 < R$. To prove continuity of $S(x)$ at $x = x_0$, we have to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $|x - x_0| < \delta$, $|S(x) - S(x_0)| < \varepsilon$. From the inequality given to us, we know there is a number $N > 0$ such that for $n > N$, $|S(x) - S_n(x)| < \frac{\varepsilon}{3}$ for $|x| < R$. We also have that for $n > N$, $|S(x_0) - S_n(x_0)| < \frac{\varepsilon}{3}$, since $-R < x_0 < R$. Then, for $n = N + 1$, there exists a $\delta > 0$ such that for all $|x - x_0| < \delta$, $|S_{N+1}(x) - S_{N+1}(x_0)| < \frac{\varepsilon}{3}$ due to the continuity of $S_n(x)$ at x_0 , since the $S_n(x)$ being polynomials in x are continuous in the stated interval, $|x| < R$. We have, for $n = N + 1$,

$$\begin{aligned} |S(x) - S(x_0)| &= |S(x) - S_{N+1}(x) + S_{N+1}(x) - S_{N+1}(x_0) + S_{N+1}(x_0) - S(x_0)| \\ &\leq |S(x) - S_{N+1}(x)| + |S_{N+1}(x) - S_{N+1}(x_0)| + |S_{N+1}(x_0) - S(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

where we have used the triangle inequality. So we have shown that $\lim_{x \rightarrow x_0} S(x) = S(x_0)$, which means $S(x)$ is continuous at $x = x_0$. Since $x = x_0$ was a general point in $|x| < R$, we have proved that $S(x)$ is continuous on $(-R, R)$.

91. Let the power series solution to the differential equation $f''(x) + f(x) = 0$ with the boundary conditions $f(0) = 0$ and $f'(0) = 1$ be given by the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

Since $f(0) = a_0 = 0$, we have that $\boxed{a_0 = 0}$.

Differentiating the series twice, assuming nonzero radius of convergence each time, we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots \\ f''(x) &= 2a_2 + 6a_3x + \cdots + n(n-1)x^{n-2} + (n+1)nx^{n-1} + (n+2)(n+1)x^n + \cdots \end{aligned}$$

Since $f'(0) = a_1 = 1$, we have $\boxed{a_1 = 1}$.

The coefficient of the x^n term of the series $f''(x) + f(x)$ must satisfy

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

for the differential equation $f''(x) + f(x) = 0$ to be satisfied. Solving, and substituting various values of n , we have

$$\begin{aligned} a_{n+2} &= -\frac{a_n}{(n+2)(n+1)} \\ a_2 &= a_{0+2} = -\frac{a_0}{(0+2)(0+1)} = 0, \text{ since } a_0 = 0; \\ a_3 &= a_{1+2} = -\frac{a_1}{(1+2)(1+1)} = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!} = -\frac{1}{3!}, \text{ since } a_1 = 1; \\ a_4 &= a_{2+2} = -\frac{a_2}{(2+2)(2+1)} = 0, \text{ since } a_2 = 0; \\ a_5 &= a_{3+2} = -\frac{a_3}{(3+2)(3+1)} = -\frac{a_3}{5 \cdot 4} = (-1)(-1)\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_1}{5!} = \frac{1}{5!} \text{ since } a_1 = 1, \end{aligned}$$

and so on. Only the odd coefficients are nonzero, with the most general term being of the form

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!} = \frac{(-1)^k}{(2k+1)!}.$$

So, the solution of the differential equation is

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

To find the radius of convergence of $f(x)$, we use the Ratio Test (noting that the terms of the series are nonzero if $x \neq 0$, and the series converges to 0 if $x = 0$):

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| &= |x|^2 \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)!} = |x|^2 \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1 \end{aligned}$$

for all values of x . So the radius of convergence of the series $\boxed{R = \infty}$.

92. The Bessel function of the first kind of order $m \geq 0$ is

$$J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+m)! k!} \left(\frac{x}{2}\right)^{2k+m}.$$

(a) We have, for $m = 0$ and $m = 1$,

$$\begin{aligned} J_0(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! k!} \left(\frac{x}{2}\right)^{2k} \\ J_1(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} \left(\frac{x}{2}\right)^{2k+1}. \end{aligned}$$

Using the product rule of differentiation, and noting that the series has a nonzero radius of convergence (seen using the Ratio Test) we get

$$\begin{aligned}
 \frac{d}{dx}(xJ_1(x)) &= J_1(x) + xJ_1'(x) \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} \left(\frac{x}{2}\right)^{2k+1} + x \cdot \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} (2k+1) \cdot \left(\frac{x}{2}\right)^{2k} \cdot \frac{1}{2} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} [1 + (2k+1)] \left(\frac{x}{2}\right)^{2k+1}; \\
 x^{-1} \frac{d}{dx}(xJ_1(x)) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} \left(\frac{2k+2}{2}\right) \left(\frac{x}{2}\right)^{2k} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} (k+1) \left(\frac{x}{2}\right)^{2k} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! k!} \left(\frac{x}{2}\right)^{2k} \\
 &= J_0(x),
 \end{aligned}$$

that is, $J_0(x) = x^{-1} \frac{d}{dx}(xJ_1(x))$, as was to be shown.

(b) Note that for $m = 2$,

$$J_2(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} \left(\frac{x}{2}\right)^{2k+2}.$$

Using the product rule of differentiation, and noting the series has a nonzero radius of convergence (seen using the Ratio Test) we get

$$\begin{aligned}
 \frac{d}{dx}(x^2 J_2(x)) &= 2xJ_2(x) + x^2 J_2'(x) \\
 &= 2x \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} \left(\frac{x}{2}\right)^{2k+2} + x^2 \cdot \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} (2k+2) \left(\frac{x}{2}\right)^{2k+1} \cdot \frac{1}{2} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} \left[\frac{x^{2k+3}}{2^{2k+1}}\right] + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} (k+1) \left[\frac{x^{2k+3}}{2^{2k+1}}\right]; \\
 x^{-2} \frac{d}{dx}(x^2 J_2(x)) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} \left(\frac{x}{2}\right)^{2k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} (k+1) \left(\frac{x}{2}\right)^{2k+1} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)! k!} [1 + (k+1)] \left(\frac{x}{2}\right)^{2k+1} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)(k+1)! k!} (k+2) \left(\frac{x}{2}\right)^{2k+1} \\
 &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)! k!} \left(\frac{x}{2}\right)^{2k+1} \\
 &= J_1(x),
 \end{aligned}$$

that is, $J_1(x) = x^{-2} \frac{d}{dx}(x^2 J_2(x))$, as was to be shown.

8.9 Infinite Series

Concepts and Vocabulary

1. *Taylor Series*: See p.613 for the full definition.
2. *Maclaurin*: See p.613 for the full definition.

Skill Building

3. To express $f(x) = \ln(1 - x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \ln(1 - x) & f(0) = \ln(1 - 0) = 0 \\
 f'(x) = -\frac{1}{1 - x} & f'(0) = -\frac{1}{1 - 0} = -1 \\
 f''(x) = -\frac{1}{(1 - x)^2} & f''(0) = -\frac{1}{(1 - 0)^2} = -1 \\
 f'''(x) = -\frac{2}{(1 - x)^3} & f'''(0) = -\frac{2}{(1 - 0)^3} = -2 = -2! \\
 f^{(4)}(x) = -\frac{3 \cdot 2}{(1 - x)^4} & f^{(4)}(0) = -\frac{3 \cdot 2}{(1 - 0)^4} = -3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= \frac{0}{0!} x^0 + \frac{-1}{1!} x^1 + \frac{-1}{2!} x^2 + \frac{-2!}{3!} x^3 + \frac{-3!}{4!} x^4 + \cdots + \frac{-(n-1)!}{n!} x^n + \cdots \\
 &= 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^n}{n} - \cdots \\
 &= -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.
 \end{aligned}$$

4. To express $f(x) = \ln(1 + x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \ln(1 + x) & f(0) = \ln(1 + 0) = 0 \\
 f'(x) = \frac{1}{1 + x} & f'(0) = \frac{1}{1 + 0} = 1 \\
 f''(x) = -\frac{1}{(1 + x)^2} & f''(0) = -\frac{1}{(1 + 0)^2} = -1 \\
 f'''(x) = \frac{2}{(1 + x)^3} & f'''(0) = \frac{2}{(1 + 0)^3} = 2 = 2! \\
 f^{(4)}(x) = -\frac{3 \cdot 2}{(1 + x)^4} & f^{(4)}(0) = -\frac{3 \cdot 2}{(1 + 0)^4} = -3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= \frac{0}{0!} x^0 + \frac{1}{1!} x^1 + \frac{-1}{2!} x^2 + \frac{2!}{3!} x^3 + \frac{-3!}{4!} x^4 + \cdots + \frac{(-1)^{n+1}(n-1)!}{n!} x^n + \cdots \\
 &= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}}.
 \end{aligned}$$

5. To express $f(x) = \frac{1}{1-x}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \frac{1}{1-x} & f(0) = \frac{1}{1-0} = 1 \\
 f'(x) = \frac{1}{(1-x)^2} & f'(0) = \frac{1}{(1-0)^2} = 1 \\
 f''(x) = \frac{2}{(1-x)^3} & f''(0) = \frac{2}{(1-0)^3} = 2 = 2! \\
 f'''(x) = \frac{3 \cdot 2}{(1-x)^4} & f'''(0) = \frac{3 \cdot 2}{(1-0)^4} = 3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= \frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{2!}{2!} x^2 + \frac{3!}{3!} x^3 + \cdots + \frac{n!}{n!} x^n + \cdots \\
 &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} x^k}.
 \end{aligned}$$

6. To express $f(x) = \frac{1}{1-3x}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \frac{1}{1-3x} & f(0) = \frac{1}{1-3 \cdot 0} = 1 \\
 f'(x) = \frac{3 \cdot 1}{(1-3x)^2} & f'(0) = \frac{3 \cdot 1}{(1-3 \cdot 0)^2} = 3 \cdot 1! \\
 f''(x) = \frac{3^2 \cdot 2 \cdot 1}{(1-3x)^3} & f''(0) = \frac{3^2 \cdot 2 \cdot 1}{(1-3 \cdot 0)^3} = 3^2 \cdot 2! \\
 f'''(x) = \frac{3^3 \cdot 3 \cdot 2!}{(1-3x)^4} & f'''(0) = \frac{3^3 \cdot 3 \cdot 2!}{(1-3 \cdot 0)^4} = 3^3 \cdot 3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 + \frac{3 \cdot 1!}{1!} x + \frac{3^2 \cdot 2!}{2!} x^2 + \frac{3^3 \cdot 3!}{3!} x^3 + \cdots + \frac{3^n \cdot n!}{n!} x^n + \cdots \\
 &= 1 + 3x + 9x^2 + 27x^3 + \cdots + 3^n x^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} 3^k x^k}
 \end{aligned}$$

7. To express $f(x) = \frac{1}{(1+x)^2}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \frac{1}{(1+x)^2} & f(0) = \frac{1}{(1+0)^2} = 1 \\
 f'(x) = -\frac{2}{(1+x)^3} & f'(0) = -\frac{2}{(1+0)^3} = -2 = -2! \\
 f''(x) = \frac{3 \cdot 2}{(1+x)^4} & f''(0) = \frac{3 \cdot 2}{(1+0)^4} = 3! \\
 f'''(x) = -\frac{4 \cdot 3 \cdot 2}{(1+x)^5} & f'''(0) = -\frac{4 \cdot 3 \cdot 2}{(1+0)^5} = -4! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 - \frac{2!}{1!} x + \frac{3!}{2!} x^2 - \frac{4!}{3!} x^3 + \cdots + (-1)^n \frac{(n+1)!}{n!} x^n + \cdots \\
 &= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n (n+1)x^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k (k+1)x^k}
 \end{aligned}$$

8. To express $f(x) = (1+x)^{-3}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = (1+x)^{-3} & f(0) = (1+0)^{-3} = 1 \\
 f'(x) = (-3)(1+x)^{-4} & f'(0) = (-3)(1+0)^{-4} = -3 \\
 f''(x) = (-3)(-4)(1+x)^{-5} & f''(0) = (-3)(-4)(1+0)^{-5} = 3 \cdot 4 \\
 f'''(x) = (-3)(-4)(-5)(1+x)^{-6} & f'''(0) = (-3)(-4)(-5)(1+0)^{-6} = -3 \cdot 4 \cdot 5 \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 - 3x + \frac{3 \cdot 4}{2!} x^2 - \frac{3 \cdot 4 \cdot 5}{3!} x^3 + \cdots + (-1)^n \frac{3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)}{n!} x^n + \cdots \\
 &= 1 - 3x + \frac{3 \cdot 4}{2} x^2 - \frac{4 \cdot 5}{2} x^3 + \cdots + (-1)^n \frac{(n+1)(n+2)}{2} x^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2} x^k}.
 \end{aligned}$$

9. To express $f(x) = \frac{1}{1+x^2}$ as a Maclaurin series, first we find the Maclaurin series for the simpler function $g(x) = \frac{1}{1+x}$ by evaluating it and its derivatives at $x = 0$.

$$\begin{array}{ll}
 g(x) = \frac{1}{1+x} & g(0) = \frac{1}{1+0} = 1 \\
 g'(x) = -\frac{1}{(1+x)^2} & g'(0) = -\frac{1}{(1+0)^2} = -1 \\
 g''(x) = \frac{2}{(1+x)^3} & g''(0) = \frac{2}{(1+0)^3} = 2 = 2! \\
 g'''(x) = -\frac{3 \cdot 2}{(1+x)^4} & g'''(0) = -\frac{3 \cdot 2}{(1+0)^4} = -3 \cdot 2 = -3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series of $g(x)$ is

$$\begin{aligned}
 g(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k \\
 &= 1 - \frac{1}{1!} x + \frac{2!}{2!} x^2 - \frac{3!}{3!} x^3 + \cdots + (-1)^n \frac{n!}{n!} x^n + \cdots \\
 &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots
 \end{aligned}$$

Then the Maclaurin expansion of $f(x) = \frac{1}{1+x^2} = g(x^2)$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} (x^2)^k \\
 &= 1 - x^2 + (x^2)^2 - (x^2)^3 + \cdots + (-1)^n (x^2)^n + \cdots \\
 &= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}}.
 \end{aligned}$$

10. To express $f(x) = \frac{1}{1+2x^3}$ as a Maclaurin series, first we find the Maclaurin series for the simpler

function $g(x) = \frac{1}{1+x}$ by evaluating it and its derivatives at $x = 0$.

$$\begin{array}{ll}
 g(x) = \frac{1}{1+x} & g(0) = \frac{1}{1+0} = 1 \\
 g'(x) = -\frac{1}{(1+x)^2} & g'(0) = -\frac{1}{(1+0)^2} = -1 \\
 g''(x) = \frac{2}{(1+x)^3} & g''(0) = \frac{2}{(1+0)^3} = 2 = 2! \\
 g'''(x) = -\frac{3 \cdot 2}{(1+x)^4} & g'''(0) = -\frac{3 \cdot 2}{(1+0)^4} = -3 \cdot 2 = -3! \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series of $g(x)$ is

$$\begin{aligned}
 g(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k \\
 &= 1 - \frac{1}{1!}x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \cdots + (-1)^n \frac{n!}{n!}x^n + \cdots \\
 &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots
 \end{aligned}$$

Then the Maclaurin expansion of $f(x) = \frac{1}{1+2x^3} = g(2x^3)$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} (2x^3)^k \\
 &= 1 - 2x^3 + (2x^3)^2 - (2x^3)^3 + \cdots + (-1)^n (2x^3)^n + \cdots \\
 &= 1 - 2x^3 + 4x^6 - 8x^9 + \cdots + (-1)^n 2^n x^{3n} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k 2^k x^{3k}}.
 \end{aligned}$$

11. To express $f(x) = e^{3x}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = e^{3x} & f(0) = e^{3 \cdot 0} = e^0 = 1 \\
 f'(x) = 3e^{3x} & f'(0) = 3e^{3 \cdot 0} = 3e^0 = 3 \\
 f''(x) = 3^2 e^{3x} & f''(0) = 3^2 e^{3 \cdot 0} = 3^2 e^0 = 3^2 \\
 f'''(x) = 3^3 e^{3x} & f'''(0) = 3^3 e^{3 \cdot 0} = 3^3 e^0 = 3^3 \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 + \frac{3}{1!}x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \cdots + \frac{3^n}{n!}x^n + \cdots \\
 &= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots + \frac{3^n}{n!}x^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{3^k x^k}{k!}}.
 \end{aligned}$$

12. To express $f(x) = e^{x/2}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = e^{x/2} & f(0) = e^{0/2} = e^0 = 1 \\
 f'(x) = \frac{1}{2}e^{x/2} & f'(0) = \frac{1}{2}e^{0/2} = \frac{1}{2}e^0 = \frac{1}{2} \\
 f''(x) = \left(\frac{1}{2}\right)^2 e^{x/2} & f''(0) = \left(\frac{1}{2}\right)^2 e^{0/2} = \left(\frac{1}{2}\right)^2 e^0 = \left(\frac{1}{2}\right)^2 \\
 f'''(x) = \left(\frac{1}{2}\right)^3 e^{x/2} & f'''(0) = \left(\frac{1}{2}\right)^3 e^{0/2} = \left(\frac{1}{2}\right)^3 e^0 = \left(\frac{1}{2}\right)^3 \\
 \vdots & \vdots
 \end{array}$$

The Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 + \frac{1}{2} \frac{x^1}{1!} + \left(\frac{1}{2}\right)^2 \frac{x^2}{2!} + \left(\frac{1}{2}\right)^3 \frac{x^3}{3!} + \cdots + \left(\frac{1}{2}\right)^n \frac{x^n}{n!} + \cdots \\
 &= 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \cdots + \left(\frac{1}{2}\right)^n \frac{x^n}{n!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{x^k}{2^k k!}}.
 \end{aligned}$$

13. To express $f(x) = \sin(\pi x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = \sin(\pi x) & f(0) = 0 \\
 f'(x) = \pi \cos(\pi x) & f'(0) = \pi \cos(0) = \pi \\
 f''(x) = -\pi^2 \sin(\pi x) & f''(0) = 0 \\
 f'''(x) = -\pi^3 \cos(\pi x) & f'''(0) = -\pi^3 \cos(0) = -\pi^3 \\
 f^{(4)}(x) = \pi^4 \sin(\pi x) & f^{(4)}(0) = 0 \\
 \vdots & \vdots
 \end{array}$$

For derivatives of odd order, $f^{(2n+1)}(0) = (-1)^n \pi^{2n+1}$. For derivatives of even order, $f^{(2n)}(0) = 0$. So the Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots + \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1} x^{2k+1}}{(2k+1)!}}.
 \end{aligned}$$

14. To express $f(x) = \cos(2x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at

$x = 0$.

$$\begin{array}{ll}
 f(x) = \cos(2x) & f(0) = 1 \\
 f'(x) = -2\sin(2x) & f'(0) = 0 \\
 f''(x) = -2^2\cos(2x) & f''(0) = -2^2 \\
 f'''(x) = 2^3\sin(2x) & f'''(0) = 0 \\
 f^{(4)}(x) = 2^4\cos(2x) & f^{(4)}(0) = 2^4 \\
 \vdots & \vdots
 \end{array}$$

For derivatives of even order, $f^{(2n)}(0) = (-1)^n 2^{2n}$. For derivatives of odd order, $f^{(2n+1)}(0) = 0$. So the Maclaurin series is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \cdots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}}.
 \end{aligned}$$

15. To find the Taylor expansion of $f(x) = e^x$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = e^x & f(1) = e \\
 f'(x) = e^x & f'(1) = e \\
 f''(x) = e^x & f''(1) = e \\
 f'''(x) = e^x & f'''(1) = e \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k \\
 &= e + e(x - 1) + \frac{e(x - 1)^2}{2!} + \frac{e(x - 1)^3}{3!} + \cdots + \frac{e(x - 1)^n}{n!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{e(x - 1)^k}{k!}}.
 \end{aligned}$$

16. To find the Taylor expansion of $f(x) = e^{2x}$ about $c = -1$, we begin by evaluating $f(x)$ and its derivatives at $x = -1$.

$$\begin{array}{ll}
 f(x) = e^{2x} & f(-1) = e^{-2} \\
 f'(x) = 2e^{2x} & f'(-1) = 2e^{-2} \\
 f''(x) = 2^2 e^{2x} & f''(-1) = 2^2 e^{-2} \\
 f'''(x) = 2^3 e^{2x} & f'''(-1) = 2^3 e^{-2} \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = -1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= e^{-2} + 2e^{-2}(x+1) + \frac{2^2 e^{-2}(x+1)^2}{2!} + \frac{2^3 e^{-2}(x+1)^3}{3!} + \cdots + \frac{2^n e^{-2}(x+1)^n}{n!} + \cdots \\
 &= \frac{1}{e^2} + \frac{2}{e^2}(x+1) + \frac{2}{e^2}(x+1)^2 + \frac{4}{3e^2}(x+1)^3 + \cdots + \frac{2^n}{e^2} \frac{(x+1)^n}{n!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{2^k (x+1)^k}{e^2 k!}}.
 \end{aligned}$$

17. To find the Taylor expansion of $f(x) = \ln x$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = \ln x & f(1) = \ln 1 = 0 \\
 f'(x) = \frac{1}{x} & f'(1) = \frac{1}{1} = 1 \\
 f''(x) = -\frac{1}{x^2} & f''(1) = -\frac{1}{1^2} = -1 \\
 f'''(x) = \frac{2}{x^3} & f'''(1) = \frac{2}{1^3} = 2 = 2! \\
 f^{(4)}(x) = -\frac{3 \cdot 2}{x^4} & f^{(4)}(1) = -\frac{3 \cdot 2}{1^4} = -3 \cdot 2 = -3! \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots + (-1)^{n+1} \frac{(n-1)!}{n!}(x-1)^n + \cdots \\
 &= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n+1} \frac{(x-1)^n}{n} + \cdots \\
 &= \boxed{\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}}.
 \end{aligned}$$

18. To find the Taylor expansion of $f(x) = \sqrt{x} = x^{1/2}$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = x^{1/2} & f(1) = 1 \\
 f'(x) = \frac{1}{2}x^{-1/2} & f'(1) = \frac{1}{2} \\
 f''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} & f''(1) = -\frac{1}{2^2} \\
 f'''(x) = \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)x^{-5/2} & f'''(1) = \frac{1 \cdot 3}{2^3} \\
 f^{(4)}(x) = \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-5/2} & f^{(4)}(1) = -\frac{1 \cdot 3 \cdot 5}{2^4} \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 1 + \frac{1}{1!}(x-1) + \frac{-\frac{1}{2^2}}{2!}(x-1)^2 + \frac{\frac{1 \cdot 3}{2^3}}{3!}(x-1)^3 + \frac{-\frac{1 \cdot 3 \cdot 5}{2^4}}{4!}(x-1)^4 + \cdots + \frac{(-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}}{n!}(x-1)^n + \cdots \\
 &= \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \cdots}
 \end{aligned}$$

19. To find the Taylor expansion of $f(x) = \frac{1}{x}$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = \frac{1}{x} & f(1) = \frac{1}{1} = 1 \\
 f'(x) = -\frac{1}{x^2} & f'(1) = -\frac{1}{1^2} = -1 \\
 f''(x) = \frac{2}{x^3} & f''(1) = \frac{2}{1^3} = 2 = 2! \\
 f'''(x) = -\frac{3 \cdot 2}{x^4} & f'''(1) = -\frac{3 \cdot 2}{1^4} = -3! \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 1 - \frac{1}{1!}(x-1)^1 + \frac{2!}{2!}(x-1)^2 - \frac{3!}{3!}(x-1)^3 + \cdots + (-1)^n \frac{n!}{n!}(x-1)^n + \cdots \\
 &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k (x-1)^k}
 \end{aligned}$$

20. To find the Taylor expansion of $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$ about $c = 4$, we begin by evaluating $f(x)$ and its derivatives at $x = 4$.

$$\begin{array}{ll}
 f(x) = x^{-1/2} & f(4) = 4^{-1/2} = (2^2)^{-1/2} = 2^{-1} \\
 f'(x) = -\frac{1}{2}x^{-3/2} & f'(4) = -\frac{1}{2}(2^2)^{-3/2} = -\frac{1}{2} \cdot 2^{-3} \\
 f''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} & f''(4) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2^2)^{-5/2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)2^{-5} \\
 f'''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} & f'''(4) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(2^2)^{-7/2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)2^{-7} \\
 \vdots & \vdots
 \end{array}$$

The Taylor expansion about $c = 4$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 2^{-1} - 1 \frac{(1)(2^{-3})(x-4)}{2 \cdot 1!} + \frac{(1 \cdot 3)(2^{-5})(x-4)^2}{2^2 \cdot 2!} - \frac{(1 \cdot 3 \cdot 5)(2^{-7})(x-4)^3}{2^3 \cdot 3!} + \cdots + \\
 &\quad + (-1)^n \frac{(1 \cdot 3 \cdot 5 \cdots (2n-1))(2^{-(2n+1)})(x-4)^n}{2^n \cdot n!} + \cdots \\
 &= \boxed{\frac{1}{2} - \frac{x-4}{16} + \frac{3(x-4)^2}{256} - \frac{5(x-4)^3}{2048} + \cdots}.
 \end{aligned}$$

21. To find the Taylor expansion of $f(x) = \sin x$ at $c = \frac{\pi}{6}$, we begin by evaluating $f(x)$ and its derivatives at $x = \frac{\pi}{6}$.

$$\begin{array}{ll}
 f(x) = \sin x & f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
 f'(x) = \cos x & f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\
 f''(x) = -\sin x & f''\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \\
 f'''(x) = -\cos x & f'''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \\
 f^{(4)}(x) = \sin x & f^{(4)}\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
 \vdots & \vdots
 \end{array}$$

For the even derivatives $f^{(2k)}\left(\frac{\pi}{6}\right) = (-1)^k \frac{1}{2}$. For the odd derivatives, $f^{(2k+1)}\left(\frac{\pi}{6}\right) = (-1)^k \frac{\sqrt{3}}{2}$. The Taylor expansion about $c = \frac{\pi}{6}$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= \frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{1!} \left(x - \frac{\pi}{6}\right) + \frac{-\frac{1}{2}}{2!} \left(x - \frac{\pi}{6}\right)^2 + \frac{-\frac{\sqrt{3}}{2}}{3!} \left(x - \frac{\pi}{6}\right)^3 + \frac{\frac{1}{2}}{4!} \left(x - \frac{\pi}{6}\right)^4 + \cdots + \\
 &\quad + \frac{(-1)^n \frac{1}{2}}{(2n)!} \left(x - \frac{\pi}{6}\right)^{2n} + \frac{(-1)^n \frac{\sqrt{3}}{2}}{(2n+1)!} \left(x - \frac{\pi}{6}\right)^{2n+1} + \cdots \\
 &= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{4} \left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48} \left(x - \frac{\pi}{6}\right)^4 + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{\sin\left(\frac{1}{6}(\pi + 3k\pi)\right) \left(x - \frac{\pi}{6}\right)^k}{k!}}.
 \end{aligned}$$

22. To find the Taylor expansion of $f(x) = \cos x$ about $c = -\frac{\pi}{2}$, we begin by evaluating $f(x)$ and its

derivatives at $x = -\frac{\pi}{2}$.

$$\begin{array}{ll}
 f(x) = \cos x & f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0 \\
 f'(x) = -\sin x & f'\left(-\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \\
 f''(x) = -\cos x & f''\left(-\frac{\pi}{2}\right) = -\cos\left(-\frac{\pi}{2}\right) = 0 \\
 f'''(x) = \sin x & f'''\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1 \\
 f^{(4)}(x) = \cos x & f^{(4)}\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0 \\
 \vdots & \vdots
 \end{array}$$

For the even derivatives $f^{(2n)}\left(-\frac{\pi}{2}\right) = 0$. For the odd derivatives $f^{(2n+1)}\left(-\frac{\pi}{2}\right) = (-1)^n$. The Taylor expansion about $c = -\frac{\pi}{2}$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 0 + \frac{1}{1!} \left(x + \frac{\pi}{2}\right) + \frac{0}{2!} \left(x + \frac{\pi}{2}\right)^2 + \frac{(-1)}{3!} \left(x + \frac{\pi}{2}\right)^3 + \frac{0}{4!} \left(x + \frac{\pi}{2}\right)^4 + \cdots + \frac{(-1)^n}{(2n+1)!} \left(x + \frac{\pi}{2}\right)^{2n+1} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{\left(x + \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}}.
 \end{aligned}$$

23. To find the Taylor expansion of $f(x) = 3x^2 + 2x^2 + 5x - 6$ about $c = 0$, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = 3x^2 + 2x^2 + 5x - 6 & f(0) = 3(0)^2 + 2(0)^2 + 5(0) - 6 = -6 \\
 f'(x) = 9x^2 + 4x + 5 & f'(0) = 9(0)^2 + 4(0) + 5 = 5 \\
 f''(x) = 18x + 4 & f''(0) = 18(0) + 4 = 4 \\
 f'''(x) = 18 & f'''(0) = 18 \\
 f^{(4)}(x) = 0 & f^{(4)}(0) = 0.
 \end{array}$$

All the higher order derivatives will be zero, so their evaluation at $x = 0$ will also be zero. The Taylor expansion about $c = 0$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= -6 + \frac{5}{1!}x + \frac{4}{2!}x^2 + \frac{18}{3!}x^3 \\
 &= -6 + 5x + 2x^2 + 3x^3 \\
 &= \boxed{3x^3 + 2x^2 + 5x - 6}.
 \end{aligned}$$

We see that the Taylor expansion of the function about $c = 0$ is identical to the function itself.

24. To find the Taylor expansion of $f(x) = 4x^4 - 2x^3 - x$ about $c = 0$, we begin by evaluating $f(x)$ and its

derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = 4x^4 - 2x^3 - x & f(0) = 4(0)^4 - 2(0)^3 - 0 = 0 \\
 f'(x) = 16x^3 - 6x^2 - 1 & f'(0) = 16(0)^3 - 6(0)^2 - 1 = -1 \\
 f''(x) = 48x^2 - 12x & f''(0) = 48(0)^2 - 12(0) = 0 \\
 f'''(x) = 96x - 12 & f'''(0) = 96(0) - 12 = -12 \\
 f^{(4)}(x) = 96 & f^{(4)}(0) = 96 \\
 f^{(5)}(x) = 0 & f^{(5)}(0) = 0.
 \end{array}$$

All the higher order derivatives will be zero, so their evaluation at $x = 0$ will also be zero. The Taylor expansion about $c = 0$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k \\
 &= 0 + \frac{-1}{1!}x + \frac{0}{2!}x^2 + \frac{-12}{3!}x^3 + \frac{96}{4!}x^4 \\
 &= -x - 2x^3 + 4x^4 \\
 &= \boxed{4x^4 - 2x^3 - x}.
 \end{aligned}$$

We see that the Taylor expansion of the function about $c = 0$ is identical to the function itself.

25. To find the Taylor expansion of $f(x) = 3x^3 + 2x^2 + 5x - 6$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = 3x^3 + 2x^2 + 5x - 6 & f(1) = 3(1)^3 + 2(1)^2 + 5(1) - 6 = 3 + 2 + 5 - 6 = 4 \\
 f'(x) = 9x^2 + 4x + 5 & f'(1) = 9(1)^2 + 4(1) + 5 = 9 + 4 + 5 = 18 \\
 f''(x) = 18x + 4 & f''(1) = 18(1) + 4 = 22 \\
 f'''(x) = 18 & f'''(1) = 18 \\
 f^{(4)}(x) = 0 & f^{(4)}(1) = 0.
 \end{array}$$

All the higher order derivatives will be zero, so their evaluation at $x = 1$ will also be zero. The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k \\
 &= 4 + \frac{18}{1!}(x - 1) + \frac{22}{2!}(x - 1)^2 + \frac{18}{3!}(x - 1)^3 \\
 &= \boxed{4 + 18(x - 1) + 11(x - 1)^2 + 3(x - 1)^3} \\
 &= 4 + 18x - 18 + 11(x^2 - 2x + 1) + 3(x^3 - 3x^2 + 3x - 1) \\
 &= 4 + 18x - 18 + 11x^2 - 22x + 11 + 3x^3 - 9x^2 + 9x - 3 \\
 &= (4 - 18 + 11 - 3) + (18x - 22x + 9x) + (11x^2 - 9x^2) + 3x^3 \\
 &= -6 + 5x + 2x^2 + 3x^3 \\
 &= 3x^3 + 2x^2 + 5x - 6.
 \end{aligned}$$

We see that the Taylor expansion of the function about $c = 1$ is identical to the function itself.

26. To find the Taylor expansion of $f(x) = 4x^4 - 2x^3 + x$ about $c = 1$, we begin by evaluating $f(x)$ and its

derivatives at $x = 1$.

$$\begin{array}{ll}
 f(x) = 4x^4 - 2x^3 + x & f(1) = 4(1)^4 - 2(1)^3 + 1 = 4 - 2 + 1 = 3 \\
 f'(x) = 16x^3 - 6x^2 + 1 & f'(1) = 16(1)^3 - 6(1)^2 + 1 = 16 - 6 + 1 = 11 \\
 f''(x) = 48x^2 - 12x & f''(1) = 48(1)^2 - 12(1) = 48 - 12 = 36 \\
 f'''(x) = 96x - 12 & f'''(1) = 96(1) - 12 = 96 - 12 = 84 \\
 f^{(4)}(x) = 96 & f^{(4)}(1) = 96 \\
 f^{(5)}(x) = 0 & f^{(5)}(1) = 0.
 \end{array}$$

All the higher order derivatives will be zero, so their evaluation at $x = 1$ will also be zero. The Taylor expansion about $c = 1$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 3 + \frac{11}{1!}(x-1) + \frac{36}{2!}(x-1)^2 + \frac{84}{3!}(x-1)^3 + \frac{96}{4!}(x-1)^4 \\
 &= \boxed{3 + 11(x-1) + 18(x-1)^2 + 14(x-1)^3 + 4(x-1)^4} \\
 &= 3 + 11x - 11 + 18(x^2 - 2x + 1) + 14(x^3 - 3x^2 + 3x - 1) + 4(x^4 - 4x^3 + 6x^2 - 4x + 1) \\
 &= (3 - 11 + 18 - 14 + 4) + (11x - 36x + 42x - 16x) + (18x^2 - 42x^2 + 24x^2) + (14x^3 - 16x^3) + 4x^4 \\
 &= 0 + x + 0 - 2x^3 + 4x^4 \\
 &= 4x^4 - 2x^3 + x.
 \end{aligned}$$

We see that the Taylor expansion of the function about $c = 1$ is identical to the function itself.

27. The function $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$ can be expressed in terms of the the exponential function, $g(x) = e^x$ whose Maclaurin series is

$$e^x = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

since all the derivatives of $g(x) = e^x$ are e^x itself, and when evaluated at $x = 0$, all the derivatives are equal in value to 1. Substituting $-x$ for x , we also have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots.$$

So we have for the Maclaurin expansion of $\sinh x$,

$$\begin{aligned}
 f(x) &= \frac{e^x - e^{-x}}{2} \\
 &= \frac{1}{2} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots \right) \right] \\
 &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
 &= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}}.
 \end{aligned}$$

28. From Problem 27, use the Maclaurin expansion

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

So we get for the Maclaurin expansion of $f(x) = e^{-x^2}$,

$$f(x) = g(-x^2) = e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}}.$$

29. The Maclaurin expansion of $f(x) = xe^x$ is obtained by multiplying the Maclaurin expansion of x , which is x , and that of $g(x) = e^x$ where (see Problem 27),

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots.$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned} f(x) &= xg(x) \\ &= x \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \right) \\ &= \boxed{x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots}. \end{aligned}$$

30. The Maclaurin expansion of $f(x) = xe^{-x}$ is obtained by multiplying the Maclaurin expansion of x , which is x , and that of $g(-x) = e^{-x}$ where (see problem 27),

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots.$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned} f(x) &= xg(-x) \\ &= x \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots \right) \\ &= \boxed{x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots}. \end{aligned}$$

31. The Maclaurin expansion of $f(x) = e^{-x} \sin x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\sin x$ together. From Problem 27, we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots.$$

From p.616 of the book, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots.$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned}
 f(x) &= e^{-x} \sin x \\
 &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right) \\
 &= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
 &= 1 \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - x \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^2}{2} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \\
 &\quad - \frac{x^3}{6} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^4}{24} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - \frac{x^5}{120} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \cdots \\
 &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(-x^2 + \frac{x^4}{6} - \frac{x^6}{120} + \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{240} - \cdots\right) + \\
 &\quad + \left(-\frac{x^4}{6} + \frac{x^6}{36} - \frac{x^8}{720} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \frac{x^9}{2880} - \cdots\right) + \left(-\frac{x^6}{120} - \cdots\right) + \cdots \\
 &= x - x^2 + x^3 \left(\frac{1}{2} - \frac{1}{6}\right) + x^4 \left(\frac{1}{6} - \frac{1}{6}\right) + x^5 \left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right) + x^6 \left(-\frac{1}{120} + \frac{1}{36} - \frac{1}{120}\right) - \cdots \\
 &= \boxed{x - x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{90}x^6 - \cdots}
 \end{aligned}$$

32. The Maclaurin expansion of $f(x) = e^{-x} \cos x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\cos x$ together. From Problem 27, we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots$$

From p.617 of the book, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned}
 f(x) &= e^{-x} \cos x \\
 &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right) \\
 &= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\
 &= 1 \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - x \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \frac{x^2}{2} \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \\
 &\quad - \frac{x^3}{6} \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \frac{x^4}{24} \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - \frac{x^5}{120} \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \cdots \\
 &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-x + \frac{x^3}{2} - \frac{x^5}{24}\right) + \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48}\right) + \\
 &\quad + \left(-\frac{x^3}{6} + \frac{x^5}{12}\right) + \left(\frac{x^4}{24} - \frac{x^6}{48}\right) + \left(-\frac{x^5}{120}\right) + \cdots \\
 &= 1 - x + x^2 \left(\frac{1}{2} - \frac{1}{2}\right) + x^3 \left(\frac{1}{2} - \frac{1}{6}\right) + x^4 \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right) + \\
 &\quad + x^5 \left(-\frac{1}{24} + \frac{1}{12} - \frac{1}{120}\right) + \cdots \\
 &= \boxed{1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} + \cdots}
 \end{aligned}$$

33. To find a Maclaurin expansion for $f(x) = \frac{1}{\sqrt{1-x^2}}$, we first find the Maclaurin expansion for the function $h(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$, by evaluating the function and its derivatives at $x = 0$.

$$\begin{array}{ll}
 h(x) = (1-x)^{-1/2} & h(0) = 1 \\
 h'(x) = -\frac{1}{2}(1-x)^{-3/2} \cdot (-1) = \frac{1}{2}(1-x)^{-3/2} & h'(0) = \frac{1}{2} \\
 h''(x) = \left(\frac{1}{2}\right) \left(-\frac{3}{2}\right) (1-x)^{-5/2} \cdot (-1) = \left(\frac{1 \cdot 3}{2^2}\right) (1-x)^{-5/2} & h''(0) = \frac{3}{4} \\
 h'''(x) = \left(\frac{1 \cdot 3 \cdot 5}{2^3}\right) (1-x)^{-7/2} & h'''(0) = \frac{15}{8} \\
 h^{(4)}(x) = \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4}\right) (1-x)^{-9/2} & h^{(4)}(0) = \frac{105}{16} \\
 \vdots & \vdots
 \end{array}$$

So the Maclaurin expansion of $h(x) = \frac{1}{\sqrt{1-x}}$ is

$$\begin{aligned}
 h(x) &= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k \\
 &= 1 + \frac{1}{2!}x + \frac{3}{4!}x^2 + \frac{15}{8!}x^3 + \frac{105}{16!}x^4 + \cdots \\
 &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots
 \end{aligned}$$

The Maclaurin expansion of $f(x)$ can now be found:

$$f(x) = \frac{1}{\sqrt{1-x^2}} = h(x^2) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots$$

Integrating this series term by term gives us the required first four terms of the Maclaurin expansion for $g(x) = \sin^{-1} x$:

$$\begin{aligned} g(x) &= \sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \int_0^x f(x) dx \\ &= \int_0^x \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots \right) dx \\ &= \boxed{x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots} \end{aligned}$$

34. To find the Maclaurin series for $f(x) = \tan x$, we begin by evaluating the function and its derivatives at $x = 0$. Let $y = f(x)$, $y_0 = f(0)$, $y' = f'(x)$, $y'_0 = f'(0)$, etc.

$y = \tan x$	$y_0 = 0$
$y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$	$y'_0 = 1 + y_0^2 = 1 + 0 = 1$
$y'' = 2yy'$	$y''_0 = 2y_0y'_0 = 2(0)(1) = 0$
$y''' = 2y'^2 + 2yy''$	$y'''_0 = 2y_0'^2 + 2y_0y''_0$ $= 2(1)^2 + 2(0)(0) = 2$
$y^{(4)} = 4y'y'' + 2y'y''' + 2y'y'''$ $= 6y'y'' + 2yy'''$	$y_0^{(4)} = 6y'_0y''_0 + 2y_0y'''_0$ $= 6(1)(0) + 2(0)(2) = 0$
$y^{(5)} = 6y''^2 + 6y'y''' + 2y'y''' + 2yy^{(4)}$ $= 6y''^2 + 8y'y''' + 2yy^{(4)}$	$y_0^{(5)} = 6y_0''^2 + 8y'_0y_0''' + 2y_0y_0^{(4)}$ $= 6(0) + 8(1)(2) + 2(0)(0) = 16$
$y^{(6)} = 12y''y''' + 8y''y''' + 8y'y^{(4)} + 2y'y^{(4)} + 2yy^{(5)}$ $= 20y''y''' + 10y'y^{(4)} + 2yy^{(5)}$	$y_0^{(6)} = 20y_0''y_0''' + 10y'_0y_0^{(4)} + 2y_0y_0^{(5)}$ $= 20(0)(2) + 10(1)(0) + 2(0)(16)$ $= 0$
$y^{(7)} = 20y'''^2 + 20y''y^{(4)} + 10y''y^{(4)} + 10y'y^{(5)} + 2y'y^{(5)} + 2yy^{(6)}$ $= 20y'''^2 + 30y''y^{(4)} + 12y'y^{(5)} + 2yy^{(6)}$	$y_0^{(7)} = 20y_0'''^2 + 30y_0''y_0^{(4)} + 12y'_0y_0^{(5)} + 2y_0y_0^{(6)}$ $= 20(2^2) + 30(0)(0) + 12(1)(16) + 2(0)(0)$ $= 80 + 192 = 272$
$\vdots$	$\vdots$

The Maclaurin series for $f(x) = \tan x$ to four nonzero terms is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 0 + \frac{1}{1!}x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5 + 0 + \frac{272}{7!}x^7 + \dots \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \end{aligned}$$

By integrating this term by term, one obtains the first four nonzero terms of the Maclaurin series for $g(x)$

because

$$\begin{aligned}
 g(x) &= \ln(\cos x) = -\ln\left(\frac{1}{\cos x}\right) \\
 &= -\ln(\sec x) \\
 &= -\int_0^x \tan x \, dx \\
 &= -\int_0^x \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right) dx \\
 &= -\left(\frac{x^2}{2} + \frac{x^4}{3 \cdot 4} + \frac{2x^6}{15 \cdot 6} + \frac{17x^8}{315 \cdot 8} + \cdots\right) \\
 &= \boxed{-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots}.
 \end{aligned}$$

35. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2}$ is given by

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^{2k} \\
 &= \binom{\frac{1}{2}}{0} x^0 + \binom{\frac{1}{2}}{1} x^2 + \binom{\frac{1}{2}}{2} x^4 + \binom{\frac{1}{2}}{3} x^6 + \cdots \\
 &= 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!} x^4 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} x^6 + \cdots \\
 &= \boxed{1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots}.
 \end{aligned}$$

Since $m = \frac{1}{2}$ satisfies $m > 0$ but m is not an integer, by the conditions of the theorem on p.620, the series converges on the closed interval $\boxed{[-1, 1]}$.

36. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$ is given by

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-1)^k x^k \\
 &= \binom{-\frac{1}{2}}{0} (-1)^0 x^0 + \binom{-\frac{1}{2}}{1} (-1)^1 x^1 + \binom{-\frac{1}{2}}{2} (-1)^2 x^2 + \binom{-\frac{1}{2}}{3} (-1)^3 x^3 + \cdots \\
 &= 1 - \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^2 - \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^3 + \cdots \\
 &= \boxed{1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots}.
 \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the series converges on the half open interval $\boxed{(-1, 1]}$.

37. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = (1+x)^{1/5}$ is given by

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \binom{\frac{1}{5}}{k} x^k \\
 &= \binom{\frac{1}{5}}{0} x^0 + \binom{\frac{1}{5}}{1} x^1 + \binom{\frac{1}{5}}{2} x^2 + \binom{\frac{1}{5}}{3} x^3 + \cdots \\
 &= 1 + \frac{1}{5}x + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)}{2!} x^2 + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{3!} x^3 + \cdots \\
 &= \boxed{1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots}
 \end{aligned}$$

Since $m > 0$ but is not an integer, by the conditions of the theorem on p.620, the series converges on the closed interval $\boxed{[-1, 1]}$.

38. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = (1-x)^{5/3}$ is given by

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \binom{\frac{5}{3}}{k} (-1)^k x^k \\
 &= \binom{\frac{5}{3}}{0} (-1)^0 x^0 + \binom{\frac{5}{3}}{1} (-1)^1 x^1 + \binom{\frac{5}{3}}{2} (-1)^2 x^2 + \binom{\frac{5}{3}}{3} (-1)^3 x^3 + \cdots \\
 &= 1 - \frac{5}{3}x + \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)}{2!} x^2 - \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!} x^3 + \cdots \\
 &= \boxed{1 - \frac{5}{3}x + \frac{5}{9}x^2 - \frac{5}{81}x^3 + \cdots}
 \end{aligned}$$

Since $m > 0$ but is not an integer, by the conditions of the theorem on p.620, the series converges on the closed interval $\boxed{[-1, 1]}$.

39. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{(1+x^2)^{1/2}} = (1+x^2)^{-1/2}$ is given by

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{2k} \\
 &= \binom{-\frac{1}{2}}{0} x^0 + \binom{-\frac{1}{2}}{1} x^2 + \binom{-\frac{1}{2}}{2} x^4 + \binom{-\frac{1}{2}}{3} x^6 + \cdots \\
 &= 1 + \left(-\frac{1}{2}\right) x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^4 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^6 + \cdots \\
 &= \boxed{1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots}
 \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the series converges on the half open interval $\boxed{(-1, 1]}$.

40. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{(1+x)^{3/4}} = (1+x)^{-3/4}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{3}{4}}{k} x^k \\ &= \binom{-\frac{3}{4}}{0} x^0 + \binom{-\frac{3}{4}}{1} x^1 + \binom{-\frac{3}{4}}{2} x^2 + \binom{-\frac{3}{4}}{3} x^3 + \dots \\ &= 1 - \frac{3}{4}x + \frac{(-\frac{3}{4})(-\frac{3}{4}-1)}{2!} x^2 + \frac{(-\frac{3}{4})(-\frac{3}{4}-1)(-\frac{3}{4}-2)}{3!} x^3 + \dots \\ &= \boxed{1 - \frac{3}{4}x + \frac{21}{32}x^2 - \frac{77}{128}x^3 + \dots} \end{aligned}$$

Since $m = -\frac{3}{4}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the series converges on the half open interval $\boxed{(-1, 1]}$.

41. $f(x) = \frac{2x}{\sqrt{1-x}}$. From Problem 36, use the binomial series for

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$$

So the expansion for $f(x)$ is

$$\begin{aligned} f(x) &= \frac{2x}{\sqrt{1-x}} \\ &= 2x \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) \\ &= \boxed{2x + x^2 + \frac{3}{4}x^3 + \frac{5}{8}x^4 + \dots} \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the binomial series $\frac{1}{\sqrt{x-1}}$ converges on the half open interval $(-1, 1]$. Since multiplying the binomial series $\frac{1}{\sqrt{x-1}}$ by $2x$ to obtain $f(x)$ will not change the interval of convergence, we conclude that the series for $f(x)$ converges in the interval $\boxed{(-1, 1]}$.

42. Write $f(x) = \frac{x}{1+x^3} = x(1+x^3)^{-1}$. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $(1+x^3)^{-1}$ is given by

$$\begin{aligned} (1+x^3)^{-1} &= \sum_{k=0}^{\infty} \binom{-1}{k} x^{3k} \\ &= \binom{-1}{0} x^0 + \binom{-1}{1} x^3 + \binom{-1}{2} x^6 + \binom{-1}{3} x^9 + \dots \\ &= 1 - x^3 + \frac{(-1)(-1-1)}{2!} x^6 + \frac{(-1)(-1-1)(-1-2)}{3!} x^9 + \dots \\ &= 1 - x^3 + \frac{2}{2!} x^6 - \frac{2 \cdot 3}{3!} x^9 + \dots \\ &= 1 - x^3 + x^6 - x^9 + \dots \end{aligned}$$

So the expansion of $f(x)$ is

$$\begin{aligned} f(x) &= \frac{x}{1+x^3} \\ &= x(1 - x^3 + x^6 - x^9 + \dots) \\ &= \boxed{x - x^4 + x^7 - x^{10} + \dots} \end{aligned}$$

Since $m = -1$ satisfies $m \leq -1$, by the conditions of the theorem on p.620, the binomial series $\frac{1}{(1+x^3)}$ converges on the open interval $(-1, 1)$. Since multiplying the binomial series $\frac{1}{(1+x^3)}$ by x to obtain $f(x)$ will not change the interval of convergence, we conclude that the series for $f(x)$ converges in the interval $\boxed{(-1, 1)}$.

Applications and Extensions

43. We write $f(x) = \sin^2 x = \frac{1 - \cos(2x)}{2} = \frac{1}{2} - \frac{1}{2} \cos(2x)$. From p.617 of the book, we see that the Maclaurin series for $\cos x$ is given by $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. So The Maclaurin series for $f(x)$ is given by

$$\begin{aligned} f(x) = \sin^2 x &= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\ &= \frac{1}{2} - \frac{1}{2} \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots \right] \\ &= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2} - \frac{2^3 x^4}{24} + \frac{2^5 x^6}{720} - \frac{2^7 x^8}{40320} + \cdots + (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!} + \cdots \\ &= \boxed{x^2 - \frac{x^4}{3} + \frac{2}{45} x^6 - \frac{1}{315} x^8 + \cdots} \end{aligned}$$

44. We write $f(x) = \cos^2 x = \frac{1 + \cos(2x)}{2} = \frac{1}{2} + \frac{1}{2} \cos(2x)$. From p.617 of the book, we see that the Maclaurin series for $\cos x$ is given by $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. So The Maclaurin series for $f(x)$ is given by

$$\begin{aligned} f(x) = \cos^2 x &= \frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\ &= \frac{1}{2} + \frac{1}{2} \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots \right] \\ &= \frac{1}{2} + \frac{1}{2} - \frac{2x^2}{2} + \frac{2^3 x^4}{24} - \frac{2^5 x^6}{720} + \frac{2^7 x^8}{40320} - \cdots + (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!} + \cdots \\ &= \boxed{1 - x^2 + \frac{x^4}{3} - \frac{2}{45} x^6 + \frac{1}{315} x^8 - \cdots} \end{aligned}$$

Check: Since $\cos^2 x = 1 - \sin^2 x$, we could also get this series directly from Problem 43:

$$\begin{aligned} \cos^2 x &= 1 - \sin^2 x \\ &= 1 - \left(x^2 - \frac{x^4}{3} + \frac{2}{45} x^6 - \frac{1}{315} x^8 + \cdots \right) \\ &= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45} x^6 + \frac{1}{315} x^8 - \cdots \end{aligned}$$

45. The Maclaurin expansion for $\sin x$, from p.616 of the book, is $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. Integrating both

sides of this, we have:

$$\begin{aligned}
 \int_0^x \sin t \, dt &= \int_0^x \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots \right] dt \\
 [\cos t] \Big|_0^x &= \left[\frac{t^2}{2} - \frac{t^4}{4 \cdot 3!} + \frac{t^6}{6 \cdot 5!} - \frac{t^8}{8 \cdot 7!} + \cdots + (-1)^n \frac{t^{2n+2}}{(2n+2) \cdot (2n+1)!} + \cdots \right]_0^x \\
 (\cos x - 1) &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!} + \cdots \\
 \cos x &= \boxed{1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}.
 \end{aligned}$$

46. We write $f(x) = \ln \frac{1}{1-x} = -\ln(1-x)$. To find the Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{array}{ll}
 f(x) = -\ln(1-x) & f(0) = -\ln(1-0) = -\ln 1 = 0 \\
 f'(x) = -\frac{1}{1-x} \cdot (-1) = (1-x)^{-1} = \frac{1}{1-x} & f'(0) = \frac{1}{1-0} = 1 \\
 f''(x) = (-1)(1-x)^{-2} \cdot (-1) = (1-x)^{-2} & f''(0) = (1-0)^{-2} = 1! \\
 f'''(x) = (-2)(1-x)^{-3} \cdot (-1) = (2)(1-x)^{-3} & f'''(0) = 2(1-0)^{-3} = 2! \\
 f^{(4)}(x) = (-3)(2)(1-x)^{-4} \cdot (-1) = (3)(2)(1-x)^{-4} & f^{(4)}(0) = 3 \cdot 2(1-0)^{-4} = 3! \\
 \vdots & \vdots \\
 f^{(n)}(x) = (2)(3) \cdots (n-1)(1-x)^{-n} \cdot (-1) = (n-1)!(1-x)^{-n} & f^{(n)}(0) = (n-1)!(1-0)^{-n} = (n-1)! \\
 \vdots & \vdots
 \end{array}$$

So the Maclaurin series of $f(x)$ is

$$\begin{aligned}
 f(x) &= \ln \frac{1}{1-x} \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 0 + \frac{1}{1!} x^1 + \frac{1!}{2!} x^2 + \frac{2!}{3!} x^3 + \frac{3!}{4!} x^4 + \cdots + \frac{(n-1)!}{n!} x^n + \cdots \\
 &= \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots}.
 \end{aligned}$$

Comparing the Maclaurin series with the power series found in Section 8.8, Example 8, p.607, we see they are identical.

47. To find the Maclaurin series for $f(x) = \sec x$, we begin by evaluating $f(x)$ and its derivatives at $x = 0$. To simplify notation, let $y = f(x) = \sec x$. Let $y_0 = f(0)$, $y' = f'(x)$, $y'_0 = f'(0)$, etc. Also note that $\frac{d}{dx} \tan x = \sec^2 x = y^2$. (We'll use this directly for the seventh and eighth derivatives to save writing.) We

compute until we get five nonzero derivatives:

$$\begin{aligned} y &= \sec x \\ y' &= \sec x \tan x \\ &= y \tan x \end{aligned}$$

$$\begin{aligned} y'' &= y' \tan x + y \sec^2 x \\ &= y' \tan x + y y^2 \\ &= y' \tan x + y^3 \end{aligned}$$

$$\begin{aligned} y''' &= y'' \tan x + y' \sec^2 x + 3y^2 y' \\ &= y'' \tan x + y' y^2 + 3y^2 y' \\ &= y'' \tan x + 4y^2 y' \end{aligned}$$

$$\begin{aligned} y^{(4)} &= y''' \tan x + y'' \sec^2 x + 8yy'^2 + 4y^2 y'' \\ &= y''' \tan x + y'' y^2 + 8yy'^2 + 4y^2 y'' \\ &= y''' \tan x + 5y^2 y'' + 8yy'^2 \end{aligned}$$

$$\begin{aligned} y^{(5)} &= y^{(4)} \tan x + y''' \sec^2 x + 10yy' y'' + 5y^2 y''' + 8y'^3 + 16yy' y'' \\ &= y^{(4)} \tan x + y''' y^2 + 10yy' y'' + 5y^2 y''' + 8y'^3 + 16yy' y'' \\ &= y^{(4)} \tan x + 6y^2 y''' + 26yy' y'' + 8y'^3 \end{aligned}$$

$$\begin{aligned} y^{(6)} &= y^{(5)} \tan x + y^{(4)} \sec^2 x + 12yy' y''' + 6y^2 y^{(4)} \\ &\quad + 26y'^2 y'' + 26yy''^2 + 26yy' y''' + 24y'^2 y'' \\ &= y^{(5)} \tan x + y^{(4)} y^2 + 12yy' y''' + 6y^2 y^{(4)} \\ &\quad + 26y'^2 y'' + 26yy''^2 + 26yy' y''' + 24y'^2 y'' \\ &= y^{(5)} \tan x + 7y^{(4)} y^2 + 38yy' y''' + \\ &\quad + 50y'^2 y'' + 26yy''^2 \end{aligned}$$

$$y_0 = \sec 0 = \boxed{1.}$$

$$\begin{aligned} y'_0 &= y_0 \tan 0 \\ &= (1)(0) \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y''_0 &= y'_0 \tan 0 + y_0^3 \\ &= (0)(0) + (1)^3 \\ &= \boxed{1.} \end{aligned}$$

$$\begin{aligned} y'''_0 &= y''_0 \tan 0 + 4y_0^2 y'_0 \\ &= (1)(0) + 4(1)^2(0) \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y^{(4)}_0 &= y'''_0 \tan 0 + 5y_0^2 y''_0 + 8y_0 y_0'^2 \\ &= (0)(0) + 5(1)^2(1) + 8(1)(0) \\ &= \boxed{5.} \end{aligned}$$

$$\begin{aligned} y^{(5)}_0 &= y^{(4)}_0 \tan 0 + 6y_0^2 y'''_0 + 26y_0 y'_0 y''_0 + 8y_0'^3 \\ &= (5)(0) + 6(1)^2(0) + 26(1)(0)(1) + 8(0)^3 \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y^{(6)}_0 &= y^{(5)}_0 \tan 0 + 7y_0^{(4)} y_0^2 + 38y_0 y'_0 y'''_0 + \\ &\quad + 50y_0'^2 y''_0 + 26y_0 y_0''^2 \\ &= (0)(0) + 7(5)(1)^2 + 38(1)(0)(0) + \\ &\quad + 50(0)^2(1) + 26(1)(1)^2 \\ &= 35 + 26 \\ &= \boxed{61.} \end{aligned}$$

$$\begin{aligned}
y^{(7)} &= y^{(6)} \tan x + y^{(5)} y^2 + 7y^{(5)} y^2 + 14y^{(4)} y y' + \\
&\quad + 38y'^2 y''' + 38y y'' y''' + 38y y' y^{(4)} + \\
&\quad + 100y' y''^2 + 50y'^2 y''' + 26y' y''^2 + 52y y'' y''' \\
&= y^{(6)} \tan x + 8y^{(5)} y^2 + 52y^{(4)} y y' + \\
&\quad + 88y''' y'^2 + 90y y'' y''' + 126y' y''^2
\end{aligned}$$

$$\begin{aligned}
y_0^{(7)} &= y_0^{(6)} \tan 0 + 8y_0^{(5)} y_0^2 + 52y_0^{(4)} y_0 y_0' + \\
&\quad + 88y_0''' y_0'^2 + 90y_0 y_0'' y_0''' + 126y_0' y_0''^2 \\
&= (61)(0) + 8(0)(1)^2 + 52(5)(1)(0) + \\
&\quad + 88(0)(0)^2 + 90(1)(1)(0) + 126(0)(1)^2 \\
&= \boxed{0}.
\end{aligned}$$

$$\begin{aligned}
y^{(8)} &= y^{(7)} \tan x + y^{(6)} y^2 + 8y^{(6)} y^2 + 16y^{(5)} y y' + \\
&\quad + 52y^{(5)} y y' + 52y^{(4)} y'^2 + 52y^{(4)} y y'' + \\
&\quad + 88y^{(4)} y'^2 + 176y''' y' y'' + 90y' y'' y''' + \\
&\quad + 90y y'''^2 + 90y y'' y^{(4)} + 126y''^3 + 252y' y'' y''' \\
&= y^{(7)} \tan x + 9y^{(6)} y^2 + 68y^{(5)} y y' + \\
&\quad + 140y^{(4)} y'^2 + 142y^{(4)} y y'' + 518y''' y'' y' + \\
&\quad + 90y y'''^2 + 126y''^3
\end{aligned}$$

$$\begin{aligned}
y_0^{(8)} &= y_0^{(7)} \tan 0 + 9y_0^{(6)} y_0^2 + 68y_0^{(5)} y_0 y_0' + \\
&\quad + 140y_0^{(4)} y_0'^2 + 142y_0^{(4)} y_0 y_0'' + 518y_0''' y_0'' y_0' + \\
&\quad + 90y_0 y_0'''^2 + 126y_0''^3 \\
&= (0)(0) + 9(61)(1) + 68(0)(1)(0) + \\
&\quad + 140(5)(0)^2 + 142(5)(1)(1) + 518(0)(1)(0) + \\
&\quad + 90(1)(0)^2 + 126(1)^3 \\
&= 9 \cdot 61 + 142 \cdot 5 + 126 \cdot 1 \\
&= 549 + 710 + 126 \\
&= \boxed{1385}.
\end{aligned}$$

So the first five nonzero terms of the Maclaurin series are

$$\begin{aligned}
f(x) &= \sec x \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= 1 + \frac{0}{1!} x + \frac{1}{2!} x^2 + \frac{0}{3!} x^3 + \frac{5}{4!} x^4 + \frac{0}{5!} x^5 + \frac{61}{6!} x^6 + \frac{0}{7!} x^7 + \frac{1385}{8!} x^8 + \cdots \\
&= \boxed{1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \frac{61}{720} x^6 + \frac{277}{8064} x^8 + \cdots}
\end{aligned}$$

48. (a) The Maclaurin series for $g(x) = e^x$ (see Problem 27) is

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

So the Maclaurin expansion of $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is given by

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} &= \frac{1}{\sqrt{2\pi}}g\left(-\frac{x^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}}\left[1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \cdots + \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n + \cdots\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots\right] \\ &= \boxed{\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k \cdot k!}}.\end{aligned}$$

(b) Integrating the series found in part (a) term by term, we obtain

$$\begin{aligned}P(a \leq z \leq b) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b \left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 2^2 \cdot 2!} - \frac{x^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot 2^n \cdot n!} + \cdots\right]_a^b \\ &= \boxed{\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{[b^{2k+1} - a^{2k+1}]}{(2k+1)2^k k!}}.\end{aligned}$$

(c) Using the first four terms of the series found in part (b), we get the desired approximation

$$\begin{aligned}P(-0.5 \leq z \leq 0.3) &\approx \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_{-0.5}^{0.3} \\ &= \frac{1}{\sqrt{2\pi}} \left[\left\{(0.3) - \frac{(0.3)^3}{6} + \frac{(0.3)^5}{40} - \frac{(0.3)^7}{336}\right\} - \left\{(-0.5) - \frac{(-0.5)^3}{6} + \frac{(-0.5)^5}{40} - \frac{(-0.5)^7}{336}\right\}\right] \\ &\approx \frac{1}{2.507} [\{0.3 - 0.0045 + 0.000061 - 0.0000006\} - \{-0.5 + 0.02083 - 0.00078 + 0.0000233\}] \\ &\approx \frac{1}{2.507} [0.7755] \\ &\approx \boxed{0.309}.\end{aligned}$$

(d) Using a CAS we find the exact value to be approximately $P(-0.5 \leq z \leq 0.3) \approx \boxed{0.309}$.

49. $\int \frac{1}{1+x^2} dx$. The Maclaurin series of the integrand from Problem 9 is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

Integrating term by term, we get

$$\begin{aligned}\int \frac{1}{1+x^2} dx &= \int [1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots] dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots + C \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}} + C,\end{aligned}$$

where C is an arbitrary constant of integration.

50. $\int \sec x \, dx$. The Maclaurin series of the integrand was worked out in Problem 47:

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots.$$

Integrating term by term, we get

$$\begin{aligned} \int \sec x \, dx &= \int \left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots \right] dx \\ &= x + \frac{1}{2} \frac{x^3}{3} + \frac{5}{24} \frac{x^5}{5} + \frac{61}{720} \frac{x^7}{7} + \frac{277}{8064} \frac{x^9}{9} + \cdots + C \\ &= \boxed{x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72,576}x^9 + \cdots + C}, \end{aligned}$$

where C is an arbitrary constant of integration.

51. $\int e^{x^{1/3}} \, dx$. The Maclaurin series of $g(x) = e^x$ was worked out in Problem 27:

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots.$$

So the integrand can be represented by the Maclaurin series

$$g(x^{1/3}) = e^{x^{1/3}} = 1 + x^{1/3} + \frac{x^{2/3}}{2!} + \frac{x^{3/3}}{3!} + \frac{x^{4/3}}{4!} + \cdots + \frac{x^{n/3}}{n!} + \cdots.$$

Integrating term by term, we get

$$\begin{aligned} \int e^{x^{1/3}} \, dx &= \int \left[1 + x^{1/3} + \frac{x^{2/3}}{2!} + \frac{x^{3/3}}{3!} + \frac{x^{4/3}}{4!} + \cdots + \frac{x^{n/3}}{n!} + \cdots \right] \\ &= x + \frac{x^{1/3+1}}{\frac{1}{3}+1} + \frac{1}{2!} \cdot \frac{x^{2/3+1}}{\frac{2}{3}+1} + \frac{1}{3!} \cdot \frac{x^{3/3+1}}{\frac{3}{3}+1} + \frac{1}{4!} \cdot \frac{x^{4/3+1}}{\frac{4}{3}+1} + \cdots + \frac{1}{n!} \cdot \frac{x^{n/3+1}}{\frac{n}{3}+1} + \cdots + C \\ &= x + \frac{3}{4}x^{4/3} + \frac{1}{2!} \cdot \frac{3}{5}x^{5/3} + \frac{1}{3!} \cdot \frac{3}{6}x^{6/3} + \frac{1}{4!} \cdot \frac{3}{7}x^{7/3} + \cdots + \frac{1}{n!} \cdot \frac{3}{n+3}x^{(n+3)/3} + \cdots + C \\ &= \boxed{\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{3}{k+3} \right) x^{(k+3)/3} + C}, \end{aligned}$$

where C is an arbitrary constant of integration.

52. $\int \ln(1+x) \, dx$. From Problem 4, we can represent the integrand as the following Maclaurin series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots.$$

Integrating term by term we get

$$\begin{aligned} \int \ln(1+x) \, dx &= \int \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \right] dx \\ &= \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \frac{x^5}{4 \cdot 5} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{n \cdot (n+1)} + \cdots + C \\ &= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{k+1}}{k(k+1)} + C}, \end{aligned}$$

where C is an arbitrary constant of integration.

53. If f is an even function, then $f(x) = f(-x)$. Taking derivatives of both sides with respect to x , $f'(x) = f'(-x) \cdot (-1) = -f'(-x)$, using the Chain Rule. So the first derivative is an odd function of x . At $x = 0$, $f'(0) = -f'(0)$, which means $f'(0) = 0$.

If $g(x)$ is an odd function of x , then $g(x) = -g(-x)$. Taking the derivatives of both sides with respect to x , $g'(x) = -g'(-x) \cdot (-1) = g'(-x)$, using the Chain Rule. So the derivative of an odd function is an even function. This means the second derivative of $f(x)$ will be an even function, and possibly evaluate to a nonzero value at $x = 0$. The third derivative will now be odd, and so forth, leading to $f^{(k)}(0) = 0$ for all odd k . This means the Maclaurin series $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ will only have even powers of x , because they are multiplied by (possibly nonzero) even order derivatives evaluated at $x = 0$.

54. If f is an odd function, then $f(x) = -f(-x)$. (Note that this rules out the x^0 term of the Maclaurin series for $f(x)$.) Taking derivatives of both sides with respect to x , $f'(x) = -f'(-x) \cdot (-1) = f'(-x)$, using the Chain Rule. So the first derivative is an even function of x . At $x = 0$, the derivative possibly evaluates to a nonzero value.

If $g(x)$ is an even function of x , then $g(x) = g(-x)$. Taking the derivatives of both sides with respect to x , $g'(x) = g'(-x) \cdot (-1) = -g'(-x)$, using the Chain Rule. Since $g(0) = -g(0)$, we have that $g(0) = 0$. So the derivative of an even function is an odd function. This means the second derivative of $f(x)$ will be an odd function, and it evaluates to zero at $x = 0$. The third derivative will now be even, possibly evaluating to a nonzero value, and so forth, leading to $f^{(k)}(0) = 0$ for all even k . This means the Maclaurin series $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ will only have odd powers of x , because they are multiplied by (possibly nonzero) odd order derivatives evaluated at $x = 0$.

55. The remainder term is given by

$$R_n(x) = \frac{f^{n+1}(u)}{(n+1)!} (x-c)^{n+1},$$

where u belongs to the interval (x, c) . Since the function $f(x) = (1+x)^m$, we have

$$\begin{aligned} f'(x) &= m(1+x)^{m-1} = 1! \cdot \binom{m}{1} (1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} = 2! \cdot \binom{m}{2} (1+x)^{m-2} \\ &\vdots \\ f^{(n+1)}(x) &= m(m-1) \cdots (m-n)(1+x)^{m-(n+1)} = (n+1)! \cdot \binom{m}{n+1} (1+x)^{m-(n+1)} \end{aligned}$$

The remainder term becomes

$$\begin{aligned} R_n(x) &= \frac{f^{n+1}(u)}{(n+1)!} (x-c)^{n+1} \\ &= \binom{m}{n+1} (1+u)^{m-(n+1)} (x-c)^{n+1} \\ &= \binom{m}{n+1} (1+u)^m \left(\frac{x-c}{1+u} \right)^{n+1}. \end{aligned}$$

Now, if m is a nonnegative integer,

$$\binom{m}{n+1} = 0 \text{ for } n+1 > m.$$

So we have $R_n(x) = 0$ if $n > m - 1$. So, as $n \rightarrow \infty$, $R_n(x) \rightarrow 0$, as required. So we have shown that $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ when m is a nonnegative integer.

56. For $m < 0$ the series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ is a series of nonzero terms if $x \neq 0$. (If $x = 0$, the series converges to 1.) Applying the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\binom{m}{n+1} x^{n+1}}{\binom{m}{n} x^n} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{\frac{m(m-1)(m-2)\cdots(m-n-1+1)}{(n+1)!}}{\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left[\frac{n!}{(n+1)!} \cdot \left| \frac{m(m-1)(m-2)\cdots(m-n)}{m(m-1)(m-2)\cdots(m-n+1)} \right| \right] \\ &= |x| \lim_{n \rightarrow \infty} \left[\frac{n!}{(n+1)n!} \cdot |m-n| \right] \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{m-n}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{\frac{m}{n} - 1}{1 + \frac{1}{n}} \right| \\ &= |x| < 1 \end{aligned}$$

for absolute convergence of the series, and $|x| > 1$ for absolute divergence of the series.

57. From p.621, Example 10 of the textbook, the Maclaurin series for $f(x) = \sin^{-1} x$ can be written as

$$f(x) = \sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{2n+1} + \cdots$$

The fractional factor multiplying the n^{th} term can be expressed as follows:

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n)}{(2 \cdot 4 \cdot 6 \cdots (2n))^2} = \frac{(2n)!}{(2^n n!)^2} = \frac{(2n)!}{4^n (n!)^2}.$$

So the series can be written as

$$\sin^{-1} x = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \frac{x^{2k+1}}{(2k+1)}.$$

Since the terms of the series are nonzero for $x \neq 0$, by the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(2(n+1))! x^{2(n+1)+1}}{4^{n+1} ((n+1)!)^2 (2(n+1)+1)}}{\frac{(2n)! x^{2n+1}}{4^n (n!)^2 (2n+1)}} \right| \\ &= |x|^2 \lim_{n \rightarrow \infty} \left[\frac{(2n+2)!}{(2n)!} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{2n+1}{2n+3} \cdot \frac{(n!)^2}{((n+1)!)^2} \right] \\ &= \frac{|x|^2}{4} \lim_{n \rightarrow \infty} \left[\frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \cdot \frac{(n!)^2}{(n+1)^2 (n!)^2} \right] \\ &= \frac{|x|^2}{4} \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \\ &= \frac{|x|^2}{4} \cdot \lim_{n \rightarrow \infty} \frac{\left(2 + \frac{2}{n}\right) \left(2 + \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)^2} \cdot \left(\frac{2+0}{2+0}\right) \\ &= \frac{|x|^2}{4} \cdot \frac{(2+0)(2+0)}{(1+0)^2} \cdot 1 \\ &= \frac{|x|^2}{4} \cdot 4 = |x|^2 < 1 \end{aligned}$$

for convergence of the series. So the series for $\sin^{-1} x$ converges for $-1 < x < 1$. For $x = 1$, the series becomes

$$\begin{aligned}\lim_{x \rightarrow 1} \sin^{-1} x &= 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} + \cdots \\ &< 1 + \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \\ &= \frac{1}{1 - \frac{1}{2}} = 2,\end{aligned}$$

using the formula for the sum of a geometric series. That is to say, the sum of the series for $\sin^{-1} 1$ is bounded from above by 2; also, since the terms of the series are all positive, the sequence of partial sums is increasing. Since the sequence of partial sums is increasing and bounded from above, it converges, so the series converges as well. (The actual sum is $\sin^{-1} 1 = \frac{\pi}{2} < 2$.)

At $x = -1$, the series becomes an alternating version of the series at $x = 1$, namely

$$\sin^{-1}(-1) = 1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} + \cdots.$$

The series of absolute values of this alternating series is precisely the series at $x = 1$, which we have shown converges. Since the alternating series converges absolutely, it converges.

So we have shown that the interval of convergence for the Maclaurin expansion of $f(x) = \sin^{-1} x$ is $[-1, 1]$.

58. (a)

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k,$$

using the sum of a geometric series of the form $\sum_{k=0}^{\infty} r^k$, with convergent sum $\frac{1}{1-r}$ whenever

$$|r| = |-x| = |x| < 1.$$

(b) Setting $x = 1$ in the above formula, we obtain

$$\frac{1}{1+1} = \frac{1}{2} = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

which is the erroneous formula arrived at by Euler.

(c) One of the conditions in writing the series representation of $\frac{1}{1+x}$ was that $|x| < 1$. So we cannot put $x = 1$ in the series representation, as it is divergent there.

Challenge Problems

59. The series given to us can be written as

$$\frac{x^3}{1(3)} - \frac{x^5}{3(5)} + \frac{x^7}{5(7)} - \frac{x^9}{7(9)} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k-1)(2k+1)} = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} x^{2k+1} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right),$$

using partial fractions. At $x = 1$, the series becomes

$$\frac{1}{1(3)} - \frac{1}{3(5)} + \frac{1}{5(7)} - \frac{1}{7(9)} + \cdots = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{2k+1} - \frac{1}{2k-1} \right).$$

Before proceeding further, let us prove that the series at $x = 1$ converges absolutely, which would allow us to rearrange the terms at will. Since the series of absolute values of the series at $x = 1$ satisfies

$$\begin{aligned} \frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \frac{1}{7(9)} + \cdots &< \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \\ &< \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2}, \end{aligned}$$

and the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series (since $p = 2 > 1$), by the Comparison Test for convergence, the series of absolute values at $x = 1$ converges, establishing absolute convergence for the original series at $x = 1$.

Now, Gregory's series for $\tan^{-1} x$ (see p.608, Section 8.8) is given by

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}.$$

At $x = 1$, we have

$$\tan^{-1} 1 = \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}.$$

So,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} - 1.$$

Consider now the sum

$$-\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k-1} = -\sum_{k'=0}^{\infty} (-1)^{k'+1} \frac{1}{2k'+1} = \sum_{k'=0}^{\infty} (-1)^{k'} \frac{1}{2k'+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = \frac{\pi}{4}.$$

(Here, the index change $k = k' + 1$ was made, and then the index was changed back to k in the final step as it is a dummy summation index.) So, the sum of the original series is

$$\frac{1}{2} \left[\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1} - \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k-1} \right] = \frac{1}{2} \left[\frac{\pi}{4} - 1 + \frac{\pi}{4} \right] = \boxed{\frac{\pi}{4} - \frac{1}{2}}.$$

60. From Problem 3, the Maclaurin expansion for $\ln(1-x)$ is

$$\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

So $\ln \frac{1}{1-x} = -\ln(1-x)$ has a Maclaurin expansion

$$\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

Integrating term by term gives us

$$\begin{aligned}
 \int_0^x \ln \frac{1}{1-t} dt &= \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{t^n}{n} + \cdots \right) dt \\
 - \int_0^x \ln(1-t) dt &= \left[\frac{t^2}{1 \cdot 2} + \frac{t^3}{2 \cdot 3} + \frac{t^4}{3 \cdot 4} + \cdots + \frac{t^n}{n(n+1)} + \cdots \right]_0^x \\
 \int_1^{1-x} \ln y dy &= \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots + \frac{x^n}{n(n+1)} + \cdots \\
 [y \ln y - y] \Big|_1^{1-x} &= \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} \\
 \{(1-x) \ln(1-x) - (1-x)\} - \{1 \ln 1 - 1\} &= \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} \\
 \text{or, } \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} &= \boxed{(1-x) \ln(1-x) + x}
 \end{aligned}$$

where the substitution $y = 1 - x$ was made, and the result $\int \ln y dy = x \ln y - y + C$ used. (This can be verified by differentiating the result or directly checked using integration by parts. (Use $u = \ln x$ and $dv = dx$.))

61. The partial sums of the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$ are

$$\begin{aligned}
 S_1 &= \frac{1}{2!} = \frac{1}{2} = \frac{2-1}{2} = \frac{2!-1}{2!} \\
 S_2 &= \frac{1}{2} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} = \frac{6-1}{6} = \frac{3!-1}{3!} \\
 S_3 &= \frac{5}{6} + \frac{3}{4!} = \frac{5}{6} + \frac{3}{24} = \frac{20}{24} + \frac{3}{24} = \frac{23}{24} = \frac{24-1}{24} = \frac{4!-1}{4!} \\
 S_4 &= \frac{23}{24} + \frac{4}{5!} = \frac{23}{24} + \frac{4}{120} = \frac{115}{120} + \frac{4}{120} = \frac{119}{120} = \frac{120-1}{120} = \frac{5!-1}{5!}, \dots
 \end{aligned}$$

In general, the n^{th} partial sum is

$$S_n = \frac{(n+1)! - 1}{(n+1)!}.$$

The sum of the series is the limit of the n^{th} partial sum:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{(n+1)! - 1}{(n+1)!} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)!} \right) = \boxed{1}.$$

62. (a) To show $n! \geq 2^{n-1}$, we write

$$n! = n(n-1)(n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n(n-1)(n-2) \cdots 4 \cdot 3 \cdot 2 \geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 2 = 2^{n-1}.$$

(b) From (a), we have

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

So

$$\begin{aligned}
 0 < s_n &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq \frac{1}{2^{1-1}} + \frac{1}{2^{2-1}} + \frac{1}{2^{3-1}} + \cdots + \frac{1}{2^{n-1}} \\
 &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n-1}.
 \end{aligned}$$

(c) Since from the definition of s_n we have

$$s_{n+1} = s_n + \frac{1}{(n+1)!},$$

we can write $0 < s_n < s_{n+1}$. Now, following the inequality used in part (b), we have

$$\begin{aligned} s_{n+1} &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!} \\ &\leq 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n \\ \text{so that } \frac{1}{2}s_{n+1} &\leq \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n+1}. \end{aligned}$$

$$\begin{aligned} \text{Now, } s_{n+1} - \frac{1}{2}s_{n+1} &= \frac{1}{2}s_{n+1} \leq 1 - \left(\frac{1}{2}\right)^{n+1} \\ \text{or, } s_{n+1} &\leq 2 \left(1 - \left(\frac{1}{2}\right)^{n+1}\right). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} s_{n+1} \leq \lim_{n \rightarrow \infty} 2 \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) = 2,$$

we have $s_{n+1} < 2$ for any n , since s_n is an increasing function of n (due to the fact that $s_n < s_{n+1}$). So finally, we have

$$0 < s_n < s_{n+1} < 2.$$

(d) $t_n = \left[1 + \frac{1}{n}\right]^n$. Applying the binomial expansion to the function t_n we get

$$\begin{aligned} \left[1 + \frac{1}{n}\right]^n &= \binom{n}{0} \left(\frac{1}{n}\right)^0 + \binom{n}{1} \left(\frac{1}{n}\right)^1 + \binom{n}{2} \left(\frac{1}{n}\right)^2 + \binom{n}{3} \left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n} \left(\frac{1}{n}\right)^n \\ &= 1 + \frac{n}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \frac{n(n-1)(n-2) \cdots (n-(n-1))}{n!} \cdot \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!} \left[1 - \frac{1}{n}\right] + \frac{1}{3!} \left[1 - \frac{1}{n}\right] \left[1 - \frac{2}{n}\right] + \cdots + \frac{1}{n!} \left[1 - \frac{1}{n}\right] \left[1 - \frac{2}{n}\right] \cdots \left[1 - \frac{n-1}{n}\right]. \end{aligned}$$

We have for large n ,

$$t_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = s_n + 1.$$

So $t_n < s_n + 1$ for finite n .

(e) Let $t(x) = \left(1 + \frac{1}{x}\right)^x$. Then $\ln t = x \ln \left(1 + \frac{1}{x}\right)$. Differentiating, we get

$$\begin{aligned} \frac{1}{t} t' &= \ln \left(1 + \frac{1}{x}\right) + x \cdot \frac{1}{1 + \frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) \\ &= \ln \left(1 + \frac{1}{x}\right) - \frac{1}{1+x} > 0 \end{aligned}$$

for $x \geq 1$. To see this is the case, proceed as follows. We require

$$\begin{aligned} \ln \left(1 + \frac{1}{x}\right) &> \frac{1}{1+x} \\ 1 + \frac{1}{x} &> e^{\frac{1}{1+x}} \\ \text{or, } e^{\frac{1}{1+x}} &< 1 + \frac{1}{x}. \end{aligned}$$

The following computation proves this last inequality:

$$\begin{aligned} e^{\frac{1}{1+x}} &= 1 + \frac{1}{1+x} + \frac{1}{2!} \frac{1}{(1+x)^2} + \frac{1}{3!} \frac{1}{(1+x)^3} + \cdots + \frac{1}{n!} \frac{1}{(1+x)^n} + \cdots \\ &< 1 + \frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} + \cdots + \frac{1}{(1+x)^n} + \cdots \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{1+x} \right)^k. \end{aligned}$$

Now, the series $\sum_{k=0}^{\infty} r^k = \sum_{k=0}^{\infty} \left(\frac{1}{1+x} \right)^k$ is a geometric series which converges to the sum of $\frac{1}{1-r}$ since $|r| = \left| \frac{1}{1+x} \right| < 1$ for $x \geq 1$. So we have

$$\begin{aligned} e^{\frac{1}{1+x}} &< \sum_{k=0}^{\infty} \left(\frac{1}{1+x} \right)^k \\ &= \frac{1}{1 - \frac{1}{1+x}} \\ &= \frac{x+1}{1+x-1} \\ &= \frac{x+1}{x} \\ &= 1 + \frac{1}{x} \end{aligned}$$

that is,

$$e^{\frac{1}{1+x}} < 1 + \frac{1}{x}$$

for $x \geq 1$ as was to be shown. So we have shown that the function $t(x)$ is an increasing function for $x \geq 1$, which means the function $t_n = t(n)$ is increasing for integers $n \geq 1$. This means

$$0 < t_n < t_{n+1} < s_{n+1} + 1 < 2 + 1 = 3.$$

Since $e = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{n} \right]^n = \lim_{n \rightarrow \infty} t_n$, we see that by the Squeeze theorem,

$$0 \leq \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} t_{n+1} \leq 3$$

or $e = \lim_{n \rightarrow \infty} t_n \leq 3$.

63. From the results of Problem 62, since t_n is an increasing function of n and $\lim_{n \rightarrow \infty} t_n = e$, for any finite value of n , we must have $t_n < e$ for $n > 0$.

64. Let $x \geq 0$. We are given that $\sin t \leq t$. By integrating both sides repeatedly we have:

$$\begin{aligned}
 \int_0^x \sin t \, dt &\leq \int_0^x t \, dt \\
 [-\cos t]_0^x &\leq \frac{x^2}{2} \\
 -\cos x + 1 &\leq \frac{x^2}{2} \\
 \text{or,} \quad 1 - \frac{x^2}{2} &\leq \cos x. \\
 \int_0^x \left(1 - \frac{t^2}{2}\right) dt &\leq \int_0^x \cos t \, dt \\
 \left[t - \frac{t^3}{6}\right]_0^x &\leq [\sin t]_0^x \\
 x - \frac{x^3}{6} &\leq \sin x. \\
 \int_0^x \left(t - \frac{t^3}{6}\right) dt &\leq \int_0^x \sin t \, dt \\
 \left[\frac{t^2}{2} - \frac{t^4}{24}\right]_0^x &\leq [-\cos t]_0^x = -\cos x + 1 \\
 \text{or,} \quad \cos x &\leq 1 - \frac{x^2}{2} + \frac{x^4}{24} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.
 \end{aligned}$$

So we conclude that

$$1 - \frac{x^2}{2!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \text{ for all } x \geq 0.$$

65. $f(x) = (1+x)^x$. Let $y = f(x)$, $y' = f'(x)$, etc, and let $y_0 = f(0)$, $y'_0 = f'(0)$, etc. Also, since $\ln y = x \ln(1+x)$, we can differentiate both sides to get $\frac{y'}{y} = \ln(1+x) + \frac{x}{1+x}$, or $y' = y \left[\ln(1+x) + \frac{x}{1+x} \right]$. To find the Maclaurin expansion of $f(x)$ to four terms we begin by evaluating the function and its

derivatives until the third nonzero derivative is reached.

$$\begin{aligned}
 y &= (1+x)^x \\
 y' &= y \left[\ln(1+x) + \frac{x}{1+x} \right] \\
 y'' &= y' \left[\ln(1+x) + \frac{x}{1+x} \right] + y \left[\frac{1}{1+x} + \frac{(1+x) \cdot 1 - x}{(1+x)^2} \right] \\
 &= y' \left[\ln(1+x) + \frac{x}{1+x} \right] + y \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] \\
 y''' &= y'' \left[\ln(1+x) + \frac{x}{1+x} \right] + y' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + \\
 &\quad + y' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + y \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3} \right] \\
 &= y'' \left[\ln(1+x) + \frac{x}{1+x} \right] + 2y' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + \\
 &\quad + y \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3} \right] \\
 y^{(4)} &= y''' \left[\ln(1+x) + \frac{x}{1+x} \right] + y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + \\
 &\quad + 2y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + 2y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3} \right] + \\
 &\quad + y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3} \right] + y \left[\frac{2}{(1+x)^3} + \frac{6}{(1+x)^4} \right] \\
 &= y''' \left[\ln(1+x) + \frac{x}{1+x} \right] + 3y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2} \right] + \\
 &\quad + 3y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3} \right] + y \left[\frac{2}{(1+x)^3} + \frac{6}{(1+x)^4} \right]
 \end{aligned}$$

$$\begin{aligned}
 y_0 &= (1)^0 = \boxed{1.} \\
 y'_0 &= y_0 \left[\ln 1 + \frac{0}{1+0} \right] \\
 &= (1)[0+0] = \boxed{0.}
 \end{aligned}$$

$$\begin{aligned}
 y''_0 &= y'_0 \left[\ln 1 + \frac{0}{1+0} \right] + y_0 \left[\frac{1}{1+0} + \frac{1}{(1+0)^2} \right] \\
 &= (0)[0+0] + (1)[1+1] = \boxed{2.}
 \end{aligned}$$

$$\begin{aligned}
 y'''_0 &= y''_0 \left[\ln 1 + \frac{0}{1+0} \right] + 2y'_0 \left[\frac{1}{1+0} + \frac{1}{(1+0)^2} \right] + \\
 &\quad + y_0 \left[-\frac{1}{(1+0)^2} - \frac{2}{(1+0)^3} \right] \\
 &= (2)[0+0] + 2(0)[1+1] + (1)[-1-2] = \boxed{-3.}
 \end{aligned}$$

$$\begin{aligned}
 y^{(4)}_0 &= y'''_0 \left[\ln 1 + \frac{0}{1+0} \right] + 3y''_0 \left[\frac{1}{1+0} + \frac{1}{(1+0)^2} \right] + \\
 &\quad + 3y'_0 \left[-\frac{1}{(1+0)^2} - \frac{2}{(1+0)^3} \right] \\
 &\quad + y_0 \left[\frac{2}{(1+0)^3} + \frac{6}{(1+0)^4} \right] \\
 &= (-3)[0+0] + 3(2)[1+1] + \\
 &\quad + 3(0)[-1-2] + (1)[2+6] = \boxed{20.}
 \end{aligned}$$

The first four nonzero terms of the Maclaurin series are

$$\begin{aligned}
 f(x) &= (1+x)^x \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
 &= 1 + \frac{0}{1!}x^1 + \frac{2}{2!}x^2 + \frac{-3}{3!}x^3 + \frac{20}{4!}x^4 + \dots \\
 &= 1 + 0 + \frac{2}{2}x^2 + \frac{-3}{6}x^3 + \frac{20}{24}x^4 + \dots \\
 &= \boxed{1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + \dots}
 \end{aligned}$$

66. Let us prove first the following result: For any positive integer n ,

$$\lim_{x \rightarrow 0} \frac{1}{x^n} e^{-1/x^2} = 0.$$

Using $y = \frac{1}{x}$ and applying L'Hôpital's rule repeatedly, we have

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \frac{1}{x^n} e^{-1/x^2} &= \lim_{y \rightarrow \infty} y^n e^{-y^2} = \lim_{y \rightarrow \infty} \frac{y^n}{e^{y^2}} \\
 &= \lim_{y \rightarrow \infty} \frac{ny^{n-1}}{2ye^{y^2}} = \lim_{y \rightarrow \infty} \frac{ny^{n-2}}{2e^{y^2}} \\
 &= \lim_{y \rightarrow \infty} \frac{n(n-2)y^{n-3}}{2 \cdot 2ye^{y^2}} = \lim_{y \rightarrow \infty} \frac{n(n-2)y^{n-4}}{2 \cdot 2e^{y^2}} \\
 &= \dots = \lim_{y \rightarrow \infty} \frac{n(n-2)(n-4) \dots 4 \cdot 2}{2 \cdot 2 \dots 2 e^{y^2}} \\
 &= 0.
 \end{aligned}$$

We also have

$$\lim_{x \rightarrow 0^-} \frac{1}{x^n} e^{-1/x^2} = \lim_{y \rightarrow -\infty} y^n e^{-y^2} = \lim_{y' \rightarrow \infty} (-y')^n e^{-y'^2} = (-1)^n \lim_{y' \rightarrow \infty} y'^n e^{-y'^2} = 0,$$

where $y' = -y$, using the result just proved above. Since we have shown the two sided limit is zero, we have shown that the limit is zero. The function $f(x) = e^{-1/x^2}$ is continuous at $x = 0$, infinitely differentiable at $x = 0$, and all its derivatives at $x = 0$ are equal to zero, due to the result proved above. For example, the first derivative evaluation at $x = 0$ is:

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \left(-\frac{1}{x^2} \right) e^{-1/x^2} = \frac{2}{x^3} e^{-1/x^2} \\
 f'(0) &= 2 \cdot \lim_{x \rightarrow 0} \frac{1}{x^3} e^{-1/x^2} = 2(0) = 0,
 \end{aligned}$$

because it is an instance of the limit result for $n = 3$. The second derivative evaluated at $x = 0$ is:

$$\begin{aligned}
 f''(x) &= -\frac{2 \cdot 3}{x^4} e^{-1/x^2} + \frac{2 \cdot 2}{x^6} e^{-1/x^2} \\
 f''(0) &= -6 \cdot \lim_{x \rightarrow 0} \frac{1}{x^4} e^{-1/x^2} + 4 \cdot \lim_{x \rightarrow 0} \frac{1}{x^6} e^{-1/x^2} = -6(0) + 4(0) = 0,
 \end{aligned}$$

because we have instances of the limit result for $n = 4$ and $n = 6$. Since all the derivatives $f^{(n)}(0) = 0$, the Maclaurin series of the function, is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0$. However, the function itself is not the zero function. So we have shown that the Maclaurin series of $f(x)$ does not converge to the function $f(x)$.

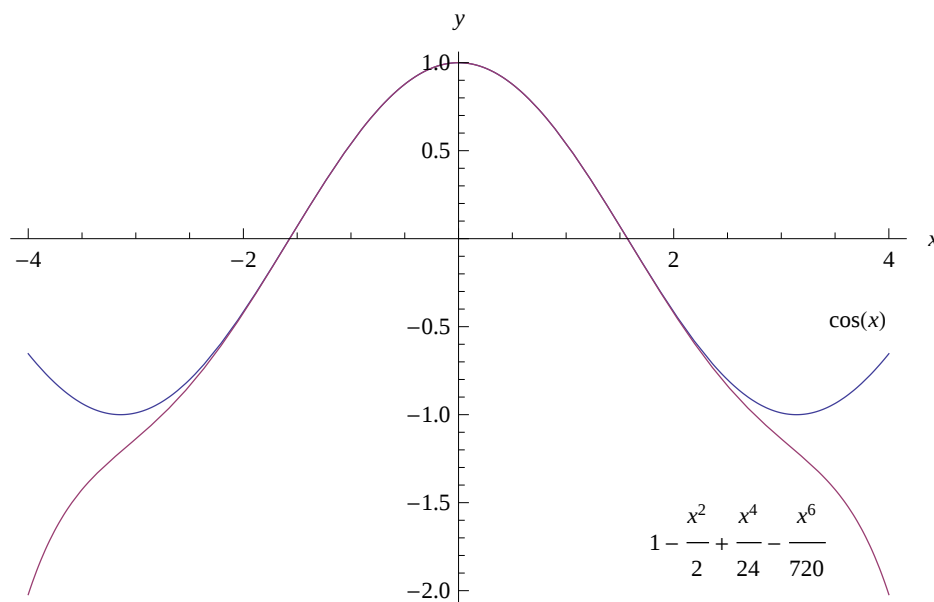
8.10 Approximations Using Taylor/Maclaurin Expansions

Skill Building

1. (a) The Maclaurin series approximation to four nonzero terms of $y = \cos x$ is

$$\begin{aligned}\cos x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ &= \boxed{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}.\end{aligned}$$

- (b) The figure below shows the graph of both the function and its Maclaurin approximation found in part (a).



- (c) Evaluating the approximation at $x = \frac{\pi}{90}$, we get

$$\cos\left(\frac{\pi}{90}\right) \approx 1 - \frac{1}{2!} \left(\frac{\pi}{90}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{90}\right)^4 - \frac{1}{6!} \left(\frac{\pi}{90}\right)^6 \approx \boxed{0.9994}.$$

- (d) Since the series is alternating, the error in the approximation is given by the next term's absolute value in the computation of part (c), which is

$$\frac{1}{8!} \left(\frac{\pi}{90}\right)^8 \approx \boxed{5.47 \times 10^{-17}}.$$

- (e) Since the error is given by $\frac{1}{8!}x^8$, and we need this to be less than 0.0001 or 10^{-4} , we solve:

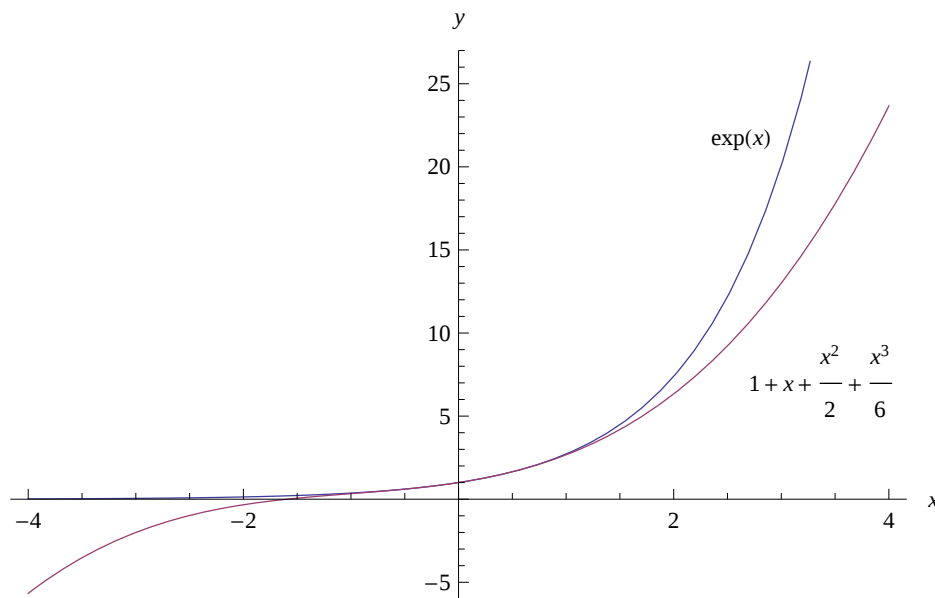
$$\begin{aligned}\frac{1}{8!}x^8 &< 10^{-4} \\ x^8 &< 8! 10^{-4}\end{aligned}$$

Taking the eighth root of both sides of this last inequality, we obtain the approximate interval for x to be $\boxed{-1.1904 < x < 1.1904}$.

2. (a) The Maclaurin series approximation to four terms of $y = e^x$ is

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ &= \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}. \end{aligned}$$

(b) The figure below shows the graph of both the function and its Maclaurin approximation found in part (a).



(c) Evaluating the approximation at $x = \frac{1}{2}$, we get

$$e^{1/2} \approx 1 + \frac{1}{2} + \frac{1}{2!} \left(\frac{1}{2}\right)^2 + \frac{1}{3!} \left(\frac{1}{2}\right)^3 \approx \boxed{1.6458}.$$

(d) The actual value of $e^{1/2} \approx 1.6487$, so the error is $\boxed{0.0029}$.

(e) For the error to be less than 0.0001, we require

$$e^x - \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right] < 0.0001.$$

Since the series is not alternating, we don't have an estimate for the error term. We proceed by trial and error to find $\boxed{|x| < 0.0838}$.

3. (a) The Maclaurin series of $f(x) = \sqrt[3]{1+x}$ is given by the Binomial Series:

$$f(x) = \sqrt[3]{1+x} = (1+x)^{1/3} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} x^k$$

$$= 1 + \frac{1}{3}x + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}x^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}x^4 + \cdots + \binom{\frac{1}{3}}{n}x^n + \cdots$$

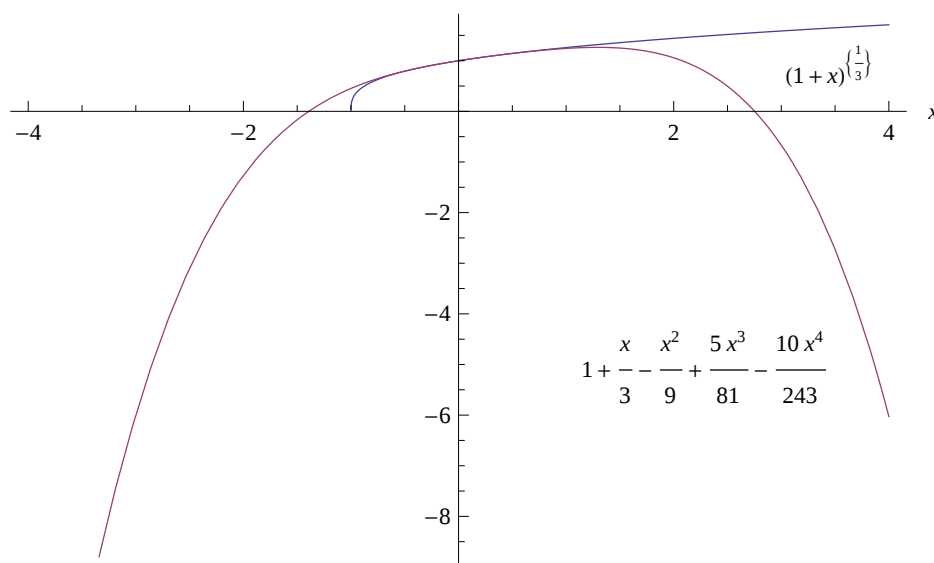
(b) By the theorem on p.620, since $m = \frac{1}{3} > 0$, but not an integer, the binomial series converges for $[-1, 1]$.

(c) Using the first five terms of the Maclaurin series found in part (a), we get

$$f(x) \approx 1 + \frac{1}{3}x + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}x^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}x^4$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4.$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) From the graphs we see that the approximation is good for values of x in the range of convergence $[-1, 1]$, and not so good outside that range.

(f) To approximate $\sqrt[3]{0.9}$, we substitute $x = -0.1$ into the approximation found in part (c):

$$\sqrt[3]{0.9} = \sqrt[3]{1-0.1} \approx 1 + \frac{1}{3}(-0.1) - \frac{1}{9}(-0.1)^2 + \frac{5}{81}(-0.1)^3 - \frac{10}{243}(-0.1)^4 \approx \boxed{0.9655}.$$

Since the series is a convergent alternating series, the error in the approximation is bounded from above by the absolute value of the next term of the approximation, which is

$$E(x) \leq \left| \frac{\binom{\frac{1}{3}}{5} x^5}{5!} \right|$$

$$= \left| \frac{22}{729} x^5 \right|.$$

So the error is less than or equal to

$$E(-0.1) = \left| \frac{22}{729}(-0.1)^5 \right| = \boxed{3.018 \times 10^{-7}}.$$

4. (a) The Maclaurin series of $y = \frac{1}{\sqrt{4+x}}$ is given by the Binomial Series:

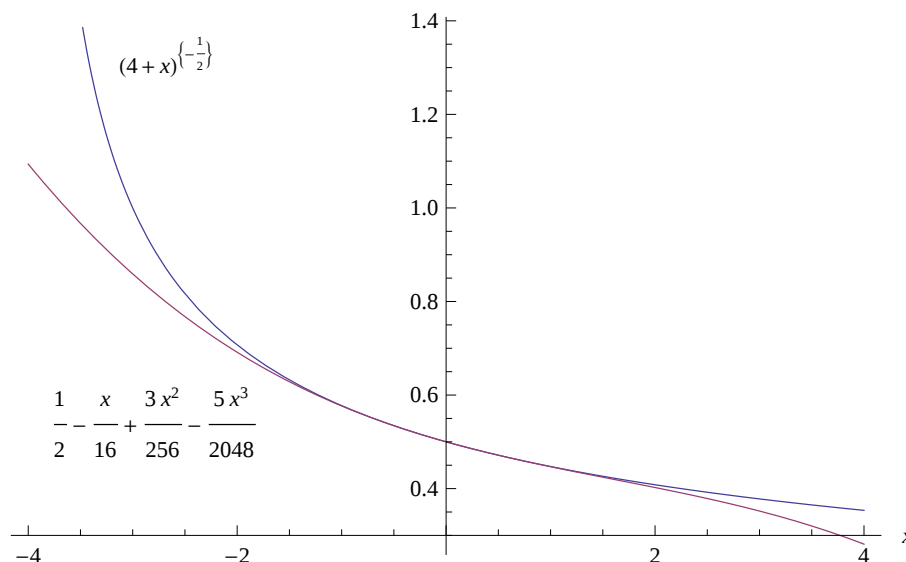
$$\begin{aligned} y &= \frac{1}{\sqrt{4+x}} = (4+x)^{-1/2} = 4^{-1/2} \cdot \left(1 + \frac{x}{4}\right)^{-1/2} \\ &= \frac{1}{2} \cdot \left(1 + \frac{x}{4}\right)^{-1/2} = \frac{1}{2} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(\frac{x}{4}\right)^k \\ &= \frac{1}{2} \left[1 + \binom{-\frac{1}{2}}{1} \frac{x}{4} + \frac{\binom{-\frac{1}{2}}{2} \left(\frac{x}{4}\right)^2}{2!} + \frac{\binom{-\frac{1}{2}}{3} \left(\frac{x}{4}\right)^3}{3!} + \cdots + \binom{-\frac{1}{2}}{n} \left(\frac{x}{4}\right)^n + \cdots \right] \\ &= \boxed{\frac{1}{2} \left[1 - \frac{x}{8} + \frac{3}{128}x^2 - \frac{5}{1024}x^3 + \cdots + \binom{-\frac{1}{2}}{n} \left(\frac{x}{4}\right)^n + \cdots \right]}. \end{aligned}$$

(b) By the theorem on p.620, since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, the interval of convergence is $(-1, 1]$.

(c) Keeping the first four terms of the Maclaurin expansion found in part (a), we have the desired approximation

$$\begin{aligned} y &= \frac{1}{\sqrt{4+x}} \approx \frac{1}{2} \left[1 - \frac{x}{8} + \frac{3}{128}x^2 - \frac{5}{1024}x^3 \right] \\ &= \boxed{\frac{1}{2} - \frac{x}{16} + \frac{3}{256}x^2 - \frac{5}{2048}x^3}. \end{aligned}$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) From the graphs we see that the approximation is good for values of x in the range of convergence $(-1, 1]$ of the series, and not so good outside that range.

(f) To approximate $\frac{1}{\sqrt{4.2}}$ we substitute $x = 0.2$ into the approximation found in part (c):

$$\frac{1}{\sqrt{4.2}} = y(0.2) \approx \frac{1}{2} - \frac{0.2}{16} + \frac{3}{256}(0.2)^2 - \frac{5}{2048}(0.2)^3 \approx \boxed{0.487949}.$$

Since the series is a converging, alternating series, the error in the approximation is bounded from above by the absolute value of the next term of the approximation, which is

$$\begin{aligned} E(x) &\leq \left| \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!} \left(\frac{x}{4}\right)^4 \right| \\ &= \left| \frac{35}{32,768} x^4 \right|. \end{aligned}$$

So the error is less than or equal to

$$E(0.2) = \left| \frac{35}{32,768} (0.2)^4 \right| \approx \boxed{1.71 \times 10^{-6}}.$$

5. (a) From p.608, Example 9, the Maclaurin series of $y = \tan^{-1} x$ is given by Gregory's series:

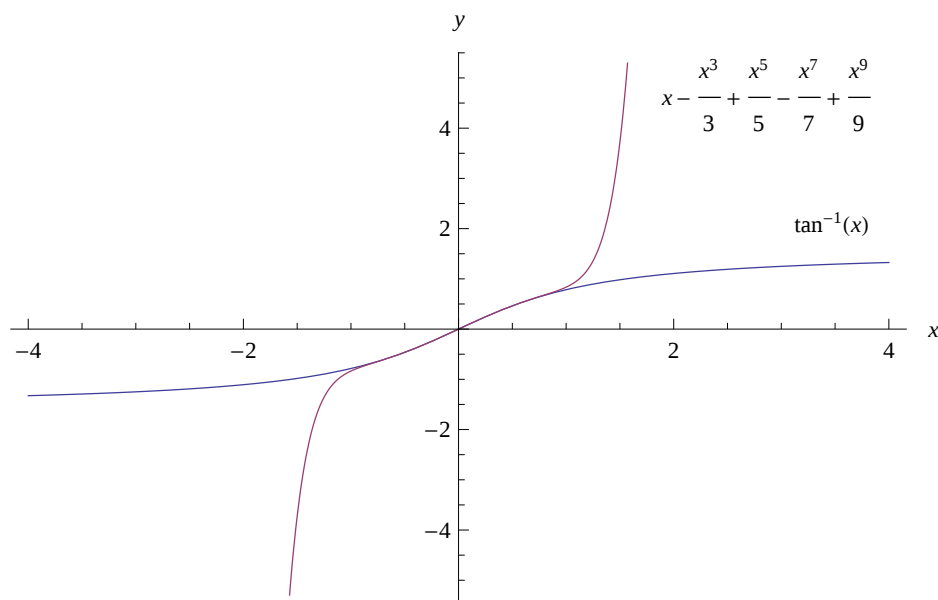
$$y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

(b) From p.608, Example 9, it is seen that the series converges in the interval $[-1, 1]$.

(c) Keeping the first five nonzero terms of Gregory's series, we get the desired approximation

$$y = \tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}.$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) We see that the approximation matches the function best in the interval of convergence $[-1, 1]$ of the Maclaurin series, and diverges rapidly from the function outside that range.

6. (a) Using the formula for the sum of geometric series, we see that the Maclaurin series is

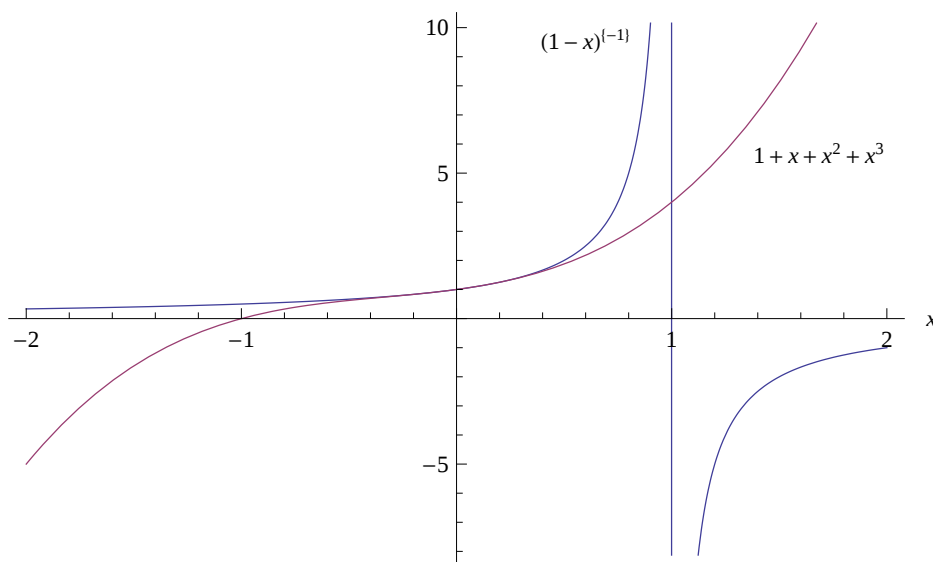
$$y = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

(b) Since the geometric series $\sum_{k=0}^{\infty} r^k$ converges if $|r| < 1$, the series above converges for $|x| < 1$, or in the open interval $\boxed{(-1, 1)}$.

(c) Using the first four nonzero terms of the Maclaurin series found in (a), we get

$$y = \frac{1}{1-x} \approx 1 + x + x^2 + x^3 .$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) We see from the graph that the approximation matches the function pretty well on the interval $(-0.5, 0.5)$. In the range $(-1, -0.5)$, which is within the range of convergence, the deviation is quite small, but in the range $(0.5, 1)$, which is also in the range of convergence, the deviation is large. Also, beyond $x = 1$, there is no agreement as to even the correct quadrant of the approximation and the function, which assumes another branch (as it is a hyperbola with asymptote $x = 1$). This is to be expected, because the series does not converge for values of $x > 1$.

7. To evaluate $\int_0^1 \sin x^2 dx$, we first find the Maclaurin series for the integrand. From p.616, we write the Maclaurin series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

So the Maclaurin series of $\sin x^2$ is

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!} + \cdots.$$

Approximating to four nonzero terms, we get

$$\sin x^2 \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}.$$

Integrating both sides we have

$$\begin{aligned} \int_0^1 \sin x^2 dx &\approx \int_0^1 \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \right] dx \\ &= \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} \\ &= \frac{258,019}{831,600} \\ &\approx \boxed{0.310}. \end{aligned}$$

8. To evaluate $\int_0^1 \cos x^2 dx$, we first find the Maclaurin series for the integrand. The Maclaurin series of $\cos x$ (see p.617) is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

So the Maclaurin series of $\cos x^2$ is

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots.$$

Approximating to four nonzero terms, we get

$$\cos x^2 \approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}.$$

Integrating both sides, we have

$$\begin{aligned} \int_0^1 \cos x^2 dx &\approx \int_0^1 \left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} \right] dx \\ &= \left[x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} \right]_0^1 \\ &= 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} \\ &= \frac{25,399}{28,080} \\ &\approx \boxed{0.905}. \end{aligned}$$

9. $\int_0^1 \frac{x^2}{1+\cos x} dx$. Write the integrand as

$$\frac{x^2}{1+\cos x} = \frac{x^2}{2 \cos^2 \left(\frac{x}{2} \right)} = \frac{x^2}{2} \sec^2 \left(\frac{x}{2} \right).$$

The Maclaurin series for $\sec x$ to five nonzero terms was worked out in **Problem 47** in Section 8.9. It is

$$\begin{aligned}\sec x &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots \\ &\approx 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6,\end{aligned}$$

where we keep the first four nonzero terms for purposes of the current problem. So the series for $\sec\left(\frac{x}{2}\right)$ to four nonzero terms is

$$\begin{aligned}\sec\left(\frac{x}{2}\right) &\approx 1 + \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{5}{24}\left(\frac{x}{2}\right)^4 + \frac{61}{720}\left(\frac{x}{2}\right)^6 \\ &= 1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}.\end{aligned}$$

To find the series approximation to four nonzero terms of $\sec^2\left(\frac{x}{2}\right)$, we multiply the series approximation of $\sec\left(\frac{x}{2}\right)$ with itself (keeping only terms up to the sixth order):

$$\begin{aligned}\sec^2\left(\frac{x}{2}\right) &= \sec\left(\frac{x}{2}\right) \cdot \sec\left(\frac{x}{2}\right) \approx \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) \cdot \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) \\ &= 1 \cdot \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) + \frac{x^2}{8} \cdot \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) + \\ &\quad + \frac{5x^4}{384} \cdot \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) + \frac{61x^6}{46,080} \cdot \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) \\ &= \left(1 + \frac{x^2}{8} + \frac{5x^4}{384} + \frac{61x^6}{46,080}\right) + \left(\frac{x^2}{8} + \frac{x^4}{64} + \frac{5x^6}{3072} + \cdots\right) + \left(\frac{5x^4}{384} + \frac{5x^6}{3072} + \cdots\right) + \\ &\quad + \left(\frac{61x^6}{46,080} + \cdots\right) \\ &\approx 1 + \left(\frac{1}{8} + \frac{1}{8}\right)x^2 + \left(\frac{5}{384} + \frac{1}{64} + \frac{5}{384}\right)x^4 + \left(\frac{61}{46,080} + \frac{5}{3072} + \frac{5}{3072} + \frac{61}{46,080}\right)x^6 \\ &= 1 + \frac{x^2}{4} + \frac{1}{24}x^4 + \frac{17}{2880}x^6.\end{aligned}$$

So the integrand, $\frac{x^2}{1+\cos x} = \frac{x^2}{2} \sec^2\left(\frac{x}{2}\right)$ can be approximated by multiplying the series for $\sec^2\left(\frac{x}{2}\right)$ by $\frac{x^2}{2}$; to four nonzero terms, this is

$$\begin{aligned}\frac{x^2}{1+\cos x} &= \frac{x^2}{2} \sec^2\left(\frac{x}{2}\right) \approx \frac{x^2}{2} \left(1 + \frac{x^2}{4} + \frac{1}{24}x^4 + \frac{17}{2880}x^6\right) \\ &= \frac{1}{2} \left(x^2 + \frac{x^4}{4} + \frac{1}{24}x^6 + \frac{17}{2880}x^8\right).\end{aligned}$$

Integrating both sides, we have

$$\begin{aligned}\int_0^1 \frac{x^2}{1+\cos x} dx &\approx \int_0^1 \frac{1}{2} \left[x^2 + \frac{x^4}{4} + \frac{x^6}{24} + \frac{17}{2880}x^8\right] dx \\ &= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^5}{5 \cdot 4} + \frac{x^7}{7 \cdot 24} + \frac{17x^9}{9 \cdot (2880)}\right]_0^1 \\ &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{20} + \frac{1}{168} + \frac{17}{25,920}\right) \\ &= \frac{70,751}{362,880} \\ &\approx \boxed{0.195}\end{aligned}$$

where we rounded the answer off to three decimal places.

10. $\int_0^{0.1} \frac{x}{\ln(2+x)} dx$. Let $\ln(2+x) = u$, which means $2+x = e^u$, or $x = e^u - 2$. Then $dx = e^u du$. When $x = 0$, $u = \ln 2$. When $x = 0.1$, $u = \ln(2.1)$. The integral becomes

$$\int_{\ln(2)}^{\ln(2.1)} \frac{e^u - 2}{u} \cdot e^u du = \int_{\ln(2)}^{\ln(2.1)} \frac{e^{2u} - 2e^u}{u} du.$$

The Maclaurin series for e^u is

$$\begin{aligned} e^u &= 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots + \frac{u^n}{n!} + \cdots \\ &= 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \cdots + \frac{u^n}{n!} + \cdots. \end{aligned}$$

The Maclaurin series for e^{2u} is

$$\begin{aligned} e^{2u} &= 1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots + \frac{(2u)^n}{n!} + \cdots \\ &= 1 + 2u + 2u^2 + \frac{4}{3}u^3 + \frac{2}{3}u^4 + \cdots + \frac{2^n}{n!}u^n + \cdots. \end{aligned}$$

So we have

$$\begin{aligned} e^{2u} - 2e^u &= \left(1 + 2u + 2u^2 + \frac{4}{3}u^3 + \frac{2}{3}u^4 + \cdots + \frac{2^n}{n!}u^n + \cdots\right) - 2\left(1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \cdots + \frac{u^n}{n!} + \cdots\right) \\ &\approx \left(1 + 2u + 2u^2 + \frac{4}{3}u^3 + \frac{2}{3}u^4\right) - \left(2 + 2u + u^2 + \frac{1}{3}u^3 + \frac{1}{12}u^4\right) \\ &= (1-2) + (2-2)u + (2-1)u^2 + \left(\frac{4}{3} - \frac{1}{3}\right)u^3 + \left(\frac{2}{3} - \frac{1}{12}\right)u^4 \\ &= -1 + u^2 + u^3 + \frac{7}{12}u^4. \end{aligned}$$

So the Maclaurin series for the integrand approximated to four nonzero terms will be

$$\begin{aligned} \frac{e^{2u} - 2e^u}{u} &\approx \frac{-1 + u^2 + u^3 + \frac{7}{12}u^4}{u} \\ &= -\frac{1}{u} + u + u^2 + \frac{7}{12}u^3. \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int_{\ln(2)}^{\ln(2.1)} \frac{e^{2u} - 2e^u}{u} du &= \int_{\ln(2)}^{\ln(2.1)} \left[-\frac{1}{u} + u + u^2 + \frac{7}{12}u^3\right] du \\ &= \left[-\ln u + \frac{u^2}{2} + \frac{u^3}{3} + \frac{7}{48}u^4\right]_{\ln(2)}^{\ln(2.1)} \\ &= \left[-\ln(\ln(2.1)) + \frac{(\ln(2.1))^2}{2} + \frac{(\ln(2.1))^3}{3} + \frac{7}{48}(\ln(2.1))^4\right] + \\ &\quad - \left[-\ln(\ln(2)) + \frac{(\ln(2))^2}{2} + \frac{(\ln(2))^3}{3} + \frac{7}{48}(\ln(2))^4\right] \\ &\approx [-(-0.29849) + (0.27524) + (0.13614) + (0.04419)] + \\ &\quad - [-(-0.36651) + (0.24023) + (0.11101) + (0.03366)] \\ &\approx [0.75406] - [0.75141] \\ &\approx 0.002650 \\ &= \boxed{2.650 \times 10^{-3}} \end{aligned}$$

correct to three decimal places.

11. The integrand in $\int_0^{0.2} \sqrt[3]{1+x^4} dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\sqrt[3]{1+x^4} &= (1+x^4)^{1/3} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} (x^4)^k \\ &= 1 + \frac{1}{3}x^4 + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^8 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}x^{12} + \dots + \binom{\frac{1}{3}}{n}x^{4n} + \dots \\ &\approx 1 + \frac{1}{3}x^4 - \frac{1}{9}x^8 + \frac{5}{81}x^{12}\end{aligned}$$

which is the integrand approximated to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{0.2} \sqrt[3]{1+x^4} dx &\approx \int_0^{0.2} \left[1 + \frac{1}{3}x^4 - \frac{1}{9}x^8 + \frac{5}{81}x^{12} \right] dx \\ &= \left[x + \frac{x^5}{5 \cdot 3} - \frac{x^9}{9 \cdot 9} + \frac{5x^{13}}{13 \cdot 81} \right]_0^{0.2} \\ &= (0.2) + \frac{(0.2)^5}{15} - \frac{(0.2)^9}{81} + \frac{5(0.2)^{13}}{1053} \\ &\approx 0.2 + 2.133 \times 10^{-5} - 6.321 \times 10^{-9} + 3.889 \times 10^{-12} \\ &\approx 0.2000213 \\ &\approx \boxed{0.2000}\end{aligned}$$

which is correct to three decimal places.

12. The integrand in $\int_0^{1/2} \sqrt[3]{1+x} dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\sqrt[3]{1+x} &= (1+x)^{1/3} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} x^k \\ &= 1 + \frac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}x^3 + \dots + \binom{\frac{1}{3}}{n}x^n + \dots \\ &\approx 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{1/2} \sqrt[3]{1+x} dx &\approx \int_0^{1/2} \left[1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 \right] dx \\ &= \left[x + \frac{x^2}{2 \cdot 3} - \frac{x^3}{3 \cdot 9} + \frac{5}{4 \cdot 81}x^4 \right]_0^{1/2} \\ &= \frac{1}{2} + \frac{1}{6} \left(\frac{1}{2} \right)^2 - \frac{1}{27} \left(\frac{1}{2} \right)^3 + \frac{5}{324} \left(\frac{1}{2} \right)^4 \\ &= \frac{2789}{5184} \\ &\approx \boxed{0.538.}\end{aligned}$$

13. The integrand in $\int_0^{1/2} \frac{1}{\sqrt[3]{1+x^2}} dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\frac{1}{\sqrt[3]{1+x^2}} &= (1+x^2)^{-1/3} = \sum_{k=0}^{\infty} \binom{-\frac{1}{3}}{k} (x^2)^k \\ &= 1 - \frac{1}{3}x^2 + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{2!}x^4 + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{3!}x^6 + \cdots + \binom{-\frac{1}{3}}{n}x^{2n} + \cdots \\ &\approx 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{1/2} \frac{1}{\sqrt[3]{1+x^2}} dx &\approx \int_0^{1/2} \left[1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 \right] dx \\ &= \left[x - \frac{x^3}{3 \cdot 3} + \frac{2}{5 \cdot 9}x^5 - \frac{14}{7 \cdot 81}x^7 \right]_0^{1/2} \\ &= \left(\frac{1}{2} \right) - \frac{1}{9} \left(\frac{1}{2} \right)^3 + \frac{2}{45} \left(\frac{1}{2} \right)^5 - \frac{14}{567} \left(\frac{1}{2} \right)^7 \\ &= \frac{12,631}{25,920} \\ &\approx \boxed{0.487}.\end{aligned}$$

14. The integrand in $\int_0^{0.2} \frac{1}{\sqrt{1+x^3}} dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\frac{1}{\sqrt{1+x^3}} &= (1+x^3)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (x^3)^k \\ &= 1 - \frac{1}{2}x^3 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}x^6 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}x^9 + \cdots + \binom{-\frac{1}{2}}{n}x^{3n} + \cdots \\ &\approx 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{0.2} \frac{1}{\sqrt{1+x^3}} dx &\approx \int_0^{0.2} \left[1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 \right] dx \\ &= \left[x - \frac{1}{2 \cdot 4}x^4 + \frac{3}{8 \cdot 7}x^7 - \frac{5}{16 \cdot 10}x^{10} \right]_0^{0.2} \\ &= 0.2 - \frac{(0.2)^4}{8} + \frac{3}{56}(0.2)^7 - \frac{5}{160}(0.2)^{10} \\ &\approx \boxed{0.1998}.\end{aligned}$$

Applications and Extensions

15. We use the recursion relation

$$\ln(N+1) = \ln N + 2 \left[\frac{1}{2N+1} + \frac{1}{3} \left(\frac{1}{2N+1} \right)^3 + \frac{1}{5} \left(\frac{1}{2N+1} \right)^5 + \cdots \right].$$

Substituting $N = 3$, and repeatedly using the recursion relation we get

$$\begin{aligned}
 \ln 4 &= \ln 3 + 2 \left[\frac{1}{2 \cdot 3 + 1} + \frac{1}{3} \left(\frac{1}{2 \cdot 3 + 1} \right)^3 + \frac{1}{5} \left(\frac{1}{2 \cdot 3 + 1} \right)^5 + \cdots \right] \\
 &= \ln 2 + 2 \left[\frac{1}{2 \cdot 2 + 1} + \frac{1}{3} \left(\frac{1}{2 \cdot 2 + 1} \right)^3 + \frac{1}{5} \left(\frac{1}{2 \cdot 2 + 1} \right)^5 + \cdots \right] + 2 \left[\frac{1}{7} + \frac{1}{3} \left(\frac{1}{7} \right)^3 + \frac{1}{5} \left(\frac{1}{7} \right)^5 + \cdots \right] \\
 &= \ln 1 + 2 \left[\frac{1}{2 \cdot 1 + 1} + \frac{1}{3} \left(\frac{1}{2 \cdot 1 + 1} \right)^3 + \frac{1}{5} \left(\frac{1}{2 \cdot 1 + 1} \right)^5 + \cdots \right] + 2 \left[\frac{1}{5} + \frac{1}{3} \left(\frac{1}{5} \right)^3 + \frac{1}{5} \left(\frac{1}{5} \right)^5 + \cdots \right] + \\
 &\quad + 2 \left[\frac{1}{7} + \frac{1}{3} \left(\frac{1}{7} \right)^3 + \frac{1}{5} \left(\frac{1}{7} \right)^5 + \cdots \right] \\
 &= 0 + 2 \left[\left\{ \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \right\} + \frac{1}{3} \left\{ \left(\frac{1}{3} \right)^3 + \left(\frac{1}{5} \right)^3 + \left(\frac{1}{7} \right)^3 \right\} + \frac{1}{5} \left\{ \left(\frac{1}{3} \right)^5 + \left(\frac{1}{5} \right)^5 + \left(\frac{1}{7} \right)^5 \right\} + \cdots \right] \\
 &\approx \boxed{1.386147}
 \end{aligned}$$

16. (a) The Maclaurin series of $\sin x$ is (see p.616, Example 3)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

So the Maclaurin series of $3 \sin \left(\frac{x}{2} \right)$ is

$$\begin{aligned}
 3 \sin \left(\frac{x}{2} \right) &= 3 \left[\left(\frac{x}{2} \right) - \frac{1}{3!} \left(\frac{x}{2} \right)^3 + \frac{1}{5!} \left(\frac{x}{2} \right)^5 - \cdots + (-1)^n \frac{1}{(2n+1)!} \left(\frac{x}{2} \right)^{2n+1} + \cdots \right] \\
 &= \boxed{\frac{3x}{2} - \frac{x^3}{16} + \frac{x^5}{1280} - \cdots + (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)!} + \cdots}
 \end{aligned}$$

(b) As shown in Example 3 on p.616, the interval of convergence of the Maclaurin series for $\sin x$ is $(-\infty, \infty)$. So the interval of convergence of the Maclaurin series for $3 \sin \left(\frac{x}{2} \right)$ is also $\boxed{(-\infty, \infty)}$.

(c) The error in using the first n nonzero terms of a convergent alternating series is bounded from above by the absolute value of the $n+1^{st}$ term. The error terms are all monomials which take their maximum values on the interval $(-2, 2)$ at the endpoints (as limits of $x \rightarrow -2^+$ or $x \rightarrow 2^-$) of the interval. We proceed by trial and error:

The error in keeping just the first term is $\left| -\frac{x^3}{16} \right|$, which as $x \rightarrow 2^-$ evaluates to $\left| \frac{2^3}{16} \right| = 0.5$.

The error in keeping the first two terms is $\left| \frac{x^5}{1280} \right|$, which as $x \rightarrow 2^-$ evaluates to $\left| \frac{2^5}{1280} \right| = 0.025$. The error is already less than the required 0.1, so that only $\boxed{two}$ terms are needed in order to approximate f on the interval $(-2, 2)$ to an error less than or equal to 0.1. (As a check, $3 \sin \left(\frac{2}{2} \right) = 3 \sin 1 \approx 2.524$, while $\frac{3x}{2} - \frac{x^3}{16}$ evaluated at $x = 2$ gives 2.5, and $|2.524 - 2.5| = 0.024 < 0.1$.)

17. (a) From p.616, we write the Maclaurin series of $\sin x$ as

$$F(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Since this series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1^{st}$ term.

For $x = 4^\circ = 4^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{45}$, keeping just the first term in the Maclaurin expansion will result in an error that is less than absolute value of the second term, or $\left| -\frac{x^3}{3!} \right|$. At $x = 4^\circ = \frac{\pi}{45}$, the error bound will be

$$\left| -\frac{\left(\frac{\pi}{45}\right)^3}{3!} \right| \approx 0.00005 < 0.001.$$

So we just need the first term to give an answer correct to three decimal places:

$$\sin 4^\circ \approx \frac{\pi}{45} \approx \boxed{0.0698}.$$

(b) From p.617, we write the Maclaurin series of $\cos x$ as

$$F(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

Since the series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1^{\text{st}}$ term.

For $x = 15^\circ = 15^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{12}$, keeping just the first two terms in the Maclaurin expansion will result in an error that is less than the absolute value of the third term, or $\left| \frac{x^4}{4!} \right|$. At $x = 15^\circ = \frac{\pi}{12}$, the error bound will be

$$\left| \frac{\left(\frac{\pi}{12}\right)^4}{4!} \right| \approx 0.0002 < 0.001.$$

So keeping the first two terms, we have an answer that is correct to three decimal places:

$$\cos 15^\circ \approx 1 - \frac{1}{2!} \left(\frac{\pi}{12}\right)^2 \approx \boxed{0.9659}.$$

(c) From p.608, using Gregory's series, we write the Maclaurin expansion of $\tan^{-1} x$ as

$$F(x) = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)} + \cdots.$$

Since the series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1^{\text{st}}$ term.

For $x = 0.05$, the error in keeping just the first two terms in the Maclaurin expansion will result in an error that is less than the absolute value of the third term, or $\left| \frac{x^5}{5} \right|$. At $x = 0.05$, the error bound will be

$$\left| \frac{(0.05)^5}{5} \right| = 6.25 \times 10^{-8} < 0.001.$$

So keeping the first two terms, we get an answer correct to three decimal places:

$$\tan^{-1} 0.05 \approx 0.05 - \frac{(0.05)^3}{3} \approx \boxed{0.04496}.$$

18. (a) We have, using the standard result (or using the substitution $u = \tan x$, $du = \sec^2 x \, dx$)

$$\int_0^1 \frac{1}{1+x^2} \, dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}.$$

(b) Using the geometric series sum, provided that $|x| < 1$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} + \cdots.$$

Integrating term by term, we have, also using the result of part (a):

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} dx &= \frac{\pi}{4} = \lim_{b \rightarrow 1^-} \int_0^b [1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} + \cdots] dx \\ &= \lim_{b \rightarrow 1^-} \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \right]_0^b \\ \text{or, } \quad &\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots}\end{aligned}$$

which is Leibniz's result.

(c) Keeping the first ten terms, we have

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} \approx \boxed{0.7604}$$

giving $\pi \approx 4(0.7604) = 3.0416$. The actual value is $\pi \approx 3.1416$, rounded to four decimal places. So we see that the agreement is pretty poor, accurate to only about

$$\left| \frac{3.1416 - 3.0416}{3.1416} \right| \approx 3\%.$$

(d) To get an approximation of π accurate to the tenth decimal place, we need the error to be smaller than 10^{-10} . Since the series for $\frac{\pi}{4}$ is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1^{\text{st}}$ term. So we need, noting that to get the series for π , we need to multiply the series for $\frac{\pi}{4}$ by 4,

$$4 \cdot \left| \frac{(-1)^n}{2n+1} \right| \leq 10^{-10}$$

or $2n+1 \geq 4 \cdot 10^{10}$, or $n \geq 2 \cdot 10^{10} - \frac{1}{2}$, or $\boxed{n \approx 2 \times 10^{10}}$ terms of the series.

19. Gregory's series is (see p. 608)

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

At $x = \frac{1}{2}$, we have, approximating with the first four terms of the series,

$$\tan^{-1} \left(\frac{1}{2} \right) \approx \frac{1}{2} - \frac{1}{3} \left(\frac{1}{2} \right)^3 + \frac{1}{5} \left(\frac{1}{2} \right)^5 - \frac{1}{7} \left(\frac{1}{2} \right)^7 = \frac{6,229}{13,440}.$$

At $x = \frac{1}{3}$, we have, approximating with the first four terms of the series,

$$\tan^{-1} \left(\frac{1}{3} \right) \approx \frac{1}{3} - \frac{1}{3} \left(\frac{1}{3} \right)^3 + \frac{1}{5} \left(\frac{1}{3} \right)^5 - \frac{1}{7} \left(\frac{1}{3} \right)^7 = \frac{24,628}{76,545}.$$

Since

$$\tan^{-1} 1 = \frac{\pi}{4} = \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{3} \right),$$

we have

$$\begin{aligned}\pi &= 4 \left[\tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{3} \right) \right] \\ &\approx 4 \left[\frac{6,229}{13,440} + \frac{24,628}{76,545} \right] \\ &= \frac{1,538,665}{489,888} \\ &\approx \boxed{3.14085}\end{aligned}$$

an answer that is correct to three decimal places.

20. The general kinetic energy formula may be expanded using the Binomial Series as follows:

$$\begin{aligned}
 K_{\text{gen.}}(v) &= mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \\
 &= mc^2 \left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) \\
 &= mc^2 \left(\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(-\frac{v^2}{c^2} \right)^k - 1 \right) \\
 &= mc^2 \left[1 - \frac{1}{2} \left(-\frac{v^2}{c^2} \right) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!} \left(-\frac{v^2}{c^2} \right)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!} \left(-\frac{v^2}{c^2} \right)^3 + \cdots - 1 \right] \\
 &= mc^2 \left[\frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \cdots \right] \\
 &= \frac{1}{2} mv^2 + \frac{3}{8} mv^2 \left(\frac{v^2}{c^2} \right) + \frac{5}{16} mv^2 \left(\frac{v^4}{c^4} \right) + \cdots \\
 &= \frac{1}{2} mv^2 \left[1 + \frac{3}{4} \left(\frac{v}{c} \right)^2 + \frac{5}{8} \left(\frac{v}{c} \right)^4 + \cdots \right] \\
 &\approx \frac{1}{2} mv^2
 \end{aligned}$$

if $\frac{v}{c} \rightarrow 0$. So we see that the general kinetic energy formula reduces to the classical one $K = \frac{1}{2}mv^2$ at speeds $v \rightarrow 0$, that is, for low speeds compared to the speed of light.

Challenge Problems

21. (a) $a_k = (-1)^{k+1} \int_0^{\pi/k} \sin(kx) dx$. Set $kx = u$. Then $x = \frac{u}{k}$, and $dx = \frac{1}{k} du$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{k}$, $u = \pi$. So the integral becomes

$$\begin{aligned}
 a_k &= (-1)^{k+1} \frac{1}{k} \int_0^{\pi} \sin u du \\
 &= \frac{(-1)^{k+1}}{k} [-\cos u] \Big|_0^{\pi} \\
 &= \frac{(-1)^{k+1}}{k} [-\cos \pi - (-\cos 0)] \\
 &= \boxed{\frac{2}{k} (-1)^{k+1}}.
 \end{aligned}$$

(b) The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2}{k} (-1)^{k+1} = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is a constant multiple of the alternating harmonic series, which converges, so it converges as well.

(c) On p.607-608, in Example 8 of Section 8.8, it is demonstrated that

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2 \approx 0.693.$$

So we have

$$\sum_{k=1}^{\infty} a_k = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \approx 2(0.693) = 1.386.$$

So we conclude that

$$1 \leq \sum_{k=1}^{\infty} a_k \leq \frac{3}{2}.$$

22. The Maclaurin expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

So the Maclaurin expansion of e^{x^3} is

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!} + \cdots.$$

The Maclaurin series of xe^{x^3} is given by

$$xe^{x^3} = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + \cdots + \frac{x^{3n+1}}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{k!}.$$

23. (a) Some terms of the Maclaurin expansion of $\sin^{-1} x$ were obtained in Example 10 of Section 8.9, p.621. To the ninth order in x , it is

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots.$$

So we have for $x = 1$,

$$\sin^{-1} 1 = \frac{\pi}{2} = 1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \cdots$$

and for $x = \frac{1}{2}$, we have

$$\sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{6} \left(\frac{1}{2} \right)^3 + \frac{3}{40} \left(\frac{1}{2} \right)^5 + \frac{5}{112} \left(\frac{1}{2} \right)^7 + \frac{35}{1152} \left(\frac{1}{2} \right)^9 + \cdots$$

$$\text{or, } \frac{\pi}{6} = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{5}{14,336} + \frac{35}{589,824} + \cdots.$$

(b) Evaluating the terms shown in the first expression, we get $\frac{\pi}{2} \approx 1.571$ correct to three decimal places.

(c) Evaluating the terms shown in the second expression, we get $\frac{\pi}{6} \approx 0.524$ correct to three decimal places.

8.11 Chapter 8 Review Exercises

1. The n^{th} term of the sequence is $s_n = \frac{(-1)^{n+1}}{n^4}$. Setting $n = 1, 2, 3, 4, 5$, we get the first five terms of the sequence to be

$$\begin{aligned}\{s_n\} &= \left\{ \frac{(-1)^{1+1}}{1^4}, \frac{(-1)^{2+1}}{2^4}, \frac{(-1)^{3+1}}{3^4}, \frac{(-1)^{4+1}}{4^4}, \frac{(-1)^{5+1}}{5^4}, \dots \right\} \\ &= \left\{ 1, -\frac{1}{2^4}, \frac{1}{3^4}, -\frac{1}{4^4}, \frac{1}{5^4}, \dots \right\}.\end{aligned}$$

So the first five terms of the sequence are

$$\boxed{1, -\frac{1}{16}, \frac{1}{81}, -\frac{1}{256}, \frac{1}{625}}.$$

2. The n^{th} term of the sequence is $s_n = \frac{2^n}{3^n}$. Setting $n = 1, 2, 3, 4, 5$, we get the first five terms of the sequence to be

$$\begin{aligned}\{s_n\} &= \left\{ \frac{2^1}{3^1}, \frac{2^2}{3^2}, \frac{2^3}{3^3}, \frac{2^4}{3^4}, \frac{2^5}{3^5}, \dots \right\} \\ &= \left\{ \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \dots \right\}.\end{aligned}$$

So the first five terms of the sequence are

$$\boxed{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}}.$$

3. The terms of the sequence

$$2, -\frac{3}{2}, \frac{9}{8}, -\frac{27}{32}, \frac{81}{128}, \dots$$

can be written as

$$(-1)^{1+1} \frac{3^0}{2^{0-1}}, (-1)^{2+1} \frac{3^1}{2^{2-1}}, (-1)^{3+1} \frac{3^2}{2^{4-1}}, (-1)^{4+1} \frac{3^3}{2^{6-1}}, (-1)^{5+1} \frac{3^4}{2^{8-1}}, \dots$$

So the n^{th} term of the sequence is

$$\begin{aligned}s_n &= (-1)^{n+1} \frac{3^{n-1}}{2^{2n-3}} \\ &= (-1)^{n-1} \cdot (-1)^2 \cdot 2 \cdot \frac{3^{n-1}}{2^{2n-2}} \\ &= (-1)^{n-1} 2 \cdot \frac{3^{n-1}}{4^{n-1}} \\ &= \boxed{2 \left(-\frac{3}{4} \right)^{n-1}}.\end{aligned}$$

4. The sequence is $\{s_n\} = \left\{ 1 + \frac{n}{n^2+1} \right\}$. Using the sum and difference property of sequences, we have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 + \frac{n}{n^2+1} \right) = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 1 + 0 = \boxed{1}.$$

5. The sequence is $\{s_n\} = \{\ln \frac{n+2}{n}\} = \{\ln b_n\}$, where $b_n = \frac{n+2}{n}$. The sequence $\{b_n\}$ converges to

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n+2}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right) = 1 + 0 = 1.$$

So by the theorem on p.543, setting $f(x) = \ln x$, which is continuous at $x = 1$ (the value of the limit of b_n), we conclude that the limit of the original sequence is

$$\ln(1) = \boxed{0.}$$

6. The sequence is $\{s_n\} = \{\tan^{-1} n\}$. Let $f(x) = \tan^{-1} x$ be a related function of the sequence. Since $f'(x) = \frac{1}{1+x^2} > 0$ for all $x \geq 1$, the function is monotonically increasing, which means the sequence is also monotonically increasing for $n \geq 1$. Also,

$$\tan^{-1} n \leq \frac{\pi}{2}$$

for all positive nonzero integers n , so the sequence $\{s_n\}$ is bounded from above. Since the sequence $\{s_n\}$ is an increasing sequence that is bounded from above, it converges, and the limit of the sequence is

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \tan^{-1} n = \boxed{\frac{\pi}{2}.}$$

7. The sequence is $\{s_n\} = \left\{\frac{(-1)^n}{(n+1)^2}\right\}$. Since

$$-\frac{1}{(n+1)^2} \leq \frac{(-1)^n}{(n+1)^2} \leq \frac{1}{(n+1)^2},$$

and the sequences $\left\{-\frac{1}{(n+1)^2}\right\}$ and $\left\{\frac{1}{(n+1)^2}\right\}$ are both convergent, and converge to the limit of 0, since $\lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} = 0$, we have by the Squeeze Theorem of sequences that

$$\lim_{n \rightarrow \infty} s_n = \boxed{0.}$$

8. The sequence is $\{s_n\} = \left\{\frac{e^n}{(n+2)^2}\right\}$. To determine if the sequence is increasing or decreasing, apply the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{\frac{e^{n+1}}{(n+3)^2}}{\frac{e^n}{(n+2)^2}} = e \left(\frac{n+2}{n+3}\right)^2.$$

We show that

$$2 \left(\frac{n+2}{n+3}\right)^2 > 1.$$

Then this will mean that $e \left(\frac{n+2}{n+3}\right)^2 > 1$ as well, since $e > 2$. Expanding the desired inequality, we get

$$\begin{aligned} 2 \left(\frac{n+2}{n+3}\right)^2 &> 1 \\ 2(n+2)^2 &> (n+3)^2 \\ 2(n^2 + 4n + 4) &> n^2 + 6n + 9 \\ 2n^2 + 8n + 8 &> n^2 + 6n + 9 \\ n^2 + 2n - 1 &> 0 \\ n^2 + 2n + 1 - 2 &> 0 \\ (n+1)^2 - 2 &> 0 \\ (n+1)^2 &> 2, \end{aligned}$$

which is true for all $n \geq 1$. So this shows that the sequence is increasing since we have shown that $\frac{s_{n+1}}{s_n} > 1$. Also the sequence is bounded from below by its first term

$$s_1 = \frac{e}{(1+2)^2} = \frac{e}{9}.$$

It is not bounded from above since

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{e^n}{(n+2)^2} = \lim_{x \rightarrow \infty} \frac{e^x}{(x+2)^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2(x+2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty,$$

where we used a related function of the n^{th} term of the sequence and applied L'Hôpital's rule to it. Since the sequence is increasing, and it is not bounded from above, it diverges.

9. The sequence $\{s_n\} = \{n!\}$ diverges since $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} n! = \infty$.

10. The sequence $\{s_n\} = \left\{\left(\frac{5}{8}\right)^n\right\}$ is of the form $\{s_n\} = \{r^n\}$ where $-1 < r = \frac{5}{8} < 1$. Since $\lim_{n \rightarrow \infty} r^n = 0$ for $-1 < r < 1$, we have $\lim_{n \rightarrow \infty} s_n = 0$, that is, the sequence converges to the limit of 0.

11. The sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ is of the form $\{s_n\} = \{r^n\}$ where $-1 < r = -\frac{1}{2} < 1$. Since $\lim_{n \rightarrow \infty} r^n = 0$ for $-1 < r < 1$, we have $\lim_{n \rightarrow \infty} s_n = 0$, that is, the sequence converges to the limit of 0.

12. The sequence is $\{s_n\} = \{(-1)^n + e^{-n}\}$. As $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} e^{-n} = 0$ but $\lim_{n \rightarrow \infty} (-1)^n$ does not exist. Since $\lim_{n \rightarrow \infty} s_n$ does not exist, the sequence diverges.

13. The sequence $\{s_n\} = \left\{1 + \frac{2}{n}\right\}$ is decreasing since using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 + \frac{2}{n+1} - 1 - \frac{2}{n} = 2 \left(\frac{1}{n+1} - \frac{1}{n} \right) = -\frac{2}{n(n+1)} < 0$$

for $n \geq 1$. It's also bounded from below by 1, since $s_n = 1 + \frac{2}{n} > 1$ for all $n \geq 1$. Since the sequence is decreasing and bounded from below, it converges.

14. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k-1}}$. The fifth partial sum of the series is

$$\begin{aligned} S_5 &= \sum_{k=1}^5 \frac{(-1)^k}{4^{k-1}} \\ &= \frac{(-1)^1}{4^{1-1}} + \frac{(-1)^2}{4^{2-1}} + \frac{(-1)^3}{4^{3-1}} + \frac{(-1)^4}{4^{4-1}} + \frac{(-1)^5}{4^{5-1}} \\ &= -\frac{1}{4^0} + \frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} \\ &= -1 + \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} \\ &= \boxed{-\frac{205}{256}}. \end{aligned}$$

15. The telescoping series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{4}{k+4} - \frac{4}{k+5} \right)$. The n^{th} partial sum of the series is

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\frac{4}{k+4} - \frac{4}{k+5} \right) \\ &= \left(\frac{4}{1+4} - \frac{4}{1+5} \right) + \left(\frac{4}{2+4} - \frac{4}{2+5} \right) + \cdots + \left(\frac{4}{n-1+4} - \frac{4}{n-1+5} \right) + \left(\frac{4}{n+4} - \frac{4}{n+5} \right) \\ &= \left(\frac{4}{5} - \frac{4}{6} \right) + \left(\frac{4}{6} - \frac{4}{7} \right) + \cdots + \left(\frac{4}{n+3} - \frac{4}{n+4} \right) + \left(\frac{4}{n+4} - \frac{4}{n+5} \right) \\ &= \frac{4}{5} - \frac{4}{n+5}. \end{aligned}$$

The limit of the sequence of partial sums is

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{4}{5} - \frac{4}{n+5} \right) = \lim_{n \rightarrow \infty} \frac{4}{5} - \lim_{n \rightarrow \infty} \frac{4}{n+5} = \frac{4}{5} - 0 = \boxed{\frac{4}{5}}$$

is the sum of the telescoping series.

16. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\cos^2(k\pi)}{k}$. Since $\cos^2(n\pi) = 1$ for any value of n , we may write the series as

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}. \text{ But this is the harmonic series which } \boxed{\text{diverges.}}$$

17. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} -(\ln 2)^k = \sum_{k=1}^{\infty} (-\ln 2)(\ln 2)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ where $a = (-\ln 2)$ and $r = \ln 2 \approx 0.693 < 1$. Since $|r| < 1$, the geometric series converges to a sum of

$$\frac{a}{1-r} = \frac{-\ln 2}{1-\ln 2} = \boxed{\frac{\ln 2}{\ln 2 - 1}}.$$

18. The series can be written $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{e}{3^k} = e + \sum_{k=1}^{\infty} \left(\frac{e}{3} \right) \left(\frac{1}{3} \right)^{k-1}$. The second term is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ where $a = \frac{e}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, the geometric series converges to a sum of

$$\frac{a}{1-r} = \frac{\frac{e}{3}}{1-\frac{1}{3}} = \frac{e/3}{2/3} = \frac{e}{2}.$$

The original series then converges to a sum of $e + \frac{e}{2} = \boxed{\frac{3e}{2}}.$

19. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (4^{1/3})^k = \sum_{k=1}^{\infty} 4^{1/3} \cdot (4^{1/3})^{k-1}$ is of the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ where $a = 4^{1/3}$ and $r = 4^{1/3} > 1$. Since $|r| > 1$, the series $\boxed{\text{diverges.}}$

20. The number $r = 0.123123123\cdots$ can be written and then summed into a rational number form, using the properties of convergent geometric series as shown below:

$$\begin{aligned}
 r &= 0.123123123123\cdots \\
 &= 0.123 + 0.123 \times 10^{-3} + 0.123 \times 10^{-6} + 0.123 \times 10^{-9} + \cdots \\
 &= 0.123 \left[1 + 10^{-3} + (10^{-3})^2 + (10^{-3})^3 + \cdots \right] \\
 &= 0.123 \cdot \sum_{k=1}^{\infty} (10^{-3})^{k-1} \\
 &= 0.123 \cdot \frac{1}{1 - 10^{-3}} \\
 &= 0.123 \cdot \frac{10^3}{10^3 - 1} = 0.123 \left(\frac{1000}{999} \right) = \frac{123}{999} = \boxed{\frac{41}{333}}.
 \end{aligned}$$

21. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k-2}{k}$. Since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3n-2}{n} = \lim_{n \rightarrow \infty} \left(3 - \frac{2}{n} \right) = \lim_{n \rightarrow \infty} 3 - \lim_{n \rightarrow \infty} \frac{2}{n} = 3 - 0 = 3 \neq 0,$$

the series *diverges* by the Divergence Test.

22. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$. A related function of the n^{th} term of the series is $f(x) = \frac{\ln x}{x^2}$, which is continuous and decreasing on $[1, \infty)$, and for which $f(k) = a_k$ for all positive nonzero integers k . To see it is decreasing, compute $f'(x)$:

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} < 0$$

for $x > \sqrt{e}$. So the series decreases for $n > 1$. By the Integral Test,

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx.$$

Let $u = \ln x$, $dv = \frac{dx}{x^2}$. Then $du = \frac{dx}{x}$, $v = -\frac{1}{x}$. So

$$\begin{aligned}
 I &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left[\left[\ln x \cdot -\frac{1}{x} \right]_1^b + \int_1^b \frac{dx}{x^2} \right] \\
 &= \lim_{b \rightarrow \infty} \left[\left(-\frac{\ln b}{b} + \frac{\ln 1}{1} \right) + \left[-\frac{1}{x} \right]_1^b \right] \\
 &= -\lim_{b \rightarrow \infty} \frac{\ln b}{b} + 0 + \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) \\
 &= -\lim_{b \rightarrow \infty} \frac{\frac{1}{b}}{1} + (-0 + 1) = -0 + (-0 + 1) = 1,
 \end{aligned}$$

where L'Hôpital's rule was used on the first term in the final line. Since the limit is a positive real number, by the Integral Test, the series *converges*.

23. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{4k^2+9}$. A related function of the n^{th} term of the series is $f(x) = \frac{1}{4x^2+9}$, which is continuous and decreasing on $[1, \infty)$, and for which $f(k) = a_k$ for all positive nonzero integers k . To

apply the integral test, we have to evaluate

$$I = \int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{dx}{4x^2 + 9}.$$

Let $x = \frac{3}{2} \tan \theta$. Then $dx = \frac{3}{2} \sec^2 \theta d\theta$. When $x = 1$, $\tan \theta = \frac{2}{3}$, or $\theta = \tan^{-1} \frac{2}{3}$. As $n \rightarrow \infty$, $\theta \rightarrow \frac{\pi}{2}$. So we have

$$\begin{aligned} I &= \int_{\tan^{-1} \frac{2}{3}}^{\frac{\pi}{2}} \frac{\frac{3}{2} \sec^2 \theta d\theta}{4 \left(\frac{9}{4} \tan^2 \theta \right) + 9} = \frac{3}{2 \cdot 9} \int_{\tan^{-1} \frac{2}{3}}^{\frac{\pi}{2}} \frac{\sec^2 \theta d\theta}{1 + \tan^2 \theta} \\ &= \frac{1}{6} \int_{\tan^{-1} \frac{2}{3}}^{\frac{\pi}{2}} \frac{\sec^2 \theta d\theta}{\sec^2 \theta} = \frac{1}{6} [\theta]_{\tan^{-1} \frac{2}{3}}^{\frac{\pi}{2}} \\ &= \frac{1}{6} \left[\frac{\pi}{2} - \tan^{-1} \frac{2}{3} \right]. \end{aligned}$$

Since $\tan^{-1} x < \frac{\pi}{2}$ for any x , the integral is a positive real number, so by the Integral Test, the series *converges*.

24. The p -series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ *converges* since $p = \frac{5}{2} > 1$. The bounds for the sum for a general convergent p -series are

$$\frac{1}{p-1} < \sum_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1},$$

so for this particular p -series they will be

$$\left(\frac{1}{\frac{5}{2}-1}, 1 + \frac{1}{\frac{5}{2}-1} \right) = \left(\frac{2}{3}, \frac{5}{3} \right).$$

So the series has a *lower bound* $\frac{2}{3}$ and *upper bound* $\frac{5}{3}$.

25. The series is $\sum_{k=5}^{\infty} a_k = \sum_{k=5}^{\infty} \left[\frac{1}{k^5} \cdot \frac{1}{2^k} \right]$. We have

$$a_n = \frac{1}{n^5} \cdot \frac{1}{2^n} < \frac{1}{n^5} = b_n$$

for all $n \geq 5$. Since the series $\sum_{k=5}^{\infty} b_k = \sum_{k=5}^{\infty} \frac{1}{k^5}$ is a convergent p -series (since $p = 5 > 1$) from the fifth term forward, and since the omission of a finite number of terms does not affect convergence or divergence properties of a p -series, the original series $\sum_{k=5}^{\infty} a_k$ also *converges* by the Comparison Test.

26. The series $\sum_{k=1}^{\infty} \left[\frac{3}{5^k} - \left(\frac{2}{3} \right)^{k-1} \right] = \sum_{k=1}^{\infty} \left[\frac{3}{5} \cdot \left(\frac{1}{5} \right)^{k-1} - \left(\frac{2}{3} \right)^{k-1} \right]$ can be written as a difference of two geometric series, $\sum_{k=1}^{\infty} \frac{3}{5} \cdot \left(\frac{1}{5} \right)^{k-1} - \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^{k-1}$ each of which converges absolutely (since $|r| = \frac{1}{5} < 1$ and $|r| = \frac{2}{3} < 1$), so the original series also *converges*.

27. The series $\sum_{k=1}^{\infty} \frac{3}{k^5}$ is a constant multiple of a convergent p -series (since $p = 5 > 1$), so it also *converges*.

28. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$. The n^{th} term of the series satisfies

$$a_n = \frac{1}{\sqrt{n+1}} > \frac{1}{n+1} = b_n$$

for $n \geq 1$ since $n+1 > \sqrt{n+1}$ for $n \geq 1$. But the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is a harmonic series from the second term forward, and since the convergence or divergence properties of a series is not affected by the omission of a finite number of terms, the series $\sum_{k=1}^{\infty} b_k$ diverges, and so we see that the original series *diverges* as well by the Comparison Test.

29. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+1}{k^{k+1}}$. The n^{th} term of the series behaves like

$$a_n = \frac{n+1}{n^{n+1}} = \frac{n+1}{n} \cdot \frac{1}{n^n} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{n^n} \approx \frac{1}{n^n} = b_n$$

for large values of n . We evaluate

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{n^{n+1}}}{\frac{1}{n^n}} = \lim_{n \rightarrow \infty} (n+1) \cdot \left(\frac{n^n}{n^{n+1}}\right) = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1.$$

Since the limit is a positive real number and the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^k}$ is the convergent “k-to-the-k” series, the original series $\sum_{k=1}^{\infty} a_k$ is also *convergent* by the Limit Comparison Test.

30. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{4}{k \cdot 3^k}$. The n^{th} term of the series satisfies

$$a_n = \frac{4}{n \cdot 3^n} \leq \frac{4}{3^n} = b_n$$

for $n \geq 1$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{4}{3^k} = \sum_{k=1}^{\infty} \frac{4}{3} \left(\frac{1}{3}\right)^{k-1}$ is a convergent geometric series (since $|r| = \frac{1}{3} < 1$), the original series *converges* by the Comparison Test.

31. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+1)}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+2}{n(n+1)} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n}}{n \left(1 + \frac{1}{n}\right)} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+3}{(n+1)(n+2)}}{\frac{n+2}{n(n+1)}} = \frac{n(n+3)}{(n+2)^2} = \frac{n^2 + 3n}{n^2 + 4n + 4} < 1.$$

So by the Alternating Series Test, the series *converges*.

To find the sum of the series to an accuracy of 0.001, we first estimate how many terms we must keep in the summation of the series. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the n^{th} term; so we require $\frac{n+2}{n(n+1)} = 0.001$. This is equivalent to

$$\begin{aligned} \frac{n+2}{n(n+1)} &= \frac{1}{1000} \\ 1000n + 2000 &= n^2 + n \\ n^2 - 999n - 2000 &= 0. \end{aligned}$$

Solving the quadratic equation, we find that the smallest number greater than the positive solution is $n = 1001$. Carrying out the sum of the first 1000 terms of the series (since $n - 1 = 1001 - 1 = 1000$), we get

$$\sum_{k=1}^{1000} (-1)^{k+1} \frac{k+2}{k(k+1)} \approx \boxed{1.079}$$

which is correct to three decimal places.

32. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^k}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{e^n} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0,$$

using L'Hôpital's rule repeatedly on a related function of the n^{th} term. By the Algebraic Ratio Test,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \left(\frac{n+1}{n} \right)^2 \cdot \frac{1}{e} < 1,$$

if $n+1 < \sqrt{e}n$ or $(\sqrt{e}-1)n > 1$, or $n > \frac{1}{\sqrt{e}-1} \approx 1.54$. So for $n \geq 2$, the series decreases. Since the convergence property does not depend on the behavior of a finite number of terms, we conclude that by the Algebraic Ratio Test, the series converges, even if its sum begins at $n = 1$.

To find the sum of the series to an accuracy of 0.001, we first estimate how many terms we must keep in the summation of the series. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the n^{th} term; so we require $\frac{n^2}{e^n} = 0.001$, or $e^n = 1000n^2$. By a process of trial and error, we find that $n \approx 12$. Carrying out the sum of the first 11 terms of the series (since $n-1 = 12-1 = 11$) we get

$$\sum_{k=1}^{11} (-1)^{k+1} \frac{k^2}{e^k} \approx \boxed{0.091}$$

which is correct to three decimal places.

33. The series is $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} (-1)^k \frac{3}{\sqrt[3]{k}}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sin \frac{3}{\sqrt[3]{n}} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3}{\sqrt[3]{n+1}}}{\frac{3}{\sqrt[3]{n}}} = \sqrt[3]{\frac{n}{n+1}} < 1,$$

for $n \geq 1$. So, by the Alternating Series Test, the series converges. This series cannot be summed by exact methods, so we have to approximate the sum by using the error estimate. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the n^{th} term; so we require

$$a_n = \frac{3}{\sqrt[3]{n}} = 10^{-3} \text{ or } \sqrt[3]{n} = 3 \times 10^3 \text{ or } n = 27 \times 10^9.$$

Using a CAS, we do the computation to these many terms, and obtain the sum of approximately $\boxed{-1.715}$, an answer that is correct to three decimal places. (The exact answer is $3(2^{2/3} - 1)\zeta(\frac{1}{3})$, where $\zeta(s)$ is the Riemann zeta function introduced in Problem 75, Section 8.3, p.575.)

34. The series is $\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2}k\right)$. The sequence of partial sums has the form $\{S_n\} = \{1, 0, -1, 0, 1, \dots\}$. It does not converge, so the series $\boxed{\text{diverges}}$.

35. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. We have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{n+1}} < 1$$

for all $n \geq 1$. So, by the Alternating Series Test, the series converges. The series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a p -series with $0 < p = \frac{1}{2} < 1$, so it is divergent. Since the original series converges, but the series of absolute values diverges, the original series *converges conditionally.*

36. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\cos k}{k^3}$. The series of absolute values is $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{|\cos k|}{k^3}$. The n^{th} term satisfies

$$|a_n| = \frac{|\cos n|}{n^3} \leq \frac{1}{n^3} = b_n$$

for all $n \geq 1$. Since the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p -series (since $p = 3 > 1$), the series of absolute values converges by the Comparison Test, and so the original series is *absolutely convergent.*

37. The n^{th} term of the series

$$\frac{1}{2} - \frac{4}{2^3 + 1} + \frac{9}{3^3 + 1} - \frac{16}{4^3 + 1} + \cdots$$

is

$$a_n = (-1)^{n+1} \frac{n^2}{n^3 + 1}.$$

It satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n^2}{n^3 + 1} = 0.$$

By the Algebraic Ratio Test,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{\frac{(n+1)^2}{(n+1)^3 + 1}}{\frac{n^2}{n^3 + 1}} = \frac{(n+1)^2}{n^2} \cdot \frac{(n^3 + 1)}{[(n+1)^3 + 1]} \\ &= \frac{(n^2 + 2n + 1)(n^3 + 1)}{n^2[(n^3 + 3n^2 + 3n + 1) + 1]} = \frac{n^5 + 2n^4 + n^3 + n^2 + 2n + 1}{n^5 + 3n^4 + 3n^3 + 2n^2} < 1, \end{aligned}$$

provided we have

$$\begin{aligned} n^5 + 2n^4 + n^3 + n^2 + 2n + 1 &< n^5 + 3n^4 + 3n^3 + 2n^2 \\ \text{or,} \quad 2n + 1 &< n^4 + 2n^3 + n^2 \\ \text{or,} \quad 2n + 1 &< n^2(n^2 + 2n + 1) \end{aligned}$$

which is true for all $n \geq 1$. So the series of absolute values decreases as well. By the Alternating Series Test, the original series converges. The n^{th} term of the series of absolute values $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{k^2}{k^3 + 1}$ behaves for large values of n like

$$|a_n| = \frac{n^2}{n^3 + 1} = \frac{n^2}{n^3 \left(1 + \frac{1}{n^3}\right)} \approx \frac{1}{n} = |b_n|.$$

Comparing with the divergent harmonic series $\sum_{k=1}^{\infty} |b_k| = \sum_{k=1}^{\infty} \frac{1}{k}$, we get

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+1} = 1.$$

Since the limit is a positive real number and the series $\sum_{k=1}^{\infty} |b_k|$ is divergent, it means that the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is also divergent by the Limit Comparison Test. Since the original series converges, but the series of absolute values diverges, the original series is *conditionally convergent*.

38. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{k!}$ is a series of nonzero terms. By the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)n!} \right| = 2 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1.$$

Since the limit is less than 1, the series *converges* by the Ratio Test.

39. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ is a series of nonzero terms. By the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}}{e^{(n^2+2n+1)}} \cdot \frac{(n+1)n!}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{e^{2n+1}} = \lim_{x \rightarrow \infty} \frac{x+1}{e^{2x+1}} = \lim_{x \rightarrow \infty} \frac{1}{2e^{2x+1}} \\ &= 0, \end{aligned}$$

applying L'Hôpital's rule to a related function of the ratio. Since the limit is less than 1, by the Ratio Test, we conclude that the series *converges*.

40. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{(k+3)^{k+1}}$ is a series of nonzero terms. By the Root Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{2^n}{(n+3)^{n+1}} \right|} = 2 \lim_{n \rightarrow \infty} \frac{1}{(n+3)^{(n+1)/n}} = 2 \lim_{n \rightarrow \infty} \frac{1}{(n+3)(n+3)^{1/n}} \\ &= 2 \lim_{n \rightarrow \infty} \frac{1}{n+3} \cdot \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} \cdot \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{3}{n}\right)^{1/n}} \\ &= 2 \cdot 0 \cdot 1 \cdot \frac{1}{(1+0)^0} \\ &= 0, \end{aligned}$$

where we have used the limit result $\lim_{n \rightarrow \infty} n^{1/n} = 1$ (see solutions to Exercise 31 of Section 8.6 for a proof). Since the limit is less than 1, by the Root Test, the series *converges*.

41. $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} (-1)^k (e^{-k} - 1)^k$ is a series of nonzero terms. Applying the Root Test,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(e^{-n} - 1)^n|} = \lim_{n \rightarrow \infty} |e^{-n} - 1| = \lim_{n \rightarrow \infty} (1 - e^{-n}) = 1.$$

Since the limit is equal to 1, the Root Test is inconclusive. However, we note

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} |(e^{-n} - 1)^n| = 1 \neq 0,$$

where we have used the fact that $\lim_{n \rightarrow \infty} r^n = 0$ if $0 < |r| < 1$; here $r = \frac{1}{e}$. Since the limit of the n^{th} term is nonzero, by the Divergence Test, the series *diverges*.

42. The series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{k+1}}{3^k} = \sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{3^k} = \sum_{k=1}^{\infty} \frac{(-2)^2}{3} \cdot \left(-\frac{2}{3}\right)^{k-1} = \sum_{k=1}^{\infty} \frac{4}{3} \left(-\frac{2}{3}\right)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{3}$ and $r = -\frac{2}{3}$. Since $0 < |r| < 1$, the series *converges*.

43. The series can be written $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty} (\ln(k+1) - \ln k)$. The n^{th} partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^n (\ln(k+1) - \ln k) \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n) \\ &= \ln(n+1) - \ln 1 = \ln(n+1) - 0 \\ &= \ln(n+1). \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln(n+1) = \infty,$$

the sequence of partial sums $\{S_n\}$ diverges, which means that the series *diverges* as well.

44. The series is $\sum_{k=5}^{\infty} a_k = \sum_{k=5}^{\infty} \frac{3}{k\sqrt{k-4}}$. The n^{th} term satisfies

$$a_n = \frac{3}{n\sqrt{n-4}} < \frac{3}{(n-4)\sqrt{n-4}} = \frac{3}{(n-4)^{3/2}} = b_n,$$

for all $n \geq 5$. Consider the series $\sum_{k=5}^{\infty} b_k = \sum_{k=5}^{\infty} \frac{3}{(k-4)^{3/2}}$. Setting $k-4 = k'$, we get $\sum_{k'=1}^{\infty} \frac{3}{k'^{3/2}}$. This is a constant multiple of a convergent p -series (since $p = \frac{3}{2} > 1$), so the series $\sum_{k=5}^{\infty} b_k$ converges. By the Comparison Test, this means that the original series $\sum_{k=5}^{\infty} a_k$ *converges* as well.

45. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\left(1 + \frac{k^2+1}{k^2}\right)^k}$ is a series of nonzero terms. Using the Root Test,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{\left(1 + \frac{n^2+1}{n^2}\right)^n} \right|} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{n^2+1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{1}{1 + \left(1 + \frac{1}{n^2}\right)} = \frac{1}{1 + (1+0)} = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series *converges* by the Root Test.

46. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$. We can write the n^{th} term as

$$a_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{(2 \cdot 4 \cdot 6 \cdots (2n))^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-1)(2n)} = \frac{[2^n \cdot n!]^2}{(2n)!} = \frac{2^{2n} (n!)^2}{(2n)!}.$$

The Ratio Test can be used since $a_n \neq 0$ for all $n \geq 1$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{2(n+1)}((n+1)!)^2}{(2(n+1))!}}{\frac{2^{2n}(n!)^2}{(2n)!}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2^{2n+2}}{2^{2n}} \cdot \left(\frac{(n+1)n!}{n!} \right)^2 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right| \\ &= 2^2 \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| \\ &= 4 \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n}\right)^2}{\left(2 + \frac{2}{n}\right)\left(2 + \frac{1}{n}\right)} \right| \\ &= 4 \cdot \left| \frac{1+0}{(2+0)(2+0)} \right| = 4 \cdot \frac{1}{4} = 1. \end{aligned}$$

Since the limit is equal to 1, the Ratio Test is inconclusive. However, observe that using

$$\left| \frac{a_{n+1}}{a_n} \right| = 4 \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right|$$

we can show that $\left| \frac{a_{n+1}}{a_n} \right| > 1$. This happens when

$$\begin{aligned} 4 \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| &> 1 \\ 4(n+1)^2 &> (2n+2)(2n+1) \\ 4(n^2 + 2n + 1) &> 4n^2 + 4n + 2n + 2 \\ 4n^2 + 8n + 4 &> 4n^2 + 6n + 2 \\ 2n + 4 &> 2 \\ \text{or,} \quad 2n &> -2, \end{aligned}$$

which is true for all $n \geq 1$. So the terms of the series are increasing. Since the first term, $a_1 = \frac{2^2(1!)}{2!} = 2$, and the terms of the series are increasing, it would mean that

$$\lim_{n \rightarrow \infty} a_n > a_1 = 2 \neq 0.$$

So the series *diverges* by the Divergence Test.

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{(1+k^3) \ln(\sqrt[3]{1+k^3})}$. Consider a related function of the n^{th} term of the series,

$$f(x) = \frac{x^2}{(1+x^3) \ln(\sqrt[3]{1+x^3})}.$$

This is continuous and defined on $[1, \infty)$, decreasing, and such that $f(k) = a_k$ for all $k \geq 1$. (To see that it is decreasing for $[\sqrt[3]{2}, \infty)$ we can use the following argument:

$$f'(x) = -\frac{[3x^4 + (x^3 - 2)x \ln(1+x^3)]}{(1+x^3)^2 (\ln(1+x^3))^2}.$$

Since denominator is always positive being the product of two squared terms, the sign of $f'(x)$ is negative if the numerator is positive. (Note the minus sign outside.) The numerator is positive at least for $x^3 - 2 \geq 0$ or $x \geq \sqrt[3]{2} \approx 1.26$.) the Using the Integral Test, we compute

$$I = \int_1^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x^2 dx}{(1+x^3) \ln(\sqrt[3]{1+x^3})}.$$

Let $u = 1 + x^3$. Then $du = 3x^2 dx$. When $x = 1$, $u = 2$. When $x = b$, $u = 1 + b^3$. The integral becomes

$$I = \lim_{b \rightarrow \infty} \int_2^{1+b^3} \frac{\frac{1}{3} du}{\frac{1}{3} u \ln u} = \lim_{b \rightarrow \infty} \int_2^{1+b^3} \frac{du}{u \ln u}.$$

Let $v = \ln u$. Then $dv = \frac{du}{u}$. When $u = 2$, $v = \ln 2$. When $u = 1 + b^3$, $v = \ln(1 + b^3)$. So we have

$$I = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln(1+b^3)} \frac{dv}{v} = \lim_{b \rightarrow \infty} [\ln |v|] \Big|_{\ln 2}^{\ln(1+b^3)} = \lim_{b \rightarrow \infty} [\ln(\ln(1 + b^3)) - \ln(\ln 2)] = \infty.$$

Since the limit of the improper integral is not finite, by the Integral Test, the series *diverges.*

48. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^{10}}{2^k}$ is a series of nonzero terms. By the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{n^{10+1}}{2^{n+1}}}{\frac{n^{10}}{2^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^n}{2 \cdot 2^n} \cdot \frac{n^{10} \cdot n}{n^{10}} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} n = \infty.$$

Since the limit is ∞ , by the Ratio Test, the series *diverges.*

49. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1 + \frac{1}{k^2})^{k^2}}{2^k}$ is a series of nonzero terms. Using the Root Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(1 + \frac{1}{n^2})^{n^2}}{2^n} \right|} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2} \right)^{n^2/n} \\ &= \frac{1}{2} \left[\lim_{n^2 \rightarrow \infty} \left(1 + \frac{1}{n^2} \right)^{n^2} \right]^{\lim_{n \rightarrow \infty} 1/n} \\ &= \frac{1}{2} \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \right]^0 \\ &= \frac{1}{2} [e]^0 = \frac{1}{2} < 1, \end{aligned}$$

where we used the substitution $m = n^2$ and the standard result that $\lim_{m \rightarrow \infty} (1 + \frac{1}{m})^m = e$. Since the limit is less than 1, by the Ratio Test, the series *converges.*

50. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{k^2+1}{k} \right)^k$ is a series of nonzero terms. Using the Root Test,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{n^2+1}{n} \right)^n \right|} = \lim_{n \rightarrow \infty} \left| \frac{n^2+1}{n} \right| = \infty.$$

Since the limit is ∞ , by the Root Test, the series *diverges.*

51. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{3^k k^k}$ is a series of nonzero terms. Using the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{3^{n+1} (n+1)^{n+1}}}{\frac{n!}{3^n n^n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)n!}{n!} \cdot \frac{n^n}{(n+1)(n+1)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{1+n} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}} \right)^n = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n} \\ &= \frac{1}{e} < 1, \end{aligned}$$

since $e > 1$, where we have used the standard result that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$. Since the limit is less than 1, by the Ratio Test, the series *converges*.

52. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{3^{k-2}}$. The absolute value of the n^{th} term satisfies

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+2}{3n-2} = \lim_{n \rightarrow \infty} \frac{1+\frac{2}{n}}{3-\frac{2}{n}} = \frac{1}{3} \neq 0.$$

Since the limit is nonzero, the series *diverges* by the Divergence Test.

53. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-3)^{3k-1}}{k^2}$ is a series of nonzero terms if $x \neq 3$. (If $x = 3$, the series sums to 0.) Applying the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{3(n+1)-1}}{(n+1)^2}}{\frac{(x-3)^{3n-1}}{n^2}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{3n-1+3}}{(x-3)^{3n-1}} \cdot \frac{n^2}{(n+1)^2} \right| = |x-3|^3 \left(\lim_{n \rightarrow \infty} \frac{n}{n+1} \right)^2 \\ &= |x-3|^3 \left(\lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} \right)^2 = |x-3|^3 \cdot \left(\frac{1}{1+0} \right)^2 \\ &= |x-3|^3 < 1 \end{aligned}$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x-3 < 1$, or $2 < x < 4$. The radius of convergence is $R = 1$.

(b) At $x = 2$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(2-3)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{3k+3}(-1)^{-4}}{k^2} = \sum_{k=1}^{\infty} \frac{[(-1)^3]^{k+1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$$

The series of absolute values $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series (since $p = 2 > 1$), so the original series is absolutely convergent, hence converges. So the power series converges at $x = 2$.

At $x = 4$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(4-3)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(1)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p -series (since $p = 2 > 1$). So the power series converges at $x = 4$. Concluding, the interval of convergence of the power series is $\boxed{2 \leq x \leq 4}$.

54. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{\sqrt[3]{k}}$ is a series of nonzero terms if $x \neq 0$. (If $x = 0$, the series sums to 0.) By the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{\sqrt[3]{n+1}}}{\frac{x^n}{\sqrt[3]{n}}} \right| \\ &= |x| \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n}{n+1}} = |x| \sqrt[3]{\lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)} \\ &= |x| \sqrt[3]{\frac{1}{1+0}} = |x| < 1 \end{aligned}$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x < 1$, and the radius of convergence is $\boxed{R = 1}$.

(b) At $x = -1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}.$$

The absolute value of the n^{th} term satisfies

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 0.$$

By the Algebraic Ratio Test, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{\sqrt[3]{n+1}}}{\frac{1}{\sqrt[3]{n}}} \right| = \sqrt[3]{\frac{n}{n+1}} < 1$$

for all $n \geq 1$. So the absolute value of the terms of the series are decreasing. By the Alterbating Series Test, the series converges. So the power series converges at $x = -1$.

At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1)^k}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}.$$

This is a p -series with $0 < p = \frac{1}{3} < 1$, so it diverges. So the power series diverges at $x = 1$.

Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1}$.

55. (a) The power series $\sum_{k=0}^{\infty} (-1)^k a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)} \left(\frac{x}{2}\right)^{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) Using the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!(n+2)} \left(\frac{x}{2}\right)^{2(n+1)+1}}{\frac{1}{n!(n+1)} \left(\frac{x}{2}\right)^{2n+1}} \right| \\ &= \left| \frac{x}{2} \right|^2 \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \cdot \frac{n!}{(n+1)n!} \right| = \left| \frac{x}{2} \right|^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n+2} \\ &= 0 \end{aligned}$$

for any value of x . So the radius of convergence of the power series is $\boxed{R = \infty}$.

(b) Since the series converges for any value of x , the interval of convergence of the series is $[-\infty < x < \infty]$.

56. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k}{(k!)^2} x^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) Using the Ratio Test, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^{n+1} x^{n+1}}{((n+1)!)^2}}{\frac{n^n x^n}{(n!)^2}} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)^n}{n^n} \cdot \left(\frac{n!}{(n+1)n!} \right)^2 \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^n \cdot \frac{1}{n+1} \right| \\ &= |x| \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= |x| \cdot e \cdot 0 = 0, \end{aligned}$$

where the standard limit result $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$ has been used. Since the limit is less than 1 for any value of x , the radius of convergence of the power series is $[R = \infty]$.

(b) Since the series converges for any value of x , the interval of convergence of the series is $[-\infty < x < \infty]$.

57. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$ is a series of nonzero terms for $x \neq 1$. (At $x = 1$, the series converges to 0.) Using the Ratio Test, we get

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}} \right| = |x-1| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = |x-1| \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right) = |x-1| \cdot \frac{1}{1+0} = |x-1| < 1$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x-1 < 1$, or $0 < x < 2$, and the radius of convergence is $[R = 1]$.

(b) At $x = 0$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(0-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

which, being a constant multiple of the alternating harmonic series (namely the alternating harmonic series multiplied by -1), converges. So the power series converges for $x = 0$.

At $x = 2$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(2-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(1)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which is the divergent harmonic series. So the power series diverges for $x = 2$.

Concluding, the interval of convergence of the power series is $[0 \leq x < 2]$.

58. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k x^k}{5^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) By the Ratio Test, we get

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{5^{n+1}}}{\frac{3^n x^n}{5^n}} \right| = \frac{3}{5} |x| < 1$$

for convergence of the series by the Ratio Test. So $|x| < \frac{5}{3}$, which means the power series converges in the interval $-\frac{5}{3} < x < \frac{5}{3}$, and its radius of convergence is $R = \frac{5}{3}$.

(b) At $x = -\frac{5}{3}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k}{5^k} \cdot \left(-\frac{5}{3}\right)^k = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums of this series is $\{S_n\} = \{1, 0, 1, 0, \dots\}$ which does not converge, and this means the series does not converge either. So the power series diverges for $x = -\frac{5}{3}$.

At $x = \frac{5}{3}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k}{5^k} \cdot \left(\frac{5}{3}\right)^k = \sum_{k=0}^{\infty} 1$$

which diverges. So the power series diverges for $x = \frac{5}{3}$.

Concluding, the interval of convergence of the power series is $-\frac{5}{3} < x < \frac{5}{3}$.

59. To express the function $f(x) = \frac{2}{x+3}$ as a power series centered at 0, we use the following function respresented as a geometric series: $g(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $-1 < x < 1$. So,

$$\begin{aligned} f(x) &= \frac{2}{3+x} = \frac{2}{3} \left(\frac{1}{1+\frac{x}{3}} \right) = \frac{2}{3} \left[\frac{1}{1 - \left(-\frac{x}{3}\right)} \right] \\ &= \frac{2}{3} \sum_{k=0}^{\infty} \left(-\frac{x}{3}\right)^k = \frac{2}{3} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{3}\right)^k, \end{aligned}$$

which converges for $\left|\frac{x}{3}\right| < 1$ or $-1 < \frac{x}{3} < 1$, or $-3 < x < 3$.

60. To express the function $f(x) = \frac{1}{1-3x}$ as a power series centered at 0, we use the following function represented as a geometric series: $g(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $-1 < x < 1$. So we have

$$f(x) = \frac{1}{1-3x} = g(3x) = \sum_{k=0}^{\infty} (3x)^k = \sum_{k=0}^{\infty} 3^k x^k,$$

which converges for $|3x| < 1$ or $-1 < 3x < 1$ or $-\frac{1}{3} < x < \frac{1}{3}$.

61. (a) From Problem 60 above, the series centered about 0 for

$$\frac{1}{1-3x^2} = f(x^2) = \sum_{k=0}^{\infty} 3^k (x^2)^k = 1 + 3x^2 + 9x^4 + 27x^6 + \dots + 3^n x^{2n} + \dots$$

Integrating both sides we get

$$\begin{aligned} \int_0^x \frac{1}{1-3t^2} dt &= \int_0^x [1 + 3t^2 + 9t^4 + 27t^6 + \dots + 3^n t^{2n} + \dots] dt \\ &= \boxed{x + x^3 + \frac{9}{5}x^5 + \frac{27}{7}x^7 + \dots + \frac{3^n}{2n+1}x^{2n+1} + \dots} \end{aligned}$$

(b) To ascertain the correct number of terms to keep, we calculate the n for which the contribution to the sum becomes less than 0.001, for $x = \frac{1}{2}$. So, setting $x = \frac{1}{2}$ in the n^{th} term, we get

$$\frac{3^n}{2n+1} \left(\frac{1}{2}\right)^{2n+1} \leq 0.001.$$

Proceeding by trial and error, we find that $n = 11$. So we have

$$\int_0^{1/2} \frac{1}{1-3x^2} dx \approx \sum_{k=0}^{11} \frac{3^k}{2k+1} \left(\frac{1}{2}\right)^{2k+1} \approx \boxed{0.758}$$

an answer that is correct to three decimal places.

62. To find the Taylor expansion of $f(x) = \frac{1}{1-2x}$ about $c = 1$, we begin by evaluating the function and its derivatives at $x = 1$.

$$\begin{array}{ll} f(x) = \frac{1}{1-2x} & f(1) = \frac{1}{1-2 \cdot 1} = -1 \\ f'(x) = \frac{2 \cdot 1}{(1-2x)^2} & f'(1) = \frac{2 \cdot 1}{(1-2 \cdot 1)^2} = 2 \cdot 1! \\ f''(x) = \frac{2^2 \cdot 2 \cdot 1}{(1-2x)^3} & f''(1) = \frac{2^2 \cdot 2!}{(1-2 \cdot 1)^3} = -2^2 \cdot 2! \\ f'''(x) = \frac{2^3 \cdot 3 \cdot 2 \cdot 1}{(1-2x)^4} & f'''(1) = \frac{2^3 \cdot 3!}{(1-2 \cdot 1)^4} = 2^3 \cdot 3! \\ \vdots & \vdots \end{array}$$

The Taylor expansion of the function about $c = 1$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\ &= \frac{-1}{0!} (x-1)^0 + \frac{2 \cdot 1!}{1!} (x-1)^1 - \frac{2^2 \cdot 2!}{2!} (x-1)^2 + \frac{2^3 \cdot 3!}{3!} (x-1)^3 - \cdots + (-1)^{n+1} \frac{2^n \cdot n!}{n!} (x-1)^n + \cdots \\ &= -1 + 2(x-1) - 2^2(x-1)^2 + 2^3(x-1)^3 - \cdots + (-1)^{n+1} 2^n (x-1)^n + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^{k+1} 2^k (x-1)^k}. \end{aligned}$$

63. To find the Taylor expansion of $f(x) = e^{x/2}$ about $c = 1$, we begin by evaluating the function and its derivatives at $x = 1$.

$$\begin{array}{ll} f(x) = e^{x/2} & f(1) = e^{1/2} \\ f'(x) = \frac{1}{2} e^{x/2} & f'(1) = \frac{1}{2} e^{1/2} \\ f''(x) = \frac{1}{2^2} e^{x/2} & f''(1) = \frac{1}{2^2} e^{1/2} \\ f'''(x) = \frac{1}{2^3} e^{x/2} & f'''(1) = \frac{1}{2^3} e^{1/2} \\ \vdots & \vdots \end{array}$$

The Taylor expansion of the function about $c = 1$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\ &= \frac{1}{0!} e^{1/2} (x-1)^0 + \frac{1}{1!} 2 e^{1/2} (x-1)^1 + \frac{1}{2!} 2^2 e^{1/2} (x-1)^2 + \frac{1}{3!} 2^3 e^{1/2} (x-1)^3 + \cdots + \frac{1}{n!} 2^n e^{1/2} (x-1)^n + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{1}{k!} 2^k e^{1/2} (x-1)^k}. \end{aligned}$$

64. To find the Maclaurin expansion of $f(x) = 2x^3 - 3x^2 + x + 5$, we begin by evaluating the function and its derivatives at $x = 0$.

$$\begin{array}{ll} f(x) = 2x^3 - 3x^2 + x + 5 & f(0) = 2(0)^3 - 3(0)^2 + 0 + 5 = 5 \\ f'(x) = 6x^2 - 6x + 1 & f'(0) = 6(0)^2 - 6(0) + 1 = 1 \\ f''(x) = 12x - 6 & f''(0) = -6 \\ f'''(x) = 12 & f'''(0) = 12 \\ f^{(4)}(x) = 0 & f^{(4)}(0) = 0. \end{array}$$

The higher derivatives and their evaluations at $x = 0$ are all 0 as well. So the Maclaurin series is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= \frac{5}{0!} x^0 + \frac{1}{1!} x^1 + \frac{-6}{2!} x^2 + \frac{12}{3!} x^3 \\ &= 5 + x - 3x^2 + 2x^3 \\ &= \boxed{2x^3 - 3x^2 + x + 5}. \end{aligned}$$

We see that the Maclaurin expansion of the function is equivalent to the function itself.

65. To find the Taylor series for $f(x) = \tan x$ about $c = \frac{\pi}{4}$, we begin by evaluating the function and its derivatives at $x = \frac{\pi}{4}$. Let $y = f(x)$, $y_1 = f\left(\frac{\pi}{4}\right)$, $y' = f'(x)$, $y'_1 = f'\left(\frac{\pi}{4}\right)$, etc.

$$\begin{array}{ll} y = \tan x & y_1 = \tan\left(\frac{\pi}{4}\right) = 1 \\ y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 & y'_1 = 1 + y_1^2 = 1 + (1)^2 = 2 \\ y'' = 2yy' & y''_1 = 2y_1 y'_1 = 2(1)(2) = 4 \\ y''' = 2y'^2 + 2yy'' & y'''_1 = 2y'^2_1 + 2y_1 y''_1 \\ & \quad = 2(2)^2 + 2(1)(4) = 16 \\ y^{(4)} = 4y' y'' + 2y' y''' & y^{(4)}_1 = 4y'_1 y''_1 + 2y'_1 y'''_1 \\ & \quad = 6y'_1 y''_1 + 2y_1 y'''_1 \\ & \quad = 6(2)(4) + 2(1)(16) = 80 \\ y^{(5)} = 6y''^2 + 6y' y''' + 2y' y^{(4)} & y^{(5)}_1 = 6y''^2_1 + 6y'_1 y'''_1 + 2y'_1 y^{(4)}_1 \\ & \quad = 6(4)^2 + 6(2)(16) + 2(1)(80) = 512 \\ \vdots & \vdots \end{array}$$

The Taylor expansion of the function about $c = \frac{\pi}{4}$ is

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \\
 &= 1 + \frac{2}{1!} \left(x - \frac{\pi}{4}\right)^1 + \frac{4}{2!} \left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!} \left(x - \frac{\pi}{4}\right)^3 + \frac{80}{4!} \left(x - \frac{\pi}{4}\right)^4 + \frac{512}{5!} \left(x - \frac{\pi}{4}\right)^5 + \cdots \\
 &= \boxed{1 + 2 \left(x - \frac{\pi}{4}\right) + 2 \left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3} \left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3} \left(x - \frac{\pi}{4}\right)^4 + \frac{64}{15} \left(x - \frac{\pi}{4}\right)^5 + \cdots}
 \end{aligned}$$

66. The Maclaurin expansion of $f(x) = e^{-x} \sin x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\sin x$ together. From Example 2, p.615 of Section 8.9, we get the Maclaurin expansion

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots.$$

So we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots.$$

From p.616 of Section 8.9 the book, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots.$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned}
 f(x) &= e^{-x} \sin x \\
 &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right) \\
 &= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
 &= 1 \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - x \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^2}{2} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \\
 &\quad - \frac{x^3}{6} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^4}{24} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - \frac{x^5}{120} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \cdots \\
 &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(-x^2 + \frac{x^4}{6} - \frac{x^6}{120} + \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{240} - \cdots\right) + \\
 &\quad + \left(-\frac{x^4}{6} + \frac{x^6}{36} - \frac{x^8}{720} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \frac{x^9}{2880} - \cdots\right) + \left(-\frac{x^6}{120} + \cdots\right) + \cdots \\
 &= x - x^2 + x^3 \left(\frac{1}{2} - \frac{1}{6}\right) + x^4 \left(\frac{1}{6} - \frac{1}{6}\right) + x^5 \left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right) + x^6 \left(-\frac{1}{120} + \frac{1}{36} - \frac{1}{120}\right) - \cdots \\
 &= \boxed{x - x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{90}x^6 - \cdots}.
 \end{aligned}$$

67. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial series of $f(x) = \frac{1}{(x+1)^4} = (1+x)^{-4}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-4}{k} x^k \\ &= \binom{-4}{0} x^0 + \binom{-4}{1} x^1 + \binom{-4}{2} x^2 + \binom{-4}{3} x^3 + \cdots + \binom{-4}{n} x^n + \cdots \\ &= 1 - 4x + \frac{(-4)(-4-1)}{2!} x^2 + \frac{(-4)(-4-1)(-4-2)}{3!} x^3 + \cdots \\ &= \boxed{1 - 4x + 10x^2 - 20x^3 + \cdots} \end{aligned}$$

Since $m = -4$ satisfies $m \leq -1$, by the conditions of the theorem on p.620, the binomial series of $\frac{1}{(x+1)^4}$ converges on the open interval $\boxed{(-1, 1)}$.

68. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial series of $f(x) = \sqrt[3]{x^2 - 1} = (-1)^{1/3} (1 - x^2)^{1/3} = -1(1 - x^2)^{1/3}$ is given by

$$\begin{aligned} f(x) &= - \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} (-x^2)^k \\ &= - \left[\binom{\frac{1}{3}}{0} (-x^2)^0 + \binom{\frac{1}{3}}{1} (-x^2)^1 + \binom{\frac{1}{3}}{2} (-x^2)^2 + \binom{\frac{1}{3}}{3} (-x^2)^3 + \cdots + \binom{\frac{1}{3}}{n} (-x^2)^n + \cdots \right] \\ &= - \left[1 - \frac{1}{3} x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} x^4 - \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} x^6 + \cdots \right] \\ &= \boxed{-1 + \frac{1}{3} x^2 + \frac{1}{9} x^4 + \frac{5}{81} x^6 + \cdots} \end{aligned}$$

Since $m = \frac{1}{3}$ satisfies $m > 0$ but m is not an integer, by the conditions of the theorem on p.620, the series converges on the closed interval $\boxed{[-1, 1]}$.

69. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-1)^k x^k \\ &= \binom{-\frac{1}{2}}{0} (-1)^0 x^0 + \binom{-\frac{1}{2}}{1} (-1)^1 x^1 + \binom{-\frac{1}{2}}{2} (-1)^2 x^2 + \binom{-\frac{1}{2}}{3} (-1)^3 x^3 + \cdots \\ &= 1 - \left(-\frac{1}{2} \right) x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} x^2 - \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} x^3 + \cdots \\ &= \boxed{1 + \frac{1}{2} x + \frac{3}{8} x^2 + \frac{5}{16} x^3 + \cdots} \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the series converges on the half open interval $\boxed{(-1, 1]}$.

70. (a) To find the Taylor expansion to four nonzero terms of $y = \cos x$ about $x = \frac{\pi}{2}$, we compute the

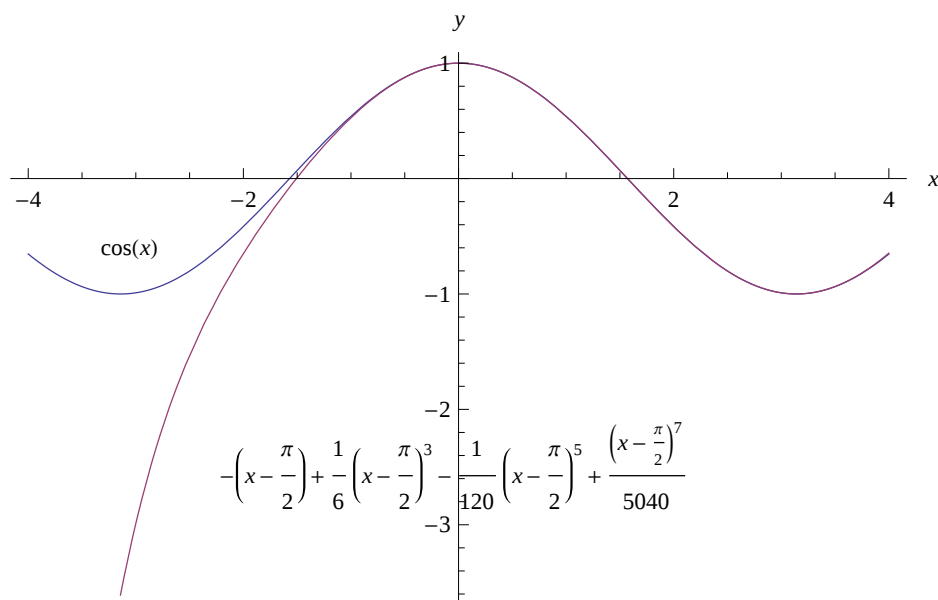
function and its derivatives at $x = \frac{\pi}{2}$ until four nonzero values result.

$y = \cos x$	$y\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$
$y'(x) = -\sin x$	$y'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = \boxed{-1}$
$y''(x) = -\cos x$	$y''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$
$y'''(x) = \sin x$	$y'''\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = \boxed{1}$
$y^{(4)}(x) = \cos x$	$y^{(4)}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$
$y^{(5)}(x) = -\sin x$	$y^{(5)}\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = \boxed{-1}$
$y^{(6)}(x) = -\cos x$	$y^{(6)}\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$
$y^{(7)}(x) = \sin x$	$y^{(7)}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = \boxed{1}$
$\vdots$	$\vdots$

So the Taylor expansion to four nonzero terms of $y = \cos x$ about $c = \frac{\pi}{2}$ is given by

$$\begin{aligned}
 y = f(x) = \cos x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k \\
 &= \frac{1}{1!} \left(x - \frac{\pi}{2}\right)^1 + \frac{1}{3!} \left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!} \left(x - \frac{\pi}{2}\right)^5 + \frac{1}{7!} \left(x - \frac{\pi}{2}\right)^7 - \dots \\
 &= \boxed{-\left(x - \frac{\pi}{2}\right) + \frac{1}{6} \left(x - \frac{\pi}{2}\right)^3 - \frac{1}{120} \left(x - \frac{\pi}{2}\right)^5 + \frac{1}{5040} \left(x - \frac{\pi}{2}\right)^7 - \dots}
 \end{aligned}$$

(b) The figure below shows the graph of both the function and its Maclaurin approximation found above in part (a).



(c) Set $x = 88^\circ = \frac{22}{45}\pi$. Then $(x - \frac{\pi}{2}) = (\frac{22}{45}\pi - \frac{\pi}{2}) = -\frac{\pi}{90}$. So

$$\begin{aligned}\cos(88^\circ) &\approx -\left(-\frac{\pi}{90}\right) + \frac{1}{6}\left(-\frac{\pi}{90}\right)^3 - \frac{1}{120}\left(-\frac{\pi}{90}\right)^5 + \frac{1}{5040}\left(-\frac{\pi}{90}\right)^7 \\ &\approx \boxed{0.0349}.\end{aligned}$$

(d) Since the series is a converging alternating one, the error in using this approximation is bounded from above by the absolute value of the next nonzero term, which is

$$\left|\frac{1}{9!}\left(x - \frac{\pi}{2}\right)^9\right|.$$

Evaluating this error bound for $x = 88^\circ$, we have

$$\text{error} \leq \left|\frac{1}{9!}\left(-\frac{\pi}{90}\right)^9\right| \approx \boxed{2.12 \times 10^{-19}}.$$

(e) We need to determine the x values for which the error term would be less than or equal to 0.0001, that is, we need

$$\begin{aligned}\left|\frac{1}{9!}\left(x - \frac{\pi}{2}\right)^9\right| &\leq 0.0001 \\ -36.288 &\leq \left(x - \frac{\pi}{2}\right)^9 \leq 36.288 \\ -1.490414 &\lesssim x - \frac{\pi}{2} \lesssim 1.490414 \\ -1.490414 + \frac{\pi}{2} &\lesssim x \lesssim 1.490414 + \frac{\pi}{2} \\ 0.08038 &\lesssim x \lesssim 3.061210 \\ 4.6^\circ &\lesssim x \lesssim 175.4^\circ.\end{aligned}$$

So the x values for which the error in the approximation would be less than or equal to 0.0001 are approximately $\boxed{4.6^\circ \leq x \leq 175.4^\circ}$.

71. The Maclaurin series for $f(x) = e^x$ is given by (see p.615) by

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

To ensure the accuracy of the approximation of $e^{0.3}$ to three decimal places, let us find the number of terms we need to keep:

$$\frac{x^n}{n!} = 0.001$$

gives, by trial and error, $n = 4$. Keeping the first five nonzero terms of the expansion, we have

$$\begin{aligned}e^{0.3} &\approx 1 + 0.3 + \frac{(0.3)^2}{2!} + \frac{(0.3)^3}{3!} + \frac{(0.3)^4}{4!} \\ &\approx 1 + 0.3 + 0.045 + 0.0045 + 0.00033 \\ &\approx \boxed{1.34983}\end{aligned}$$

The exact value is $e^{0.3} \approx 1.34986$ so the value found using the first five terms is accurate to at least three decimal places.

72. The integrand in $\int_0^{1/2} \frac{1}{\sqrt{1-x^3}} dx$ can be expressed as a binomial series:

$$\begin{aligned} \frac{1}{\sqrt{1-x^3}} &= (1-x^3)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-x^3)^k \\ &= 1 - \frac{1}{2}(-x^3) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x^3)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(-x^3)^3 + \cdots + \binom{-\frac{1}{2}}{n}(-x^3)^n + \cdots \\ &\approx 1 + \frac{1}{2}x^3 + \frac{3}{8}x^6 + \frac{5}{16}x^9 \end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides between the limits, we have

$$\begin{aligned} \int_0^{1/2} \frac{1}{\sqrt{1-x^3}} dx &\approx \int_0^{1/2} \left[1 + \frac{1}{2}x^3 + \frac{3}{8}x^6 + \frac{5}{16}x^9 \right] dx \\ &= \left[x + \frac{1}{2 \cdot 4}x^4 + \frac{3}{8 \cdot 7}x^7 + \frac{5}{16 \cdot 10}x^{10} \right]_0^{1/2} \\ &= (0.5) + \frac{1}{8}(0.5)^4 + \frac{3}{56}(0.5)^7 + \frac{1}{32}(0.5)^{10} \\ &\approx \boxed{0.5083.} \end{aligned}$$

73. The Maclaurin expansion of e^x is (see p.615)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

So the Maclaurin expansion of e^{x^2} is

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots .$$

The integrand is expanded to four nonzero terms for the approximation:

$$\begin{aligned} \int_0^{1/2} e^{x^2} dx &\approx \int_0^{1/2} \left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} \right] dx \\ &= \left[x + \frac{x^3}{3} + \frac{x^5}{2! \cdot 5} + \frac{x^7}{3! \cdot 7} \right]_0^{1/2} \\ &= 0.5 + \frac{(0.5)^3}{3} + \frac{(0.5)^5}{10} + \frac{(0.5)^7}{42} \\ &\approx \boxed{0.545.} \end{aligned}$$

Chapter 9

Parametric Equations; Polar Equations

9.1 Parametric Equations

Concepts and Vocabulary

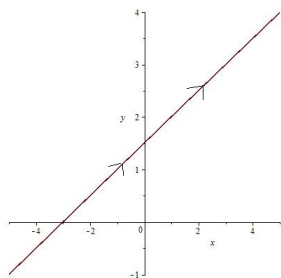
1. Let $x = x(t)$ and $y = y(t)$ be two functions whose common domain is some interval I . The collection of points defined by $(x, y) = (x(t), y(t))$ is called a plane **curve**. The variable t is called a **parameter**.
2. The parametric equations $x(t) = 2 \sin t$, $y(t) = 3 \cos t$ define an **(c) ellipse**.
3. The parametric equations $x(t) = a \sin t$, $y(t) = a \cos t$ define a **(d) circle**.
4. If a circle rolls along a horizontal line without slipping, a point P on the circle traces out a curve called a **cycloid**.
5. **False**. A curve can be defined parametrically in an infinite number of ways.
6. **True**. As the parameter increases, we plot the corresponding points in a certain order which defines the orientation of the curve.

Skill Building

7. (a) Solving for t in the $x(t)$ equation, we have $t = \frac{1}{2}(x - 1)$. Plug this into $y(t)$: $y = \frac{1}{2}(x - 1) + 2$ or

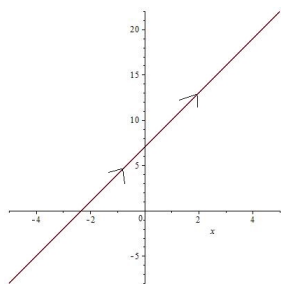
$y = \frac{1}{2}x + \frac{3}{2}$. Since t varies from $-\infty$ to $+\infty$, the rectangular equation is defined for all x from $-\infty$ to ∞ .

(b)



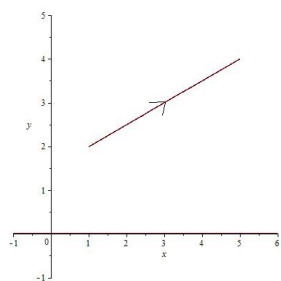
8. (a) Solving for t in the $x(t)$ equation, we have $t = x + 2$. Plug this into $y(t)$: $y = 3(x + 2) + 1$ or $y = 3x + 7$. Since t varies from $-\infty$ to $+\infty$, the rectangular equation is defined for all x from $-\infty$ to ∞ .

(b)



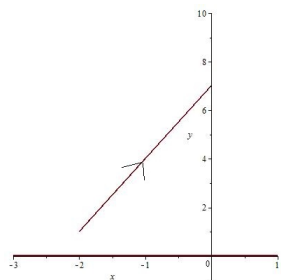
9. (a) Solving for t in the $y(t)$ equation, we have $t = y - 2$. Plug this into $x(t)$: $x = 2(y - 2) + 1$ or $x = 2y - 3$. Since t varies from 0 to 2, we have $1 \leq x \leq 5$ and y varies from $y(0) = 2$ to $y(2) = 4$.

(b)



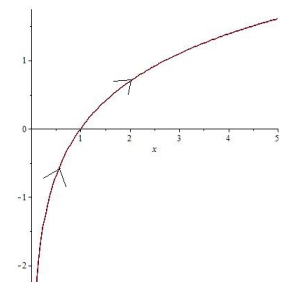
10. (a) Solving for t in the $x(t)$ equation, we have $t = x + 2$. Plug this into $y(t)$: $y = 3(x + 2) + 1$ or $y = 3x + 7$. Since t varies from 0 to 2, x varies from $x(0) = -2$ to $x(2) = 0$.

(b)



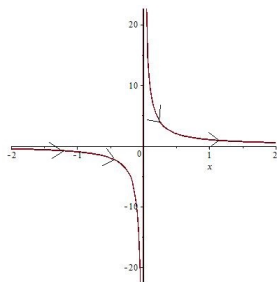
11. (a) Plug $y = t$ into $x(t)$: $x = e^y$. This can be rewritten as $y = \ln x$. Since $-\infty < t < \infty$, we see that $0 < x(t) < \infty$.

(b)



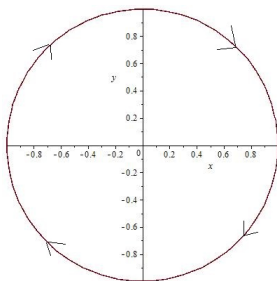
12. (a) Plug $x = t$ into $y(t)$: $y = \frac{1}{x}$. Since $x = t$, the rectangular equation is defined for all $x \neq 0$.

(b)



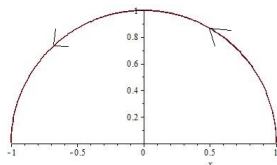
13. (a) Notice that $[x(t)]^2 + [y(t)]^2 = \sin^2 t + \cos^2 t = 1$, which means our parametrization is that of a circle of radius 1 centered at $(0, 0)$, namely $x^2 + y^2 = 1$. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. We find, respectively, the points $(0, 1), (1, 0), (0, -1), (-1, 0)$, and $(0, 1)$. Plotted in this order, we trace a circle starting at the point $(0, 1)$ and moving clockwise until we arrive back at $(0, 1)$.

(b)



14. (a) Notice that $[x(t)]^2 + [y(t)]^2 = \cos^2 t + \sin^2 t = 1$, which means our parametrization is that of a circle of radius 1 centered at $(0, 0)$, namely $x^2 + y^2 = 1$. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi$. We find, respectively, the points $(1, 0), (0, 1)$, and $(-1, 0)$. Plotted in this order, we trace a semicircle, starting at the point $(1, 0)$ and moving counterclockwise until we end at the point $(-1, 0)$.

(b)



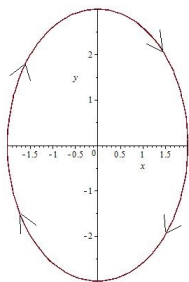
15. (a) We eliminate the parameter t using the Pythagorean Theorem $\sin^2 t + \cos^2 t = 1$. Since $\sin t = \frac{x}{2}$ and $\cos t = \frac{y}{3}$,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1$$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$$

which is an ellipse centered at the origin. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. We find, respectively, the points $(0, 3), (2, 0), (0, -3), (-2, 0)$, and $(0, 3)$. Plotted in this order, we trace an ellipse oriented clockwise.

(b)



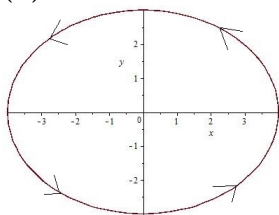
16. (a) We eliminate the parameter t using the Pythagorean Theorem $\cos^2 t + \sin^2 t = 1$. Since $\cos t = \frac{x}{4}$ and $\sin t = \frac{y}{3}$,

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{9} = 1}$$

which is an ellipse centered at the origin. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. We find, respectively, the points $(4, 0), (0, 3), (-4, 0), (0, -3)$, and $(4, 0)$. Plotted in this order, we trace an ellipse oriented counterclockwise.

(b)



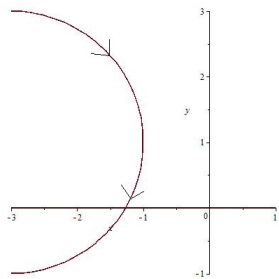
17. (a) Notice that $2 \sin t = x + 3$ and $2 \cos t = y - 1$. If we square and add both equations, then we have

$$(x + 3)^2 + (y - 1)^2 = 4 \sin^2 t + 4 \cos^2 t = 4$$

$$\boxed{\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{4} = 1},$$

which is a circle centered at $(-3, 1)$ with radius 2. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}$, and π . We find, respectively, the points $(-3, 3), (-1, 1)$, and $(-3, -1)$. Plotted in this order, we trace the right half of the circle centered at $(-3, 1)$ with radius 2 oriented clockwise.

(b)

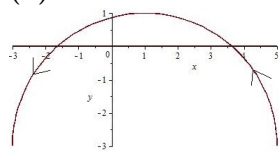


18. (a) Notice that $4 \cos t = x - 1$ and $4 \sin t = y + 3$. If we square and add both equations, then we have

$$\boxed{(x - 1)^2 + (y + 3)^2 = 16 \cos^2 t + 16 \sin^2 t = 16},$$

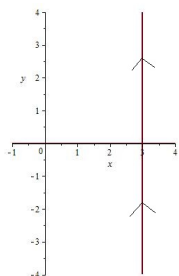
which is a circle centered at $(1, -3)$ with radius 4. To find its starting point, ending point, and orientation, plot the points $t = 0, \frac{\pi}{2}$, and π . We find, respectively, the points $(5, -3), (1, 1)$, and $(-3, -3)$. Plotted in this order, we trace the top half of the circle centered at $(1, -3)$ with radius 4 oriented counterclockwise.

(b)



19. (a) Since $x(t) = 3$, every point on our plane curve has x -coordinate equal to 3. These points describe the line $x = 3$. Since t goes from $-\infty$ to ∞ , so does y so the orientation of the curve is upward.

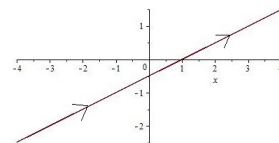
(b)



20. (a) Solving for t in the $y(t)$ equation, we have $t = \frac{y}{2}$. Plug this into $x(t)$: $x = 4\left(\frac{y}{2}\right) + 1$ or

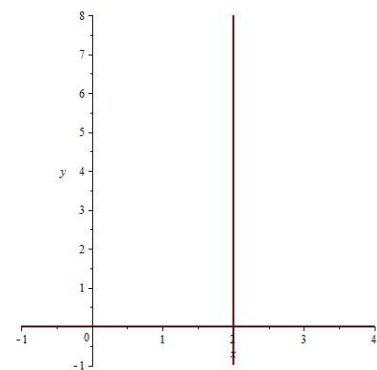
$y = \frac{1}{2}(x - 1)$. Since t varies from $-\infty$ to ∞ , so does x .

(b)



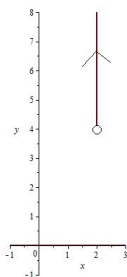
21. (a) Since $x(t) = 2$, the rectangular equation is $x = 2$.

(b)



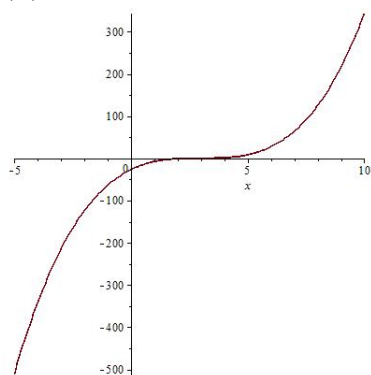
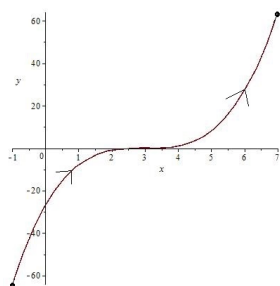
(c) Since $t > 0$, we know that $y > 4$.

(d)



22. (a) Solving for t in the $x(t)$ equation, we have $t = x - 3$. Plug this into the $y(t)$ equation: $y = (x - 3)^3$.

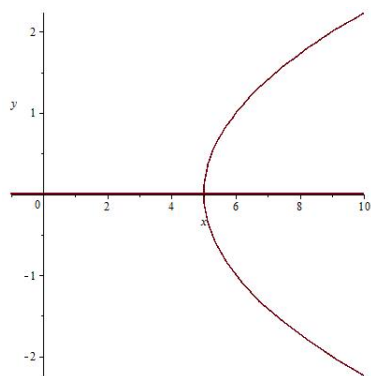
(b)

(c) Since $-4 \leq t \leq 4$, we see that $-1 \leq x \leq 7$ and $-64 \leq y \leq 64$.

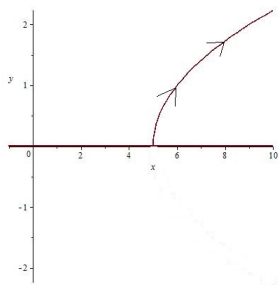
(d)

23. (a) Solving for t in the $y(t)$ equation, we have $t = y^2$. Plug this into the $x(t)$ equation: $x = y^2 + 5$.

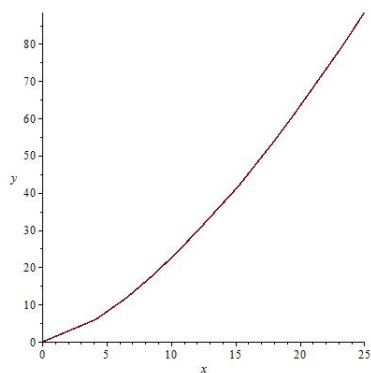
(b)

(c) Since $t \geq 0$, we see that the parametric curve is defined only for $x \geq 5$ and $y \geq 0$.

(d)

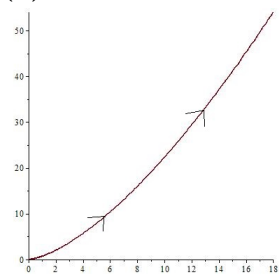
24. (a) Solve for t in the $y(t)$ equation: $t = (y/2)^{1/3}$. Plug this into the $x(t)$ equation: $x = 2[(y/2)^{1/3}]^2$. Simplifying, $x = 2^{1/3}y^{2/3}$.

(b)



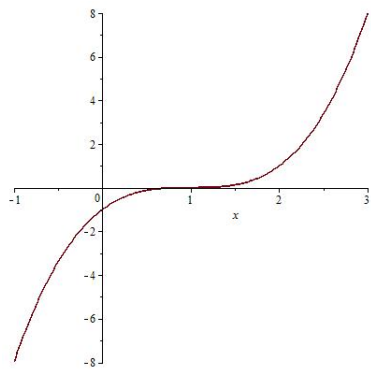
(c) Since $0 \leq t \leq 3$, we see that $0 \leq x \leq 18$ and $0 \leq y \leq 54$.

(d)



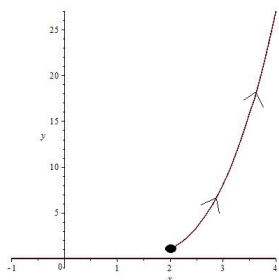
25. (a) Solving for t in the $y(t)$ equation: $t = y^{2/3}$. Plug this into the $x(t)$ equation: $x = (y^{2/3})^{1/2} + 1$ which is the same as $x = y^{1/3} + 1$, or, equivalently, $y = (x - 1)^3$.

(b)



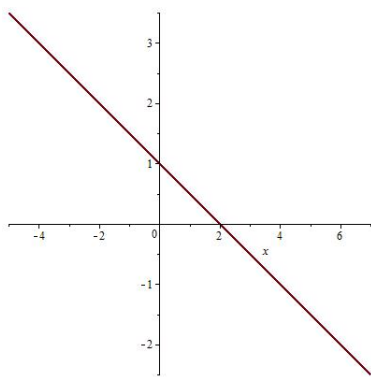
(c) Since $t \geq 1$, we see that $x \geq 2$ and $y \geq 1$.

(d)



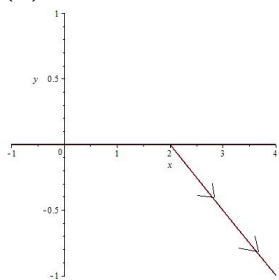
26. (a) Solving for e^t in the $x(t)$ equation: $e^t = \frac{x}{2}$. Plug this into the $y(t)$ equation: $y = 1 - \frac{x}{2}$.

(b)



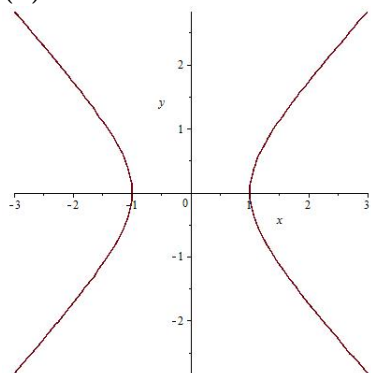
(c) For $t \geq 0$, it must be that $x \geq 2 \cdot e^0 = 2$ and $y \leq 1 - e^0 = 0$.

(d)



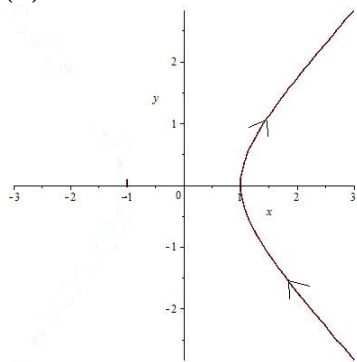
27. (a) Solving for t we have y equation, we have $t = \tan^{-1} y$, so that $x = \sec(\tan^{-1} y)$.

(b)



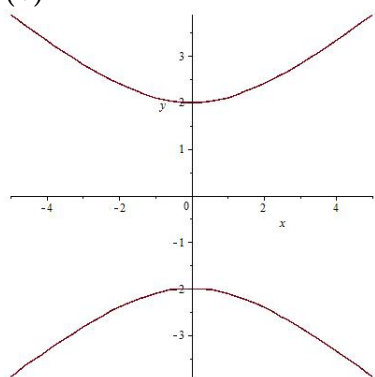
(c) Since $\sec t \geq 1$ for all $t \geq 1$, we have $x \geq 1$, so we only have the right-hand portion of the graph in part (b).

(d)



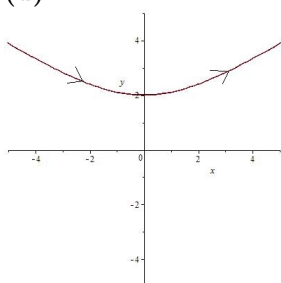
28. (a) Using the hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$, we see that $\left(\frac{y}{2}\right)^2 - \left(\frac{x}{3}\right)^2 = 1$.

(b)



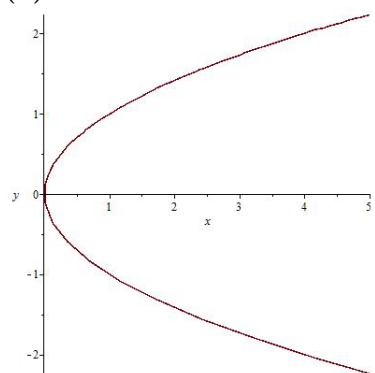
(c) Since $\cosh t > 1$ for all $-\infty < t < \infty$, the parametric graph only represents the top half of the figure in part (b), so $y \geq 2$.

(d)



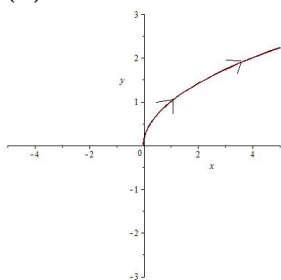
29. Since $x = t^4$ and $y^2 = (t^2)^2 = t^4$, we see that $x = y^2$.

(b)



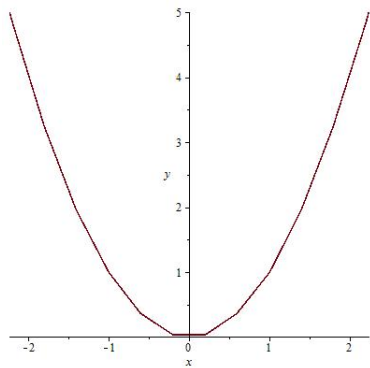
(c) There are no restrictions on t , so we assume $-\infty < t < \infty$. Since $y(t)$ is an even power of t , we have that $y \geq 0$ and also $x \geq 0$.

(d)



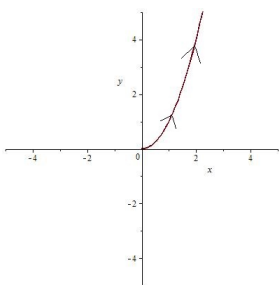
30. Since $x^2 = (t^2)^2 = t^4$ and $y = t^2$, we see that $y = x^2$.

(b)



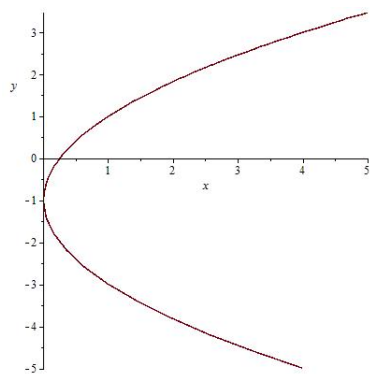
(c) There are no restrictions on t , so we assume $-\infty < t < \infty$. Since $x(t)$ and $y(t)$ are both even powers of t , we see that $x \geq 0$ and $y \geq 0$ for all values of t .

(d)



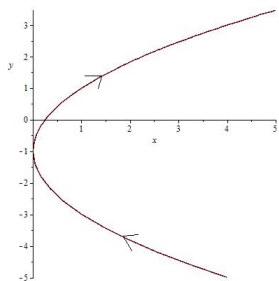
31. Solving for t in the $y(t)$ equation: $t = \frac{y+1}{2}$. Plug this into the $x(t)$ equation: $x = \left(\frac{y+1}{2}\right)^2$.

(b)



(c) Since $x \geq 0$, there are no restrictions on the plane curve.

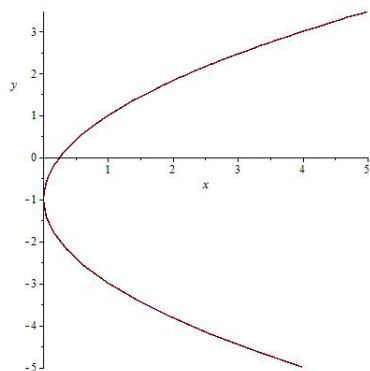
(d)



32. Notice that this parametrization is the same as in Problem 31 replacing t with e^t . This means the plane

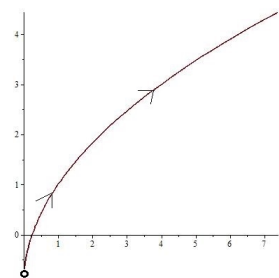
curve equation is the same: $x = \left(\frac{y+1}{2}\right)^2$.

(b)



(c) Since $x(t) = e^{2t}$, we see that $x(t) > 0$ for all t . Also, $y(t) > 0 - 1 = -1$ since $e^t > 0$ for all t .

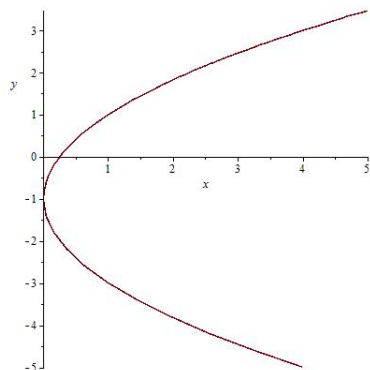
(d)



33. Notice that this parametrization is the same as in Problem 31 replacing t with $\frac{1}{t}$. This means the plane

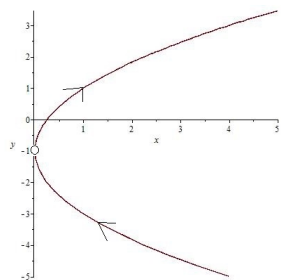
curve is the same: $x = \left(\frac{y+1}{2}\right)^2$.

(b)



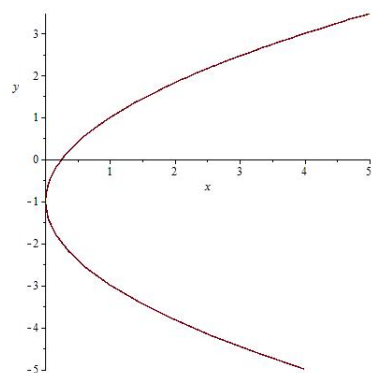
(c) Since $t \neq 0$, $x > 0$ and $y \neq -1$.

(d)



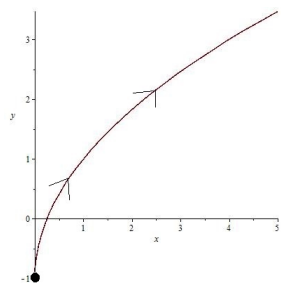
34. Notice that this parametrization is the same as in Problem 31 replacing t with t^2 . This means the plane curve is the same: $x = \left(\frac{y+1}{2}\right)^2$.

(b)



(c) There are no restrictions on t . Since $t^2 \geq 0$, we see that $y \geq -1$.

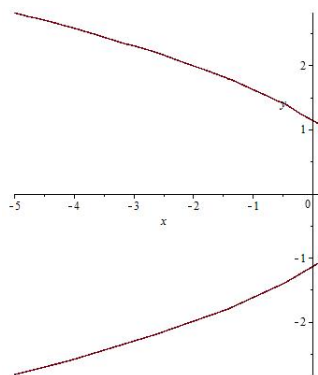
(d)



35. (a) Solving for $\sin^2 t$ in $x(t)$ we have $\sin^2 t = \frac{x+2}{3}$. Since $\cos^2 t = \left(\frac{y}{2}\right)^2$ we see that

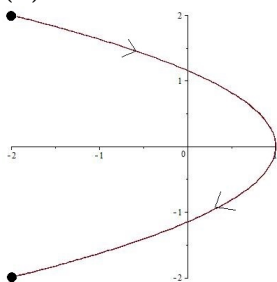
$$\frac{x+2}{3} + \frac{y^2}{4} = 1.$$

(b)



(c) For $0 \leq t \leq \pi$, we have $0 \leq \sin t \leq 1$ and $-1 \leq \cos t \leq 1$ so we see that $-2 \leq x \leq 1$ and $-2 \leq y \leq 2$.

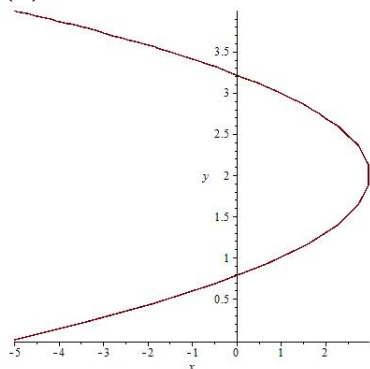
(d)



36. (a) Solving for $\sin^2 t$ in $x(t)$ we have $\sin^2 t = \frac{x-1}{2}$ and solving for $\cos t$ in $y(t)$ we have $\cos t = 2-y$. This means $\cos^2 t = (2-y)^2$ and we see that

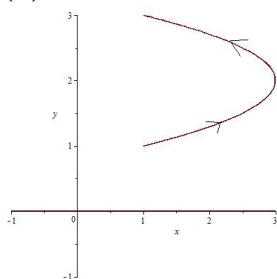
$$\frac{x-1}{2} + (2-y)^2 = 1.$$

(b)



(c) For $0 \leq t \leq 2\pi$ we have $0 \leq \sin^2 t \leq 1$ and $-1 \leq \cos t \leq 1$ so we see that $x \geq 1$ and $1 \leq y \leq 3$.

(d)



37. Solving for t^2 , we have $t^2 = 1/x$. Now plug this into y :

$$y = \frac{2}{1/x + 1} = \frac{2x}{1 + x}.$$

As t increases, x approaches 0 and so does y . Also $x \geq 0$ and $0 < y \leq 2$.

38. If we square both $x(t)$ and $y(t)$ and add them together, we have

$$x^2 + y^2 = \frac{9t^2}{t^2 + 1} + \frac{9}{t^2 + 1} = \frac{9(t^2 + 1)}{t^2 + 1} = 9$$

$$x^2 + y^2 = 9$$

which is a circle centered at the origin of radius 3. At $t = 0$, we are at the point $(0, 3)$. As t increases, the x values approach 3 and the y values approach 0. Since $y \geq 0$ for all t , the object is traveling along the quarter circle from $(0, 3)$ to $(3, 0)$ as t goes from 0 to ∞ . Also $-3 < x < 3$ and $0 < y < 3$.

39. Solving for t^2 in the $x(t)$ equation, we have

$$\begin{aligned} x^2 &= \frac{16}{4 - t^2} \\ 4 - t^2 &= \frac{16}{x^2} \\ t^2 &= 4 - \frac{16}{x^2}. \end{aligned}$$

Taking the square root to solve for t and plugging both t and t^2 into the $y(t)$ equation, we have

$$\begin{aligned} y &= \frac{4 \left(\sqrt{4 - \frac{16}{x^2}} \right)}{\sqrt{4 - \left(4 - \frac{16}{x^2} \right)}} \\ y &= \frac{4 \sqrt{4 - \frac{16}{x^2}}}{\sqrt{\frac{16}{x^2}}} \\ y &= \frac{4 \sqrt{4 - \frac{16}{x^2}}}{\frac{4}{x}} \\ y &= x \sqrt{4 - \frac{16}{x^2}} \\ y &= 2\sqrt{x^2 - 4}. \end{aligned}$$

As t approaches 2, the denominators of $x(t)$ and $y(t)$ are getting close to 0, so $x(t)$ and $y(t)$ are approaching ∞ . The points are going upward from the point $(2, 0)$ along the curve $y = 2\sqrt{x^2 - 4}$. Also $x \geq 2$ and $y \geq 0$.

40. Squaring $x(t)$ and solving for t , we have $t = x^2 + 3$. Plug this into $y(t)$:

$$y = \sqrt{x^2 + 4}.$$

As t increases, so do x and y along the curve $y = \sqrt{x^2 + 4}$ starting at the point $(0, 2)$. Also $x \geq 0$ and $y \geq 2$.

41. Solving for $\sin t$ and $\cos t$ in $x(t)$ and $y(t)$, respectively, and then squaring, we have

$$\begin{aligned}\sin t &= x + 2 \\ \cos t &= \frac{y - 4}{-2} \\ \boxed{(x + 2)^2 + \frac{(y - 4)^2}{4} = 1},\end{aligned}$$

which is an ellipse centered at $(-2, 4)$. At $t = 0$, the object is at the point $(-2, 2)$; at $t = \frac{\pi}{2}$, the object is at the point $(-1, 4)$. This means the object is tracing this ellipse in a counterclockwise manner. Also $\boxed{-3 \leq x \leq -1}$ and $\boxed{2 \leq y \leq 6}$.

42. Solving for $\tan t$ and $\sec t$ in $x(t)$ and $y(t)$, respectively, and then squaring, we have

$$\begin{aligned}\tan t &= x - 2 \\ \sec t &= \frac{y - 3}{-2} \\ 1 + (x - 2)^2 &= \frac{(y - 3)^2}{9} \\ \boxed{\frac{(y - 3)^2}{9} - (x - 2)^2 = 1},\end{aligned}$$

where we used the Pythagorean Identity $1 + \tan^2 t = \sec^2 t$. Since $\sec t \geq 1$ for all t , we have $y = 3 - 2 \sec t \leq 3 - 2 = 1$, so $y \leq 1$. An object moving along the parametrized curve follows the bottom half of the hyperbola $\frac{(y - 3)^2}{9} - (x - 2)^2 = 1$ moving from left to right. Also $\boxed{-\infty < x < \infty}$ and $\boxed{y \leq 1}$.

43. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \ y(t) = 4t - 2, \ -\infty < t < \infty}$$

$$\boxed{x(t) = t^3, \ y(t) = 4t^3 - 2, \ -\infty < t < \infty}$$

44. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \ y(t) = -8t + 3, \ -\infty < t < \infty}$$

$$\boxed{x(t) = t^3, \ y(t) = -8t^3 + 3, \ -\infty < t < \infty}$$

45. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \ y(t) = -2t^2 + 1, \ -\infty < t < \infty}$$

$$\boxed{x(t) = t + 1, \ y(t) = -2(t + 1)^2 + 1, \ -\infty < t < \infty}$$

46. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \ y(t) = t^2 + 1, \ -\infty < t < \infty}$$

$$\boxed{x(t) = -t, \ y(t) = t^2 + 1, \ -\infty < t < \infty}$$

47. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \ y(t) = 4t^3, \ -\infty < t < \infty}$$

$$x(t) = t^3, y(t) = 4t, -\infty < t < \infty$$

48. Answers will vary. Here are two parameterizations.

$$x(t) = t, y(t) = 2t^2, -\infty < t < \infty$$

$$x(t) = -t, y(t) = 2t^2, -\infty < t < \infty$$

49. Answers will vary. Here are two parameterizations.

$$y(t) = t, x(t) = \frac{1}{3}\sqrt{t} - 3, t \geq 0$$

$$y(t) = t^3, x(t) = \frac{1}{3}t^{1/6} - 3, t \geq 0$$

50. Answers will vary. Here are two parameterizations.

$$y(t) = t, x(t) = t^{3/2}, t \geq 0$$

$$x(t) = t, y(t) = t^{2/3}, t \geq 0$$

51. Answers will vary. One such answer would be the following: The line through the given segment is $y = x - 2$ so one pair of parametric equations would be $x(t) = t, y(t) = t - 2; 2 \leq t \leq 7$.

52. Answers will vary. One such answer would be the following: The line through the given segment is $y = -x + 1$ so one pair of parametric equations would be $x(t) = t, y(t) = -t + 1; -1 \leq t \leq 3$.

53. Answers will vary. One such answer would be the following: The ellipse can be parametrized by $x(t) = 3 \cos t, y(t) = 2 \sin t$. To start at the left end of the ellipse, we can begin with $t = \pi$. At $t = \frac{3\pi}{2}$, we are at the bottom of the ellipse (which agrees with the counterclockwise orientation), so an appropriate range on t would be $\pi \leq t \leq \frac{5\pi}{2}$.

54. Answers will vary. One such answer would be the following: The ellipse can be parametrized by $x(t) = \cos t, y(t) = 4 \sin t$. To start at the bottom of the ellipse, we can begin with $t = -\frac{\pi}{2}$. At $t = 0$, we are at the right end of the ellipse (which agrees with the counterclockwise orientation), so an appropriate range on t would be $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

For problems 55-58, the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ can be parametrized by either $x(t) = 3 \cos(\omega t), y(t) = 2 \sin(\omega t)$ (counterclockwise orientation) or $x(t) = 3 \sin(\omega t), y(t) = 2 \cos(\omega t)$ (clockwise orientation) for some real number $\omega > 0$ and for $0 \leq t \leq \frac{2\pi}{\omega}$.

55. For counterclockwise orientation, we choose $x(t) = 3 \cos(\omega t), y(t) = 2 \sin(\omega t)$. If 1 revolution takes 3 seconds, the period is $\frac{2\pi}{\omega} = 3$, so $\omega = \frac{2\pi}{3}$. The parametrization is

$$x(t) = 3 \cos\left(\frac{2\pi}{3}t\right), y(t) = 2 \sin\left(\frac{2\pi}{3}t\right).$$

To begin at $(3, 0)$ the interval on t would be $0 \leq t \leq 3$.

56. For clockwise orientation, we choose $x(t) = 3 \sin(\omega t), y(t) = 2 \cos(\omega t)$. If 1 revolution takes 3 seconds, the period is $\frac{2\pi}{\omega} = 3$, so $\omega = \frac{2\pi}{3}$. The parametrization is

$$x(t) = 3 \sin\left(\frac{2\pi}{3}t\right), y(t) = 2 \cos\left(\frac{2\pi}{3}t\right).$$

To begin at $(3, 0)$, we need to find t such that $3 \sin\left(\frac{2\pi}{3}t\right) = 3$, or $\frac{2\pi}{3}t = \frac{\pi}{2}$. The value $t = \frac{3}{4}$ solves this equation. To complete one revolution, we need the interval on t to be $\frac{3}{4} \leq t \leq 3 + \frac{3}{4}$. To start our parametrization at $t = 0$ we introduce the time-shift $t \mapsto t + \frac{3}{4}$. The parametrization is then

$$x(t) = 3 \sin\left(\frac{2\pi}{3}\left(t + \frac{3}{4}\right)\right), \quad y(t) = 2 \cos\left(\frac{2\pi}{3}\left(t + \frac{3}{4}\right)\right), \quad 0 \leq t \leq 3.$$

57. For clockwise orientation, we choose $x(t) = 3 \sin(\omega t)$, $y(t) = 2 \cos(\omega t)$. If 1 revolution takes 2 seconds, the period is $\frac{2\pi}{\omega} = 2$, so $\omega = \pi$. The parametrization is

$$x(t) = 3 \sin(\pi t), \quad y(t) = 2 \cos(\pi t).$$

To begin at $(0, 2)$ the interval on t would be $0 \leq t \leq 2$.

58. For counterclockwise orientation, we choose $x(t) = 3 \cos(\omega t)$, $y(t) = 2 \sin(\omega t)$. If 1 revolution takes 1 second, the period is $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. The parametrization is

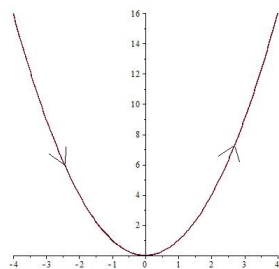
$$x(t) = 3 \cos(2\pi t), \quad y(t) = 2 \sin(2\pi t).$$

To begin at $(0, 2)$, we need to find t such that $2 \sin(2\pi t) = 2$, or $2\pi t = \frac{\pi}{2}$. The value $t = \frac{1}{4}$ solves this equation. To complete one revolution, we need the interval on t to be $\frac{1}{4} \leq t \leq 1 + \frac{1}{4}$. To start the parametrization at $t = 0$, we introduce the time-shift $t \mapsto t + \frac{1}{4}$. The parametrization is then

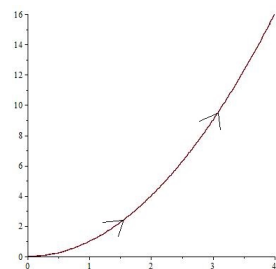
$$x(t) = 3 \cos\left(2\pi\left(t + \frac{1}{4}\right)\right), \quad y(t) = 2 \sin\left(2\pi\left(t + \frac{1}{4}\right)\right), \quad 0 \leq t \leq 1.$$

59. Notice that the parametrization in (a) is the plane curve $y = x^2$ for $-4 \leq x \leq 4$. Each of the parameterizations in (b), (c), and (d) are some variation of the parametrization in (a). Part (b) is the same plane curve as (a) if we replace t with $\sqrt{t}$, but now we can only consider non-negative x -values since in (b) $x(t) = \sqrt{t} \geq 0$; there are no restrictions on x in (a). Part (c) is the same plane curve as (a) if we replace t with e^t , but again there are included restrictions since in (b) $x > 0$. Part (d) is the same plane curve as (a) if we replace t with $\cos t$ (with no included restrictions) except that the orientation is opposite of that in (a) and $-1 \leq x \leq 1$ whereas in (a) $-4 \leq x \leq 4$.

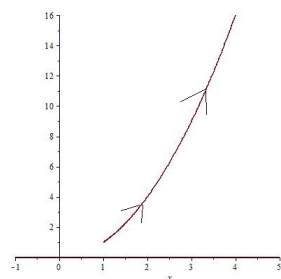
(a)



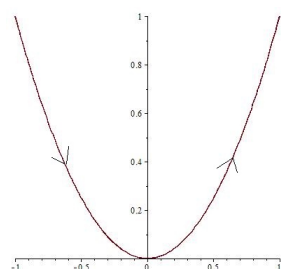
(b)



(c)

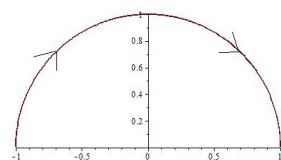


(d)

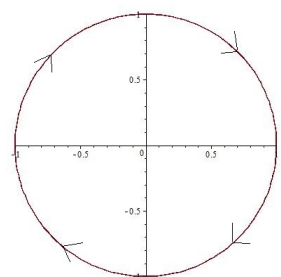


60. For each of the four parameterizations, $[x(t)]^2 + [y(t)]^2 = 1$ so each parametrization's plane curve is the unit circle. Part (a) is the top half of the unit circle oriented clockwise. Part (b) is the full unit circle oriented clockwise. Part (c) is the full circle oriented counterclockwise. Part (d) is the right half of the unit circle oriented counterclockwise.

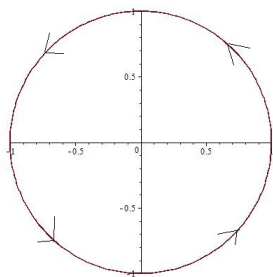
(a)



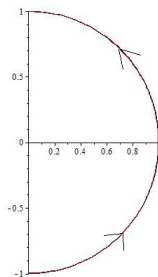
(b)



(c)



(d)

**Applications and Extensions**

61. I $\rightarrow$ (d) counterclockwise, II $\rightarrow$ (a) counterclockwise, III $\rightarrow$ (b) counterclockwise, IV $\rightarrow$ (c) counterclockwise

62. I $\rightarrow$ (c) downward, II $\rightarrow$ (a) downward, III $\rightarrow$ (b) downward, IV $\rightarrow$ (d) downward

63. I $\rightarrow$ (c) from $(1, 0)$ to $(-1, 0)$, II $\rightarrow$ (b) from $(-1, 0)$ to $(1, 0)$, III $\rightarrow$ (a) clockwise,

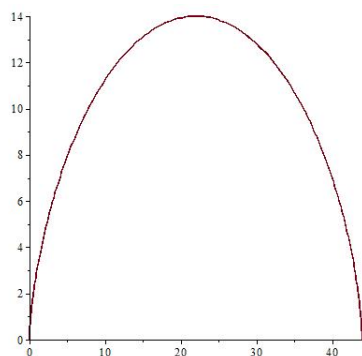
IV $\rightarrow$ (d) from $\left(-\frac{\sqrt{2}}{2}, 1\right)$ to $(1, 0)$

64. I $\rightarrow$ (b) from $(0, -1)$ to $\left(-\frac{\sqrt{3}}{2}, 1\right)$ to $\left(-\frac{\sqrt{3}}{2}, -1\right)$ to $(0, 1)$,

II $\rightarrow$ (c) from $(0, 1)$ to $(1, 0)$, III $\rightarrow$ (a) from $(0, 1)$ to $\left(\frac{1}{2}, 0\right)$ to $\left(\frac{\sqrt{3}}{2}, -1\right)$ to $(0, 1)$ to $\left(\frac{\sqrt{3}}{2}, 1\right)$ to $\left(\frac{1}{2}, 0\right)$,

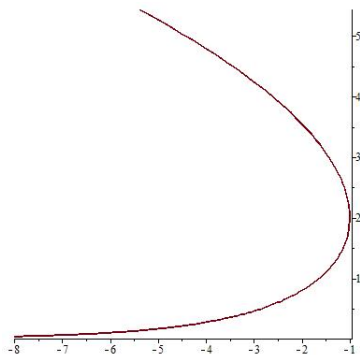
IV $\rightarrow$ (d) counterclockwise

65. (a)



(b) $(x(2.1), y(2.1)) \approx (8.66, 10.53)$

66. (a)



(b) $(x(1.5), y(1.5)) \approx (-2.98, 4.23)$

67. Answers will vary.

68. Answers will vary.

69. (a) Using the given formulas $x(t) = (125 \cos 40^\circ)t \approx 95.8t$ and $y(t) = -\frac{1}{2} \cdot 32t^2 + (125 \sin 40^\circ)t + 3 \approx -16t^2 + 80.3t + 3$

(b) The height of the ball after 2 seconds is $y(2) = -\frac{1}{2} \cdot 32(2)^2 + (125 \sin 40^\circ) \cdot 2 + 3 \approx 99.6$ feet.

(c) The horizontal distance the ball has traveled after 2 seconds is

$$x(2) - x(0) = (125 \cos 40^\circ) \cdot 2 - 0 \approx 191.6 \text{ feet}.$$

(d) To find how long it takes, solve $x(t) = 300$ for t .

$$(125 \cos 40^\circ)t = 300$$

$$t = \frac{300}{125 \cos 40^\circ}$$

$$t \approx 3.13 \text{ seconds}.$$

(e) The height at $t \approx 3.13$ seconds is $y(3.1) \approx 97.6$ feet.

(f) To find the time when the ball hits the ground, we need to solve $y(t) = 0$ (i.e. the height of the is 0).

$$-\frac{1}{2} \cdot 32t^2 + (125 \sin 40^\circ)t + 3 = 0.$$

Using the quadratic formula, we find the two solutions $t \approx -0.04$ seconds and $t \approx 5.1$ seconds. Since $t \geq 0$, it must be ≈ 5.1 seconds that the ball is in the air.

(g) The ball has traveled $x(5.1) \approx 488.6$ feet horizontally before it hits the ground.

70. (a) Using the given formulas $x(t) = (145 \cos 20^\circ)t$ and $y(t) = -\frac{1}{2} \cdot 32t^2 + (145 \sin 20^\circ)t + 5$.

(b) The height of the ball after $\frac{1}{2}$ second is $y\left(\frac{1}{2}\right) = -\frac{1}{2} \cdot 32 \cdot \left(\frac{1}{2}\right)^2 + (145 \sin 20^\circ) \cdot \frac{1}{2} + 5 \approx 25.8$ feet.

(c) The horizontal distance the ball has traveled after $\frac{1}{2}$ second is

$$x\left(\frac{1}{2}\right) - x(0) = (145 \cos 20^\circ) \cdot \frac{1}{2} - 0 \approx 68.1 \text{ feet}.$$

(d) To find how long it takes, solve $x(t) = 60$ for t .

$$(145 \cos 20^\circ)t = 60$$

$$t = \frac{60}{145 \cos 20^\circ}$$

$$t \approx 0.4 \text{ seconds}.$$

(e) The height of the ball at $t \approx 0.4$ seconds is $y(0.4) \approx 23.7$ feet.

71. (a) Using the given formulas $x(t) = (80 \cos 35^\circ)t \approx 65.5t$ and $y(t) = -\frac{1}{2} \cdot 32t^2 + (80 \sin 35^\circ)t + 6 \approx -16t^2 + 45.9t + 6$.

(b) The height of the football after 1 second is $y(1) = -\frac{1}{2} \cdot 32 + (80 \sin 35^\circ) + 6 \approx 35.9$ feet.

(c) The horizontal distance the ball has traveled after 1 second is $x(1) - x(0) = (80 \cos 35^\circ) - 0 \approx 65.5$ feet.

(d) To find how long it takes, solve $x(t) = 120$ for t .

$$(80 \cos 35^\circ)t = 120$$

$$t = \frac{120}{80 \cos 35^\circ}$$

$$t \approx 1.8 \text{ seconds}.$$

(e) The height of the ball at $t \approx 1.8$ seconds is $y(1.8) \approx 36.8$ feet.

72. Both parameterizations yield the plane curve $y = x^2$ with orientation left to right as t increases, but they do not trace out the curve at the same rate. For example, when time $t = 0$, both parameterizations are at the origin $(0, 0)$. However, if we consider time $t = 10$, the parametrization (t, t^2) has only traced the curve $y = x^2$ from the origin to the point $(10, 100)$ while the parametrization (t^2, t^4) has traced the curve $y = x^2$ from the origin to the point $(100, 10000)$ in the same amount of time.

73. The parametrization $(2 \cos \theta, 2 \sin \theta)$, $0 \leq \theta \leq 2\pi$, traces a circle of radius 2 centered at the origin starting at $(2, 0)$ in the counterclockwise direction while the parametrization $(2 \sin \theta, 2 \cos \theta)$, $0 \leq \theta \leq 2\pi$, traces a circle of radius 2 centered at the origin starting at $(0, 2)$ in the clockwise direction.

74. Let $t = 0$ denote the time (in seconds) when the train leaves the station, $s_T(t)$ the position function (in miles) of the train, and $s_M(t)$ the position function (in miles) of Mary. Since t is in seconds, we convert the train's acceleration to mi/sec² and Mary's running speed to mi/sec. Doing such we find that the train accelerates at $3 \cdot \left(\frac{1}{3600}\right)^2 = \frac{1}{4320000}$ mi/sec² and Mary runs at $6 \cdot \left(\frac{1}{3600}\right) = \frac{1}{600}$ mi/sec. Using the hint, we know that

$$s_T(t) = \frac{1}{2} \cdot \frac{1}{4320000} \cdot t^2 = \frac{1}{8640000} t^2 \text{ miles}.$$

Since Mary does not arrive at the station until $t = 2$, we can assume that $s_M(t) = 0$ for $t < 2$. For $t \geq 2$, Mary runs at a constant 6 mi/h, so $s_M(2) = 0$, $s_M(3) = \frac{1}{600}$ miles, $s_M(4) = \frac{2}{600}$ miles, $s_M(5) = \frac{3}{600}$ miles, etc. Since Mary's speed is constant, her position increases linearly which means that $s_M(t) = \frac{1}{600}(t - 2)$ miles for $t \geq 2$.

Challenge Problems 75. The line through $(-R, 0)$ with slope m has equation $y = m(x + R)$. We want to find the point $P = (x, y)$ on the line $y = m(x + R)$ that intersects the circle $x^2 + y^2 = R^2$. By plugging $y = m(x + R) = mx + mR$ into the circle equation to solve for x , we have

$$\begin{aligned} x^2 + (mx + mR)^2 &= R^2 \\ x^2 + m^2x^2 + 2m^2Rx + m^2R^2 &= R^2 \\ (m^2 + 1)x^2 + (2m^2R)x + (m^2 - 1)R^2 &= 0. \end{aligned}$$

Using the quadratic equation to solve for x , the solutions are

$$\begin{aligned}
 x &= \frac{-(2m^2R) \pm \sqrt{(2m^2R)^2 - 4(m^2+1)(m^2-1)R^2}}{2(m^2+1)} \\
 &= \frac{-2m^2R \pm \sqrt{4m^4R^2 - 4(m^4-1)R^2}}{2(m^2+1)} \\
 &= \frac{-2m^2R \pm \sqrt{4R^2}}{2(m^2+1)} \\
 &= \frac{-2m^2R \pm 2R}{2(m^2+1)} \\
 &= \frac{-m^2 \pm 1}{m^2+1} \cdot R.
 \end{aligned}$$

Writing the two solutions separately, they are

$$x = \frac{-m^2+1}{m^2+1} \cdot R, \quad x = \frac{-m^2-1}{m^2+1} \cdot R = -R.$$

The second solution $x = -R$ is the original point we wrote the equation of the line through, so the new point $P = (x, y)$ has as its x -coordinate

$$x = \frac{-m^2+1}{m^2+1} \cdot R.$$

To find the y -coordinate, we know that P lies on the line $y = m(x + R)$. Plugging in x -coordinate, we have

$$y = m \left(\frac{-m^2+1}{m^2+1} \cdot R + R \right) = \frac{2mR}{m^2+1}.$$

This means we can parametrize a circle using the slope m through the point $(-R, 0)$ using the equations

$$\boxed{x(m) = \frac{R(1-m^2)}{1+m^2} \quad y(m) = \frac{2Rm}{1+m^2}}$$

for $-\infty < m < \infty$.

76. The line through $(1, 1)$ with slope m has equation $y - 1 = m(x - 1)$ or $y = mx - m + 1$. We want to find the point $P = (x, y)$ on the line $y = mx - m + 1$ that intersects the parabola $y = x^2$. By plugging $y = mx - m + 1$ into the equation $y = x^2$ to solve for x , we have

$$\begin{aligned}
 mx - m + 1 &= x^2 \\
 x^2 - mx + (m - 1) &= 0.
 \end{aligned}$$

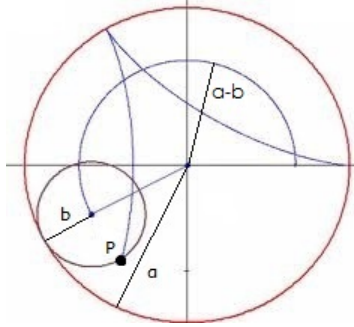
Factoring to solve for x (or using the quadratic formula), we have

$$(x - 1)(x - m + 1) = 0,$$

which means $x = 1$ or $x = m - 1$. The solution $x = 1$ corresponds to the initial point $(1, 1)$ that we wrote the equation of the line through. The other point of intersection is at $P = (x, y)$, where $x = m - 1$. This then gives $y = x^2 = (m - 1)^2$ so the parametrization of $y = x^2$ using the slope m through the point $(1, 1)$ is

$$\boxed{x(m) = m - 1, \quad y(m) = (m - 1)^2, \quad -\infty < m < \infty}.$$

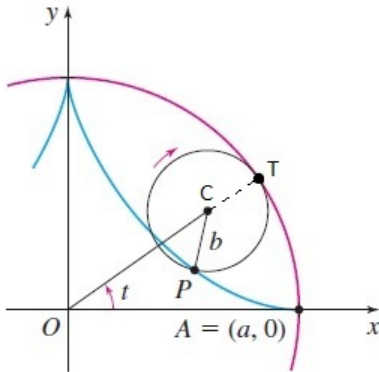
77. By referring to the picture, the center of the smaller circle traces out a circle centered at the origin of radius $a - b$ in a counterclockwise manner.



The center of the smaller circle $(x_c(t), y_c(t))$ is parametrized by

$$\begin{aligned}x_c(t) &= (a - b) \cos t \\y_c(t) &= (a - b) \sin t,\end{aligned}$$

for $0 \leq t \leq 2\pi$. Now refer to the following picture.



Note that arc AT is equal to arc TP since the inner circle is rolling along the outer circle. Recall that in a circle of radius R , the measure of an arc with central angle θ is $(2\pi R) \cdot \frac{\theta}{2\pi} = R \cdot \theta$. With this in hand, we know that arc AT is $a \cdot t$. This means that arc TP is also $a \cdot t$ so the central angle PCT has measure satisfying the equation

$$b \cdot m\angle PCT = a \cdot t.$$

This means

$$m\angle PCT = \frac{a}{b}t.$$

Now picture the inner circle rotated so that CT is parallel to the x -axis. [Keep in mind that this amounts to a clockwise rotation of t radians.] Then since the center of the inner circle is located at (x_c, y_c) with radius b and rotates clockwise, this circle could be parametrized by $(x_c + b \cos t, y_c - b \sin t)$. The point P would be located on the circle after we moved through an angle with measure $m\angle PCT = \frac{a}{b}t$, so it would be located at

$$\left(x_c + b \cos\left(\frac{a}{b}t\right), y_c - b \sin\left(\frac{a}{b}t\right)\right).$$

But to find where P is *actually* located (remember we rotated the inner circle t radians clockwise), we need to rotate the inner circle back t radians (i.e. subtract t radians inside the sine and cosine). The parametrization of the point P , as it travels around the hypocycloid, is given by

$$\begin{aligned}x(t) &= x_c + b \cos\left(\frac{a}{b}t - t\right) = (a - b) \cos t + b \cos\left(\frac{a - b}{b}t\right) \\y(t) &= y_c - b \sin\left(\frac{a}{b}t - t\right) = (a - b) \sin t - b \sin\left(\frac{a - b}{b}t\right),\end{aligned}$$

for $0 \leq t \leq 2\pi$.

78. Using the parametric equations developed in Problem 77 with $a = 4b$, we find that

$$\begin{aligned}x(t) &= 3b \cos t + b \cos(3t) \\y(t) &= 3b \sin t - b \sin(3t).\end{aligned}$$

We will need the following *triple* angle formulas for cosine and sine (which can be proved using angle sum formulas for cosine and sine):

$$\begin{aligned}\cos(3t) &= 4 \cos^3 t - 3 \cos t \\ \sin(3t) &= -4 \sin^3 t + 3 \sin t.\end{aligned}$$

Plugging these formulas into $x(t)$ and $y(t)$, we have the parametric equations

$$\begin{aligned}x(t) &= 3b \cos t + b(4 \cos^3 t - 3 \cos t) = 4b \cos^3 t \\y(t) &= 3b \sin t - b(-4 \sin^3 t + 3 \sin t) = 4b \sin^3 t.\end{aligned}$$

If we raise both $x(t)$ and $y(t)$ to the $\frac{2}{3}$ power, we have

$$\begin{aligned}x(t)^{2/3} &= (4b \cos^3 t)^{2/3} = (4b)^{2/3} \cos^2 t \\y(t)^{2/3} &= (4b \sin^3 t)^{2/3} = (4b)^{2/3} \sin^2 t.\end{aligned}$$

If we add these together, then we can factor out the $(4b)^{2/3}$ and we have

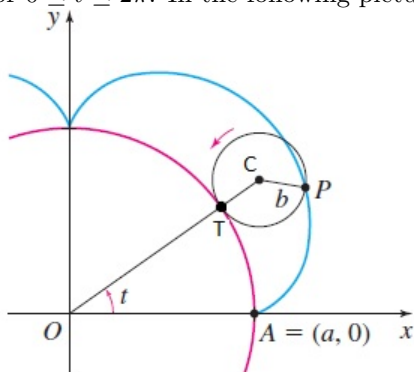
$$\begin{aligned}x(t)^{2/3} + y(t)^{2/3} &= (4b)^{2/3}(\cos^2 t + \sin^2 t) \\&= (4b)^{2/3} \\&= a^{2/3},\end{aligned}$$

since $a = 4b$.

79. It will be helpful to look through the solution to Problem 77 before reading this solution. The smaller circle's center traces out a circle centered at $(0,0)$ with radius $a + b$ in a counterclockwise manner. The center of the smaller circle $(x_c(t), y_c(t))$, oriented counterclockwise, is parametrized by

$$\begin{aligned}x_c(t) &= (a + b) \cos t \\y_c(t) &= (a + b) \sin t,\end{aligned}$$

for $0 \leq t \leq 2\pi$. In the following picture,



we have that arc AT is equal to arc TP . Reasoning as in Problem 77, we find that $m\angle PCT = \frac{a}{b}t$. If we rotate the smaller circle about its center so that TC is parallel to the x -axis with T to the right of C (which is a clockwise rotation of $\pi + t$ radians), then the smaller circle (also oriented counterclockwise) can be parametrized by $(x_c + b \cos t, y_c + b \sin t)$. The point P would be located on this circle after we moved through an angle with measure $m\angle PCT = \frac{a}{b}t$, so it would be located at

$$\left(x_c + b \cos\left(\frac{a}{b}t\right), y_c + b \sin\left(\frac{a}{b}t\right)\right).$$

If we undo the original clockwise rotation of $\pi + t$ radians (this would amount to adding $\pi + t$ radians inside the sine and cosine), we find that the parametrization of the point P on the epicycloid is given by

$$\begin{aligned}x(t) &= x_c + b \cos\left(\frac{a}{b}t + (\pi + t)\right) = (a + b) \cos t - b \cos\left(\frac{a + b}{b}t\right) \\y(t) &= y_c + b \sin\left(\frac{a}{b}t + (\pi + t)\right) = (a + b) \sin t - b \sin\left(\frac{a + b}{b}t\right),\end{aligned}$$

for $0 \leq t \leq 2\pi$ where we used that $\cos(\theta + \pi) = -\cos \theta$ and $\sin(\theta + \pi) = -\sin \theta$ for any angle θ .

9.2 Tangent Lines; Arc Length

Concepts and Vocabulary

1. Let C denote a curve represented by the parametric equations $x = x(t), y = y(t), a \leq t \leq b$, where each function $x(t)$ and $y(t)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

If both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous and never simultaneously 0 on (a, b) , then C is called a (a) smooth curve.

2. True.

3. If in the formula for the slope of a tangent line, $\frac{dy}{dt} = 0$ (but $\frac{dx}{dt} \neq 0$), then the curve has a horizontal

tangent line at the point $(x(t), y(t))$. If $\frac{dx}{dt} = 0$ (but $\frac{dy}{dt} \neq 0$), then the curve has a vertical tangent line at the point $(x(t), y(t))$.

4. False. The formula should be

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

which means under the square root we should square the first derivatives of x and y , not take the second derivative of x and y .

Skill Building

For problems 5-12, we will use the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

5. $\frac{dx}{dt} = e^t \cos t - e^t \sin t$ and $\frac{dy}{dt} = e^t \sin t + e^t \cos t$, so

$$\frac{dy}{dx} = \frac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t} = \frac{\sin t + \cos t}{\cos t - \sin t}.$$

6. $\frac{dx}{dt} = -e^{-t}$ and $\frac{dy}{dt} = 3e^{3t}$, so

$$\frac{dy}{dx} = \frac{3e^{3t}}{-e^{-t}} = -3e^{4t}.$$

7. $\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1$, so

$$\frac{dy}{dx} = \frac{1}{1 - \frac{1}{t^2}}.$$

8. $\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1 + \frac{1}{t^2}$, so

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}.$$

9. $\frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$ and $\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$, so

$$\frac{dy}{dx} = \frac{t \sin t}{t \cos t} = \tan t.$$

10. $\frac{dx}{dt} = -3 \cos^2 t \sin t$ and $\frac{dy}{dt} = 3 \sin^2 t \cos t$, so

$$\frac{dy}{dx} = \frac{3 \sin^2 t \cos t}{-3 \cos^2 t \sin t} = -\frac{\sin t}{\cos t} = -\tan t.$$

11. $\frac{dx}{dt} = -2 \cot t \csc^2 t$ and $\frac{dy}{dt} = -\csc^2 t$, so

$$\frac{dy}{dx} = \frac{-\csc^2 t}{-2 \cot t \csc^2 t} = \boxed{\frac{1}{2 \cot t}}.$$

12. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2 \sec t \cdot \sec t \tan t$, so

$$\frac{dy}{dx} = \frac{2 \sec^2 t \tan t}{\cos t} = \boxed{\frac{2 \tan t}{\cos^3 t}}.$$

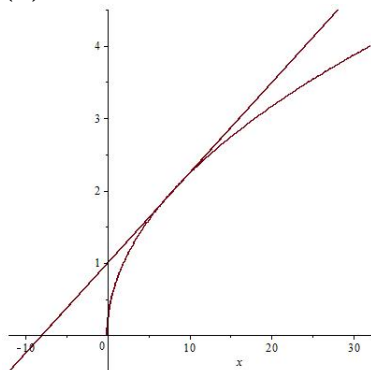
In Problems 13-26, we use the fact that the slope of the tangent at a value t is found by evaluating the derivative $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at t .

13. (a) The slope of the tangent line is $\left[\frac{1}{4t}\right]_{t=2} = \frac{1}{8}$. The equation of the tangent line at $(x(2), y(2)) = (8, 2)$ is

$$y - 2 = \frac{1}{8}(x - 8)$$

$$\boxed{y = \frac{1}{8}x + 1}.$$

(b)

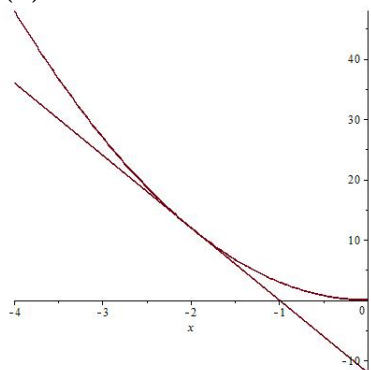


14. (a) The slope of the tangent line is $\left[\frac{6t}{1}\right]_{t=-2} = -12$. The equation of the tangent line at $(x(-2), y(-2)) = (-2, 12)$ is

$$y - 12 = -12(x + 2)$$

$$\boxed{y = -12x - 12}.$$

(b)

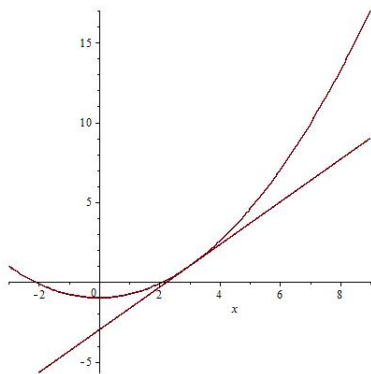


15. (a) The slope of the tangent line is $\left[\frac{4t}{3}\right]_{t=1} = \frac{4}{3}$. The equation of the tangent line at $(x(1), y(1)) = (3, 1)$ is

$$y - 1 = \frac{4}{3}(x - 3)$$

$$\boxed{y = \frac{4}{3}x - 3}.$$

(b)

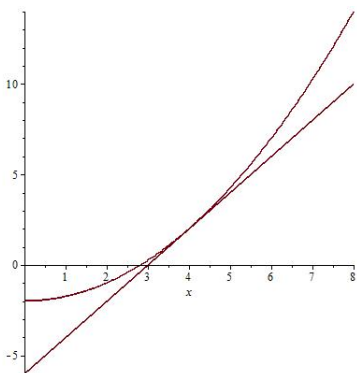


16. (a) The slope of the tangent line is $\left[\frac{2t}{2}\right]_{t=2} = 2$. The equation of the tangent line at $(x(2), y(2)) = (4, 2)$ is

$$y - 2 = 2(x - 4)$$

$$\boxed{y = 2x - 6}.$$

(b)

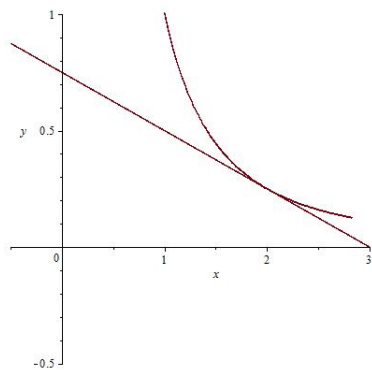


17. (a) The slope of the tangent line is $\left[\frac{-t^{-2}}{\frac{1}{2}t^{-1/2}}\right]_{t=4} = \left[\frac{-2}{t^{3/2}}\right]_{t=4} = -\frac{1}{4}$. The equation of the tangent line at $(x(4), y(4)) = (2, 1/4)$ is

$$y - \frac{1}{4} = -\frac{1}{4}(x - 2)$$

$$\boxed{y = -\frac{1}{4}x + \frac{3}{4}}.$$

(b)

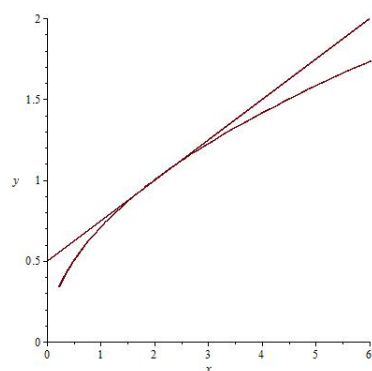


18. (a) The slope of the tangent line is $\left[\frac{-t^{-2}}{-4t^{-3}} \right]_{t=1} = \frac{-1}{-4} = \frac{1}{4}$. The equation of the tangent line at $(x(1), y(1)) = (2, 1)$ is

$$y - 1 = \frac{1}{4}(x - 2)$$

$$\boxed{y = \frac{1}{4}x + \frac{1}{2}}.$$

(b)



19. (a) The slope of the tangent line is $\left[\frac{-4(t+2)^{-2}}{2(t+2)^{-2}} \right]_{t=0} = -2$. The equation of the tangent line at $(x(0), y(0)) = (0, 2)$ is

$$y - 2 = -2(x - 0)$$

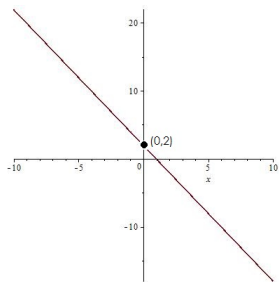
$$\boxed{y = -2x + 2}.$$

If we add $x(t)$ and $y(t)$ together, then

$$\begin{aligned} x(t) + y(t) &= \frac{t}{t+2} + \frac{4}{t+2} = \frac{t+4}{t+2} = \frac{t+2+2}{t+2} \\ &= \frac{t+2}{t+2} + \frac{2}{t+2} = 1 + \frac{1}{2}y(t). \end{aligned}$$

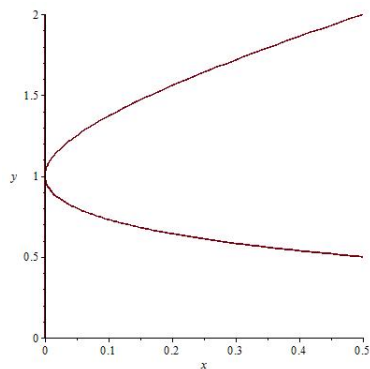
So $x + y = 1 + \frac{1}{2}y$, or $y = -2x + 2$, is the equation of the plane curve. This means that the tangent line is the plane curve itself.

(b)



20. (a) The slope of the tangent line is $\left[\frac{-(1+t)^{-2}}{\frac{2t+t^2}{(1+t)^2}} \right]_{t=0} = \left[-\frac{1}{2t+t^2} \right]_{t=0}$ which is undefined. This means the tangent line is vertical. The equation of the tangent line at $(x(0), y(0)) = (0, 1)$ is $\boxed{x = 0}$.

(b)

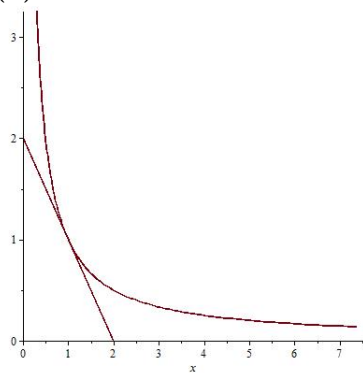


21. (a) The slope of the tangent line is $\left[\frac{-e^{-t}}{e^t} \right]_{t=0} = -1$. The equation of the tangent line at $(x(0), y(0)) = (1, 1)$ is

$$y - 1 = -(x - 1)$$

$$\boxed{y = -x + 2}.$$

(b)

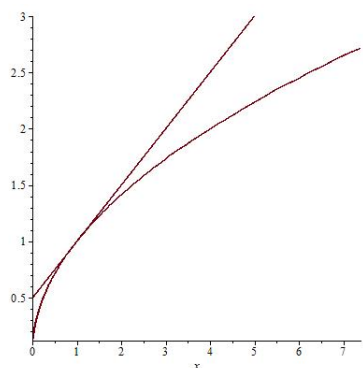


22. (a) The slope of the tangent line is $\left[\frac{e^t}{2e^{2t}} \right]_{t=0} = \frac{1}{2}$. The equation of the tangent line at $(x(0), y(0)) = (1, 1)$ is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}.$$

(b)

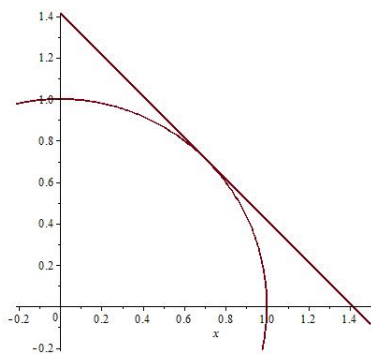


23. (a) The slope of the tangent line is $\left[\frac{-\sin t}{\cos t} \right]_{t=\pi/4} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$. The equation of the tangent line at $(x(\pi/4), y(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$ is

$$y - \frac{\sqrt{2}}{2} = - \left(x - \frac{\sqrt{2}}{2} \right)$$

$$\boxed{y = -x + \sqrt{2}}.$$

(b)

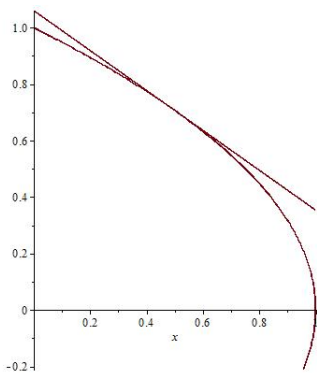


24. (a) The slope of the tangent line is $\left[\frac{-\sin t}{2 \sin t \cos t} \right]_{t=\pi/4} = \frac{-\frac{\sqrt{2}}{2}}{1} = -\frac{\sqrt{2}}{2}$. The equation of the tangent line at $(x(\pi/4), y(\pi/4)) = (1/2, \sqrt{2}/2)$ is

$$y - \frac{1}{2} = -\frac{\sqrt{2}}{2} \left(x - \frac{1}{2} \right)$$

$$\boxed{y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{4} + \frac{1}{2}}.$$

(b)

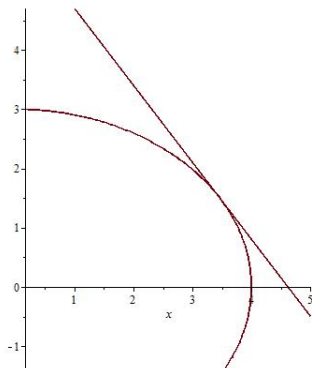


25. (a) The slope of the tangent line is $\left[\frac{-3 \sin t}{4 \cos t} \right]_{t=\pi/3} = \frac{-3\sqrt{3}/2}{4 \cdot 1/2} = -\frac{3\sqrt{3}}{4}$. The equation of the tangent line at $(x(\pi/3), y(\pi/3)) = (2\sqrt{3}, 3/2)$ is

$$y - \frac{3}{2} = -\frac{3\sqrt{3}}{4} (x - 2\sqrt{3})$$

$$y = -\frac{3\sqrt{3}}{4}x + 6.$$

(b)

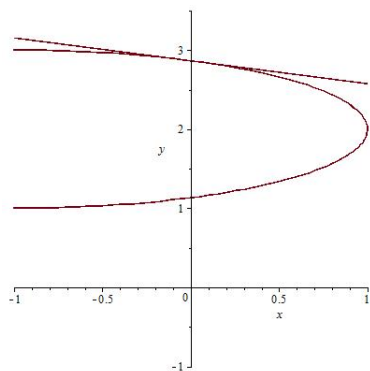


26. (a) The slope of the tangent line is $\left[\frac{-\sin t}{2 \cos t} \right]_{t=\pi/6} = \frac{-1/2}{\sqrt{3}} = -\frac{\sqrt{3}}{6}$. The equation of the tangent line at $(x(\pi/6), y(\pi/6)) = (0, \sqrt{3}/2 + 2)$ is

$$y - \left(\frac{\sqrt{3}}{2} + 2 \right) = -\frac{\sqrt{3}}{6} (x - 0)$$

$$y = -\frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{2} + 2.$$

(b)



In Problems 27-30, we use the facts

$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ implies a horizontal tangent line at t

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ implies a vertical tangent line at t .

27.

$$\begin{aligned}\frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 3t^2 - 4\end{aligned}$$

$\frac{dy}{dt} = 0$ when $t = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3}$. At both values, $\frac{dx}{dt} \neq 0$, so we have horizontal tangent lines at $t = \pm \frac{2\sqrt{3}}{3}$

which correspond to the points $\left(\frac{4}{3}, -\frac{16\sqrt{3}}{9}\right)$ and $\left(\frac{4}{3}, \frac{16\sqrt{3}}{9}\right)$. Next, $\frac{dx}{dt} = 0$ when $t = 0$. At $t = 0$,

$\frac{dy}{dt} \neq 0$, so we have a vertical tangent line at $t = 0$ which corresponds to the point $(0, 0)$.

28.

$$\begin{aligned}\frac{dx}{dt} &= 3t^2 - 9 \\ \frac{dy}{dt} &= 2t\end{aligned}$$

$\frac{dy}{dt} = 0$ when $t = 0$. At $t = 0$, $\frac{dx}{dt} \neq 0$, so we have a horizontal tangent line at $t = 0$ which corresponds to the point $(0, 0)$. Next, $\frac{dx}{dt} = 0$ when $t = \pm\sqrt{3}$. At $t = \pm\sqrt{3}$, $\frac{dy}{dt} \neq 0$, so we have vertical tangent lines at both values which correspond to the points $(-6\sqrt{3}, 3)$ and $(6\sqrt{3}, 3)$.

29.

$$\begin{aligned}\frac{dx}{dt} &= \sin t \\ \frac{dy}{dt} &= -\cos t\end{aligned}$$

On the interval $0 \leq t \leq 2\pi$, $\frac{dy}{dt} = 0$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$. At both values, $\frac{dx}{dt} \neq 0$, so we have horizontal tangent lines at $t = \frac{\pi}{2}, \frac{3\pi}{2}$ which correspond to the points $(1, 0)$ and $(1, 2)$. Next, on the interval $0 \leq t \leq 2\pi$, $\frac{dx}{dt} = 0$ when $t = 0, \pi, 2\pi$. At all three values, $\frac{dy}{dt} \neq 0$, so we have vertical tangent lines at $t = 0, \pi, 2\pi$ which correspond to the points $(0, 1)$ and $(2, 1)$.

30.

$$\begin{aligned}\frac{dx}{dt} &= 3 \sin t - 3 \sin(3t) \\ \frac{dy}{dt} &= \cos t\end{aligned}$$

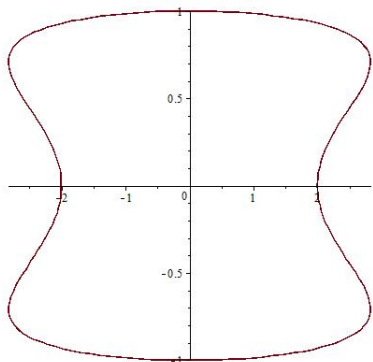
On the interval $0 \leq t \leq 2\pi$, $\frac{dy}{dt} = 0$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$. At both values, $\frac{dx}{dt} \neq 0$, so we have horizontal tangent lines at $t = \frac{\pi}{2}, \frac{3\pi}{2}$ which correspond to the points $(0, 1)$ and $(0, -1)$. Next, $\frac{dx}{dt} = 0$ when $\sin t = \sin(3t)$. If we use the triple angle formula for $\sin(3t)$, which says

$$\sin(3t) = -4 \sin^3 t + 3 \sin t,$$

then we need to solve the equation $\sin t = -4 \sin^3 t + 3 \sin t$. This equation can be solved as follows

$$\begin{aligned}\sin t &= -4 \sin^3 t + 3 \sin t \\ 4 \sin^3 t - 2 \sin t &= 0 \\ 2 \sin t(2 \sin^2 t - 1) &= 0 \\ 2 \sin t = 0 \quad \text{or} \quad 2 \sin^2 t - 1 = 0.\end{aligned}$$

On the interval $0 \leq t \leq 2\pi$, the left equation has solutions $t = 0, \pi, 2\pi$. The right equation can be simplified further to $\sin^2 t = \frac{1}{2}$ or $\sin t = \pm \frac{\sqrt{2}}{2}$. The values for t on the interval $0 \leq t \leq 2\pi$ that satisfy this set of equations is $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. At each of the values $t = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, $\frac{dy}{dt} \neq 0$, so each yields a point where we have a vertical tangent line. Those points are $(-2, 0), (2, 0), (-2\sqrt{2}, \sqrt{2}/2), (2\sqrt{2}, \sqrt{2}/2), (2\sqrt{2}, -\sqrt{2}/2)$, and $(-2\sqrt{2}, -\sqrt{2}/2)$. A picture of the plane curve is as follows.



For Problem 31-38, we will use the formula $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ to find the length of the parametrized curve $(x(t), y(t))$ over the interval $a \leq t \leq b$.

31. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 3t^2 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \leq t \leq 2$. Using the arc length formula, we have

$$\begin{aligned}s &= \int_0^2 \sqrt{(3t^2)^2 + (2t)^2} dt \\ &= \int_0^2 \sqrt{9t^4 + 4t^2} dt \\ &= \int_0^2 t\sqrt{9t^2 + 4} dt.\end{aligned}$$

We use the substitution $u = 9t^2 + 4$. Then $du = 18t \, dt$, or equivalently, $t \, dt = \frac{du}{18}$. Changing the limits of integration, we find that when $t = 0$, then $u = 4$, and when $t = 2$, then $u = 40$. The arc length s is

$$\begin{aligned} s &= \int_0^2 t \sqrt{9t^2 + 4} \, dt = \int_4^{40} \sqrt{u} \frac{du}{18} = \frac{1}{18} \int_4^{40} u^{1/2} \, du = \frac{1}{18} \left[\frac{u^{3/2}}{3/2} \right]_4^{40} \\ &= \frac{1}{27} \left[40^{3/2} - 4^{3/2} \right] = \boxed{\frac{1}{27} [80\sqrt{10} - 8]}. \end{aligned}$$

32. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 6t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

The curve is smooth for $0 \leq t \leq 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(6t)^2 + (3t^2)^2} \, dt \\ &= \int_0^2 \sqrt{36t^2 + 9t^4} \, dt \\ &= \int_0^2 3t \sqrt{4 + t^2} \, dt. \end{aligned}$$

We use the substitution $u = 4 + t^2$. Then $du = 2t \, dt$, or equivalently, $3t \, dt = \frac{3du}{2}$. Changing the limits of integration, we find that when $t = 0$, then $u = 4$, and when $t = 2$, then $u = 8$. The arc length s is

$$\begin{aligned} s &= \int_0^2 3t \sqrt{4 + t^2} \, dt = \int_4^8 \sqrt{u} \frac{3du}{2} = \frac{3}{2} \int_4^8 u^{1/2} \, du = \frac{3}{2} \left[\frac{u^{3/2}}{3/2} \right]_4^8 \\ &= 8^{3/2} - 4^{3/2} = \boxed{16\sqrt{2} - 8}. \end{aligned}$$

33. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = t$$

The curve is smooth for $0 \leq t \leq 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(1)^2 + (t)^2} \, dt \\ &= \int_0^2 \sqrt{1 + t^2} \, dt. \end{aligned}$$

To compute this integral, we use the Table of Integrals 49 with $a = 1$. Then

$$\begin{aligned} s &= \left[\frac{t}{2} \sqrt{1 + t^2} + \frac{1}{2} \ln |t + \sqrt{1 + t^2}| \right]_0^2 \\ &= \sqrt{5} + \frac{1}{2} \ln |2 + \sqrt{5}| - 0 \\ &= \boxed{\sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5})}. \end{aligned}$$

34. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 2$$

The curve is smooth for $1 \leq t \leq 3$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_1^3 \sqrt{(2t)^2 + (2)^2} dt \\ &= 2 \int_1^3 \sqrt{t^2 + 1} dt. \end{aligned}$$

This is the same integrand we found in Problem 33. As in Problem 33, we use the Table of Integrals 49 with $a = 1$. Then

$$\begin{aligned} s &= 2 \left[\frac{t}{2} \sqrt{1+t^2} + \frac{1}{2} \ln |t + \sqrt{1+t^2}| \right]_1^3 \\ &= 2 \left[\frac{3}{2} \sqrt{10} + \frac{1}{2} \ln |3 + \sqrt{10}| \right] - 2 \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln |1 + \sqrt{2}| \right] \\ &= \boxed{3\sqrt{10} - \sqrt{2} + \ln \left(\frac{\sqrt{10} + 3}{\sqrt{2} + 1} \right)}. \end{aligned}$$

35. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 4 \cos t \quad \text{and} \quad \frac{dy}{dt} = -4 \sin t$$

The curve is smooth for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_{-\pi/2}^{\pi/2} \sqrt{(4 \cos t)^2 + (-4 \sin t)^2} dt \\ &= \int_{-\pi/2}^{\pi/2} 4 dt \\ &= [4t]_{-\pi/2}^{\pi/2} \\ &= 4 \left(\frac{\pi}{2} \right) - 4 \left(-\frac{\pi}{2} \right) \\ &= \boxed{4\pi}. \end{aligned}$$

36. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 6 \cos t \quad \text{and} \quad \frac{dy}{dt} = -6 \sin t$$

The curve is smooth for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_{-\pi/2}^{\pi/2} \sqrt{(6 \cos t)^2 + (-6 \sin t)^2} dt \\ &= \int_{-\pi/2}^{\pi/2} 6 dt \\ &= [6t]_{-\pi/2}^{\pi/2} \\ &= 6 \left(\frac{\pi}{2} \right) - 6 \left(-\frac{\pi}{2} \right) \\ &= \boxed{6\pi}. \end{aligned}$$

37. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2 \cos t \quad \text{and} \quad \frac{dy}{dt} = -2 \sin t$$

The curve is smooth for $0 \leq t \leq 2\pi$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(2 \cos t)^2 + (-2 \sin t)^2} dt \\ &= \int_0^{2\pi} 2 dt \\ &= [2t]_0^{2\pi} \\ &= 2(2\pi) - 2(0) \\ &= \boxed{4\pi}. \end{aligned}$$

38. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \sin t + e^t \cos t \quad \text{and} \quad \frac{dy}{dt} = e^t \cos t - e^t \sin t$$

The curve is smooth for $0 \leq t \leq \pi$. Using the arc length formula, we have

$$s = \int_0^{\pi} \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \sin t - e^t \cos t)^2} dt.$$

We will simplify the expression under the square root first.

$$\begin{aligned} (e^t \sin t + e^t \cos t)^2 + (e^t \sin t - e^t \cos t)^2 &= [e^t(\sin t + \cos t)]^2 + [e^t(\sin t - \cos t)]^2 \\ &= e^{2t}[\sin^2 t + 2 \sin t \cos t + \cos^2 t] + e^{2t}[\sin^2 t - 2 \sin t \cos t + \cos^2 t] \\ &= e^{2t}[1 + 2 \sin t \cos t] + e^{2t}[1 - 2 \sin t \cos t] \\ &= 2e^{2t}. \end{aligned}$$

Then the arc length is

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{2e^{2t}} dt = \sqrt{2} \int_0^{\pi} e^t dt \\ &= \left[\sqrt{2}e^t \right]_0^{\pi} = \boxed{\sqrt{2}e^{\pi} - \sqrt{2}}. \end{aligned}$$

39. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

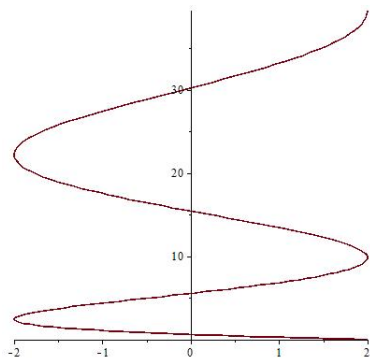
$$\frac{dx}{dt} = -4 \sin(2t) \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \leq t \leq 2\pi$. Using the arc length formula, we have

$$s = \int_0^{2\pi} \sqrt{(-4 \sin(2t))^2 + (2t)^2} dt.$$

(b) Using a Computer Algebra system to compute this integral, we find that $\boxed{s \approx 44.527}$.

(c)



40. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

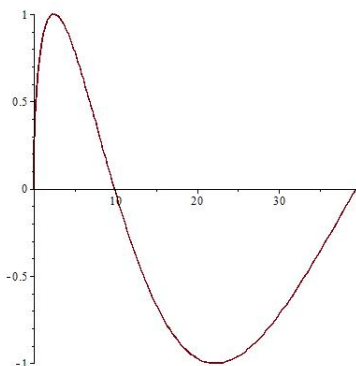
$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = \cos t$$

The curve is smooth for $0 \leq t \leq 2\pi$. Using the arc length formula, we have

$$s = \int_0^{2\pi} \sqrt{(2t)^2 + (\cos t)^2} dt.$$

(b) Using a Computer Algebra system to compute this integral, we find that $s \approx 40.051$.

(c)



41. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2}(t+2)^{-1/2}$$

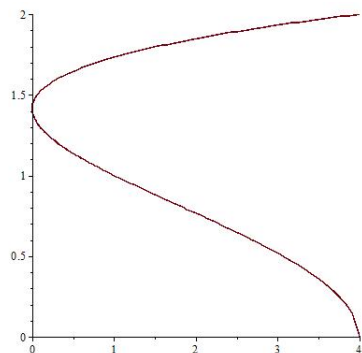
The curve is smooth for $-2 \leq t \leq 2$. Using the arc length formula, we have

$$s = \int_{-2}^2 \sqrt{(2t)^2 + \left(\frac{1}{2\sqrt{t+2}}\right)^2} dt.$$

Note that this is an improper integral because of the discontinuity at $t = -2$.

(b) Using a Computer Algebra system to compute this integral, we find that $s \approx 8.429$.

(c)



42. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

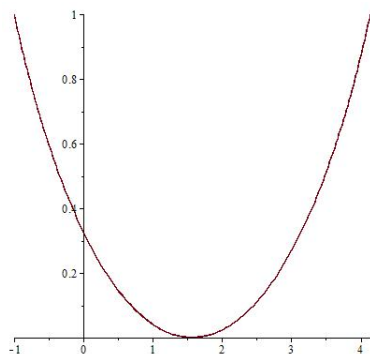
$$\frac{dx}{dt} = 1 + \sin t \quad \text{and} \quad \frac{dy}{dt} = -\cos t$$

The curve is smooth for $0 \leq t \leq \pi$. Using the arc length formula, we have

$$s = \int_0^\pi \sqrt{(1 + \sin t)^2 + (-\cos t)^2} dt.$$

(b) Using a Computer Algebra system to compute this integral, we find that $s = 4\sqrt{2}$.

(c)



43. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

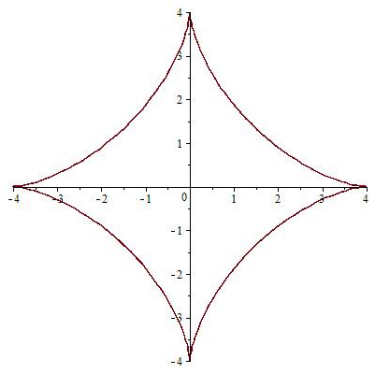
$$\frac{dx}{dt} = -3\sin t - 3\sin(3t) \quad \text{and} \quad \frac{dy}{dt} = 3\cos t - 3\cos(3t)$$

By using a Computer Algebra system, we find that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously 0 on $0 < t < 2\pi$ when $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ so the curve is not smooth on $[0, 2\pi]$. To find the arc length, we will exploit symmetry and find the arc length on the interval $\left[0, \frac{\pi}{2}\right]$ (where the curve is smooth) and then quadruple it. Using the arc length formula, the full arc length is

$$s = 4 \cdot \int_0^{\pi/2} \sqrt{(-3\sin t - 3\sin(3t))^2 + (3\cos t - 3\cos(3t))^2} dt.$$

(b) Using a Computer Algebra system to compute this integral, we find that $s = 24$.

(c)



44. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

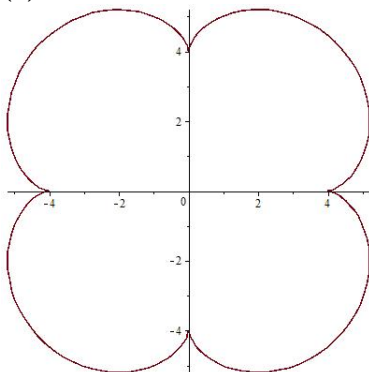
$$\frac{dx}{dt} = -5 \sin t + 5 \sin(5t) \quad \text{and} \quad \frac{dy}{dt} = 5 \cos t - 5 \cos(5t)$$

By using a Computer Algebra system, we find that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously 0 on $0 < t < 2\pi$ when $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ so the curve is not smooth on $[0, 2\pi]$. To find the arc length, we will exploit symmetry and find the arc length on the interval $\left[0, \frac{\pi}{2}\right]$ (where the curve is smooth) and then quadruple it. Using the arc length formula, we have

$$s = 4 \cdot \int_0^{\pi/2} \sqrt{(-5 \sin t + 5 \sin(5t))^2 + (5 \cos t - 5 \cos(5t))^2} dt.$$

(b) Using a Computer Algebra system to compute this integral, we find that $s = 40$.

(c)



Applications and Extensions

45. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2 - 4$$

The vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. Notice that this is when $t = 0$, which corresponds to the point $(2, 0)$. The horizontal tangent lines occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. This is when

$$\begin{aligned} 3t^2 - 4 &= 0 \\ t^2 &= \frac{4}{3} \\ t &= \pm \frac{2}{\sqrt{3}}. \end{aligned}$$

For both of these values, $\frac{dx}{dt} \neq 0$, so the horizontal tangent lines occur at the points

$$(x(2/\sqrt{3}), y(2/\sqrt{3})) = \left(\frac{10}{3}, -\frac{16\sqrt{3}}{9}\right) \text{ and } (x(-2/\sqrt{3}), y(-2/\sqrt{3})) = \left(\frac{10}{3}, \frac{16\sqrt{3}}{9}\right).$$

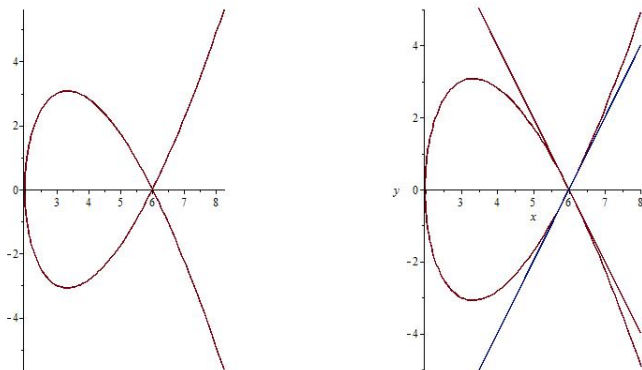
(b) The point $(6, 0)$ corresponds to the values $t = \pm 2$. Each of these values produces a slope of a tangent line at $(6, 0)$, namely

$$\frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right]_{t=2} = \frac{3(2)^2 - 4}{2(2)} = 2 \text{ and } \frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right]_{t=-2} = \frac{3(-2)^2 - 4}{2(-2)} = -2.$$

(c) The equations of the two tangent lines at $(6, 0)$ are

$$\begin{aligned} y &= 2x - 12 \\ y &= -2x + 12. \end{aligned}$$

(d)



46. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 4t \quad \text{and} \quad \frac{dy}{dt} = 8 - 3t^2$$

The vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. Notice that this is when $t = 0$, which corresponds to the point $(0, 0)$. The horizontal tangent lines occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. This is when

$$\begin{aligned} 8 - 3t^2 &= 0 \\ t^2 &= \frac{8}{3} \\ t &= \pm \frac{2\sqrt{2}}{\sqrt{3}} = \pm \frac{2\sqrt{6}}{3}. \end{aligned}$$

For both of these values, $\frac{dx}{dt} \neq 0$, so the horizontal tangent lines occur at the points

$$(x(2\sqrt{6}/3), y(2\sqrt{6}/3)) = \left(\frac{16}{3}, \frac{32\sqrt{6}}{9}\right) \text{ and } (x(-2\sqrt{6}/3), y(-2\sqrt{6}/3)) = \left(\frac{16}{3}, -\frac{32\sqrt{6}}{9}\right).$$

(b) The point $(16, 0)$ corresponds to the values satisfying $2t^2 = 16$, or $t = \pm 2\sqrt{2}$. Each of these values produces a slope of a tangent line at $(16, 0)$, namely

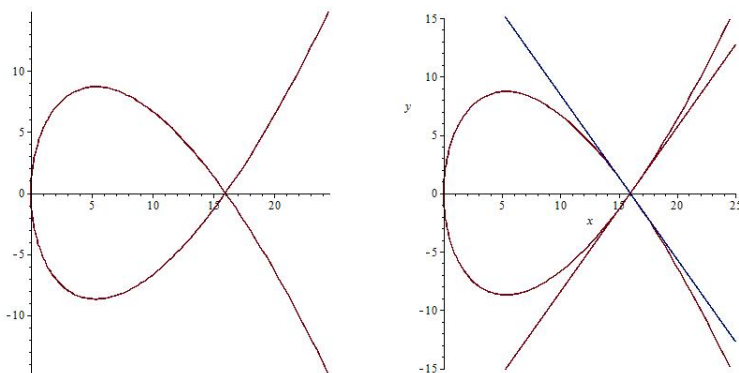
$$\frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right]_{t=2\sqrt{2}} = \frac{8 - 3(2\sqrt{2})^2}{4(2\sqrt{2})} = -\sqrt{2} \text{ and } \frac{dy}{dx} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right]_{t=-2\sqrt{2}} = \frac{8 - 3(-2\sqrt{2})^2}{4(-2\sqrt{2})} = \sqrt{2}.$$

(c) The equations of the two tangent lines at $(16, 0)$ are

$$y = -\sqrt{2}(x - 16)$$

$$y = \sqrt{2}(x - 16).$$

(d)



47. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 3b \sin^2 t \cos t \quad \text{and} \quad \frac{dy}{dt} = -3b \cos^2 t \sin t$$

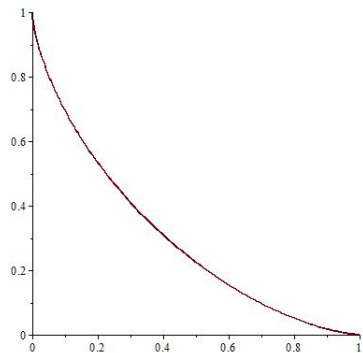
The curve is smooth for $0 \leq t \leq \frac{\pi}{2}$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{(3b \sin^2 t \cos t)^2 + (-3b \cos^2 t \sin t)^2} dt \\ &= \int_0^{\pi/2} \sqrt{9b^2 \sin^4 t \cos^2 t + 9b^2 \cos^4 t \sin^2 t} dt \\ &= \int_0^{\pi/2} \sqrt{9b^2 \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t)} dt \\ &= \int_0^{\pi/2} 3b \sin t \cos t dt. \end{aligned}$$

Using the substitution $u = \sin t$, we have $du = \cos t dt$ and the new limits of integration are $u = 0$ to $u = 1$. So the arc length is

$$\begin{aligned} s &= 3b \int_0^1 u du \\ &= \left[\frac{3b}{2} u^2 \right]_0^1 = \boxed{\frac{3b}{2}}. \end{aligned}$$

(b) The following graph shows the case $b = 1$.



48. (a) We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \cos t - t \sin t \quad \text{and} \quad \frac{dy}{dt} = \sin t + t \cos t$$

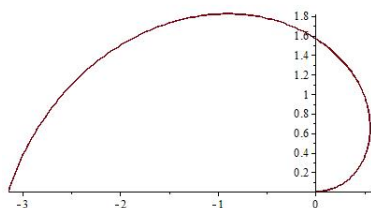
Using a Computer Algebra system, we find that $\frac{dx}{dt} = 0$ on $0 < t < \pi$ when $t \approx 0.86$ and $\frac{dy}{dt} = 0$ on $0 < t < \pi$ when $t \approx 2.03$. From these solutions, we see that the curve is smooth for $0 \leq t \leq \pi$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^\pi \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} dt \\ &= \int_0^\pi \sqrt{\cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t + \sin^2 t + 2t \cos t \sin t + t^2 \cos^2 t} dt \\ &= \int_0^\pi \sqrt{1 + t^2} dt. \end{aligned}$$

Using the Table of Integrals 49 with $a = 1$, we find that

$$\begin{aligned} s &= \left[\frac{t}{2} \sqrt{1 + t^2} + \frac{1}{2} \ln \left| t + \sqrt{1 + t^2} \right| \right]_0^\pi \\ &= \left[\frac{\pi}{2} \sqrt{1 + \pi^2} + \frac{1}{2} \ln \left| \sqrt{1 + \pi^2} + \pi \right| \right] - 0 \\ &= \boxed{\frac{\pi}{2} \sqrt{1 + \pi^2} + \frac{1}{2} \ln \left(\sqrt{1 + \pi^2} + \pi \right)}. \end{aligned}$$

(b)



49. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

The curve is smooth for $0 \leq t \leq 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$s = \int_0^2 \sqrt{(3)^2 + (2t)^2} dt = \int_0^2 \sqrt{4\left(\frac{9}{4} + t^2\right)} dt = 2 \int_0^2 \sqrt{\frac{9}{4} + t^2} dt.$$

Using the Table of Integrals 49 with $a = \frac{3}{2}$, we find that

$$\begin{aligned} s &= 2 \left[\frac{t}{2} \sqrt{\frac{9}{4} + t^2} + \frac{9}{8} \ln \left| t + \sqrt{\frac{9}{4} + t^2} \right| \right]_0^2 \\ &= 2 \left[\frac{5}{2} + \frac{9}{8} \ln \left| \frac{9}{2} \right| \right] - 2 \left[\frac{9}{8} \ln \left| \frac{3}{2} \right| \right] \\ &= \boxed{5 + \frac{9}{4} \ln(3)}. \end{aligned}$$

50. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3$$

The curve is smooth for $0 \leq t \leq 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$s = \int_0^2 \sqrt{(2t)^2 + (3)^2} dt.$$

Notice that this is the same integral as Problem 49. This means the distance traveled is $\boxed{5 + \frac{9}{4} \ln(3)}$.

51. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = t \quad \text{and} \quad \frac{dy}{dt} = \sqrt{2t+3}$$

The curve is smooth for $0 \leq t \leq 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(t)^2 + (\sqrt{2t+3})^2} dt \\ &= \int_0^2 \sqrt{t^2 + 2t + 3} dt \\ &= \int_0^2 \sqrt{(t+1)^2 + 2} dt, \end{aligned}$$

where in the last step we completed the square. If we make a substitution $u = t + 1$, then $du = dt$ and changing the limits of integration

$$s = \int_1^3 \sqrt{u^2 + 2} du.$$

Using the Table of Integrals 49 with $a = \sqrt{2}$, we find that

$$\begin{aligned} s &= \left[\frac{u}{2} \sqrt{u^2 + 2} + \ln \left| u + \sqrt{u^2 + 2} \right| \right]_1^3 \\ &= \left[\frac{3}{2} \sqrt{11} + \ln \left| 3 + \sqrt{11} \right| \right] - \left[\frac{1}{2} \sqrt{3} + \ln \left| 1 + \sqrt{3} \right| \right] \\ &= \boxed{\frac{3\sqrt{11} - \sqrt{3}}{2} + \ln \left(\frac{3 + \sqrt{11}}{1 + \sqrt{3}} \right)}. \end{aligned}$$

52. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -a \sin t \quad \text{and} \quad \frac{dy}{dt} = a \cos t$$

The curve is smooth for $0 \leq t \leq \pi$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^\pi \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt \\ &= \int_0^\pi \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt \\ &= \int_0^\pi a dt \\ &= [at]_0^\pi \\ &= \boxed{a\pi}. \end{aligned}$$

53. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -2 \sin(2t) \quad \text{and} \quad \frac{dy}{dt} = 2 \sin t \cos t$$

The curve is smooth for $0 \leq t \leq \frac{\pi}{2}$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{(-2 \sin(2t))^2 + (2 \sin t \cos t)^2} dt \\ &= \int_0^{\pi/2} \sqrt{4 \sin^2(2t) + (2 \sin t \cos t)^2} dt. \end{aligned}$$

Using the double angle formula $\sin(2t) = 2 \sin t \cos t$, we have

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{4 \sin^2(2t) + \sin^2(2t)} dt \\ &= \int_0^{\pi/2} \sqrt{5} \sin(2t) dt = \left[-\frac{\sqrt{5}}{2} \cos(2t) \right]_0^{\pi/2} \\ &= -\frac{\sqrt{5}}{2}(-1) + \frac{\sqrt{5}}{2}(1) \\ &= \boxed{\sqrt{5}}. \end{aligned}$$

54. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -t^{-2} \quad \text{and} \quad \frac{dy}{dt} = t^{-1}$$

The curve is smooth for $1 \leq t \leq 2$ so the distance traveled by the particle over the time interval is equal to the arc length over that interval. Using the arc length formula, we have

$$s = \int_1^2 \sqrt{(-t^{-2})^2 + (t^{-1})^2} dt = \int_1^2 \sqrt{t^{-4} + t^{-2}} dt = \int_1^2 \sqrt{\frac{1+t^2}{t^4}} dt.$$

Using the Table of Integrals 54 with $a = 1$, we find that

$$\begin{aligned} s &= \left[-\frac{\sqrt{1+t^2}}{t} + \ln |t + \sqrt{1+t^2}| \right]_1^2 \\ &= \left[-\frac{\sqrt{5}}{2} + \ln |2 + \sqrt{5}| \right] - \left[-\sqrt{2} + \ln |1 + \sqrt{2}| \right] \\ &= \boxed{-\frac{\sqrt{5}}{2} + \sqrt{2} + \ln \left(\frac{2 + \sqrt{5}}{1 + \sqrt{2}} \right)}. \end{aligned}$$

For Problems 55-57, we recall that the speed of an object is the absolute value of the rate of change of s with respect to t , or $\left| \frac{ds}{dt} \right|$. Using the formula for s and the Fundamental Theorem of Calculus, we have

$$\left| \frac{ds}{dt} \right| = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}.$$

55. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 20 \quad \text{and} \quad \frac{dy}{dt} = -32t$$

So

$$\left| \frac{ds}{dt} \right| = \sqrt{(20)^2 + (-32t)^2} = \boxed{\sqrt{400 + 1024t^2}}.$$

56. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 1 - \sin t \quad \text{and} \quad \frac{dy}{dt} = 2 - \cos t$$

So

$$\left| \frac{ds}{dt} \right| = \sqrt{(1 - \sin t)^2 + (2 - \cos t)^2} = \boxed{\sqrt{6 - 2 \sin t - 4 \cos t}}.$$

57. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 40 \cos(2t) \quad \text{and} \quad \frac{dy}{dt} = -6 \sin t$$

So

$$\left| \frac{ds}{dt} \right| = \sqrt{(40 \cos(2t))^2 + (-6 \sin t)^2} = \boxed{\sqrt{1600 \cos^2(2t) + 36 \sin^2 t}}.$$

58. A circle of radius r can be parametrized by $x(t) = r \cos t$ and $y(t) = r \sin t$ for $0 \leq t \leq 2\pi$. Then the arc length s of an arc subtended by a central angle of θ radians is the arc length of the above parametrization over the interval $0 \leq t \leq \theta$. Notice that $\sqrt{[x'(t)]^2 + [y'(t)]^2} = r$ which gives the length of the arc

$$\begin{aligned} s &= \int_0^\theta \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_0^\theta r dt = [rt]_0^\theta \\ &= \boxed{r\theta}. \end{aligned}$$

59. For a function $y = f(x)$ on $a \leq x \leq b$ representing a smooth curve C , we can parametrize C by choosing $x(t) = t$ and $y(t) = f(t)$ on the interval $a \leq t \leq b$. It is given that C is smooth on $a \leq t \leq b$, so we compute the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t)$$

The arc length s of C on $a \leq t \leq b$ using the arc length formula is

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{1 + [f'(t)]^2} dt = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For Problems 60-63, we note that since ds approximates s for nearby points, we will compute ds by the formula $ds = \sqrt{(dx)^2 + (dy)^2}$ where dx and dy represent that changes in x and y , respectively, between the nearby points.

60. We start by computing dx and dy .

$$\begin{aligned} dx &= x(1.1) - x(1) = 1.1^{1/3} - 1^3 = 1.1^{1/3} - 1 \\ dy &= y(1.1) - y(1) = 1.1^2 - 1^2 = 0.21. \end{aligned}$$

The arc length s is approximately

$$s \approx ds = \sqrt{(1.1^{1/3} - 1)^2 + (0.21)^2} \approx 0.212466.$$

To see how this estimate compares to the actual value of

$$s = \int_1^{1.1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

using a CAS, we find that $s \approx 0.212472$.

61. We start by computing dx and dy .

$$\begin{aligned} dx &= x(1.2) - x(1) = \sqrt{1.2} - \sqrt{1} = \sqrt{1.2} - 1 \\ dy &= y(1.2) - y(1) = 1.2^3 - 1^3 = 0.728. \end{aligned}$$

The arc length s is approximately

$$s \approx ds = \sqrt{(\sqrt{1.2} - 1)^2 + (0.728)^2} \approx 0.7342.$$

To see how this estimate compares to the actual value of

$$s = \int_1^{1.2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

using a CAS, we find that $s \approx 0.7343$.

62. We start by computing dx and dy .

$$\begin{aligned} dx &= x(0.1) - x(0) = a \sin(0.1) - a \sin(0) = a \sin(0.1) \\ dy &= y(0.1) - y(0) = b \cos(0.1) - b \cos(0) = b \cos(0.1) - b. \end{aligned}$$

So the arc length is

$$s \approx ds = \sqrt{(a \sin(0.1))^2 + (b \cos(0.1) - b)^2},$$

for given values of a and b .

63. We start by computing dx and dy .

$$\begin{aligned} dx &= x(0.2) - x(0) = e^{0.2a} - e^{0a} = e^{0.2a} - 1 \\ dy &= y(0.2) - y(0) = e^{0.2b} - e^{0b} = e^{0.2b} - 1. \end{aligned}$$

The arc length s is approximately

$$s \approx ds = \sqrt{(e^{0.2a} - 1)^2 + (e^{0.2b} - 1)^2} = \sqrt{e^{0.4a} - 2e^{0.2a} + e^{0.4b} - 2e^{0.2b} + 2},$$

for given values of a and b .

Challenge Problems

64. If $\frac{dx}{dt}$ is never equal to 0, then $x(t)$ is always increasing or always decreasing, which means that $x(t)$ passes the horizontal line test. Since $x(t)$ passes the horizontal line test, we know an inverse of $x = f(t)$ exists, namely $t = f^{-1}(x)$. If we substitute $t = f^{-1}(x)$ into $y = g(t)$, then

$$y = g(f^{-1}(x)).$$

By defining the function $F = g \circ f^{-1}$, we see that $y = F(x)$. Furthermore, we know that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and

since the curve C is smooth (and $\frac{dx}{dt} \neq 0$), $\frac{dy}{dx}$ exists, making F differentiable.

65. An equivalent question would be to find the value d such that the arc length s along $(x(t), y(t))$ on $0 \leq t \leq d$ equals $\frac{80\sqrt{2} - 40}{3}$. Then

$$s = \int_0^d \sqrt{(4t^2 + 6t)^2 + (3t^2 - 8t)^2} dt = \int_0^d \sqrt{25t^4 + 100t^2} dt = \int_0^d 5t\sqrt{t^2 + 4} dt.$$

Using the u -substitution $u = t^2 + 4$ and changing the limits of integration, we find that

$$s = \frac{5}{2} \int_4^{d^2+4} u^{1/2} du = \left[\frac{5}{3} u^{3/2} \right]_4^{d^2+4} = \frac{5}{3} (d^2 + 4)^{3/2} - \frac{40}{3}.$$

If we want the value of d so that $s = \frac{80\sqrt{2} - 40}{3}$, then

$$\begin{aligned} \frac{5}{3} (d^2 + 4)^{3/2} - \frac{40}{3} &= \frac{80\sqrt{2} - 40}{3} \\ \frac{5}{3} (d^2 + 4)^{3/2} &= \frac{80\sqrt{2}}{3} \\ (d^2 + 4)^{3/2} &= 16\sqrt{2} \\ d^2 + 4 &= (16\sqrt{2})^{2/3} = 8 \\ d &= \pm 2. \end{aligned}$$

Since we started at $t = 0$, the points on the curve that have arc length $\frac{80\sqrt{2} - 40}{3}$ from $(0, 0)$ are

$$(x(2), y(2)) = \left(\frac{68}{3}, -8 \right) \quad \text{and} \quad (x(-2), y(-2)) = \left(\frac{4}{3}, -24 \right).$$

66. In this section, we developed the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, assuming $\frac{dx}{dt} \neq 0$, using the Chain Rule. We knew that $y = F(x)$, and that x is a function of t to write $y(t) = F(x(t))$. Then we found the derivative using the

Chain Rule:

$$\frac{dy}{dt} = F'(x(t)) \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

where we noted that $F'(x(t)) = F'(x) = \frac{dy}{dx}$. Take the equation $\frac{dy}{dt} = F'(x(t)) \cdot \frac{dx}{dt}$ and differentiate again with respect to t (remember to use the product rule!).

$$\begin{aligned} \frac{d}{dt} \left(\frac{dy}{dt} \right) &= \frac{d}{dt} \left(F'(x(t)) \cdot \frac{dx}{dt} \right) \\ \frac{d^2 y}{dt^2} &= \left[F''(x(t)) \cdot \frac{dx}{dt} \right] \cdot \frac{dx}{dt} + F'(x(t)) \cdot \frac{d^2 x}{dt^2}. \end{aligned}$$

Noting that $F'(x(t)) = \frac{dy}{dx}$ and $F''(x(t)) = \frac{d^2 y}{dx^2}$, we have

$$\begin{aligned} \frac{d^2 y}{dt^2} &= \frac{d^2 y}{dx^2} \left(\frac{dx}{dt} \right)^2 + \frac{dy}{dx} \cdot \frac{d^2 x}{dt^2} \\ \frac{d^2 y}{dx^2} &= \frac{\frac{d^2 y}{dt^2} - \frac{dy}{dx} \cdot \frac{d^2 x}{dt^2}}{\left(\frac{dx}{dt} \right)^2}. \end{aligned}$$

Using the formula above for $\frac{dy}{dx}$, we have

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{\frac{d^2 y}{dt^2} - \frac{dy}{dx} \cdot \frac{d^2 x}{dt^2}}{\left(\frac{dx}{dt} \right)^2} \\ &= \boxed{\frac{\frac{d^2 y}{dt^2} \cdot \frac{dx}{dt} - \frac{dy}{dt} \cdot \frac{d^2 x}{dt^2}}{\left(\frac{dx}{dt} \right)^3}}. \end{aligned}$$

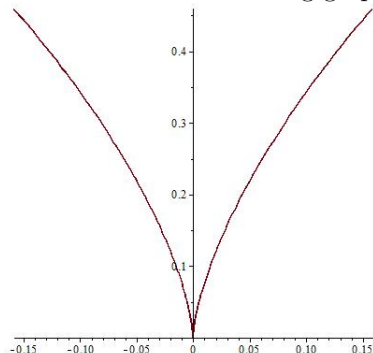
67. Using the formula developed in Problem 66, we start by finding the first and second derivatives of x and y with respect to θ .

$$\begin{aligned} \frac{dx}{d\theta} &= -3a \cos^2 \theta \sin \theta \\ \frac{dy}{d\theta} &= 3a \sin^2 \theta \cos \theta \\ \frac{d^2 x}{d\theta^2} &= 6a \sin^2 \theta \cos \theta - 3a \cos^3 \theta \\ \frac{d^2 y}{d\theta^2} &= 6a \cos^2 \theta \sin \theta - 3a \sin^3 \theta. \end{aligned}$$

So

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(6a \cos^2 \theta \sin \theta - 3a \sin^3 \theta)(-3a \cos^2 \theta \sin \theta) - (3a \sin^2 \theta \cos \theta)(6a \sin^2 \theta \cos \theta - 3a \cos^3 \theta)}{(-3a \cos^2 \theta \sin \theta)^3} \\
 &= \frac{-18a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta - 18a^2 \sin^4 \theta \cos^2 \theta + 9a^2 \sin^2 \theta \cos^4 \theta}{-27a^3 \cos^6 \theta \sin^3 \theta} \\
 &= \frac{-18a^2 \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta) + 9a^2 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)}{-27a^3 \cos^6 \theta \sin^3 \theta} \\
 &= \frac{-9a^2 \sin^2 \theta \cos^2 \theta}{-27a^3 \cos^6 \theta \sin^3 \theta} \\
 &= \boxed{\frac{1}{3a \cos^4 \theta \sin \theta}}.
 \end{aligned}$$

68. Consider the following graph of the cycloid where $a = 1$.



The value $t = 0$ corresponds to $(0,0)$. As we approach the origin from the left, the slopes of the tangent lines are approaching $-\infty$ and as we approach the origin from the right, the slopes of the tangent lines are approaching $+\infty$. This can be verified by determining $\frac{dy}{dx}$, which is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t},$$

and computing the left and right hand limits at $t = 0$. From the left,

$$\lim_{t \rightarrow 0^-} \frac{dy}{dx} = \frac{0}{0},$$

so using L'Hopital's rule

$$\begin{aligned}
 \lim_{t \rightarrow 0^-} \frac{dy}{dx} &= \lim_{t \rightarrow 0^-} \frac{\cos t}{\sin t} \\
 &= \lim_{t \rightarrow 0^-} \cot t = -\infty.
 \end{aligned}$$

From the right,

$$\lim_{t \rightarrow 0^+} \frac{dy}{dx} = \frac{0}{0},$$

so using L'Hopital's rule

$$\begin{aligned}
 \lim_{t \rightarrow 0^+} \frac{dy}{dx} &= \lim_{t \rightarrow 0^+} \frac{\cos t}{\sin t} \\
 &= \lim_{t \rightarrow 0^+} \cot t = +\infty.
 \end{aligned}$$

9.3 Surface Area of a Solid of Revolution

Concepts and Vocabulary

1. **False**. The formula should be

$$S = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

2. The surface area S of a solid of revolution generated by revolving the smooth curve C represented by

$$x = x(t), y = y(t), a \leq t \leq b, \text{ where } x(t) \geq 0, \text{ about the } y\text{-axis is } S = \boxed{2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt}.$$

Skill Building

3. We use formula (2) with $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6$. Then

$$S = 2\pi \int_0^1 6t \sqrt{(6t)^2 + (6)^2} dt = 2\pi \int_0^1 36t \sqrt{t^2 + 1} dt.$$

Using the substitution $u = t^2 + 1$, then $du = 2t dt$ and the new limits are $u = (0)^2 + 1 = 1$ to $u = (1)^2 + 1 = 2$, so we have

$$\begin{aligned} S &= 36\pi \int_1^2 \sqrt{u} du = 36\pi \left[\frac{2}{3} u^{3/2} \right]_1^2 \\ &= 36\pi \left(\frac{2}{3} (2)^{3/2} - \frac{2}{3} \right) = 36\pi \left(\frac{4\sqrt{2} - 2}{3} \right) \\ &= \boxed{24\pi(2\sqrt{2} - 1)}. \end{aligned}$$

4. We use formula (2) with $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2$. Then

$$S = 2\pi \int_0^3 2t \sqrt{(2t)^2 + (2)^2} dt = 2\pi \int_0^3 4t \sqrt{t^2 + 1} dt.$$

Using the substitution $u = t^2 + 1$, then $du = 2t dt$ and the new limits are $u = (0)^2 + 1 = 1$ to $u = (3)^2 + 1 = 10$, so we have

$$\begin{aligned} S &= 4\pi \int_1^{10} \sqrt{u} du = 4\pi \left[\frac{2}{3} u^{3/2} \right]_1^{10} \\ &= 4\pi \left(\frac{2}{3} (10)^{3/2} - \frac{2}{3} \right) = \boxed{4\pi \left(\frac{20\sqrt{10} - 2}{3} \right)}. \end{aligned}$$

5. We use formula (2) with $\frac{dx}{d\theta} = -3 \cos^2 \theta \sin \theta$ and $\frac{dy}{d\theta} = 3 \sin^2 \theta \cos \theta$. Then

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} \sin^3 \theta \sqrt{(-3 \cos^2 \theta \sin \theta)^2 + (3 \sin^2 \theta \cos \theta)^2} d\theta \\ &= 2\pi \int_0^{\pi/2} \sin^3 \theta \sqrt{9 \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)} d\theta \\ &= 2\pi \int_0^{\pi/2} \sin^3 \theta \cdot 3 \cos \theta \sin \theta d\theta = 6\pi \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta. \end{aligned}$$

Using the substitution $u = \sin \theta$, then $du = \cos \theta d\theta$ and the new limits are $u = \sin(0) = 0$ to $u = \sin(\pi/2) = 1$, so we have

$$\begin{aligned} S &= 6\pi \int_0^1 u^4 du = 6\pi \left[\frac{1}{5} u^5 \right]_0^1 \\ &= 6\pi \left(\frac{1}{5} (1)^5 - 0 \right) \\ &= \boxed{\frac{6\pi}{5}}. \end{aligned}$$

6. We use formula (2) with $\frac{dx}{dt} = 1 - \cos t$ and $\frac{dy}{dt} = \sin t$. Then

$$\begin{aligned} S &= 2\pi \int_0^\pi (1 - \cos t) \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\ &= 2\pi \int_0^\pi (1 - \cos t) \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} dt \\ &= 2\pi \int_0^\pi (1 - \cos t) \sqrt{2 - 2\cos t} dt \\ &= 2\pi \int_0^\pi \sqrt{2}(1 - \cos t)^{3/2} dt. \end{aligned}$$

Make the substitution $u = 1 - \cos t$ so $du = \sin t dt$, or $\frac{du}{\sin t} = dt$, and the new limits are $u = 1 - \cos(0) = 0$ to $u = 1 - \cos(\pi) = 2$. We need to rewrite $\sin t$ in terms of u . If we solve $u = 1 - \cos t$ for $\cos t$, then $\cos t = 1 - u$ which means $\cos^2 t = (1 - u)^2$. Using the Pythagorean Identity, we have

$$1 - \sin^2 t = \cos^2 t = (1 - u)^2 = 1 - 2u + u^2$$

so

$$\sin t = \sqrt{2u - u^2}.$$

This means the equation above with differentials becomes

$$\frac{du}{\sqrt{2u - u^2}} = dt.$$

With this substitution,

$$S = 2\sqrt{2}\pi \int_0^2 u^{3/2} \frac{du}{\sqrt{2u - u^2}} = 2\sqrt{2}\pi \int_0^2 \frac{u^{3/2}}{\sqrt{u}\sqrt{2 - u}} du = 2\sqrt{2}\pi \int_0^2 \frac{u}{\sqrt{2 - u}} du.$$

Now use the substitution $w = 2 - u$ where $dw = -du$ and the new limits of integration are $w = 2 - (0) = 2$ to $w = 2 - (2) = 0$. Then

$$\begin{aligned} S &= 2\sqrt{2}\pi \int_2^0 -\frac{2-w}{w^{1/2}} dw = 2\sqrt{2}\pi \int_0^2 \left(2w^{-1/2} - w^{1/2} \right) dw \\ &= 2\sqrt{2}\pi \left[4w^{1/2} - \frac{2}{3}w^{3/2} \right]_0^2 = 2\sqrt{2}\pi \left[4\sqrt{2} - \frac{4}{3}\sqrt{2} \right] \\ &= \boxed{\frac{32}{3}\pi}. \end{aligned}$$

7. We use formula (3) with $f(x) = x^3$ and $f'(x) = 3x^2$. Then

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + [3x^2]^2} dx = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} dx.$$

Using the substitution $u = 1 + 9x^4$, then $du = 36x^3 dx$ and the new limits are $u = 1 + 9(0)^2 = 1$ to $u = 1 + 9(1)^2 = 10$, so we have

$$\begin{aligned} S &= \frac{\pi}{18} \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_1^{10} \\ &= \frac{\pi}{18} \left(\frac{2}{3} (10)^{3/2} - \frac{2}{3} \right) = \frac{\pi}{18} \left(\frac{20\sqrt{10} - 2}{3} \right) \\ &= \boxed{\frac{\pi(10\sqrt{10} - 1)}{27}}. \end{aligned}$$

8. We use formula (3) with $f(x) = 4x^3$ and $f'(x) = 12x^2$. Then

$$S = 2\pi \int_0^2 4x^3 \sqrt{1 + [12x^2]^2} dt = 2\pi \int_0^2 4x^3 \sqrt{1 + 144x^4} dx.$$

Using the substitution $u = 1 + 144x^4$, then $du = 576x^3 dx$ and the new limits are $u = 1 + 144(0)^4 = 1$ to $u = 1 + 144(2)^4 = 2305$, so we have

$$\begin{aligned} S &= \frac{\pi}{72} \int_1^{2305} \sqrt{u} du = \frac{\pi}{72} \left[\frac{2}{3} u^{3/2} \right]_1^{2305} \\ &= \frac{\pi}{72} \left(\frac{2}{3} (2305)^{3/2} - \frac{2}{3} \right) = \frac{\pi}{72} \left(\frac{4610\sqrt{2305} - 2}{3} \right) \\ &= \boxed{\frac{\pi(2305\sqrt{2305} - 1)}{108}}. \end{aligned}$$

9. We use formula (3) with $f(x) = \frac{x^4}{8} + \frac{1}{4x^2} = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ and $f'(x) = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$. Then

$$\begin{aligned} S &= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2} \right) \sqrt{1 + \left[\frac{1}{2}x^3 - \frac{1}{2}x^{-3} \right]^2} dx \\ &= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2} \right) \sqrt{1 + \left[\frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6} \right]} dx \\ &= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2} \right) \sqrt{\frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}} dx \\ &= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2} \right) \sqrt{\left[\frac{1}{2}x^3 + \frac{1}{2}x^{-3} \right]^2} dx \\ &= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2} \right) \left[\frac{1}{2}x^3 + \frac{1}{2}x^{-3} \right] dx \\ &= 2\pi \int_1^2 \left(\frac{1}{16}x^7 + \frac{3}{16}x + \frac{1}{8}x^{-5} \right) dx \\ &= 2\pi \left[\frac{1}{128}x^8 + \frac{3}{32}x^2 - \frac{1}{32}x^{-4} \right]_1^2 \\ &= 2\pi \left[\frac{1}{128}(2)^8 + \frac{3}{32}(2)^2 - \frac{1}{32}(2)^{-4} \right] - 2\pi \left[\frac{1}{128} + \frac{3}{32} - \frac{1}{32} \right] \\ &= \boxed{\frac{1179\pi}{256}}. \end{aligned}$$

10. We use formula (3) with $f(x) = \sqrt{x} = x^{1/2}$ and $f'(x) = \frac{1}{2}x^{-1/2}$. Then

$$\begin{aligned} S &= 2\pi \int_1^9 x^{1/2} \sqrt{1 + \left[\frac{1}{2}x^{-1/2}\right]^2} dx = 2\pi \int_1^9 \sqrt{x} \sqrt{1 + \frac{1}{4}x^{-1}} dx \\ &= 2\pi \int_1^9 \sqrt{x + \frac{1}{4}} dx. \end{aligned}$$

Using the substitution $u = x + \frac{1}{4}$, then $du = dx$ and the new limits are $u = (1) + \frac{1}{4} = \frac{5}{4}$ to $u = (9) + \frac{1}{4} = \frac{37}{4}$, so we have

$$\begin{aligned} S &= 2\pi \int_{5/4}^{37/4} \sqrt{u} du = 2\pi \left[\frac{2}{3} u^{3/2} \right]_{5/4}^{37/4} \\ &= \boxed{2\pi \left(\frac{37^{3/2}}{12} - \frac{5^{3/2}}{12} \right)}. \end{aligned}$$

11. We use formula (3) with $f(x) = e^x$ and $f'(x) = e^x$. Then

$$S = 2\pi \int_0^1 e^x \sqrt{1 + [e^x]^2} dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} dx.$$

Using the substitution $u = e^x$, then $du = e^x dx$ and the new limits are $u = e^0 = 1$ to $u = e^1 = e$, so we have

$$S = 2\pi \int_1^e \sqrt{1 + u^2} du.$$

Using the Table of Integrals 49 with $a = 1$, we have

$$\begin{aligned} S &= 2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_1^e \\ &= 2\pi \left[\frac{e}{2} \sqrt{1 + e^2} + \frac{1}{2} \ln |e + \sqrt{1 + e^2}| \right] - 2\pi \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln |1 + \sqrt{2}| \right] \\ &= \boxed{\pi \left[e \sqrt{1 + e^2} - \sqrt{2} + \ln \left(\frac{e + \sqrt{1 + e^2}}{1 + \sqrt{2}} \right) \right]}. \end{aligned}$$

12. We use formula (3) with $f(x) = e^{-x}$ and $f'(x) = -e^{-x}$. Then

$$S = 2\pi \int_0^1 e^{-x} \sqrt{1 + [-e^{-x}]^2} dx = 2\pi \int_0^1 e^{-x} \sqrt{1 + e^{-2x}} dx.$$

Using the substitution $u = e^{-x}$, then $du = -e^{-x} dx$ and the new limits are $u = e^{-0} = 1$ to $u = e^{-1}$, so we have

$$S = 2\pi \int_1^{e^{-1}} -\sqrt{1 + u^2} du = -2\pi \int_1^{e^{-1}} \sqrt{1 + u^2} du.$$

Using the Table of Integrals 49 with $a = 1$, we have

$$\begin{aligned} S &= -2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_1^{e^{-1}} \\ &= -2\pi \left[\frac{e^{-1}}{2} \sqrt{1 + e^{-2}} + \frac{1}{2} \ln |e^{-1} + \sqrt{1 + e^{-2}}| \right] + 2\pi \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln |1 + \sqrt{2}| \right] \\ &= \boxed{-\pi \left[e^{-1} \sqrt{1 + e^{-2}} - \sqrt{2} + \ln \left(\frac{e^{-1} + \sqrt{1 + e^{-2}}}{1 + \sqrt{2}} \right) \right]}. \end{aligned}$$

13. We start by computing the derivative of $f(x) = \sqrt{a^2 - x^2} = (a^2 - x^2)^{1/2}$, namely $f'(x) = \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) = -x(a^2 - x^2)^{-1/2}$. Then

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left[-x(a^2 - x^2)^{-1/2} \right]^2 = 1 + x^2(a^2 - x^2)^{-1} \\ &= 1 + \frac{x^2}{a^2 - x^2} \\ &= \frac{a^2}{a^2 - x^2}. \end{aligned}$$

This means

$$f(x)\sqrt{1 + [f'(x)]^2} = (a^2 - x^2)^{1/2} \cdot \frac{a}{(a^2 - x^2)^{1/2}} = a.$$

Using formula (3)

$$\begin{aligned} S &= 2\pi \int_{-a}^a f(x)\sqrt{1 + [f'(x)]^2} dx \\ &= 2\pi \int_{-a}^a a dx = 2\pi [ax]_{-a}^a \\ &= \boxed{4\pi a^2}. \end{aligned}$$

[Note that $y = \sqrt{a^2 - x^2}$ from $-a \leq x \leq a$ rotated about the x -axis gives a sphere of radius a , so our answer gives the surface area of a sphere of radius a .]

14. The derivative of $f(x) = \frac{a}{2} (e^{x/a} + e^{-x/a})$ is $f'(x) = \frac{a}{2} \left(e^{x/a} \cdot \frac{1}{a} - e^{-x/a} \cdot \frac{1}{a} \right) = \frac{1}{2} (e^{x/a} - e^{-x/a})$.

Then

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left[\frac{1}{2} (e^{x/a} - e^{-x/a}) \right]^2 \\ &= 1 + \frac{1}{4} (e^{2x/a} - 2 + e^{-2x/a}) \\ &= \frac{1}{4} (e^{2x/a} + 2 + e^{-2x/a}) \\ &= \frac{1}{4} (e^{x/a} + e^{-x/a})^2. \end{aligned}$$

Using formula (3)

$$\begin{aligned} S &= 2\pi \int_0^a \frac{a}{2} (e^{x/a} + e^{-x/a}) \sqrt{\frac{1}{4} (e^{x/a} + e^{-x/a})^2} dx \\ &= \pi \int_0^a \frac{a}{2} (e^{x/a} + e^{-x/a})^2 dx \\ &= \frac{\pi a}{2} \int_0^a (e^{2x/a} + 2 + e^{-2x/a}) dx \\ &= \frac{\pi a}{2} \left[\frac{a}{2} e^{2x/a} + 2x - \frac{a}{2} e^{-2x/a} \right]_0^a \\ &= \frac{\pi a}{2} \left[\frac{a}{2} e^2 + 2a - \frac{a}{2} e^{-2} \right] - \frac{\pi a}{2} \left[\frac{a}{2} - \frac{a}{2} \right] \\ &= \boxed{\frac{\pi a^2}{4} (e^2 + 4 - e^{-2})}. \end{aligned}$$

15. We use the formula following Example 1 with $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$. Then

$$\begin{aligned} S &= 2\pi \int_0^1 3t^2 \sqrt{[6t]^2 + [6t^2]^2} dt = 2\pi \int_0^1 3t^2 \sqrt{36t^2 + 36t^4} dt \\ &= 2\pi \int_0^1 18t^3 \sqrt{1+t^2} dt = 36\pi \int_0^1 t^3 \sqrt{1+t^2} dt. \end{aligned}$$

Using the substitution $u = 1 + t^2$, we have $du = 2t dt$ and the new limits are $u = 1 + (0)^2 = 1$ to $u = 1 + (1)^2 = 2$. Solving for t^2 in the first equation, we have $t^2 = u - 1$ and so

$$(u - 1) du = (t^2) \cdot 2t dt = 2t^3 dt \text{ or } \frac{u-1}{2} du = t^3 dt.$$

With this substitution, our surface area integral becomes

$$\begin{aligned} S &= 18\pi \int_1^2 (u-1)\sqrt{u} du = 18\pi \int_1^2 (u^{3/2} - u^{1/2}) du \\ &= 18\pi \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^2 = 18\pi \left[\frac{2}{5}(2)^{5/2} - \frac{2}{3}(2)^{3/2} \right] - 18\pi \left[\frac{2}{5} - \frac{2}{3} \right] \\ &= 18\pi \left[\frac{8}{5}\sqrt{2} - \frac{4}{3}\sqrt{2} \right] + \frac{24}{5}\pi = \boxed{\frac{24}{5}\pi(\sqrt{2}+1)} \end{aligned}$$

16. We use the formula following Example 1 with $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2t$. Then

$$\begin{aligned} S &= 2\pi \int_0^3 (2t+1)\sqrt{[2]^2 + [2t]^2} dt = 2\pi \int_0^3 (2t+1)\sqrt{4(1+t^2)} dt = 4\pi \int_0^3 (2t+1)\sqrt{1+t^2} dt \\ &= 4\pi \int_0^3 2t\sqrt{1+t^2} dt + 4\pi \int_0^3 \sqrt{1+t^2} dt. \end{aligned}$$

To compute the first integral, use the substitution $u = 1 + t^2$. Then $du = 2t dt$ and the new limits are $u = 1 + (0)^2 = 1$ to $u = 1 + (3)^2 = 10$, so we have

$$\begin{aligned} 4\pi \int_0^3 2t\sqrt{1+t^2} dt &= 4\pi \int_1^{10} \sqrt{u} du = 4\pi \left[\frac{2}{3}u^{3/2} \right]_1^{10} \\ &= \frac{8}{3}\pi(10)^{3/2} - \frac{8\pi}{3}(1)^{3/2} = \frac{8}{3}\pi(10\sqrt{10} - 1). \end{aligned}$$

To compute the second integral, we use the Table of Integrals 49 with $a = 1$. Then

$$\begin{aligned} 4\pi \int_0^3 \sqrt{1+t^2} dt &= 4\pi \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln|t + \sqrt{1+t^2}| \right]_0^3 \\ &= 4\pi \left(\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln(3 + \sqrt{10}) \right) - 0. \end{aligned}$$

Putting these two integrals together,

$$\begin{aligned} S &= \frac{8}{3}\pi(10\sqrt{10} - 1) + 4\pi \left(\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln(3 + \sqrt{10}) \right) \\ &= \boxed{\frac{8}{3}\pi(10\sqrt{10} - 1) + 2\pi(3\sqrt{10} + \ln(3 + \sqrt{10}))}. \end{aligned}$$

17. We use the formula following Example 1 with $\frac{dx}{dt} = 2 \cos t$ and $\frac{dy}{dt} = -2 \sin t$. Then

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 2 \sin t \sqrt{[2 \cos t]^2 + [-2 \sin t]^2} dt = 8\pi \int_0^{\pi/2} \sin t dt \\ &= 8\pi [-\cos t]_0^{\pi/2} = 8\pi [0 - (-1)] \\ &= \boxed{8\pi}. \end{aligned}$$

18. We use the formula following Example 1 with $\frac{dx}{dt} = -3 \sin t$ and $\frac{dy}{dt} = 2 \cos t$. Then

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 3 \cos t \sqrt{[-3 \sin t]^2 + [2 \cos t]^2} dt = 6\pi \int_0^{\pi/2} \cos t \sqrt{9 \sin^2 t + 4 \cos^2 t} dt \\ &= 6\pi \int_0^{\pi/2} \cos t \sqrt{9 \sin^2 t + 4(1 - \sin^2 t)} dt = 6\pi \int_0^{\pi/2} \cos t \sqrt{5 \sin^2 t + 4} dt. \end{aligned}$$

Introduce the substitution $u = \sin t$, where $du = \cos t$ and the new limits are $u = \sin(0) = 0$ to $u = \sin(\pi/2) = 1$. Then

$$S = 6\pi \int_0^1 \sqrt{5u^2 + 4} du = 6\sqrt{5}\pi \int_0^1 \sqrt{u^2 + \frac{4}{5}} du.$$

Now we use the Table of Integrals 49 with $a = \frac{2}{\sqrt{5}}$. Then

$$\begin{aligned} S &= 6\sqrt{5}\pi \left[\frac{u}{2} \sqrt{u^2 + \frac{4}{5}} + \frac{4/5}{2} \ln \left| u + \sqrt{u^2 + \frac{4}{5}} \right| \right]_0^1 \\ &= 6\sqrt{5}\pi \left[\frac{1}{2} \frac{3}{\sqrt{5}} + \frac{2}{5} \ln \left| 1 + \frac{3}{\sqrt{5}} \right| \right] - 6\sqrt{5}\pi \left[0 + \frac{2}{5} \ln \left| \frac{2}{\sqrt{5}} \right| \right] \\ &= \boxed{9\pi + \frac{12\sqrt{5}\pi}{5} \ln \left(\frac{\sqrt{5} + 3}{2} \right)}. \end{aligned}$$

19. We can parametrize the given curve with

$$\begin{aligned} y(t) &= t \\ x(t) &= \frac{1}{4}t^2, \end{aligned}$$

for $0 \leq t \leq 2$. Using the formula following Example 1 with $\frac{dx}{dt} = \frac{1}{2}t$ and $\frac{dy}{dt} = 1$, the surface area is

$$S = 2\pi \int_0^2 \frac{1}{4}t^2 \sqrt{\left[\frac{1}{2}t\right]^2 + [1]^2} dt = 2\pi \int_0^2 \frac{1}{4}t^2 \sqrt{\frac{1}{4}t^2 + 1} dt.$$

Using the u -substitution $u = \frac{1}{4}t^2$, we have $du = \frac{1}{2}t dt$ which, using the u equation, can be rewritten as $dt = \frac{1}{\sqrt{u}} du$, and the new limits of integration are $u = \frac{1}{4}(0)^2 = 0$ to $u = \frac{1}{4}(2)^2 = 1$. Then the surface area integral becomes

$$\begin{aligned} S &= 2\pi \int_0^1 u \sqrt{1+u} \frac{1}{\sqrt{u}} du \\ &= 2\pi \int_0^1 \sqrt{u^2 + u} du \\ &= 2\pi \int_0^1 \sqrt{\left(u + \frac{1}{2}\right)^2 - \frac{1}{4}} du, \end{aligned}$$

where we completed the square in the last step. Now we make the substitution $w = u + \frac{1}{2}$, with $dw = du$ and new limits $w = (0) + \frac{1}{2} = \frac{1}{2}$ to $w = (1) + \frac{1}{2} = \frac{3}{2}$, and then

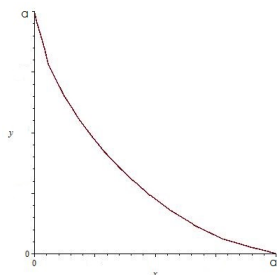
$$\begin{aligned}
 S &= 2\pi \int_{1/2}^{3/2} \sqrt{w^2 - \frac{1}{4}} dw \\
 &= 2\pi \left[\frac{w}{2} \sqrt{w^2 - \frac{1}{4}} - \frac{1/4}{2} \ln \left| w + \sqrt{w^2 - \frac{1}{4}} \right| \right]_{1/2}^{3/2} \\
 &= 2\pi \left(\frac{3}{4} \sqrt{2} - \frac{1}{8} \ln \left| \frac{3}{2} + \sqrt{2} \right| \right) - 2\pi \left(0 - \frac{1}{8} \ln \left| \frac{1}{2} \right| \right) \\
 &= \boxed{2\pi \left(\frac{3}{4} \sqrt{2} + \frac{1}{8} \ln \left(\frac{1}{3 + 2\sqrt{2}} \right) \right)},
 \end{aligned}$$

where we used the Table of Integrals 49 with $a = \frac{1}{2}$ to compute the integral.

20. Using Problem 78 from Section 9.1, we recognize that the given equation is a hypocycloid where $a = 4b$. Using the formula developed in Section 9.1 Problem 77 parametric equations for the given plane curve are

$$\begin{aligned}
 x(t) &= (4b - b) \cos t + b \cos \left(\frac{4b - b}{b} t \right) = 3b \cos t + b \cos(3t) \\
 y(t) &= (4b - b) \sin t - b \sin \left(\frac{4b - b}{b} t \right) = 3b \sin t - b \sin(3t),
 \end{aligned}$$

for $0 \leq t \leq 2\pi$. The inequalities $x \geq 0$ and $0 \leq y \leq a$ restrict our attention to the hypocycloid in the first quadrant, so $0 \leq t \leq \frac{\pi}{2}$. The following picture shows the hypocycloid in the first quadrant.



Using the formula following Example 1 with

$$\begin{aligned}
 \frac{dx}{dt} &= -3b \sin t - 3b \sin(3t) \\
 \frac{dy}{dt} &= 3b \cos t - 3b \cos(3t),
 \end{aligned}$$

the surface area S is

$$S = 2\pi \int_0^{\pi/2} (3b \cos t + b \cos(3t)) \sqrt{(-3b \sin t - 3b \sin(3t))^2 + (3b \cos t - 3b \cos(3t))^2} dt.$$

Using the triple angle formulas

$$\begin{aligned}
 \sin(3t) &= -4 \sin^3 t + 3 \sin t \\
 \cos(3t) &= 4 \cos^3 t - 3 \cos t,
 \end{aligned}$$

we have

$$\begin{aligned}
 S &= 2\pi \int_0^{\pi/2} (3b \cos t + 4b \cos^3 t - 3b \cos t) \sqrt{(-3b \sin t - 3b(-4 \sin^3 t + 3 \sin t))^2 + (3b \cos t - 3b(4 \cos^3 t - 3 \cos t))^2} dt \\
 &= 2\pi \int_0^{\pi/2} 4b \cos^3 t \sqrt{(-12b \sin t + 12b \sin^3 t)^2 + (12b \cos t - 12b \cos^3 t)^2} dt \\
 &= 2\pi \int_0^{\pi/2} 4b \cos^3 t \sqrt{(-12b \sin t)^2(1 - \sin^2 t)^2 + (12b \cos t)^2(1 - \cos^2 t)^2} dt \\
 &= 8\pi b \int_0^{\pi/2} \cos^3 t \sqrt{144b^2 \sin^2 t \cos^4 t + 144b^2 \cos^2 t \sin^4 t} dt \\
 &= 8\pi b \int_0^{\pi/2} \cos^3 t \sqrt{144b^2 \sin^2 t \cos^2 t} dt \\
 &= 8\pi b \int_0^{\pi/2} (12b \cos^4 t \sin t) dt \\
 &= 96b^2 \pi \int_0^{\pi/2} \cos^4 t \sin t dt.
 \end{aligned}$$

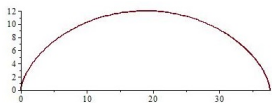
If we let $u = \cos t$, then $du = -\sin t dt$ and the new limits are $u = \cos(0) = 1$ to $u = \cos(\pi/2) = 0$, so the surface area integral becomes

$$\begin{aligned}
 S &= 96b^2 \pi \int_1^0 -u^4 du \\
 &= 96b^2 \pi \int_0^1 u^4 du \\
 &= \frac{96b^2 \pi}{5}.
 \end{aligned}$$

Finally, since $b = \frac{a}{4}$ we have

$$S = \boxed{\frac{6a^2 \pi}{5}}.$$

21. The graph of the given cycloid over one period $0 \leq t \leq 2\pi$ is



Using formula (2) with $\frac{dx}{dt} = 6(1 - \cos t)$ and $\frac{dy}{dt} = 6 \sin t$, the surface area S of the cycloid over the interval $0 \leq t \leq 2\pi$ is

$$\begin{aligned}
 S &= 2\pi \int_0^{2\pi} 6(1 - \cos t) \sqrt{[6(1 - \cos t)]^2 + [6 \sin t]^2} dt = 72\pi \int_0^{2\pi} (1 - \cos t) \sqrt{(1 - \cos t)^2 + \sin^2 t} dt \\
 &= 72\pi \int_0^{2\pi} (1 - \cos t) \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt = 72\pi \int_0^{2\pi} (1 - \cos t) \sqrt{2 - 2 \cos t} dt \\
 &= 72\sqrt{2}\pi \int_0^{2\pi} (1 - \cos t)^{3/2} dt.
 \end{aligned}$$

To compute this integral, we make the substitution $u = 1 - \cos t$, so $du = \sin t dt$, or $\frac{du}{\sin t} = dt$. Solving for $\cos t$ in the first equation and squaring both sides, we have $\cos^2 t = (u - 1)^2$ or, using the Pythagorean

Identity,

$$\begin{aligned} 1 - \sin^2 t &= (u - 1)^2 \\ \sin^2 t &= 2u - u^2 \\ \sin t &= \pm \sqrt{2u - u^2}. \end{aligned}$$

On the interval $[0, \pi]$ we have $\sin t \geq 0$ and on $[\pi, 2\pi]$, we have $\sin t \leq 0$, so to calculate our original integral, we will split it into two: one over $[0, \pi]$ and the other over $[\pi, 2\pi]$. Then

$$\int_0^{2\pi} (1 - \cos t)^{3/2} dt = \underbrace{\int_0^{\pi} (1 - \cos t)^{3/2} dt}_A + \underbrace{\int_{\pi}^{2\pi} (1 - \cos t)^{3/2} dt}_B.$$

Label the integral over $[0, \pi]$ A and the integral over $[\pi, 2\pi]$ B .

To calculate A , we use the substitution $u = 1 - \cos t$, so $\frac{du}{\sin t} = dt$, or $\frac{du}{\sqrt{2u - u^2}} = dt$ with new limits of integration $u = 1 - \cos(0) = 0$ to $u = 1 - \cos(\pi) = 2$. To calculate B , we use the substitution $u = 1 - \cos t$, so $\frac{du}{\sin t} = dt$, or $\frac{du}{-\sqrt{2u - u^2}} = dt$ with new limits of integration $u = 1 - \cos(\pi) = 2$ to $u = 1 - \cos(2\pi) = 0$. Then

$$\begin{aligned} A &= \int_0^2 u^{3/2} \cdot \frac{1}{\sqrt{2u - u^2}} du \\ B &= \int_2^0 u^{3/2} \frac{1}{-\sqrt{2u - u^2}} du, \end{aligned}$$

and we notice that $A = B$. To compute A , we have

$$A = \int_0^2 u^{3/2} \frac{du}{\sqrt{2u - u^2}} = \int_0^2 \frac{u^{3/2}}{\sqrt{u}\sqrt{2-u}} du = \int_0^2 \frac{u}{\sqrt{2-u}} du.$$

Now use the substitution $w = 2 - u$ where $dw = -du$ and the new limits of integration are $w = 2 - (0) = 2$ to $w = 2 - (2) = 0$. Then

$$\begin{aligned} A &= \int_2^0 -\frac{2-w}{w^{1/2}} dw = \int_0^2 (2w^{-1/2} - w^{1/2}) dw \\ &= \left[4w^{1/2} - \frac{2}{3}w^{3/2} \right]_0^2 = \left[4\sqrt{2} - \frac{4}{3}\sqrt{2} \right] \\ &= \frac{8\sqrt{2}}{3}. \end{aligned}$$

This means $A = B = \frac{8\sqrt{2}}{3}$ and if we put this altogether, we have

$$\begin{aligned} S &= 72\sqrt{2}\pi(A + B) = 72\sqrt{2}\pi \cdot \frac{16\sqrt{2}}{3} \\ &= \boxed{768\pi}. \end{aligned}$$

22. Using formula (3) with $f(x) = \ln x$ and $f'(x) = \frac{1}{x}$, the surface area S of the generated solid is

$$S = 2\pi \int_1^{10} \ln x \sqrt{1 + \left(\frac{1}{x}\right)^2} dx.$$

Using a CAS, we find that $\boxed{S \approx 90.157}$.

Applications and Extensions

23. (a) Using formula (3) with $f(x) = \frac{1}{x} = x^{-1}$ and $f'(x) = -x^{-2}$, the surface area of Gabriel's Horn is

$$S = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + [-x^{-2}]^2} dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Note that if $x \geq 1$, then $\sqrt{1 + \frac{1}{x^4}} \geq 1$ so

$$\frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \geq \frac{1}{x}.$$

This implies that

$$S \geq 2\pi \int_1^\infty \frac{1}{x} dx.$$

However we know this integral is divergent, so S must be infinite by the Comparison Test for Improper Integrals.

(b) To find the volume of Gabriel's horn, we will use the disk method. The radius of a slice for $x \geq 1$ is $\frac{1}{x}$. Since $x \rightarrow \infty$, the integral representing volume will be an improper integral. The volume V is

$$\begin{aligned} V &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx \\ &= \lim_{b \rightarrow \infty} \int_1^b \pi x^{-2} dx = \lim_{b \rightarrow \infty} [\pi (-x^{-1})]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\pi \cdot \frac{1}{b} + \pi\right] = \boxed{\pi}. \end{aligned}$$

24. Using formula (3) with $f(x) = e^{-x}$ and $f'(x) = -e^{-x}$ for $x \geq 0$ and noting that the integral representing surface area S is an improper integral, we find that

$$S = 2\pi \int_0^\infty e^{-x} \sqrt{1 + [-e^{-x}]^2} dx = \lim_{b \rightarrow \infty} 2\pi \int_0^b e^{-x} \sqrt{1 + e^{-2x}} dx.$$

Using the substitution $u = e^{-x}$ so $du = -e^{-x} dx$ and our new lower limit is $e^{-0} = 1$ and our new upper limit is e^{-b} , and we have

$$S = \lim_{b \rightarrow \infty} 2\pi \int_1^{e^{-b}} -\sqrt{1 + u^2} du = \lim_{b \rightarrow \infty} -2\pi \int_1^{e^{-b}} \sqrt{1 + u^2} du.$$

Using the Table of Integrals 49 with $a = 1$, we have

$$\begin{aligned} S &= \lim_{b \rightarrow \infty} -2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_1^{e^{-b}} \\ &= \lim_{b \rightarrow \infty} \left(-2\pi \left[\frac{e^{-b}}{2} \sqrt{1 + e^{-2b}} + \frac{1}{2} \ln |e^{-b} + \sqrt{1 + e^{-2b}}| \right] + 2\pi \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln |1 + \sqrt{2}| \right] \right). \end{aligned}$$

Noting that $\lim_{b \rightarrow \infty} e^{-b} = 0$, then

$$S = 2\pi \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln |1 + \sqrt{2}| \right] = \boxed{\pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}.$$

25. To find the surface area S of the catenoid, we use formula (3) with $f(x) = \cosh x$ and $f'(x) = \sinh x$ on the interval $a \leq x \leq b$. Then

$$S = 2\pi \int_a^b \cosh x \sqrt{1 + [\sinh x]^2} dx.$$

Using the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$, we can rewrite our integral for S .

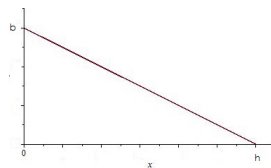
$$\begin{aligned} S &= 2\pi \int_a^b \cosh x \sqrt{\cosh^2 x} \, dx = 2\pi \int_a^b \cosh^2 x \, dx \\ &= 2\pi \left[\frac{\sinh(2x)}{4} + \frac{x}{2} \right]_a^b = 2\pi \left[\frac{\sinh(2b)}{4} + \frac{b}{2} \right] - 2\pi \left[\frac{\sinh(2a)}{4} + \frac{a}{2} \right] \\ &= \boxed{\frac{\pi}{2} [\sinh(2b) - \sinh(2a)] + \pi(b-a)}, \end{aligned}$$

where we used the Table of Integrals 138 for $\int \cosh^2 x \, dx$.

26. The surface area of a sphere is found by taking the top half of a circle of radius R , parametrized by the equations $x(t) = R \cos t$ and $y(t) = R \sin t$ for $0 \leq t \leq \pi$, and rotating it about the x -axis. Using formula (2), we find that the surface area S of the sphere is

$$\begin{aligned} S &= 2\pi \int_0^\pi R \sin t \sqrt{(-R \sin t)^2 + (R \cos t)^2} \, dt = 2\pi \int_0^\pi R^2 \sin t \, dt \\ &= 2\pi R^2 [-\cos t]_0^\pi = \boxed{4\pi R^2}. \end{aligned}$$

27. A right circular cone can be created by taking a line with negative slope in the first quadrant and rotating it about the x -axis. Consider the following picture of the line through the points $(0, b)$ and $(h, 0)$ where $b, h > 0$.



The line through these points is $y = -\frac{b}{h}x + b$. If we rotate this line from $0 \leq x \leq h$ about the x -axis, we will have a right circular cone with altitude h and base of radius b . Using formula (3) with $f(x) = -\frac{b}{h}x + b$ and $f'(x) = -\frac{b}{h}$, the surface area S of a right circular cone is

$$\begin{aligned} S &= 2\pi \int_0^h \left(-\frac{b}{h}x + b \right) \sqrt{1 + \left(-\frac{b}{h} \right)^2} \, dx \\ &= 2\pi \sqrt{1 + \frac{b^2}{h^2}} \int_0^h \left(-\frac{b}{h}x + b \right) \, dx \\ &= 2\pi \sqrt{1 + \frac{b^2}{h^2}} \left[-\frac{b}{2h}x^2 + bx \right]_0^h \\ &= 2\pi \sqrt{1 + \frac{b^2}{h^2}} \left(\left[-\frac{b}{2h} \cdot (h)^2 + bh \right] - [0] \right) \\ &= 2\pi \sqrt{1 + \frac{b^2}{h^2}} \left(\frac{1}{2}bh \right) = \pi b \sqrt{h^2 + b^2} \\ &= \boxed{\pi b \sqrt{h^2 + b^2}}. \end{aligned}$$

Challenge Problems

28. To create the reflector, we rotate the top part of a parabola $x = cy^2$ about the x -axis for $0 \leq x \leq \frac{1}{4}$ where c is some positive constant. We are given that the reflector is 1 meter tall when it is $\frac{1}{4}$ meters deep.

These specifications mean that the points $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{1}{4}, -\frac{1}{2}\right)$ lie on the parabola $x = cy^2$. To find c , we substitute the first point in.

$$\begin{aligned} x &= cy^2 \\ \frac{1}{4} &= c\left(\frac{1}{2}\right)^2 \\ c &= 1. \end{aligned}$$

Solving the equation $x = y^2$ for y , we have $y = \sqrt{x}$. Using formula (3) to find the surface area S of rotating $y = \sqrt{x}$ about the x -axis from $0 \leq x \leq \frac{1}{4}$, we find that

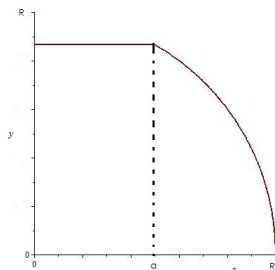
$$\begin{aligned} S &= 2\pi \int_0^{1/4} \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx = 2\pi \int_0^{1/4} \sqrt{x \left(1 + \frac{1}{4x}\right)} dx \\ &= 2\pi \int_0^{1/4} \frac{1}{2} \sqrt{4x + 1} dx. \end{aligned}$$

Using the substitution $u = 4x + 1$, with $du = 4 dx$ and the new limits $u = 4(0) + 1 = 1$ to $u = 4\left(\frac{1}{4}\right) + 1 = 2$, we have

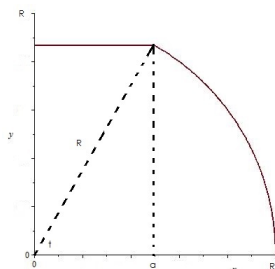
$$\begin{aligned} S &= \frac{\pi}{4} \int_1^2 \sqrt{u} du = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_1^2 \\ &= \frac{\pi}{6} (2\sqrt{2} - 1). \end{aligned}$$

Then the reflector has surface area $\boxed{\frac{\pi}{6} (2\sqrt{2} - 1) \approx 0.957}$ square meters.

29. Utilizing symmetry, we will find the surface area of the bead on the top hemisphere of the bead, and then double it to find the full surface area. Consider the following picture.



To find the surface area of the top half of the bead, we can take the top half of a circle of radius R centered at the origin and rotate it about the y -axis for $a \leq x \leq R$. Instead of working with the equation of a circle in the form $y = f(x)$ to rotate, we will parametrize it using $x(t) = R \cos t$ and $y(t) = R \sin t$. Before calculating the surface area, we must find the bounds on t . Consider now the following picture.



The piece of the circle we will rotate about the y -axis ranges from $t = 0$ to the value of t satisfying $\cos t = \frac{a}{R}$, or equivalently $\sin t = \frac{\sqrt{R^2 - a^2}}{R}$ which means $t = \sin^{-1} \left(\frac{\sqrt{R^2 - a^2}}{R} \right)$. Using the formula after Example 1, we find that the surface area S of the top half of the bead is

$$\begin{aligned} S &= 2\pi \int_0^{\sin^{-1} \left(\frac{\sqrt{R^2 - a^2}}{R} \right)} R \cos t \sqrt{(-R \sin t)^2 + (R \cos t)^2} dt = 2\pi \int_0^{\sin^{-1} \left(\frac{\sqrt{R^2 - a^2}}{R} \right)} R^2 \cos t dt \\ &= 2\pi R^2 [\sin t]_0^{\sin^{-1} \left(\frac{\sqrt{R^2 - a^2}}{R} \right)} = 2\pi R^2 \sin \left(\sin^{-1} \left(\frac{\sqrt{R^2 - a^2}}{R} \right) \right) - 2\pi R^2 \sin(0) \\ &= 2\pi R^2 \left(\frac{\sqrt{R^2 - a^2}}{R} \right). \end{aligned}$$

Finally, we find that the surface area of the bead is

$$4\pi R^2 \left(\frac{\sqrt{R^2 - a^2}}{R} \right) = \boxed{4\pi R \sqrt{R^2 - a^2}}.$$

30. Notice that the surface area of the plug is the surface area of the full sphere minus the surface area we found in Problem 29. In Problem 26, we found that the surface area of a sphere of radius R is $4\pi R^2$. This means that the surface area of the plug is

$$4\pi R^2 - 4\pi R^2 \left(\frac{\sqrt{R^2 - a^2}}{R} \right) = \boxed{4\pi R^2 \left(1 - \frac{\sqrt{R^2 - a^2}}{R} \right)}.$$

9.4 Polar Coordinates

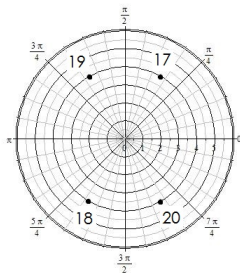
Concepts and Vocabulary

1. In a polar coordinate system, the origin is called the **pole**, and the **polar axis** coincides with the positive x -axis of the rectangular coordinate system.
2. **False**. The point $\left(2, \frac{\pi}{3}\right)$ is in the first quadrant whereas the point $\left(2, \frac{4\pi}{3}\right)$ is in the third quadrant.
3. **False**. A point in polar coordinates has infinitely many representations in polar coordinates by adding multiples of 2π to the angle to obtain a new, equivalent point.
4. **True**. Every point in rectangular coordinates has a unique representation.
5. **True**. If $r < 0$, then we move r units from the origin in the opposite direction of θ to plot the point.
6. **True**.
7. To convert the point (r, θ) in polar coordinates to a point (x, y) in rectangular coordinates, use the formulas $x = r \cos \theta$ and $y = r \sin \theta$.
8. An equation whose variables are polar coordinates is called a **polar equation**.

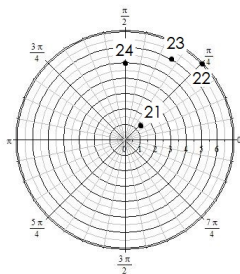
Skill Building

9. A
10. B
11. C
12. C
13. B
14. D
15. A
16. D

For Problems 17-20, refer to the following graph.



For Problems 21-24, refer to the following graph.



For Problems 25-32, we use the facts that $(r, \theta) = (-r, \theta \pm \pi)$ and $(r, \theta) = (r, \theta \pm 2\pi k)$ for all non-negative integers k .

25.

$$(a) \left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} - 2\pi\right) = \left(5, -\frac{4\pi}{3}\right)$$

$$(b) \left(5, \frac{2\pi}{3}\right) = \left(-5, \frac{2\pi}{3} + \pi\right) = \left(-5, \frac{5\pi}{3}\right)$$

$$(c) \left(5, \frac{2\pi}{3}\right) = \left(5, \frac{2\pi}{3} + 2\pi\right) = \boxed{\left(5, \frac{8\pi}{3}\right)}$$

26.

$$(a) \left(4, \frac{3\pi}{4}\right) = \left(4, \frac{3\pi}{4} - 2\pi\right) = \boxed{\left(4, -\frac{5\pi}{4}\right)}$$

$$(b) \left(4, \frac{3\pi}{4}\right) = \left(-4, \frac{3\pi}{4} + \pi\right) = \boxed{\left(-4, \frac{7\pi}{4}\right)}$$

$$(c) \left(4, \frac{3\pi}{4}\right) = \left(4, \frac{3\pi}{4} + 2\pi\right) = \boxed{\left(4, \frac{11\pi}{4}\right)}$$

27.

$$(a) (-2, 3\pi) = (2, 3\pi - \pi) = (2, 2\pi) = (2, 2\pi - 4\pi) = \boxed{(2, -2\pi)}$$

$$(b) (-2, 3\pi) = (-2, 3\pi - 2\pi) = \boxed{(-2, \pi)}$$

$$(c) (-2, 3\pi) = (2, 3\pi - \pi) = \boxed{(2, 2\pi)}$$

28.

$$(a) (-3, 4\pi) = (3, 4\pi - \pi) = (3, 3\pi) = (3, 3\pi - 4\pi) = \boxed{(3, -\pi)}$$

$$(b) (-3, 4\pi) = (-3, 4\pi - 4\pi) = \boxed{(-3, 0)}$$

$$(c) (-3, 4\pi) = (3, 4\pi - \pi) = \boxed{(3, 3\pi)}$$

29.

$$(a) \left(1, \frac{\pi}{2}\right) = \left(1, \frac{\pi}{2} - 2\pi\right) = \boxed{\left(1, -\frac{3\pi}{2}\right)}$$

$$(b) \left(1, \frac{\pi}{2}\right) = \left(-1, \frac{\pi}{2} + \pi\right) = \boxed{\left(-1, \frac{3\pi}{2}\right)}$$

$$(c) \left(1, \frac{\pi}{2}\right) = \left(1, \frac{\pi}{2} + 2\pi\right) = \boxed{\left(1, \frac{5\pi}{2}\right)}$$

30.

$$(a) (2, \pi) = (2, \pi - 2\pi) = \boxed{(2, -\pi)}$$

$$(b) (2, \pi) = (-2, \pi - \pi) = \boxed{(-2, 0)}$$

$$(c) (2, \pi) = (2, \pi + 2\pi) = \boxed{(2, 3\pi)}$$

31.

$$(a) \left(-3, -\frac{\pi}{4}\right) = \left(3, -\frac{\pi}{4} - \pi\right) = \boxed{\left(3, -\frac{5\pi}{4}\right)}$$

$$(b) \left(-3, -\frac{\pi}{4}\right) = \left(-3, -\frac{\pi}{4} + 2\pi\right) = \boxed{\left(-3, \frac{7\pi}{4}\right)}$$

$$(c) \left(-3, -\frac{\pi}{4}\right) = \left(3, -\frac{\pi}{4} + \pi\right) = \left(3, \frac{3\pi}{4}\right) = \left(3, \frac{3\pi}{4} + 2\pi\right) = \boxed{\left(3, \frac{11\pi}{4}\right)}$$

32.

$$(a) \left(-2, -\frac{2\pi}{3}\right) = \left(2, -\frac{2\pi}{3} - \pi\right) = \boxed{\left(2, -\frac{5\pi}{3}\right)}$$

$$(b) \left(-2, -\frac{2\pi}{3}\right) = \left(-2, -\frac{2\pi}{3} + 2\pi\right) = \boxed{\left(-2, \frac{4\pi}{3}\right)}$$

$$(c) \left(-2, -\frac{2\pi}{3}\right) = \left(2, -\frac{2\pi}{3} + \pi\right) = \left(2, \frac{\pi}{3}\right) = \left(2, \frac{\pi}{3} + 2\pi\right) = \boxed{\left(2, \frac{7\pi}{3}\right)}$$

33.

$$\begin{aligned}x &= r \cos \theta = 6 \cos \left(\frac{\pi}{6} \right) = 3\sqrt{3} \\y &= r \sin \theta = 6 \sin \left(\frac{\pi}{6} \right) = 3\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(3\sqrt{3}, 3)}$.

34.

$$\begin{aligned}x &= r \cos \theta = -6 \cos \left(\frac{\pi}{6} \right) = -3\sqrt{3} \\y &= r \sin \theta = -6 \sin \left(\frac{\pi}{6} \right) = -3\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(-3\sqrt{3}, -3)}$.

35.

$$\begin{aligned}x &= r \cos \theta = -6 \cos \left(-\frac{\pi}{6} \right) = -3\sqrt{3} \\y &= r \sin \theta = -6 \sin \left(-\frac{\pi}{6} \right) = 3\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(-3\sqrt{3}, 3)}$.

36.

$$\begin{aligned}x &= r \cos \theta = 6 \cos \left(-\frac{\pi}{6} \right) = 3\sqrt{3} \\y &= r \sin \theta = 6 \sin \left(-\frac{\pi}{6} \right) = -3\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(3\sqrt{3}, -3)}$.

37.

$$\begin{aligned}x &= r \cos \theta = 5 \cos \left(\frac{\pi}{2} \right) = 0 \\y &= r \sin \theta = 5 \sin \left(\frac{\pi}{2} \right) = 5\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(0, 5)}$.

38.

$$\begin{aligned}x &= r \cos \theta = 8 \cos \left(\frac{\pi}{4} \right) = 4\sqrt{2} \\y &= r \sin \theta = 8 \sin \left(\frac{\pi}{4} \right) = 4\sqrt{2}\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(4\sqrt{2}, 4\sqrt{2})}$.

39.

$$\begin{aligned}x &= r \cos \theta = 2\sqrt{2} \cos \left(-\frac{\pi}{4} \right) = 2 \\y &= r \sin \theta = 2\sqrt{2} \sin \left(-\frac{\pi}{4} \right) = -2\end{aligned}$$

The rectangular coordinates of the point are $\boxed{(2, -2)}$.

40.

$$\begin{aligned}x &= r \cos \theta = -5 \cos \left(-\frac{\pi}{3}\right) = -\frac{5}{2} \\y &= r \sin \theta = -5 \sin \left(-\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{2}\end{aligned}$$

The rectangular coordinates of the point are $\boxed{\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)}$.

For Problems 41-45, the graph of these points follows Problem 45.

41. The point $(5, 0)$ is located on the polar axis $\theta = 0$, 5 units from the origin, so the polar coordinates of the point are $\boxed{(5, 0)}$.

42. The point $(2, -2)$ is in the fourth quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2)^2} = 2\sqrt{2}$ from the origin. The angle θ satisfies $\frac{3\pi}{2} \leq \theta < 2\pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi = \tan^{-1}\left(\frac{-2}{2}\right) + 2\pi = \tan^{-1}(-1) + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}.$$

The polar coordinates of the point are $\boxed{\left(2\sqrt{2}, \frac{7\pi}{4}\right)}$.

43. The point $(-2, 2)$ is in the second quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$ from the origin. The angle θ satisfies $\frac{\pi}{2} \leq \theta \leq \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{2}{-2}\right) + \pi = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

The polar coordinates of the point are $\boxed{\left(2\sqrt{2}, \frac{3\pi}{4}\right)}$.

44. The point $(-2, -2\sqrt{3})$ is in the third quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$ from the origin. The angle θ satisfies $\pi \leq \theta \leq \frac{3\pi}{2}$ where

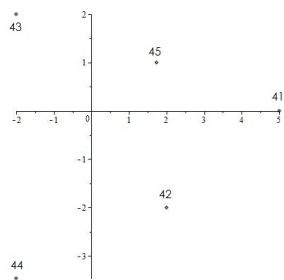
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) + \pi = \tan^{-1}(\sqrt{3}) + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}.$$

The polar coordinates of the point are $\boxed{\left(4, \frac{4\pi}{3}\right)}$.

45. The point $(\sqrt{3}, 1)$ is in the first quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ from the origin. The angle θ satisfies $0 \leq \theta \leq \frac{\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

The polar coordinates of the point are $\boxed{\left(2, \frac{\pi}{6}\right)}$.



For Problems 46-50, the graph of these points follows Problem 50.

46. The point $(0, -3)$ is on the negative y -axis three units down. So $r = 3$ and $\theta = \frac{3\pi}{2}$. The polar coordinates of the point are $\left(3, \frac{3\pi}{2}\right)$.

47. The point $(-\sqrt{3}, 1)$ is in the second quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = 2$ from the origin. The angle θ satisfies $\frac{\pi}{2} \leq \theta \leq \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

The polar coordinates of the point are $\left(2, \frac{5\pi}{6}\right)$.

48. The point $(3\sqrt{2}, -3\sqrt{2})$ is in the fourth quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(3\sqrt{2})^2 + (-3\sqrt{2})^2} = 6$ from the origin. The angle θ satisfies $\frac{3\pi}{2} \leq \theta < 2\pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi = \tan^{-1}\left(\frac{-3\sqrt{2}}{3\sqrt{2}}\right) + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}.$$

The polar coordinates of the point are $\left(6, \frac{7\pi}{4}\right)$.

49. The point $(3, 2)$ is in the first quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$ from the origin. The angle θ satisfies $0 \leq \theta \leq \frac{\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{3}\right).$$

Using a calculator $\tan^{-1}\left(\frac{2}{3}\right) \approx 0.588$ radians. The polar coordinates of the point are

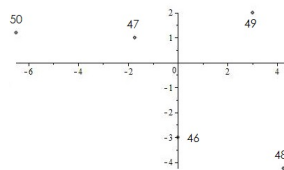
$$\left(\sqrt{13}, \tan^{-1}\left(\frac{2}{3}\right)\right) \approx (\sqrt{13}, 0.588).$$

50. The point $(-6.5, 1.2)$ is in the second quadrant a distance of $r = \sqrt{x^2 + y^2} = \sqrt{(-6.5)^2 + (1.2)^2} = \sqrt{43.69}$ from the origin. The angle θ satisfies $\frac{\pi}{2} \leq \theta \leq \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{1.2}{-6.5}\right) + \pi.$$

Using a calculator $\tan^{-1}\left(\frac{1.2}{-6.5}\right) + \pi \approx 2.959$ radians. The polar coordinates of the point are

$$\left(\sqrt{43.69}, \tan^{-1} \left(\frac{1.2}{-6.5} \right) + \pi \right) \approx \left(\sqrt{43.69}, 2.959 \right).$$



51. If r is fixed at 2, then the set of points are all a distance of 2 units from the origin, forming a circle of radius 2 centered at the origin, which matches graph $\boxed{(E)}$.

52. If θ is fixed at $\frac{\pi}{4}$ and r is allowed to vary, the result is a line containing the pole, making an angle of $\frac{\pi}{4}$ with the polar axis. Such a line has slope $\tan \frac{\pi}{4} = 1$, which has rectangular equation $y = x$ and matches graph $\boxed{(A)}$.

53. If we multiply both sides by r , the equation becomes $r^2 = 2r \cos \theta$. Now use the formulas $r^2 = x^2 + y^2$ and $x = r \cos \theta$ to convert the equation to rectangular coordinates.

$$\begin{aligned} r^2 &= 2r \cos \theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ (x - 1)^2 + y^2 &= 1 \end{aligned}$$

This is a circle of radius 1 centered at $(1, 0)$, which matches graph $\boxed{(F)}$.

54. Using the formula $x = r \cos \theta$, the polar equation becomes $x = 2$, which matches graph $\boxed{(B)}$.

55. If we multiply both sides by r , the equation becomes $r^2 = -2r \cos \theta$. Now use the formulas $r^2 = x^2 + y^2$ and $x = r \cos \theta$ to convert the equation to rectangular coordinates.

$$\begin{aligned} r^2 &= -2r \cos \theta \\ x^2 + y^2 &= -2x \\ x^2 + 2x + y^2 &= 0 \\ (x + 1)^2 + y^2 &= 1 \end{aligned}$$

This is a circle of radius 1 centered at $(-1, 0)$, which matches graph $\boxed{(H)}$.

56. If we multiply both sides by r , the equation becomes $r^2 = 2r \sin \theta$. Now use the formulas $r^2 = x^2 + y^2$ and $y = r \sin \theta$ to convert the equation to rectangular coordinates.

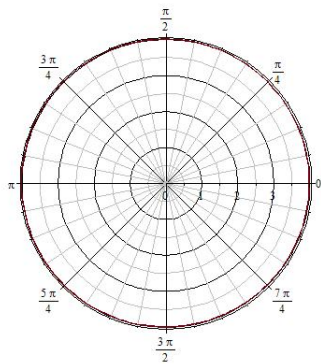
$$\begin{aligned} r^2 &= 2r \sin \theta \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + (y - 1)^2 &= 1 \end{aligned}$$

This is a circle of radius 1 centered at $(0, 1)$, which matches graph $\boxed{(G)}$.

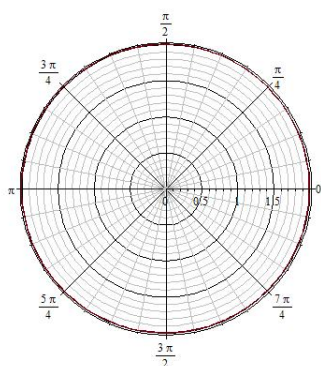
57. If θ is fixed at $\frac{3\pi}{4}$ and r is allowed to vary, the result is a line containing the pole, making an angle of $\frac{3\pi}{4}$ with the polar axis. Such a line has slope $\tan \frac{3\pi}{4} = -1$, which has rectangular equation $y = -x$ and matches graph $\boxed{(D)}$.

58. Using the formula $y = r \sin \theta$, the polar equation becomes $y = 2$, which matches graph $\boxed{(C)}$.

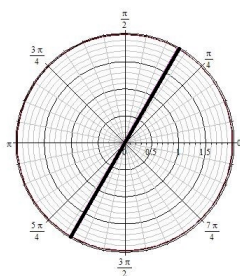
59. $r = 4$ is a circle centered at $(0, 0)$ of radius 4.



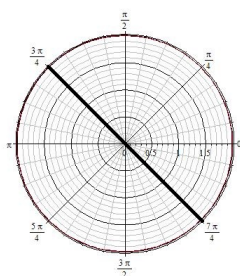
60. $r = 2$ is a circle centered at $(0, 0)$ of radius 2.



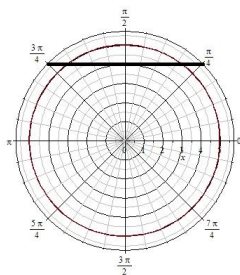
61. $\theta = \frac{\pi}{3}$ is a line through the origin making an angle of $\frac{\pi}{3}$ with the positive x -axis.



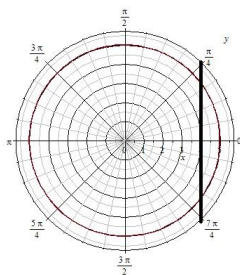
62. $\theta = -\frac{\pi}{4}$ is a line through the origin making an angle of $-\frac{\pi}{4}$ with the positive x -axis. The angle $-\frac{\pi}{4}$ creates the same angle as $\frac{7\pi}{4}$ with the positive x -axis.



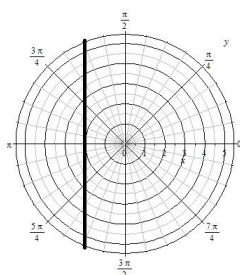
63. $r \sin \theta = 4$ is the equivalent of $y = 4$.



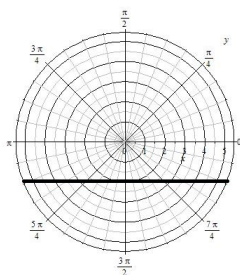
64. $r \cos \theta = 4$ is the equivalent of $x = 4$.



65. $r \cos \theta = -2$ is the equivalent of $x = -2$.



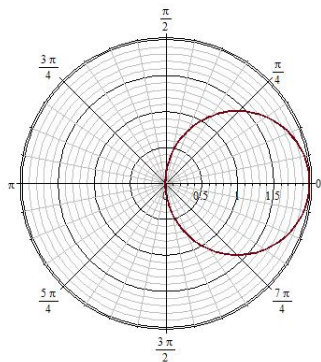
66. $r \sin \theta = -2$ is the equivalent of $y = -2$.



67. If we multiply both sides of the equation by r , then we have $r^2 = 2r \cos \theta$. Since $r^2 = x^2 + y^2$ and $x = r \cos \theta$, the equation becomes $x^2 + y^2 = 2x$. By moving the $2x$ term to the left side and completing the square, we have

$$(x - 1)^2 + y^2 = 1,$$

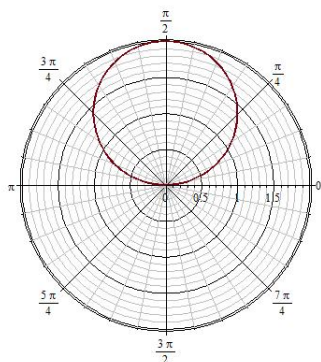
which is a circle centered at $(1, 0)$ with radius 1.



68. If we multiply both sides of the equation by r , then we have $r^2 = 2r \sin \theta$. Since $r^2 = x^2 + y^2$ and $y = r \sin \theta$, the equation becomes $x^2 + y^2 = 2y$. By moving the $2y$ term to the left side and completing the square, we have

$$x^2 + (y - 1)^2 = 1,$$

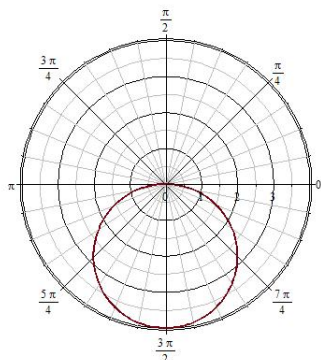
which is a circle centered at $(0, 1)$ with radius 1.



69. If we multiply both sides of the equation by r , then we have $r^2 = -4r \sin \theta$. Since $r^2 = x^2 + y^2$ and $y = r \sin \theta$, the equation becomes $x^2 + y^2 = -4y$. By moving the $-4y$ term to the left side and completing the square, we have

$$x^2 + (y + 2)^2 = 4,$$

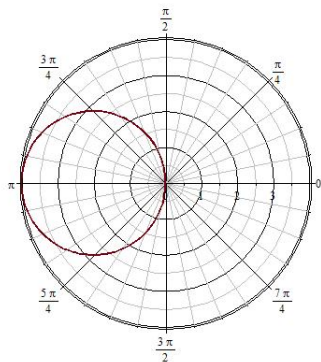
which is a circle centered at $(0, -2)$ with radius 2.



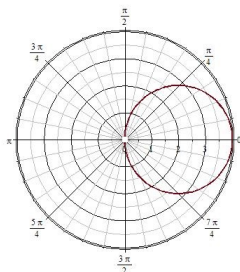
70. If we multiply both sides of the equation by r , then we have $r^2 = -4r \cos \theta$. Since $r^2 = x^2 + y^2$ and $x = r \cos \theta$, the equation becomes $x^2 + y^2 = -4x$. By moving the $-4x$ term to the left side and completing the square, we have

$$(x + 2)^2 + y^2 = 4,$$

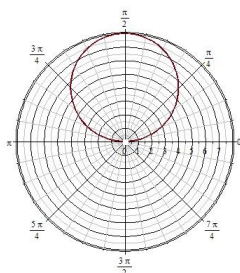
which is a circle centered at $(-2, 0)$ with radius 2.



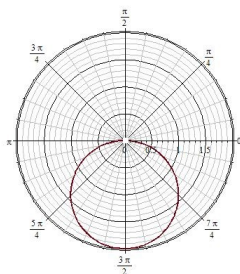
- 71.** The polar equation $r \sec \theta = 4$ can be rewritten as $r = 4 \cos \theta$ when $\cos \theta \neq 0$. The θ values where $\cos \theta = 0$ are $\theta = \frac{\pi}{2} + k\pi$ for some integer k . This is a circle centered at $(2, 0)$ with radius 2, excluding the points where $\theta = \frac{\pi}{2} + k\pi$ which is the pole.



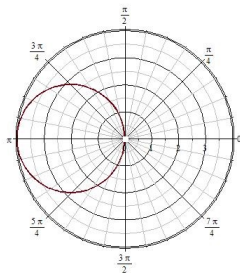
- 72.** The polar equation $r \csc \theta = 8$ can be rewritten as $r = 8 \sin \theta$ when $\sin \theta \neq 0$. The θ values where $\sin \theta = 0$ are $\theta = k\pi$ for some integer k . This is a circle centered at $(0, 4)$ with radius 4, excluding the points where $\theta = k\pi$ which is the pole.



- 73.** The polar equation $r \csc \theta = -2$ can be rewritten as $r = -2 \sin \theta$ when $\sin \theta \neq 0$. The θ values where $\sin \theta = 0$ are $\theta = k\pi$ for some integer k . This is a circle centered at $(0, -1)$ with radius 1, excluding the points where $\theta = k\pi$ which is the pole.



- 74.** The polar equation $r \sec \theta = -4$ can be rewritten as $r = -4 \cos \theta$ when $\cos \theta \neq 0$. The θ values where $\cos \theta = 0$ are $\theta = \frac{\pi}{2} + k\pi$ for some integer k . This is a circle centered at $(-2, 0)$ with radius 2, excluding the points where $\theta = \frac{\pi}{2} + k\pi$ which is the pole.



75. Substituting $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} \frac{(r \cos \theta)^2}{4} + \frac{(r \sin \theta)^2}{9} &= 1 \\ 9r^2 \cos^2 \theta + 4r^2 \sin^2 \theta &= 36 \\ r^2 &= \frac{36}{9 \cos^2 \theta + 4 \sin^2 \theta} \\ r &= 6 \frac{\sqrt{9 \cos^2 \theta + 4 \sin^2 \theta}}{9 \cos^2 \theta + 4 \sin^2 \theta}. \end{aligned}$$

76. Substituting $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} r \cos \theta - 4r \sin \theta &= -4 \\ r &= -\frac{4}{\cos \theta - 4 \sin \theta}. \end{aligned}$$

77. Substituting $r^2 = x^2 + y^2$ and $x = r \cos \theta$, the equation becomes

$$\begin{aligned} r^2 - 4r \cos \theta &= 0 \\ r &= 4 \cos \theta. \end{aligned}$$

78. Substituting $y = r \sin \theta$, the equation becomes

$$r \sin \theta = -6.$$

79. Substituting $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} (r \cos \theta)^2 &= 1 - 4r \sin \theta \\ r^2 \cos^2 \theta + 4 \cos \theta - 1 &= 0. \end{aligned}$$

80. Substituting $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$(r \sin \theta)^2 = 1 - 4r \cos \theta.$$

81. Substituting $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} (r \cos \theta)(r \sin \theta) &= 1 \\ r^2 &= \frac{1}{\cos \theta \sin \theta} \\ r &= \frac{\sqrt{\cos \theta \sin \theta}}{\cos \theta \sin \theta}. \end{aligned}$$

82. Substituting $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} r^2 &= 2r \cos \theta - 4r \sin \theta \\ r &= 2 \cos \theta - 4 \sin \theta. \end{aligned}$$

83. Multiplying both sides by r , we have $r^2 = r \cos \theta$. Using $r^2 = x^2 + y^2$ and $x = r \cos \theta$, the equation becomes

$$\begin{aligned} x^2 + y^2 &= x \\ \left(x - \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}, \end{aligned}$$

where we completed the square in the final step.

84. First we rewrite the equation

$$\begin{aligned} r - \cos \theta &= 2 \\ r^2 - r \cos \theta &= 2r. \end{aligned}$$

Now we use the equations $r^2 = x^2 + y^2$ and $x = r \cos \theta$.

$$x^2 + y^2 - x = 2\sqrt{x^2 + y^2}.$$

85. Multiply both sides by r and use the equations $r = \sqrt{x^2 + y^2}$ and $y = r \sin \theta$.

$$\begin{aligned} r^3 &= r \sin \theta \\ \left(\sqrt{x^2 + y^2}\right)^3 &= y \\ (x^2 + y^2)^{3/2} &= y. \end{aligned}$$

86. Multiply both sides by r and use the equations $r = \sqrt{x^2 + y^2}$ and $y = r \sin \theta$.

$$\begin{aligned} r^3 &= r - r \sin \theta \\ \left(\sqrt{x^2 + y^2}\right)^3 &= \sqrt{x^2 + y^2} - y. \end{aligned}$$

87. Multiply both sides by $1 - \cos \theta$ and use the equations $r = \sqrt{x^2 + y^2}$ and $x = r \cos \theta$.

$$\begin{aligned} r - r \cos \theta &= 4 \\ \sqrt{x^2 + y^2} - x &= 4. \end{aligned}$$

88. Multiply both sides by $3 - \cos \theta$ and use the equations $r = \sqrt{x^2 + y^2}$ and $x = r \cos \theta$.

$$\begin{aligned} 3r - r \cos \theta &= 3 \\ 3\sqrt{x^2 + y^2} - x &= 3. \end{aligned}$$

89. Substituting $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned} x^2 + y^2 &= \tan^{-1}\left(\frac{y}{x}\right) \\ \tan(x^2 + y^2) &= \frac{y}{x} \\ y &= x \tan(x^2 + y^2). \end{aligned}$$

90. Substituting $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned}\tan^{-1}\left(\frac{y}{x}\right) &= -\frac{\pi}{4} \\ \frac{y}{x} &= \tan\left(-\frac{\pi}{4}\right) \\ y &= -1 \cdot x \\ \boxed{y} &= \boxed{-x}.\end{aligned}$$

91. Substituting $r = \sqrt{x^2 + y^2}$ the equation becomes

$$\begin{aligned}\sqrt{x^2 + y^2} &= 2 \\ \boxed{y\sqrt{4 - x^2}} &.\end{aligned}$$

92. Substituting $r = \sqrt{x^2 + y^2}$ the equation becomes

$$\begin{aligned}\sqrt{x^2 + y^2} &= -5 \\ \boxed{x^2 + y^2} &= \boxed{25}.\end{aligned}$$

93. Substituting $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned}\tan\left(\tan^{-1}\left(\frac{y}{x}\right)\right) &= 4 \\ \frac{y}{x} &= 4 \\ \boxed{y} &= \boxed{4x}.\end{aligned}$$

94. Rewriting the equation as $\frac{1}{\tan \theta} = 3$ and substituting $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ the equation becomes

$$\begin{aligned}\frac{1}{\tan\left(\tan^{-1}\left(\frac{y}{x}\right)\right)} &= 3 \\ \frac{1}{y/x} &= 3 \\ y &= \frac{1}{3}x.\end{aligned}$$

Applications and Extensions

95. (a) Wrigley Field resides at the point $\boxed{(-10, 36)}$ in rectangular coordinates.

(b) The point $(-10, 36)$ lies in the second quadrant so the angle θ satisfies $\frac{\pi}{2} \leq \theta \leq \pi$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{36}{-10}\right) + \pi \approx 1.842 \text{ radians.}$$

Wrigley Field is $r = \sqrt{(-10)^2 + (36)^2} = \sqrt{1396} = 2\sqrt{349} \approx 37.363$ blocks from the intersection of Madison and State Streets, so in polar coordinates, Wrigley Field is located approximately at $\boxed{(37.363, 1.842)}$.

(c) U.S. Cellular Field resides at the point $\boxed{(-3, -35)}$ in rectangular coordinates.

(d) The point $(-3, -35)$ lies in the third quadrant so the angle θ satisfies $\pi \leq \theta \leq \frac{3\pi}{2}$ where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{-35}{-3}\right) + \pi \approx 4.627 \text{ radians.}$$

U.S. Cellular Field is $r = \sqrt{(-3)^2 + (-35)^2} = \sqrt{1234} \approx 35.128$ blocks from the intersection of Madison and State Streets, so in polar coordinates, U.S. Cellular Field is located approximately at $\boxed{(35.128, 4.627)}$.

96. If we convert P_1 and P_2 to rectangular coordinates, then

$$\begin{aligned} P_1 &= (r_1 \cos \theta_1, r_1 \sin \theta_1) \\ P_2 &= (r_2 \cos \theta_2, r_2 \sin \theta_2). \end{aligned}$$

Using the rectangular distance formula, the distance d between P_1 and P_2 is

$$\begin{aligned} d &= \sqrt{(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2} \\ &= \sqrt{r_1^2 \cos^2 \theta_1 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_2^2 \cos^2 \theta_2 + r_1^2 \sin^2 \theta_1 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_2^2 \sin^2 \theta_2} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}, \end{aligned}$$

where in the final step we used the angle difference formula for cosine.

97. Since $y = r \sin \theta$, the polar equation $r \sin \theta = a$ converts to $y = a$ in rectangular coordinates, which is a horizontal line a units above the origin (i.e. above the pole) if $a > 0$ and $|a|$ units below the origin (i.e. below the pole) if $a < 0$.

98. Since $x = r \cos \theta$, the polar equation $r \cos \theta = a$ converts to $x = a$ in rectangular coordinates, which is a vertical line a units to the right of the origin (i.e. right of the pole) if $a > 0$ and $|a|$ units to the left of the origin (i.e. left of the pole) if $a < 0$.

99. If we multiply the equation on both sides by r , then we have $r^2 = 2ar \sin \theta$. If we use the equations $r^2 = x^2 + y^2$ and $y = r \sin \theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= 2ay \\ x^2 + (y - a)^2 &= a^2 \end{aligned}$$

in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(0, a)$ with radius a .

100. If we multiply the equation on both sides by r , then we have $r^2 = -2ar \sin \theta$. If we use the equations $r^2 = x^2 + y^2$ and $y = r \sin \theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= -2ay \\ x^2 + (y + a)^2 &= a^2 \end{aligned}$$

in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(0, -a)$ with radius a .

101. If we multiply the equation on both sides by r , then we have $r^2 = 2ar \cos \theta$. If we use the equations $r^2 = x^2 + y^2$ and $x = r \cos \theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= 2ax \\ (x - a)^2 + y^2 &= a^2 \end{aligned}$$

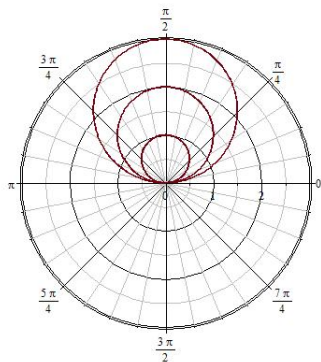
in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(a, 0)$ with radius a .

102. If we multiply the equation on both sides by r , then we have $r^2 = -2ar \cos \theta$. If we use the equations $r^2 = x^2 + y^2$ and $x = r \cos \theta$, then the polar equation becomes

$$\begin{aligned} x^2 + y^2 &= -2ax \\ (x + a)^2 + y^2 &= a^2 \end{aligned}$$

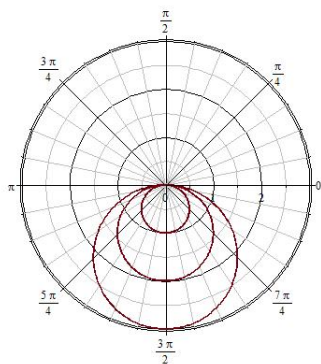
in rectangular coordinates. Since $a > 0$, then this is the equation of a circle centered at $(-a, 0)$ with radius a .

103. (a) The smallest circle is r_1 , the middle circle is r_2 , and the largest circle is r_3 .



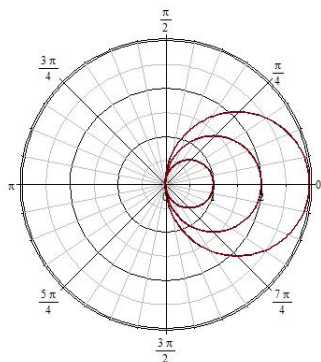
(b) Answers will vary.

(c) The smallest circle is r_1 , the middle circle is r_2 , and the largest circle is r_3 .



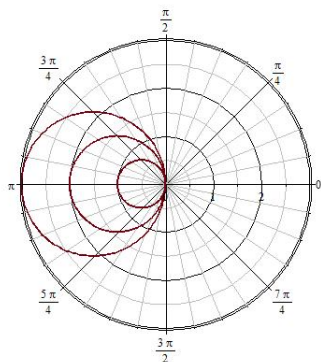
(d) Answers will vary.

104. (a) The smallest circle is r_1 , the middle circle is r_2 , and the largest circle is r_3 .



(b) Answers will vary.

(c) The smallest circle is r_1 , the middle circle is r_2 , and the largest circle is r_3 .



(d) Answers will vary.

Challenge Problems

105. If we multiply both sides of the equation by r , then the equation becomes $r^2 = ar \sin \theta + br \cos \theta$. Now if we use the equations $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, then the polar equation becomes

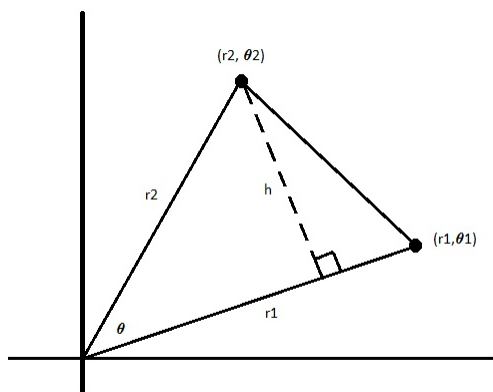
$$\begin{aligned} x^2 + y^2 &= ay + bx \\ x^2 - bx + y^2 - ay &= 0 \\ x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4} + y^2 - ay + \frac{a^2}{4} - \frac{a^2}{4} &= 0 \\ \left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 &= \frac{a^2 + b^2}{4}. \end{aligned}$$

This is the equation of a circle centered at $\left(\frac{b}{2}, \frac{a}{2}\right)$ of radius $\frac{\sqrt{a^2 + b^2}}{2}$.

106. If we use the double-angle formula for cosine, then the equation becomes $r^2 = 2 \cos^2 \theta - 1$. If we use the equations $r^2 = x^2 + y^2$ and $x = r \cos \theta$ (rewritten as $\frac{x}{r} = \cos \theta$), then the equation becomes

$$\begin{aligned} x^2 + y^2 &= 2 \left(\frac{x}{r}\right)^2 - 1 \\ x^2 + y^2 &= \frac{2x^2}{r^2} - 1 \\ x^2 + y^2 &= \frac{2x^2}{x^2 + y^2} - 1 = \frac{2x^2 - x^2 - y^2}{x^2 + y^2} \\ (x^2 + y^2)^2 &= x^2 - y^2. \end{aligned}$$

107. Consider the following picture.



From the point (r_2, θ_2) , drop a perpendicular to the segment connecting the pole and the point (r_1, θ_1) . If we denote the length of the perpendicular by h , then the area of the triangle is $\frac{1}{2}r_1h$. The angle $\theta = \theta_2 - \theta_1$ and then we have the equation $\sin \theta = \frac{h}{r_2}$ or $h = r_2 \sin \theta$. Using these equations, the area of the triangle is

$$\frac{1}{2}r_1r_2 \sin(\theta_2 - \theta_1).$$

9.5 Polar Equations; Parametric Equations of Polar Equations; Arc Length of Polar Equations

Concepts and Vocabulary

- True**.
- The equations for cardioids and limaçons are very similar. They all have the form

$$r = a \pm b \cos \theta \quad r = a \pm b \sin \theta, \quad a > 0, b > 0$$

The equations represent a limaçon with an inner loop if **(a)** $a < b$; a cardioid if **(c)** $a = b$; and a limaçon without an inner loop if **(b)** $a > b$.

- True**. The rose has eight petals.
- The rose $r = \cos(3\theta)$ has **three** petals.

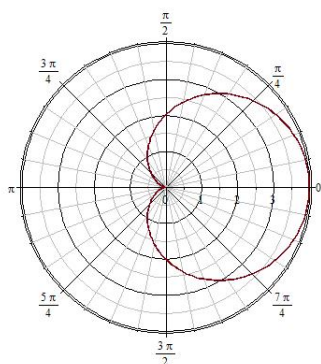
Skill Building

- (a)** The polar equation $r = 2 + 2 \cos \theta$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (2 + 2 \cos \theta, \theta)$, and trace out the graph, beginning at the point $(4, 0)$ and ending at $(4, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(4, 0)$	$(2 + \sqrt{3}, \frac{\pi}{6})$	$(2 + \sqrt{2}, \frac{\pi}{4})$	$(3, \frac{\pi}{3})$	$(2, \frac{\pi}{2})$	$(1, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(2 - \sqrt{2}, \frac{3\pi}{4})$	$(2 - \sqrt{3}, \frac{5\pi}{6})$	$(0, \pi)$	$(2 - \sqrt{3}, \frac{7\pi}{6})$	$(2 - \sqrt{2}, \frac{5\pi}{4})$	$(1, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(2, \frac{3\pi}{2})$	$(3, \frac{5\pi}{3})$	$(2 + \sqrt{2}, \frac{7\pi}{4})$	$(2 + \sqrt{3}, \frac{11\pi}{6})$	$(4, 2\pi)$



- Parametric equations for $r = 2 + 2 \cos \theta$:

$$x = r \cos \theta = 2(1 + \cos \theta) \cos \theta \quad y = r \sin \theta = 2(1 + \cos \theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

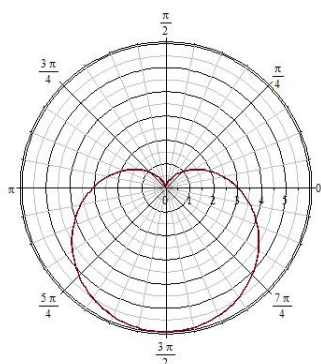
- (a)** The polar equation $r = 3 - 3 \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (3 - 3 \sin \theta, \theta)$, and trace out the graph,

beginning at the point $(3, 0)$ and ending at $(3, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(3, 0)$	$\left(\frac{3}{2}, \frac{\pi}{6}\right)$	$\left(3 - \frac{3}{2}\sqrt{2}, \frac{\pi}{4}\right)$	$\left(3 - \frac{3}{2}\sqrt{3}, \frac{\pi}{3}\right)$	$\left(0, \frac{\pi}{2}\right)$	$\left(3 - \frac{3}{2}\sqrt{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(3 - \frac{3}{2}\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(\frac{3}{2}, \frac{5\pi}{6}\right)$	$(3, \pi)$	$\left(\frac{9}{2}, \frac{7\pi}{6}\right)$	$\left(3 + \frac{3}{2}\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(3 + \frac{3}{2}\sqrt{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(6, \frac{3\pi}{2}\right)$	$\left(3 + \frac{3}{2}\sqrt{3}, \frac{5\pi}{3}\right)$	$\left(3 + \frac{3}{2}\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(\frac{9}{2}, \frac{11\pi}{6}\right)$	$(3, 2\pi)$



(b) Parametric equations for $r = 3 - 3 \sin \theta$:

$$x = r \cos \theta = (3 - 3 \sin \theta) \cos \theta \quad y = r \sin \theta = (3 - 3 \sin \theta) \sin \theta$$

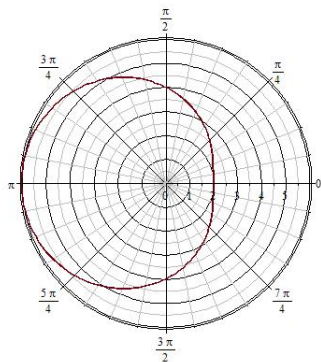
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

7. (a) The polar equation $r = 4 - 2 \cos \theta$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (4 - 2 \cos \theta, \theta)$, and trace out the graph, beginning at the point $(2, 0)$ and ending at $(2, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(2, 0)$	$\left(4 - \sqrt{3}, \frac{\pi}{6}\right)$	$\left(4 - \sqrt{2}, \frac{\pi}{4}\right)$	$\left(3, \frac{\pi}{3}\right)$	$\left(4, \frac{\pi}{2}\right)$	$\left(5, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(4 + \sqrt{2}, \frac{3\pi}{4}\right)$	$\left(4 + \sqrt{3}, \frac{5\pi}{6}\right)$	$(6, \pi)$	$\left(4 + \sqrt{3}, \frac{7\pi}{6}\right)$	$\left(4 + \sqrt{2}, \frac{5\pi}{4}\right)$	$\left(5, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(4, \frac{3\pi}{2}\right)$	$\left(3, \frac{5\pi}{3}\right)$	$\left(4 - \sqrt{2}, \frac{7\pi}{4}\right)$	$\left(4 - \sqrt{3}, \frac{11\pi}{6}\right)$	$(2, 2\pi)$



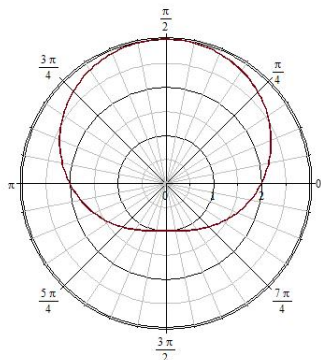
(b) Parametric equations for $r = 4 - 2 \cos \theta$:

$$x = r \cos \theta = (4 - 2 \cos \theta) \cos \theta \quad y = r \sin \theta = (4 - 2 \cos \theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

8. (a) The polar equation $r = 2 + \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (2 + \sin \theta, \theta)$, and trace out the graph, beginning at the point $(2, 0)$ and ending at $(2, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(2, 0)$	$(\frac{5}{2}, \frac{\pi}{6})$	$(2 + \frac{\sqrt{2}}{2}, \frac{\pi}{4})$	$(2 + \frac{\sqrt{3}}{2}, \frac{\pi}{3})$	$(3, \frac{\pi}{2})$	$(2 + \frac{\sqrt{3}}{2}, \frac{2\pi}{3})$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(2 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4})$	$(\frac{5}{2}, \frac{5\pi}{6})$	$(2, \pi)$	$(\frac{3}{2}, \frac{7\pi}{6})$	$(2 - \frac{\sqrt{2}}{2}, \frac{5\pi}{4})$	$(2 - \frac{\sqrt{3}}{2}, \frac{4\pi}{3})$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$(1, \frac{3\pi}{2})$	$(2 - \frac{\sqrt{3}}{2}, \frac{5\pi}{3})$	$(2 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4})$	$(\frac{3}{2}, \frac{11\pi}{6})$	$(2, 2\pi)$	



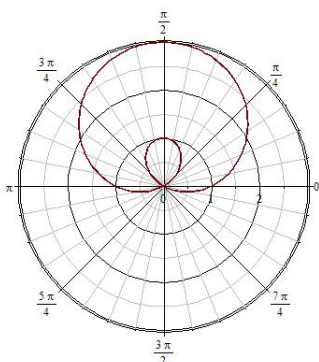
(b) Parametric equations for $r = 2 + \sin \theta$:

$$x = r \cos \theta = (2 + \sin \theta) \cos \theta \quad y = r \sin \theta = (2 + \sin \theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

9. (a) The polar equation $r = 1 + 2 \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (1 + 2 \sin \theta, \theta)$, and trace out the graph, beginning at the point $(1, 0)$ and ending at $(1, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$(2, \frac{\pi}{6})$	$(1 + \sqrt{2}, \frac{\pi}{4})$	$(1 + \sqrt{3}, \frac{\pi}{3})$	$(3, \frac{\pi}{2})$	$(1 + \sqrt{3}, \frac{2\pi}{3})$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(1 + \sqrt{2}, \frac{3\pi}{4})$	$(2, \frac{5\pi}{6})$	$(1, \pi)$	$(0, \frac{7\pi}{6})$	$(1 - \sqrt{2}, \frac{5\pi}{4})$	$(1 - \sqrt{3}, \frac{4\pi}{3})$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$(-1, \frac{3\pi}{2})$	$(1 - \sqrt{3}, \frac{5\pi}{3})$	$(1 - \sqrt{2}, \frac{7\pi}{4})$	$(0, \frac{11\pi}{6})$	$(1, 2\pi)$	



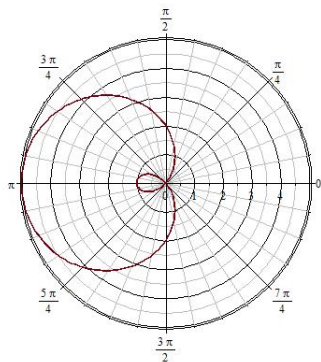
(b) Parametric equations for $r = 1 + 2 \sin \theta$:

$$x = r \cos \theta = (1 + 2 \sin \theta) \cos \theta \quad y = r \sin \theta = (1 + 2 \sin \theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

10. (a) The polar equation $r = 2 - 3 \cos \theta$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (2 - 3 \cos \theta, \theta)$, and trace out the graph, beginning at the point $(-1, 0)$ and ending at $(-1, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(-1, 0)$	$(2 - \frac{3\sqrt{3}}{2}, \frac{\pi}{6})$	$(2 - \frac{3\sqrt{2}}{2}, \frac{\pi}{4})$	$(\frac{1}{2}, \frac{\pi}{3})$	$(2, \frac{\pi}{2})$	$(\frac{7}{2}, \frac{2\pi}{3})$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(2 + \frac{3\sqrt{2}}{2}, \frac{3\pi}{4})$	$(2 + \frac{3\sqrt{3}}{2}, \frac{5\pi}{6})$	$(5, \pi)$	$(2 + \frac{3\sqrt{3}}{2}, \frac{7\pi}{6})$	$(2 + \frac{3\sqrt{2}}{2}, \frac{5\pi}{4})$	$(\frac{7}{2}, \frac{4\pi}{3})$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$(2, \frac{3\pi}{2})$	$(\frac{1}{2}, \frac{5\pi}{3})$	$(2 - \frac{3\sqrt{2}}{2}, \frac{7\pi}{4})$	$(2 - \frac{3\sqrt{3}}{2}, \frac{11\pi}{6})$	$(-1, 2\pi)$	



(b) Parametric equations for $r = 2 - 3 \cos \theta$:

$$x = r \cos \theta = (2 - 3 \cos \theta) \cos \theta \quad y = r \sin \theta = (2 - 3 \cos \theta) \sin \theta$$

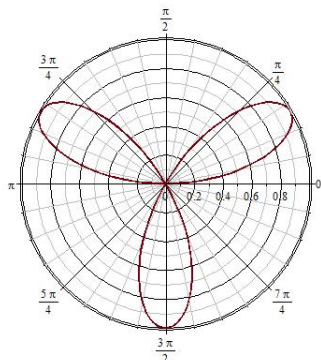
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

11. The polar equation $r = \sin(3\theta)$ contains $\sin(3\theta)$, which has period $\frac{2\pi}{3}$. So we construct a table of common values for θ that range from $0 \leq \theta \leq 2\pi$, noting that the values $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$ and $\frac{4\pi}{3} \leq \theta \leq 2\pi$ repeat the values for $0 \leq \theta \leq \frac{2\pi}{3}$. Then we plot the points and trace out the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(0, 0)$	$(1, \frac{\pi}{6})$	$(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$	$(0, \frac{\pi}{3})$	$(-1, \frac{\pi}{2})$	$(0, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(\frac{\sqrt{2}}{2}, \frac{3\pi}{4})$	$(1, \frac{5\pi}{6})$	$(0, \pi)$	$(-1, \frac{7\pi}{6})$	$(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4})$	$(0, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(1, \frac{3\pi}{2})$	$(0, \frac{5\pi}{3})$	$(\frac{\sqrt{2}}{2}, \frac{7\pi}{4})$	$(1, \frac{11\pi}{6})$	$(0, 2\pi)$



(b) Parametric equations for $r = \sin(3\theta)$:

$$x = r \cos \theta = \sin(3\theta) \cos \theta \quad y = r \sin \theta = \sin(3\theta) \sin \theta$$

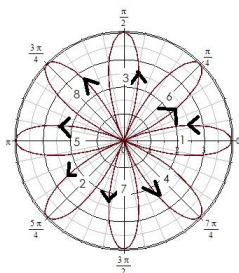
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

12. The polar equation $r = 4\cos(4\theta)$ contains $\cos(4\theta)$, which has period $\frac{\pi}{2}$. So we construct a table of common values for θ that range from $0 \leq \theta \leq 2\pi$, noting that the values $\frac{\pi}{2} \leq \theta \leq \pi$, $\pi \leq \theta \leq \frac{3\pi}{2}$, and $\frac{3\pi}{2} \leq \theta \leq 2\pi$ repeat the values for $0 \leq \theta \leq \frac{\pi}{2}$. Then we plot the points and trace out the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(4, 0)$	$(-2, \frac{\pi}{6})$	$(-4, \frac{\pi}{4})$	$(-2, \frac{\pi}{3})$	$(4, \frac{\pi}{2})$	$(-2, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(-4, \frac{3\pi}{4})$	$(-2, \frac{5\pi}{6})$	$(4, \pi)$	$(-2, \frac{7\pi}{6})$	$(-4, \frac{5\pi}{4})$	$(-2, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(4, \frac{3\pi}{2})$	$(-2, \frac{5\pi}{3})$	$(-4, \frac{7\pi}{4})$	$(-2, \frac{11\pi}{6})$	$(4, 2\pi)$



(b) Parametric equations for $r = 4\cos(4\theta)$:

$$x = r \cos \theta = 4 \cos(4\theta) \cos \theta \quad y = r \sin \theta = 4 \cos(4\theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the direction drawn on the graph starting at $(4, 0)$ and ending at $(4, 2\pi)$. [The numbers inside each rose indicate the order in which the roses are drawn.]

13. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

$$\begin{aligned} 8 \cos \theta &= 2 \sec \theta \\ 4 \cos^2 \theta &= 1 \\ \cos \theta &= \pm \frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}. \end{aligned}$$

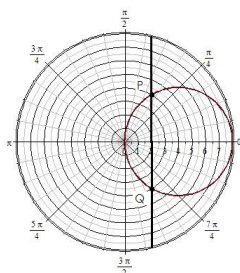
The points of intersection are

$$\begin{aligned} \left(8 \cos \left(\frac{\pi}{3} \right), \frac{\pi}{3} \right) &= \left(4, \frac{\pi}{3} \right) \\ \left(8 \cos \left(\frac{2\pi}{3} \right), \frac{2\pi}{3} \right) &= \left(-4, \frac{2\pi}{3} \right) \\ \left(8 \cos \left(\frac{4\pi}{3} \right), \frac{4\pi}{3} \right) &= \left(-4, \frac{4\pi}{3} \right) \\ \left(8 \cos \left(\frac{5\pi}{3} \right), \frac{5\pi}{3} \right) &= \left(4, \frac{5\pi}{3} \right). \end{aligned}$$

9.5. POLAR EQUATIONS; PARAMETRIC EQUATIONS OF POLAR EQUATIONS; ARC LENGTH OF POLAR EQUATIONS

Notice that the first and third points are the same, and the second and fourth points are the same, so the two unique points of intersection are

$$P = \left(4, \frac{\pi}{3}\right) \text{ and } Q = \left(4, \frac{5\pi}{3}\right).$$



14. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

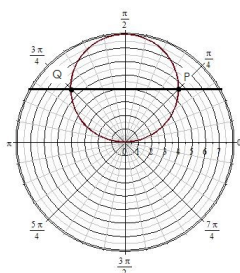
$$\begin{aligned} 8 \sin \theta &= 4 \csc \theta \\ 2 \sin^2 \theta &= 1 \\ \sin \theta &= \pm \frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \end{aligned}$$

The points of intersection are

$$\begin{aligned} \left(8 \sin \left(\frac{\pi}{4}\right), \frac{\pi}{4}\right) &= \left(4\sqrt{2}, \frac{\pi}{4}\right) \\ \left(8 \sin \left(\frac{3\pi}{4}\right), \frac{3\pi}{4}\right) &= \left(4\sqrt{2}, \frac{3\pi}{4}\right) \\ \left(8 \sin \left(\frac{5\pi}{4}\right), \frac{5\pi}{4}\right) &= \left(-4\sqrt{2}, \frac{5\pi}{4}\right) \\ \left(8 \sin \left(\frac{7\pi}{4}\right), \frac{7\pi}{4}\right) &= \left(-4\sqrt{2}, \frac{7\pi}{4}\right). \end{aligned}$$

Notice that the first and third points are the same, and the second and fourth points are the same, so the two unique points of intersection are

$$P = \left(4\sqrt{2}, \frac{\pi}{4}\right) \text{ and } Q = \left(4\sqrt{2}, \frac{3\pi}{4}\right).$$

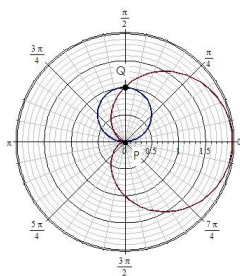


15. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

$$\begin{aligned}
 \sin \theta &= 1 + \cos \theta \\
 \sin \theta - \cos \theta &= 1 \\
 (\sin \theta - \cos \theta)^2 &= 1^2 \\
 \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta &= 1 \\
 \sin \theta \cos \theta &= 0 \\
 \sin \theta = 0 &\text{ or } \cos \theta = 0 \\
 \theta &= 0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}.
 \end{aligned}$$

By squaring both sides in the third step, we may have introduced extraneous solutions, so we need to check all of them. The values $\theta = 0, 2\pi, \frac{3\pi}{2}$ do not make $\sin \theta = 1 + \cos \theta$, so our only values of intersection are when $\theta = \pi, \frac{\pi}{2}$. The points of intersection are

$$\begin{aligned}
 P = (\sin(\pi), \pi) &= \boxed{(0, \pi)} \\
 Q = \left(\sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}\right) &= \boxed{\left(1, \frac{\pi}{2}\right)}.
 \end{aligned}$$

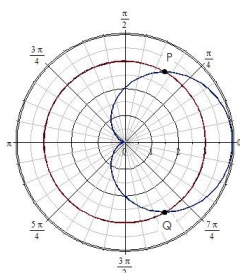


16. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

$$\begin{aligned}
 3 &= 2 + 2 \cos \theta \\
 \cos \theta &= \frac{1}{2} \\
 \theta &= \frac{\pi}{3}, \frac{5\pi}{3}.
 \end{aligned}$$

The points of intersection are

$$\begin{aligned}
 P &= \boxed{\left(3, \frac{\pi}{3}\right)} \\
 Q &= \boxed{\left(3, \frac{5\pi}{3}\right)}.
 \end{aligned}$$

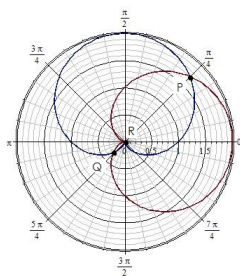


17. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

$$\begin{aligned} 1 + \sin \theta &= 1 + \cos \theta \\ \sin \theta &= \cos \theta \\ \tan \theta &= 1 \\ \theta &= \frac{\pi}{4}, \frac{5\pi}{4}. \end{aligned}$$

We also note that both polar curves go through the pole, so the pole is another point of intersection. The points of intersection are the pole, R , and

$$\begin{aligned} P &= \left(1 + \sin\left(\frac{\pi}{4}\right), \frac{\pi}{4}\right) = \boxed{\left(1 + \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)} \\ Q &= \left(1 + \sin\left(\frac{5\pi}{4}\right), \frac{5\pi}{4}\right) = \boxed{\left(1 - \frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)}. \end{aligned}$$

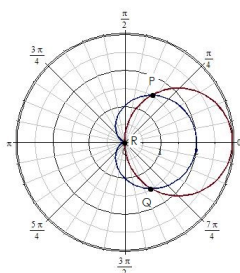


18. To find the points of intersection, where $0 \leq \theta \leq 2\pi$, set both polar equations equal.

$$\begin{aligned} 1 + \cos \theta &= 3 \cos \theta \\ \cos \theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{5\pi}{3}. \end{aligned}$$

We also note that both polar curves go through the pole, so the pole is another point of intersection. The points of intersection are the pole, R , and

$$\begin{aligned} P &= \left(1 + \cos\left(\frac{\pi}{3}\right), \frac{\pi}{3}\right) = \boxed{\left(\frac{3}{2}, \frac{\pi}{3}\right)} \\ Q &= \left(1 + \cos\left(\frac{5\pi}{3}\right), \frac{5\pi}{3}\right) = \boxed{\left(\frac{3}{2}, \frac{5\pi}{3}\right)}. \end{aligned}$$



19. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = e^{\theta/2}$. Then $\frac{dr}{d\theta} = \frac{1}{2}e^{\theta/2}$ and

$$\begin{aligned} s &= \int_0^2 \sqrt{(e^{\theta/2})^2 + \left(\frac{1}{2}e^{\theta/2}\right)^2} d\theta = \int_0^2 \sqrt{\frac{5}{4}e^{\theta}} d\theta = \frac{\sqrt{5}}{2} \int_0^2 e^{\theta/2} d\theta \\ &= \frac{\sqrt{5}}{2} \left[2e^{\theta/2} \right]_0^2 = \boxed{\sqrt{5}(e-1)}. \end{aligned}$$

20. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = e^{2\theta}$. Then $\frac{dr}{d\theta} = 2e^{2\theta}$ and

$$\begin{aligned} s &= \int_0^2 \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} d\theta = \int_0^2 \sqrt{5e^{4\theta}} d\theta = \sqrt{5} \int_0^2 e^{2\theta} d\theta \\ &= \sqrt{5} \left[\frac{1}{2}e^{2\theta} \right]_0^2 = \boxed{\frac{\sqrt{5}}{2}(e^4-1)}. \end{aligned}$$

21. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = \cos^2 \frac{\theta}{2}$. Then $\frac{dr}{d\theta} = -\cos \frac{\theta}{2} \sin \frac{\theta}{2}$ and

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{\left(\cos^2 \frac{\theta}{2}\right)^2 + \left(-\cos \frac{\theta}{2} \sin \frac{\theta}{2}\right)^2} d\theta = \int_0^{\pi} \sqrt{\cos^4 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}} d\theta \\ &= \int_0^{\pi} \sqrt{\cos^2 \frac{\theta}{2} \left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right)} d\theta = \int_0^{\pi} \cos \frac{\theta}{2} d\theta \\ &= \left[2 \sin \frac{\theta}{2} \right]_0^{\pi} = \boxed{2}, \end{aligned}$$

where in the second line we are using that $\cos \frac{\theta}{2} \geq 0$ on $[0, \pi]$ when taking the square root.

22. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = \sin^2 \frac{\theta}{2}$. Then $\frac{dr}{d\theta} = \cos \frac{\theta}{2} \sin \frac{\theta}{2}$ and

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{\left(\sin^2 \frac{\theta}{2}\right)^2 + \left(\cos \frac{\theta}{2} \sin \frac{\theta}{2}\right)^2} d\theta = \int_0^{\pi} \sqrt{\sin^4 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}} d\theta \\ &= \int_0^{\pi} \sqrt{\sin^2 \frac{\theta}{2} \left(\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}\right)} d\theta = \int_0^{\pi} \sin \frac{\theta}{2} d\theta \\ &= \left[-2 \cos \frac{\theta}{2} \right]_0^{\pi} = \boxed{2}. \end{aligned}$$

Applications and Extensions

23. $\boxed{r = 3 + 3 \cos \theta}$

24. $\boxed{r = 3 - 3 \cos \theta}$

25. $\boxed{r = 4 + \sin \theta}$

26. $\boxed{r = 1 + 4 \sin \theta}$

In Problems 27-32, use the parameterizations $x(\theta) = r \cos \theta = f(\theta) \cos \theta$ and $y(\theta) = r \sin \theta = f(\theta) \sin \theta$ to

find $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. Do not forget to use the product rule!

27. Since $r = f(\theta) = 2 \cos(3\theta)$, we have the parametrization $x(\theta) = 2 \cos(3\theta) \cos \theta$ and $y(\theta) = 2 \cos(3\theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= -6 \sin(3\theta) \cos \theta - 2 \cos(3\theta) \sin \theta \\ \frac{dy}{d\theta} &= -6 \sin(3\theta) \sin \theta + 2 \cos(3\theta) \cos \theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx} \right]_{\theta=\pi/6} = \left[\frac{-6 \sin(3\theta) \sin \theta + 2 \cos(3\theta) \cos \theta}{-6 \sin(3\theta) \cos \theta - 2 \cos(3\theta) \sin \theta} \right]_{\theta=\pi/6} = \frac{\sqrt{3}}{3}.$$

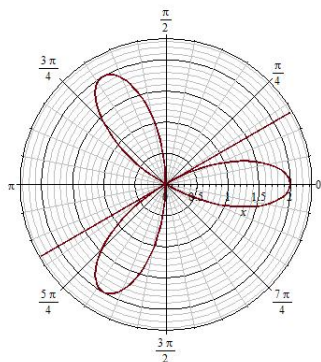
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{6}\right) &= 0 \\ y\left(\frac{\pi}{6}\right) &= 0.\end{aligned}$$

The equation of the tangent line is

$$y - 0 = \frac{\sqrt{3}}{3}(x - 0)$$

$$y = \frac{\sqrt{3}}{3}x.$$



28. Since $r = f(\theta) = 3 \sin(3\theta)$, we have the parametrization $x(\theta) = 3 \sin(3\theta) \cos \theta$ and $y(\theta) = 3 \sin(3\theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= 9 \cos(3\theta) \cos \theta - 3 \sin(3\theta) \sin \theta \\ \frac{dy}{d\theta} &= 9 \cos(3\theta) \sin \theta + 3 \sin(3\theta) \cos \theta.\end{aligned}$$

The slope of the tangent line is then

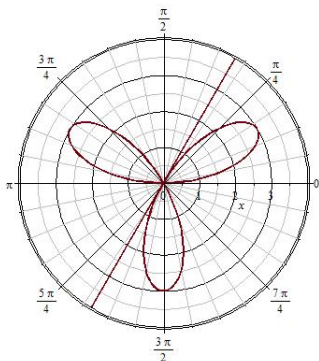
$$\left[\frac{dy}{dx} \right]_{\theta=\pi/3} = \left[\frac{9 \cos(3\theta) \sin \theta + 3 \sin(3\theta) \cos \theta}{9 \cos(3\theta) \cos \theta - 3 \sin(3\theta) \sin \theta} \right]_{\theta=\pi/3} = \sqrt{3}.$$

The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{3}\right) &= 0 \\ y\left(\frac{\pi}{3}\right) &= 0.\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y - 0 &= \sqrt{3}(x - 0) \\ y &= \sqrt{3}x.\end{aligned}$$



29. Since $r = f(\theta) = 2 + \cos \theta$, we have the parametrization $x(\theta) = (2 + \cos \theta) \cos \theta$ and $y(\theta) = (2 + \cos \theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= -\sin \theta \cos \theta - (2 + \cos \theta) \sin \theta \\ \frac{dy}{d\theta} &= -\sin \theta \sin \theta + (2 + \cos \theta) \cos \theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx} \right]_{\theta=\pi/4} = \left[\frac{-\sin \theta \sin \theta + (2 + \cos \theta) \cos \theta}{-\sin \theta \cos \theta - (2 + \cos \theta) \sin \theta} \right]_{\theta=\pi/4} = \sqrt{2} - 2.$$

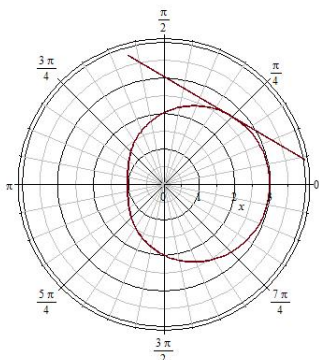
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= \sqrt{2} + \frac{1}{2} \\ y\left(\frac{\pi}{4}\right) &= \sqrt{2} + \frac{1}{2}.\end{aligned}$$

The equation of the tangent line is

$$y - \left(\sqrt{2} + \frac{1}{2}\right) = (\sqrt{2} - 2) \left(x - \left(\sqrt{2} + \frac{1}{2}\right)\right)$$

$$\boxed{y = (\sqrt{2} - 2)x + \frac{5\sqrt{2}}{2} - \frac{1}{2}}.$$



30. Since $r = f(\theta) = 3 - \sin \theta$, we have the parametrization $x(\theta) = (3 - \sin \theta) \cos \theta$ and $y(\theta) = (3 - \sin \theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= -\cos \theta \cos \theta - (3 - \sin \theta) \sin \theta \\ \frac{dy}{d\theta} &= -\cos \theta \sin \theta + (3 - \sin \theta) \cos \theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx} \right]_{\theta=\pi/4} = \left[\frac{-\cos \theta \sin \theta + (3 - \sin \theta) \cos \theta}{-\cos \theta \cos \theta - (3 - \sin \theta) \sin \theta} \right]_{\theta=\pi/4} = -\frac{\sqrt{3}}{2}.$$

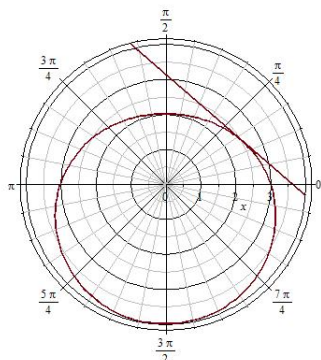
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= \frac{5\sqrt{3}}{4} \\ y\left(\frac{\pi}{4}\right) &= \frac{5}{4}.\end{aligned}$$

The equation of the tangent line is

$$y - \frac{5}{4} = -\frac{\sqrt{3}}{2} \left(x - \frac{5\sqrt{3}}{4} \right)$$

$$y = -\frac{\sqrt{3}}{2}x + \frac{25}{8}.$$



31. Since $r = f(\theta) = 4 + 5 \sin \theta$, we have the parametrization $x(\theta) = (4 + 5 \sin \theta) \cos \theta$ and $y(\theta) = (4 + 5 \sin \theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned}\frac{dx}{d\theta} &= 5 \cos \theta \cos \theta - (4 + 5 \sin \theta) \sin \theta \\ \frac{dy}{d\theta} &= 5 \cos \theta \sin \theta + (4 + 5 \sin \theta) \cos \theta.\end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx} \right]_{\theta=\pi/4} = \left[\frac{5 \cos \theta \sin \theta + (4 + 5 \sin \theta) \cos \theta}{5 \cos \theta \cos \theta - (4 + 5 \sin \theta) \sin \theta} \right]_{\theta=\pi/4} = -\frac{5\sqrt{2}}{4} - 1.$$

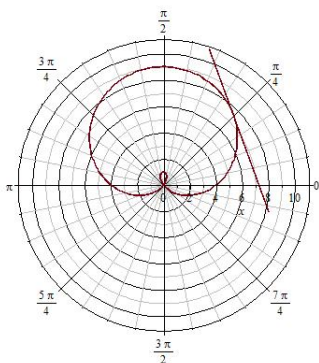
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned}x\left(\frac{\pi}{4}\right) &= 2\sqrt{2} + \frac{5}{2} \\ y\left(\frac{\pi}{4}\right) &= 2\sqrt{2} + \frac{5}{2}.\end{aligned}$$

The equation of the tangent line is

$$y - \left(2\sqrt{2} + \frac{5}{2}\right) = \left(-\frac{5\sqrt{2}}{4} - 1\right) \left(x - \left(2\sqrt{2} + \frac{5}{2}\right)\right)$$

$$\boxed{y = -\left(1 + \frac{5\sqrt{2}}{4}\right)x + \frac{57\sqrt{2}}{8} + 10}.$$



32. Since $r = f(\theta) = 1 - 2 \cos \theta$, we have the parametrization $x(\theta) = (1 - 2 \cos \theta) \cos \theta$ and $y(\theta) = (1 - 2 \cos \theta) \sin \theta$. Now we find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ using the product rule.

$$\begin{aligned} \frac{dx}{d\theta} &= 2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta \\ \frac{dy}{d\theta} &= 2 \sin \theta \sin \theta + (1 - 2 \cos \theta) \cos \theta. \end{aligned}$$

The slope of the tangent line is then

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/4} = \left[\frac{2 \sin \theta \sin \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}\right]_{\theta=\pi/4} = \frac{1 + 2\sqrt{2}}{7}.$$

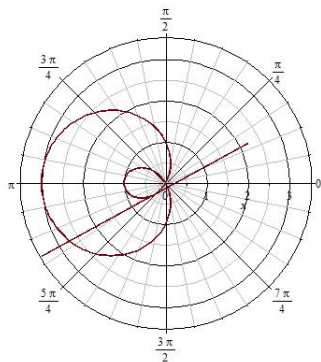
The rectangular coordinates of the point (x, y) to write the equation of the tangent line at are

$$\begin{aligned} x\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} - 1 \\ y\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} - 1. \end{aligned}$$

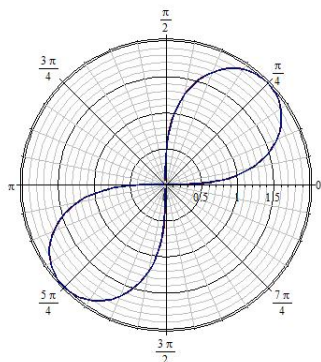
The equation of the tangent line is

$$y - \left(\frac{\sqrt{2}}{2} - 1\right) = \frac{1 + 2\sqrt{2}}{7} \left(x - \left(\frac{\sqrt{2}}{2} - 1\right)\right)$$

$$\boxed{y = \frac{1 + 2\sqrt{2}}{7}x + \frac{1}{7}(5\sqrt{2} - 8)}.$$



33. (a)

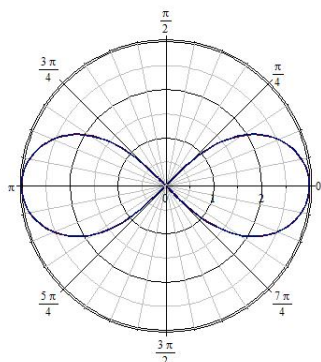


(b) The equation $r^2 = 4 \sin(2\theta)$ can be rewritten as $r = \pm 2\sqrt{\sin(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

$$x = r \cos \theta = 2\sqrt{\sin(2\theta)} \cos \theta \quad y = r \sin \theta = 2\sqrt{\sin(2\theta)} \sin \theta$$

where θ is the parameter.

34. (a)

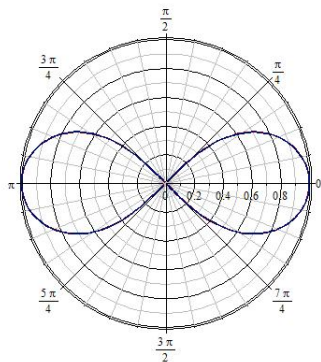


(b) The equation $r^2 = 9 \cos(2\theta)$ can be rewritten as $r = \pm 3\sqrt{\cos(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

$$x = r \cos \theta = 3\sqrt{\cos(2\theta)} \cos \theta \quad y = r \sin \theta = 3\sqrt{\cos(2\theta)} \sin \theta$$

where θ is the parameter.

35. (a)

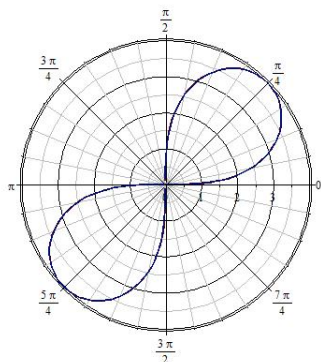


(b) The equation $r^2 = \cos(2\theta)$ can be rewritten as $r = \pm\sqrt{\cos(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

$$x = r \cos \theta = \sqrt{\cos(2\theta)} \cos \theta \quad y = r \sin \theta = \sqrt{\cos(2\theta)} \sin \theta$$

where θ is the parameter.

36. (a)



(b) The equation $r^2 = 16 \sin(2\theta)$ can be rewritten as $r = \pm 4\sqrt{\sin(2\theta)}$. Besides orientation, the $\pm$ does not affect the graph of the lemniscate, so one set of parametric equations would be:

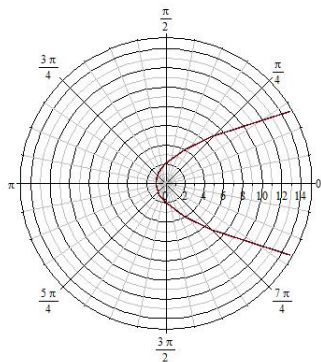
$$x = r \cos \theta = 4\sqrt{\sin(2\theta)} \cos \theta \quad y = r \sin \theta = 4\sqrt{\sin(2\theta)} \sin \theta$$

where θ is the parameter.

For Problem 37-48, to find parametric equations, we will use the formulas $x = r \cos \theta$ and $y = r \sin \theta$.

37. (a) The polar equation $r = \frac{2}{1 - \cos \theta}$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding these values since r is not defined there) and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	undefined	$\left(\frac{4}{2 - \sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{4}{2 - \sqrt{2}}, \frac{\pi}{4}\right)$	$\left(4, \frac{\pi}{3}\right)$	$\left(2, \frac{\pi}{2}\right)$	$\left(\frac{4}{3}, \frac{2\pi}{3}\right)$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{4}{\sqrt{2} + 2}, \frac{3\pi}{4}\right)$	$\left(\frac{4}{\sqrt{3} + 2}, \frac{5\pi}{6}\right)$	$(1, \pi)$	$\left(\frac{4}{\sqrt{3} + 2}, \frac{7\pi}{6}\right)$	$\left(\frac{4}{\sqrt{2} + 2}, \frac{5\pi}{4}\right)$	$\left(\frac{4}{3}, \frac{4\pi}{3}\right)$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$\left(2, \frac{3\pi}{2}\right)$	$\left(4, \frac{5\pi}{3}\right)$	$\left(\frac{4}{2 - \sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{4}{2 - \sqrt{3}}, \frac{11\pi}{6}\right)$	undefined	



(b) Parametric equations for the polar equation $r = \frac{2}{1 - \cos \theta}$ are

$$x = \frac{2 \cos \theta}{1 - \cos \theta}$$

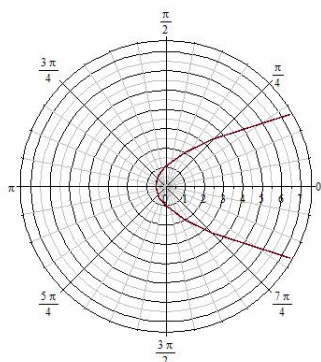
$$y = \frac{2 \sin \theta}{1 - \cos \theta}.$$

38. (a) The polar equation $r = \frac{1}{1 - \cos \theta}$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding these values since r is not defined there) and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	undefined	$\left(\frac{2}{2 - \sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{2}{2 - \sqrt{2}}, \frac{\pi}{4}\right)$	$\left(2, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{2}{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{2}{\sqrt{2} + 2}, \frac{3\pi}{4}\right)$	$\left(\frac{2}{\sqrt{3} + 2}, \frac{5\pi}{6}\right)$	$\left(\frac{1}{2}, \pi\right)$	$\left(\frac{2}{\sqrt{3} + 2}, \frac{7\pi}{6}\right)$	$\left(\frac{2}{\sqrt{2} + 2}, \frac{5\pi}{4}\right)$	$\left(\frac{2}{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(1, \frac{3\pi}{2}\right)$	$\left(2, \frac{5\pi}{3}\right)$	$\left(\frac{2}{2 - \sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{2}{2 - \sqrt{3}}, \frac{11\pi}{6}\right)$	undefined



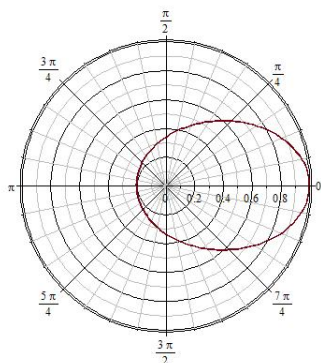
(b) Parametric equations for the polar equation $r = \frac{1}{1 - \cos \theta}$ are

$$x = \frac{\cos \theta}{1 - \cos \theta}$$

$$y = \frac{\sin \theta}{1 - \cos \theta}.$$

39. (a) The polar equation $r = \frac{1}{3 - 2 \cos \theta}$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding the values of θ such that $3 - 2 \cos \theta = 0$, however there are none) and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$\left(\frac{1}{3 - \sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{1}{3 - \sqrt{2}}, \frac{\pi}{4}\right)$	$\left(\frac{1}{2}, \frac{\pi}{3}\right)$	$\left(\frac{1}{3}, \frac{\pi}{2}\right)$	$\left(\frac{1}{4}, \frac{2\pi}{3}\right)$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{1}{3 + \sqrt{2}}, \frac{3\pi}{4}\right)$	$\left(\frac{1}{3 + \sqrt{3}}, \frac{5\pi}{6}\right)$	$\left(\frac{1}{5}, \pi\right)$	$\left(\frac{1}{3 + \sqrt{3}}, \frac{7\pi}{6}\right)$	$\left(\frac{1}{3 + \sqrt{2}}, \frac{5\pi}{4}\right)$	$\left(\frac{1}{4}, \frac{4\pi}{3}\right)$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$\left(\frac{1}{3}, \frac{3\pi}{2}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{3}\right)$	$\left(\frac{1}{3 - \sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{1}{3 - \sqrt{3}}, \frac{11\pi}{6}\right)$	$(1, 2\pi)$	



(b) Parametric equations for the polar equation $r = \frac{1}{3 - 2 \cos \theta}$ are

$$x = \frac{\cos \theta}{3 - 2 \cos \theta}$$

$$y = \frac{\sin \theta}{3 - 2 \cos \theta}.$$

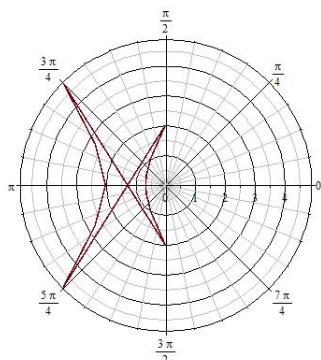
40. (a) The polar equation $r = \frac{2}{1 - 2 \cos \theta}$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding the values $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ since r is not defined there) and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(-2, 0)$	$\left(\frac{2}{1 - \sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{2}{1 - \sqrt{2}}, \frac{\pi}{4}\right)$	undefined	$\left(2, \frac{\pi}{2}\right)$	$\left(1, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{2}{1+\sqrt{2}}, \frac{3\pi}{4}\right)$	$\left(\frac{2}{1+\sqrt{3}}, \frac{5\pi}{6}\right)$	$\left(\frac{2}{3}, \pi\right)$	$\left(\frac{2}{1+\sqrt{3}}, \frac{7\pi}{6}\right)$	$\left(\frac{2}{1+\sqrt{2}}, \frac{5\pi}{4}\right)$	$\left(1, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(2, \frac{3\pi}{2}\right)$	undefined	$\left(\frac{2}{1-\sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{2}{1-\sqrt{3}}, \frac{11\pi}{6}\right)$	$(-2, 2\pi)$

The asymptotes of the hyperbola are drawn in.



(b) Parametric equations for the polar equation $r = \frac{2}{1-2\cos\theta}$ are

$$x = \frac{2\cos\theta}{1-2\cos\theta}$$

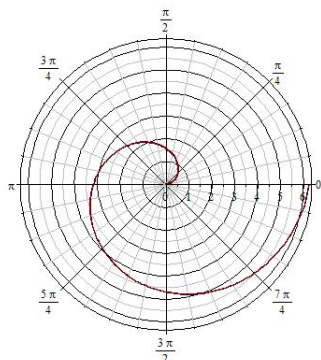
$$y = \frac{2\sin\theta}{1-2\cos\theta}.$$

41. (a) Constructing a table of common values for $\theta > 0$, we find some points on the polar graph $r = \theta$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{3\pi}{4}, \frac{3\pi}{4}\right)$	$\left(\frac{5\pi}{6}, \frac{5\pi}{6}\right)$	(π, π)	$\left(\frac{7\pi}{6}, \frac{7\pi}{6}\right)$	$\left(\frac{5\pi}{4}, \frac{5\pi}{4}\right)$	$\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$	$\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$	$\left(\frac{7\pi}{4}, \frac{7\pi}{4}\right)$	$\left(\frac{11\pi}{6}, \frac{11\pi}{6}\right)$	$(2\pi, 2\pi)$



(b) Parametric equations for the polar equation $r = \theta$ are

$$x = \theta \cos \theta$$

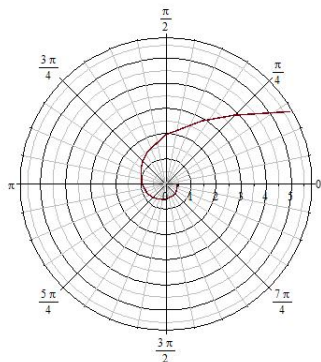
$$y = \theta \sin \theta.$$

42. (a) Constructing a table of common values for $\theta > 0$, we find some points on the polar graph $r = \frac{3}{\theta}$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$\left(\frac{18}{\pi}, \frac{\pi}{6}\right)$	$\left(\frac{12}{\pi}, \frac{\pi}{4}\right)$	$\left(\frac{9}{\pi}, \frac{\pi}{3}\right)$	$\left(\frac{6}{\pi}, \frac{\pi}{2}\right)$	$\left(\frac{9}{2\pi}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{4}{\pi}, \frac{3\pi}{4}\right)$	$\left(\frac{18}{5\pi}, \frac{5\pi}{6}\right)$	$\left(\frac{3}{\pi}, \pi\right)$	$\left(\frac{18}{7\pi}, \frac{7\pi}{6}\right)$	$\left(\frac{12}{5\pi}, \frac{5\pi}{4}\right)$	$\left(\frac{9}{4\pi}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(\frac{2}{\pi}, \frac{3\pi}{2}\right)$	$\left(\frac{9}{5\pi}, \frac{5\pi}{3}\right)$	$\left(\frac{12}{7\pi}, \frac{7\pi}{4}\right)$	$\left(\frac{18}{11\pi}, \frac{11\pi}{6}\right)$	$\left(\frac{3}{2\pi}, 2\pi\right)$



(b) Parametric equations for the polar equation $r = \frac{3}{\theta}$ are

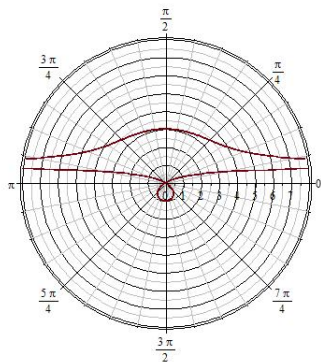
$$x = \frac{3 \cos \theta}{\theta}$$

$$y = \frac{3 \sin \theta}{\theta}.$$

43. Constructing a table of common values for θ between 0 and 2π (excluding 0, π , and 2π which make r undefined), we find some points on the polar graph $r = \csc \theta - 2$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
(r, θ)	$\left(0, \frac{\pi}{6}\right)$	$\left(-2 + \sqrt{2}, \frac{\pi}{4}\right)$	$\left(\frac{2\sqrt{3}}{3} - 2, \frac{\pi}{3}\right)$	$\left(-1, \frac{\pi}{2}\right)$	$\left(\frac{2\sqrt{3}}{3} - 2, \frac{2\pi}{3}\right)$	$\left(-2 + \sqrt{2}, \frac{3\pi}{4}\right)$	$\left(0, \frac{5\pi}{6}\right)$

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
(r, θ)	$\left(-4, \frac{7\pi}{6}\right)$	$\left(-2 - \sqrt{2}, \frac{5\pi}{4}\right)$	$\left(-\frac{2\sqrt{3}}{3} - 2, \frac{4\pi}{3}\right)$	$\left(-3, \frac{3\pi}{2}\right)$	$\left(-\frac{2\sqrt{3}}{3} - 2, \frac{5\pi}{3}\right)$	$\left(-2 - \sqrt{2}, \frac{7\pi}{4}\right)$	$\left(-4, \frac{11\pi}{6}\right)$



(b) Parametric equations for the polar equation $r = \csc \theta - 2$ are

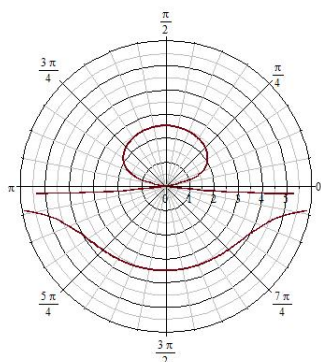
$$x = (\csc \theta - 2) \cos \theta$$

$$y = (\csc \theta - 2) \sin \theta.$$

44. Constructing a table of common values for θ between 0 and 2π (excluding the values 0, π , and 2π since r is not defined there), we find some points on the polar graph $r = 3 - \frac{1}{2} \csc \theta$.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
(r, θ)	$(2, \frac{\pi}{6})$	$(3 - \frac{\sqrt{2}}{2}, \frac{\pi}{4})$	$(3 - \frac{\sqrt{3}}{3}, \frac{\pi}{3})$	$(\frac{5}{2}, \frac{\pi}{2})$	$(3 - \frac{\sqrt{3}}{3}, \frac{2\pi}{3})$	$(3 - \frac{\sqrt{2}}{2}, \frac{3\pi}{4})$	$(2, \frac{5\pi}{6})$

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
(r, θ)	$(4, \frac{7\pi}{6})$	$(3 + \frac{\sqrt{2}}{2}, \frac{5\pi}{4})$	$(3 + \frac{\sqrt{3}}{3}, \frac{4\pi}{3})$	$(\frac{7}{2}, \frac{3\pi}{2})$	$(3 + \frac{\sqrt{3}}{3}, \frac{5\pi}{3})$	$(3 + \frac{\sqrt{2}}{2}, \frac{7\pi}{4})$	$(4, \frac{11\pi}{6})$



(b) Parametric equations for the polar equation $r = 3 - \frac{1}{2} \csc \theta$ are

$$x = \left(3 - \frac{1}{2} \csc \theta\right) \cos \theta$$

$$y = \left(3 - \frac{1}{2} \csc \theta\right) \sin \theta.$$

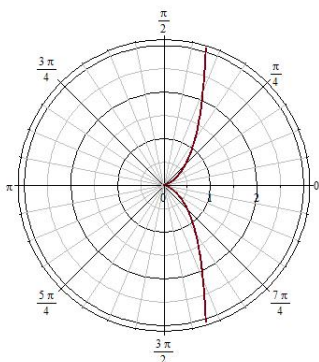
45. (a) The polar equation $r = \sin \theta \tan \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding the values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ since r is not defined there)

and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	(0, 0)	$\left(\frac{\sqrt{3}}{6}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$	$\left(\frac{3}{2}, \frac{\pi}{3}\right)$	undefined	$\left(-\frac{3}{2}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{6}, \frac{5\pi}{6}\right)$	(0, π)	$\left(-\frac{\sqrt{3}}{6}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{4}\right)$	$\left(-\frac{3}{2}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	undefined	$\left(\frac{3}{2}, \frac{5\pi}{3}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$	$\left(\frac{\sqrt{3}}{6}, \frac{11\pi}{6}\right)$	(0, 2π)



(b) Parametric equations for the polar equation $r = \sin \theta \tan \theta$ are

$$x = \sin \theta \tan \theta \cos \theta = \sin^2 \theta$$

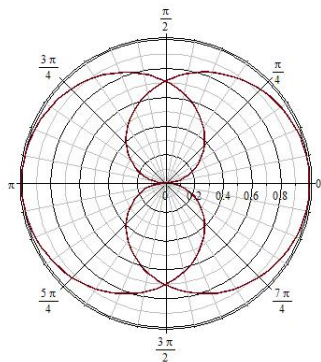
$$y = \sin \theta \tan \theta \sin \theta = \sin^2 \theta \tan \theta.$$

46. (a) The polar equation $r = \cos \frac{\theta}{2}$ contains $\cos \frac{\theta}{2}$, which has the period 4π . We construct a table of common values of θ that range from 0 to 4π and plot the points (r, θ) . The points with decimal coordinates are approximate values of r .

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
(r, θ)	(1, 0)	$\left(0.924, \frac{\pi}{4}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)$	$\left(0.383, \frac{3\pi}{4}\right)$	(0, π)	$\left(-0.383, \frac{5\pi}{4}\right)$

θ	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$
(r, θ)	$\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{2}\right)$	$\left(-0.924, \frac{7\pi}{4}\right)$	(-1, 2π)	$\left(-0.924, \frac{9\pi}{4}\right)$	$\left(-\frac{\sqrt{2}}{2}, \frac{5\pi}{2}\right)$	$\left(-0.383, \frac{11\pi}{4}\right)$

θ	3π	$\frac{13\pi}{4}$	$\frac{7\pi}{2}$	$\frac{15\pi}{4}$	4π
(r, θ)	(0, 3π)	$\left(0.383, \frac{13\pi}{4}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{7\pi}{2}\right)$	$\left(0.924, \frac{15\pi}{4}\right)$	(1, 4π)



(b) Parametric equations for the polar equation $r = \cos \frac{\theta}{2}$ are

$$x = \cos \frac{\theta}{2} \cos \theta$$

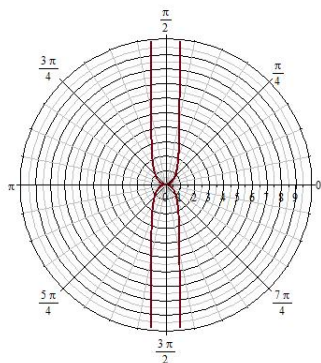
$$y = \cos \frac{\theta}{2} \sin \theta.$$

47. (a) The polar equation $r = \tan \theta$ contains $\tan \theta$, which has the period π . We construct a table of common values of θ that range from 0 to 2π (excluding $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ since r is not defined there), noting that $\pi \leq \theta \leq 2\pi$ duplicates the values taken on $0 \leq \theta \leq \pi$, and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	(0, 0)	$\left(\frac{\sqrt{3}}{3}, \frac{\pi}{6}\right)$	$\left(1, \frac{\pi}{4}\right)$	$\left(\sqrt{3}, \frac{\pi}{3}\right)$	undefined	$\left(-\sqrt{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(-1, \frac{3\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{3}, \frac{5\pi}{6}\right)$	(0, π)	$\left(\frac{\sqrt{3}}{3}, \frac{7\pi}{6}\right)$	$\left(1, \frac{5\pi}{4}\right)$	$\left(\sqrt{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	undefined	$\left(-\sqrt{3}, \frac{5\pi}{3}\right)$	$\left(-1, \frac{7\pi}{4}\right)$	$\left(-\frac{\sqrt{3}}{3}, \frac{11\pi}{6}\right)$	(0, 2π)



(b) Parametric equations for the polar equation $r = \tan \theta$ are

$$x = \tan \theta \cos \theta = \sin \theta$$

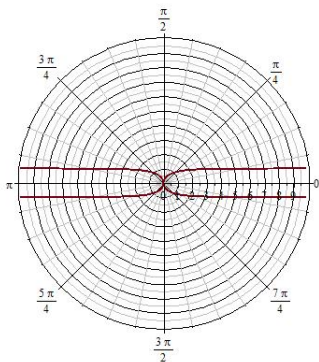
$$y = \sin \theta \tan \theta.$$

48. (a) The polar equation $r = \cot \theta$ contains $\cot \theta$, which has the period π . We construct a table of common values of θ that range from 0 to 2π (excluding $\theta = 0, \pi$ and 2π since r is not defined there), noting that $\pi \leq \theta \leq 2\pi$ duplicates the values taken on $0 \leq \theta \leq \pi$, and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	undefined	$(\sqrt{3}, \frac{\pi}{6})$	$(1, \frac{\pi}{4})$	$(\frac{\sqrt{3}}{3}, \frac{\pi}{3})$	$(0, \frac{\pi}{2})$	$(-\frac{\sqrt{3}}{3}, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(-1, \frac{3\pi}{4})$	$(-\sqrt{3}, \frac{5\pi}{6})$	undefined	$(\sqrt{3}, \frac{7\pi}{6})$	$(1, \frac{5\pi}{4})$	$(\frac{\sqrt{3}}{3}, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(0, \frac{3\pi}{2})$	$(-\frac{\sqrt{3}}{3}, \frac{5\pi}{3})$	$(-1, \frac{7\pi}{4})$	$(-\sqrt{3}, \frac{11\pi}{6})$	undefined



(b) Parametric equations for the polar equation $r = \cot \theta$ are

$$\begin{aligned} x &= \cot \theta \cos \theta \\ y &= \cot \theta \sin \theta. \end{aligned}$$

49. First we recall that for a polar point (r, θ) , by adding π to the angle θ and changing the sign of r , we arrive at the same point. In other words, $(r, \theta) = (-r, \theta + \pi)$. To show that $r_1 = 4(\cos \theta + 1)$ has the same graph as $r_2 = 4(\cos \theta - 1)$, we will show that every point on r_1 is also on r_2 , and vice versa.

Recalling the angle addition formula for cosine, we have

$$\cos(\theta + \pi) = \cos \theta \cos \pi - \sin \theta \sin \pi = -\cos \theta.$$

If (r, θ) is a point on the graph of r_1 , then $r_1(\theta) = r$ and

$$r_2(\theta + \pi) = 4(\cos(\theta + \pi) - 1) = 4(-\cos \theta - 1) = -4(\cos \theta + 1) = -r_1(\theta) = -r$$

and so $(-r, \theta + \pi) = (r, \theta)$ is also a point on the graph of r_2 .

If (r, θ) is a point on the graph of r_2 , then $r_2(\theta) = r$ and

$$r_1(\theta + \pi) = 4(\cos(\theta + \pi) + 1) = 4(-\cos \theta + 1) = -4(\cos \theta - 1) = -r_2(\theta) = -r$$

and so $(-r, \theta + \pi) = (r, \theta)$ is also a point on the graph of r_1 .

50. First we recall that for a polar point (r, θ) , by adding π to the angle θ and changing the sign of r , we arrive at the same point. In other words, $(r, \theta) = (-r, \theta + \pi)$. To show that $r_1 = 5(\sin \theta + 1)$ has the same graph as $r_2 = 5(\sin \theta - 1)$, we will show that every point on r_1 is also on r_2 , and vice versa.

Recalling the angle addition formula for sine, we have

$$\sin(\theta + \pi) = \sin \theta \cos \pi + \sin \pi \cos \theta = -\sin \theta.$$

If (r, θ) is a point on the graph of r_1 , then $r_1(\theta) = r$ and

$$r_2(\theta + \pi) = 5(\sin(\theta + \pi) - 1) = 5(-\sin \theta - 1) = -5(\sin \theta + 1) = -r_1(\theta) = -r$$

and so $(-r, \theta + \pi) = (r, \theta)$ is also a point on the graph of r_2 .

If (r, θ) is a point on the graph of r_2 , then $r_2(\theta) = r$ and

$$r_1(\theta + \pi) = 5(\sin(\theta + \pi) + 1) = 5(-\sin \theta + 1) = -5(\sin \theta - 1) = -r_2(\theta) = -r$$

and so $(-r, \theta + \pi) = (r, \theta)$ is also a point on the graph of r_1 .

51. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = \theta$. Then $\frac{dr}{d\theta} = 1$ and

$$s = \int_0^{2\pi} \sqrt{(\theta)^2 + (1)^2} d\theta = \left[\frac{\theta}{2} \sqrt{\theta^2 + 1} + \frac{1}{2} \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right]_0^{2\pi},$$

where we have used the Table of Integrals 49 with $a = 1$. Then

$$s = \boxed{\pi \sqrt{4\pi^2 + 1} + \frac{1}{2} \ln \left(2\pi + \sqrt{4\pi^2 + 1} \right)}.$$

52. We use the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = 3\theta$. Then $\frac{dr}{d\theta} = 3$ and

$$s = \int_0^{2\pi} \sqrt{(3\theta)^2 + (3)^2} d\theta = 3 \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta = 3 \left[\frac{\theta}{2} \sqrt{\theta^2 + 1} + \frac{1}{2} \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right]_0^{2\pi},$$

where we have used the Table of Integrals 49 with $a = 1$. Then

$$s = \boxed{3\pi \sqrt{4\pi^2 + 1} + \frac{3}{2} \ln \left(2\pi + \sqrt{4\pi^2 + 1} \right)}.$$

53. The perimeter of the cardioid is found by calculating the arc length of the cardioid over the interval $[-\pi, \pi]$. We will exploit symmetry and find the length s of the top half of the cardioid using the arc length

formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ with $r = 1 - \cos \theta$ and then double it. With the function $r = 1 - \cos \theta$, we have $\frac{dr}{d\theta} = \sin \theta$ and

$$s = \int_0^{\pi} \sqrt{(1 - \cos \theta)^2 + (\sin \theta)^2} d\theta = \int_0^{\pi} \sqrt{2 - 2 \cos \theta} d\theta.$$

Let $u = 2 - 2 \cos \theta$ and then $du = 2 \sin \theta d\theta$. Solving for $\cos \theta$ in the first equation, we have

$$\cos \theta = \frac{2 - u}{2}.$$

Using the Pythagorean Identity $\cos^2 \theta + \sin^2 \theta = 1$, on the interval $0 \leq \theta \leq \pi$ we find that

$$\sin \theta = \frac{\sqrt{4u - u^2}}{2} = \frac{\sqrt{u}\sqrt{4 - u}}{2},$$

and so

$$\begin{aligned} du &= 2 \sin \theta \, d\theta \\ \frac{1}{\sqrt{u}\sqrt{4-u}} du &= d\theta. \end{aligned}$$

Changing the limits of integration to $u = 2 - 2 \cos(0) = 0$ to $u = 2 - 2 \cos \pi = 4$, the integral for s becomes

$$s = \int_0^4 \sqrt{u} \cdot \frac{1}{\sqrt{u}\sqrt{4-u}} du = \int_0^4 \frac{1}{\sqrt{4-u}} du.$$

Another substitution, $w = 4 - u$ with $dw = -du$, the limits of integration change to $w = 4 - (0) = 4$ to $w = 4 - (4) = 0$, and we have

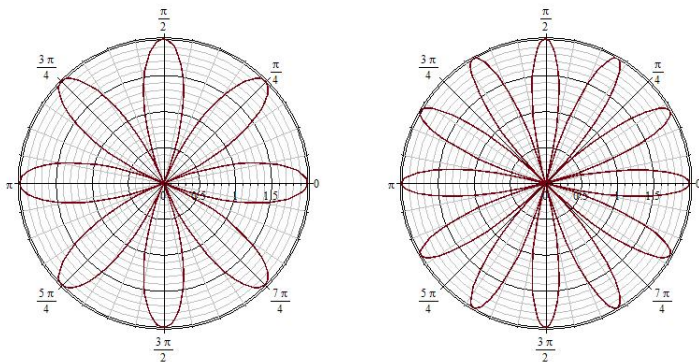
$$s = \int_4^0 -\frac{1}{\sqrt{w}} dw = \int_0^4 \frac{1}{\sqrt{w}} dw.$$

Notice that this is an improper integral. So

$$\begin{aligned} s &= \lim_{b \rightarrow 0^+} \int_b^4 \frac{1}{\sqrt{w}} dw \\ &= \lim_{b \rightarrow 0^+} [2\sqrt{w}]_b^4 = \lim_{b \rightarrow 0^+} [4 - \sqrt{b}] \\ &= 4. \end{aligned}$$

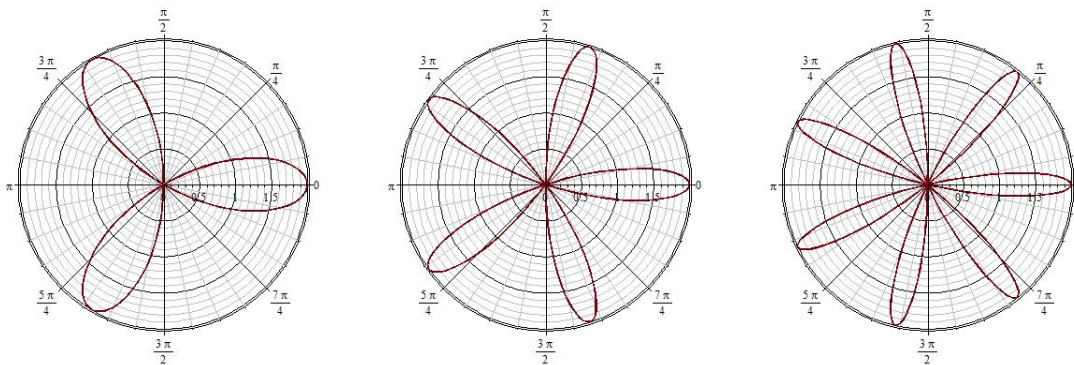
Since the top half of the cardioid has length 4, the full cardioid has perimeter equal to $\boxed{8}$.

54. (a) The left graph is r_1 and the right graph is r_2 .

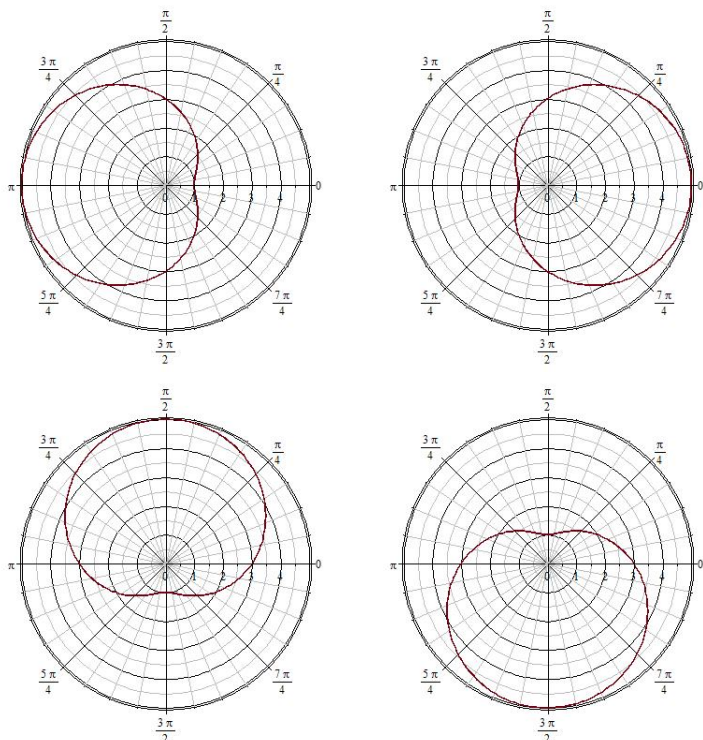


r_1 has 8 petals; r_2 has 12.

(b) Answers will vary. The graphs below, from left to right, are r_1, r_2, r_3 .



55. The graphs below, from left to right and top to bottom, are r_1, r_2, r_3, r_4 .



For Problems 56-61, using the parameterizations $x(\theta) = r \cos \theta = f(\theta) \cos \theta$ and $y(\theta) = r \sin \theta = f(\theta) \sin \theta$ we can find $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. Then we have a horizontal tangent line if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$, and we have a vertical tangent line if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

56. Parametric equations for $r = 1 - \sin \theta$ are

$$x = (1 - \sin \theta) \cos \theta \quad y = (1 - \sin \theta) \sin \theta$$

and

$$\frac{dy}{d\theta} = (-\cos \theta) \sin \theta + (1 - \sin \theta) \cos \theta = \cos \theta (1 - 2 \sin \theta)$$

$$\frac{dx}{d\theta} = (-\cos \theta) \cos \theta - (1 - \sin \theta) \sin \theta = -\cos^2 \theta - \sin \theta + \sin^2 \theta = 2 \sin^2 \theta - \sin \theta - 1 = (2 \sin \theta + 1)(\sin \theta - 1).$$

Horizontal Tangent Lines on $0 \leq \theta \leq 2\pi$:

$$\begin{aligned} \frac{dy}{d\theta} &= 0 \\ \cos \theta (1 - 2 \sin \theta) &= 0 \\ \cos \theta &= 0 \quad \text{or} \quad 1 - 2 \sin \theta = 0 \\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

Notice that $\theta = \frac{\pi}{2}$ makes $\frac{dx}{d\theta} = 0$ so we must exclude it but the others do not make $\frac{dx}{d\theta}$ zero. The y -coordinates of the points corresponding to $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ are $-2, \frac{1}{4}, \frac{1}{4}$, respectively. The horizontal tangent

lines are then $y = -2$ and $y = \frac{1}{4}$.

Vertical Tangent Lines on $0 \leq \theta \leq 2\pi$:

$$\begin{aligned}\frac{dx}{d\theta} &= 0 \\ (2 \sin \theta + 1)(\sin \theta - 1) &= 0 \\ 2 \sin \theta + 1 = 0 &\text{ or } \sin \theta - 1 = 0 \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}.\end{aligned}$$

Notice that $\theta = \frac{\pi}{2}$ makes $\frac{dy}{d\theta} = 0$ so we must exclude it where the others do not make $\frac{dy}{d\theta}$ zero. The x -coordinates of the points corresponding to $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ are $-\frac{3\sqrt{3}}{4}$ and $\frac{3\sqrt{3}}{4}$, respectively. The vertical

tangent lines are then $x = -\frac{3\sqrt{3}}{4}$ and $x = \frac{3\sqrt{3}}{4}$.

57. Parametric equations for $r = 3 + 3 \cos \theta$ are

$$x = (3 + 3 \cos \theta) \cos \theta \quad y = (3 + 3 \cos \theta) \sin \theta$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= (-3 \sin \theta) \sin \theta + (3 + 3 \cos \theta) \cos \theta = -3 \sin^2 \theta + 3 \cos \theta + 3 \cos^2 \theta = 6 \cos^2 \theta + 3 \cos \theta - 3 \\ &= 3(\cos \theta + 1)(2 \cos \theta - 1) \\ \frac{dx}{d\theta} &= (-3 \sin \theta) \cos \theta - (3 + 3 \cos \theta) \sin \theta = -3 \sin \theta \cos \theta - 3 \sin \theta - 3 \cos \theta \sin \theta \\ &= -\sin \theta(6 \cos \theta + 3).\end{aligned}$$

Horizontal Tangent Lines on $0 \leq \theta \leq 2\pi$:

$$\begin{aligned}\frac{dy}{d\theta} &= 0 \\ 3(\cos \theta + 1)(2 \cos \theta - 1) &= 0 \\ \cos \theta + 1 = 0 &\text{ or } 2 \cos \theta - 1 = 0 \\ \theta &= \pi, \frac{\pi}{3}, \frac{5\pi}{3}.\end{aligned}$$

Notice that $\theta = \pi$ makes $\frac{dx}{d\theta} = 0$ so we must exclude it but the others do not make $\frac{dx}{d\theta}$ zero. The y -coordinates of the points corresponding to $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$ are $\frac{9\sqrt{3}}{4}$ and $-\frac{9\sqrt{3}}{4}$, respectively. The horizontal

tangent lines are then $y = \frac{9\sqrt{3}}{4}$ and $y = -\frac{9\sqrt{3}}{4}$.

Vertical Tangent Lines on $0 \leq \theta \leq 2\pi$:

$$\begin{aligned}\frac{dx}{d\theta} &= 0 \\ -\sin \theta(6 \cos \theta + 3) &= 0 \\ -\sin \theta = 0 &\text{ or } 6 \cos \theta + 3 = 0 \\ \theta &= 0, \pi, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}.\end{aligned}$$

Notice that $\theta = \pi$ makes $\frac{dy}{d\theta} = 0$ so we must exclude it but the others do not make $\frac{dy}{d\theta}$ zero. The x -coordinates of the points corresponding to $\theta = 0, 2\pi, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ are 6, 6, $-\frac{3}{4}$, and $-\frac{3}{4}$, respectively. The

vertical tangent lines are then $x = 6$ and $x = -\frac{3}{4}$.

58. Parametric equations for $r = 1 + 2 \cos \theta$ are

$$x = (1 + 2 \cos \theta) \cos \theta \quad y = (1 + 2 \cos \theta) \sin \theta$$

and

$$\begin{aligned} \frac{dy}{d\theta} &= (-2 \sin \theta) \sin \theta + (1 + 2 \cos \theta) \cos \theta \\ \frac{dx}{d\theta} &= (-2 \sin \theta) \cos \theta - (1 + 2 \cos \theta) \sin \theta. \end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$, we find that $\theta \approx 0.936, 2.574, 3.709, 5.347$.

None of these values make $\frac{dx}{d\theta} = 0$. The y -coordinates of the points corresponding to $\theta \approx 0.936, 2.574, 3.709, 5.347$ are approximately 1.76, -0.369, 0.369, and -1.76, respectively. The horizontal tangent lines are then $y \approx \pm 1.76$ and $y \approx \pm 0.369$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$, we find that $\theta = 0, \pi, 2\pi$ and $\theta \approx 1.823, 4.46$.

None of these values make $\frac{dy}{d\theta} = 0$. The x -coordinates of the points corresponding to $\theta = 0, \pi, 2\pi$ and $\theta \approx 1.823, 4.46$ are 3, 1, 3, and approximately -0.125, -0.125, respectively. The vertical tangent lines are then $x = 1$, $x = 3$ and $x \approx -0.125$.

59. Parametric equations for $r = 2 \cos(2\theta)$ are

$$x = (2 \cos(2\theta)) \cos \theta \quad y = (2 \cos(2\theta)) \sin \theta$$

and

$$\begin{aligned} \frac{dy}{d\theta} &= (-4 \sin(2\theta)) \sin \theta + (2 \cos(2\theta)) \cos \theta \\ \frac{dx}{d\theta} &= (-4 \sin(2\theta)) \cos \theta - (2 \cos(2\theta)) \sin \theta. \end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$, we find that

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right), -\tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right), -\tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right) + \pi, \tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right) - \pi. \text{ None of these values}$$

make $\frac{dx}{d\theta} = 0$. The y -coordinates of the points corresponding to

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right), -\tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right), -\tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right) + \pi, \tan^{-1} \left(\frac{\sqrt{6}}{\sqrt{30}} \right) - \pi \text{ are } -2, 2, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}},$$

respectively. The horizontal tangent lines are then $y = \pm 2$ and $y \pm \frac{4}{3\sqrt{6}}$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$, we find that $\theta = 0, \pi, 2\pi, \tan^{-1}(\sqrt{5}), -\tan^{-1}(\sqrt{5}) +$

$\pi, -\tan^{-1}(\sqrt{5}), \tan^{-1}(\sqrt{5}) - \pi$. None of these values make $\frac{dy}{d\theta} = 0$. The x -coordinates of the points corresponding to $\theta = 0, \pi, 2\pi, \tan^{-1}(\sqrt{5}), -\tan^{-1}(\sqrt{5}) + \pi, -\tan^{-1}(\sqrt{5}), \tan^{-1}(\sqrt{5}) - \pi$ are

$$2, -2, 2, \frac{4}{3\sqrt{6}}, \frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}, -\frac{4}{3\sqrt{6}}, \text{ respectively. The vertical tangent lines are then } x = \pm 2 \text{ and } x = \pm \frac{4}{3\sqrt{6}}.$$

60. Parametric equations for $r = e^{\theta/5}$ are

$$x = (e^{\theta/5}) \cos \theta \quad y = (e^{\theta/5}) \sin \theta$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= \left(\frac{1}{5}e^{\theta/5}\right)\sin\theta + \left(e^{\theta/5}\right)\cos\theta \\ \frac{dx}{d\theta} &= \left(\frac{1}{5}e^{\theta/5}\right)\cos\theta - \left(e^{\theta/5}\right)\sin\theta.\end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$ on $0 \leq \theta \leq 2\pi$, we find that $\theta = \tan^{-1}\left(\frac{1}{5}\right), \tan^{-1}\left(\frac{1}{5}\right) + \pi$. Neither of these values make $\frac{dx}{d\theta} = 0$. The y -coordinates of the points corresponding to $\theta = \tan^{-1}\left(\frac{1}{5}\right), \tan^{-1}\left(\frac{1}{5}\right) + \pi$ are $\frac{\sqrt{26}}{26}e^{1/5 \tan^{-1}(1/5)}, \frac{\sqrt{26}}{26}e^{1/5 \tan^{-1}(1/5+1/5\pi)}$, respectively.

The horizontal tangent lines are then $y = \frac{\sqrt{26}}{26}e^{1/5 \tan^{-1}(1/5)}$ and $y = \frac{\sqrt{26}}{26}e^{1/5 \tan^{-1}(1/5+1/5\pi)}$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$ on $-\pi \leq \theta \leq \pi$, we find that $\theta = -\tan^{-1}(5), -\tan^{-1}(5) + \pi$. Neither of these values make $\frac{dy}{d\theta} = 0$. The x -coordinates of the points corresponding to $\theta = -\tan^{-1}(5), -\tan^{-1}(5) + \pi$ are $\frac{\sqrt{26}}{26}e^{-1/5 \tan^{-1}(5)}, \frac{\sqrt{26}}{26}e^{-1/5 \tan^{-1}(5)+1/5\pi}$, respectively. The vertical tangent lines are then

$$x = \frac{\sqrt{26}}{26}e^{-1/5 \tan^{-1}(5)} \text{ and } x = \frac{\sqrt{26}}{26}e^{-1/5 \tan^{-1}(5)+1/5\pi}.$$

61. In Problem 33, we found parametric equations for $r^2 = 4\sin(2\theta)$.

$$x = r \cos \theta = 2\sqrt{\sin(2\theta)} \cos \theta \quad y = r \sin \theta = 2\sqrt{\sin(2\theta)} \sin \theta, \quad \sin(2\theta) > 0.$$

Using a CAS to find the derivatives,

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{2 \cos \theta (4 \cos^2 \theta - 3)}{\sqrt{\sin(2\theta)}} \\ \frac{dy}{d\theta} &= \frac{2 \sin \theta (4 \cos^2 \theta - 1)}{\sqrt{\sin(2\theta)}}.\end{aligned}$$

Horizontal Tangent Lines: Using a CAS to find where $\frac{dy}{d\theta} = 0$ on $0 \leq \theta \leq 2\pi$ where $\sin(2\theta) > 0$, we find that $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. Neither of these values make $\frac{dx}{d\theta} = 0$. The y -coordinates of the points corresponding to $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$ are $3^{3/4}\frac{\sqrt{2}}{2}$ and $-3^{3/4}\frac{\sqrt{2}}{2}$, respectively. The horizontal tangent lines are then $y = \pm \frac{3^{3/4}}{\sqrt{2}}$.

Vertical Tangent Lines: Using a CAS to find where $\frac{dx}{d\theta} = 0$ on $0 \leq \theta \leq 2\pi$ where $\sin(2\theta) > 0$, we find that $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. Neither of these values make $\frac{dy}{d\theta} = 0$. The x -coordinates of the points corresponding to $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$ are $3^{3/4}\frac{\sqrt{2}}{2}$ and $-3^{3/4}\frac{\sqrt{2}}{2}$, respectively. The vertical tangent lines are then $x = \pm \frac{3^{3/4}}{\sqrt{2}}$.

62. (a) Answers will vary.

(b) Since $\cos \theta$ is an even function, $\cos \theta = \cos(-\theta)$. So $3 + 2\cos(-\theta) = 3 + 2\cos \theta$ and is symmetric with respect to the polar axis.

63. (a) Answers will vary.

(b) Since $r^2 = (-r)^2$, we have $(-r)^2 = r^2 = 4\sin(2\theta)$ and so the lemniscate is symmetric with respect to the pole.

64. (a) Answers will vary.

(b) Using the angle subtraction formula for cosine,

$$\begin{aligned} 2 \cos(2(\pi - \theta)) &= 2 \cos(2\pi - 2\theta) \\ &= 2 [\cos(2\pi) \cos(2\theta) + \sin(2\pi) \sin(2\theta)] \\ &= 2 \cos(2\theta), \end{aligned}$$

and so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Challenge Problems

65. Symmetry with respect to the pole: Substituting $\theta + \pi$ for θ and using the angle addition formula for sine, we have

$$\begin{aligned} r &= \sin(2(\theta + \pi)) = \sin(2\theta + 2\pi) \\ r &= \sin(2\theta) \cos(2\pi) + \sin(2\pi) \cos(2\theta) \\ r &= \sin(2\theta), \end{aligned}$$

and an equivalent equation results implying that the graph is symmetric with respect to the pole.

Symmetry with respect to the polar axis: Substituting $-\theta$ for θ and recalling that $\sin \theta$ is an odd function, we have

$$\begin{aligned} r &= \sin(-2\theta) \\ r &= -\sin(2\theta), \end{aligned}$$

which is not an equivalent equation so the test fails.

66. Following the hint, parametric equations are $x = a \sin^3 \frac{\theta}{3} \cos \theta$ and $y = a \sin^3 \frac{\theta}{3} \sin \theta$. We will use the

arc length formula $s = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$ from Section 9.2. Then

$$\begin{aligned} \frac{dx}{d\theta} &= a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3} \cos \theta - a \sin^3 \frac{\theta}{3} \sin \theta \\ \frac{dy}{d\theta} &= a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3} \sin \theta + a \sin^3 \frac{\theta}{3} \cos \theta. \end{aligned}$$

Then, after expanding and using the Pythagorean Identity $\cos^2 \alpha + \sin^2 \alpha = 1$ for any angle α ,

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3} \cos \theta - a \sin^3 \frac{\theta}{3} \sin \theta\right)^2 + \left(a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3} \sin \theta + a \sin^3 \frac{\theta}{3} \cos \theta\right)^2 \\ &= a^2 \sin^4 \frac{\theta}{3}. \end{aligned}$$

The function $\sin^3 \frac{\theta}{3}$ is traced out exactly once on the interval $[0, 3\pi]$. The period of $\sin^3 \frac{\theta}{3}$ is 6π , but the point (r, θ) for $0 \leq \theta \leq 3\pi$ is the same as the point $(-r, \theta + 3\pi)$. Since $a > 0$, $\sqrt{a^2} = a$ and the arc length is then

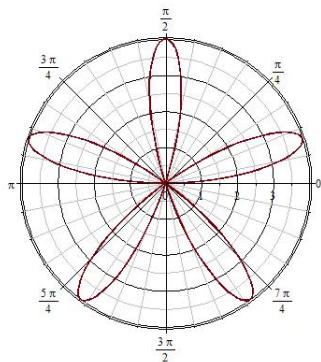
$$s = \int_0^{3\pi} a \sin^2 \frac{\theta}{3} d\theta.$$

Making the substitution $u = \frac{\theta}{3}$ so $du = \frac{1}{3} d\theta$, and changing the limits $u = \frac{(0)}{3} = 0$ to $u = \frac{3\pi}{3} = \pi$, we have

$$\begin{aligned} s &= \int_0^\pi 3a \sin^2 u du \\ &= 3a \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^\pi \\ &= \boxed{\frac{3}{2}\pi a}, \end{aligned}$$

where to compute the integral we used Table of Integrals 84.

67. (a)



(b) The graph intersects the pole when $r = 0$. Setting $4 \sin(5\theta) = 0$ we see that $5\theta = \pi n$ for any integer n , or

$$\theta = \frac{n}{5}\pi.$$

We seek the smallest positive value of θ , which is when $n = 1$. So $\alpha = \frac{1}{5}\pi$.

(c) The arc length of the petal is given by

$$s = \int_0^{\pi/5} \sqrt{(4 \sin(5\theta))^2 + (20 \cos(5\theta))^2} d\theta.$$

Using a CAS, we find that $s \approx \boxed{8.404}$.

9.6 Area in Polar Coordinates

Note: Throughout this assignment, we will be integrating $\sin^2(a\theta)$ or $\cos^2(a\theta)$ frequently. To integrate them we use the formulas $\sin^2(a\theta) = \frac{1 - \cos(2a\theta)}{2}$ and $\cos^2(a\theta) = \frac{1 + \cos(2a\theta)}{2}$. With these formulas, we have

$$\begin{aligned}\int \sin^2(a\theta) \, d\theta &= \frac{1}{2}\theta - \frac{1}{4a} \sin(2a\theta) = \frac{1}{2}\theta - \frac{1}{2a} \cos(a\theta) \sin(a\theta) \\ \int \cos^2(a\theta) \, d\theta &= \frac{1}{2}\theta + \frac{1}{4a} \sin(2a\theta) = \frac{1}{2}\theta + \frac{1}{2a} \cos(a\theta) \sin(a\theta)\end{aligned}$$

In each problem where the above equations are used, the problem will be marked with an asterisk (*).

Concepts and Vocabulary

1. The area A of a sector of a circle of radius r and central angle θ is $A = \boxed{\frac{1}{2}r^2\theta}$.

2. **True**.

3. **False**. The area is given by $A = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta$.

4. **True**.

Skill Building

5*. The area of the shaded region in quadrant I equals one-fourth of the area A of the shaded region. The area of the shaded region in quadrant I is equal to the area of the non-shaded region inside the rose in quadrant I which is swept out starting at $\theta = 0$ and, by symmetry, ends at $\theta = \frac{\pi}{4}$. The area of the

non-shaded region in quadrant I is given by $\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta$, and the area A we seek is 4 times this area,

$$\begin{aligned}A &= 4 \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta = 2 \int_0^{\pi/4} \cos^2(2\theta) \, d\theta \\ &= \left[\theta + \frac{1}{2} \cos(2\theta) \sin(2\theta) \right]_0^{\pi/4} = \boxed{\frac{\pi}{4}}.\end{aligned}$$

6*. The area A of the shaded region is equal to the area of the non-shaded region in quadrant I. The non-shaded region in quadrant I is swept out beginning with the ray $\theta = 0$ and ending at the point $(0, \theta)$, $0 < \theta < \frac{\pi}{2}$, where θ is the solution of the following equation.

$$\begin{aligned}2 \sin(3\theta) &= 0 \\ \sin(3\theta) &= 0 \\ 3\theta &= n\pi \quad (n \text{ any integer}) \\ \theta &= \frac{n}{3}\pi.\end{aligned}$$

Since $0 < \theta < \frac{\pi}{2}$, it must be that $\theta = \frac{\pi}{3}$. The area of the shaded region is then equal to $\int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta$.

$$\begin{aligned}A &= \int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/3} \frac{1}{2} (2 \sin(3\theta))^2 \, d\theta = 2 \int_0^{\pi/3} \sin^2(3\theta) \, d\theta \\ &= \left[\theta - \frac{1}{3} \sin(3\theta) \cos(3\theta) \right]_0^{\pi/3} = \boxed{\frac{\pi}{3}}.\end{aligned}$$

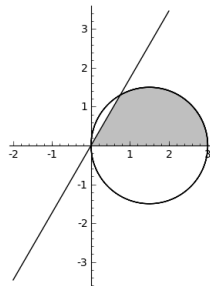
7*. The shaded region is swept out beginning with the ray $\theta = 0$ and ending with the ray $\theta = \pi$. The area A of the shaded region is then equal to $\int_0^\pi \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^\pi \frac{1}{2} r^2 d\theta = \int_0^\pi \frac{1}{2} (2 + 2 \sin \theta)^2 d\theta = 2 \int_0^\pi (1 + 2 \sin \theta + \sin^2 \theta) d\theta \\ &= [-4 \cos \theta + 3\theta - \sin \theta \cos \theta]_0^\pi = \boxed{8 + 3\pi}. \end{aligned}$$

8*. The shaded region is swept out beginning with the ray $\theta = -\frac{\pi}{2}$ and ending with the ray $\theta = \frac{\pi}{2}$. The area S of the shaded region is then equal to $\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (3 - 3 \cos \theta)^2 d\theta = \frac{9}{2} \int_{-\pi/2}^{\pi/2} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \left[-9 \sin \theta + \frac{27}{4} \theta + \frac{9}{4} \sin \theta \cos \theta \right]_{-\pi/2}^{\pi/2} = \boxed{\frac{27}{4} \pi - 18}. \end{aligned}$$

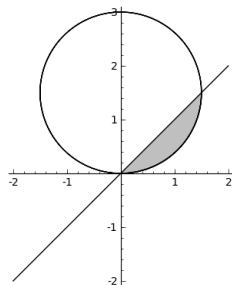
9*.



The area A of the shaded region is equal to $\int_0^{\pi/3} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} r^2 d\theta = \int_0^{\pi/3} \frac{1}{2} (3 \cos \theta)^2 d\theta = \frac{9}{2} \int_0^{\pi/3} \cos^2 \theta d\theta \\ &= \left[\frac{9}{4} \theta + \frac{9}{4} \cos \theta \sin \theta \right]_0^{\pi/3} = \frac{9\sqrt{3}}{16} + \frac{3\pi}{4} \\ &= \boxed{\frac{3}{16} (3\sqrt{3} + 4\pi)}. \end{aligned}$$

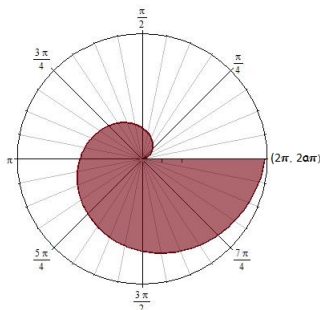
10*.



The area A of the shaded region is equal to $\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = \int_0^{\pi/4} \frac{1}{2} (3 \sin \theta)^2 d\theta = \frac{9}{2} \int_0^{\pi/4} \sin^2 \theta d\theta \\ &= \left[\frac{9}{4} \theta - \frac{9}{4} \cos \theta \sin \theta \right]_0^{\pi/4} = \boxed{-\frac{9}{8} + \frac{9\pi}{16}}. \end{aligned}$$

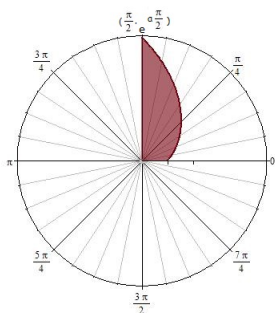
11.



[The graph shown is when $a = 1$.] The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (a\theta)^2 d\theta = \frac{a^2}{2} \int_0^{2\pi} \theta^2 d\theta \\ &= \left[\frac{a^2}{6} \theta^3 \right]_0^{2\pi} = \boxed{\frac{4\pi^3 a^2}{3}}. \end{aligned}$$

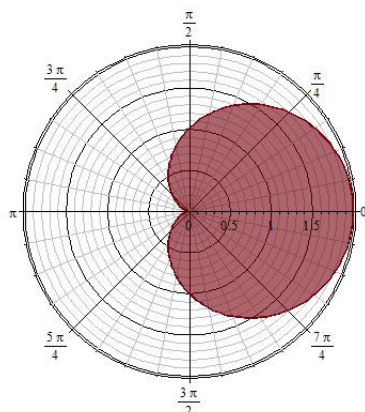
12.



[The graph shown is when $a = 1$.] The area A of the shaded region is equal to $\int_0^{\pi/2} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} (e^{a\theta})^2 d\theta \\ &= \left[\frac{1}{4a} e^{2a\theta} \right]_0^{\pi/2} = \boxed{\frac{e^{\pi a} - 1}{4a}}. \end{aligned}$$

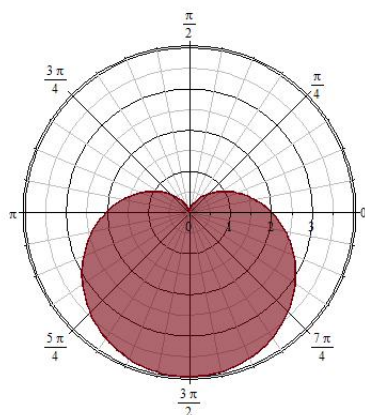
13*.



The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \left[\frac{3}{4} \theta + \sin \theta + \frac{1}{4} \cos \theta \sin \theta \right]_0^{2\pi} = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

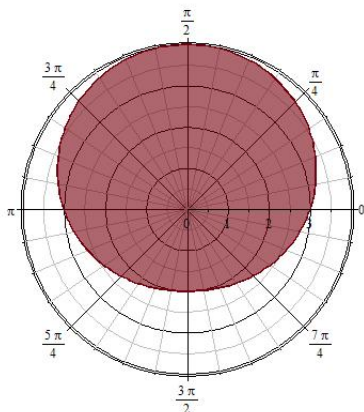
14*.



The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (2 - 2 \sin \theta)^2 d\theta = 2 \int_0^{2\pi} (1 - 2 \sin \theta + \sin^2 \theta) d\theta \\ &= [3\theta + 4 \cos \theta - \cos \theta \sin \theta]_0^{2\pi} = \boxed{6\pi}. \end{aligned}$$

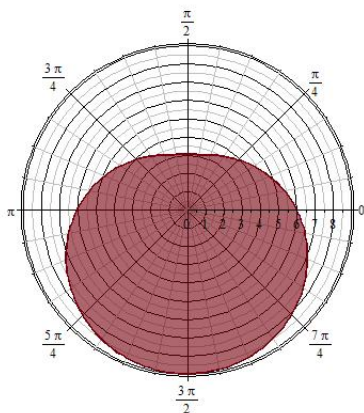
15*.



The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (3 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (9 + 6 \sin \theta + \sin^2 \theta) d\theta \\ &= \left[\frac{19}{4} \theta - 3 \cos \theta - \frac{1}{4} \cos \theta \sin \theta \right]_0^{2\pi} = \boxed{\frac{19\pi}{2}}. \end{aligned}$$

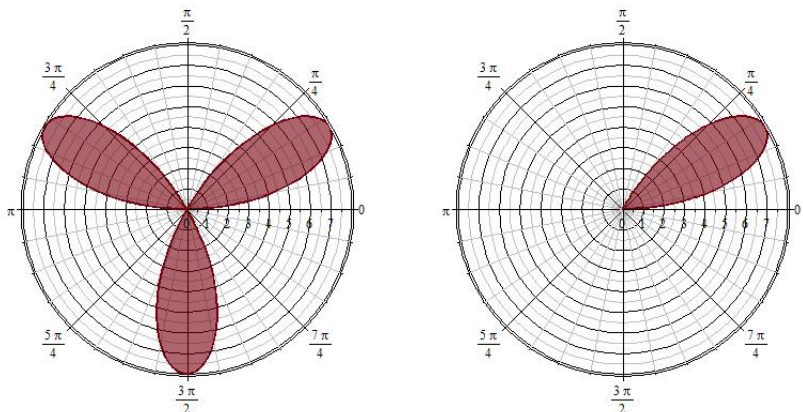
16*.



The area A of the shaded region is equal to $\int_0^{2\pi} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (3(2 - \sin \theta))^2 d\theta = \frac{9}{2} \int_0^{2\pi} (4 - 4 \sin \theta + \sin^2 \theta) d\theta \\ &= \left[\frac{81}{4} \theta + 18 \cos \theta - \frac{9}{4} \cos \theta \sin \theta \right]_0^{2\pi} = \boxed{\frac{81\pi}{2}}. \end{aligned}$$

17*.

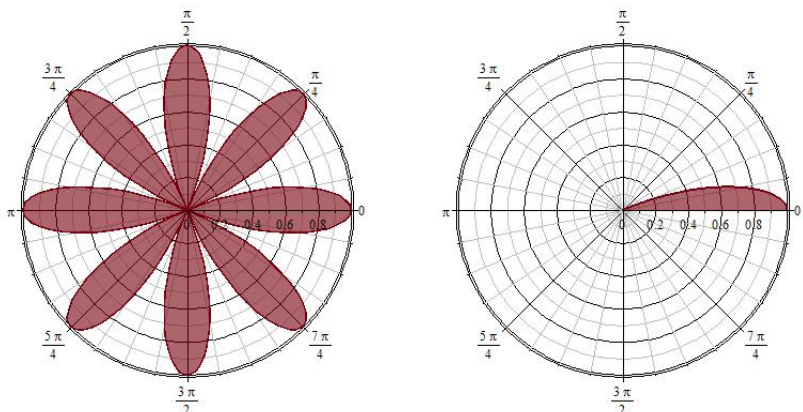


Exploiting symmetry, we will find the area A of the petal in quadrant I (see the second picture) which is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{3}$ and then triple it to find the full area of the rose. The area A of the petal in quadrant I is equal to $\int_0^{\pi/3} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} r^2 d\theta = \int_0^{\pi/3} \frac{1}{2} (8 \sin(3\theta))^2 d\theta = 32 \int_0^{\pi/3} \sin^2(3\theta) d\theta \\ &= \left[16\theta - \frac{16}{3} \sin(3\theta) \cos(3\theta) \right]_0^{\pi/3} = \frac{16\pi}{3}. \end{aligned}$$

The total area of the full rose is then $3 \cdot \frac{16\pi}{3} = \boxed{16\pi}$.

18*.

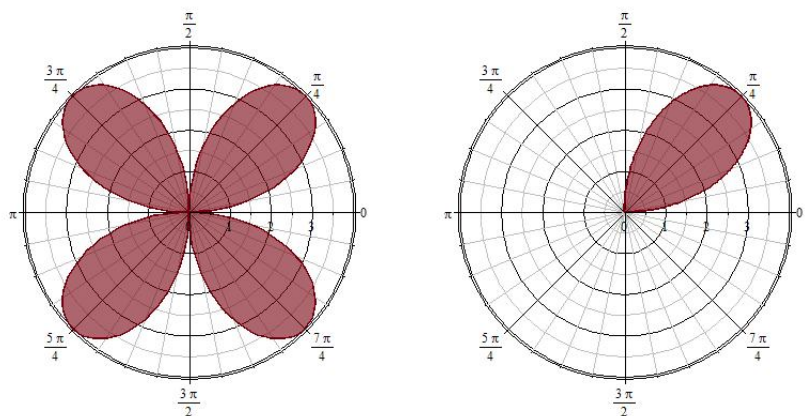


Exploiting symmetry, we will find the area A of the half-petal in quadrant I (see the second picture) which is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{8}$ (which is the first angle $\theta > 0$ such that $\cos(4\theta) = 0$) and then multiply it by 16 to get the full area of the rose. The area A of the half-petal in quadrant I is equal to $\int_0^{\pi/8} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/8} \frac{1}{2} r^2 d\theta = \int_0^{\pi/8} \frac{1}{2} (\cos(4\theta))^2 d\theta \\ &= \left[\frac{1}{4} \theta + \frac{1}{16} \sin(4\theta) \cos(4\theta) \right]_0^{\pi/8} = \frac{\pi}{32}. \end{aligned}$$

The total area of the full rose is then $16 \cdot \frac{\pi}{32} = \boxed{\frac{\pi}{2}}$.

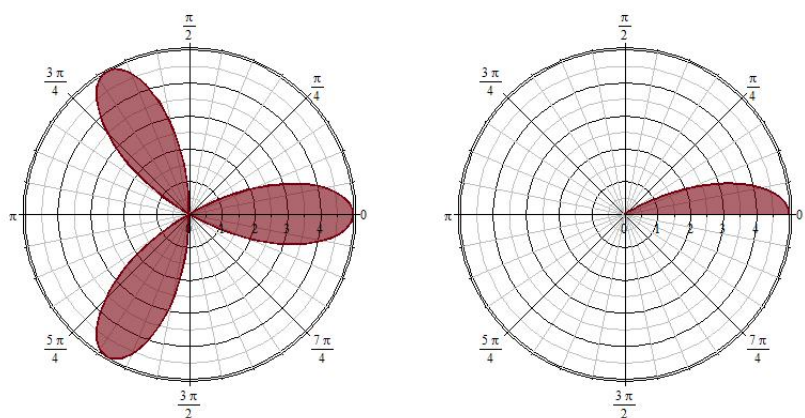
19*.



The area A of the petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{2}$ so A is equal to $\int_0^{\pi/2} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} (4 \sin(2\theta))^2 d\theta = 8 \int_0^{\pi/2} \sin^2(2\theta) d\theta \\ &= [4\theta - 2 \sin(2\theta) \cos(2\theta)]_0^{\pi/2} = \boxed{2\pi}. \end{aligned}$$

20*.

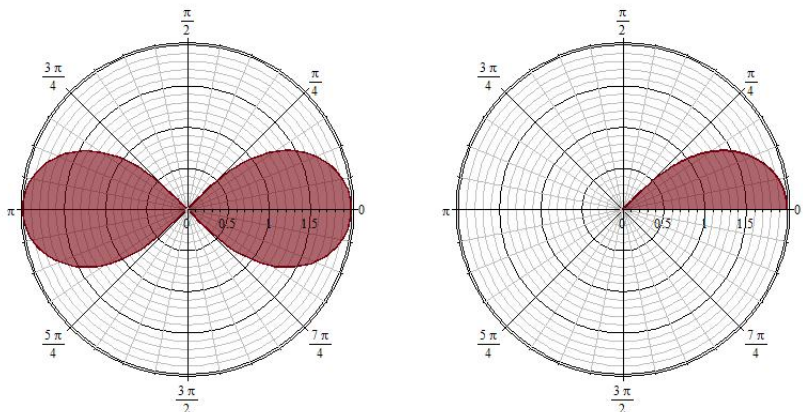


The area A of the half-petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{6}$ so A is equal to $\int_0^{\pi/6} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/6} \frac{1}{2} r^2 d\theta = \int_0^{\pi/6} \frac{1}{2} (5 \cos(3\theta))^2 d\theta = \frac{25}{2} \int_0^{\pi/6} \cos^2(3\theta) d\theta \\ &= \left[\frac{25}{4} \theta + \frac{25}{12} \sin(3\theta) \cos(3\theta) \right]_0^{\pi/6} = \frac{25\pi}{24}. \end{aligned}$$

Then a full petal has area $2 \cdot \frac{25\pi}{24} = \boxed{\frac{25\pi}{12}}$.

21.

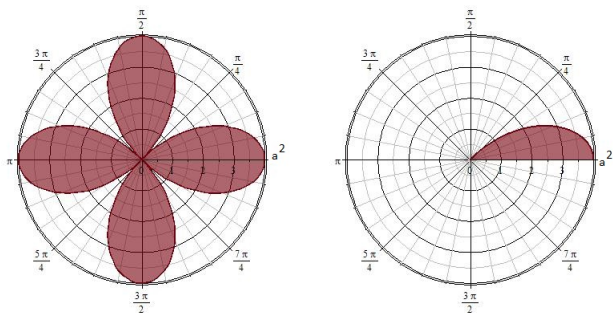


The area A of the half-petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{4}$ so A is equal to $\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = \int_0^{\pi/4} \frac{1}{2} (4 \cos(2\theta))^2 d\theta = 2 \int_0^{\pi/4} \cos^2(2\theta) d\theta \\ &= [\sin(2\theta)]_0^{\pi/4} = 1. \end{aligned}$$

Then a full petal has area $2 \cdot 1 = \boxed{2}$.

22*.

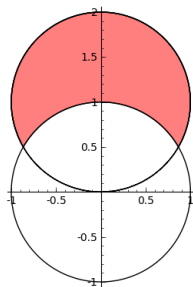


The area A of the half-petal in quadrant I (see the second picture) is swept out beginning with the ray $\theta = 0$ and ending at the ray $\theta = \frac{\pi}{4}$ so A is equal to $\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = \int_0^{\pi/4} \frac{1}{2} (a^2 \cos(2\theta))^2 d\theta = \frac{a^4}{2} \int_0^{\pi/4} \cos^2(2\theta) d\theta \\ &= \frac{a^4}{2} \left[\frac{1}{2} \theta + \frac{1}{4} \cos(2\theta) \sin(2\theta) \right]_0^{\pi/4} = \frac{a^4 \pi}{16}. \end{aligned}$$

Then a full petal has area $2 \cdot \frac{a^4 \pi}{16} = \boxed{\frac{a^4 \pi}{8}}$.

23*.



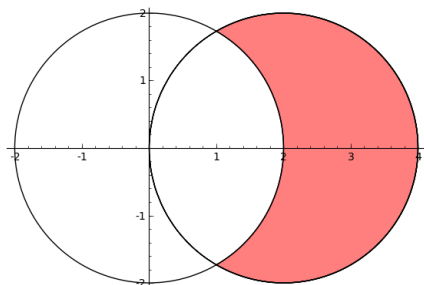
Referring to the picture we need to find the intersection points of the two circles, so we set them equal.

$$\begin{aligned} 2 \sin \theta &= 1 \\ \sin \theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}. \end{aligned}$$

Then

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2} (2 \sin \theta)^2 d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} (1)^2 d\theta \\ &= \left[-\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \theta \right]_{\pi/6}^{5\pi/6} - \left[\frac{1}{2} \theta \right]_{\pi/6}^{5\pi/6} \\ &= \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\pi}{3} = \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}. \end{aligned}$$

24*.



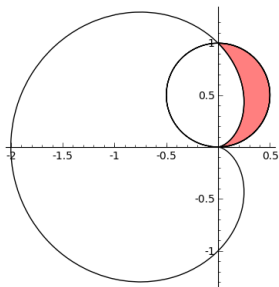
Referring to the picture we need to find the intersection points of the two circles, so we set them equal.

$$\begin{aligned} 4 \cos \theta &= 2 \\ \cos \theta &= \frac{1}{2} \\ \theta &= -\frac{\pi}{3}, \frac{\pi}{3}. \end{aligned}$$

Then

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} (4 \cos \theta)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2)^2 d\theta \\ &= [4 \sin \theta \cos \theta + 4\theta]_{-\pi/3}^{\pi/3} - [2\theta]_{-\pi/3}^{\pi/3} \\ &= 2\sqrt{3} + \frac{8\pi}{3} - \frac{4\pi}{3} = \boxed{2\sqrt{3} + \frac{4\pi}{3}}. \end{aligned}$$

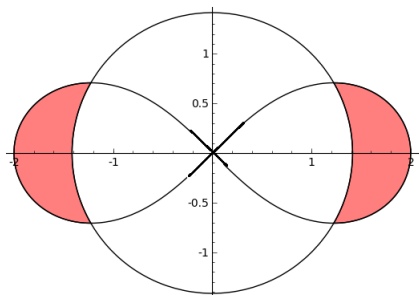
25*.



Referring to the picture we can see that the shaded region is swept out beginning with $\theta = 0$ and ending at $\theta = \frac{\pi}{2}$. Then the area A of the shaded region is

$$\begin{aligned}
 A &= \int_0^{\pi/2} \frac{1}{2}(\sin \theta)^2 d\theta - \int_0^{\pi/2} \frac{1}{2}(1 - \cos \theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} (\sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} (-\cos(2\theta) - 1 + 2 \cos \theta) d\theta \\
 &= \frac{1}{2} \left[-\frac{1}{2} \sin(2\theta) - \theta + 2 \sin \theta \right]_0^{\pi/2} \\
 &= \boxed{1 - \frac{\pi}{4}}.
 \end{aligned}$$

26.



Referring to the picture we need to find the intersection points of the polar curves, so we set $r^2 = r^2$.

$$\begin{aligned}
 4 \cos(2\theta) &= (\sqrt{2})^2 \\
 \cos(2\theta) &= \frac{1}{2} \\
 \theta &= -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}.
 \end{aligned}$$

Exploiting symmetry we will find the area of the shaded region in quadrant I and then quadruple it to find the full shaded area. The area A in quadrant I is swept out beginning with $\theta = 0$ and ending with $\theta = \frac{\pi}{6}$. Then

$$\begin{aligned}
 A &= \int_0^{\pi/6} \frac{1}{2}(4 \cos(2\theta)) d\theta - \int_0^{\pi/6} \frac{1}{2}(\sqrt{2})^2 d\theta \\
 &= \frac{\sqrt{3}}{2} - \frac{\pi}{6}.
 \end{aligned}$$

The full shaded area is then $4 \cdot \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) = \boxed{2\sqrt{3} - \frac{2\pi}{3}}$.

27*. Using formula (1) with $f(\theta) = \sin \theta$ and $f'(\theta) = \cos \theta$, the surface area S is

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} (\sin \theta) \sin \theta \sqrt{[\sin \theta]^2 + [\cos \theta]^2} d\theta \\ &= 2\pi \int_0^{\pi/2} \sin^2 \theta d\theta \\ &= 2\pi \left[\frac{1}{2}\theta - \frac{1}{2}\cos \theta \sin \theta \right]_0^{\pi/2} \\ &= 2\pi \left(\frac{1}{4}\pi \right) = \boxed{\frac{\pi^2}{2}}. \end{aligned}$$

28. Using formula (1) with $f(\theta) = 1 + \cos \theta$ and $f'(\theta) = -\sin \theta$, the surface area S is

$$\begin{aligned} S &= 2\pi \int_0^\pi (1 + \cos \theta) \sin \theta \sqrt{[1 + \cos \theta]^2 + [-\sin \theta]^2} d\theta \\ &= 2\pi \int_0^\pi (1 + \cos \theta) \sin \theta \sqrt{2(1 + \cos \theta)} d\theta = 2\sqrt{2}\pi \int_0^\pi (1 + \cos \theta)^{3/2} \sin \theta d\theta. \end{aligned}$$

Using the substitution $u = 1 + \cos \theta$, then $-du = \sin \theta d\theta$ and the new limits are $u = 1 + \cos(0) = 2$ to $u = 1 + \cos(\pi) = 0$, we have

$$\begin{aligned} S &= 2\sqrt{2}\pi \int_2^0 -u^{3/2} du = 2\sqrt{2}\pi \int_0^2 u^{3/2} du \\ &= 2\sqrt{2}\pi \left[\frac{2}{5}u^{5/2} \right]_0^2 = \boxed{\frac{32\pi}{5}}. \end{aligned}$$

29. Using formula (1) with $f(\theta) = e^\theta$ and $f'(\theta) = e^\theta$, the surface area S is

$$\begin{aligned} S &= 2\pi \int_0^\pi (e^\theta) \sin \theta \sqrt{[e^\theta]^2 + [e^\theta]^2} d\theta = 2\sqrt{2}\pi \int_0^\pi e^{2\theta} \sin \theta d\theta \\ &= 2\sqrt{2}\pi \left[\frac{e^{2\theta}}{5} (2 \sin \theta - \cos \theta) \right]_0^\pi = \boxed{\frac{2\sqrt{2}\pi}{5} (e^{2\pi} + 1)}, \end{aligned}$$

where we used Table of Integrals 125.

30. Using formula (1) with $f(\theta) = 2a \cos \theta$ and $f'(\theta) = -2a \sin \theta$, the surface area S is

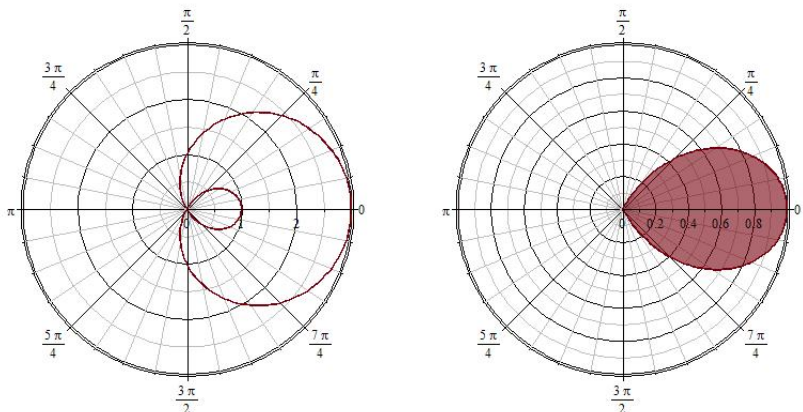
$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} (2a \cos \theta) \sin \theta \sqrt{[2a \cos \theta]^2 + [-2a \sin \theta]^2} d\theta = 8a^2\pi \int_0^{\pi/2} \cos \theta \sin \theta d\theta \\ &= 8a^2\pi \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = \boxed{4a^2\pi}. \end{aligned}$$

Applications and Extensions

31*. To find the area of the small loop, we need to find the values of θ that sweep out the inner loop. We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 1 + 2 \cos \theta &= 0 \\ \cos \theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

The two graphs show the full limaçon and just the small loop (zoomed in).



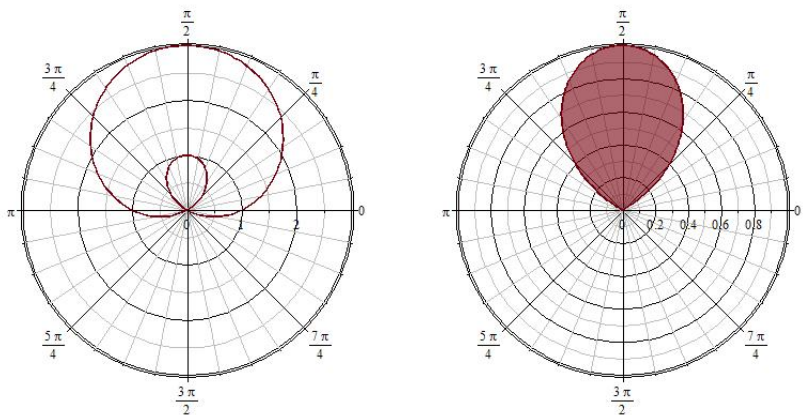
The area A of the small loop is

$$\begin{aligned}
 A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2} (1 + 2\cos\theta)^2 d\theta = \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta) d\theta \\
 &= \left[\frac{3}{2}\theta + 2\sin\theta + \cos\theta \sin\theta \right]_{2\pi/3}^{4\pi/3} = \boxed{\pi - \frac{3\sqrt{3}}{2}}.
 \end{aligned}$$

32*. To find the area of the small loop, we need to find the values of θ that sweep out the inner loop. We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned}
 1 + 2\sin\theta &= 0 \\
 \sin\theta &= -\frac{1}{2} \\
 \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}.
 \end{aligned}$$

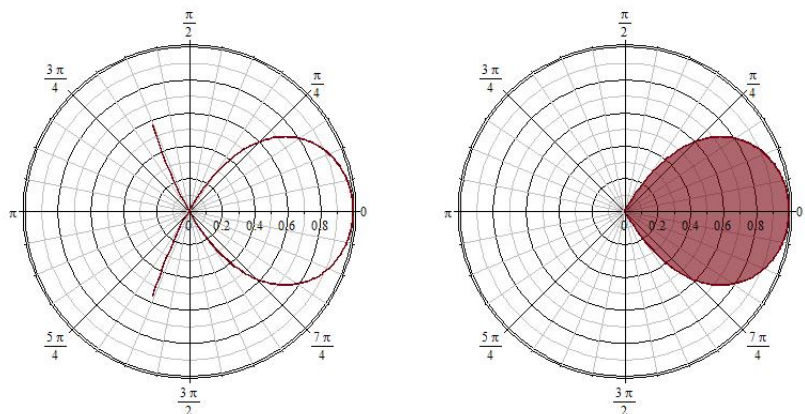
The two graphs show the full limaçon and just the small loop (zoomed in).



The area A of the small loop is

$$\begin{aligned}
 A &= \int_{7\pi/6}^{11\pi/6} \frac{1}{2} (1 + 2\sin\theta)^2 d\theta = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 4\sin\theta + 4\sin^2\theta) d\theta \\
 &= \left[\frac{3}{2}\theta - 2\cos\theta - \cos\theta \sin\theta \right]_{7\pi/6}^{11\pi/6} = \boxed{\pi - \frac{3\sqrt{3}}{2}}.
 \end{aligned}$$

33. The two graphs show the full graph and the loop.



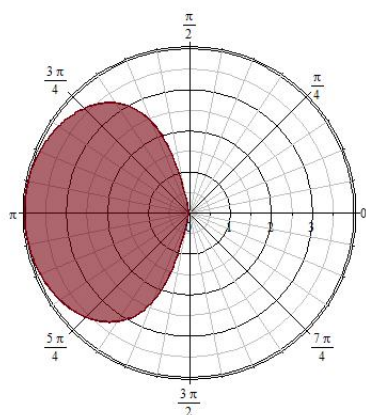
To find the area of loop, we need to find the values of θ that sweep out the loop. We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 2 - \sec \theta &= 0 \\ \sec \theta &= 2 \\ \theta &= \pm \frac{\pi}{3}. \end{aligned}$$

The area A of the small loop is

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2 - \sec \theta)^2 d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4 - 4 \sec \theta + \sec^2 \theta) d\theta \\ &= \left[2\theta - 2 \ln |\sec \theta + \tan \theta| + \frac{1}{2} \tan \theta \right]_{-\pi/3}^{\pi/3} \\ &= \left(\frac{2\pi}{3} - 2 \ln (2 + \sqrt{3}) + \frac{\sqrt{3}}{2} \right) - \left(-\frac{2\pi}{3} - 2 \ln (2 - \sqrt{3}) - \frac{\sqrt{3}}{2} \right) \\ &= \frac{4\pi}{3} - 2 \ln \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right) + \sqrt{3} = \frac{4\pi}{3} - 2 \ln ((2 + \sqrt{3})^2) + \sqrt{3} \\ &= \boxed{\frac{4\pi}{3} - 4 \ln (2 + \sqrt{3}) + \sqrt{3}}. \end{aligned}$$

34.



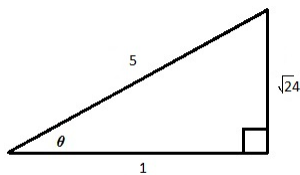
We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 5 + \sec \theta &= 0 \\ \cos \theta &= -\frac{1}{5} \\ \theta &= \cos^{-1}\left(-\frac{1}{5}\right), 2\pi - \cos^{-1}\left(-\frac{1}{5}\right). \end{aligned}$$

The area A of the loop is

$$\begin{aligned} A &= \int_{\cos^{-1}(-1/5)}^{2\pi - \cos^{-1}(-1/5)} \frac{1}{2}(5 + \sec \theta)^2 d\theta = \int_{\cos^{-1}(-1/5)}^{2\pi - \cos^{-1}(-1/5)} \left(\frac{25}{2} + 5 \sec \theta + \frac{1}{2} \sec^2 \theta\right) d\theta \\ &= \left[\frac{25}{2}\theta + 5 \ln |\sec \theta + \tan \theta| + \frac{1}{2} \tan \theta\right]_{\cos^{-1}(-1/5)}^{2\pi - \cos^{-1}(-1/5)}. \end{aligned}$$

Since $\sec(2\pi + \theta) = \sec \theta$ and $\tan(2\pi + \theta) = \tan \theta$, when substituting in the upper limit $2\pi - \cos^{-1}\left(-\frac{1}{5}\right)$ to the secant and tangent functions, we need only substitute in $-\cos^{-1}\left(-\frac{1}{5}\right)$. To compute A , we will need to simplify the following expressions using the picture where $\cos \theta = -\frac{1}{5}$ or, equivalently, $\theta = \cos^{-1}\left(-\frac{1}{5}\right)$ which means $\frac{\pi}{2} < \theta < \pi$.

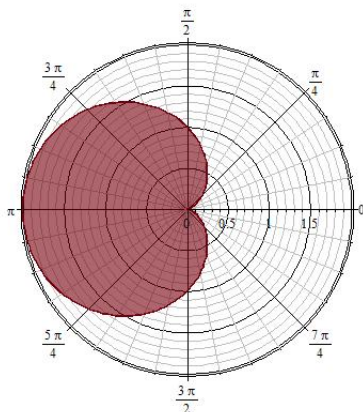


$$\begin{aligned} \sec\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) &= -5 \quad \text{and} \quad \sec\left(-\cos^{-1}\left(-\frac{1}{5}\right)\right) = -5 \\ \tan\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) &= -\sqrt{24} = -2\sqrt{6} \quad \text{and} \quad \tan\left(-\cos^{-1}\left(-\frac{1}{5}\right)\right) = \sqrt{24} = 2\sqrt{6} \end{aligned}$$

So

$$\begin{aligned} A &= \frac{25}{2} \left(2\pi - \cos^{-1}\left(-\frac{1}{5}\right)\right) + 5 \ln \left[\sec\left(-\cos^{-1}\left(-\frac{1}{5}\right)\right) + \tan\left(-\cos^{-1}\left(-\frac{1}{5}\right)\right) \right] + \frac{1}{2} \tan\left(-\cos^{-1}\left(-\frac{1}{5}\right)\right) \\ &\quad - \frac{25}{2} \left(\cos^{-1}\left(-\frac{1}{5}\right)\right) + 5 \ln \left[\sec\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) + \tan\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) \right] + \frac{1}{2} \tan\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) \\ &= \frac{25}{2} \left(2\pi - \cos^{-1}\left(-\frac{1}{5}\right)\right) + 5 \ln |-5 + 2\sqrt{6}| + \frac{1}{2}(2\sqrt{6}) - \frac{25}{2} \cos^{-1}\left(-\frac{1}{5}\right) - 5 \ln |-5 - 2\sqrt{6}| - \frac{1}{2}(-2\sqrt{6}) \\ &= \boxed{25\pi - 25 \cos^{-1}\left(-\frac{1}{5}\right) + 5 \ln \left[\frac{-5 + 2\sqrt{6}}{-5 - 2\sqrt{6}} \right] + 2\sqrt{6}}. \end{aligned}$$

35.



Exploiting symmetry, we will find the area A of the regions in the first and second quadrant, and then double that area. The area in the first two quadrants is swept out starting at the ray $\theta = 0$ and ending at $\theta = \pi$. Then

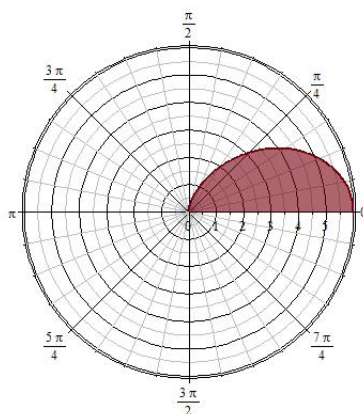
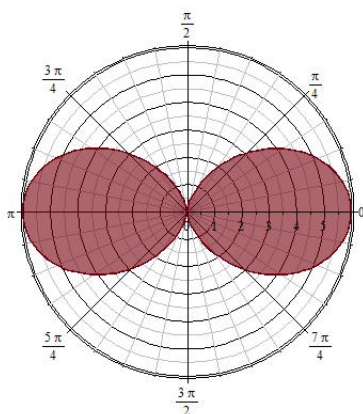
$$A = \int_0^\pi \frac{1}{2} \left[2 \sin^2 \frac{\theta}{2} \right]^2 d\theta = 2 \int_0^\pi \sin^4 \frac{\theta}{2} d\theta.$$

Using the substitution $u = \frac{\theta}{2}$ and changing the limits of integration,

$$\begin{aligned} A &= 4 \int_0^{\pi/2} \sin^4 u \, du \\ &= 4 \left[-\frac{1}{4} \sin^3 u \cos u - \frac{3}{8} \cos u \sin u + \frac{3}{8} u \right]_0^{\pi/2} = \frac{3\pi}{4}, \end{aligned}$$

where to compute the integral we used Table of Integrals 84 & 90. Then the full shaded region has area equal to $2 \cdot \frac{3\pi}{4} = \boxed{\frac{3\pi}{2}}$.

36.

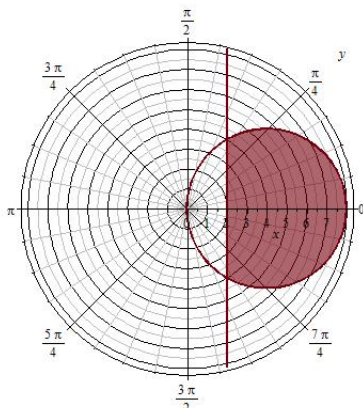


Exploiting symmetry, we will find the area A of the region in quadrant I and then quadruple it to find the full shaded region. The area in quadrant I is swept out beginning with the ray $\theta = 0$ and ending at $\theta = \frac{\pi}{2}$. Then

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2} [6 \cos^2 \theta]^2 d\theta = 18 \int_0^{\pi/2} \cos^4 \theta d\theta \\ &= \left[\frac{9}{2} \cos^3 \theta \sin \theta + \frac{27}{4} \cos \theta \sin \theta + \frac{27}{4} \theta \right]_0^{\pi/2} = \boxed{\frac{27\pi}{8}}, \end{aligned}$$

where to compute the integral we use Table of Integrals 85 & 91.

37*.



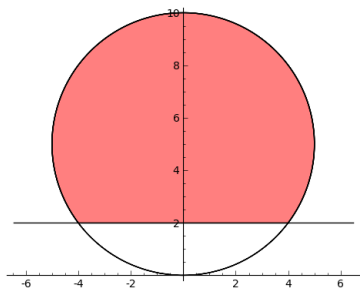
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} 8 \cos \theta &= 2 \sec \theta \\ \cos^2 \theta &= \frac{1}{4} \\ \cos \theta &= \pm \frac{1}{2} \\ \theta &= \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}. \end{aligned}$$

The shaded region is swept out beginning with $\theta = -\frac{\pi}{3}$ and ending at $\theta = \frac{\pi}{3}$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} (8 \cos \theta)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2 \sec \theta)^2 d\theta = \int_{-\pi/3}^{\pi/3} (32 \cos^2 \theta - 2 \sec^2 \theta) d\theta \\ &= [16 \cos \theta \sin \theta + 16\theta - 2 \tan \theta]_{-\pi/3}^{\pi/3} = \boxed{4\sqrt{3} + \frac{32\pi}{3}}. \end{aligned}$$

38*.



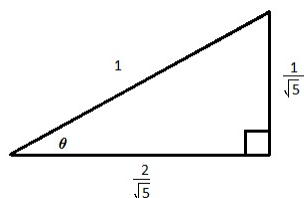
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} 10 \sin \theta &= 2 \csc \theta \\ \sin^2 \theta &= \frac{1}{5} \\ \sin \theta &= \pm \frac{1}{\sqrt{5}} \\ \theta &= \pm \sin^{-1} \left(\frac{1}{\sqrt{5}} \right), \pm \sin^{-1} \left(\frac{1}{\sqrt{5}} \right) + \pi. \end{aligned}$$

The shaded region is swept out starting at $\theta = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and ending at $\theta = \pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{\sin^{-1}(1/\sqrt{5})}^{\pi - \sin^{-1}(1/\sqrt{5})} \frac{1}{2} (10 \sin \theta)^2 d\theta - \int_{\sin^{-1}(1/\sqrt{5})}^{\pi - \sin^{-1}(1/\sqrt{5})} \frac{1}{2} (2 \csc \theta)^2 d\theta \\ &= \int_{\sin^{-1}(1/\sqrt{5})}^{\pi - \sin^{-1}(1/\sqrt{5})} (50 \sin^2 \theta - 2 \csc^2 \theta) d\theta. \end{aligned}$$

To compute A , we will need to simplify the following expressions using the picture where $\sin \theta = \frac{1}{\sqrt{5}}$ or, equivalently, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$.

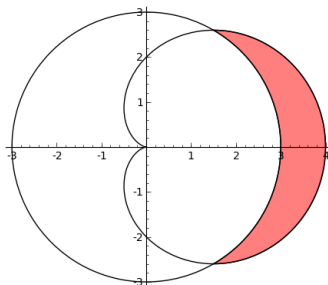


$$\begin{aligned} \cos \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) &= \frac{2}{\sqrt{5}} \\ \cos\left(\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) &= \cos \pi \cos \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin \pi \sin \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} \\ \sin \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) &= \frac{1}{\sqrt{5}} \\ \sin\left(\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) &= \sin \pi \cos \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \cos \pi \sin \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}. \end{aligned}$$

Then

$$A = [-25 \cos \theta \sin \theta + 25\theta + 2 \cot \theta]_{\sin^{-1}(1/\sqrt{5})}^{\pi - \sin^{-1}(1/\sqrt{5})} = \boxed{12 - 50 \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + 25\pi}.$$

39*.



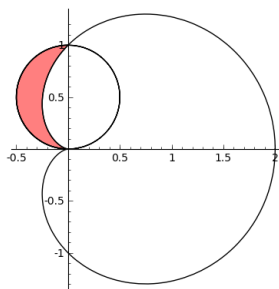
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} 2 + 2 \cos \theta &= 3 \\ \cos \theta &= \frac{1}{2} \\ \theta &= \pm \frac{\pi}{3}. \end{aligned}$$

The shaded region is swept out beginning with $\theta = -\frac{\pi}{3}$ and ending at $\theta = \frac{\pi}{3}$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2 + 2 \cos \theta)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} (3)^2 d\theta = \int_{-\pi/3}^{\pi/3} \left(-\frac{5}{2} + 4 \cos \theta + 2 \cos^2 \theta \right) d\theta \\ &= \left[-\frac{3}{2} \theta + 4 \sin \theta + \cos \theta \sin \theta \right]_{-\pi/3}^{\pi/3} = \boxed{\frac{9\sqrt{3}}{2} - \pi}. \end{aligned}$$

40*.



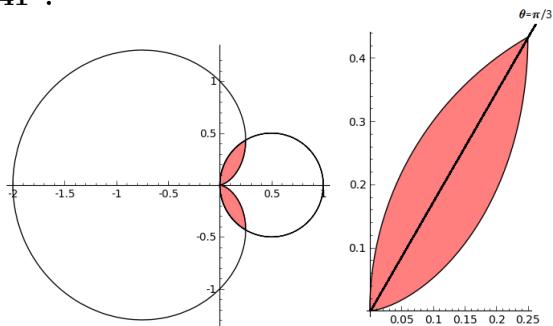
We need to find the angles that sweep out the shaded region so we set the equations equal.

$$\begin{aligned} \sin \theta &= 1 + \cos \theta \\ \sin \theta - \cos \theta &= 1 \\ \theta &= \frac{\pi}{2}, \pi. \end{aligned}$$

The shaded region is swept out beginning with $\theta = \frac{\pi}{2}$ and ending at $\theta = \pi$. The area A of the shaded region is then

$$\begin{aligned} A &= \int_{\pi/2}^{\pi} \frac{1}{2} (\sin \theta)^2 d\theta - \int_{\pi/2}^{\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \int_{\pi/2}^{\pi} \left(\frac{1}{2} \sin^2 \theta - \frac{1}{2} - \cos \theta - \frac{1}{2} \cos^2 \theta \right) d\theta \\ &= \left[-\frac{1}{2} \sin \theta \cos \theta - \frac{1}{2} \theta - \sin \theta \right]_{\pi/2}^{\pi} = \boxed{1 - \frac{\pi}{4}}. \end{aligned}$$

41*.



Exploiting symmetry, we will find the area of the shaded region in quadrant I and then double it to find the full shaded area. Referring to the second picture, the shaded region in quadrant I can be divided by the ray from the pole through the intersection point of the two curves. The angle that represents the ray through the intersection point can be found by setting the equations equal.

$$\begin{aligned} \cos \theta &= 1 - \cos \theta \\ \cos \theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}. \end{aligned}$$

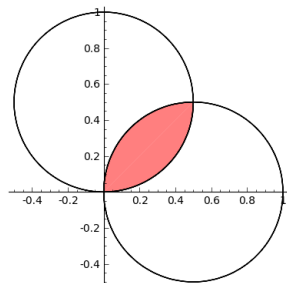
The shaded region in quadrant I can then be found by finding the area of the shaded region on $\left[0, \frac{\pi}{3}\right]$ of the cardioid and then adding the area of the shaded region on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ of the circle. This means the area A in quadrant I is

$$\begin{aligned}
 A &= \int_0^{\pi/3} \frac{1}{2}(1 - \cos \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos \theta)^2 d\theta \\
 &= \int_0^{\pi/3} \left(\frac{1}{2} - \cos \theta + \frac{1}{2} \cos^2 \theta \right) d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} \cos^2 \theta d\theta \\
 &= \left[\frac{3}{4}\theta - \sin \theta + \frac{1}{4} \cos \theta \sin \theta \right]_0^{\pi/3} + \left[\frac{1}{4}\theta + \frac{1}{4} \cos \theta \sin \theta \right]_{\pi/3}^{\pi/2} \\
 &= \left[\frac{\pi}{4} - \frac{7\sqrt{3}}{16} \right] + \left[\frac{\pi}{24} - \frac{\sqrt{3}}{16} \right] = \frac{7\pi}{24} - \frac{\sqrt{3}}{2}.
 \end{aligned}$$

Then the full shaded area is

$$2 \cdot \left(\frac{7\pi}{24} - \frac{\sqrt{3}}{2} \right) = \boxed{\frac{7\pi}{12} - \sqrt{3}}.$$

42*.

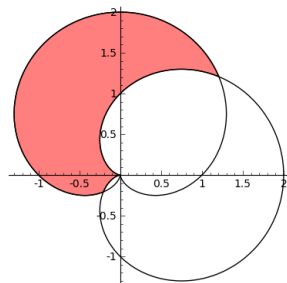


Exploiting symmetry, we will find the area A inside the (top) circle $r = \sin \theta$ from $\left[0, \frac{\pi}{4}\right]$ and then double it. Then

$$\begin{aligned}
 A &= \int_0^{\pi/4} \frac{1}{2}(\sin \theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/4} \sin^2 \theta d\theta \\
 &= \left[\frac{1}{4}\theta - \frac{1}{4} \sin \theta \cos \theta \right]_0^{\pi/4} = \frac{\pi}{16} - \frac{1}{8}.
 \end{aligned}$$

This means the full area of the shaded region is $2 \cdot \left(\frac{\pi}{16} - \frac{1}{8} \right) = \boxed{\frac{\pi}{8} - \frac{1}{4}}.$

43*.



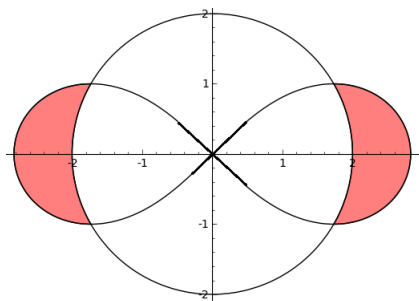
First we find the intersection points by setting the equations equal.

$$\begin{aligned} 1 + \cos \theta &= 1 + \sin \theta \\ \cos \theta &= \sin \theta \\ \theta &= \frac{\pi}{4}, \frac{5\pi}{4}. \end{aligned}$$

The area A of the shaded region is swept out beginning with the ray $\theta = \frac{\pi}{4}$ and ending with the ray $\theta = \frac{5\pi}{4}$. Then

$$\begin{aligned} A &= \int_{\pi/4}^{5\pi/4} \frac{1}{2}(1 + \sin \theta)^2 d\theta - \int_{\pi/4}^{5\pi/4} \frac{1}{2}(1 + \cos \theta)^2 d\theta = \int_{\pi/4}^{5\pi/4} \left(\sin \theta - \cos \theta + \frac{1}{2} \sin^2 \theta - \frac{1}{2} \cos^2 \theta \right) d\theta \\ &= \left[-\cos \theta - \sin \theta - \frac{1}{2} \sin \theta \cos \theta \right]_{\pi/4}^{5\pi/4} = \boxed{2\sqrt{2}}. \end{aligned}$$

44.



Exploiting symmetry, we will find the area A of the shaded region in quadrant I and then quadruple it to find the full shaded area. First we find the intersection in quadrant I by setting $r^2 = r^2$.

$$\begin{aligned} 8 \cos(2\theta) &= 4 \\ \cos(2\theta) &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3} \\ \theta &= \frac{\pi}{6}. \end{aligned}$$

Then

$$\begin{aligned} A &= \int_0^{\pi/6} \frac{1}{2} \cdot 8 \cos(2\theta) d\theta - \int_0^{\pi/6} \frac{1}{2}(2)^2 d\theta = \int_0^{\pi/6} (4 \cos(2\theta) - 2) d\theta \\ &= [2 \sin(2\theta) - 2\theta]_0^{\pi/6} = \sqrt{3} - \frac{\pi}{3}. \end{aligned}$$

This means the full shaded region has area $4 \cdot \left(\sqrt{3} - \frac{\pi}{3} \right) = \boxed{4\sqrt{3} - \frac{4\pi}{3}}$.

45. Using the area formula, the enclosed area A is

$$\begin{aligned} A &= \int_0^1 \frac{1}{2}(e^{-\theta})^2 d\theta = \int_0^1 \frac{1}{2}e^{-2\theta} d\theta \\ &= \left[-\frac{1}{4}e^{-2\theta} \right]_0^1 = \boxed{\frac{1 - e^{-2}}{4}}. \end{aligned}$$

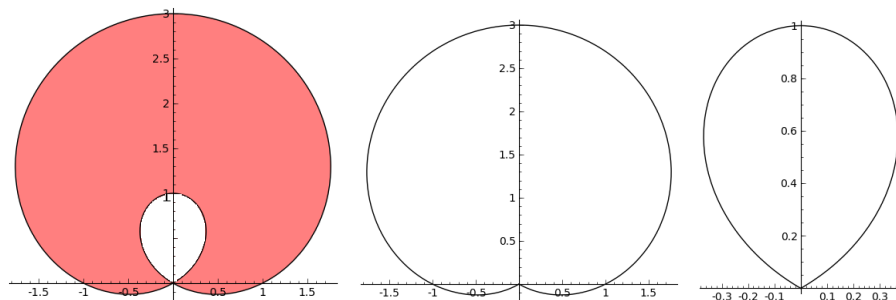
46. Using the area formula, the enclosed area A is

$$\begin{aligned} A &= \int_0^1 \frac{1}{2} (e^\theta)^2 d\theta = \int_0^1 \frac{1}{2} e^{2\theta} d\theta \\ &= \left[\frac{1}{4} e^{2\theta} \right]_0^1 = \boxed{\frac{1}{4} e^2 - \frac{1}{4}}. \end{aligned}$$

47. Using the area formula, the area A enclosed is

$$\begin{aligned} A &= \int_1^\pi \frac{1}{2} \left(\frac{1}{\theta} \right)^2 d\theta = \int_1^\pi \frac{1}{2} \theta^{-2} d\theta \\ &= \left[-\frac{1}{2} \theta^{-1} \right]_1^\pi = \boxed{\frac{\pi - 1}{2\pi}}. \end{aligned}$$

48*.



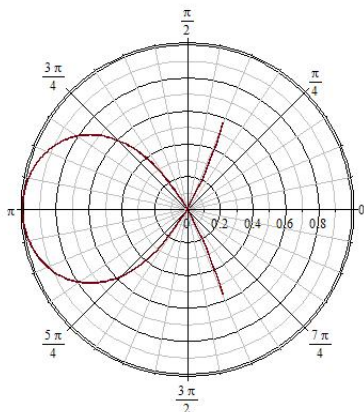
First we find when the polar curve intersects the pole by setting the equation equal to zero.

$$\begin{aligned} 1 + 2 \sin \theta &= 0 \\ \sin \theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

The left-most graph represents the full curve; the middle graph represents the curve on $-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$ (notice we chose the ray $\theta = -\frac{\pi}{6}$ and not $\theta = \frac{11\pi}{6}$ so we could start with a ray whose angle was smaller than $\frac{7\pi}{6}$); the right-most graph represents the inner loop of the curve on $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$ (zoomed in). The area A we seek is the difference between area B of the middle graph and the area C of the right-most graph. Then

$$\begin{aligned} A &= B - C = \int_{-\pi/6}^{7\pi/6} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta - \int_{7\pi/6}^{11\pi/6} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \left(\frac{1}{2} + 2 \sin \theta + 2 \sin^2 \theta \right) d\theta - \int_{7\pi/6}^{11\pi/6} \left(\frac{1}{2} + 2 \sin \theta + 2 \sin^2 \theta \right) d\theta \\ &= \left[\frac{3}{2} \theta - 2 \cos \theta - \sin \theta \cos \theta \right]_{-\pi/6}^{7\pi/6} - \left[\frac{3}{2} \theta - 2 \cos \theta - 2 \sin \theta \cos \theta \right]_{7\pi/6}^{11\pi/6} = \left(\frac{3}{2} \sqrt{3} + 2\pi \right) - \left(-\frac{3}{2} \sqrt{3} + \pi \right) \\ &= \boxed{3\sqrt{3} + \pi}. \end{aligned}$$

49.



We need to find when the polar curve intersects with the pole, so we set $r = 0$.

$$\begin{aligned} 2 + \sec \theta &= 0 \\ \cos \theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

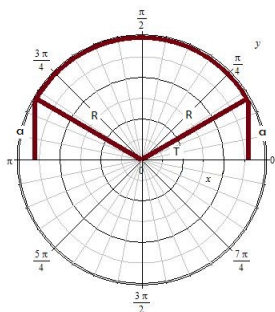
The area A of the loop is

$$\begin{aligned} A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2} (2 + \sec \theta)^2 d\theta = \int_{2\pi/3}^{4\pi/3} \left(\frac{1}{2} \sec^2 \theta + 2 \sec \theta + 2 \right) d\theta \\ &= \left[\frac{1}{2} \tan \theta + 2 \ln |\sec \theta + \tan \theta| + 2\theta \right]_{2\pi/3}^{4\pi/3} = \sqrt{3} + \frac{4\pi}{3} + 2 \ln \left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}} \right) \\ &= \boxed{\sqrt{3} + \frac{4\pi}{3} - 4 \ln (2 + \sqrt{3})}. \end{aligned}$$

50. By rotating the top half of the polar curve $r = R$ about the polar axis, we have a sphere. If we use formula (1) with $f(\theta) = R$, $f'(\theta) = 0$, and $0 \leq \theta \leq \pi$, then the surface area S of a sphere is

$$\begin{aligned} S &= 2\pi \int_0^\pi R \sin \theta \sqrt{[R]^2 + [0]^2} d\theta = 2\pi R^2 \int_0^\pi \sin \theta d\theta \\ &= 2\pi R^2 [-\cos \theta]_0^\pi = \boxed{4\pi R^2}. \end{aligned}$$

51. (a)



If we rotate the top half of the circle $r = R$ about the polar axis for $T \leq \theta \leq \pi - T$, then we will have the surface of the bead. Using formula (1) with $f(\theta) = R$, $f'(\theta) = 0$, and $T \leq \theta \leq \pi - T$, the surface area S of the bead is

$$\begin{aligned} S &= 2\pi \int_T^{\pi-T} R \sin \theta \sqrt{[R]^2 + [0]^2} d\theta = 2\pi R^2 \int_T^{\pi-T} \sin \theta d\theta \\ &= 2\pi R^2 [-\cos \theta]_T^{\pi-T} = 2\pi R^2 [-\cos(\pi - T) + \cos(T)]. \end{aligned}$$

From the picture, we see that $\cos(T) = \frac{\sqrt{R^2 - a^2}}{R}$ and so $-\cos(\pi - T) = \frac{\sqrt{R^2 - a^2}}{R}$. Then

$$S = 2\pi R^2 \cdot \frac{2\sqrt{R^2 - a^2}}{R} = \boxed{4\pi R\sqrt{R^2 - a^2}}.$$

(b) Answers will vary.

52. (a) Using Problems 50 and 51, we can find the area left by taking the difference between the full surface area of the sphere and subtracting the surface area of the bead. This means the surface area of the plug is

$$4\pi R^2 - 4\pi R\sqrt{R^2 - a^2}.$$

(b) Answers will vary.

53*. We first find where the curve intersects the pole by setting $r = 0$.

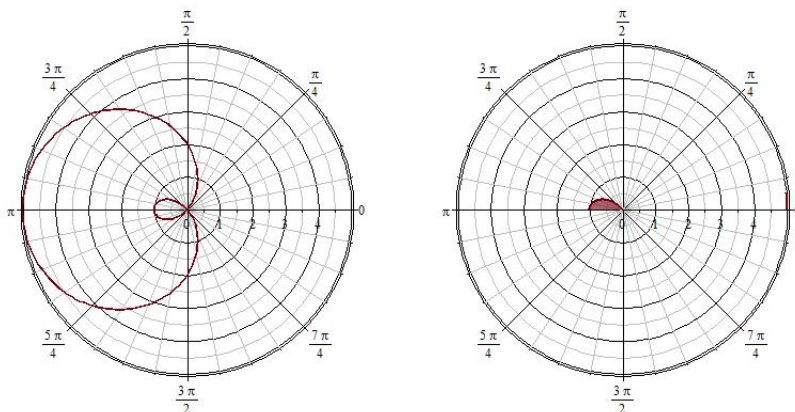
$$\begin{aligned}\sec \theta - 2 \cos \theta &= 0 \\ 1 - 2 \cos^2 \theta &= 0 \\ \cos^2 \theta &= \frac{1}{2} \\ \cos \theta &= \pm \frac{\sqrt{2}}{2} \\ \theta &= -\frac{\pi}{4}, \frac{\pi}{4}.\end{aligned}$$

From the picture we see that the shaded region is swept out beginning with $\theta = -\frac{\pi}{4}$ and ending with $\theta = \frac{\pi}{4}$. So the area A of the shaded region is

$$\begin{aligned}A &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1}{2} (\sec \theta - 2 \cos \theta)^2 d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \sec^2 \theta - 2 + 2 \cos^2 \theta \right) d\theta \\ &= \left[\frac{1}{2} \tan \theta - \theta + \sin \theta \cos \theta \right]_{-\pi/4}^{\pi/4} = \boxed{2 - \frac{\pi}{2}}.\end{aligned}$$

54.

(a)



(b) By setting $r = 0$, we see that the curve intersects the pole when $\theta = \cos^{-1}(2/3), 2\pi - \cos^{-1}(2/3)$. The second graph depicts the curve when $2\pi - \cos^{-1}(2/3) \leq \theta \leq 2\pi$. If we find the area A of this piece and double it, we will have the full area of the inner loop. Then using a CAS

$$A = \int_{2\pi - \cos^{-1}(2/3)}^{2\pi} \frac{1}{2} (2 - 3 \cos \theta)^2 d\theta \approx 0.2204,$$

which means the full area of the inner loop is approximately $2 \cdot 0.2204 \approx \boxed{0.441}$.

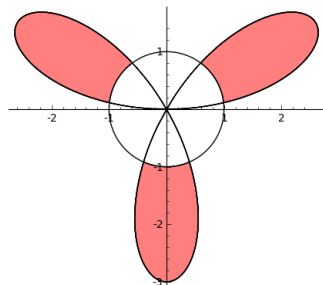
Challenge Problems

55. The area enclosed by the graph $r\theta = a$ and the rays $\theta = \theta_1$ and $\theta = \theta_2$ is

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \frac{1}{2} \left(\frac{a}{\theta} \right)^2 d\theta &= \frac{a^2}{2} \int_{\theta_1}^{\theta_2} \theta^{-2} d\theta \\ &= \frac{a^2}{2} [-\theta^{-1}]_{\theta_1}^{\theta_2} = \frac{a^2}{2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_2} \right) = \frac{a}{2} \left(\frac{a}{\theta_1} - \frac{a}{\theta_2} \right) \\ &= \frac{a}{2} (r_1 - r_2). \end{aligned}$$

Since $\frac{a}{2}$ is a constant, we see that the area is a constant multiple of (i.e. proportional to) $r_1 - r_2$.

56*.



Exploiting symmetry, we will find the shaded area A in quadrant I and triple it. We must first find the intersection points of the rose and circle in quadrant I.

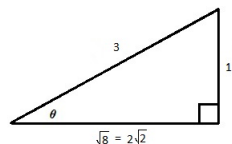
$$\begin{aligned} 3 \sin(3\theta) &= 1 \\ 3\theta &= \sin^{-1}\left(\frac{1}{3}\right), \pi - \sin^{-1}\left(\frac{1}{3}\right) \\ \theta &= \frac{1}{3} \sin^{-1}\left(\frac{1}{3}\right), \frac{\pi}{3} - \frac{1}{3} \sin^{-1}\left(\frac{1}{3}\right). \end{aligned}$$

Then

$$\begin{aligned} A &= \int_{1/3 \sin^{-1}(1/3)}^{\pi/3 - 1/3 \sin^{-1}(1/3)} \frac{1}{2} (3 \sin(3\theta))^2 d\theta - \int_{1/3 \sin^{-1}(1/3)}^{\pi/3 - 1/3 \sin^{-1}(1/3)} \frac{1}{2} (1)^2 d\theta \\ &= \int_{1/3 \sin^{-1}(1/3)}^{\pi/3 - 1/3 \sin^{-1}(1/3)} \left(\frac{9}{2} \sin^2(3\theta) - \frac{1}{2} \right) d\theta \\ &= \left[-\frac{3}{4} \sin(3\theta) \cos(3\theta) + \frac{7}{4} \theta \right]_{1/3 \sin^{-1}(1/3)}^{\pi/3 - 1/3 \sin^{-1}(1/3)} \\ &= \left[-\frac{3}{8} \sin(6\theta) + \frac{7}{4} \theta \right]_{1/3 \sin^{-1}(1/3)}^{\pi/3 - 1/3 \sin^{-1}(1/3)}. \end{aligned}$$

To compute A , we need to simplify the following expressions using the picture where $\sin \theta = \frac{1}{3}$ or, equivalently,

$$\theta = \sin^{-1}\left(\frac{1}{3}\right).$$

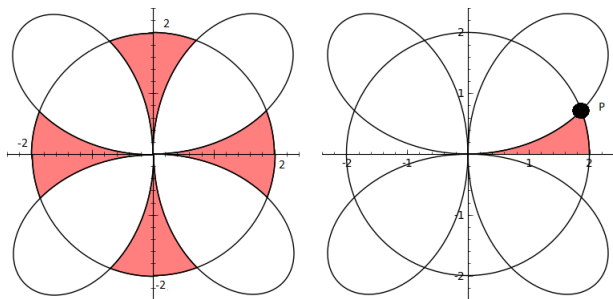


$$\begin{aligned}
\sin\left(6\left(\frac{1}{3}\sin^{-1}\left(\frac{1}{3}\right)\right)\right) &= \sin\left(2\sin^{-1}\left(\frac{1}{3}\right)\right) = 2\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\sin^{-1}\frac{1}{3}\right) \\
&= 2 \cdot \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{9} \\
\sin\left(6\left(\frac{\pi}{3} - \frac{1}{3}\sin^{-1}\left(\frac{1}{3}\right)\right)\right) &= \sin\left(2\pi - 2\sin^{-1}\left(\frac{1}{3}\right)\right) = \sin(2\pi)\cos\left(2\sin^{-1}\left(\frac{1}{3}\right)\right) - \sin\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)\cos(2\pi) \\
&= -\frac{4\sqrt{2}}{9}.
\end{aligned}$$

Using this, we find that

$$\begin{aligned}
A &= \left[-\frac{3}{8}\left(-\frac{4\sqrt{2}}{9}\right) + \frac{7}{4}\left(\frac{\pi}{3} - \frac{1}{3}\sin^{-1}\left(\frac{1}{3}\right)\right) \right] - \left[-\frac{3}{8}\left(\frac{4\sqrt{2}}{9}\right) + \frac{7}{4}\left(\frac{1}{3}\sin^{-1}\left(\frac{1}{3}\right)\right) \right] \\
&= \boxed{\frac{\sqrt{2}}{3} + \frac{7\pi}{12} - \frac{7}{6}\sin^{-1}\left(\frac{1}{3}\right)}.
\end{aligned}$$

57.



To find the area of the shaded region in the first figure, we will exploit symmetry and find the area A of the shaded region in the second figure and multiply it by eight to find the area we seek. We must first find the intersection point P of the rose and circle.

$$\begin{aligned}
3\sin(2\theta) &= 2 \\
\sin(2\theta) &= \frac{2}{3} \\
2\theta &= \sin^{-1}\left(\frac{2}{3}\right) \\
\theta &= \frac{1}{2}\sin^{-1}\left(\frac{2}{3}\right).
\end{aligned}$$

Then the area A is swept out beginning at $\theta = 0$ and ending at $\theta = \frac{1}{2}\sin^{-1}\left(\frac{2}{3}\right)$, so

$$\begin{aligned}
A &= \int_0^{1/2\sin^{-1}(2/3)} \frac{1}{2}(2)^2 d\theta - \int_0^{1/2\sin^{-1}(2/3)} \frac{1}{2}(3\sin(2\theta))^2 d\theta = \int_0^{1/2\sin^{-1}(2/3)} \left(2 - \frac{9}{2}\sin^2(2\theta)\right) d\theta \\
&= \left[-\frac{1}{4}\theta + \frac{9}{8}\sin(2\theta)\cos(2\theta) \right]_0^{1/2\sin^{-1}(2/3)} = \frac{\sqrt{5}}{4} - \frac{1}{8}\sin^{-1}\left(\frac{2}{3}\right).
\end{aligned}$$

Multiplying this value by eight, we have the area we sought.

$$8\left(\frac{\sqrt{5}}{4} - \frac{1}{8}\sin^{-1}\left(\frac{2}{3}\right)\right) = \boxed{2\sqrt{5} - \sin^{-1}\left(\frac{2}{3}\right)}.$$

9.7 The Polar Equation of a Conic

Concepts and Vocabulary

1. A **(a) parabola** is the set of points P in the plane for which the distance from a fixed point called the focus P equals the distance from a fixed line called the directrix.

2. **Answers will vary.**

3. **In both cases, $e = 1$, so both graphs are parabolas. Answers will vary.**

4. **False**. The polar equation should be $r = \frac{ep}{1 - e \cos \theta}$.

Skill Building

5. Using Table 8,

$$r = \frac{1}{1 + \cos \theta}$$

which means $e = 1$ and $p = 1$ so we have a **parabola** where the directrix is perpendicular to the polar axis 1 unit to the right of the pole.

6. Using Table 8,

$$r = \frac{3}{1 - \sin \theta}$$

which means $e = 1$ and $p = 3$ so we have a **parabola** where the directrix is parallel to the polar axis 3 units below the pole.

7. Using Table 8,

$$r = \frac{4}{2 - 3 \sin \theta} = \frac{2}{1 - \frac{3}{2} \sin \theta}$$

which means $e = \frac{3}{2}$ and $ep = 2$ so $p = \frac{4}{3}$ which means we have a **hyperbola** where the directrix is parallel to the polar axis $\frac{4}{3}$ units below the pole.

8. Using Table 8,

$$r = \frac{2}{1 + 2 \cos \theta}$$

which means $e = 2$ and $ep = 2$ so $p = 1$ which means we have a **hyperbola** where the directrix is perpendicular to the polar axis 1 unit to the right of the pole.

9. Using Table 8,

$$r = \frac{3}{4 - 2 \cos \theta} = \frac{\frac{3}{4}}{1 - \frac{1}{2} \cos \theta}$$

which means $e = \frac{1}{2}$ and $ep = \frac{3}{4}$ so $p = \frac{3}{2}$ which means we have an **ellipse** where the directrix is perpendicular to the polar axis $\frac{3}{2}$ units to the left of the pole.

10. Using Table 8,

$$r = \frac{6}{8 + 2 \cos \theta} = \frac{\frac{3}{4}}{1 + \frac{1}{4} \cos \theta}$$

which means $e = \frac{1}{4}$ and $ep = \frac{3}{4}$ so $p = 3$ which means we have an **ellipse** where the directrix is perpendicular to the polar axis 3 units to the right of the pole.

11. Using Table 8,

$$r = \frac{4}{3 + 3 \sin \theta} = \frac{\frac{4}{3}}{1 + \sin \theta}$$

which means $e = 1$ and $ep = \frac{4}{3}$ so $p = \frac{4}{3}$ which means we have a parabola where the directrix is parallel to the polar axis $\frac{4}{3}$ units above the pole.

12. Using Table 8,

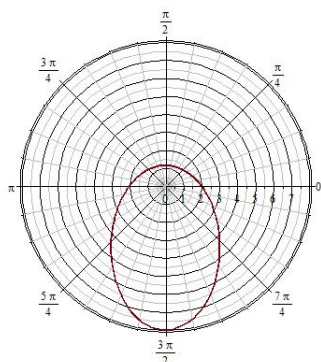
$$r = \frac{1}{6 + 2 \sin \theta} = \frac{\frac{1}{6}}{1 + \frac{1}{3} \sin \theta}$$

which means $e = \frac{1}{3}$ and $ep = \frac{1}{6}$ so $p = \frac{1}{2}$ which means we have an ellipse where the directrix is parallel to the polar axis $\frac{1}{2}$ unit above the pole.

13. (a) Using Table 8,

$$r = \frac{8}{4 + 3 \sin \theta} = \frac{2}{1 + \frac{3}{4} \sin \theta}$$

which means $e = \frac{3}{4}$ and $ep = 2$, so $p = \frac{8}{3}$. Since $e < 1$, this is an ellipse with directrix parallel to the polar axis $\frac{8}{3}$ units above the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 2)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y -intercepts are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, which gives rise to the polar points $\left(\frac{\pi}{2}, \frac{8}{7}\right)$ and $\left(\frac{3\pi}{2}, 8\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned} r &= \frac{8}{4 + 3 \sin \theta} \\ 4r + 3r \sin \theta &= 8 \\ 4r &= 8 - 3r \sin \theta \\ 16r^2 &= (8 - 3r \sin \theta)^2 \\ 16(x^2 + y^2) &= (8 - 3y)^2 \\ 16x^2 + 16y^2 &= 64 - 48y + 9y^2 \\ 16x^2 + 7\left(y^2 + \frac{48}{7}y\right) &= 64 \\ 16x^2 + 7\left(y + \frac{24}{7}\right)^2 &= 64 + 7\left(\frac{24}{7}\right)^2 \\ \boxed{16x^2 + 7\left(y + \frac{24}{7}\right)^2} &= \boxed{\frac{1024}{7}}. \end{aligned}$$

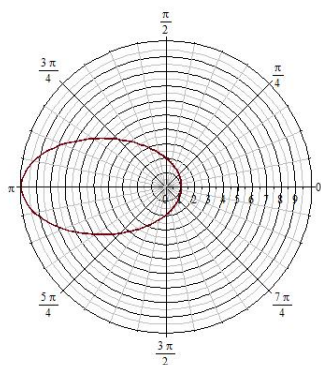
(c) Parametric equations are

$$x = \frac{8 \cos \theta}{4 + 3 \sin \theta}, \quad y = \frac{8 \sin \theta}{4 + 3 \sin \theta}.$$

14. (a) Using Table 8,

$$r = \frac{10}{5 + 4 \cos \theta} = \frac{2}{1 + \frac{4}{5} \cos \theta}$$

which means $e = \frac{4}{5}$ and $ep = 2$, so $p = \frac{5}{2}$. Since $e < 1$, this is an ellipse with directrix perpendicular to the polar axis $\frac{5}{2}$ units to the right of the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $\left(0, \frac{10}{9}\right)$ and $(\pi, 10)$, respectively, which are vertices of the ellipse. The y -intercepts are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, which gives rise to the polar points $\left(\frac{\pi}{2}, 2\right)$ and $\left(\frac{3\pi}{2}, 2\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned} r &= \frac{10}{5 + 4 \cos \theta} \\ 5r + 4r \cos \theta &= 10 \\ 5r &= 10 - 4r \cos \theta \\ 25r^2 &= (10 - 4r \cos \theta)^2 \\ 25(x^2 + y^2) &= (10 - 4x)^2 \\ 25x^2 + 25y^2 &= 100 - 80x + 16x^2 \\ 9\left(x^2 + \frac{80}{9}x\right) + 25y^2 &= 100 \\ 9\left(x + \frac{40}{9}\right)^2 + 25y^2 &= 100 + 9\left(\frac{40}{9}\right)^2 \\ \boxed{9\left(x + \frac{40}{9}\right)^2 + 25y^2} &= \boxed{\frac{2500}{9}}. \end{aligned}$$

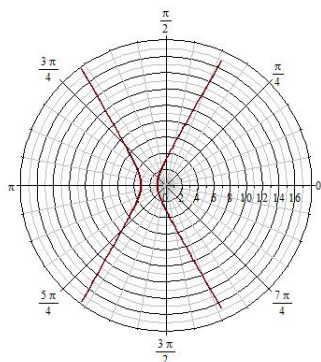
(c) Parametric equations are

$$x = \frac{10 \cos \theta}{5 + 4 \cos \theta}, \quad y = \frac{10 \sin \theta}{5 + 4 \cos \theta}.$$

15. (a) Using Table 8,

$$r = \frac{9}{3 - 6 \cos \theta} = \frac{3}{1 - 2 \cos \theta}$$

which means $e = 2$. Since $e > 1$, this is a hyperbola.



(b) To obtain a rectangular equation of the hyperbola, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
 r &= \frac{9}{3 - 6 \cos \theta} \\
 3r - 6r \cos \theta &= 9 \\
 3r &= 9 + 6r \cos \theta \\
 9r^2 &= (9 + 6r \cos \theta)^2 \\
 9(x^2 + y^2) &= (9 + 6x)^2 \\
 9x^2 + 9y^2 &= 81 + 108x + 36x^2 \\
 27(x^2 + 4x) - 9y^2 &= -81 \\
 27(x + 2)^2 - 9y^2 &= -81 + 27(4) \\
 \boxed{3(x + 2)^2 - y^2} &= 3.
 \end{aligned}$$

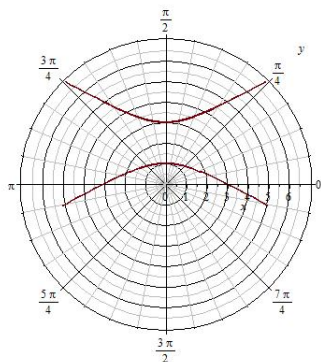
(c) Parametric equations are

$$\boxed{x = \frac{9 \cos \theta}{3 - 6 \cos \theta}, \quad y = \frac{9 \sin \theta}{3 - 6 \cos \theta}}.$$

16. (a) Using Table 8,

$$r = \frac{12}{4 + 8 \sin \theta} = \frac{3}{1 + 2 \sin \theta}$$

which means $e = 2$. Since $e > 1$, this is a hyperbola.



(b) To obtain a rectangular equation of the hyperbola, we eliminate the fraction and then square the

resulting polar equation.

$$\begin{aligned}
 r &= \frac{12}{4 + 8 \sin \theta} \\
 4r &= 12 - 8r \sin \theta \\
 16r^2 &= (12 - 8r \sin \theta)^2 \\
 16(x^2 + y^2) &= (12 - 8y)^2 \\
 16x^2 + 16y^2 &= 144 - 192y + 64y^2 \\
 48y^2 - 192y - 16x^2 &= -144 \\
 48(y^2 - 4y) - 16x^2 &= -144 \\
 48(y - 2)^2 - 16x^2 &= -144 + 48(2)^2 \\
 \boxed{48(y - 2)^2 - 16x^2 &= 48}.
 \end{aligned}$$

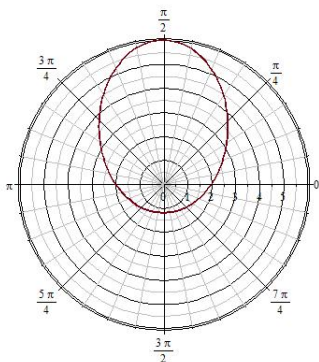
(c) Parametric equations are

$$\boxed{x = \frac{12 \cos \theta}{4 + 8 \sin \theta}, \quad y = \frac{12 \sin \theta}{4 + 8 \sin \theta}}.$$

17. (a) Using Table 8,

$$r = \frac{6}{3 - 2 \sin \theta} = \frac{2}{1 - \frac{2}{3} \sin \theta}$$

which means $e = \frac{2}{3}$ and $ep = 2$, so $p = 3$. Since $e < 1$, this is an ellipse with directrix parallel to the polar axis 3 units below the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 2)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y -intercepts are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, which gives rise to the points $\left(\frac{\pi}{2}, 6\right)$ and $\left(\frac{3\pi}{2}, \frac{6}{5}\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting

polar equation.

$$\begin{aligned}
 3r - 2r \sin \theta &= 6 \\
 9r^2 &= (3r)^2 = (6 + 2r \sin \theta)^2 \\
 9(x^2 + y^2) &= (6 + 2y)^2 \\
 9x^2 + 9y^2 &= 36 + 24y + 4y^2 \\
 9x^2 + 5\left(y^2 - \frac{24}{5}y\right) &= 36 \\
 9x^2 + 5\left(y - \frac{12}{5}\right)^2 &= 36 + 5\left(\frac{12}{5}\right)^2 \\
 \boxed{9x^2 + 5\left(y - \frac{12}{5}\right)^2} &= \frac{324}{5}.
 \end{aligned}$$

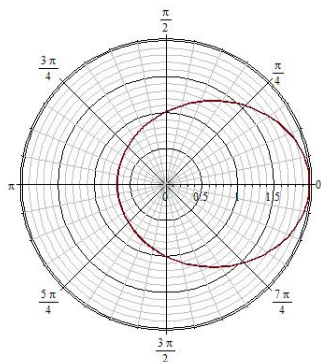
(c) Parametric equations are

$$\boxed{x = \frac{6 \cos \theta}{3 - 2 \sin \theta}, \quad y = \frac{6 \sin \theta}{3 - 2 \sin \theta}}.$$

18. (a) Using Table 8,

$$r = \frac{2}{2 - \cos \theta} = \frac{1}{1 - \frac{1}{2} \cos \theta}$$

which means $e = \frac{1}{2}$ and $ep = 1$, so $p = 2$. Since $e < 1$, this is an ellipse with directrix perpendicular to the polar axis 2 units to the left of the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 2)$ and $\left(\pi, \frac{2}{3}\right)$, respectively, which are vertices of the ellipse. The y -intercepts are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, which gives rise to the points $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, 1\right)$.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting

polar equation.

$$\begin{aligned}
 2r - r \cos \theta &= 2 \\
 4r^2 &= (2 + r \cos \theta)^2 \\
 4(x^2 + y^2) &= (2 + x)^2 \\
 4x^2 + 4y^2 &= 4 + 4x + x^2 \\
 3x^2 - 4x + 4y^2 &= 4 \\
 3\left(x^2 - \frac{4}{3}x\right) + 4y^2 &= 4 \\
 3\left(x - \frac{2}{3}\right)^2 + 4y^2 &= 4 + 3\left(\frac{2}{3}\right)^2 \\
 \boxed{3\left(x - \frac{2}{3}\right)^2 + 4y^2 = \frac{16}{3}}.
 \end{aligned}$$

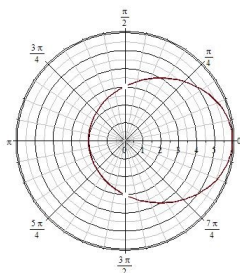
(c) Parametric equations are

$$x = \frac{2 \cos \theta}{2 - \cos \theta}, \quad y = \frac{2 \sin \theta}{2 - \cos \theta}.$$

19. (a) Using Table 8,

$$r = \frac{6 \sec \theta}{2 \sec \theta - 1} = \frac{6}{2 - \cos \theta} = \frac{3}{1 - \frac{1}{2} \cos \theta}, \quad \theta \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

which means $e = \frac{1}{2}$ and $ep = 3$, so $p = 6$. Since $e < 1$, this is an ellipse with directrix perpendicular to the polar axis 6 units to the left of the pole. At $t = 0$ and $t = \pi$, the polar points corresponding to these values are $(0, 6)$ and $(\pi, 2)$, respectively, which are vertices of the ellipse. The y -intercepts are undefined since the function is undefined at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.



(b) To obtain a rectangular equation of the hyperbola, we eliminate the fraction and then square the resulting polar equation.

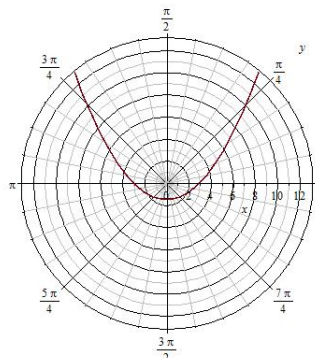
$$\begin{aligned}
 2r - r \cos \theta &= 6 \\
 4r^2 &= (6 + r \cos \theta)^2 \\
 4(x^2 + y^2) &= (6 + x)^2 \\
 4x^2 + 4y^2 &= 36 + 12x + x^2 \\
 3(x^2 - 4x) + 4y^2 &= 36 \\
 3(x - 2)^2 + 4y^2 &= 36 + 3(2)^2 \\
 \boxed{3(x - 2)^2 + 4y^2 = 48}.
 \end{aligned}$$

(c) Parametric equations are

$$x = \frac{6 \cos \theta}{2 - \cos \theta}, \quad y = \frac{6 \sin \theta}{2 - \cos \theta}.$$

20. (a) Using Table 8,

$$r = \frac{3 \csc \theta}{\csc \theta - 1} = \frac{3}{1 - \sin \theta}, \quad \theta \neq 0, \pi, 2\pi,$$



which means $e = 1$. Since $e = 1$, this is a parabola.

(b) To obtain a rectangular equation of the parabola, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned} r - r \sin \theta &= 3 \\ r^2 &= (3 + r \sin \theta)^2 \\ x^2 + y^2 &= (3 + y)^2 = 9 + 6y + y^2 \\ \boxed{x^2} &= 9 + 6y. \end{aligned}$$

(c) Parametric equations are

$$\boxed{x = \frac{3}{1 - \sin \theta} \cos \theta, \quad y = \frac{3}{1 - \sin \theta} \sin \theta}.$$

Applications and Extensions

For Problems 21-26, we will use the parametric form of each graph $x = r \cos \theta$ and $y = r \sin \theta$ along with the formula $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ to determine the slope at the given value of θ .

21. Parametric equations for $r = \frac{9}{4 - \cos \theta}$ are

$$\begin{aligned} y &= \frac{9 \sin \theta}{4 - \cos \theta} \\ x &= \frac{9 \cos \theta}{4 - \cos \theta}, \end{aligned}$$

so

$$\begin{aligned} \left[\frac{dy}{d\theta} \right]_{\theta=0} &= \left[\frac{(4 - \cos \theta)(9 \cos \theta) - (9 \sin \theta)(\sin \theta)}{(4 - \cos \theta)^2} \right]_{\theta=0} = 3 \\ \left[\frac{dx}{d\theta} \right]_{\theta=0} &= \left[\frac{(4 - \cos \theta)(-9 \sin \theta) - (9 \cos \theta)(\sin \theta)}{(4 - \cos \theta)^2} \right]_{\theta=0} = 0. \end{aligned}$$

Since $\frac{dx}{d\theta} = 0$, this means $\frac{dy}{dx}$ is undefined which means the tangent line at $\theta = 0$ is vertical so the slope is undefined.

22. Parametric equations for $r = \frac{3}{1 - \sin \theta}$ are

$$\begin{aligned} y &= \frac{3 \sin \theta}{1 - \sin \theta} \\ x &= \frac{3 \cos \theta}{1 - \sin \theta}, \end{aligned}$$

so

$$\begin{aligned}\left[\frac{dy}{d\theta}\right]_{\theta=0} &= \left[\frac{(1-\sin\theta)(3\cos\theta) - (3\sin\theta)(-\cos\theta)}{(1-\sin\theta)^2}\right]_{\theta=0} = 3 \\ \left[\frac{dx}{d\theta}\right]_{\theta=0} &= \left[\frac{(1-\sin\theta)(-3\sin\theta) - (3\cos\theta)(-\cos\theta)}{(1-\sin\theta)^2}\right]_{\theta=0} = 3.\end{aligned}$$

Then the slope of the tangent line at $\theta = 0$ is

$$\left[\frac{dy}{dx}\right]_{\theta=0} = \frac{3}{3} = \boxed{1}.$$

23. Parametric equations for $r = \frac{8}{4 + \sin\theta}$ are

$$\begin{aligned}y &= \frac{8\sin\theta}{4 + \sin\theta} \\ x &= \frac{8\cos\theta}{4 + \sin\theta},\end{aligned}$$

so

$$\begin{aligned}\left[\frac{dy}{d\theta}\right]_{\theta=\pi/2} &= \left[\frac{(4 + \sin\theta)(8\cos\theta) - (8\sin\theta)(\cos\theta)}{(4 + \sin\theta)^2}\right]_{\theta=\pi/2} = 0 \\ \left[\frac{dx}{d\theta}\right]_{\theta=\pi/2} &= \left[\frac{(4 + \sin\theta)(-8\sin\theta) - (8\cos\theta)(\cos\theta)}{(4 + \sin\theta)^2}\right]_{\theta=\pi/2} = -\frac{40}{25}.\end{aligned}$$

Then

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/2} = \frac{0}{-40/25} = 0$$

which means the tangent line has slope $\boxed{0}$ at $\theta = \frac{\pi}{2}$.

24. Parametric equations for $r = \frac{10}{5 + 4\sin\theta}$ are

$$\begin{aligned}y &= \frac{10\sin\theta}{5 + 4\sin\theta} \\ x &= \frac{10\cos\theta}{5 + 4\sin\theta},\end{aligned}$$

so

$$\begin{aligned}\left[\frac{dy}{d\theta}\right]_{\theta=\pi} &= \left[\frac{(5 + 4\sin\theta)(10\cos\theta) - (10\sin\theta)(4\cos\theta)}{(5 + 4\sin\theta)^2}\right]_{\theta=\pi} = -2 \\ \left[\frac{dx}{d\theta}\right]_{\theta=\pi} &= \left[\frac{(5 + 4\sin\theta)(-10\sin\theta) - (10\cos\theta)(4\cos\theta)}{(5 + 4\sin\theta)^2}\right]_{\theta=\pi} = -\frac{8}{5}.\end{aligned}$$

Then the slope of the tangent line at $\theta = \pi$ is

$$\frac{dy}{dx} = \frac{-2}{-8/5} = \boxed{\frac{5}{4}}.$$

25. Parametric equations for $r = \frac{4}{2 + \cos\theta}$ are

$$\begin{aligned}y &= \frac{4\sin\theta}{2 + \cos\theta} \\ x &= \frac{4\cos\theta}{2 + \cos\theta},\end{aligned}$$

so

$$\begin{aligned}\left[\frac{dy}{d\theta}\right]_{\theta=\pi} &= \left[\frac{(2+\cos\theta)(4\cos\theta) - (4\sin\theta)(-\sin\theta)}{(2+\cos\theta)^2}\right]_{\theta=\pi} = -4 \\ \left[\frac{dx}{d\theta}\right]_{\theta=\pi} &= \left[\frac{(2+\cos\theta)(-4\sin\theta) - (4\cos\theta)(-\sin\theta)}{(2+\cos\theta)^2}\right]_{\theta=\pi} = 0.\end{aligned}$$

Since $\frac{dx}{d\theta} = 0$, this means $\frac{dy}{dx}$ is undefined which means the tangent line at $\theta = \pi$ is vertical so the slope is undefined.

26. Parametric equations for $r = \frac{6}{3-2\sin\theta}$ are

$$\begin{aligned}y &= \frac{6\sin\theta}{3-2\sin\theta} \\ x &= \frac{6\cos\theta}{3-2\sin\theta},\end{aligned}$$

so

$$\begin{aligned}\left[\frac{dy}{d\theta}\right]_{\theta=\pi/2} &= \left[\frac{(3-2\sin\theta)(6\cos\theta) - (6\sin\theta)(-2\cos\theta)}{(3-2\sin\theta)^2}\right]_{\theta=\pi/2} = 0 \\ \left[\frac{dx}{d\theta}\right]_{\theta=\pi/2} &= \left[\frac{(3-2\sin\theta)(-6\sin\theta) - (6\cos\theta)(-2\cos\theta)}{(3-2\sin\theta)^2}\right]_{\theta=\pi/2} = -6\end{aligned}$$

Then the slope of the tangent line at $\theta = \frac{\pi}{2}$ is

$$\left[\frac{dy}{dx}\right]_{\theta=\pi/2} = \frac{0}{-6} = \boxed{0}.$$

27. It is given that $e = \frac{4}{5}$ and $p = 3$, so using Table 8, we have

$$r = \frac{\frac{4}{5} \cdot 3}{1 - \frac{4}{5} \cos\theta} = \frac{12}{5 - 4 \cos\theta}.$$

28. It is given that $e = \frac{2}{3}$ and $p = 3$, so using Table 8, we have

$$r = \frac{\frac{2}{3} \cdot 3}{1 + \frac{2}{3} \sin\theta} = \frac{6}{3 + 2 \sin\theta}.$$

29. It is given that $e = 1$ and $p = 1$, so using Table 8, we have

$$r = \frac{1 \cdot 1}{1 + \sin\theta} = \frac{1}{1 + \sin\theta}.$$

30. It is given that $e = 1$ and $p = 2$, so using Table 8, we have

$$r = \frac{1 \cdot 2}{1 - \sin\theta} = \frac{2}{1 - \sin\theta}.$$

31. It is given that $e = 6$ and $p = 2$, so using Table 8, we have

$$r = \frac{6 \cdot 2}{1 - 6 \sin\theta} = \frac{12}{1 - 6 \sin\theta}.$$

32. It is given that $e = 5$ and $p = 3$, so using Table 8, we have

$$r = \frac{5 \cdot 3}{1 + 5 \cos \theta} = \frac{15}{1 + 5 \cos \theta}.$$

33. (a) $e = 0.967$.

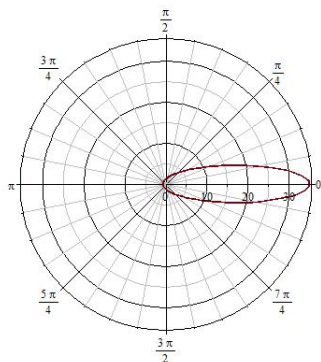
(b) The shortest distance occurs when r is a minimum which occurs when $\cos \theta = -1$. The shortest distance is then

$$r = \frac{1.155}{1 - 0.967(-1)} \approx 0.587 \text{ AU}.$$

(c) The greatest distance occurs when r is a maximum which occurs when $\cos \theta = 1$. The shortest distance is then

$$r = \frac{1.155}{1 - 0.967(1)} = 35 \text{ AU}.$$

(d)



34. (a) $e = 0.206$.

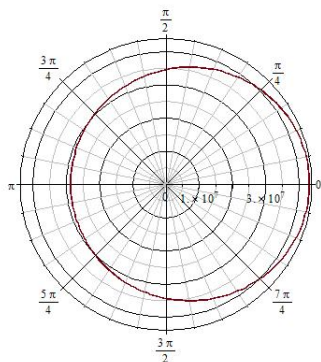
(b) The shortest distance occurs when r is a minimum which occurs when $\cos \theta = -1$. The shortest distance is then

$$r = \frac{(3.442)10^7}{1 - 0.206(-1)} \approx 28,540,630.182 \text{ miles}.$$

(c) The greatest distance occurs when r is a maximum which occurs when $\cos \theta = 1$. The shortest distance is then

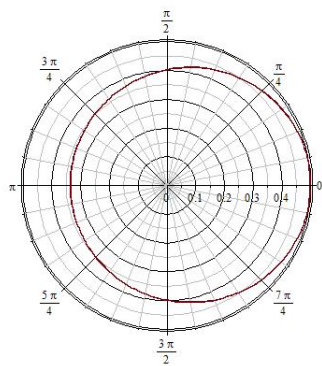
$$r = \frac{(3.442)10^7}{1 - 0.206(1)} \approx 43,350,125.945 \text{ miles}.$$

(d)

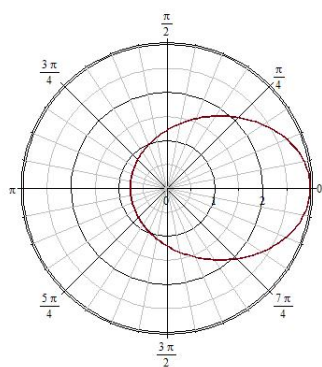


35. (a)

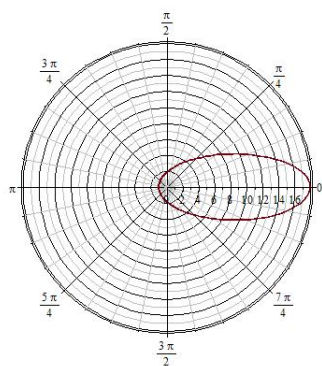
(i)



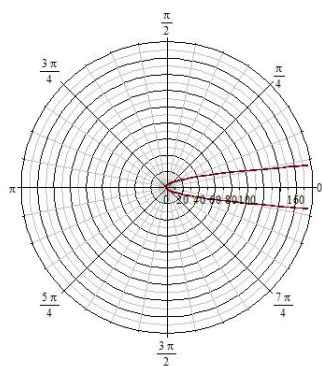
(ii)



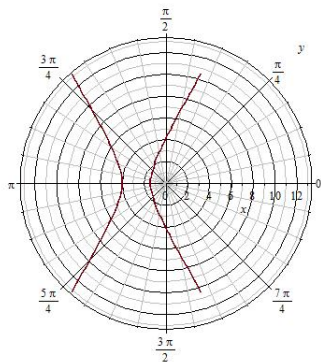
(iii)



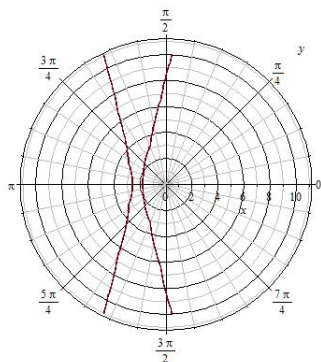
(iv)



(v)



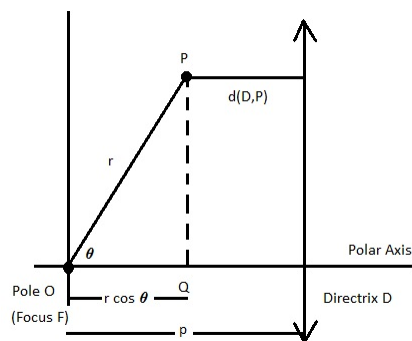
(vi)



(b) Answers will vary.

(c) Answers will vary.

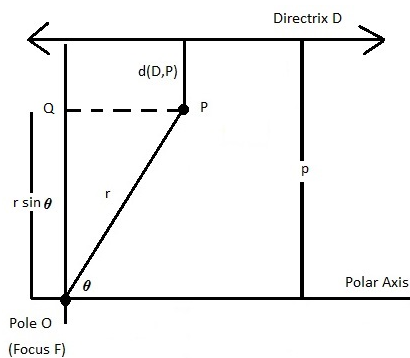
36.



If we drop a perpendicular from point P to the polar axis, the intersection point of the perpendicular and the polar axis is labeled Q . The distance from the pole to Q is $r \cos \theta$. Since the distance from the pole to the directrix D is p , the distance $d(D, P) = p - r \cos \theta$. Then since $d(F, P) = r$, the equation of the conic is

$$\begin{aligned} \frac{d(F, P)}{d(D, P)} &= e \\ \frac{r}{p - r \cos \theta} &= e \\ r &= ep - re \cos \theta \\ r(1 + e \cos \theta) &= ep \\ r &= \frac{ep}{1 + e \cos \theta}. \end{aligned}$$

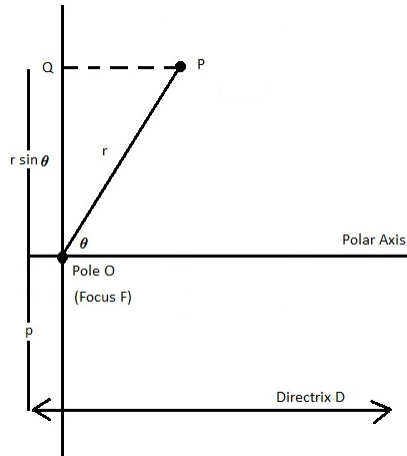
37.



If we drop a perpendicular from point P to the ray $\theta = \frac{\pi}{2}$, the intersection point of the perpendicular and the ray is labeled Q . The distance from the pole to Q is $r \sin \theta$. Since the distance from the pole to the directrix D is p , the distance $d(D, P) = p - r \sin \theta$. Then since $d(F, P) = r$, the equation of the conic is

$$\begin{aligned} \frac{d(F, P)}{d(D, P)} &= e \\ \frac{r}{p - r \sin \theta} &= e \\ r &= ep - re \sin \theta \\ r(1 + e \sin \theta) &= ep \\ r &= \frac{ep}{1 + e \sin \theta}. \end{aligned}$$

38.



If we drop a perpendicular from point P to the ray $\theta = \frac{\pi}{2}$, the intersection point of the perpendicular and the ray is labeled Q . The distance from the pole to Q is $r \sin \theta$. Since the distance from the pole to the directrix D is p , the distance $d(D, P) = p + r \sin \theta$. Then since $d(F, P) = r$, the equation of the conic is

$$\begin{aligned} \frac{d(F, P)}{d(D, P)} &= e \\ \frac{r}{p + r \sin \theta} &= e \\ r &= ep + re \sin \theta \\ r(1 - e \sin \theta) &= ep \\ r &= \frac{ep}{1 - e \sin \theta}. \end{aligned}$$

Challenge Problems

39. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the value $c = \sqrt{a^2 - b^2}$ is the focal distance from the center and $e = \frac{c}{a}$ is the eccentricity. If we use the parametric equations $x = a \cos \theta$ and $y = b \sin \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, then this parametrizes the ellipse in the first quadrant. If we rotate this part of the ellipse about the x -axis to find the surface area S , then

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = 2\pi \int_0^{\pi/2} b \sin \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \\ &= 2\pi b \int_0^{\pi/2} \sin \theta \sqrt{a^2(1 - \cos^2 \theta) + b^2 \cos^2 \theta} d\theta \\ &= 2\pi b \int_0^{\pi/2} a \sin \theta \sqrt{1 - \left(\frac{a^2 - b^2}{a^2} \cos^2 \theta\right)} d\theta. \end{aligned}$$

Now we use the fact that $e = \frac{\sqrt{a^2 - b^2}}{a}$. Then

$$\begin{aligned} S &= 2\pi ab \int_0^{\pi/2} \sin \theta \sqrt{1 - e^2 \cos^2 \theta} d\theta \\ &= 2\pi ab \int_e^0 -\frac{1}{e} \sqrt{1 - u^2} du, \end{aligned}$$

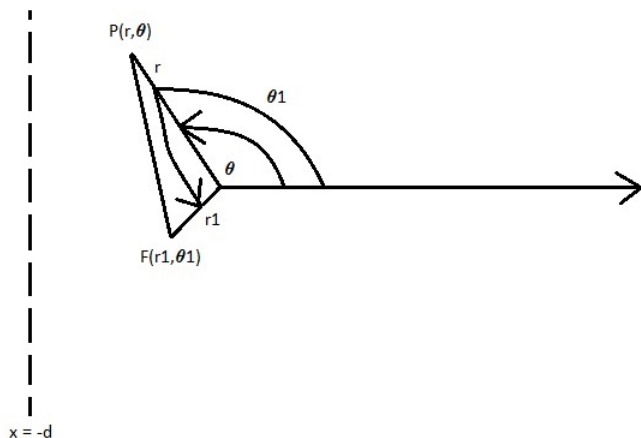
where we used the substitution $u = e \cos \theta$ with $du = -e \sin \theta d\theta$ and new limits of integration $u = e \cos(0) = e$ to $u = e \cos(\pi/2) = 0$. To calculate this integral, we use Table of Integrals 62 with $a = 1$.

$$\begin{aligned} S &= \frac{2\pi ab}{e} \int_0^e \sqrt{1 - u^2} du \\ &= \frac{2\pi ab}{e} \left[\frac{u}{2} \sqrt{1 - u^2} + \frac{1}{2} \sin^{-1} u \right]_0^e \\ &= \frac{2\pi ab}{e} \left(\frac{e}{2} \sqrt{1 - e^2} + \frac{1}{2} \sin^{-1} e \right) \\ &= \pi b^2 + \frac{\pi ab}{e} \sin^{-1} e, \end{aligned}$$

where in the last step we used that fact that

$$\sqrt{1 - e^2} = \sqrt{1 - \frac{a^2 - b^2}{a^2}} = \sqrt{\frac{a^2 - a^2 + b^2}{a^2}} = \frac{b}{a}.$$

40. First, let $\theta_0 = 0$. Then the directrix is $r \cos \theta = -d$ or, in rectangular coordinates, $x = -d$, the vertical line parallel to the y -axis d units to the left (since $d > 0$). See the following picture.



We see that $d(D, P) = r \cos \theta + d$, so

$$e = \frac{d(F, P)}{d(D, P)}.$$

Using the Law of Cosines,

$$d(P, F) = \sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1)}.$$

This means when $\theta_0 = 0$, the general equation for the conic is

$$e = \frac{\sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1)}}{r \cos \theta + d}.$$

For the general case, rotate this by $-\theta_0$ to yield

$$e = \frac{\sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta + \theta_0 - \theta_1)}}{r \cos(\theta + \theta_0) + d}.$$

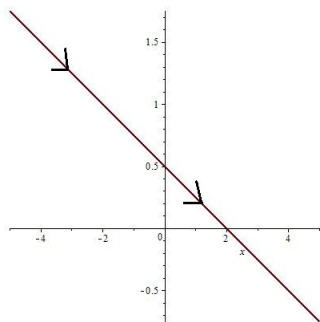
Chapter Review

1. (a) Solving for t in the y equation, we have $t = 1 - y$. Then

$$x = 4(1 - y) - 2$$

$$y = -\frac{1}{4}x + \frac{1}{2}.$$

(b)

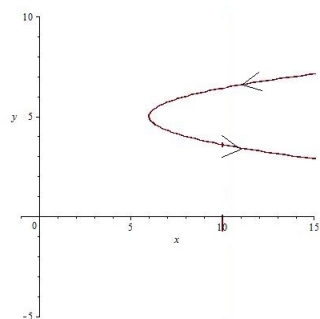


(c) There are no restrictions.

2. (a) Solving for t in the y equation, we have $t = 5 - y$. Then

$$x = 2(5 - y)^2 + 6.$$

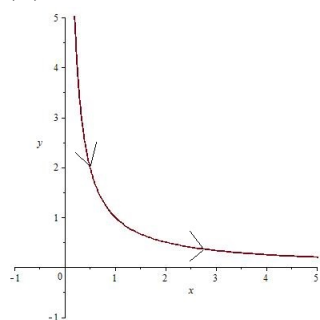
(b)



(c) There are no restrictions.

3. (a) Since $x = e^t$ and $y = (e^t)^{-1}$, we have $y = \frac{1}{x}$.

(b)



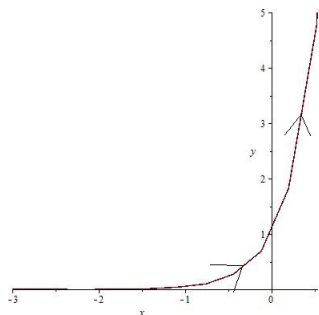
(c) Since $e^t > 0$ and $e^{-t} > 0$, we have $x > 0$ and $y > 0$.

4. (a) Solving for t in the x equation, we have $t = e^x$. Then

$$y = (e^x)^3$$

$$\boxed{y = e^{3x}}.$$

(b)

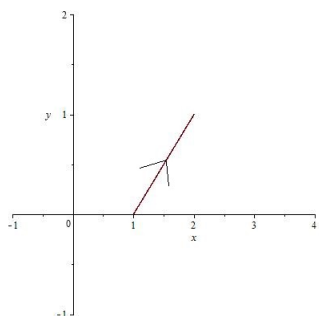


(c) There are $\boxed{\text{no restrictions}}$.

5. (a) Since $\tan^2 t + 1 = \sec^2 t$, the rectangular equation is

$$\boxed{y + 1 = x}.$$

(b)



(c) On $0 \leq t \leq \frac{\pi}{4}$, $1 \leq \sec t \leq \sqrt{2}$ and $0 \leq \tan t \leq 1$ so

$$1 \leq \sec^2 t \leq 2$$

$$0 \leq \tan^2 t \leq 1.$$

The restrictions on x and y are then

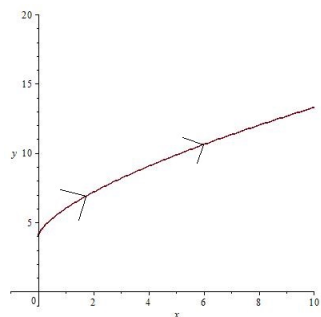
$$\boxed{1 \leq x \leq 2}$$

$$\boxed{0 \leq y \leq 1}.$$

6. (a) Solving for t in the y equation, we have $t = \frac{y-4}{2}$. Then

$$\boxed{x = \left(\frac{y-4}{2}\right)^{3/2}}.$$

(b)



(c) There are no restrictions.

7. (a)

$$\begin{aligned}\frac{dy}{dt} &= 1 \\ \frac{dx}{dt} &= 2t\end{aligned}$$

The slope of the tangent line is

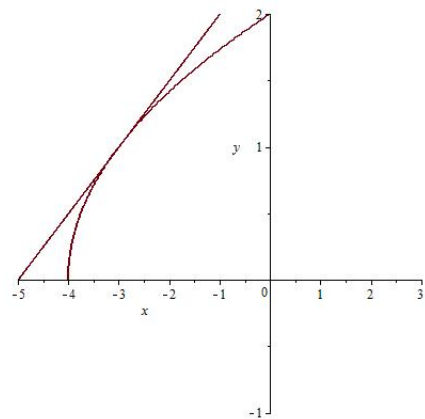
$$\left[\frac{dy}{dx}\right]_{t=1} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=1} = \left[\frac{1}{2t}\right]_{t=1} = \frac{1}{2}.$$

The point of tangency is $(x(1), y(1)) = (-3, 1)$. The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x + 3)$$

$$\boxed{y = \frac{1}{2}x + \frac{5}{2}}.$$

(b)



8. (a)

$$\begin{aligned}\frac{dy}{dt} &= -4 \sin t \\ \frac{dx}{dt} &= 3 \cos t\end{aligned}$$

The slope of the tangent line is

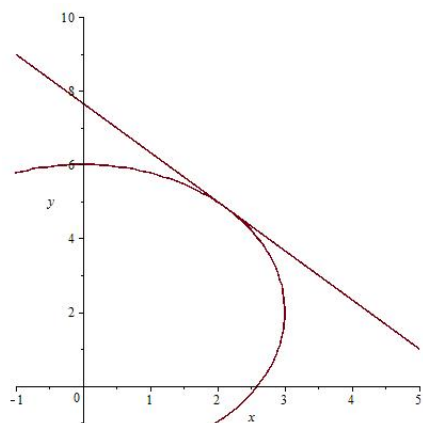
$$\left[\frac{dy}{dx}\right]_{t=\pi/4} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=\pi/4} = \left[\frac{-4 \sin t}{3 \cos t}\right]_{t=\pi/4} = -\frac{4}{3}.$$

The point of tangency is $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2} + 2\right)$. The equation of the tangent line is

$$y - (2\sqrt{2} + 2) = -\frac{4}{3} \left(x - \frac{3\sqrt{2}}{2}\right)$$

$$\boxed{y = -\frac{4}{3}x + 4\sqrt{2} + 2.}$$

(b)



9. (a)

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{2}(t^2 + 1)^{-1/2}(2t) = t(t^2 + 1)^{-1/2} \\ \frac{dx}{dt} &= -2t^{-3}\end{aligned}$$

The slope of the tangent line is

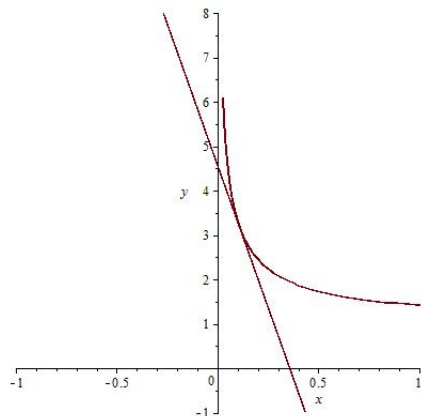
$$\left[\frac{dy}{dx}\right]_{t=3} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=3} = \left[\frac{t(t^2 + 1)^{-1/2}}{-2t^{-3}}\right]_{t=3} = \frac{3/\sqrt{10}}{-2/27} = -\frac{81\sqrt{10}}{20}.$$

The point of tangency is $(x(3), y(3)) = \left(\frac{1}{9}, \sqrt{10}\right)$. The equation of the tangent line is

$$y - \sqrt{10} = -\frac{81\sqrt{10}}{20} \left(x - \frac{1}{9}\right)$$

$$\boxed{y = -\frac{81}{2\sqrt{10}}x + \frac{9}{2\sqrt{10}} + \sqrt{10}.}$$

(b)



10. (a)

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{(1+t)^2} \\ \frac{dx}{dt} &= \frac{2t+t^2}{(1+t)^2}\end{aligned}$$

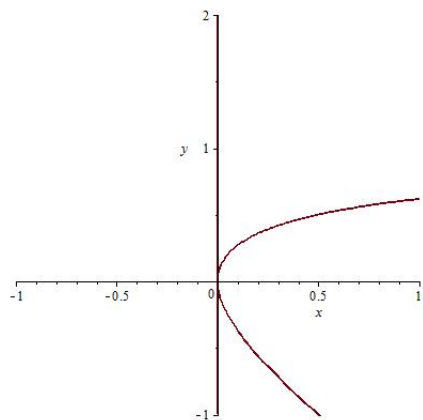
The slope of the tangent line is

$$\left[\frac{dy}{dx}\right]_{t=0} = \left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=0} = \left[\frac{\frac{1}{(1+t)^2}}{\frac{2t+t^2}{(1+t)^2}}\right]_{t=0},$$

which is undefined, so the tangent line is vertical. The point of tangency is $(x(0), y(0)) = (0, 0)$. The equation of the tangent line is

$$\boxed{x = 0}.$$

(b)



11. Answers will vary. Here are two parameterizations.

$$\boxed{x(t) = t, \quad y(t) = -2t + 4, \quad -\infty < t < \infty}$$

$$\boxed{x(t) = t - 2, \quad y(t) = -2t + 8, \quad -\infty < t < \infty}.$$

12. Answers will vary. Here are two parameterizations.

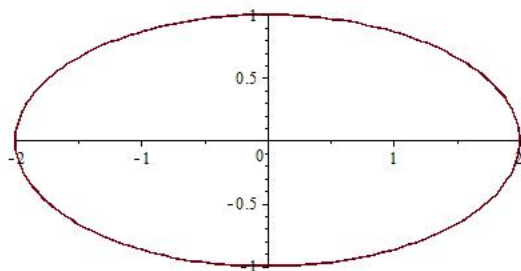
$$\boxed{x(t) = t, \quad y(t) = 2t, \quad -\infty < t < \infty}$$

$$x(t) = t^3, \quad y(t) = 2t^3, \quad -\infty < t < \infty.$$

13. We eliminate the parameter t by using the Pythagorean Identity $\cos^2 t + \sin^2 t = 1$. Since $\cos t = \frac{x}{2}$ and $\sin t = y$, we have

$$\frac{x^2}{4} + y^2 = 1.$$

The plane curve is below.



When $t = 0$, the object is at the point $(2, 0)$. As t increases, the object moves around the ellipse in a counterclockwise direction, taking $t = 2\pi$ seconds to complete one revolution.

For the graphs of Problems 14-17, see the graph below Problem 17.

14. Converting to rectangular coordinates

$$\begin{aligned} x &= r \cos \theta = 3 \cos \left(\frac{\pi}{6} \right) = \frac{3\sqrt{3}}{2} \\ y &= r \sin \theta = 3 \sin \left(\frac{\pi}{6} \right) = \frac{3}{2}. \end{aligned}$$

The rectangular coordinates are $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2} \right)$.

15. Converting to rectangular coordinates

$$\begin{aligned} x &= r \cos \theta = -2 \cos \left(\frac{4\pi}{3} \right) = 1 \\ y &= r \sin \theta = -2 \sin \left(\frac{4\pi}{3} \right) = \sqrt{3}. \end{aligned}$$

The rectangular coordinates are $(1, \sqrt{3})$.

16. Converting to rectangular coordinates

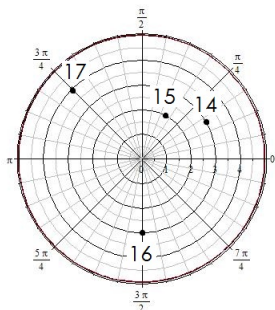
$$\begin{aligned} x &= r \cos \theta = 3 \cos \left(-\frac{\pi}{2} \right) = 0 \\ y &= r \sin \theta = 3 \sin \left(-\frac{\pi}{2} \right) = -3. \end{aligned}$$

The rectangular coordinates are $(0, -3)$.

17. Converting to rectangular coordinates

$$\begin{aligned}x &= r \cos \theta = -4 \cos \left(-\frac{\pi}{4}\right) = -2\sqrt{2} \\y &= r \sin \theta = -4 \sin \left(-\frac{\pi}{4}\right) = 2\sqrt{2}.\end{aligned}$$

The rectangular coordinates are $\boxed{(-2\sqrt{2}, 2\sqrt{2})}$.



18. Converting to polar coordinates

$$\begin{aligned}r &= \sqrt{2^2 + 0^2} = 2 \\ \theta &= \tan^{-1} \left(\frac{0}{2}\right) = 0.\end{aligned}$$

One pair of polar coordinates is $\boxed{(2, 0)}$. Another pair is $(-2, 0 + \pi) = \boxed{(-2, \pi)}$.

19. Converting to polar coordinates

$$\begin{aligned}r &= \sqrt{3^2 + 4^2} = 5 \\ \theta &= \tan^{-1} \left(\frac{4}{3}\right).\end{aligned}$$

One pair of polar coordinates is $\boxed{\left(5, \tan^{-1} \left(\frac{4}{3}\right)\right) \approx (5, 0.927)}$. Another pair is $\boxed{\left(-5, \tan^{-1} \left(\frac{4}{3}\right) + \pi\right) \approx (-5, 4.068)}$.

20. Noting that the given point is in quadrant II, when we convert to polar coordinates we have

$$\begin{aligned}r &= \sqrt{(-5)^2 + 12^2} = 13 \\ \theta &= \tan^{-1} \left(-\frac{12}{5}\right) + \pi.\end{aligned}$$

One pair of polar coordinates is $\boxed{\left(13, \tan^{-1} \left(-\frac{12}{5}\right) + \pi\right) \approx (13, 1.966)}$. Another pair is

$$\boxed{\left(-13, \tan^{-1} \left(-\frac{12}{5}\right) + 2\pi\right) \approx (-13, 5.107)}.$$

21. Noting that the given point is in quadrant II, when we convert to polar coordinates we have

$$\begin{aligned}r &= \sqrt{(-3)^2 + 3^2} = 3\sqrt{2} \\ \theta &= \tan^{-1} \left(-\frac{3}{3}\right) + \pi = \frac{3\pi}{4}.\end{aligned}$$

One pair of polar coordinates is $\boxed{\left(3\sqrt{2}, \frac{3\pi}{4}\right)}$. Another pair is $\boxed{\left(-3\sqrt{2}, \frac{7\pi}{4}\right)}$.

22. Use the double angle formula for sine, $\sin(2\theta) = 2 \sin \theta \cos \theta$, and multiply the equation through by r^2 .

$$\begin{aligned} r^3 &= 8r \sin(\theta) r \cos(\theta) \\ \left(\sqrt{x^2 + y^2}\right)^3 &= 8xy \\ \sqrt{x^2 + y^2} &= 2x^{1/3}y^{1/3} \\ \boxed{x^2 + y^2} &= 4x^{2/3}y^{2/3}. \end{aligned}$$

23. Using the formulas $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the equation becomes

$$\begin{aligned} \sqrt{x^2 + y^2} &= e^{\tan^{-1}(y/x)/2} \\ x^2 + y^2 &= e^{\tan^{-1}(y/x)} \\ \frac{y}{x} &= \tan(\ln(x^2 + y^2)) \\ \boxed{y} &= x \tan(\ln(x^2 + y^2)). \end{aligned}$$

24. First clear fractions.

$$\begin{aligned} r + 2r \cos \theta &= 1 \\ \sqrt{x^2 + y^2} + 2x &= 1 \\ x^2 + y^2 &= (1 - 2x)^2 \\ \boxed{-3x^2 + 4x + y^2 - 1} &= 0. \end{aligned}$$

25. Multiply the equation through by r .

$$\begin{aligned} r^2 &= ar - r \sin \theta \\ \boxed{x^2 + y^2} &= a\sqrt{x^2 + y^2} - y. \end{aligned}$$

26. Use the double angle formula for cosine, $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$, and multiply the equation through by r^2 .

$$\begin{aligned} r^2 &= 4 \cos^2 \theta - 4 \sin^2 \theta \\ r^4 &= 4r^2 \cos^2 \theta - 4r^2 \sin^2 \theta \\ \boxed{(x^2 + y^2)^2} &= 4x^2 - 4y^2. \end{aligned}$$

27.

$$\begin{aligned} \sqrt{x^2 + y^2} &= \tan^{-1}\left(\frac{y}{x}\right) \\ \boxed{y} &= x \tan\left(\sqrt{x^2 + y^2}\right). \end{aligned}$$

28. Using the formulas $r^2 = x^2 + y^2$ and $x = r \cos \theta$, the equation becomes $r^2 = r \cos \theta$, or $\boxed{r = \cos \theta}$.

29. Using the formulas $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, the equation becomes

$$\begin{aligned} (r^2)^2 &= (r \cos \theta)^2 - (r \sin \theta)^2 \\ r^4 &= r^2(\cos^2 \theta - \sin^2 \theta) \quad (\text{for } r \neq 0) \\ \boxed{r^2} &= \cos^2 \theta - \sin^2 \theta \quad (\text{for } r \neq 0). \end{aligned}$$

30. Using the formulas $r^2 = x^2 + y^2$, $x = r \cos \theta$, and $y = r \sin \theta$, the equation becomes

$$(r \sin \theta)^2 = r^2 \cos^2 \theta$$

$$\boxed{\sin^2 \theta = \cos^2 \theta}.$$

31. Using the formulas $x = r \cos \theta$ and $y = r \sin \theta$, the equation becomes

$$\frac{(r \cos \theta)^2}{4} + \frac{(r \sin \theta)^2}{9} = 1$$

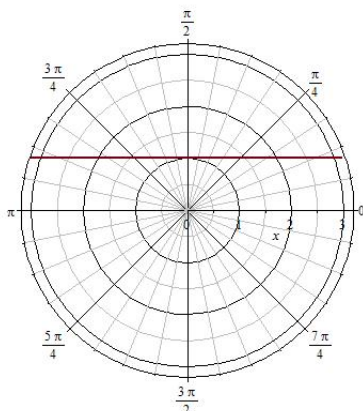
$$r^2 \left(\frac{\cos^2 \theta}{4} + \frac{\sin^2 \theta}{9} \right) = 1$$

$$r^2 \left(\frac{9 \cos^2 \theta + 4 \sin^2 \theta}{36} \right) = 1$$

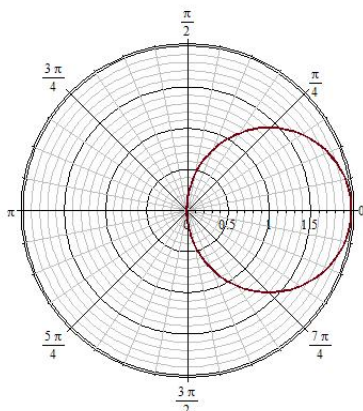
$$r = \frac{6}{\sqrt{9 \cos^2 \theta + 4 \sin^2 \theta}}$$

$$\boxed{r = \frac{6\sqrt{9 \cos^2 \theta + 4 \sin^2 \theta}}{9 \cos^2 \theta + 4 \sin^2 \theta}}.$$

32. In rectangular coordinates, the equation is $y = 1$.

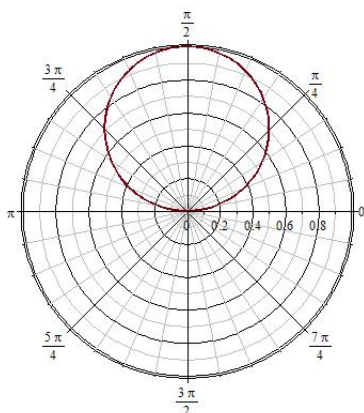


33. Multiplying by $r \cos \theta$, the equation becomes $r^2 = 2r \cos \theta$ which in rectangular coordinates is $x^2 + y^2 = 2x$. Upon completing the square, the equation becomes $(x - 1)^2 + y^2 = 1$. We recognize this as a circle centered at $(1, 0)$ with radius 1.

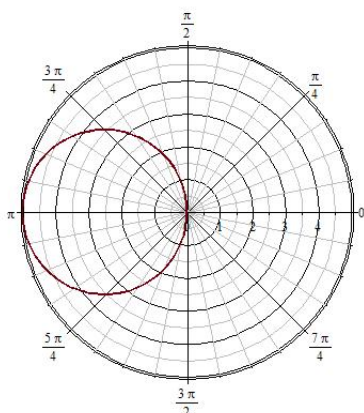


34. Multiplying by r , the equation becomes $r^2 = r \sin \theta$ which in rectangular coordinates is $x^2 + y^2 = y$. Upon completing the square, the equations becomes $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. We recognize this as a circle

centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$.



35. Multiplying by r , the equation becomes $r^2 = -5r \cos \theta$ which in rectangular coordinates is $x^2 + y^2 = -5x$. Upon completing the square, the equation becomes $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$. We recognize this as a circle centered at $\left(-\frac{5}{2}, 0\right)$ with radius $\frac{5}{2}$.

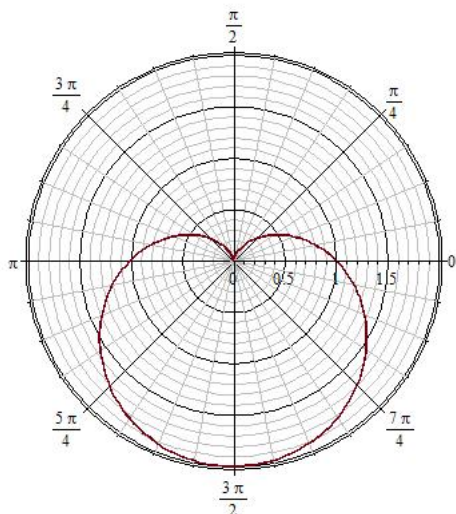


36. (a) The polar equation $r = 1 - \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (1 - \sin \theta, \theta)$, and trace out the graph, beginning at the point $(1, 0)$ and ending at $(1, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(1 - \frac{1}{2}\sqrt{2}, \frac{\pi}{4}\right)$	$\left(1 - \frac{1}{2}\sqrt{3}, \frac{\pi}{3}\right)$	$\left(0, \frac{\pi}{2}\right)$	$\left(1 - \frac{1}{2}\sqrt{3}, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(1, \pi)$	$\left(\frac{3}{2}, \frac{7\pi}{6}\right)$	$\left(1 + \frac{1}{2}\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(1 + \frac{1}{2}\sqrt{3}, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(2, \frac{3\pi}{2}\right)$	$\left(1 + \frac{1}{2}\sqrt{3}, \frac{5\pi}{3}\right)$	$\left(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(\frac{3}{2}, \frac{11\pi}{6}\right)$	$(1, 2\pi)$



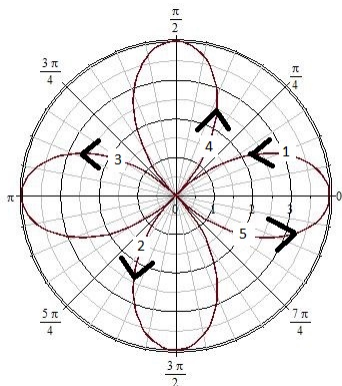
(b) Parametric equations for $r = 1 - \sin \theta$:

$$x = r \cos \theta = (1 - \sin \theta) \cos \theta \quad y = r \sin \theta = (1 - \sin \theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

37. (a) The polar equation $r = 4 \cos(2\theta)$ contains $\cos(2\theta)$, which has period π . We construct a table of common values of θ that range from 0 to 2π noting that the values from π to 2π duplicate those from 0 to π .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(4, 0)$	$(2, \frac{\pi}{6})$	$(0, \frac{\pi}{4})$	$(-2, \frac{\pi}{3})$	$(-4, \frac{\pi}{2})$	$(-2, \frac{2\pi}{3})$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(0, \frac{3\pi}{4})$	$(2, \frac{5\pi}{6})$	$(4, \pi)$	$(2, \frac{7\pi}{6})$	$(0, \frac{5\pi}{4})$	$(-2, \frac{4\pi}{3})$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$(-4, \frac{3\pi}{2})$	$(-2, \frac{5\pi}{3})$	$(0, \frac{7\pi}{4})$	$(2, \frac{11\pi}{6})$	$(4, 2\pi)$	



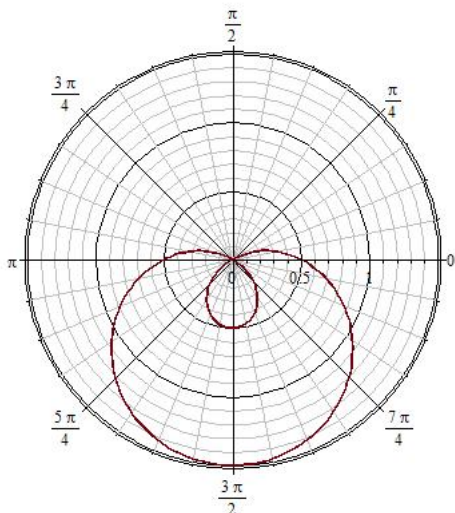
(b) Parametric equations for $r = 4 \cos(2\theta)$:

$$x = r \cos \theta = 4 \cos(2\theta) \cos \theta \quad y = r \sin \theta = 4 \cos(2\theta) \sin \theta$$

where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once starting at $(4, 0)$ and following the arrows in increasing order.

38. (a) The polar equation $r = \frac{1}{2} - \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = \left(\frac{1}{2} - \sin \theta, \theta\right)$, and trace out the graph, beginning at the point $\left(\frac{1}{2}, 0\right)$ and ending at $\left(\frac{1}{2}, 2\pi\right)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$\left(\frac{1}{2}, 0\right)$	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{1}{2} - \frac{1}{2}\sqrt{2}, \frac{\pi}{4}\right)$	$\left(\frac{1}{2} - \frac{1}{2}\sqrt{3}, \frac{\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{\pi}{2}\right)$	$\left(\frac{1}{2} - \frac{1}{2}\sqrt{3}, \frac{2\pi}{3}\right)$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{1}{2} - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(0, \frac{5\pi}{6}\right)$	$\left(\frac{1}{2}, \pi\right)$	$\left(1, \frac{7\pi}{6}\right)$	$\left(\frac{1}{2} + \frac{1}{2}\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}, \frac{4\pi}{3}\right)$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$\left(\frac{3}{2}, \frac{3\pi}{2}\right)$	$\left(\frac{1}{2} + \frac{1}{2}\sqrt{3}, \frac{5\pi}{3}\right)$	$\left(\frac{1}{2} + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(1, \frac{11\pi}{6}\right)$	$\left(\frac{1}{2}, 2\pi\right)$	



(b) Parametric equations for $r = \frac{1}{2} - \sin \theta$:

$$x = r \cos \theta = \left(\frac{1}{2} - \sin \theta\right) \cos \theta \quad y = r \sin \theta = \left(\frac{1}{2} - \sin \theta\right) \sin \theta$$

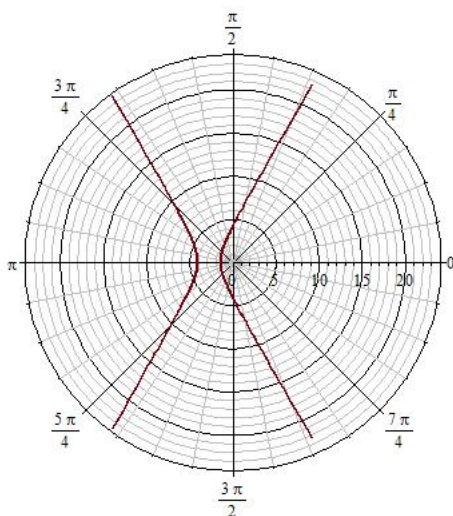
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

39. (a) The polar equation $r = \frac{4}{1 - 2\cos \theta}$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π (excluding the values $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ since r is not defined there) and plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(-4, 0)$	$\left(\frac{4}{1 - \sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\frac{4}{1 - \sqrt{2}}, \frac{\pi}{4}\right)$	undefined	$\left(4, \frac{\pi}{2}\right)$	$\left(2, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{4}{1+\sqrt{2}}, \frac{3\pi}{4}\right)$	$\left(\frac{4}{1+\sqrt{3}}, \frac{5\pi}{6}\right)$	$\left(\frac{4}{3}, \pi\right)$	$\left(\frac{4}{1+\sqrt{3}}, \frac{7\pi}{6}\right)$	$\left(\frac{4}{1+\sqrt{2}}, \frac{5\pi}{4}\right)$	$\left(2, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(4, \frac{3\pi}{2}\right)$	undefined	$\left(\frac{4}{1-\sqrt{2}}, \frac{7\pi}{4}\right)$	$\left(\frac{4}{1-\sqrt{3}}, \frac{11\pi}{6}\right)$	$(-4, 2\pi)$



(b) Parametric equations for the polar equation $r = \frac{4}{1 - 2\cos\theta}$ are

$$x = \frac{4\cos\theta}{1 - 2\cos\theta}$$

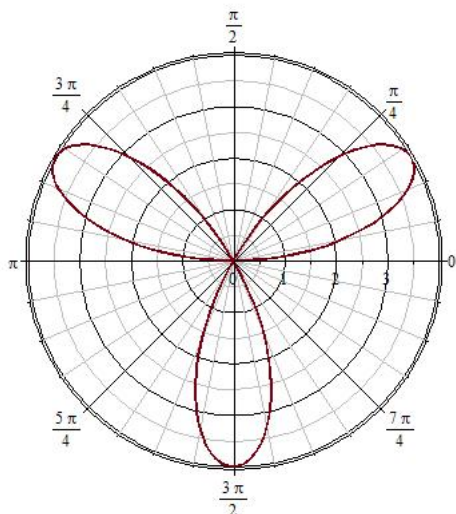
$$y = \frac{4\sin\theta}{1 - 2\cos\theta}.$$

40. The polar equation $r = 4\sin(3\theta)$ contains $\sin(3\theta)$, which has period $\frac{2\pi}{3}$. So we construct a table of common values for θ that range from $0 \leq \theta \leq 2\pi$, noting that the values $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$ and $\frac{4\pi}{3} \leq \theta \leq 2\pi$ repeat the values for $0 \leq \theta \leq \frac{2\pi}{3}$. Then we plot the points and trace out the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	(0, 0)	$\left(4, \frac{\pi}{6}\right)$	$\left(2\sqrt{2}, \frac{\pi}{4}\right)$	$\left(0, \frac{\pi}{3}\right)$	$\left(-4, \frac{\pi}{2}\right)$	$\left(0, \frac{2\pi}{3}\right)$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(2\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(4, \frac{5\pi}{6}\right)$	(0, π)	$\left(-4, \frac{7\pi}{6}\right)$	$\left(-2\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(0, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(4, \frac{3\pi}{2}\right)$	$\left(0, \frac{5\pi}{3}\right)$	$\left(-2\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(-4, \frac{11\pi}{6}\right)$	(0, 2π)



(b) Parametric equations for $r = 4 \sin(3\theta)$:

$$x = r \cos \theta = 4 \sin(3\theta) \cos \theta \quad y = r \sin \theta = 4 \sin(3\theta) \sin \theta$$

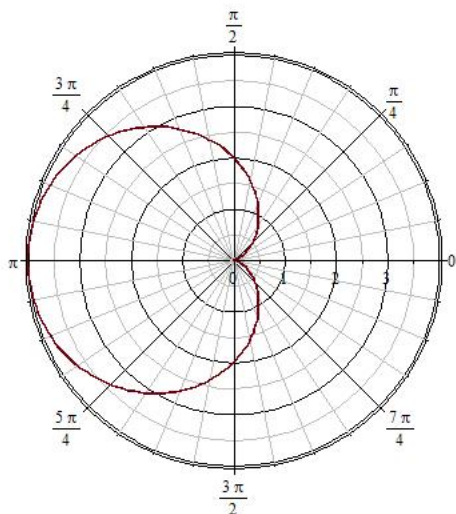
where θ is the parameter.

41. (a) The polar equation $r = 2 - 2 \cos \theta$ contains $\cos \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (2 - 2 \cos \theta, \theta)$, and trace out the graph, beginning at the point $(0, 0)$ and ending at $(0, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(0, 0)$	$(2 - \sqrt{3}, \frac{\pi}{6})$	$(2 - \sqrt{2}, \frac{\pi}{4})$	$(1, \frac{\pi}{3})$	$(2, \frac{\pi}{2})$	$(3, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(2 + \sqrt{2}, \frac{3\pi}{4})$	$(2 + \sqrt{3}, \frac{5\pi}{6})$	$(4, \pi)$	$(2 + \sqrt{3}, \frac{7\pi}{6})$	$(2 + \sqrt{2}, \frac{5\pi}{4})$	$(3, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(2, \frac{3\pi}{2})$	$(1, \frac{5\pi}{3})$	$(2 - \sqrt{2}, \frac{7\pi}{4})$	$(2 - \sqrt{3}, \frac{11\pi}{6})$	$(0, 2\pi)$



(b) Parametric equations for $r = 2 - 2 \cos \theta$:

$$x = r \cos \theta = (2 - 2 \cos \theta) \cos \theta \quad y = r \sin \theta = (2 - 2 \cos \theta) \sin \theta$$

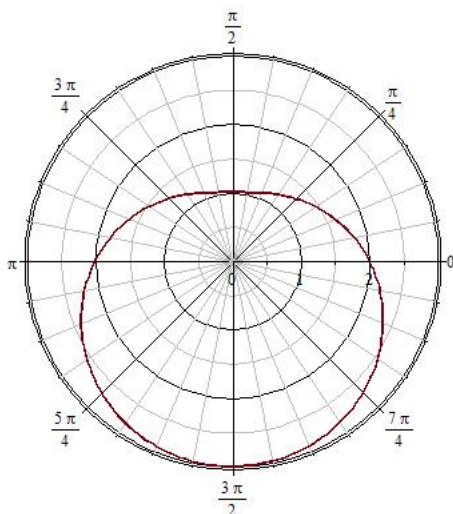
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

42. (a) The polar equation $r = 2 - \sin \theta$ contains $\sin \theta$, which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points $(r, \theta) = (2 - \sin \theta, \theta)$, and trace out the graph, beginning at the point $(2, 0)$ and ending at $(2, 2\pi)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(2, 0)$	$(\frac{3}{2}, \frac{\pi}{6})$	$(2 - \frac{1}{2}\sqrt{2}, \frac{\pi}{4})$	$(2 - \frac{1}{2}\sqrt{3}, \frac{\pi}{3})$	$(1, \frac{\pi}{2})$	$(2 - \frac{1}{2}\sqrt{3}, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$(2 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4})$	$(\frac{3}{2}, \frac{5\pi}{6})$	$(2, \pi)$	$(\frac{5}{2}, \frac{7\pi}{6})$	$(2 + \frac{1}{2}\sqrt{2}, \frac{5\pi}{4})$	$(2 + \frac{1}{2}\sqrt{3}, \frac{4\pi}{3})$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$(3, \frac{3\pi}{2})$	$(2 + \frac{1}{2}\sqrt{3}, \frac{5\pi}{3})$	$(2 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4})$	$(\frac{5}{2}, \frac{11\pi}{6})$	$(2, 2\pi)$



(b) Parametric equations for $r = 2 - \sin \theta$:

$$x = r \cos \theta = (2 - \sin \theta) \cos \theta \quad y = r \sin \theta = (2 - \sin \theta) \sin \theta$$

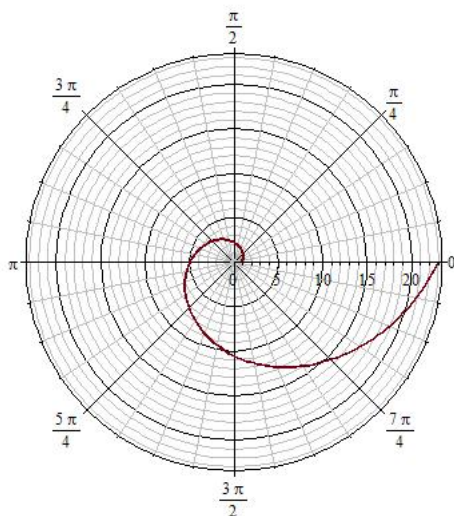
where θ is the parameter, and if $0 \leq \theta \leq 2\pi$, then the graph is traced out exactly once in the counterclockwise direction.

43. (a) The polar equation $r = e^{0.5\theta}$ increases in radius as the angle increases. We construct a table of common values of θ that range from 0 to 2π , plot the points (r, θ) , and trace out the graph. All decimals in the table are approximate values of r .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$(1.299, \frac{\pi}{6})$	$(1.481, \frac{\pi}{4})$	$(1.688, \frac{\pi}{3})$	$(2.193, \frac{\pi}{2})$	$(2.850, \frac{2\pi}{3})$

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(3.248, \frac{3\pi}{4}\right)$	$\left(3.702, \frac{5\pi}{6}\right)$	$(4.810, \pi)$	$\left(6.250, \frac{7\pi}{6}\right)$	$\left(7.124, \frac{5\pi}{4}\right)$	$\left(8.121, \frac{4\pi}{3}\right)$

θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
(r, θ)	$\left(10.551, \frac{3\pi}{2}\right)$	$\left(13.708, \frac{5\pi}{3}\right)$	$\left(15.625, \frac{7\pi}{4}\right)$	$\left(17.811, \frac{11\pi}{6}\right)$	$(23.141, 2\pi)$

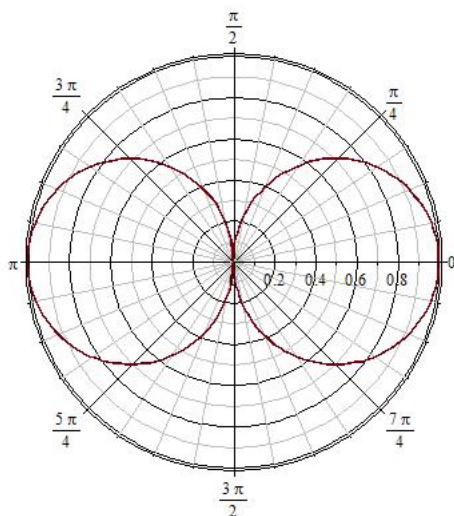


(b) Parametric equations for $r = e^{0.5\theta}$:

$$x = r \cos \theta = e^{0.5\theta} \cos \theta \quad y = r \sin \theta = e^{0.5\theta} \sin \theta$$

where θ is the parameter.

44. (a) The equation can be rewritten as $r^2 = \cos^2 \theta$, or $r = \pm \cos \theta$. The equation $r = \cos \theta$ is a circle centered at $\left(\frac{1}{2}, 0\right)$ of radius $\frac{1}{2}$ and the equation $r = -\cos \theta$ is a circle centered at $\left(-\frac{1}{2}, 0\right)$ of radius $\frac{1}{2}$. Put these two graphs together to obtain the graph of the given equation.



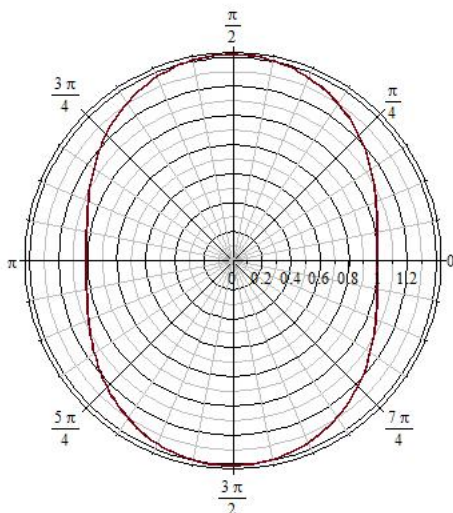
(b) Parametric equations for $r = \sqrt{1 - \sin^2 \theta}$:

$$x = r \cos \theta = \sqrt{1 - \sin^2 \theta} \cos \theta \quad y = r \sin \theta = \sqrt{1 - \sin^2 \theta} \sin \theta$$

where θ is the parameter.

45. The equation $r^2 = 1 + \sin^2 \theta$, or $r = \sqrt{1 + \sin^2 \theta}$, contains $\sin \theta$ which has the period 2π . We construct a table of common values of θ that range from 0 to 2π , plot the points (r, θ) .

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
(r, θ)	$(1, 0)$	$\left(\frac{\sqrt{5}}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{\pi}{4}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{\pi}{3}\right)$	$\left(\sqrt{2}, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{2\pi}{3}\right)$
θ	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
(r, θ)	$\left(\frac{\sqrt{6}}{2}, \frac{3\pi}{4}\right)$	$\left(\frac{\sqrt{5}}{2}, \frac{5\pi}{6}\right)$	$(1, \pi)$	$\left(\frac{\sqrt{5}}{2}, \frac{7\pi}{6}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{5\pi}{4}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{4\pi}{3}\right)$
θ	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π	
(r, θ)	$\left(\sqrt{2}, \frac{3\pi}{2}\right)$	$\left(\frac{\sqrt{7}}{2}, \frac{5\pi}{3}\right)$	$\left(\frac{\sqrt{6}}{2}, \frac{7\pi}{4}\right)$	$\left(\frac{\sqrt{5}}{2}, \frac{11\pi}{6}\right)$	$(1, 2\pi)$	

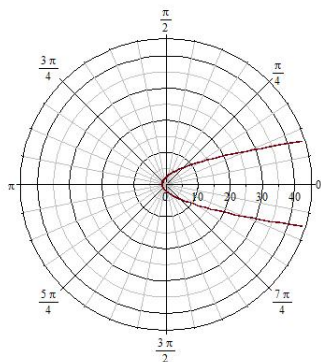


(b) Parametric equations for $r = \sqrt{1 + \sin^2 \theta}$:

$$x = r \cos \theta = \sqrt{1 + \sin^2 \theta} \cos \theta \quad y = r \sin \theta = \sqrt{1 + \sin^2 \theta} \sin \theta$$

where θ is the parameter.

46. (a) Since $e = 1$, this is a parabola.



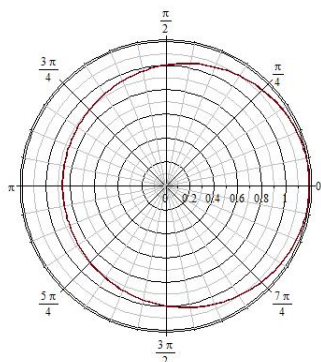
(b) To obtain a rectangular equation of the parabola, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
 r &= \frac{2}{1 - \cos \theta} \\
 r - r \cos \theta &= 2 \\
 r &= 2 + r \cos \theta \\
 r^2 &= (2 + r \cos \theta)^2 \\
 x^2 + y^2 &= (2 + x)^2 \\
 x^2 + y^2 &= 4 + 4x + x^2 \\
 \boxed{y^2} &= 4 + 4x.
 \end{aligned}$$

(c) Parametric equations are

$$\boxed{x = \frac{2}{1 - \cos \theta} \cos \theta, \quad y = \frac{2}{1 - \cos \theta} \sin \theta.}$$

47. (a) Since $e = \frac{1}{6} < 1$, this is an ellipse.



(b) To obtain a rectangular equation of the ellipse, we eliminate the fraction and then square the resulting polar equation.

$$\begin{aligned}
 r &= \frac{1}{1 - \frac{1}{6} \cos \theta} \\
 r - \frac{1}{6} r \cos \theta &= 1 \\
 r^2 &= \left(1 + \frac{1}{6} r \cos \theta\right)^2 \\
 x^2 + y^2 &= \left(1 + \frac{1}{6} x\right)^2 \\
 x^2 + y^2 &= 1 + \frac{1}{3} x + \frac{1}{36} x^2 \\
 \frac{35}{36} \left(x^2 - \frac{12}{35} x\right) + y^2 &= 1 \\
 \frac{35}{36} \left(x - \frac{6}{35}\right)^2 + y^2 &= 1 + \frac{35}{36} \left(\frac{6}{35}\right)^2 \\
 \boxed{35 \left(x - \frac{6}{35}\right)^2 + 36 y^2} &= \frac{1296}{35}.
 \end{aligned}$$

(c) Parametric equations are

$$x = \frac{6 \cos \theta}{6 - \cos \theta}, \quad y = \frac{6 \sin \theta}{6 - \cos \theta}.$$

48. For counterclockwise orientation, the ellipse can be parametrized by $x(t) = 4 \cos(\omega t)$, $y(t) = 3 \sin(\omega t)$. If it takes 4 seconds for one revolution, then the period is $\frac{2\pi}{\omega} = 4$, or $\omega = \frac{\pi}{2}$. The parametrization is then

$$x(t) = 4 \cos\left(\frac{\pi}{2}t\right), y(t) = 3 \sin\left(\frac{\pi}{2}t\right), \quad 0 \leq t \leq 4$$

which has counterclockwise orientation, and does indeed start at $(4, 0)$ when $t = 0$.

49. For clockwise orientation, the ellipse can be parametrized by $x(t) = 4 \sin(\omega t)$, $y(t) = 3 \cos(\omega t)$. If it takes 5 seconds for one revolution, then the period is $\frac{2\pi}{\omega} = 5$, or $\omega = \frac{2\pi}{5}$. The parametrization is then

$$x(t) = 4 \sin\left(\frac{2\pi}{5}t\right), y(t) = 3 \cos\left(\frac{2\pi}{5}t\right), \quad 0 \leq t \leq 5$$

which has clockwise orientation, and does indeed start at $(0, 3)$ when $t = 0$.

50. First we find the derivatives.

$$\begin{aligned} \frac{dx}{dt} &= 3t^2 \\ \frac{dy}{dt} &= 4t \end{aligned}$$

Since the only solution to $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ is $t = 0$, there are no horizontal nor vertical tangent lines.

51. First we find the derivatives.

$$\begin{aligned} \frac{dx}{dt} &= -\cos t \\ \frac{dy}{dt} &= -3 \sin t \end{aligned}$$

To find the horizontal tangent lines, we set $\frac{dy}{dt} = 0$.

$$\begin{aligned} -3 \sin t &= 0 \\ t &= 0, \pi, 2\pi. \end{aligned}$$

None of these values make $\frac{dx}{dt} = 0$, so we have horizontal tangent lines at each value. The horizontal tangent lines occur at the points

$$\begin{aligned} (x(0), y(0)) = (x(2\pi), y(2\pi)) &= (1, 5) \\ (x(\pi), y(\pi)) &= (1, -1). \end{aligned}$$

To find the vertical tangent lines, we set $\frac{dx}{dt} = 0$.

$$\begin{aligned} -\cos t &= 0 \\ t &= \frac{\pi}{2}, \frac{3\pi}{2}. \end{aligned}$$

None of these values make $\frac{dy}{dt} = 0$, so we have vertical tangent lines at each value. The vertical tangent lines occur at the points

$$\begin{aligned} \left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) &= \boxed{(0, 2)} \\ \left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) &= \boxed{(2, 2)}. \end{aligned}$$

52. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{2}(1+t^2)^{-1/2} \cdot 2t = \frac{t}{\sqrt{1+t^2}}$$

The curve is smooth for $0 \leq t \leq 1$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^1 \sqrt{\left(\frac{1}{\sqrt{1+t^2}}\right)^2 + \left(\frac{t}{\sqrt{1+t^2}}\right)^2} dt \\ &= \int_0^1 dt \\ &= [t]_0^1 = \boxed{1}. \end{aligned}$$

53. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \sec^2 t \quad \text{and} \quad \frac{dy}{dt} = \frac{2}{3} \sec t \sec t \tan t = \frac{2}{3} \sec^2 t \tan t$$

The curve is smooth for $0 \leq t \leq \frac{\pi}{4}$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^{\pi/4} \sqrt{(\sec^2 t)^2 + \left(\frac{2}{3} \sec^2 t \tan t\right)^2} dt \\ &= \int_0^{\pi/4} \sqrt{\sec^4 t \left(1 + \frac{4}{9} \tan^2 t\right)} dt \\ &= \int_0^{\pi/4} \sec^2 t \sqrt{1 + \frac{4}{9} \tan^2 t} dt. \end{aligned}$$

Using the substitution $u = \tan t$, then $du = \sec^2 t$ and the new limits are $u = \tan(0) = 0$ to $u = \tan\left(\frac{\pi}{4}\right) = 1$, then

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + \frac{4}{9}u^2} du = \int_0^1 \frac{2}{3} \sqrt{\frac{9}{4} + u^2} du \\ &= \frac{2}{3} \left[\frac{u}{2} \sqrt{\frac{9}{4} + u^2} + \frac{9/4}{2} \ln \left| u + \sqrt{\frac{9}{4} + u^2} \right| \right]_0^1 \\ &= \boxed{\frac{1}{3} \frac{\sqrt{13}}{2} + \frac{9}{8} \ln \left(1 + \frac{\sqrt{13}}{2} \right) - \frac{9}{8} \ln \left(\frac{\sqrt{13}}{2} \right)}, \end{aligned}$$

where to compute the integral we used Table of Integrals 49 with $a = \frac{3}{2}$.

54. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \quad \text{and} \quad \frac{dy}{dt} = e^{2t} - \frac{1}{4}$$

The curve is smooth for $0 \leq t \leq 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_0^2 \sqrt{(e^t)^2 + \left(e^{2t} - \frac{1}{4}\right)^2} dt = \int_0^2 \sqrt{e^{2t} + e^{4t} - \frac{1}{2}e^{2t} + \frac{1}{16}} dt \\ &= \int_0^2 \sqrt{\frac{1}{16}(16e^{4t} + 8e^{2t} + 1)} dt = \frac{1}{4} \int_0^2 \sqrt{(4e^{2t} + 1)^2} dt \\ &= \frac{1}{4} [2e^{2t} + t]_0^2 = \frac{1}{2}e^4 + \frac{1}{2} - \frac{1}{2} = \boxed{\frac{1}{2}e^4}. \end{aligned}$$

55. The given curve can be parametrized by

$$\begin{aligned} y(t) &= t \\ x(t) &= \frac{1}{2}t^2 - \frac{1}{4}\ln t \end{aligned}$$

on $1 \leq t \leq 2$. We begin by finding the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = t - \frac{1}{4t} \quad \text{and} \quad \frac{dy}{dt} = 1$$

The curve is smooth for $1 \leq t \leq 2$. Using the arc length formula, we have

$$\begin{aligned} s &= \int_1^2 \sqrt{[1]^2 + \left[t - \frac{1}{4t}\right]^2} dt = \int_1^2 \sqrt{\frac{1}{16}t^{-2} + \frac{1}{2} + t^2} dt \\ &= \int_1^2 \sqrt{\left[\frac{1}{4}t^{-1} + t\right]^2} dt = \int_1^2 \left(\frac{1}{4}t^{-1} + t\right) dt \\ &= \left[\frac{1}{4}\ln|t| + \frac{1}{2}t^2\right]_1^2 = 2 + \frac{1}{4}\ln(2) - \frac{1}{2} \\ &= \boxed{\frac{3}{2} + \frac{1}{4}\ln(2)}. \end{aligned}$$

56. Using the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ from Section 9.5 with $r = 2 \sin \theta$ and $\frac{dr}{d\theta} = 2 \cos \theta$, the arc length is

$$\begin{aligned} s &= \int_0^{\pi} \sqrt{(2 \sin \theta)^2 + (2 \cos \theta)^2} d\theta = \int_0^{\pi} 2 d\theta \\ &= [2\theta]_0^{\pi} = \boxed{2\pi}. \end{aligned}$$

57. Using the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ from Section 9.5 with $r = e^{-\theta}$ and $\frac{dr}{d\theta} = -e^{-\theta}$, the arc length is

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2} d\theta = \int_0^{2\pi} \sqrt{2}e^{-\theta} d\theta \\ &= [-\sqrt{2}e^{-\theta}]_0^{2\pi} = \boxed{\sqrt{2}(1 - e^{-2\pi})}. \end{aligned}$$

58. Using the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ from Section 9.5 with $r = 3\theta$ and $\frac{dr}{d\theta} = 3$, the arc length is

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(3\theta)^2 + (3)^2} d\theta = \int_0^{2\pi} 3\sqrt{\theta^2 + 1} d\theta \\ &= 3 \left[\frac{\theta}{2} \sqrt{\theta^2 + 1} + \frac{1}{2} \ln \left| \theta + \sqrt{\theta^2 + 1} \right| \right]_0^{2\pi} \\ &= 3 \left[\pi \sqrt{4\pi^2 + 1} + \frac{1}{2} \ln \left(2\pi + \sqrt{4\pi^2 + 1} \right) \right] \\ &= \boxed{3\pi \sqrt{4\pi^2 + 1} + \frac{3}{2} \ln \left(2\pi + \sqrt{4\pi^2 + 1} \right)}, \end{aligned}$$

where we used Table of Integrals 49 to calculate the integral.

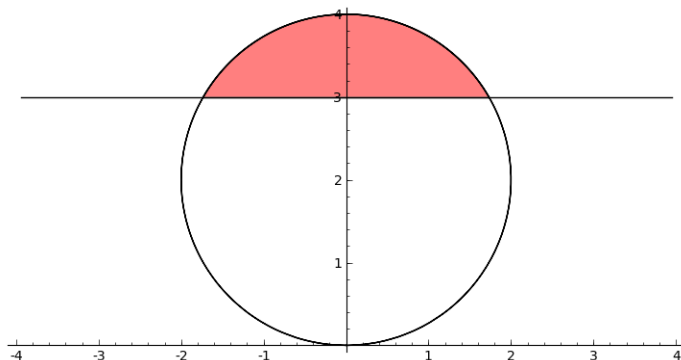
59. Using the arc length formula $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ from Section 9.5 with $r = 2 \sin^2 \frac{\theta}{2}$ and $\frac{dr}{d\theta} = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, we will exploit symmetry and find the arc length s of the curve from $\theta = 0$ to $\theta = \frac{\pi}{2}$ and then double it. Then

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(2 \sin^2 \frac{\theta}{2}\right)^2 + \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2} d\theta = \int_0^{\pi/2} \sqrt{4 \sin^4 \frac{\theta}{2} + \left(4 \sin^4 \frac{\theta}{2} \cos^2 \frac{\theta}{2}\right)} d\theta \\ &= \int_0^{\pi/2} 2 \sin \frac{\theta}{2} d\theta = \left[-4 \cos \frac{\theta}{2} \right]_0^{\pi/2} \\ &= 4 - 2\sqrt{2}. \end{aligned}$$

The full arc length is then

$$2(4 - 2\sqrt{2}) = \boxed{8 - 4\sqrt{2}}.$$

60.



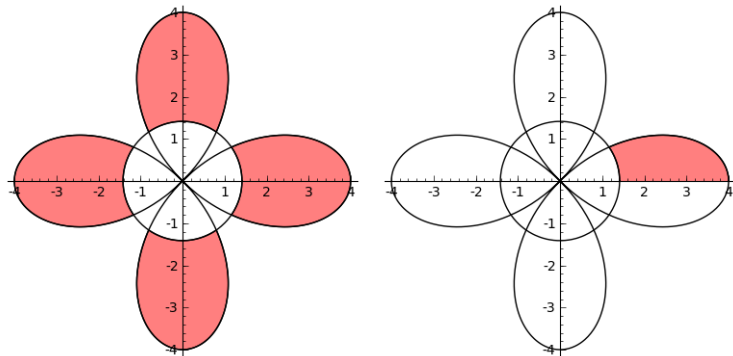
We find the intersection points of the curves by setting their equations equal.

$$\begin{aligned} 4 \sin \theta &= 3 \csc \theta \\ \sin^2 \theta &= \frac{3}{4} \\ \sin \theta &= \frac{\sqrt{3}}{2} \\ \theta &= \frac{\pi}{3}, \frac{2\pi}{3}. \end{aligned}$$

In the third step, we only need the positive square root since we are working only in the first and second quadrants, where sine is positive. The area A of the shaded region is swept out beginning with $\theta = \frac{\pi}{3}$ and ending with $\theta = \frac{2\pi}{3}$. Then

$$\begin{aligned} A &= \int_{\pi/3}^{2\pi/3} \frac{1}{2} ((4 \sin \theta)^2 - (3 \csc \theta)^2) d\theta = \int_{\pi/3}^{2\pi/3} \left(8 \sin^2 \theta - \frac{9}{2} \csc^2 \theta \right) d\theta \\ &= \left[4\theta - 4 \sin \theta \cos \theta + \frac{9}{2} \cot \theta \right]_{\pi/3}^{2\pi/3} = \boxed{\frac{4\pi}{3} - \sqrt{3}}. \end{aligned}$$

61.



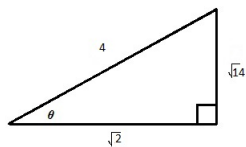
Exploiting symmetry, we will find the area A of the half-petal in quadrant I (see second figure), and then multiply it by 8 to find the full shaded region (see first figure). The area of the half-petal is swept out starting at $\theta = 0$ and ending at the first intersection point of the curves.

$$\begin{aligned} 4 \cos(2\theta) &= \sqrt{2} \\ 2\theta &= \cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \\ \theta &= \frac{1}{2} \cos^{-1} \left(\frac{\sqrt{2}}{4} \right). \end{aligned}$$

Then

$$\begin{aligned} A &= \int_0^{1/2 \cos^{-1}(\sqrt{2}/4)} \frac{1}{2} ((4 \cos(2\theta))^2 - (\sqrt{2})^2) d\theta = \int_0^{1/2 \cos^{-1}(\sqrt{2}/4)} (8 \cos^2(2\theta) - 1) d\theta \\ &= [3\theta + 2 \cos(2\theta) \sin(2\theta)]_0^{1/2 \cos^{-1}(\sqrt{2}/4)}. \end{aligned}$$

In the following picture, we have $\cos \theta = \frac{\sqrt{2}}{4}$ or, equivalently, $\theta = \cos^{-1} \left(\frac{\sqrt{2}}{4} \right)$.



Then

$$\begin{aligned} \cos \left(\cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) &= \frac{\sqrt{2}}{4} \\ \sin \left(\cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) &= \frac{\sqrt{14}}{4}. \end{aligned}$$

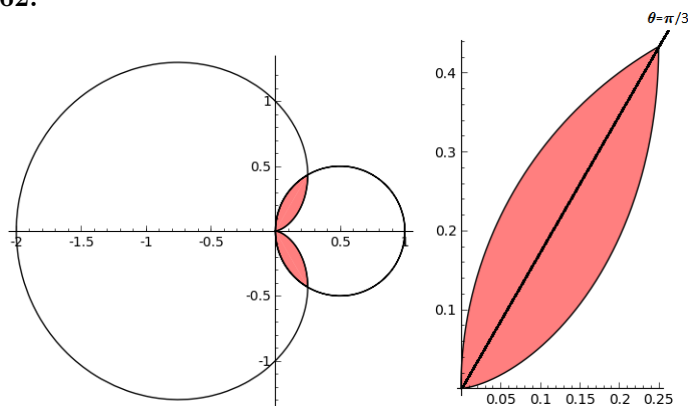
With these values, we have

$$\begin{aligned} A &= \left[3 \left(\frac{1}{2} \cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) + 2 \cos \left(\cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) \sin \left(\cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) \right] \\ &= \frac{3}{2} \cos^{-1} \left(\frac{\sqrt{2}}{4} \right) + \frac{\sqrt{7}}{4}. \end{aligned}$$

Then the full shaded region has area

$$8 \left(\frac{\sqrt{7}}{4} + \frac{3}{2} \cos^{-1} \left(\frac{\sqrt{2}}{4} \right) \right) = \boxed{2\sqrt{7} + 12 \cos^{-1} \left(\frac{\sqrt{2}}{4} \right)}.$$

62.



Exploiting symmetry, we will find the area of the shaded region in quadrant I and then double it to find the full shaded area. Referring to the second picture, the shaded region in quadrant I can be divided by the ray from the pole through the intersection point of the two curves. The angle that represents the ray through the intersection point can be found by setting the equations equal.

$$\begin{aligned} \cos \theta &= 1 - \cos \theta \\ \cos \theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}. \end{aligned}$$

The shaded region in quadrant I can then be found by finding the area of the shaded region on $\left[0, \frac{\pi}{3}\right]$ of the cardioid and then adding the area of the shaded region on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ of the circle. This means the area A in quadrant I is

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} (1 - \cos \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} (\cos \theta)^2 d\theta \\ &= \int_0^{\pi/3} \left(\frac{1}{2} - \cos \theta + \frac{1}{2} \cos^2 \theta \right) d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} \cos^2 \theta d\theta \\ &= \left[\frac{3}{4} \theta - \sin \theta + \frac{1}{4} \cos \theta \sin \theta \right]_0^{\pi/3} + \left[\frac{1}{4} \theta + \frac{1}{4} \cos \theta \sin \theta \right]_{\pi/3}^{\pi/2} \\ &= \left[\frac{\pi}{4} - \frac{7\sqrt{3}}{16} \right] + \left[\frac{\pi}{24} - \frac{\sqrt{3}}{16} \right] = \frac{7\pi}{24} - \frac{\sqrt{3}}{2}. \end{aligned}$$

Then the full shaded area is

$$2 \left(\frac{7\pi}{24} - \frac{\sqrt{3}}{2} \right) = \boxed{\frac{7\pi}{12} - \sqrt{3}}.$$

63. We start by finding $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\sqrt{t^2+1}} \\ \frac{dy}{dt} &= \frac{t}{\sqrt{t^2+1}}\end{aligned}$$

Using formula (2) in Section 9.3, the surface area S is

$$\begin{aligned}S &= 2\pi \int_0^1 \sqrt{t^2+1} \sqrt{\left(\frac{1}{\sqrt{1+t^2}}\right)^2 + \left(\frac{t}{\sqrt{1+t^2}}\right)^2} dt = 2\pi \int_0^1 \sqrt{t^2+1} dt \\ &= 2\pi \left[\frac{t}{2} \sqrt{t^2+1} + \frac{1}{2} \ln |t + \sqrt{t^2+1}| \right]_0^1 = \boxed{\pi (\sqrt{2} + \ln(1 + \sqrt{2}))},\end{aligned}$$

where the integral was computed using the Table of Integrals 49.

64. We start by finding $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\begin{aligned}\frac{dx}{dt} &= e^t - 1 \\ \frac{dy}{dt} &= 2e^{t/2}\end{aligned}$$

Using formula (2) in Section 9.3, the surface area S is

$$\begin{aligned}S &= 2\pi \int_0^1 4e^{t/2} \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt = 2\pi \int_0^1 4e^{t/2} \sqrt{(e^t + 1)^2} dt = 2\pi \int_0^1 (4e^{3/2t} + 4e^{t/2}) d\theta \\ &= 2\pi \left[\frac{8}{3} e^{3/2t} + 8e^{t/2} \right]_0^1 = \boxed{\frac{16\pi}{3} e^{3/2} + 16\pi e^{1/2} - \frac{64\pi}{3}}.\end{aligned}$$

65. The given curve can be parametrized by

$$\begin{aligned}y(t) &= t \\ x(t) &= t^2 + 1\end{aligned}$$

for $0 \leq t \leq 2$. Using the formula following Example 1 in Section 9.5, the surface area S is

$$S = 2\pi \int_0^2 (t^2 + 1) \sqrt{[1]^2 + [2t]^2} dt = 2\pi \int_0^2 (t^2 + 1) \sqrt{1 + 4t^2} dt.$$

To compute this integral, we use the substitution $t = \frac{\tan u}{2}$ since then $\sqrt{1 + 4t^2} = \sqrt{1 + \tan^2 u} = \sec u$. With this substitution, $dt = \frac{1}{2} \sec^2 u du$, $u = \tan^{-1}(2t)$, and the new limits of integration are $u = \tan^{-1}(2(0)) = 0$ to $u = \tan^{-1}(2(2)) = \tan^{-1}(4)$. Then

$$S = 2\pi \int_0^{\tan^{-1}(4)} \left(\frac{1}{4} \tan^2 u + 1 \right) \cdot \sec u \cdot \frac{1}{2} \sec^2 u du = 2\pi \int_0^{\tan^{-1}(4)} \left(\frac{1}{8} \sec^5 u + \frac{3}{8} \sec^3 u \right) du,$$

where we used the identity $\tan^2 u = \sec^2 u - 1$. From here we use the reduction formula for secant, Table of Integrals 94, multiple times. Then

$$S = 2\pi \left[\frac{1}{32} \tan u \sec^3 u + \frac{3}{64} \tan u \sec u + \frac{3}{64} \ln |\sec u + \tan u| + \frac{3}{16} \tan u \sec u + \frac{3}{16} \ln |\sec u + \tan u| \right]_0^{\tan^{-1}(4)}.$$

If $u = \tan^{-1}(4)$, then $\tan u = 4 = \frac{4}{1}$ so $\sec u = \sqrt{17}$, or $\sec(\tan^{-1}(4)) = \sqrt{17}$. Then

$$\begin{aligned} S &= 2\pi \left[\frac{17\sqrt{17}}{8} + \frac{3\sqrt{17}}{16} + \frac{3}{64} \ln |\sqrt{17} + 4| + \frac{3\sqrt{17}}{4} + \frac{3}{16} \ln |\sqrt{17} + 4| \right] - 2\pi [0] \\ &= \boxed{\frac{49\sqrt{17}\pi}{8} + \frac{15\pi}{32} \ln (\sqrt{17} + 4)}. \end{aligned}$$

66. Using the formula at the end of Section 9.3, the surface area S is

$$S = 2\pi \int_0^\pi \cos \frac{x}{2} \sqrt{1 + \left[-\frac{1}{2} \sin \frac{x}{2}\right]^2} dx.$$

Using the substitution $u = \frac{1}{2} \sin \frac{x}{2}$, then $du = \frac{1}{4} \cos \frac{x}{2} dx$ and the new limits of integration are $u = \frac{1}{2} \sin(0) = 0$ to $u = \frac{1}{2} \sin \frac{\pi}{2} = \frac{1}{2}$, and then

$$\begin{aligned} S &= 8\pi \int_0^{1/2} \sqrt{1 + [-u]^2} du \\ &= 8\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_0^{1/2} \\ &= \boxed{\sqrt{5}\pi + 4\pi \ln \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)}, \end{aligned}$$

where to compute the integral we used Table of Integrals 49.

67. Using formula (1) from Section 9.6, the surface area S is

$$\begin{aligned} S &= 2\pi \int_0^{\pi/3} 4 \sin \theta \sqrt{[4]^2 + [0]^2} d\theta = 32\pi \int_0^{\pi/3} \sin \theta d\theta \\ &= 32\pi [-\cos \theta]_0^{\pi/3} = \boxed{16\pi}. \end{aligned}$$

Chapter 10: Vectors: Lines, Planes, and Quadric Surfaces in Space

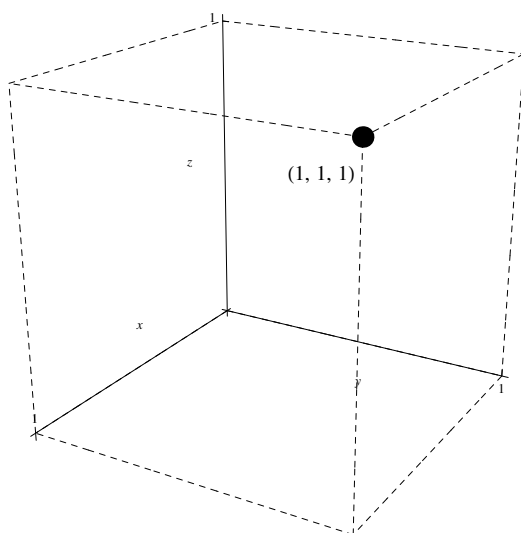
10.1: Rectangular Coordinates in Space

Concepts and Vocabulary

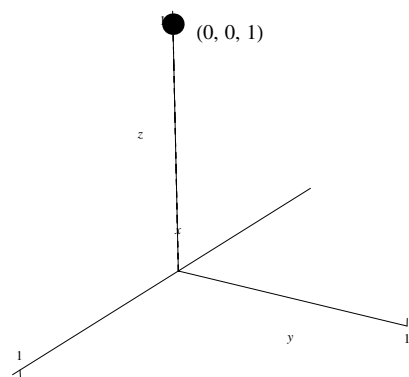
1. True. See objective 1.
2. False. Points on the z -axis have their x - and y -coordinates equal to zero. The point given lies on the y -axis.
3. True. See objective 1.
4. True. It is the plane parallel to the xz -plane, 3 units above that plane.
5. In space, the set of all points a fixed distance R from a fixed point (x_0, y_0, z_0) is called a *sphere*.
6. The equation $x^2 + y^2 + z^2 = 8$ describes a sphere whose center is at the point $(0, 0, 0)$. See the theorem in objective 3.

Skill Building

7.

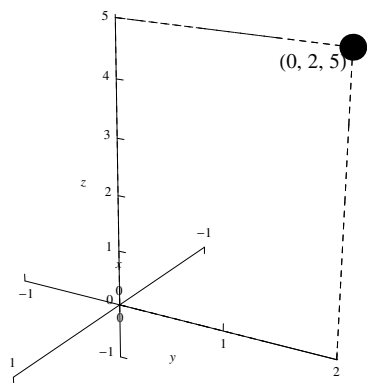


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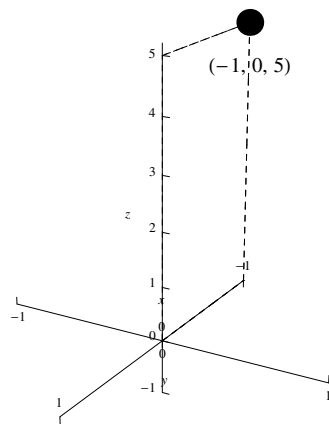


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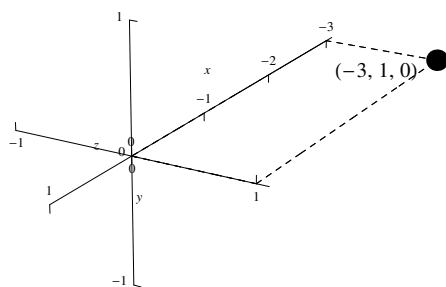
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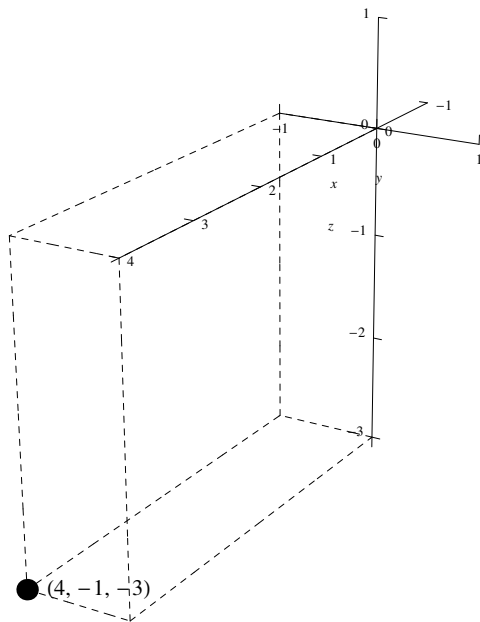
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11.



12.



13. Using Figure 3 as a model, the other vertices of the box are $(2, 1, 3)$ with one or more of its components set to zero, so they are $(2, 1, 0)$, $(2, 0, 3)$, $(0, 1, 3)$, $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 3)$.

14. Using Figure 3 as a model, the other vertices of the box are $(4, 2, 2)$ with one or more of its components set to zero, so they are $(4, 2, 0)$, $(4, 0, 2)$, $(0, 2, 2)$, $(4, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

15. Each x -coordinate of a vertex must be either 1 or 3; each y -coordinate must be either 2 or 4, and each z -coordinate must be either 3 or 5. Therefore the other six vertices are

$$(1, 2, 5), (1, 4, 3), (3, 2, 3), (3, 4, 3), (3, 2, 5), \text{ and } (1, 4, 5).$$

16. Each x -coordinate of a vertex must be either 5 or 3; each y -coordinate must be either 6 or 8, and each z -coordinate must be either 1 or 2. Therefore the other six vertices are

$$(5, 6, 2), (5, 8, 1), (3, 6, 1), (3, 8, 1), (3, 6, 2), \text{ and } (5, 8, 2).$$

17. Each x -coordinate of a vertex must be either -1 or 4 ; each y -coordinate must be either 0 or 2 , and each z -coordinate must be either 2 or 5 . Therefore the other six vertices are

$$(-1, 0, 5), (-1, 2, 2), (4, 0, 2), (-1, 2, 5), (4, 0, 5), \text{ and } (4, 2, 2).$$

18. Each x -coordinate of a vertex must be either -2 or -6 ; each y -coordinate must be either -3 or 7 , and each z -coordinate must be either 0 or 1 . Therefore the other six vertices are

$$(-2, -3, 1), (-2, 7, 0), (-6, -3, 0), (-2, 7, 1), (-6, -3, 1), \text{ and } (-6, 7, 0).$$

19. x and z are free variables. So this is a plane parallel to the xz plane lying two units to the negative side in the y -direction of that plane (passing through the point $(0, -2, 0)$).

20. x and y are free variables. So this is a plane parallel to the xy plane lying 3 units below it in the z -direction (passing through the point $(0, 0, -3)$).

21. This is the yz plane.

22. x and y are free variables. So this is a plane parallel to the xy plane lying 5 units above it in the z -direction (passing through the point $(0, 0, 5)$).
23. This is a line parallel to the z -axis through the point $(1, 0, 0)$, since it is the set of points of the form $(1, 0, z)$ where z is a free variable.
24. This is the set of points of the form $(x, x, 0)$, where x is a free variable, which is the line $y = x$ in the xy -plane.
25. y and z are free variables. So this is the region of space where $x > -2A$ and the y and z values are any real number.
26. x and z are free variables. So this is the region of space between (and including) the planes $y = 0$ and $y = 4$. The plane $y = 0$ is the xz plane, while the plane $y = 4$ is parallel to $y = 0$ and 4 units in the positive y -direction from that plane.
27. Since $x^2 + y^2 + z^2 = 1$ is the sphere of radius 1 centered at the origin, $x^2 + y^2 + z^2 \leq 1$ is that sphere together with its interior. That is, it is all points whose distance from the origin is at most 1.
28. Since $x^2 + y^2 + z^2 = 25$ is the sphere of radius 5 centered at the origin, $x^2 + y^2 + z^2 \geq 25$ is that sphere together with its exterior. That is, it is all points whose distance from the origin is at least 5.
29. With $P_1 = (3, 2, 5)$ and $P_2 = (-1, 2, 2)$, we have

$$|P_1 P_2| = \sqrt{(3 - (-1))^2 + (2 - 2)^2 + (5 - 2)^2} = \sqrt{16 + 0 + 9} = \boxed{5}.$$

30. With $P_1 = (12, -5, 16)$ and $P_2 = (21, 15, 4)$, we have

$$|P_1 P_2| = \sqrt{(12 - 21)^2 + (-5 - 15)^2 + (16 - 4)^2} = \sqrt{81 + 400 + 144} = \sqrt{625} = \boxed{25}.$$

31. With $P_1 = (-1, 2, -3)$ and $P_2 = (4, -2, 1)$, we have

$$|P_1 P_2| = \sqrt{(-1 - 4)^2 + (2 - (-2))^2 + (-3 - 1)^2} = \sqrt{25 + 16 + 16} = \boxed{\sqrt{57}}.$$

32. With $P_1 = (1.0, 3.2, 4.5)$ and $P_2 = (1.4, 1.0, 6.5)$, we have

$$|P_1 P_2| = \sqrt{(1.0 - 1.4)^2 + (3.2 - 1.0)^2 + (4.5 - 6.5)^2} = \sqrt{0.16 + 4.84 + 4} = \sqrt{9} = \boxed{3}.$$

33. With $P_1 = (4, -2, -2)$ and $P_2 = (3, 2, 1)$, we have

$$|P_1 P_2| = \sqrt{(4 - 3)^2 + (-2 - 2)^2 + (-2 - 1)^2} = \sqrt{1 + 16 + 9} = \boxed{\sqrt{26}}.$$

34. With $P_1 = (2, -3, -3)$ and $P_2 = (4, 1, -1)$, we have

$$|P_1 P_2| = \sqrt{(2 - 4)^2 + (-3 - 1)^2 + (-3 - (-1))^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = \boxed{2\sqrt{6}}.$$

35. Using the theorem in objective 3 giving the equation of a sphere, we get

$$\boxed{(x - 3)^2 + (y - 1)^2 + (z - 1)^2 = 1^2 = 1}.$$

36. Using the theorem in objective 3 giving the equation of a sphere, we get

$$\boxed{(x - 1)^2 + (y - 2)^2 + (z - 2)^2 = 2^2 = 4}.$$

37. Using the theorem in objective 3 giving the equation of a sphere, we get

$$(x - (-1))^2 + (y - 1)^2 + (z - 2)^2 = 3^2, \text{ or } \boxed{(x + 1)^2 + (y - 1)^2 + (z - 2)^2 = 9}.$$

38. Using the theorem in objective 3 giving the equation of a sphere, we get

$$(x - (-3))^2 + (y - 1)^2 + (z - (-1))^2 = 1^2, \text{ or } \boxed{(x + 3)^2 + (y - 1)^2 + (z + 1)^2 = 1}.$$

39. As in the text, we complete the square:

$$\begin{aligned} x^2 + y^2 + z^2 + 2x - 2y &= 2 \\ (x^2 + 2x) + (y^2 - 2y) + z^2 &= 2 \\ (x^2 + 2x + 1) + (y^2 - 2y + 1) + z^2 &= 2 + 1 + 1 \\ (x + 1)^2 + (y - 1)^2 + z^2 &= 4. \end{aligned}$$

This is $\boxed{\text{a sphere with radius } \sqrt{4} = 2 \text{ and center } (-1, 1, 0)}.$

40. As in the text, we complete the square:

$$\begin{aligned} x^2 + y^2 + z^2 + 2x - 2z &= -1 \\ (x^2 + 2x) + y^2 + (z^2 - 2z) &= -1 \\ (x^2 + 2x + 1) + y^2 + (z^2 - 2z + 1) &= -1 + 1 + 1 \\ (x + 1)^2 + y^2 + (z - 1)^2 &= 1. \end{aligned}$$

This is $\boxed{\text{a sphere with radius } \sqrt{1} = 1 \text{ and center } (-1, 0, 1)}.$

41. As in the text, we complete the square:

$$\begin{aligned} x^2 + y^2 + z^2 + 4x - 4y + 2z &= 0 \\ (x^2 + 4x) + (y^2 - 4y) + (z^2 + 2z) &= 0 \\ (x^2 + 4x + 4) + (y^2 - 4y + 4) + (z^2 + 2z + 1) &= 0 + 4 + 4 + 1 \\ (x + 2)^2 + (y - 2)^2 + (z + 1)^2 &= 9. \end{aligned}$$

This is $\boxed{\text{a sphere with radius } \sqrt{9} = 3 \text{ and center } (-2, 2, -1)}.$

42. As in the text, we complete the square:

$$\begin{aligned} x^2 + y^2 + z^2 - 4x &= 0 \\ (x^2 - 4x) + y^2 + z^2 &= 0 \\ (x^2 - 4x + 4) + y^2 + z^2 &= 0 + 4 \\ (x - 2)^2 + y^2 + z^2 &= 4. \end{aligned}$$

This is $\boxed{\text{a sphere with radius } \sqrt{4} = 2 \text{ and center } (2, 0, 0)}.$

43. As in the text, we complete the square:

$$\begin{aligned}
 2x^2 + 2y^2 + 2z^2 - 8x + 5z + 1 &= 0 \\
 x^2 + y^2 + z^2 - 4x + \frac{5}{2}z &= -\frac{1}{2} \\
 (x^2 - 4x) + y^2 + \left(z^2 + \frac{5}{2}z\right) &= -\frac{1}{2} \\
 (x^2 - 4x + 4) + y^2 + \left(z^2 + \frac{5}{2}z + \frac{25}{16}\right) &= -\frac{1}{2} + 4 + \frac{25}{16} \\
 (x - 2)^2 + y^2 + \left(z + \frac{5}{4}\right)^2 &= \frac{81}{16}.
 \end{aligned}$$

This is a sphere with radius $\sqrt{\frac{81}{16}} = \frac{9}{4}$ and center $(2, 0, -\frac{5}{4})$.

44. As in the text, we complete the square:

$$\begin{aligned}
 3x^2 + 3y^2 + 3z^2 + 6x - y - 3 &= 0 \\
 x^2 + y^2 + z^2 + 2x - \frac{1}{3}y &= 1 \\
 (x^2 + 2x) + \left(y^2 - \frac{1}{3}y\right) + z^2 &= 1 \\
 (x^2 + 2x + 1) + \left(y^2 - \frac{1}{3}y + \frac{1}{36}\right) + z^2 &= 1 + 1 + \frac{1}{36} \\
 (x + 1)^2 + \left(y - \frac{1}{6}\right)^2 + z^2 &= \frac{73}{36}.
 \end{aligned}$$

This is a sphere with radius $\sqrt{\frac{73}{36}} = \frac{\sqrt{73}}{6}$ and center $(-1, \frac{1}{6}, 0)$.

Applications and Extensions

45. If the given points are the endpoints of a diameter, then the center of the circle must be the point halfway between the given points, which is

$$P = \left(\frac{-2+2}{2}, \frac{0+6}{2}, \frac{4+8}{2}\right) = (0, 3, 6).$$

The radius of the circle is the distance from P to either of the given points, say $P_1 = (-2, 0, 4)$:

$$R = |PP_1| = \sqrt{(0 - (-2))^2 + (3 - 0)^2 + (6 - 4)^2} = \sqrt{4 + 9 + 4} = \sqrt{17}.$$

So the equation of the sphere is

$$x^2 + (y - 3)^2 + (z - 6)^2 = R^2 = 17.$$

46. If the given points are the endpoints of a diameter, then the center of the circle must be the point halfway between the given points, which is

$$P = \left(\frac{1-3}{2}, \frac{3+1}{2}, \frac{6+4}{2}\right) = (-1, 2, 5).$$

The radius of the circle is the distance from P to either of the given points, say $P_1 = (1, 3, 6)$:

$$R = |PP_1| = \sqrt{(-1 - 1)^2 + (2 - 3)^2 + (5 - 6)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}.$$

So the equation of the sphere is

$$(x+1)^2 + (y-2)^2 + (z-5)^2 = R^2 = 6.$$

47. Since the center is $P = (-3, 2, 1)$ and the sphere passes through the point $P_1 = (4, -1, 3)$, the radius is

$$R = |PP_1| = \sqrt{(-3-4)^2 + (2-(-1))^2 + (1-3)^2} = \sqrt{49+9+4} = \sqrt{62}.$$

Therefore the equation of the sphere is

$$(x-(-3))^2 + (y-2)^2 + (z-1)^2 = R^2, \text{ or } (x+3)^2 + (y-2)^2 + (z-1)^2 = 62.$$

48. Since the center is $P = (0, -3, 4)$ and the sphere passes through the point $P_1 = (2, 1, 1)$, the radius is

$$R = |PP_1| = \sqrt{(0-2)^2 + (-3-1)^2 + (4-1)^2} = \sqrt{4+16+9} = \sqrt{29}.$$

Therefore the equation of the sphere is

$$x^2 + (y-(-3))^2 + (z-4)^2 = R^2, \text{ or } x^2 + (y+3)^2 + (z-4)^2 = 29.$$

49. The point of tangency is on the xy -plane, so it has the form $(x, y, 0)$. The point of tangency must lie directly below the center in the z -direction, so it must be the point $(2, 1, 0)$. Then the radius is

$$|(2, 1, -2) - (2, 1, 0)| = \sqrt{(2-2)^2 + (1-1)^2 + (-2-0)^2} = \sqrt{4} = 2.$$

Therefore the equation of the sphere is

$$(x-2)^2 + (y-1)^2 + (z-(-2))^2 = 2^2, \text{ or } (x-2)^2 + (y-1)^2 + (z+2)^2 = 4.$$

50. The point of tangency is on the yz -plane, so it has the form $(0, y, z)$. It must have the same y and z -coordinates as the given point, since it is tangent to the sphere. Therefore the point of tangency is $(0, 5, 4)$. Then the radius is

$$|(1, 5, 4) - (0, 5, 4)| = \sqrt{(1-0)^2 + (5-5)^2 + (4-4)^2} = 1.$$

Therefore the equation of the sphere is

$$(x-1)^2 + (y-5)^2 + (z-4)^2 = 1^2 = 1.$$

51. (a) Since one vertex is at the origin, one point is $(0, 0, 0)$. The sides are 0.287 nm long, so (measuring in nanometers), three other vertices of the cube are at $(0.287, 0, 0)$, $(0, 0.287, 0)$, and $(0, 0, 0.287)$. The other four vertices of the cube are then

$$(0.287, 0.287, 0), (0, 0.287, 0.287), (0.287, 0, 0.287), (0.287, 0.287, 0.287).$$

Finally, the atom in the center is at the midpoint of the line segment joining the points $(0, 0, 0)$ and $(0.287, 0.287, 0.287)$, so its coordinates must be $(\frac{0.287}{2}, \frac{0.287}{2}, \frac{0.287}{2}) = (0.1435, 0.1435, 0.1435)$.

- (b) The atom that is furthest from the origin is the one at the opposite corner of the cube, with coordinates $(0.287, 0.287, 0.287)$.

- (c) From part (a), the coordinates of the atom at the center are $(0.1435, 0.1435, 0.1435)$.

- (d) The atom in the xz plane that is farthest from the origin is the opposite corner of the square in the xz plane, which is the point $(0.287, 0, 0.287)$. The distance from this atom to the atom in the center is

$$\sqrt{(0.287 - 0.1435)^2 + (0 - 0.1435)^2 + (0.287 - 0.1435)^2} \approx \sqrt{3 \cdot 0.1435^2} \approx 0.249 \text{ nm}.$$

52. If it is an equilateral triangle, then the distance from each of these points to the others must be equal. Let $A = (2, 4, 2)$, $B = (2, 1, 5)$, and $C = (5, 1, 2)$. Then

$$\begin{aligned} |AB| &= \sqrt{(2-2)^2 + (4-1)^2 + (2-5)^2} = \sqrt{0+9+9} = \sqrt{18} = 3\sqrt{2} \\ |AC| &= \sqrt{(2-5)^2 + (4-1)^2 + (2-2)^2} = \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2} \\ |BC| &= \sqrt{(2-5)^2 + (1-1)^2 + (5-2)^2} = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}. \end{aligned}$$

Since all three sides have length $3\sqrt{2}$, the triangle is equilateral.

53. (a) From the formula in objective 3, the equation of this sphere is

$$(x-4)^2 + (y-0)^2 + (z-(-2))^2 = 5^2 \quad \text{or} \quad \boxed{(x-4)^2 + y^2 + (z+2)^2 = 25}.$$

- (b) Answers will vary.

54. (a) To show this is a right triangle, we must show that the lengths of its sides satisfy the Pythagorean theorem. Let $A = (-2, 6, 0)$, $B = (4, 9, 1)$, and $C = (-3, 2, 18)$. Then

$$\begin{aligned} |AB| &= \sqrt{(-2-4)^2 + (6-9)^2 + (0-1)^2} = \sqrt{36+9+1} = \sqrt{46} \\ |AC| &= \sqrt{(-2-(-3))^2 + (6-2)^2 + (0-18)^2} = \sqrt{1+16+324} = \sqrt{341} \\ |BC| &= \sqrt{(4-(-3))^2 + (9-2)^2 + (1-18)^2} = \sqrt{49+49+289} = \sqrt{387}. \end{aligned}$$

Since $|AB|^2 + |AC|^2 = 46 + 341 = 387 = |BC|^2$, these three points form a right triangle with hypotenuse BC .

- (b) Note that a circle whose diameter is the hypotenuse also passes through the third vertex of the triangle (Euclid: a triangle inscribed in a circle with one edge the diameter is a right triangle). This will be an equator of the desired sphere. Since BC is the hypotenuse, the center of the sphere must be at the point $P = \frac{B+C}{2} = \left(\frac{4-3}{2}, \frac{9+2}{2}, \frac{1+18}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}, \frac{19}{2}\right)$. The radius of the sphere is half the length of the hypotenuse, so

$$R = \frac{1}{2} |BC| = \frac{1}{2} \sqrt{387}.$$

Therefore the equation of the sphere is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{11}{2}\right)^2 + \left(z - \frac{19}{2}\right)^2 = R^2 = \frac{387}{4}.$$

We can simplify this by multiplying both sides by 4 to get

$$\begin{aligned} 4\left(x - \frac{1}{2}\right)^2 + 4\left(y - \frac{11}{2}\right)^2 + 4\left(z - \frac{19}{2}\right)^2 &= 387 \\ \boxed{(2x-1)^2 + (2y-11)^2 + (2z-19)^2} &= 387. \end{aligned}$$

55. A point on the xy plane has the form $A = (x, y, 0)$, so the distance from the point $P = (4, 2, -1)$ to the xy plane is

$$|PA| = \sqrt{(x-4)^2 + (y-2)^2 + (0-1)^2} = \sqrt{(x-4)^2 + (y-2)^2 + 1}.$$

Now, both $(x-4)^2$ and $(y-2)^2$ are nonnegative, so the distance is at a minimum when they are both zero, which is when $x = 4$ and $y = 2$. So the closest point to P on the xy plane is $(4, 2, 0)$, at a distance of $\boxed{1 \text{ unit}}$ from P .

56. We know how to find the midpoint of a segment in the plane: If (a_1, b_1) and (a_2, b_2) are two points, then the midpoint is $M = \left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$. Note that the x -coordinate of M is the average of the x -coordinates and the y -coordinate of M is the average of the y -coordinates. Extend this notion to points in space $P_1 = (a_1, b_1, c_1)$ and $P_2 = (a_2, b_2, c_2)$. Each coordinate of the midpoint of the segment between P_1 and P_2 will be the average of the corresponding coordinates, so the point is

$$\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}, \frac{c_1 + c_2}{2}\right).$$

Challenge Problems

57. Let $A = (3, 0, 0)$, $B = (0, 0, 1)$, $C = (-1, -3, 1)$, and $D = (2, 0, 1)$. Let $P = (a, b, c)$ be the center of the sphere, and R its radius. Since both A and B lie on the sphere, we must have

$$\begin{aligned}(3-a)^2 + (0-b)^2 + (0-c)^2 &= R^2 = (0-a)^2 + (0-b)^2 + (1-c)^2 \\ 9 - 6a + a^2 + b^2 + c^2 &= a^2 + b^2 + 1 - 2c + c^2 \\ 9 - 6a &= 1 - 2c \\ 2c &= 6a - 8 \\ c &= 3a - 4.\end{aligned}$$

So $P = (a, b, 3a - 4)$. Now, since both A and C lie on the sphere, we must have

$$\begin{aligned}(3-a)^2 + (0-b)^2 + (0-(3a-4))^2 &= R^2 = (-1-a)^2 + (-3-b)^2 + (1-(3a-4))^2 \\ (9-6a+a^2) + b^2 + (9a^2-24a+16) &= (1+2a+a^2) + (9+6b+b^2) + (5-3a)^2 \\ (9-6a+a^2) + b^2 + (9a^2-24a+16) &= (1+2a+a^2) + (9+6b+b^2) + 25-30a+9a^2 \\ 9-6a-24a+16 &= 1+2a+9+6b+25-30a \\ 2a &= -6b-10 \\ a &= -3b-5.\end{aligned}$$

Since $a = -3b - 5$ and $c = 3a - 4$, we have $c = 3(-3b - 5) - 4 = -9b - 19$. Therefore the center of the sphere is $P = (-3b - 5, b, -9b - 19)$. Finally, we use the fourth point: since both A and D lie on the sphere, we must have

$$\begin{aligned}(3-(-3b-5))^2 + (0-b)^2 + (0-(-9b-19))^2 &= R^2 = (2-(-3b-5))^2 + (0-b)^2 + (1-(-9b-19))^2 \\ (3b+8)^2 + b^2 + (9b+19)^2 &= (3b+7)^2 + b^2 + (9b+20)^2 \\ (9b^2+48b+64) + b^2 + (81b^2+342b+361) &= (9b^2+42b+49) + b^2 + (81b^2+360b+400) \\ 91b^2+390b+425 &= 91b^2+402b+449 \\ 12b &= -24 \\ b &= -2.\end{aligned}$$

So the center of the sphere is $P = (-3b - 5, b, -9b - 19) = (1, -2, -1)$. Then the radius of the sphere is

$$R = |AP| = \sqrt{(3-1)^2 + (0-(-2))^2 + (0-(-1))^2} = \sqrt{4+4+1} = 3.$$

So the equation of the sphere is

$$(x-1)^2 + (y+2)^2 + (z+1)^2 = 9.$$

58. (a) Since $A > 0$, first divide through by A , giving

$$x^2 + y^2 + z^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A}z = \frac{G}{A}.$$

Now write P for $\frac{D}{A}$, Q for $\frac{E}{A}$, and so forth, giving

$$x^2 + y^2 + z^2 + Px + Qy + Rz = S.$$

This has the same graph as the original equation. Completing the square gives

$$\begin{aligned} (x^2 + Px) + (y^2 + Qy) + (z^2 + Rz) &= S \\ \left(x^2 + Px + \frac{P^2}{4}\right) + \left(y^2 + Qy + \frac{Q^2}{4}\right) + \left(z^2 + Rz + \frac{R^2}{4}\right) &= S + \frac{P^2}{4} + \frac{Q^2}{4} + \frac{R^2}{4} \\ \left(x + \frac{P}{2}\right)^2 + \left(y + \frac{Q}{2}\right)^2 + \left(z + \frac{R}{2}\right)^2 &= S + \frac{P^2}{4} + \frac{Q^2}{4} + \frac{R^2}{4}. \end{aligned}$$

The left-hand side is a sum of three squares, so it is greater than or equal to zero. So if the right-hand side is negative, the equation has no solutions, so the graph is the empty set. If the right-hand side is zero, then the left-hand side is zero when each of the terms is zero, which happens only at the point $\left(-\frac{P}{2}, -\frac{Q}{2}, -\frac{R}{2}\right)$. Therefore the graph consists of a single point. Finally, if the right-hand side is positive, then this equation is the equation of a sphere centered at $\left(-\frac{P}{2}, -\frac{Q}{2}, -\frac{R}{2}\right)$ with radius $\sqrt{S + \frac{P^2}{4} + \frac{Q^2}{4} + \frac{R^2}{4}}$.

(b) See part (a).

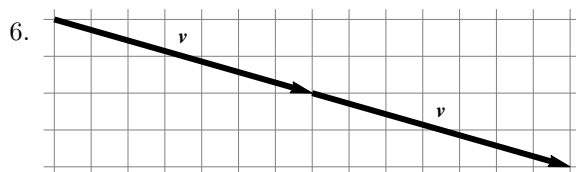
10.2: Introduction to Vectors

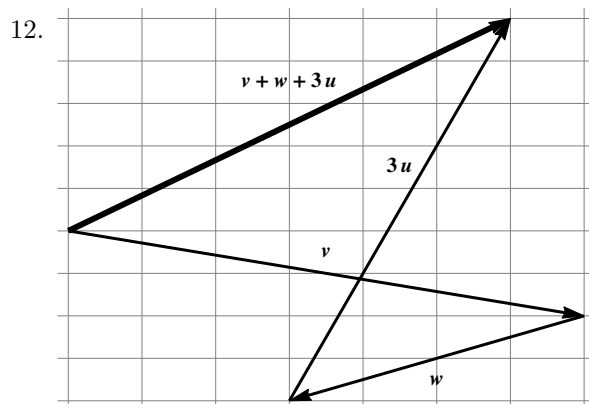
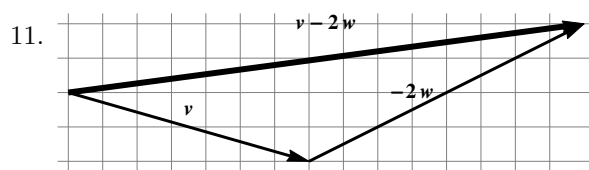
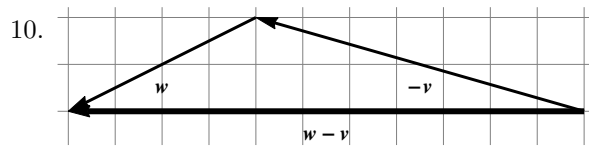
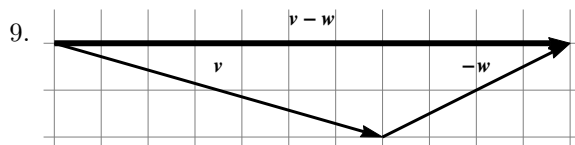
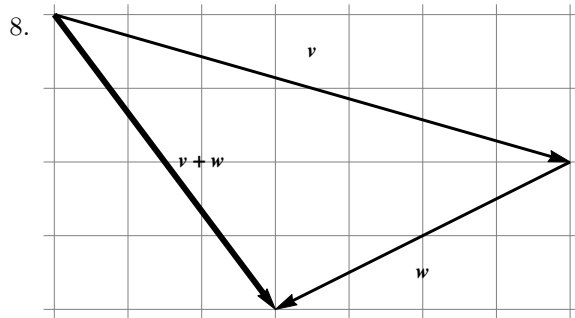
Concepts and Vocabulary

1. True. See the first sentence of this section.
2. Scalars are quantities that have only magnitude. See the introductory material in this section.
3. The product of a scalar a and a vector $\mathbf{v}$ is called a *scalar multiple* of $\mathbf{v}$.
4. The vectors $-\mathbf{v}$ and $\mathbf{v}$ have the same magnitude ((a)) but point in opposite directions ((d)).

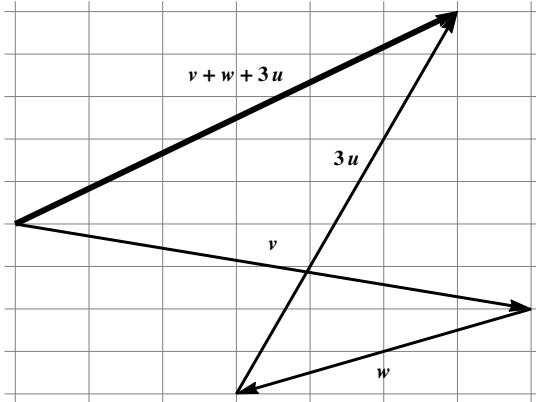
Skill Building

5. (a) Volume is a scalar; it has magnitude, but no direction.
 (b) Speed is a scalar; it has magnitude, but no direction (compare with velocity, in part (h)).
 (c) Force is a vector; one applies an amount of force in a given direction.
 (d) Work is a scalar; it has no direction.
 (e) Mass is a scalar; it has magnitude, but no direction.
 (f) Distance is a scalar. The distance between two things is some number of meters. (Note: one could also say that one object is 3 meters to the northwest of another, in which case this would be a vector, but that is not the usual meaning of the word “distance”.)
 (g) Age is a scalar; it has magnitude, but no direction.
 (h) Velocity is a vector. It measures speed (a magnitude) in a given direction.

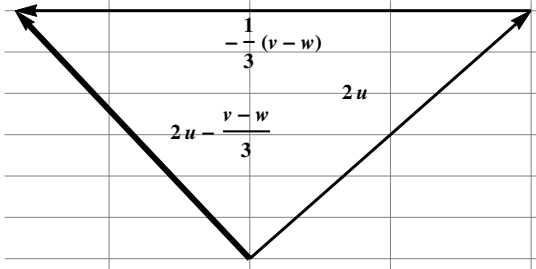




13.



14.



15. From the diagram, since the terminal point of $\mathbf{A}$ is at the initial point of $\mathbf{B}$, we see that $\mathbf{A} + \mathbf{B} = \mathbf{F}$, so that $\mathbf{x} = \boxed{\mathbf{A}}$.

16. From the diagram, since the terminal point of $\mathbf{C}$ is at the initial point of $\mathbf{H}$, we see that $\mathbf{C} + \mathbf{H} = \mathbf{K}$, so that $\mathbf{C} = -\mathbf{H} + \mathbf{K}$. Therefore $\mathbf{x} = \boxed{-\mathbf{H}}$.

17. Since the vectors $\mathbf{D}$ and $\mathbf{E}$ have their terminal points at the initial points of $\mathbf{E}$ and $\mathbf{F}$ respectively, we have $\boxed{\mathbf{C} = -\mathbf{D} - \mathbf{E} - \mathbf{F}}$.

18. Since the initial point of $\mathbf{K}$ is the initial point of $\mathbf{C}$, the terminal point of $-\mathbf{K}$ is at the initial point of $\mathbf{C}$. Since the terminal points of $\mathbf{C}$ and $\mathbf{D}$ are the initial points of $\mathbf{D}$ and $\mathbf{E}$ respectively, the initial point of $-\mathbf{K}$ is the initial point of $\mathbf{G}$, and the terminal point of $\mathbf{E}$ is the terminal point of $\mathbf{G}$, we have

$$\boxed{\mathbf{G} = -\mathbf{K} + \mathbf{C} + \mathbf{D} + \mathbf{E}}.$$

19. The initial point of $-\mathbf{D}$ is the initial point of $\mathbf{E}$, and its terminal point is the initial point of $\mathbf{H}$. Further, the terminal point of $\mathbf{H}$ is the initial point of $\mathbf{G}$, and the terminal point of $\mathbf{G}$ is the terminal point of $\mathbf{E}$. Therefore

$$\boxed{\mathbf{E} = -\mathbf{D} + \mathbf{H} + \mathbf{G}}.$$

20. Reversing all of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ exchanges all of their initial and terminal points; then the initial point of $-\mathbf{D}$ is the initial point of $\mathbf{E}$, and the terminal point of $-\mathbf{A}$ is the terminal point of $\mathbf{E}$, so that

$$\boxed{\mathbf{E} = -\mathbf{D} - \mathbf{C} - \mathbf{B} - \mathbf{A}}.$$

21. $\mathbf{A} + \mathbf{B} = \mathbf{F}$. Then $\mathbf{F} + \mathbf{K} = -\mathbf{G}$. So

$$\mathbf{A} + \mathbf{B} + \mathbf{K} + \mathbf{G} = (\mathbf{A} + \mathbf{B}) + \mathbf{K} + \mathbf{G} = (\mathbf{F} + \mathbf{K}) + \mathbf{G} = -\mathbf{G} + \mathbf{G} = \boxed{\mathbf{0}}.$$

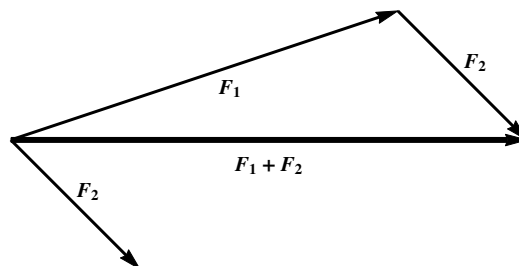
Alternatively, note that following the four vectors in order returns to the original starting point, so their sum must be the zero vector.

22. Following the five vectors in order returns to the original starting point, so their sum must be the zero vector. Alternatively,

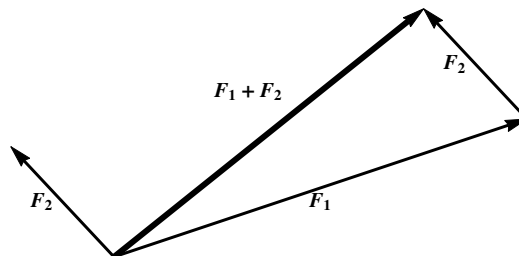
$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{H} + \mathbf{G} = \mathbf{F} + \mathbf{C} + \mathbf{H} + \mathbf{G} = (\mathbf{F} + \mathbf{C}) + \mathbf{H} + \mathbf{G} = (-\mathbf{G} - \mathbf{H}) + \mathbf{H} + \mathbf{G} = \boxed{\mathbf{0}}.$$

Applications and Extensions

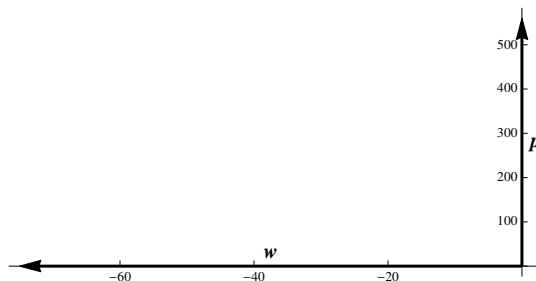
23. Translate $\mathbf{F}_2$ so that its initial point is at the terminal point of $\mathbf{F}_1$; then their sum is the terminal point of the translated vector.



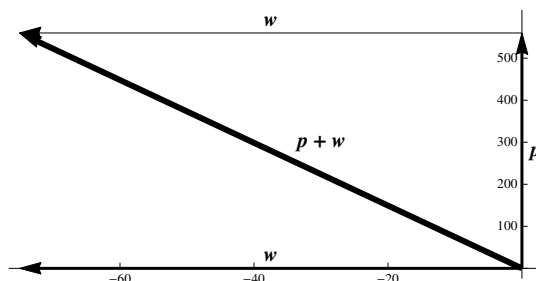
24. Translate $\mathbf{F}_2$ so that its initial point is at the terminal point of $\mathbf{F}_1$; then their sum is the terminal point of the translated vector.



25. (a) A diagram is below, where $\mathbf{p}$ represents the airspeed vector of the airplane, and $\mathbf{w}$ the wind (note that the diagram is not to scale, since it would be very tall):

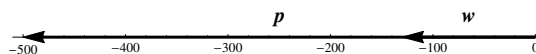


- (b) To compute the resultant vector, translate $\mathbf{w}$ so that its initial point is at the terminal point of $\mathbf{p}$; then the terminal point of $\mathbf{w}$ is the sum $\mathbf{p} + \mathbf{w}$:

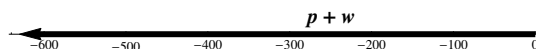


- (c) Answers will vary.

26. (a) A diagram is below, where $\mathbf{p}$ represents the airspeed vector of the airplane, and $\mathbf{w}$ the wind. Note that $\mathbf{p}$ overlaps with $\mathbf{w}$; they have the same initial point of 0.

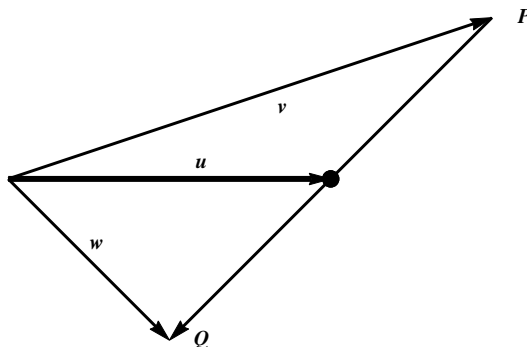


- (b) To compute the resultant vector, translate $\mathbf{w}$ so that its initial point is at the terminal point of $\mathbf{p}$; then the terminal point of $\mathbf{w}$ is the sum $\mathbf{p} + \mathbf{w}$:



- (c) As a result of the wind, the airplane's groundspeed vector is still due west, since the wind and the airspeed point in the same direction, but the groundspeed is higher than the airspeed.

27. The situation is shown below:



The midpoint of $\overrightarrow{PQ}$ is $\mathbf{v}$ plus one half of the vector from P to Q , so it is $\mathbf{v} + \frac{1}{2}\overrightarrow{PQ}$. But $\overrightarrow{PQ} = \mathbf{w} - \mathbf{v}$, so that the vector from the origin to the midpoint is

$$\mathbf{v} + \frac{1}{2}\overrightarrow{PQ} = \mathbf{v} + \frac{1}{2}(\mathbf{w} - \mathbf{v}) = \boxed{\frac{1}{2}(\mathbf{v} + \mathbf{w})}.$$

28. Expand and simplify:

$$\mathbf{0} = 4\mathbf{v} + a(\mathbf{v} - \mathbf{w}) + b(\mathbf{v} + \mathbf{w}) = 4\mathbf{v} + a\mathbf{v} - a\mathbf{w} + b\mathbf{v} + b\mathbf{w} = (4 + a + b)\mathbf{v} + (b - a)\mathbf{w}.$$

Since this equation must hold for every pair of vectors $\mathbf{v}$ and $\mathbf{w}$, start with $\mathbf{w} = \mathbf{0}$ and $\mathbf{v}$ some nonzero vector. Then $\mathbf{0} = (4 + a + b)\mathbf{v}$, so that $4 + a + b = 0$. Similarly, if we let $\mathbf{v} = \mathbf{0}$ and $\mathbf{w}$ be any nonzero vector, we get $\mathbf{0} = (b - a)\mathbf{w}$, so that $b - a = 0$. So we get the pair of equations $a + b = -4$ and $-a + b = 0$, which have the solution $\boxed{a = b = -2}$.

10.3: Vectors in the Plane and in Space

Concepts and Vocabulary

1. In space, the standard basis vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.
2. A vector whose magnitude is 1 is called a *unit* vector.
3. v_1 , v_2 , and v_3 are called the *components* of $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.
4. True. Since $4 > 0$, $\mathbf{w} = 4\mathbf{v}$ points in the same direction as $\mathbf{v}$.
5. False. The magnitude of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by $\sqrt{v_1^2 + v_2^2 + v_3^2}$, and similarly for vectors with other numbers of components.
6. True. See the theorem at the beginning of objective 5.
7. Since the magnitude of $\mathbf{v}$ is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\frac{1}{\sqrt{3}}\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}.$$

8. False. See the Theorem at the end of objective 3, Magnitude of the Scalar Multiple of a Vector.

Skill Building

9. (a) $\mathbf{v} = \langle 6 - 2, -2 - 3 \rangle = \langle 4, -5 \rangle$ (b) $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j}$.

10. (a) $\mathbf{v} = \langle -3 - 4, 2 - (-1) \rangle = \langle -7, 3 \rangle$ (b) $\mathbf{v} = -7\mathbf{i} + 3\mathbf{j}$.

11. (a) $\mathbf{v} = \langle -1 - 0, 6 - 5 \rangle = \langle -1, 1 \rangle$ (b) $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

12. (a) $\mathbf{v} = \langle 3 - (-2), 2 - 0 \rangle = \langle 5, 2 \rangle$ (b) $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$.

13. (a) $\mathbf{v} = \langle 3 - 6, 0 - 2, 2 - 1 \rangle = \langle -3, -2, 1 \rangle$ (b) $\mathbf{v} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

14. (a) $\mathbf{v} = \langle 0 - 4, 5 - 7, 6 - 0 \rangle = \langle -4, -2, 6 \rangle$ (b) $\mathbf{v} = -4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

15. (a) $\mathbf{v} = \langle 2 - (-1), 0 - 0, 0 - 1 \rangle = \langle 3, 0, -1 \rangle$ (b) $\mathbf{v} = 3\mathbf{i} - \mathbf{k}$.

16. (a) $\mathbf{v} = \langle 2 - 6, 6 - 2, 2 - 2 \rangle = \langle -4, 4, 0 \rangle$ (b) $\mathbf{v} = -4\mathbf{i} + 4\mathbf{j}$.

17. $2\mathbf{v} - \mathbf{w} = 2\langle 3, -1 \rangle - \langle -3, 2 \rangle = \langle 2 \cdot 3 - (-3), 2 \cdot (-1) - 2 \rangle = \langle 9, -4 \rangle$.

18. $\mathbf{v} + 5\mathbf{w} = \langle 3, -1 \rangle + 5\langle -3, 2 \rangle = \langle 3 + 5 \cdot (-3), -1 + 5 \cdot 2 \rangle = \langle -12, 9 \rangle$.

19. $\frac{1}{3}\mathbf{v} + \frac{1}{2}\mathbf{w} = \frac{1}{3}\langle 3, -1 \rangle + \frac{1}{2}\langle -3, 2 \rangle = \langle \frac{1}{3} \cdot 3 + \frac{1}{2} \cdot (-3), \frac{1}{3} \cdot (-1) + \frac{1}{2} \cdot 2 \rangle = \left\langle -\frac{1}{2}, \frac{2}{3} \right\rangle$.

20. $\frac{2}{3}\mathbf{v} - \frac{1}{2}\mathbf{w} = \frac{2}{3}\langle 3, -1 \rangle - \frac{1}{2}\langle -3, 2 \rangle = \langle \frac{2}{3} \cdot 3 - \frac{1}{2} \cdot (-3), \frac{2}{3} \cdot (-1) - \frac{1}{2} \cdot 2 \rangle = \left\langle \frac{7}{2}, -\frac{5}{3} \right\rangle$.

21. $\|\mathbf{v}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$.

22. $\|\mathbf{w}\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$.

23. Using Problem 17, $\|2\mathbf{v} - \mathbf{w}\| = \|\langle 9, -4 \rangle\| = \sqrt{9^2 + (-4)^2} = \sqrt{97}$.

24. $\|\mathbf{v} + \mathbf{w}\| = \|\langle 3, -1 \rangle + \langle -3, 2 \rangle\| = \|\langle 0, 1 \rangle\| = \sqrt{0^2 + 1^2} = 1$.

25. $3\mathbf{v} - 2\mathbf{w} = 3\langle 3, 0, 8 \rangle - 2\langle -3, -1, 2 \rangle = \langle 3 \cdot 3 - 2 \cdot (-3), 3 \cdot 0 - 2 \cdot (-1), 3 \cdot 8 - 2 \cdot 2 \rangle = \langle 15, 2, 20 \rangle$.

26. $-2\mathbf{v} + \mathbf{w} = -2\langle 3, 0, 8 \rangle + \langle -3, -1, 2 \rangle = \langle -2 \cdot 3 + (-3), -2 \cdot 0 + (-1), -2 \cdot 8 + 2 \rangle = \langle -9, -1, -14 \rangle$.

27.

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w} &= 2\langle 1, 1, -6 \rangle - 3\langle 3, 0, 8 \rangle + 4\langle -3, -1, 2 \rangle \\ &= \langle 2 \cdot 1 - 3 \cdot 3 + 4 \cdot (-3), 2 \cdot 1 - 3 \cdot 0 + 4 \cdot (-1), 2 \cdot (-6) - 3 \cdot 8 + 4 \cdot 2 \rangle \\ &= \langle -19, -2, -28 \rangle. \end{aligned}$$

28.

$$\begin{aligned} \mathbf{u} + 3(\mathbf{v} - 4\mathbf{w}) &= \mathbf{u} + 3\mathbf{v} - 12\mathbf{w} \\ &= \langle 1, 1, -6 \rangle + 3\langle 3, 0, 8 \rangle - 12\langle -3, -1, 2 \rangle \\ &= \langle 1 + 3 \cdot 3 - 12 \cdot (-3), 1 + 3 \cdot 0 - 12 \cdot (-1), -6 + 3 \cdot 8 - 12 \cdot 2 \rangle \\ &= \langle 46, 13, -6 \rangle. \end{aligned}$$

29. $\|\mathbf{u}\| = \sqrt{1^2 + 1^2 + (-6)^2} = \boxed{\sqrt{38}}.$

30. $\|\mathbf{v}\| = \sqrt{3^2 + 0^2 + 8^2} = \boxed{\sqrt{73}}.$

31. First compute $5\mathbf{u} - \mathbf{v} + \mathbf{w}$:

$$\begin{aligned} 5\mathbf{u} - \mathbf{v} + \mathbf{w} &= 5\langle 1, 1, -6 \rangle - \langle 3, 0, 8 \rangle + \langle -3, -1, 2 \rangle \\ &= \langle 5 \cdot 1 - 3 + (-3), 5 \cdot 1 - 0 + (-1), 5 \cdot (-6) - 8 + 2 \rangle = \langle -1, 4, -36 \rangle. \end{aligned}$$

Then $\|5\mathbf{u} - \mathbf{v} + \mathbf{w}\| = \|\langle -1, 4, -36 \rangle\| = \sqrt{(-1)^2 + 4^2 + (-36)^2} = \boxed{\sqrt{1313}}.$

32. Using Problems 29 and 30,

$$\|5\mathbf{u}\| - \|\mathbf{v}\| + \|\mathbf{w}\| = 5\|\mathbf{u}\| - \|\mathbf{v}\| + \sqrt{(-3)^2 + (-1)^2 + 2^2} = \boxed{5\sqrt{38} - \sqrt{73} + \sqrt{14}}.$$

33. $\|4\mathbf{i} - 3\mathbf{j}\| = \sqrt{4^2 + (-3)^2} = \boxed{5}.$

34. $\|\mathbf{i} - 12\mathbf{j}\| = \sqrt{1^2 + (-12)^2} = \boxed{\sqrt{145}}.$

35. $\|\mathbf{i} + \mathbf{j}\| = \sqrt{1^2 + 1^2} = \boxed{\sqrt{2}}.$

36. $\|\mathbf{i} - \mathbf{j}\| = \sqrt{1^2 + (-1)^2} = \boxed{\sqrt{2}}.$

37. $\|4\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| = \sqrt{4^2 + 2^2 + 1^2} = \boxed{\sqrt{21}}.$

38. $\|\mathbf{i} - \mathbf{j} + \mathbf{k}\| = \sqrt{1^2 + (-1)^2 + 1^2} = \boxed{\sqrt{3}}.$

39. $\|\mathbf{j} + \mathbf{k}\| = \sqrt{0^2 + 1^2 + 1^2} = \boxed{\sqrt{2}}.$

40. $\|2\mathbf{i} - \mathbf{k}\| = \sqrt{2^2 + 0^2 + (-1)^2} = \boxed{\sqrt{5}}.$

41. $\|(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \boxed{1}.$

42. $\|(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + \mathbf{k}\| = \sqrt{\cos^2 \theta + \sin^2 \theta + 1} = \boxed{\sqrt{2}}.$

43. Since $\|\mathbf{v}\| = \sqrt{5^2 + 12^2} = 13$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $-\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = \boxed{\left\langle -\frac{5}{13}, -\frac{12}{13} \right\rangle}.$

44. Since $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $-\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \boxed{\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle}.$

45. Since $\|\mathbf{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \boxed{\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}}.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $\boxed{-\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}}.$

46. Since $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \boxed{\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}}.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $\boxed{-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}}.$

47. Since $\|\mathbf{v}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$, we see that $\boxed{\mathbf{v} \text{ is already a unit vector}}$. The unit vector

in the direction opposite to $\mathbf{v}$ is $-\mathbf{v} = \boxed{-\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}}.$

48. Since $\|\mathbf{v}\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$, we see that $\boxed{\mathbf{v} \text{ is already a unit vector}}$. The unit

vector in the direction opposite to $\mathbf{v}$ is $-\mathbf{v} = \boxed{-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}.$

49. Since $\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \boxed{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle}.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $-\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \boxed{\left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle}.$

50. Since $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \boxed{\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle}.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $-\left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle = \boxed{\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle}.$

51. Since $\|\mathbf{v}\| = \sqrt{\cos^2 \theta + \sin^2 \theta + (2\sqrt{2})^2} = \sqrt{1 + 8} = 3$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \boxed{\frac{1}{3}\cos\theta\mathbf{i} + \frac{1}{3}\sin\theta\mathbf{j} + \frac{2}{3}\sqrt{2}\mathbf{k}}.$$

The unit vector in the direction opposite to $\mathbf{v}$ is $\boxed{-\frac{1}{3}\cos\theta\mathbf{i} - \frac{1}{3}\sin\theta\mathbf{j} - \frac{2}{3}\sqrt{2}\mathbf{k}}.$

52. Since $\|\mathbf{v}\| = \sqrt{\sin^2 \theta + \cos^2 \theta + 1^2} = \sqrt{1+1} = \sqrt{2}$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left[\frac{1}{\sqrt{2}} \sin \theta \mathbf{i} + \frac{1}{\sqrt{2}} \cos \theta \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k} \right].$$

The unit vector in the direction opposite to $\mathbf{v}$ is $\left[-\frac{1}{\sqrt{2}} \sin \theta \mathbf{i} - \frac{1}{\sqrt{2}} \cos \theta \mathbf{j} - \frac{1}{\sqrt{2}} \mathbf{k} \right].$

53. Since $\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2 + 12^2} = \sqrt{169} = 13$, the normalization of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left[\frac{3}{13} \mathbf{i} - \frac{4}{13} \mathbf{j} + \frac{12}{13} \mathbf{k} \right].$$

The unit vector in the direction opposite to $\mathbf{v}$ is $\left[-\frac{3}{13} \mathbf{i} + \frac{4}{13} \mathbf{j} - \frac{12}{13} \mathbf{k} \right].$

54. Since

$$\|\mathbf{v}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} = 1,$$

$\mathbf{v}$ is already a unit vector. The unit vector in the direction opposite to $\mathbf{v}$ is $-\mathbf{v} = \left[-\frac{1}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} - \frac{\sqrt{2}}{2} \mathbf{k} \right].$

55. $2\mathbf{v} - \mathbf{w} = 2(2\mathbf{i} - 3\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 6\mathbf{j} - \mathbf{i} - 2\mathbf{j} = \left[3\mathbf{i} - 8\mathbf{j} \right].$

56. $\mathbf{v} + 5\mathbf{w} = (2\mathbf{i} - 3\mathbf{j}) + 5(\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{i} + 10\mathbf{j} = \left[7\mathbf{i} + 7\mathbf{j} \right].$

57. $\frac{1}{3}\mathbf{v} + \frac{1}{2}\mathbf{w} = \frac{1}{3}(2\mathbf{i} - 3\mathbf{j}) + \frac{1}{2}(\mathbf{i} + 2\mathbf{j}) = \frac{2}{3}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{i} + \mathbf{j} = \left[\frac{7}{6}\mathbf{i} \right].$

58. $\frac{2}{3}\mathbf{v} - \frac{1}{2}\mathbf{w} = \frac{2}{3}(2\mathbf{i} - 3\mathbf{j}) - \frac{1}{2}(\mathbf{i} + 2\mathbf{j}) = \frac{4}{3}\mathbf{i} - 2\mathbf{j} - \frac{1}{2}\mathbf{i} - \mathbf{j} = \left[\frac{5}{6}\mathbf{i} - 3\mathbf{j} \right].$

59. $\|\mathbf{v} - \mathbf{w}\| = \|(2\mathbf{i} - 3\mathbf{j}) - (\mathbf{i} + 2\mathbf{j})\| = \|\mathbf{i} - 5\mathbf{j}\| = \sqrt{1^2 + 5^2} = \left[\sqrt{26} \right].$

60. $\|\mathbf{v} + \mathbf{w}\| = \|(2\mathbf{i} - 3\mathbf{j}) + (\mathbf{i} + 2\mathbf{j})\| = \|3\mathbf{i} - \mathbf{j}\| = \sqrt{3^2 + 1^2} = \left[\sqrt{10} \right].$

61. $3\mathbf{v} - 2\mathbf{w} = 3(2\mathbf{i} + \mathbf{j} - \mathbf{k}) - 2(6\mathbf{i} + \mathbf{j} + \mathbf{k}) = 9\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - 12\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = \left[-3\mathbf{i} + \mathbf{j} - 5\mathbf{k} \right].$

62. $-2\mathbf{v} + \mathbf{w} = -2(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + (6\mathbf{i} + \mathbf{j} + \mathbf{k}) = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + 6\mathbf{i} + \mathbf{j} + \mathbf{k} = \left[-\mathbf{j} + 3\mathbf{k} \right].$

63. $2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w} = 2(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - 3(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(6\mathbf{i} + \mathbf{j} + \mathbf{k}) = \left[17\mathbf{i} - 3\mathbf{j} + 13\mathbf{k} \right].$

64. $5\mathbf{u} - \mathbf{v} + \mathbf{w} = 5(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - (2\mathbf{i} + \mathbf{j} - \mathbf{k}) + (6\mathbf{i} + \mathbf{j} + \mathbf{k}) = \left[8\mathbf{i} - 10\mathbf{j} + 17\mathbf{k} \right].$

65. $\|\mathbf{v} - \mathbf{w}\| = \|(3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (6\mathbf{i} + \mathbf{j} + \mathbf{k})\| = \|-3\mathbf{i} - 2\mathbf{k}\| = \sqrt{(-3)^2 + (-2)^2} = \left[\sqrt{13} \right].$

66. $\|2\mathbf{v} + \mathbf{w}\| = \|2(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + (6\mathbf{i} + \mathbf{j} + \mathbf{k})\| = \|12\mathbf{i} + 3\mathbf{j} - \mathbf{k}\| = \sqrt{12^2 + 3^2 + (-1)^2} = \left[\sqrt{154} \right].$

67.

$$\begin{aligned} \|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\| &= \|\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\| + \|3\mathbf{i} + \mathbf{j} - \mathbf{k}\| + \|6\mathbf{i} + \mathbf{j} + \mathbf{k}\| \\ &= \sqrt{1^2 + (-2)^2 + 3^2} + \sqrt{3^2 + 1^2 + (-1)^2} + \sqrt{6^2 + 1^2 + 1^2} \\ &= \left[\sqrt{14} + \sqrt{11} + \sqrt{38} \right]. \end{aligned}$$

$$68. \|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} + \mathbf{j} - \mathbf{k}) + (6\mathbf{i} + \mathbf{j} + \mathbf{k})\| = \|10\mathbf{i} + 3\mathbf{k}\| = \sqrt{10^2 + 3^2} = \boxed{\sqrt{109}}.$$

$$69. \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{w}\|} = \frac{(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} + \mathbf{j} - \mathbf{k})}{\|6\mathbf{i} + \mathbf{j} + \mathbf{k}\|} = \frac{4\mathbf{i} - \mathbf{j} + 2\mathbf{k}}{\sqrt{6^2 + 1^2 + 1^2}} = \boxed{\frac{4}{\sqrt{38}}\mathbf{i} - \frac{1}{\sqrt{38}}\mathbf{j} + \frac{2}{\sqrt{38}}\mathbf{k}}.$$

$$70. \frac{\mathbf{u}}{\|\mathbf{v} + \mathbf{w}\|} = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\|(3\mathbf{i} + \mathbf{j} - \mathbf{k}) + (6\mathbf{i} + \mathbf{j} + \mathbf{k})\|} = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\|9\mathbf{i} + 2\mathbf{j}\|} = \frac{\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}}{\sqrt{9^2 + 2^2}} = \boxed{\frac{1}{\sqrt{85}}\mathbf{i} - \frac{2}{\sqrt{85}}\mathbf{j} + \frac{3}{\sqrt{85}}\mathbf{k}}.$$

71. Since $\frac{3}{2}\mathbf{v} = \frac{3}{2} \cdot 2\mathbf{i} + \frac{3}{2} \cdot 3\mathbf{j} = 3\mathbf{i} + \frac{9}{2}\mathbf{j} = \mathbf{w}$, the two vectors are parallel. Since $\frac{3}{2} > 0$, they point in the same direction.

72. Since $3 \cdot 2 = 6$, but $2 \cdot 2 = 4$ rather than -4 , these two vectors are not parallel: $\mathbf{w}$ is not a scalar multiple of $\mathbf{v}$.

73. Since the first two components of $\mathbf{v}$ are -2 times the corresponding components of $\mathbf{w}$, but the third component is not, these two vectors are not scalar multiples of one another, so they are not parallel.

74. These vectors are not parallel. Their second components are the same, so if one were a multiple of another, the multiple would have to be 1. But they are not equal.

75. These vectors are parallel, since $-3\mathbf{w} = \langle -3 \cdot (-4), -3 \cdot (-1), -3 \cdot 3 \rangle = \langle 12, 3, -9 \rangle = \mathbf{v}$. Since $-3 < 0$, the vectors point in opposite directions.

76. These vectors are not parallel. The first two components of $\mathbf{v}$ are 2 times the corresponding components of $\mathbf{w}$, but this is not the case for the third component.

Applications and Extensions

77. The magnitude of the given vector is $\|2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}\| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$, so that a unit vector in the direction of the given vector is

$$\frac{2}{\sqrt{29}}\mathbf{i} - \frac{3}{\sqrt{29}}\mathbf{j} + \frac{4}{\sqrt{29}}\mathbf{k}.$$

So a vector of magnitude 4 in that direction is

$$4 \left(\frac{2}{\sqrt{29}}\mathbf{i} - \frac{3}{\sqrt{29}}\mathbf{j} + \frac{4}{\sqrt{29}}\mathbf{k} \right) = \boxed{\frac{8}{\sqrt{29}}\mathbf{i} - \frac{12}{\sqrt{29}}\mathbf{j} + \frac{16}{\sqrt{29}}\mathbf{k}}.$$

78. The magnitude of the given vector is $\|-\mathbf{i} + \mathbf{j} + 2\mathbf{k}\| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$, so that a unit vector in the direction of the given vector is

$$-\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}.$$

So a vector of magnitude 2 in that direction is

$$2 \left(-\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k} \right) = \boxed{-\frac{2}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{4}{\sqrt{6}}\mathbf{k}}.$$

79. The magnitudes of the two vectors are

$$\begin{aligned} \|2\mathbf{i} + \mathbf{j} - \mathbf{k}\| &= \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6} \\ \|2a\mathbf{i} + \mathbf{j} + \mathbf{k}\| &= \sqrt{(2a)^2 + 1^2 + 1^2} = \sqrt{4a^2 + 2}. \end{aligned}$$

So we want to find a such that $4a^2 + 2 = 6$, so that $4a^2 = 4$ and therefore $a = 1$ or $a = -1$.

80. The given magnitude is

$$\|a\mathbf{i} + (a+1)\mathbf{j} + 2\mathbf{k}\| = \sqrt{a^2 + (a+1)^2 + 4} = \sqrt{2a^2 + 2a + 5}.$$

Since we want the magnitude to be 3, we want $2a^2 + 2a + 5 = 9$, so that $2a^2 + 2a - 4 = 2(a+2)(a-1) = 0$.

So $\boxed{a = 1 \text{ and } a = -2}$ satisfy the given condition.

81. $\mathbf{v} + \mathbf{w} = (2\mathbf{i} + \mathbf{j}) + (x\mathbf{i} + 3\mathbf{j}) = (2+x)\mathbf{i} + 4\mathbf{j}$, so that

$$\|\mathbf{v} + \mathbf{w}\| = \|(2+x)\mathbf{i} + 4\mathbf{j}\| = \sqrt{(2+x)^2 + 4^2} = \sqrt{x^2 + 4x + 20} = 5.$$

So we want $x^2 + 4x + 20 = 25$, or $x^2 + 4x - 5 = (x+5)(x-1) = 0$. Therefore $\boxed{x = 1 \text{ and } x = -5}$ satisfy the given condition.

82. The vector $\mathbf{v} = \langle x - (-3), 4 - 1 \rangle = \langle x + 3, 3 \rangle$, so that $\|\mathbf{v}\| = \sqrt{(x+3)^2 + 3^2} = \sqrt{x^2 + 6x + 18} = 5$. So we want $x^2 + 6x + 18 = 25$, or $x^2 + 6x - 7 = (x+7)(x-1) = 0$. Therefore $\boxed{x = 1 \text{ and } x = -7}$ satisfy the given condition.

83. From the theorem in objective 5, $\mathbf{v}$ satisfies

$$\mathbf{v} = \|\mathbf{v}\| (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = 4 (\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = \boxed{2\sqrt{3}\mathbf{i} + 2\mathbf{j}}.$$

84. From the theorem in objective 5, $\mathbf{v}$ satisfies

$$\mathbf{v} = \|\mathbf{v}\| (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = 2(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}) = \boxed{\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}}.$$

85. Since the $\mathbf{i}$ component is twice the $\mathbf{j}$ component, the vector is of the form $\mathbf{v} = 2k\mathbf{i} + k\mathbf{j}$. Then

$$5 = \|\mathbf{v}\| = \sqrt{(2k)^2 + k^2} = \sqrt{5k^2} = |k| \sqrt{5},$$

so that $k = \pm\sqrt{5}$. So the vectors are $\boxed{\pm(2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j})}$.

86. Since the $\mathbf{i}$ component equals the $\mathbf{j}$ component, the vector is of the form $\mathbf{v} = k\mathbf{i} + k\mathbf{j}$. Then

$$5 = \|\mathbf{v}\| = \sqrt{k^2 + k^2} = \sqrt{2k^2} = |k| \sqrt{2},$$

so that $|k| = \frac{5}{\sqrt{2}}$ and therefore $k = \pm\frac{5}{\sqrt{2}} = \pm\frac{5\sqrt{2}}{2}$. So the vectors are $\boxed{\pm\left(\frac{5\sqrt{2}}{2}\mathbf{i} + \frac{5\sqrt{2}}{2}\mathbf{j}\right)}$.

87. From the theorem in objective 5, $\mathbf{v}$ satisfies $\mathbf{v} = \|\mathbf{v}\| (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$. Since the $\mathbf{j}$ component is 1, we have $\|\mathbf{v}\| \sin 135^\circ = \|\mathbf{v}\| \cdot \frac{\sqrt{2}}{2} = 1$. Therefore $\|\mathbf{v}\| = \sqrt{2}$, so that

$$\mathbf{v} = \sqrt{2}(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}) = \sqrt{2}\left(-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) = \boxed{-\mathbf{i} + \mathbf{j}}.$$

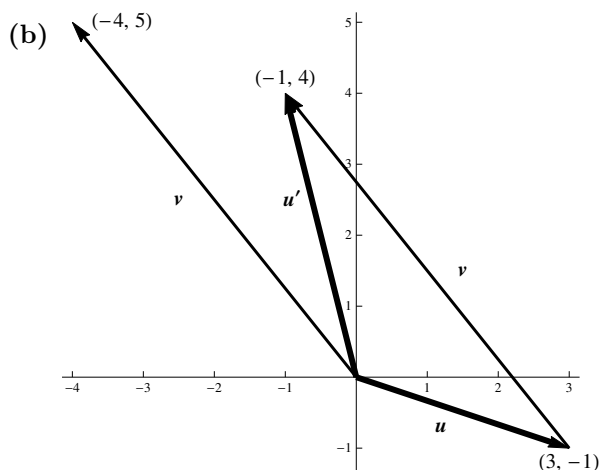
88. From the theorem in objective 5, $\mathbf{v}$ satisfies $\mathbf{v} = \|\mathbf{v}\| (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$. Since the $\mathbf{i}$ component is -3 , we have $\|\mathbf{v}\| \cos 210^\circ = \|\mathbf{v}\| \cdot \left(-\frac{\sqrt{3}}{2}\right) = -3$, so that $\|\mathbf{v}\| = -3 \cdot \left(-\frac{2}{\sqrt{3}}\right) = 2\sqrt{3}$. Therefore

$$\mathbf{v} = 2\sqrt{3}(\cos 210^\circ \mathbf{i} + \sin 210^\circ \mathbf{j}) = 2\sqrt{3}\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) = \boxed{-3\mathbf{i} - \sqrt{3}\mathbf{j}}.$$

89. The magnitude of the force is $F = \|\mathbf{F}\| = \sqrt{10^2 + (-15)^2 + 25^2} = \sqrt{950} = 5\sqrt{38}$ N. Using the formula

$$F = ma, \text{ we get } a = \frac{F}{m} = \frac{5\sqrt{38}}{10} = \boxed{\sqrt{\frac{19}{2}} \text{ N/kg}}.$$

90. (a) The new vector will be $\langle 3, -1 \rangle + \mathbf{v} = \langle 3, -1 \rangle + \langle -4, 5 \rangle = \boxed{\langle -1, 4 \rangle}$.



91. (a) The new vertices are

$$\begin{aligned}\langle -3, 0 \rangle + \langle 3, -2 \rangle &= \langle 0, -2 \rangle, & \langle -1, -2 \rangle + \langle 3, -2 \rangle &= \langle 2, -4 \rangle, \\ \langle 3, 1 \rangle + \langle 3, -2 \rangle &= \langle 6, -1 \rangle, & \langle 1, 3 \rangle + \langle 3, -2 \rangle &= \langle 4, 1 \rangle.\end{aligned}$$

Therefore $\boxed{A' = (0, -2), B' = (2, -4), C' = (6, -1), \text{ and } D' = (4, 1)}$.

- (b) Since $\frac{1}{2}\mathbf{v} = \langle \frac{3}{2}, -1 \rangle$, the new vertices are

$$\begin{aligned}\langle -3, 0 \rangle + \left\langle \frac{3}{2}, -1 \right\rangle &= \left\langle -\frac{3}{2}, -1 \right\rangle, & \langle -1, -2 \rangle + \left\langle \frac{3}{2}, -1 \right\rangle &= \left\langle \frac{1}{2}, -3 \right\rangle, \\ \langle 3, 1 \rangle + \left\langle \frac{3}{2}, -1 \right\rangle &= \left\langle \frac{9}{2}, 0 \right\rangle, & \langle 1, 3 \rangle + \left\langle \frac{3}{2}, -1 \right\rangle &= \left\langle \frac{5}{2}, 2 \right\rangle.\end{aligned}$$

Therefore $\boxed{A' = \left(-\frac{3}{2}, -1\right), B' = \left(\frac{1}{2}, -3\right), C' = \left(\frac{9}{2}, 0\right), \text{ and } D' = \left(\frac{5}{2}, 2\right)}$.

92. The plane's velocity relative to the air is $500\mathbf{i}$. The windspeed is 60 towards the northwest, so the wind velocity is

$$\mathbf{v} = \|\mathbf{v}\| (\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}) = -30\sqrt{2}\mathbf{i} + 30\sqrt{2}\mathbf{j}.$$

Therefore the ground speed of the airplane is

$$\begin{aligned}\left\| 500\mathbf{i} + (-30\sqrt{2}\mathbf{i} + 30\sqrt{2}\mathbf{j}) \right\| &= \left\| (500 - 30\sqrt{2})\mathbf{i} + 30\sqrt{2}\mathbf{j} \right\| \\ &= \sqrt{(500 - 30\sqrt{2})^2 + (30\sqrt{2})^2} = \boxed{\approx 459.5 \text{ km/h}}.\end{aligned}$$

93. The first leg of the trip has displacement $\mathbf{u} = -200\mathbf{i}$; the second trip is a vector of length 150 at an angle 60° north of west, which is an angle of 120° to the positive x axis, so that vector is

$$\mathbf{v} = \|\mathbf{v}\| (\cos 120^\circ \mathbf{i} + \sin 120^\circ \mathbf{j}) = 150 \left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \right) = -75\mathbf{i} + 75\sqrt{3}\mathbf{j}.$$

Therefore the position of the airplane is

$$\mathbf{u} + \mathbf{v} = -200\mathbf{i} + (-75\mathbf{i} + 75\sqrt{3}\mathbf{j}) = \boxed{-275\mathbf{i} + 75\sqrt{3}\mathbf{j}}.$$

94. Let $\mathbf{v}$ be the plane's air velocity vector. The vector of the wind is (recall that a northwest wind blows southeast)

$$\mathbf{w} = \|\mathbf{w}\| (\cos(-45^\circ) \mathbf{i} + \sin(-45^\circ) \mathbf{j}) = 20 \left(\frac{\sqrt{2}}{2} \mathbf{i} - \frac{\sqrt{2}}{2} \mathbf{j} \right) = 10\sqrt{2} \mathbf{i} - 10\sqrt{2} \mathbf{j}.$$

Then since the plane's final position after one hour is $-200\mathbf{j}$, its average ground velocity was $-200\mathbf{j}$ mi/h, so we have

$$-200\mathbf{j} = \mathbf{v} + \mathbf{w} = \mathbf{v} + 10\sqrt{2} \mathbf{i} - 10\sqrt{2} \mathbf{j}.$$

Therefore

$$\mathbf{v} = -200\mathbf{j} - (10\sqrt{2} \mathbf{i} - 10\sqrt{2} \mathbf{j}) = (-200 - 10\sqrt{2})\mathbf{i} + 10\sqrt{2} \mathbf{j},$$

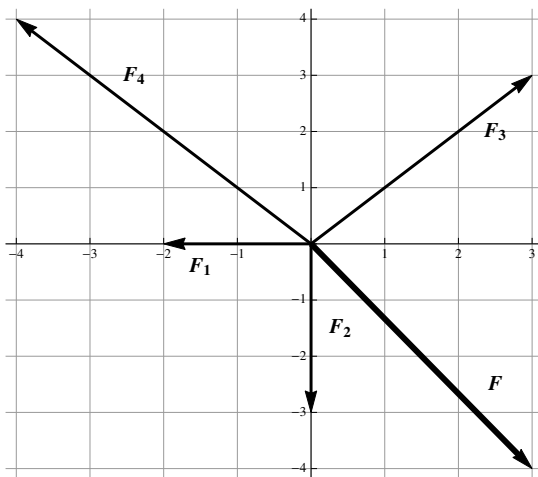
so that the average airspeed was

$$\|\mathbf{v}\| = \sqrt{(-200 - 10\sqrt{2})^2 + (10\sqrt{2})^2} \approx 214.6 \text{ mi/h}.$$

95. We have $\mathbf{F}_1 = \langle -2, 0 \rangle$, $\mathbf{F}_2 = \langle 0, -3 \rangle$, $\mathbf{F}_3 = \langle 3, 3 \rangle$, and $\mathbf{F}_4 = \langle -4, 4 \rangle$. Then the total force acting on the object is

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = \langle -2, 0 \rangle + \langle 0, -3 \rangle + \langle 3, 3 \rangle + \langle -4, 4 \rangle = \langle -3, 4 \rangle.$$

So the force $\mathbf{F}$ required to keep the object from moving is $-\langle -3, 4 \rangle = \langle 3, -4 \rangle$:



96. The total force being applied to the car is

$$\mathbf{F}_1 + \mathbf{F}_2 = 200\mathbf{i} + 350\mathbf{j} + 100\mathbf{k} + 250\mathbf{i} + 300\mathbf{j} + 150\mathbf{k} = 450\mathbf{i} + 650\mathbf{j} + 250\mathbf{k}.$$

Therefore the opposing force is

$$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2) = \boxed{-450\mathbf{i} - 650\mathbf{j} - 250\mathbf{k}}.$$

97. Place the origin at the point where the cables meet the object. Let $\mathbf{F}_1$ be the force vector of the right cable (pointing upwards and to the right), and $\mathbf{F}_2$ be the force vector of the left cable (pointing upwards and to the left). Then the two tensions are $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$. Note that the right-hand cable makes an angle of 25° with the ceiling, so it makes an angle of 25° with the positive x -axis. Likewise, the left-hand cable makes an angle of 40° with the ceiling, so an angle of 40° with the negative x -axis, so it makes an angle of $180^\circ - 40^\circ = 140^\circ$ with the positive x -axis. Then

$$\mathbf{F}_1 = \|\mathbf{F}_1\| (\cos 25^\circ \mathbf{i} + \sin 25^\circ \mathbf{j})$$

$$\mathbf{F}_2 = \|\mathbf{F}_2\| (\cos 140^\circ \mathbf{i} + \sin 140^\circ \mathbf{j})$$

$$\mathbf{F}_3 = -1000\mathbf{j}.$$

For the object to be in equilibrium, the sum of the forces must be zero, so that

$$\begin{aligned}\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &= \|\mathbf{F}_1\| (\cos 25^\circ \mathbf{i} + \sin 25^\circ \mathbf{j}) + \|\mathbf{F}_2\| (\cos 140^\circ \mathbf{i} + \sin 140^\circ \mathbf{j}) - 1000\mathbf{j} \\ &= (\|\mathbf{F}_1\| \cos 25^\circ + \|\mathbf{F}_2\| \cos 140^\circ) \mathbf{i} + (\|\mathbf{F}_1\| \sin 25^\circ + \|\mathbf{F}_2\| \sin 140^\circ - 1000) \mathbf{j} = \mathbf{0}.\end{aligned}$$

Considering the $\mathbf{i}$ and $\mathbf{j}$ components separately, we get the two equations in $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$:

$$\begin{aligned}\cos 25^\circ \|\mathbf{F}_1\| + \cos 140^\circ \|\mathbf{F}_2\| &= 0 \\ \sin 25^\circ \|\mathbf{F}_1\| + \sin 140^\circ \|\mathbf{F}_2\| &= 1000.\end{aligned}$$

Solving the first equation for $\|\mathbf{F}_1\|$ gives

$$\|\mathbf{F}_1\| = -\frac{\cos 140^\circ}{\cos 25^\circ} \|\mathbf{F}_2\|.$$

Substitute this for $\|\mathbf{F}_1\|$ in the second equation, giving

$$\begin{aligned}\sin 25^\circ \left(-\frac{\cos 140^\circ}{\cos 25^\circ} \|\mathbf{F}_2\| \right) + \sin 140^\circ \|\mathbf{F}_2\| &= 1000 \\ (-\tan 25^\circ \cos 140^\circ + \sin 140^\circ) \|\mathbf{F}_2\| &= 1000 \\ \|\mathbf{F}_2\| &= \frac{1000}{-\tan 25^\circ \cos 140^\circ + \sin 140^\circ} \boxed{\approx 1000 \text{ lb}}.\end{aligned}$$

(In fact, this is exactly 1000^1 .) Then

$$\|\mathbf{F}_1\| = -\frac{\cos 140^\circ}{\cos 25^\circ} \|\mathbf{F}_2\| = -\frac{1000 \cos 140^\circ}{\cos 25^\circ} \boxed{\approx 845.237 \text{ lb}}.$$

98. Place the origin at the point where the cables meet the object. Let $\mathbf{F}_1$ be the force vector of the right cable (pointing upwards and to the right), and $\mathbf{F}_2$ be the force vector of the left cable (pointing upwards and to the left). Then the two tensions are $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$. Note that the right-hand cable makes an angle of 50° with the ceiling, so it makes an angle of 50° with the positive x -axis. Likewise, the left-hand cable makes an angle of 35° with the ceiling, so an angle of 35° with the negative x -axis, so it makes an angle of $180^\circ - 35^\circ = 145^\circ$ with the positive x -axis. Then

$$\begin{aligned}\mathbf{F}_1 &= \|\mathbf{F}_1\| (\cos 50^\circ \mathbf{i} + \sin 50^\circ \mathbf{j}) \\ \mathbf{F}_2 &= \|\mathbf{F}_2\| (\cos 145^\circ \mathbf{i} + \sin 145^\circ \mathbf{j}) \\ \mathbf{F}_3 &= -800\mathbf{j}.\end{aligned}$$

For the object to be in equilibrium, the sum of the forces must be zero, so that

$$\begin{aligned}\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &= \|\mathbf{F}_1\| (\cos 50^\circ \mathbf{i} + \sin 50^\circ \mathbf{j}) + \|\mathbf{F}_2\| (\cos 145^\circ \mathbf{i} + \sin 145^\circ \mathbf{j}) - 800\mathbf{j} \\ &= (\|\mathbf{F}_1\| \cos 50^\circ + \|\mathbf{F}_2\| \cos 145^\circ) \mathbf{i} + (\|\mathbf{F}_1\| \sin 50^\circ + \|\mathbf{F}_2\| \sin 145^\circ - 800) \mathbf{j} = \mathbf{0}.\end{aligned}$$

Considering the $\mathbf{i}$ and $\mathbf{j}$ components separately, we get the two equations in $\|\mathbf{F}_1\|$ and $\|\mathbf{F}_2\|$:

$$\begin{aligned}\cos 50^\circ \|\mathbf{F}_1\| + \cos 145^\circ \|\mathbf{F}_2\| &= 0 \\ \sin 50^\circ \|\mathbf{F}_1\| + \sin 145^\circ \|\mathbf{F}_2\| &= 800.\end{aligned}$$

Solving the first equation for $\|\mathbf{F}_1\|$ gives

$$\|\mathbf{F}_1\| = -\frac{\cos 145^\circ}{\cos 50^\circ} \|\mathbf{F}_2\|.$$

¹ $\tan 25^\circ = \tan\left(\frac{50}{2}\right)^\circ = \frac{\sin 50^\circ}{1 + \cos 50^\circ}$. Further, $\cos 140^\circ = -\cos 40^\circ = -\sin 50^\circ$ and $\sin 140^\circ = \sin 40^\circ = \cos 50^\circ$. Putting this all together, we get

$$-\tan 25^\circ \cos 140^\circ + \sin 140^\circ = -\frac{\sin 50^\circ}{1 + \cos 50^\circ}(-\sin 50^\circ) + \cos 50^\circ = \frac{\sin^2 50^\circ + \cos^2 50^\circ + \cos 50^\circ}{1 + \cos 50^\circ} = \frac{1 + \cos 50^\circ}{1 + \cos 50^\circ} = 1.$$

Substitute this for $\|\mathbf{F}_1\|$ in the second equation, giving

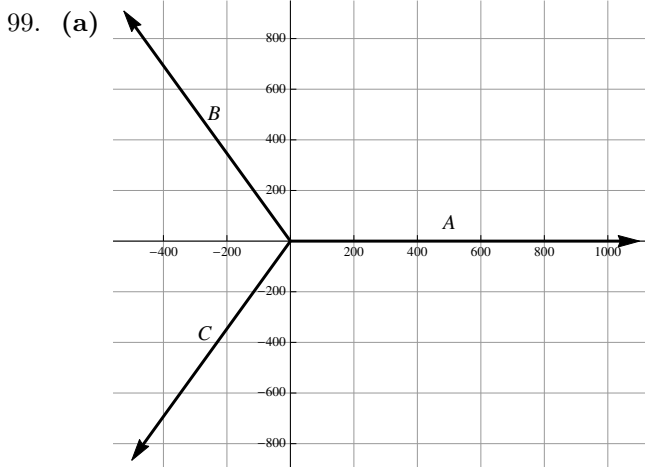
$$\sin 50^\circ \left(-\frac{\cos 145^\circ}{\cos 50^\circ} \|\mathbf{F}_2\| \right) + \sin 145^\circ \|\mathbf{F}_2\| = 800$$

$$(-\tan 50^\circ \cos 145^\circ + \sin 145^\circ) \|\mathbf{F}_2\| = 800$$

$$\|\mathbf{F}_2\| = \frac{800}{-\tan 50^\circ \cos 145^\circ + \sin 145^\circ} \approx 516.194 \text{ lb}.$$

Then

$$\|\mathbf{F}_1\| = -\frac{\cos 145^\circ}{\cos 50^\circ} \|\mathbf{F}_2\| \approx -\frac{516.194 \cos 145^\circ}{\cos 50^\circ} \approx 657.825 \text{ lb}.$$



(b) The three forces are

$$\mathbf{A} = 1100\mathbf{i}$$

$$\mathbf{B} = 1050(\cos 120^\circ \mathbf{i} + \sin 120^\circ \mathbf{j}) = 1050 \left(-\frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{j} \right) = -525\mathbf{i} + 525\sqrt{3}\mathbf{j}$$

$$\mathbf{C} = 1000(\cos 240^\circ \mathbf{i} + \sin 240^\circ \mathbf{j}) = 1000 \left(-\frac{1}{2} \mathbf{i} - \frac{\sqrt{3}}{2} \mathbf{j} \right) = -500\mathbf{i} - 500\sqrt{3}\mathbf{j}.$$

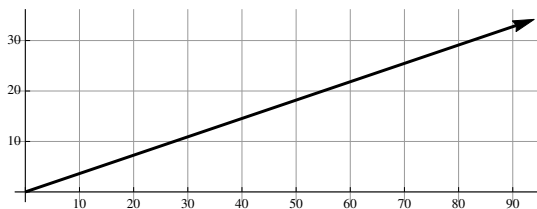
The net force is then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = 1100\mathbf{i} - 525\mathbf{i} + 525\sqrt{3}\mathbf{j} - 500\mathbf{i} - 500\sqrt{3}\mathbf{j} = \boxed{75\mathbf{i} + 25\sqrt{3}\mathbf{j}}.$$

100. Place the x -axis along the ground with the y -axis pointing up. The vector is

$$\mathbf{v} = 100(\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j}).$$

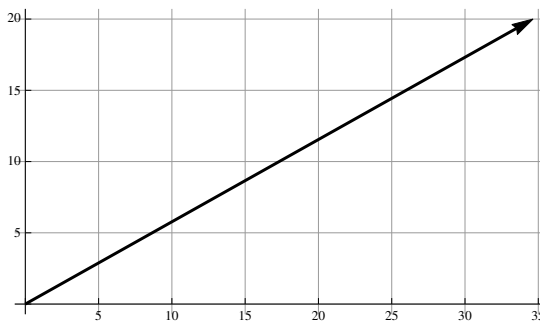
The force vector is below:



101. Place the x -axis along the ground with the y -axis pointing up. The vector is

$$\mathbf{v} = 40(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 20\sqrt{3}\mathbf{i} + 20\mathbf{j}.$$

Then the force vector is below:



102. With the axes as shown in the diagram, the force exerted by the pulley must be $25\mathbf{i} + 40\mathbf{j}$, so the magnitude of that force, which is the tension, is $\sqrt{25^2 + 40^2} = \sqrt{625 + 1600} = \sqrt{2225} = \boxed{5\sqrt{89} \approx 47.2 \text{ N}}$.
103. (a) The eight vertices of the cube are

$$\begin{aligned} &\boxed{(0, 0, 0), \quad (0, 0.408, 0), \quad (0, 0, 0.408), \quad (0, 0.408, 0.408),} \\ &\boxed{(0.408, 0, 0), \quad (0.408, 0.408, 0), \quad (0.408, 0, 0.408), \quad (0.408, 0.408, 0.408)} \end{aligned}$$

Each of the other atoms is in the center of one of the faces, so two of its coordinates are $\frac{0.408}{2} = 0.204$ and the third is either 0.408 (if it does not lie in one of the coordinate planes) or 0 (if it does). Therefore these six atoms have coordinates

$$\begin{aligned} &\boxed{(0, 0.204, 0.204), \quad (0.204, 0, 0.204), \quad (0.204, 0.204, 0)} \\ &\boxed{(0.408, 0.204, 0.204), \quad (0.204, 0.408, 0.204), \quad (0.204, 0.204, 0.408)} \end{aligned}$$

- (b) (i) The atom furthest from the origin has coordinates $(0.408, 0.408, 0.408)$, so the vector corresponding to that atom is $\boxed{0.408\mathbf{i} + 0.408\mathbf{j} + 0.408\mathbf{k}}$.
- (ii) The xz -plane is the plane $y = 0$, so we are looking for the atom in the center of the face on the plane $y = 0.408$. From part (a), this point has coordinates $(0.204, 0.408, 0.204)$, so its vector is $\boxed{0.204\mathbf{i} + 0.408\mathbf{j} + 0.204\mathbf{k}}$.
- (iii) The xy -plane is the plane $z = 0$, so we are looking for the atom in the center of the face on the plane $z = 0.408$. From part (a), this point has coordinates $(0.204, 0.204, 0.408)$, so its vector is $\boxed{0.204\mathbf{i} + 0.204\mathbf{j} + 0.408\mathbf{k}}$.

(c)

$$\begin{aligned} \|\langle 0.408, 0.408, 0.408 \rangle\| &= \sqrt{0.408^2 + 0.408^2 + 0.408^2} \approx 0.707 \text{ nm}, \\ \|\langle 0.204, 0.408, 0.204 \rangle\| &= \sqrt{0.204^2 + 0.408^2 + 0.204^2} \approx 0.500 \text{ nm}, \\ \|\langle 0.204, 0.204, 0.408 \rangle\| &= \sqrt{0.204^2 + 0.204^2 + 0.408^2} \approx 0.500 \text{ nm}. \end{aligned}$$

104. Let $\mathbf{v}$ be the plane's air velocity vector. The vector of the wind is (recall that an easterly wind blows west) $-50\mathbf{i}$. Then since the plane's velocity with respect to the ground is 250 km/h to the northeast, its velocity vector with respect to the ground is

$$250(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}) = 125\sqrt{2}\mathbf{i} + 125\sqrt{2}\mathbf{j},$$

so that

$$125\sqrt{2}\mathbf{i} + 125\sqrt{2}\mathbf{j} = \mathbf{v} - 50\mathbf{i}.$$

Therefore

$$\mathbf{v} = 125\sqrt{2}\mathbf{i} + 125\sqrt{2}\mathbf{j} + 50\mathbf{i} = (125\sqrt{2} + 50)\mathbf{i} + 125\sqrt{2}\mathbf{j}.$$

Therefore the airspeed of the plane, which is the number we are looking for, is

$$\|\mathbf{v}\| = \sqrt{(125\sqrt{2} + 50)^2 + (125\sqrt{2})^2} \approx 287.537 \text{ km/h}.$$

105. Let $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$ represent the velocity vector of the wind. The woman's velocity is $8\mathbf{i}$. We know that $8\mathbf{i} + \mathbf{w}$ is a vector that comes from the north (i.e., it has no $\mathbf{i}$ component). But $8\mathbf{i} + \mathbf{w} = (8 + w_1)\mathbf{i} + w_2\mathbf{j}$, so that $w_1 = -8$ and $\mathbf{w} = -8\mathbf{i} + w_2\mathbf{j}$. Next, if the woman doubles her velocity to $16\mathbf{i}$, then the apparent velocity of the wind is $16\mathbf{i} + \mathbf{w} = 8\mathbf{i} + w_2\mathbf{j}$. Since this is apparently from the northeast, its $\mathbf{i}$ and $\mathbf{j}$ components must be equal, so that $w_2 = 8$. Therefore the velocity of the wind is $-8\mathbf{i} + 8\mathbf{j}$. So the wind is blowing at a speed of $\|-8\mathbf{i} + 8\mathbf{j}\| = 8\sqrt{2}$ from the northwest.

106. Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $\mathbf{w} = \langle w_1, w_2 \rangle$. Then

Commutative property of vector addition:

$$\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle = \mathbf{v} + \mathbf{u}.$$

Associative property of vector addition:

$$\begin{aligned} \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2 \rangle + (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle) \\ &= \langle u_1, u_2 \rangle + \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2) \rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2 \rangle \\ &= \langle u_1 + v_1, u_2 + v_2 \rangle + \langle w_1, w_2 \rangle \\ &= (\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) + \langle w_1, w_2 \rangle \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}. \end{aligned}$$

Additive identity $\mathbf{0}$:

$$\begin{aligned} \mathbf{v} + \mathbf{0} &= \langle v_1, v_2 \rangle + \langle 0, 0 \rangle = \langle v_1 + 0, v_2 + 0 \rangle = \langle v_1, v_2 \rangle = \mathbf{v} \\ \mathbf{0} + \mathbf{v} &= \langle 0, 0 \rangle + \langle v_1, v_2 \rangle = \langle 0 + v_1, 0 + v_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}. \end{aligned}$$

Additive inverse property:

$$\mathbf{v} + (-\mathbf{v}) = \langle v_1, v_2 \rangle + (-\langle v_1, v_2 \rangle) = \langle v_1, v_2 \rangle + \langle -v_1, -v_2 \rangle = \langle v_1 - v_1, v_2 - v_2 \rangle = \langle 0, 0 \rangle = \mathbf{0}.$$

Distributive property of scalar multiplication over vector addition:

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a(\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) \\ &= a\langle u_1 + v_1, u_2 + v_2 \rangle \\ &= \langle a(u_1 + v_1), a(u_2 + v_2) \rangle \\ &= \langle au_1 + av_1, au_2 + av_2 \rangle \\ &= \langle au_1, au_2 \rangle + \langle av_1, av_2 \rangle \\ &= a\langle u_1, u_2 \rangle + a\langle v_1, v_2 \rangle \\ &= a\mathbf{u} + a\mathbf{v}. \end{aligned}$$

Distributive property of scalar multiplication over scalar addition:

$$\begin{aligned} (a + b)\mathbf{v} &= (a + b)\langle v_1, v_2 \rangle \\ &= \langle (a + b)v_1, (a + b)v_2 \rangle \\ &= \langle av_1 + bv_1, av_2 + bv_2 \rangle \\ &= \langle av_1, av_2 \rangle + \langle bv_1, bv_2 \rangle \\ &= a\langle v_1, v_2 \rangle + b\langle v_1, v_2 \rangle \\ &= a\mathbf{v} + b\mathbf{v}. \end{aligned}$$

Linearity property of scalar multiplication:

$$\begin{aligned}
 a(b\mathbf{v}) &= a(b\langle v_1, v_2 \rangle) \\
 &= a\langle bv_1, bv_2 \rangle \\
 &= \langle a(bv_1), a(bv_2) \rangle \\
 &= \langle (ab)v_1, (ab)v_2 \rangle \\
 &= (ab)\langle v_1, v_2 \rangle \\
 &= (ab)\mathbf{v}.
 \end{aligned}$$

Scalar multiplication identity:

$$1\mathbf{v} = 1\langle v_1, v_2 \rangle = \langle 1 \cdot v_1, 1 \cdot v_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

107. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. Then

Commutative property of vector addition:

$$\begin{aligned}
 \mathbf{u} + \mathbf{v} &= \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle \\
 &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\
 &= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \\
 &= \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \\
 &= \mathbf{v} + \mathbf{u}.
 \end{aligned}$$

Associative property of vector addition:

$$\begin{aligned}
 \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle + (\langle v_1, v_2, v_3 \rangle + \langle w_1, w_2, w_3 \rangle) \\
 &= \langle u_1, u_2, u_3 \rangle + \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\
 &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3) \rangle \\
 &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3 \rangle \\
 &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle + \langle w_1, w_2, w_3 \rangle \\
 &= (\langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle) + \langle w_1, w_2, w_3 \rangle \\
 &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}.
 \end{aligned}$$

Additive identity 0:

$$\begin{aligned}
 \mathbf{v} + \mathbf{0} &= \langle v_1, v_2, v_3 \rangle + \langle 0, 0, 0 \rangle = \langle v_1 + 0, v_2 + 0, v_3 + 0 \rangle = \langle v_1, v_2, v_3 \rangle = \mathbf{v} \\
 \mathbf{0} + \mathbf{v} &= \langle 0, 0, 0 \rangle + \langle v_1, v_2, v_3 \rangle = \langle 0 + v_1, 0 + v_2, 0 + v_3 \rangle = \langle v_1, v_2, v_3 \rangle = \mathbf{v}.
 \end{aligned}$$

Additive inverse property:

$$\begin{aligned}
 \mathbf{v} + (-\mathbf{v}) &= \langle v_1, v_2, v_3 \rangle + (-\langle v_1, v_2, v_3 \rangle) \\
 &= \langle v_1, v_2, v_3 \rangle + \langle -v_1, -v_2, -v_3 \rangle \\
 &= \langle v_1 - v_1, v_2 - v_2, v_3 - v_3 \rangle \\
 &= \langle 0, 0, 0 \rangle \\
 &= \mathbf{0}.
 \end{aligned}$$

Distributive property of scalar multiplication over vector addition:

$$\begin{aligned}
 a(\mathbf{u} + \mathbf{v}) &= a(\langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle) \\
 &= a\langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\
 &= \langle a(u_1 + v_1), a(u_2 + v_2), a(u_3 + v_3) \rangle \\
 &= \langle au_1 + av_1, au_2 + av_2, au_3 + av_3 \rangle \\
 &= \langle au_1, au_2, au_3 \rangle + \langle av_1, av_2, av_3 \rangle \\
 &= a\langle u_1, u_2, u_3 \rangle + a\langle v_1, v_2, v_3 \rangle \\
 &= a\mathbf{u} + a\mathbf{v}.
 \end{aligned}$$

Distributive property of scalar multiplication over scalar addition:

$$\begin{aligned}
(a+b)\mathbf{v} &= (a+b)\langle v_1, v_2, v_3 \rangle \\
&= \langle (a+b)v_1, (a+b)v_2, (a+b)v_3 \rangle \\
&= \langle av_1 + bv_1, av_2 + bv_2, av_3 + bv_3 \rangle \\
&= \langle av_1, av_2, av_3 \rangle + \langle bv_1, bv_2, bv_3 \rangle \\
&= a\langle v_1, v_2, v_3 \rangle + b\langle v_1, v_2, v_3 \rangle \\
&= a\mathbf{v} + b\mathbf{v}.
\end{aligned}$$

Linearity property of scalar multiplication:

$$\begin{aligned}
a(b\mathbf{v}) &= a(b\langle v_1, v_2, v_3 \rangle) \\
&= a\langle bv_1, bv_2, bv_3 \rangle \\
&= \langle a(bv_1), a(bv_2), a(bv_3) \rangle \\
&= \langle (ab)v_1, (ab)v_2, (ab)v_3 \rangle \\
&= (ab)\langle v_1, v_2, v_3 \rangle \\
&= (ab)\mathbf{v}.
\end{aligned}$$

Scalar multiplication identity:

$$1\mathbf{v} = 1\langle v_1, v_2, v_3 \rangle = \langle 1 \cdot v_1, 1 \cdot v_2, 1 \cdot v_3 \rangle = \langle v_1, v_2, v_3 \rangle = \mathbf{v}.$$

108. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Computing $\|a\mathbf{v}\|$ gives

$$\begin{aligned}
\|a\mathbf{v}\| &= \|a\langle v_1, v_2, v_3 \rangle\| \\
&= \|\langle av_1, av_2, av_3 \rangle\| \\
&= \sqrt{(av_1)^2 + (av_2)^2 + (av_3)^2} \\
&= \sqrt{a^2(v_1^2 + v_2^2 + v_3^2)} \\
&= |a| \sqrt{v_1^2 + v_2^2 + v_3^2} \\
&= |a| \|\mathbf{v}\|.
\end{aligned}$$

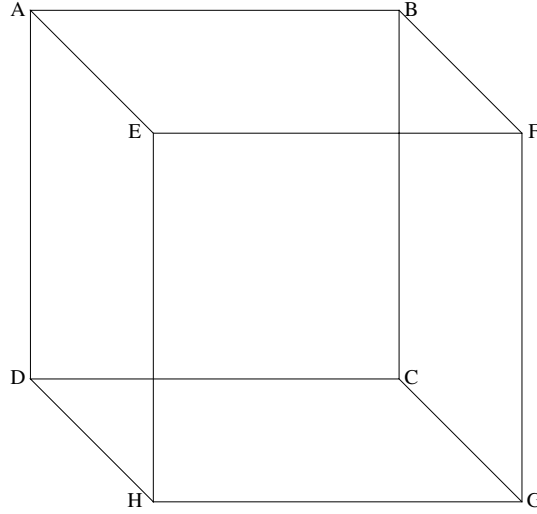
109. Since $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$, we have

$$\begin{aligned}
v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} &= v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle \\
&= \langle v_1 \cdot 1, v_1 \cdot 0, v_1 \cdot 0 \rangle + \langle v_2 \cdot 0, v_2 \cdot 1, v_2 \cdot 0 \rangle + \langle v_3 \cdot 0, v_3 \cdot 0, v_3 \cdot 1 \rangle \\
&= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\
&= \langle v_1 + 0 + 0, 0 + v_2 + 0, 0 + 0 + v_3 \rangle \\
&= \langle v_1, v_2, v_3 \rangle.
\end{aligned}$$

Challenge Problems

110. Since $\mathbf{b} + \overrightarrow{BD} = \mathbf{d}$, we have $\boxed{\overrightarrow{BD} = \mathbf{d} - \mathbf{b}}$. Since $\mathbf{c} + \overrightarrow{CD} = \mathbf{d}$, we have $\boxed{\overrightarrow{CD} = \mathbf{d} - \mathbf{c}}$. Finally, since $\mathbf{b} + \overrightarrow{BC} = \mathbf{c}$, we have $\boxed{\overrightarrow{BC} = \mathbf{c} - \mathbf{b}}$.

111. Each face is a parallelogram; they are $ABCD$, $ABFE$, $AEHD$, $BFGC$, $CDHG$, and $EFGH$:



$$\begin{aligned}
 \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{b} + \mathbf{d} \\
 \overrightarrow{AF} &= \overrightarrow{AB} + \overrightarrow{BF} = \overrightarrow{AB} + \overrightarrow{AE} = \mathbf{b} + \mathbf{e} \\
 \overrightarrow{AG} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{BF} = \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE} = \mathbf{b} + \mathbf{d} + \mathbf{e} \\
 \overrightarrow{FG} &= \overrightarrow{BC} = \overrightarrow{AD} = \mathbf{d} \\
 \overrightarrow{EG} &= \overrightarrow{EF} + \overrightarrow{FG} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{b} + \mathbf{d}.
 \end{aligned}$$

112. Let the position vector of P be $\mathbf{r}$. Then $\overrightarrow{P_1P} = \mathbf{r} - \mathbf{r}_1$ and $\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1$. If $t = \frac{\overrightarrow{P_1P}}{\overrightarrow{P_1P_2}}$, then $t\overrightarrow{P_1P_2} = \overrightarrow{P_1P}$ and therefore

$$t(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{r} - \mathbf{r}_1; \text{ rearranging gives } \mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)$$

as desired. Then

$$\begin{aligned}
 \langle x, y, z \rangle &= P = \langle x_1, y_1, z_1 \rangle + t(\langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle) \\
 &= \langle x_1, y_1, z_1 \rangle + t \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\
 &= \langle x_1, y_1, z_1 \rangle + \langle t(x_2 - x_1), t(y_2 - y_1), t(z_2 - z_1) \rangle \\
 &= \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1) \rangle.
 \end{aligned}$$

113. Suppose $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. Then we want to show that for any vector $\mathbf{w} = \langle w_1, w_2 \rangle$ there are unique scalars s and t such that

$$\mathbf{w} = \langle w_1, w_2 \rangle = s\mathbf{u} + t\mathbf{v} = s \langle u_1, u_2 \rangle + t \langle v_1, v_2 \rangle = \langle su_1 + tv_1, su_2 + tv_2 \rangle.$$

This amounts to showing that the pair of linear equations in s and t

$$\begin{aligned}
 su_1 + tv_1 &= w_1 \\
 su_2 + tv_2 &= w_2
 \end{aligned}$$

has a unique solution. Multiply the first equation by u_2 and the second by u_1 and subtract, giving

$$t(u_1v_2 - u_2v_1) = u_1w_2 - u_2w_1, \quad \text{so that} \quad t = \frac{u_1w_2 - u_2w_1}{u_1v_2 - u_2v_1}.$$

Note that $u_1v_2 - u_2v_1 \neq 0$ since the two vectors are nonparallel, for if $u_1v_2 = u_2v_1$ then $\frac{u_1}{v_1} = \frac{u_2}{v_2}$ so that $\mathbf{u}$ and $\mathbf{v}$ would be multiples of each other. Therefore t is well-defined, and we can substitute back into the first equation to get

$$\begin{aligned} su_1 + \frac{u_1w_2 - u_2w_1}{u_1v_2 - u_2v_1}v_1 &= w_1 \\ su_1 &= w_1 - \frac{u_1v_1w_2 - u_2v_1w_1}{u_1v_2 - u_2v_1} \\ su_1 &= \frac{u_1v_2w_1 - u_2v_1w_1 - u_1v_1w_2 + u_2v_1w_1}{u_1v_2 - u_2v_1} \\ su_1 &= \frac{u_1(v_2w_1 - v_1w_2)}{u_1v_2 - u_2v_1} \\ s &= \frac{v_2w_1 - v_1w_2}{u_1v_2 - u_2v_1}. \end{aligned}$$

Therefore s and t exist and are unique. Geometrically, this means that any vector in the plane can be written uniquely as a sum of multiples of two fixed nonparallel vectors. The two vectors we normally use, $\mathbf{e}_1$ and $\mathbf{e}_2$, are one example of this.

10.4: The Dot Product

Concepts and Vocabulary

1. False. If $\mathbf{v}$ is any vector, then $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$. This is the Magnitude Property.
2. False. $\mathbf{v} \cdot \mathbf{w} = 0$ if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, not if they are parallel. This is the Orthogonal Vector theorem.
3. If θ is the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, then $\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$. This is the Angle Between Vectors theorem.
4. True. See the theorem in objective 4.
5. False. The dot product is sometimes called the *scalar* product.
6. True. The vector parallel to $\mathbf{w}$ is $\text{proj}_{\mathbf{w}} \mathbf{v}$; see the Vector Projection theorem.

Skill Building

7. (a) $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 - 3 \cdot (-1) + 1 \cdot 1 = 2 + 3 + 1 = \boxed{6}$.
 (b) Since $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$ and $\|\mathbf{w}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \frac{6}{\sqrt{14}\sqrt{3}} \boxed{\approx 0.388}.$$
8. (a) $\mathbf{v} \cdot \mathbf{w} = -3 \cdot 2 + 2 \cdot 1 - 1 \cdot (-1) = -6 + 2 + 1 = \boxed{-3}$.
 (b) Since $\|\mathbf{v}\| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$ and $\|\mathbf{w}\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \left(-\frac{3}{\sqrt{14}\sqrt{6}} \right) \boxed{\approx 1.904}.$$
9. (a) $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 0 - 1 \cdot 1 + 0 \cdot 1 = \boxed{-1}$.
 (b) Since $\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$ and $\|\mathbf{w}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \left(-\frac{1}{\sqrt{2}\sqrt{2}} \right) = \cos^{-1} \left(-\frac{1}{2} \right) = \boxed{\frac{2\pi}{3}}.$$

10. (a) $\mathbf{v} \cdot \mathbf{w} = 0 \cdot 1 + 1 \cdot 0 - 1 \cdot 1 = \boxed{-1}$.

(b) Since $\|\mathbf{v}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$ and $\|\mathbf{w}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \left(-\frac{1}{\sqrt{2}\sqrt{2}} \right) = \cos^{-1} \left(-\frac{1}{2} \right) = \boxed{\frac{2\pi}{3}}.$$

11. (a) $\mathbf{v} \cdot \mathbf{w} = 3 \cdot (-2) + 1 \cdot (-1) - 1 \cdot 1 = \boxed{-8}$.

(b) Since $\|\mathbf{v}\| = \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}$ and $\|\mathbf{w}\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \left(-\frac{8}{\sqrt{11}\sqrt{6}} \right) \boxed{\approx 2.967}.$$

12. (a) $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 4 + 3 + 12 = \boxed{19}$.

(b) Since $\|\mathbf{v}\| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$ and $\|\mathbf{w}\| = \sqrt{4^2 + 1^2 + 3^2} = \sqrt{26}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \frac{19}{\sqrt{26}\sqrt{26}} \boxed{\approx 0.751}.$$

13. (a) $\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 - 1 \cdot 1 = \boxed{0}$.

(b) Since the dot product is zero, the vectors are orthogonal, so the angle between them is $\boxed{\frac{\pi}{2}}$.

14. (a) $\mathbf{v} \cdot \mathbf{w} = 3 \cdot (-6) + 4 \cdot (-8) = -18 - 32 = \boxed{-50}$.

(b) Since $\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$ and $\|\mathbf{w}\| = \sqrt{6^2 + 8^2} = 10$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1} \left(-\frac{50}{5 \cdot 10} \right) = \cos^{-1}(-1) = \boxed{\pi}.$$

15. For $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal, we must have

$$\mathbf{v} \cdot \mathbf{w} = 2a \cdot 1 + 1 \cdot (-1) - 1 \cdot 1 = 2a - 2 = 0,$$

so that $\boxed{a = 1}$.

16. For $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal, we must have

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 1 + 2a \cdot (-1) - 1 \cdot 1 = -2a = 0,$$

so that $\boxed{a = 0}$.

17. For $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal, we must have

$$\mathbf{v} \cdot \mathbf{w} = a \cdot 1 + 1 \cdot a + 1 \cdot 4 = 2a + 4 = 0,$$

so that $\boxed{a = -2}$.

18. For $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal, we must have

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 2a - a \cdot 1 + 2 \cdot 1 = a + 2 = 0,$$

so that $\boxed{a = -2}$.

19. (a) We have

$$\|\mathbf{v}\| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{49} = 7$$

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{3}{7}$$

$$\cos \beta = \frac{v_2}{\|\mathbf{v}\|} = -\frac{6}{7}$$

$$\cos \gamma = \frac{v_3}{\|\mathbf{v}\|} = -\frac{2}{7}.$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{7 \left(\frac{3}{7} \mathbf{i} - \frac{6}{7} \mathbf{j} - \frac{2}{7} \mathbf{k} \right)}.$$

20. (a) We have

$$\|\mathbf{v}\| = \sqrt{6^2 + 12^2 + 4^2} = \sqrt{196} = 14$$

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|} = -\frac{6}{14} = -\frac{3}{7}$$

$$\cos \beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{12}{14} = \frac{6}{7}$$

$$\cos \gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{14} = \frac{2}{7}.$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{14 \left(-\frac{3}{7} \mathbf{i} + \frac{6}{7} \mathbf{j} + \frac{2}{7} \mathbf{k} \right)}.$$

21. (a) We have

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}$$

$$\cos \beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}$$

$$\cos \gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}.$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{3} \left(\frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k} \right)}.$$

22. (a) We have

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}$$

$$\cos \beta = \frac{v_2}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{3}}$$

$$\cos \gamma = \frac{v_3}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{3}}.$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{3} \left(\frac{1}{\sqrt{3}} \mathbf{i} - \frac{1}{\sqrt{3}} \mathbf{j} - \frac{1}{\sqrt{3}} \mathbf{k} \right)}.$$

23. (a) We have

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2} \\ \cos \alpha &= \frac{v_1}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}} \\ \cos \beta &= \frac{v_2}{\|\mathbf{v}\|} = 0 \\ \cos \gamma &= \frac{v_3}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}.\end{aligned}$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{2} \left(\frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{k} \right)}.$$

24. (a) We have

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \\ \cos \alpha &= \frac{v_1}{\|\mathbf{v}\|} = 0 \\ \cos \beta &= \frac{v_2}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}} \\ \cos \gamma &= \frac{v_3}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}.\end{aligned}$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{2} \left(\frac{1}{\sqrt{2}} \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k} \right)}.$$

25. (a) We have

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38} \\ \cos \alpha &= \frac{v_1}{\|\mathbf{v}\|} = \frac{3}{\sqrt{38}} \\ \cos \beta &= \frac{v_2}{\|\mathbf{v}\|} = -\frac{5}{\sqrt{38}} \\ \cos \gamma &= \frac{v_3}{\|\mathbf{v}\|} = \frac{2}{\sqrt{38}}.\end{aligned}$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{38} \left(\frac{3}{\sqrt{38}} \mathbf{i} - \frac{5}{\sqrt{38}} \mathbf{j} + \frac{2}{\sqrt{38}} \mathbf{k} \right)}.$$

26. (a) We have

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \\ \cos \alpha &= \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}} \\ \cos \beta &= \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{29}} \\ \cos \gamma &= \frac{v_3}{\|\mathbf{v}\|} = -\frac{4}{\sqrt{29}}.\end{aligned}$$

$$(b) \quad \mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{29} \left(\frac{2}{\sqrt{29}} \mathbf{i} + \frac{3}{\sqrt{29}} \mathbf{j} - \frac{4}{\sqrt{29}} \mathbf{k} \right)}.$$

27. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{2 \cdot 1 - 3 \cdot (-1) + 1 \cdot 1}{1 \cdot 1 - 1 \cdot (-1) + 1 \cdot 1} (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \frac{6}{3} (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \boxed{2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = -\mathbf{j} - \mathbf{k}. \end{aligned}$$

28. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{-3 \cdot 2 + 2 \cdot 1 - 1 \cdot (-1)}{2 \cdot 2 + 1 \cdot 1 - 1 \cdot (-1)} (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = -\frac{3}{6} (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \boxed{-\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = -\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - \left(-\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}\right) = -2\mathbf{i} + \frac{5}{2}\mathbf{j} - \frac{3}{2}\mathbf{k}. \end{aligned}$$

29. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{1 \cdot 0 - 1 \cdot 1 + 0 \cdot 1}{0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1} (\mathbf{j} + \mathbf{k}) = -\frac{1}{2} (\mathbf{j} + \mathbf{k}) = \boxed{-\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = -\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (\mathbf{i} - \mathbf{j}) - \left(-\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}\right) = \mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}. \end{aligned}$$

30. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{0 \cdot 1 + 1 \cdot 0 - 1 \cdot 1}{1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1} (\mathbf{i} + \mathbf{k}) = -\frac{1}{2} (\mathbf{i} + \mathbf{k}) = \boxed{-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (\mathbf{j} - \mathbf{k}) - \left(-\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{k}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}. \end{aligned}$$

31. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{3 \cdot (-2) + 1 \cdot (-1) - 1 \cdot 1}{-2 \cdot (-2) - 1 \cdot (-1) + 1 \cdot 1} (-2\mathbf{i} - \mathbf{j} + \mathbf{k}) = -\frac{4}{3} (-2\mathbf{i} - \mathbf{j} + \mathbf{k}) = \boxed{\frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{4}{3}\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = \frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{4}{3}\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - \left(\frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{4}{3}\mathbf{k}\right) = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}. \end{aligned}$$

32. (a) $\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{1 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3}{4 \cdot 4 - 1 \cdot (-1) + 3 \cdot 3} (4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \frac{19}{26} (4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \boxed{\frac{38}{13}\mathbf{i} - \frac{19}{26}\mathbf{j} + \frac{57}{26}\mathbf{k}}.$

(b) Since $\text{proj}_{\mathbf{w}} \mathbf{v}$ is parallel to $\mathbf{w}$, set

$$\begin{aligned} \mathbf{v}_1 &= \text{proj}_{\mathbf{w}} \mathbf{v} = \frac{38}{13}\mathbf{i} - \frac{19}{26}\mathbf{j} + \frac{57}{26}\mathbf{k} \\ \mathbf{v}_2 &= \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} = (\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) - \left(\frac{38}{13}\mathbf{i} - \frac{19}{26}\mathbf{j} + \frac{57}{26}\mathbf{k}\right) = -\frac{25}{13}\mathbf{i} - \frac{59}{26}\mathbf{j} + \frac{47}{26}\mathbf{k}. \end{aligned}$$

33. The angle θ between the vectors satisfies

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{a \cdot 1 + 1 \cdot a + 1 \cdot 1}{\sqrt{a^2 + 2} \sqrt{a^2 + 2}} = \frac{2a + 1}{a^2 + 2}.$$

For θ to equal $\frac{\pi}{3}$, we must have $\frac{2a+1}{a^2+2} = \frac{1}{2}$, so that (clearing fractions) $4a + 2 = a^2 + 2$, or $a^2 - 4a = 0$. Therefore $\boxed{a = 0 \text{ and } a = 4}$ satisfy the given condition.

34. The angle θ between the vectors satisfies

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{a \cdot 1 - 1 \cdot (-1) + 1 \cdot a}{\sqrt{a^2 + 2} \sqrt{a^2 + 2}} = \frac{2a + 1}{a^2 + 2}.$$

For θ to equal $\frac{2\pi}{3}$, we must have $\frac{2a+1}{a^2+2} = -\frac{1}{2}$, so that (clearing fractions) $4a + 2 = -a^2 - 2$, or $a^2 + 4a + 4 = (a + 2)^2 = 0$. The only solution is $\boxed{a = -2}$.

Applications and Extensions

35. The velocity of the plane with respect to the air is $500\mathbf{j}$. The wind, which is blowing toward the southeast at 60 km/h, has a velocity vector of $60(\cos(-45^\circ)\mathbf{i} + \sin(-45^\circ)\mathbf{j}) = 30\sqrt{2}\mathbf{i} - 30\sqrt{2}\mathbf{j}$. Therefore the velocity vector of the plane with respect to the ground is

$$\mathbf{v}_g = 500\mathbf{j} + 30\sqrt{2}\mathbf{i} - 30\sqrt{2}\mathbf{j} = 30\sqrt{2}\mathbf{i} + (500 - 30\sqrt{2})\mathbf{j} \approx 42.426\mathbf{i} + 457.574\mathbf{j}.$$

The speed of the plane with respect to the ground is therefore

$$\|\mathbf{v}_g\| \approx \sqrt{42.426^2 + 457.574^2} \approx 459.536 \text{ km/h}.$$

Its direction is given by the angle θ that it makes with the x axis, which is

$$\theta = \cos^{-1} \frac{\mathbf{v}_g \cdot \mathbf{i}}{\|\mathbf{v}_g\| \|\mathbf{i}\|} \approx \cos^{-1} \frac{30\sqrt{2}}{459.537} \approx 1.478 \approx 84.703^\circ,$$

or about 5.297° east of north.

36. The velocity of the bird with respect to the air is $-12\mathbf{i}$. The wind, which is blowing toward the northwest at 5 km/h, has a velocity vector of $5(\cos 135^\circ\mathbf{i} + \sin 135^\circ\mathbf{j}) = -\frac{5}{2}\sqrt{2}\mathbf{i} + \frac{5}{2}\sqrt{2}\mathbf{j}$. Therefore the velocity vector of the bird with respect to the ground is

$$\mathbf{v}_g = -12\mathbf{i} - \frac{5}{2}\sqrt{2}\mathbf{i} + \frac{5}{2}\sqrt{2}\mathbf{j} = -\left(12 + \frac{5}{2}\sqrt{2}\right)\mathbf{i} + \frac{5}{2}\sqrt{2}\mathbf{j} \approx -15.536\mathbf{i} + 3.536\mathbf{j}.$$

The speed of the bird with respect to the ground is therefore

$$\|\mathbf{v}_g\| \approx \sqrt{15.536^2 + 3.536^2} \approx 15.933 \text{ km/h}.$$

Its direction is given by the angle θ that it makes with the x axis, which is

$$\theta = \cos^{-1} \frac{\mathbf{v}_g \cdot \mathbf{i}}{\|\mathbf{v}_g\| \|\mathbf{i}\|} \approx \cos^{-1} \frac{-12 - \frac{5}{2}\sqrt{2}}{15.933} \approx 2.918 \approx 167.175^\circ.$$

37. Set up a coordinate system with the origin at the point where the boat starts, x -axis pointing directly across the river, and the y -axis pointing up the river, so that the current's velocity is $-5\mathbf{j}$. Let θ be the angle that the boat's velocity with respect to the water makes with the x -axis. Then the boat's velocity with respect to the water is

$$\mathbf{v} = 15(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}),$$

so that the velocity of the boat with respect to the ground is

$$\mathbf{v} - 5\mathbf{j} = 15 \cos \theta \mathbf{i} + (-5 + 15 \sin \theta)\mathbf{j}.$$

Now, we want the velocity of the boat with respect to the ground to be orthogonal to $\mathbf{j}$ (i.e., straight across the river), so we want

$$(15 \cos \theta \mathbf{i} + (-5 + 15 \sin \theta)\mathbf{j}) \cdot \mathbf{j} = -5 + 15 \sin \theta = 0.$$

This gives $\sin \theta = \frac{1}{3}$, so that

$$\theta = \sin^{-1} \frac{1}{3} \approx 0.340 \approx 19.471^\circ.$$

The boat should be pointed about 19.471° upriver from the landing point; this is an angle of about $\boxed{70.529^\circ}$ to the shore.

38. Set up a coordinate system with the origin at the point where the swimmer starts, x -axis pointing directly across the river, and the y -axis pointing up the river, so that the current's velocity is $-1\mathbf{j}$ km/h. Let θ be the angle that the swimmer's velocity with respect to the water makes with the x -axis. Then the swimmer's velocity with respect to the water is

$$\mathbf{v} = 2(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}),$$

so that the velocity of the swimmer with respect to the ground is

$$\mathbf{v} - \mathbf{j} = 2 \cos \theta \mathbf{i} + (-1 + 2 \sin \theta)\mathbf{j}.$$

Now, we want the velocity of the swimmer with respect to the ground to be orthogonal to $\mathbf{j}$ (i.e., straight across the river), so we want

$$(2 \cos \theta \mathbf{i} + (-1 + 2 \sin \theta)\mathbf{j}) \cdot \mathbf{j} = -1 + 2 \sin \theta = 0.$$

This gives $\sin \theta = \frac{1}{2}$, so that

$$\theta = \sin^{-1} \frac{1}{2} = \frac{\pi}{6} = 30^\circ.$$

The swimmer should aim 30° upriver from the landing point, or at an angle of $\boxed{60^\circ}$ to the shore.

39. Let $\mathbf{v}$ be the vertical vector of gravitational force, so that $\mathbf{v} = -5300\mathbf{j}$, and let $\mathbf{w}$ be the vector describing the grade of the street, so that $\mathbf{w} = \cos 8^\circ \mathbf{i} + \sin 8^\circ \mathbf{j}$. Then the force required to keep the car from rolling down the hill is $\text{proj}_{\mathbf{w}} \mathbf{v}$ (the component parallel to $\mathbf{w}$), while the force perpendicular to the street is $\mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v}$.

$$\begin{aligned} \text{proj}_{\mathbf{w}} \mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{-5300 \sin 8^\circ}{\cos^2 8^\circ + \sin^2 8^\circ} (\cos 8^\circ \mathbf{i} + \sin 8^\circ \mathbf{j}) \\ &= -5300 \sin 8^\circ (\cos 8^\circ \mathbf{i} + \sin 8^\circ \mathbf{j}) \approx -730.439\mathbf{i} - 102.657\mathbf{j} \\ \mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v} &= -5300\mathbf{j} - (-730.439\mathbf{i} - 102.657\mathbf{j}) = 730.439\mathbf{i} - 5197.343\mathbf{j}. \end{aligned}$$

So the magnitude of the force required to keep the car from rolling downhill is

$$\|\text{proj}_{\mathbf{w}} \mathbf{v}\| \approx \sqrt{730.439^2 + 102.657^2} \approx 737.618 \text{ lb},$$

and the magnitude of the force perpendicular to the street is

$$\|\mathbf{v} - \text{proj}_{\mathbf{w}} \mathbf{v}\| \approx \sqrt{730.439^2 + 5197.343^2} \approx 5248.420 \text{ lb}.$$

40. The amount of force that Billy must exert is the magnitude of the component of the piano's 250 pound gravitational force vector, which is $\mathbf{v} = -250\mathbf{j}$, in the direction of the ramp. The ramp is described by $\mathbf{w} = \cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j}$, and the component in the direction of the ramp is the projection. So the force required is

$$\begin{aligned}\text{proj}_{\mathbf{w}} \mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{-250 \sin 20^\circ}{\cos^2 20^\circ + \sin^2 20^\circ} (\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j}) \\ &= -250 \sin 20^\circ (\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j}) \approx -80.348\mathbf{i} - 29.244\mathbf{j}.\end{aligned}$$

The magnitude of this force is

$$\|\text{proj}_{\mathbf{w}} \mathbf{v}\| = \sqrt{80.348^2 + 29.244^2} \approx 85.505 \text{ lbs}.$$

41. If $\mathbf{F} = 20(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 10\sqrt{3}\mathbf{i} + 10\mathbf{j}$, then the amount of work done in moving the wagon 100 feet horizontally, which has the vector $\overrightarrow{AB} = 100\mathbf{i}$, is

$$\mathbf{F} \cdot \overrightarrow{AB} = (10\sqrt{3}\mathbf{i} + 10\mathbf{j}) \cdot 100\mathbf{i} = 1000\sqrt{3} \approx 1732.051 \text{ ft-lbs}.$$

42. The magnitude of the given direction vector for the force is $\sqrt{2^2 + 2^2 + 1^2} = 3$, so that a unit vector in the direction of the force is $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$. Therefore the force is

$$\mathbf{F} = \|\mathbf{F}\| \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) = 1 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

We want to move the object through the vector $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, so that the work is

$$\mathbf{F} \cdot \overrightarrow{AB} = \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{2}{3} \cdot 1 + \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 2 = \frac{8}{3} \text{ J}.$$

43. The magnitude of the given direction vector for the force is $\sqrt{2^2 + 1^2 + 2^2} = 3$, so that a unit vector in the direction of the force is $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$. Therefore the force is

$$\mathbf{F} = \|\mathbf{F}\| \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \right) = 3 \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \right) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

We want to move the object through the vector $\overrightarrow{AB} = 2\mathbf{j}$, so that the work is

$$\mathbf{F} \cdot \overrightarrow{AB} = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{j}) = 1 \cdot 2 = 2 \text{ J}.$$

44. Gravity is $\mathbf{v}_g = mg\mathbf{j}$. To move from $(0, b)$ to $(b, 0)$, we are moving along the vector $\overrightarrow{AB}_1 = b\mathbf{i} - b\mathbf{j}$; to move from $(b, 0)$ to $(0, 0)$, we are moving along the vector $\overrightarrow{AB}_2 = -b\mathbf{i}$; to move from $(0, 0)$ to $(0, b)$, we are moving along the vector $\overrightarrow{AB}_3 = b\mathbf{j}$. Therefore the work done by gravity along each side of the triangle is

$$\begin{aligned}W_1 &= \mathbf{v}_g \cdot \overrightarrow{AB}_1 = mg\mathbf{j} \cdot (b\mathbf{i} - b\mathbf{j}) = -mgb \\ W_2 &= \mathbf{v}_g \cdot \overrightarrow{AB}_2 = mg\mathbf{j} \cdot (-b\mathbf{i}) = 0 \\ W_3 &= \mathbf{v}_g \cdot \overrightarrow{AB}_3 = mg\mathbf{j} \cdot (b\mathbf{j}) = mgb.\end{aligned}$$

So the total work done by gravity is $-mgb + 0 + mgb = 0$.

45. (a) The two workers exert force vectors of

$$\begin{aligned}\mathbf{F}_A &= 850(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 425\sqrt{3}\mathbf{i} + 425\mathbf{j} \\ \mathbf{F}_B &= 850(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}) = 425\sqrt{2}\mathbf{i} + 425\sqrt{2}\mathbf{j}.\end{aligned}$$

The work done by each worker in moving the crate from $(0, 0, 0)$ to $(10, 0, 0)$ (i.e., along the vector $10\mathbf{i}$) is therefore

$$\begin{aligned}\mathbf{F}_A \cdot \overrightarrow{AB} &= (425\sqrt{3}\mathbf{i} + 425\mathbf{j}) \cdot 10\mathbf{i} = 4250\sqrt{3} \approx 7361.216 \text{ J} \\ \mathbf{F}_B \cdot \overrightarrow{AB} &= (425\sqrt{2}\mathbf{i} + 425\sqrt{2}\mathbf{j}) \cdot 10\mathbf{i} = 4250\sqrt{2} \approx 6010.408 \text{ J}.\end{aligned}$$

(b) Answers will vary.

46. (a) The force vector applied by the horse is $\mathbf{F} = 725(\cos 15^\circ \mathbf{i} + \sin 15^\circ \mathbf{j})$, and we wish to move the plow through the vector $\overrightarrow{AB} = 15\mathbf{j}$, so the work is

$$\mathbf{F} \cdot \overrightarrow{AB} = 725(\cos 15^\circ \mathbf{i} + \sin 15^\circ \mathbf{j}) \cdot 15\mathbf{j} = \boxed{10875 \sin 15^\circ \approx 2814.657 \text{ J}}.$$

(b) Friction is a force vector equal to $-625\mathbf{j}$, so the work done by friction is

$$-625\mathbf{j} \cdot 15\mathbf{j} = \boxed{-9375 \text{ J}}.$$

(The sign is negative to indicate that this work subtracts from the work done by the horse; it reflects a force in the opposite direction.)

47. Let $A = (2, 2, 2)$, $B = (-1, 3, 3)$, $C = (0, 1, 2)$, and $D = (3, 0, 1)$. Then claim $ABCD$ is a parallelogram. First,

$$\begin{aligned}\overrightarrow{AB} &= \langle -1 - 2, 3 - 2, 3 - 2 \rangle = \langle -3, 1, 1 \rangle & \overrightarrow{BA} &= \langle 3, -1, -1 \rangle \\ \overrightarrow{BC} &= \langle 0 - (-1), 1 - 3, 2 - 3 \rangle = \langle 1, -2, -1 \rangle & \overrightarrow{CB} &= \langle -1, 2, 1 \rangle \\ \overrightarrow{CD} &= \langle 3 - 0, 0 - 1, 1 - 2 \rangle = \langle 3, -1, -1 \rangle & \overrightarrow{DC} &= \langle -3, 1, 1 \rangle \\ \overrightarrow{DA} &= \langle 2 - 3, 2 - 0, 2 - 1 \rangle = \langle -1, 2, 1 \rangle & \overrightarrow{AD} &= \langle 1, -2, -1 \rangle.\end{aligned}$$

Then

$$\begin{aligned}\|\overrightarrow{AB}\| &= \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11} \\ \|\overrightarrow{BC}\| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \\ \|\overrightarrow{CD}\| &= \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11} \\ \|\overrightarrow{DA}\| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.\end{aligned}$$

Therefore the pairs of opposite sides AB and CD , and BC and DA , are equal in length. The angles between pairs of adjacent sides satisfy

$$\begin{aligned}\cos \angle(\overrightarrow{BA}, \overrightarrow{BC}) &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\| \|\overrightarrow{BC}\|} = \frac{3 \cdot 1 - 1 \cdot (-2) - 1 \cdot (-1)}{\sqrt{11}\sqrt{6}} = \frac{6}{\sqrt{66}} \\ \cos \angle(\overrightarrow{CB}, \overrightarrow{CD}) &= \frac{\overrightarrow{CB} \cdot \overrightarrow{CD}}{\|\overrightarrow{CB}\| \|\overrightarrow{CD}\|} = \frac{-1 \cdot 3 + 2 \cdot (-1) + 1 \cdot (-1)}{\sqrt{6}\sqrt{11}} = -\frac{6}{\sqrt{66}} \\ \cos \angle(\overrightarrow{DC}, \overrightarrow{DA}) &= \frac{\overrightarrow{DC} \cdot \overrightarrow{DA}}{\|\overrightarrow{DC}\| \|\overrightarrow{DA}\|} = \frac{-3 \cdot (-1) + 1 \cdot 2 + 1 \cdot 1}{\sqrt{6}\sqrt{11}} = \frac{6}{\sqrt{66}} \\ \cos \angle(\overrightarrow{AD}, \overrightarrow{AB}) &= \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{\|\overrightarrow{AD}\| \|\overrightarrow{AB}\|} = \frac{1 \cdot (-3) - 2 \cdot 1 - 1 \cdot 1}{\sqrt{6}\sqrt{11}} = -\frac{6}{\sqrt{66}}.\end{aligned}$$

So the angles at C and A are equal, and the angles at B and D are equal. Therefore $ABCD$ is indeed a parallelogram.

48. Let $A = (2, 2, 2)$, $B = (2, 0, 1)$, $C = (4, 1, -1)$, and $D = (4, 3, 0)$. Claim $ABCD$ is a rectangle. First,

$$\begin{aligned}\overrightarrow{AB} &= \langle 2 - 2, 0 - 2, 1 - 2 \rangle = \langle 0, -2, -1 \rangle \\ \overrightarrow{BC} &= \langle 4 - 2, 1 - 0, -1 - 1 \rangle = \langle 2, 1, -2 \rangle \\ \overrightarrow{CD} &= \langle 4 - 4, 3 - 1, 0 - (-1) \rangle = \langle 0, 2, 1 \rangle \\ \overrightarrow{DA} &= \langle 2 - 4, 2 - 3, 2 - 0 \rangle = \langle -2, -1, 2 \rangle.\end{aligned}$$

we have

$$\begin{aligned}\|\overrightarrow{AB}\| &= \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5} \\ \|\overrightarrow{BC}\| &= \sqrt{2^2 + 1^2 + 2^2} = 3 \\ \|\overrightarrow{CD}\| &= \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5} \\ \|\overrightarrow{DA}\| &= \sqrt{2^2 + 1^2 + 2^2} = 3.\end{aligned}$$

Therefore opposite sides are equal. Also,

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{BC} &= 0 \cdot 2 - 2 \cdot 1 - 1 \cdot (-2) = 0 \\ \overrightarrow{BC} \cdot \overrightarrow{CD} &= 2 \cdot 0 + 1 \cdot 2 - 2 \cdot 1 = 0 \\ \overrightarrow{CD} \cdot \overrightarrow{DA} &= 0 \cdot (-2) + 2 \cdot (-1) + 1 \cdot 2 = 0 \\ \overrightarrow{DA} \cdot \overrightarrow{AB} &= -2 \cdot 0 - 1 \cdot (-2) + 2 \cdot (-1) = 0,\end{aligned}$$

so that all four vertices are right angles. Therefore $ABCD$ is a rectangle.

49. Let $A = (-2, 6, 0)$, $B = (4, 9, 1)$, and $C = (-3, 2, 18)$. Then

$$\begin{aligned}\overrightarrow{AB} &= \langle 4 - (-2), 9 - 6, 1 - 0 \rangle = \langle 6, 3, 1 \rangle, \\ \overrightarrow{BC} &= \langle -3 - 4, 2 - 9, 18 - 1 \rangle = \langle -7, -7, 17 \rangle, \\ \overrightarrow{AC} &= \langle -2 - (-3), 6 - 2, 0 - 18 \rangle = \langle 1, 4, -18 \rangle.\end{aligned}$$

Then

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 6 \cdot 1 + 3 \cdot 4 + 1 \cdot (-18) = 0,$$

so that the angle at A is a right angle, so that ABC is a right triangle.

50. We want the triangle to have its right angle at C , so we must have $\overrightarrow{CB}$ orthogonal to $\overrightarrow{CA}$. Now,

$$\overrightarrow{CB} \cdot \overrightarrow{CA} = \langle -2 - 1, 2 - 2, 1 - c \rangle \cdot \langle 1 - 1, -1 - 2, 0 - c \rangle = \langle -3, 0, 1 - c \rangle \cdot \langle 0, -3, -c \rangle = -c(1 - c).$$

The dot product is zero, meaning that $\overrightarrow{CB}$ and $\overrightarrow{CA}$ are orthogonal, for $\boxed{c = 0 \text{ and } c = 1}$. So these are the desired values of c .

51. Let θ be the angle that the force vector $\mathbf{F}$ makes with the x -axis. Since $\mathbf{F}$ has magnitude one, we have $\mathbf{F} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$. Then the work done by $\mathbf{F}$ is

$$\mathbf{F} \cdot 4\mathbf{i} = 4 \cos \theta = 2,$$

so that $\cos \theta = \frac{1}{2}$ and $\theta = \frac{\pi}{3} = 60^\circ$. The force vector makes an angle of $\boxed{\frac{\pi}{3}}$ with the positive x -axis.

52. If $\mathbf{F}$ is orthogonal to $\mathbf{AB}$, then $\mathbf{F} \cdot \overrightarrow{AB} = 0$, so $\boxed{\text{no work is done by } \mathbf{F}}$.

53. Using the Magnitude Property, we have

$$\left\| \frac{1}{2} \mathbf{u} - \mathbf{w} \right\|^2 = \left(\frac{1}{2} \mathbf{u} - \mathbf{w} \right) \cdot \left(\frac{1}{2} \mathbf{u} - \mathbf{w} \right) = \frac{1}{4} \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} = \frac{1}{4} \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{w} + \|\mathbf{w}\|^2.$$

Since both $\mathbf{u}$ and $\mathbf{w}$ are unit vectors, they each have norm 1, so this simplifies to

$$\frac{1}{4} - \mathbf{u} \cdot \mathbf{w} + 1 = \frac{5}{4} - \mathbf{u} \cdot \mathbf{w} = \frac{5}{4} - \|\mathbf{u}\| \|\mathbf{w}\| \cos \theta = \frac{5}{4} - \cos \theta$$

using the formula for the cosine of the angle between two vectors. Taking square roots gives

$$\left\| \frac{1}{2} \mathbf{u} - \mathbf{w} \right\| = \sqrt{\frac{5}{4} - \cos \theta}.$$

54. The vector projection of $\mathbf{v}$ on $\mathbf{i}$ is

$$\text{proj}_{\mathbf{i}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{i}\|^2} \mathbf{i} = (\mathbf{v} \cdot \mathbf{i}) \mathbf{i},$$

since $\|\mathbf{i}\| = 1$. For the second part, suppose that $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then

$$(\mathbf{v} \cdot \mathbf{i}) \mathbf{i} + (\mathbf{v} \cdot \mathbf{j}) \mathbf{j} + (\mathbf{v} \cdot \mathbf{k}) \mathbf{k} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{v}.$$

55. We have

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| \cos \theta + \|\mathbf{w}\|^2, \\ \|\mathbf{v} - \mathbf{w}\|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - 2\|\mathbf{v}\| \|\mathbf{w}\| \cos \theta + \|\mathbf{w}\|^2 \end{aligned}$$

by the Magnitude Property and the formula for the angle between vectors. Now, since $\theta = \frac{\pi}{3}$, we have $\cos \theta = \frac{1}{2}$, so these simplify to

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= 2^2 + 2 \cdot 2 \cdot 6 \cdot \frac{1}{2} + 6^2 = 52 \\ \|\mathbf{v} - \mathbf{w}\|^2 &= 2^2 - 2 \cdot 2 \cdot 6 \cdot \frac{1}{2} + 6^2 = 28. \end{aligned}$$

Taking square roots gives

$$\|\mathbf{v} + \mathbf{w}\| = \sqrt{52} = 2\sqrt{13}, \quad \|\mathbf{v} - \mathbf{w}\| = \sqrt{28} = 2\sqrt{7}.$$

56. Compute the dot product and the two magnitudes:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \langle 2a, -2, 1 \rangle \cdot \langle b, 2, 2 \rangle = 2ab - 2 \\ \|\mathbf{v}\| &= \sqrt{(2a)^2 + 2^2 + 1^2} = \sqrt{4a^2 + 5} \\ \|\mathbf{w}\| &= \sqrt{b^2 + 2^2 + 2^2} = \sqrt{b^2 + 8}. \end{aligned}$$

Since the vectors are orthogonal, we must have $2ab = 2$, so that $b = \frac{1}{a}$; since they have the same magnitude, we must have $4a^2 + 5 = b^2 + 8$, so that $b^2 = 4a^2 - 3$. Now substitute $b = \frac{1}{a}$ into this equation, giving

$$\left(\frac{1}{a} \right)^2 = 4a^2 - 3, \quad \text{so that} \quad \frac{1}{a^2} = 4a^2 - 3 \quad \text{and then} \quad 4a^4 - 3a^2 - 1 = 0.$$

Now, both $a = 1$ and $a = -1$ are roots of this quartic, so it factors as

$$(4a^4 - 3a^2 - 1) = (a - 1)(a + 1)(4a^2 + 1),$$

and the remaining quadratic has no real roots. So the possible values of a are $a = 1$ or $a = -1$; the corresponding values of b are $b = \frac{1}{1} = 1$ and $b = \frac{1}{-1} = -1$. So either

$$a = b = 1 \quad \text{or} \quad a = b = -1.$$

57. Compute the dot product of $\mathbf{w}$ and $\mathbf{v} - a\mathbf{w}$.

$$\mathbf{w} \cdot (\mathbf{v} - a\mathbf{w}) = \mathbf{w} \cdot \mathbf{v} - a\mathbf{w} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w} \cdot \mathbf{w}.$$

By the Magnitude Property, $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$, so this simplifies to

$$\mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} = 0$$

since the dot product is commutative. Therefore $\mathbf{w}$ and $\mathbf{v} - a\mathbf{w}$ are orthogonal.

58. The dot product of these two vectors is

$$(\|\mathbf{w}\| \mathbf{v} + \|\mathbf{v}\| \mathbf{w}) \cdot (\|\mathbf{w}\| \mathbf{v} - \|\mathbf{v}\| \mathbf{w}) = \|\mathbf{w}\|^2 \mathbf{v} \cdot \mathbf{v} - \|\mathbf{w}\| \|\mathbf{v}\| \mathbf{v} \cdot \mathbf{w} + \|\mathbf{v}\| \|\mathbf{w}\| \mathbf{w} \cdot \mathbf{v} - \|\mathbf{v}\|^2 \mathbf{w} \cdot \mathbf{w}.$$

By the Magnitude Property, $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ and similarly for $\mathbf{w}$. Also, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, so this becomes

$$\begin{aligned} \|\mathbf{w}\|^2 \mathbf{v} \cdot \mathbf{v} - \|\mathbf{w}\| \|\mathbf{v}\| \mathbf{v} \cdot \mathbf{w} + \|\mathbf{v}\| \|\mathbf{w}\| \mathbf{w} \cdot \mathbf{v} - \|\mathbf{v}\|^2 \mathbf{w} \cdot \mathbf{w} \\ = \|\mathbf{w}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{w}\| \|\mathbf{v}\| \mathbf{v} \cdot \mathbf{w} + \|\mathbf{v}\| \|\mathbf{w}\| \mathbf{v} \cdot \mathbf{w} - \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = 0. \end{aligned}$$

Therefore the two vectors are indeed orthogonal.

59. Let θ be the angle between $\mathbf{w}$ and $\mathbf{u}$. Then

$$\cos \theta = \frac{\mathbf{w} \cdot \mathbf{u}}{\|\mathbf{w}\| \|\mathbf{u}\|}, \quad \text{so that} \quad \mathbf{w} \cdot \mathbf{u} = \|\mathbf{w}\| \|\mathbf{u}\| \cos \theta = \|\mathbf{w}\| \cos \theta,$$

since $\mathbf{u}$ is a unit vector. So the dot product is maximized when $\cos \theta = 1$; this means that the angle between $\mathbf{w}$ and $\mathbf{u}$ has cosine 1, so that $\theta = 0$ and therefore $\mathbf{w}$ and $\mathbf{u}$ point in the same direction.

60. Let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}, \quad \text{so that} \quad \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta = |\mathbf{v} \cdot \mathbf{w}|.$$

But norms are always nonnegative, so we can pull both norms outside of the absolute value sign, giving

$$\|\mathbf{v}\| \|\mathbf{w}\| |\cos \theta| = |\mathbf{v} \cdot \mathbf{w}|.$$

Since $|\cos \theta| \leq 1$, we see that

$$|\mathbf{v} \cdot \mathbf{w}| = \|\mathbf{v}\| \|\mathbf{w}\| |\cos \theta| \leq \|\mathbf{v}\| \|\mathbf{w}\|,$$

which is the Cauchy-Schwarz inequality. This inequality is an equality when $\cos \theta = 1$ or $\cos \theta = -1$; that is, when $\mathbf{v}$ and $\mathbf{w}$ point either in the same direction or in opposite directions.

61. We have

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} \\ &\leq \mathbf{v} \cdot \mathbf{v} + 2|\mathbf{v} \cdot \mathbf{w}| + \mathbf{w} \cdot \mathbf{w} \\ &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2. \end{aligned}$$

Since norms are nonnegative, we take the square root of both sides, giving

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

62. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos \beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos \gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

Then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{v_1^2}{\|\mathbf{v}\|^2} + \frac{v_2^2}{\|\mathbf{v}\|^2} + \frac{v_3^2}{\|\mathbf{v}\|^2} = \frac{v_1^2 + v_2^2 + v_3^2}{v_1^2 + v_2^2 + v_3^2} = 1.$$

63. From Problem 62, we know that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \cos^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{3} + \cos^2 \gamma = \frac{1}{4} + \frac{1}{4} + \cos^2 \gamma = 1,$$

so that $\cos^2 \gamma = \frac{1}{2}$. Then $\cos \gamma = \frac{\sqrt{2}}{2}$, so that $\boxed{\gamma = \frac{\pi}{4} = 45^\circ}$.

64. From Problem 62, we know that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \cos^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{4} + \cos^2 \gamma = \frac{1}{4} + \frac{1}{2} + \cos^2 \gamma = 1,$$

so that $\cos^2 \gamma = \frac{1}{4}$. Then $\cos \gamma = \pm \frac{1}{2}$; since γ is in the first quadrant, we use the plus sign, so that $\gamma = \frac{\pi}{3} = 60^\circ$. Then

$$\mathbf{v} = 3(\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = 3 \left(\frac{1}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} + \frac{1}{2} \mathbf{k} \right) = \boxed{\frac{3}{2} \mathbf{i} + \frac{3\sqrt{2}}{2} \mathbf{j} + \frac{3}{2} \mathbf{k}}.$$

65. From Problem 62, we know that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \cos^2 \frac{\pi}{4} + \cos^2 \beta + \cos^2 \frac{\pi}{3} = \frac{1}{2} + \cos^2 \beta + \frac{1}{4} = 1,$$

so that $\cos^2 \beta = \frac{1}{4}$. Then $\cos \beta = \pm \frac{1}{2}$; since β is in the second quadrant, we use the minus sign, so that $\beta = \frac{2\pi}{3} = 120^\circ$. Then

$$\mathbf{v} = 2(\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = 2 \left(\frac{\sqrt{2}}{2} \mathbf{i} - \frac{1}{2} \mathbf{j} + \frac{1}{2} \mathbf{k} \right) = \boxed{\sqrt{2} \mathbf{i} - \mathbf{j} + \mathbf{k}}.$$

66. Suppose all three direction angles are equal to θ . From Problem 62, we know that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3 \cos^2 \theta = 1,$$

so that $\cos^2 \theta = \frac{1}{3}$. Then $\cos \theta = \pm \frac{\sqrt{3}}{3}$; since we want $\mathbf{v}$ to have all positive components, we choose the plus sign in each component. Then

$$\mathbf{v} = 3(\cos \theta \mathbf{i} + \cos \theta \mathbf{j} + \cos \theta \mathbf{k}) = 3 \left(\frac{\sqrt{3}}{3} \mathbf{i} + \frac{\sqrt{3}}{3} \mathbf{j} + \frac{\sqrt{3}}{3} \mathbf{k} \right) = \boxed{\sqrt{3} \mathbf{i} + \sqrt{3} \mathbf{j} + \sqrt{3} \mathbf{k}}.$$

67. From Problem 62, we know that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \cos^2 \alpha + \frac{1}{16} + \frac{7}{8} = 1,$$

so that $\cos^2 \alpha = \frac{1}{16}$. Then $\cos \alpha = \frac{1}{4}$ since we are given that $\cos \alpha > 0$. Then

$$\mathbf{v} = \frac{1}{2}(\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \frac{1}{2} \left(\frac{1}{4} \mathbf{i} + \frac{1}{4} \mathbf{j} + \sqrt{\frac{7}{8}} \mathbf{k} \right) = \boxed{\frac{1}{8} \mathbf{i} + \frac{1}{8} \mathbf{j} + \frac{1}{2} \sqrt{\frac{7}{8}} \mathbf{k}}.$$

68. Suppose that $\langle a, b, c \rangle$ is orthogonal to both $\mathbf{v} = \langle 2, -1, 1 \rangle$ and $\mathbf{w} = \langle 4, 2, -1 \rangle$. Then

$$\begin{aligned} 2a - b + c &= 0 \\ 4a + 2b - c &= 0. \end{aligned}$$

Adding these equations gives $6a + b = 0$, so that $b = -6a$. Substitute that value for b back into the first equation to get $2a - (-6a) + c = 8a + c = 0$, so that $c = -8a$. So the vector $\langle a, -6a, -8a \rangle$ is orthogonal to both vectors for any value of a ; for example, with $a = 1$, we get $\langle 1, -6, -8 \rangle$, and

$$\begin{aligned} \langle 2, -1, 1 \rangle \cdot \langle 1, -6, -8 \rangle &= 2 \cdot 1 - 1 \cdot (-6) + 1 \cdot (-8) = 0 \\ \langle 4, 2, -1 \rangle \cdot \langle 1, -6, -8 \rangle &= 4 \cdot 1 + 2 \cdot (-6) - 1 \cdot (-8) = 0. \end{aligned}$$

69. Suppose $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then

$$\begin{aligned} \mathbf{v} \cdot \mathbf{i} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{i} = a = 0, \\ \mathbf{v} \cdot \mathbf{j} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{j} = b = 0, \\ \mathbf{v} \cdot \mathbf{k} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{k} = c = 0. \end{aligned}$$

Therefore $a = b = c = 0$, so that $\mathbf{v} = \mathbf{0}$.

70. First take the dot product of $\mathbf{b}$ with both sides of the equation, giving

$$a\mathbf{x} \cdot \mathbf{b} + (\mathbf{x} \cdot \mathbf{b})\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{b}.$$

Now factor out $\mathbf{x} \cdot \mathbf{b}$ from the left-hand side, giving

$$(\mathbf{x} \cdot \mathbf{b})(a + \mathbf{a} \cdot \mathbf{b}) = \mathbf{c} \cdot \mathbf{b}.$$

Since $a + \mathbf{a} \cdot \mathbf{b} \neq 0$, we can divide both sides by that expression, giving

$$\mathbf{x} \cdot \mathbf{b} = \frac{\mathbf{c} \cdot \mathbf{b}}{a + \mathbf{a} \cdot \mathbf{b}}.$$

Next, substitute that value for $\mathbf{x} \cdot \mathbf{b}$ in the original equation, giving

$$a\mathbf{x} + \left(\frac{\mathbf{c} \cdot \mathbf{b}}{a + \mathbf{a} \cdot \mathbf{b}} \right) \mathbf{a} = \mathbf{c},$$

so that (since $a \neq 0$)

$$\mathbf{x} = \frac{1}{a} \left(\mathbf{c} - \left(\frac{\mathbf{c} \cdot \mathbf{b}}{a + \mathbf{a} \cdot \mathbf{b}} \right) \mathbf{a} \right) = \frac{a\mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}}{a(a + \mathbf{a} \cdot \mathbf{b})}.$$

71. (a) If there were, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \left(\frac{\sqrt{3}}{2} \right)^2 + \left(\frac{\sqrt{2}}{2} \right)^2 + \cos^2 \gamma = \frac{3}{4} + \frac{1}{2} + \cos^2 \gamma > 1.$$

But this contradicts Problem 62, so this is impossible.

(b) Suppose that $\alpha + \beta < \frac{\pi}{2}$. Then $\beta < \frac{\pi}{2} - \alpha$, so that $\sin \beta < \sin \left(\frac{\pi}{2} - \alpha \right) = \cos \alpha$ and therefore $\sin^2 \beta < \cos^2 \alpha$. But then

$$\cos^2 \alpha + \cos^2 \beta = \cos^2 \alpha + 1 - \sin^2 \beta > \cos^2 \alpha + 1 - \cos^2 \alpha = 1.$$

Since $\cos^2 \alpha + \cos^2 \beta > 1$ contradicts Problem 62, this is impossible, so that $\alpha + \beta \geq \frac{\pi}{2}$.

72. (a) Since any three points lie in at least one plane, the terminal points of the three vectors lie in a plane. The lines between those points form the sides of a triangle if the sum of any two of the lengths is greater than the third. The lengths of these lines are

$$\|\mathbf{u} - \mathbf{v}\| = \|4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}\| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{56} = 2\sqrt{14} \approx 7.48$$

$$\|\mathbf{v} - \mathbf{w}\| = \|-5\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}\| = \sqrt{5^2 + 5^2 + 10^2} = \sqrt{150} = 5\sqrt{6} \approx 12.25$$

$$\|\mathbf{u} - \mathbf{w}\| = \|\mathbf{-i} + 3\mathbf{j} + 4\mathbf{k}\| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26} \approx 5.10.$$

Since the sum of the two shorter sides is $7.48 + 5.10 = 12.58 > 12.25$, these three vectors form a triangle.

- (b) The midpoints of the three sides are:

$$\mathbf{M}_1 \text{ (midpoint of side } \mathbf{u} - \mathbf{v}) : \quad \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{1}{2}(2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{M}_2 \text{ (midpoint of side } \mathbf{v} - \mathbf{w}) : \quad \frac{1}{2}(\mathbf{v} + \mathbf{w}) = \frac{1}{2}(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{3}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\mathbf{M}_3 \text{ (midpoint of side } \mathbf{w} - \mathbf{u}) : \quad \frac{1}{2}(\mathbf{u} + \mathbf{w}) = \frac{1}{2}(7\mathbf{i} - \mathbf{j} - 8\mathbf{k}) = \frac{7}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - 4\mathbf{k}.$$

Then the lengths of the three medians are

$$\|\mathbf{w} - \mathbf{M}_1\| = \|3\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}\| = \sqrt{3^2 + 4^2 + 7^2} = \sqrt{74}$$

$$\|\mathbf{u} - \mathbf{M}_2\| = \left\| \frac{3}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k} \right\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2} = \frac{1}{2}\sqrt{14}$$

$$\|\mathbf{v} - \mathbf{M}_3\| = \left\| -\frac{9}{2}\mathbf{i} + \frac{7}{2}\mathbf{j} + 8\mathbf{k} \right\| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + 8^2} = \frac{1}{2}\sqrt{386}.$$

73. (a) We have

$$\|\mathbf{u}_r\| = \sqrt{(\cos \theta)^2 + (\sin \theta)^2} = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1$$

$$\|\mathbf{u}_\theta\| = \sqrt{(-\sin \theta)^2 + (\cos \theta)^2} = \sqrt{\sin^2 \theta + \cos^2 \theta} = \sqrt{1} = 1.$$

- (b) Their dot product is

$$\mathbf{u}_r \cdot \mathbf{u}_\theta = (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0.$$

- (c) The ray from the origin to (r, θ) makes an angle of θ with the positive x -axis, so if regarded as a vector, it is a multiple of $\cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, so it is in the same direction as $\mathbf{u}_r$. For $\mathbf{u}_\theta$, note that

$$\cos\left(\theta + \frac{\pi}{2}\right) = \cos \theta \cos \frac{\pi}{2} - \sin \theta \sin \frac{\pi}{2} = -\sin \theta$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2} = \cos \theta,$$

so that $\mathbf{r}_\theta = \cos\left(\theta + \frac{\pi}{2}\right) \mathbf{i} + \sin\left(\theta + \frac{\pi}{2}\right) \mathbf{j}$. Therefore it points in the same direction as the ray at an angle of $\theta + \frac{\pi}{2}$ to the x -axis, which is a 90° rotation from $\mathbf{u}_r$.

74. There are many examples. For instance, let $\mathbf{a} = \langle 1, 1 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$, and $\mathbf{c} = \langle 3, 2 \rangle$. Then

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 + 1 \cdot 3 = 5 = 1 \cdot 3 + 1 \cdot 2 = \mathbf{a} \cdot \mathbf{c},$$

but neither dot product is zero and $\mathbf{b} \neq \mathbf{c}$. There is no cancellation property for dot products, nor are there any simple rules regarding the relationship among the vectors that gives cancellation in some situations.

75. Since $\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$, we have

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right)^2 = 1 - \frac{(\mathbf{v} \cdot \mathbf{w})^2}{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2} = \frac{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2}{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2}.$$

76. From the previous problem,

$$\begin{aligned} \sin^2 \theta &= \frac{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \\ &= \frac{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \\ &= \frac{u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 - u_1^2 v_1^2 - u_2^2 v_2^2 - u_3^2 v_3^2 - 2u_1 u_2 v_1 v_2 - 2u_1 u_3 v_1 v_3 - 2u_2 u_3 v_2 v_3}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \\ &= \frac{u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 - 2u_1 u_2 v_1 v_2 - 2u_1 u_3 v_1 v_3 - 2u_2 u_3 v_2 v_3}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \\ &= \frac{u_2^2 v_3^2 - 2u_2 u_3 v_2 v_3 + u_3^2 v_2^2 + u_1^2 v_3^2 - 2u_1 u_3 v_1 v_3 + u_3^2 v_1^2 + u_1^2 v_2^2 - 2u_1 u_2 v_1 v_2 + u_2^2 v_1^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \\ &= \frac{(u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \end{aligned}$$

77. Take the dot product of the given equation first with $\mathbf{u}_1$ and then with $\mathbf{u}_2$, giving the system of equations in c_1 and c_2

$$\begin{aligned} \mathbf{v} \cdot \mathbf{u}_1 &= c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_1 \\ \mathbf{v} \cdot \mathbf{u}_2 &= c_1 \mathbf{u}_1 \cdot \mathbf{u}_2 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_2. \end{aligned}$$

To eliminate c_2 from the equations, multiply the first equation by $\mathbf{u}_2 \cdot \mathbf{u}_2$ and the second by $\mathbf{u}_2 \cdot \mathbf{u}_1$, then subtract to get

$$(\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{u}_1) = c_1((\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{u}_1 \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{u}_1)),$$

then solve for c_1 :

$$c_1 = \frac{(\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{u}_1 \cdot \mathbf{u}_2)(\mathbf{u}_2 \cdot \mathbf{u}_1)} = \frac{(\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{u}_1 \cdot \mathbf{u}_2)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{u}_1 \cdot \mathbf{u}_2)^2}.$$

Substitute this value for c_1 into either equation and solve for c_2 to get

$$c_2 = \frac{(\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{u}_1 \cdot \mathbf{u}_1) - (\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{u}_1 \cdot \mathbf{u}_2)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2) - (\mathbf{u}_1 \cdot \mathbf{u}_2)^2}.$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors, then $\mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 1$ and $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so the above equations become

$$c_1 = \mathbf{v} \cdot \mathbf{u}_1, \quad c_2 = \mathbf{v} \cdot \mathbf{u}_2.$$

78. In the plane, if $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $\mathbf{w} = \langle w_1, w_2 \rangle$, then

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2 \rangle \cdot \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= \langle u_1(v_1 + w_1), u_2(v_2 + w_2) \rangle \\ &= \langle u_1 v_1 + u_1 w_1, u_2 v_2 + u_2 w_2 \rangle \\ &= \langle u_1 v_1, u_2 v_2 \rangle + \langle u_1 w_1, u_2 w_2 \rangle \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}. \end{aligned}$$

In space, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\ &= \langle u_1(v_1 + w_1), u_2(v_2 + w_2), u_3(v_3 + w_3) \rangle \\ &= \langle u_1v_1 + u_1w_1, u_2v_2 + u_2w_2, u_3v_3 + u_3w_3 \rangle \\ &= \langle u_1v_1, u_2v_2, u_3v_3 \rangle + \langle u_1w_1, u_2w_2, u_3w_3 \rangle \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.\end{aligned}$$

79. In the plane, if $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and a is a scalar, then

$$\begin{aligned}a(\mathbf{u} \cdot \mathbf{v}) &= a(\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle) \\ &= a(u_1v_1 + u_2v_2) \\ &= (au_1)v_1 + (au_2)v_2 \\ &= \langle au_1, au_2 \rangle \cdot \langle v_1, v_2 \rangle \\ &= (a \langle u_1, u_2 \rangle) \cdot \langle v_1, v_2 \rangle \\ &= (a\mathbf{u}) \cdot \mathbf{v}.\end{aligned}$$

In space, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and a is a scalar, then

$$\begin{aligned}a(\mathbf{u} \cdot \mathbf{v}) &= a(\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle) \\ &= a(u_1v_1 + u_2v_2 + u_3v_3) \\ &= (au_1)v_1 + (au_2)v_2 + (au_3)v_3 \\ &= \langle au_1, au_2, au_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (a \langle u_1, u_2, u_3 \rangle) \cdot \langle v_1, v_2, v_3 \rangle \\ &= (a\mathbf{u}) \cdot \mathbf{v}.\end{aligned}$$

80. In the plane, let $\mathbf{v} = \langle v_1, v_2 \rangle$; then $\mathbf{0} = \langle 0, 0 \rangle$, and

$$\mathbf{0} \cdot \mathbf{v} = \langle 0, 0 \rangle \cdot \langle v_1, v_2 \rangle = 0 \cdot v_1 + 0 \cdot v_2 = 0.$$

In space, let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$; then $\mathbf{0} = \langle 0, 0, 0 \rangle$, and

$$\mathbf{0} \cdot \mathbf{v} = \langle 0, 0, 0 \rangle \cdot \langle v_1, v_2, v_3 \rangle = 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 = 0.$$

81. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \langle v_1, v_2 \rangle \cdot \langle u_1, u_2 \rangle = \mathbf{v} \cdot \mathbf{u}.$$

82. Let $\mathbf{v} = \langle v_1, v_2 \rangle$. Then

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 = v_1v_1 + v_2v_2 = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle = \mathbf{v} \cdot \mathbf{v}.$$

83. Let $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$, and $\mathbf{w} = \langle w_1, w_2, \dots, w_n \rangle$; let a be a scalar. Then

Magnitude Property:

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + \dots + v_n^2 = v_1v_1 + v_2v_2 + \dots + v_nv_n = \langle v_1, v_2, \dots, v_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle = \mathbf{v} \cdot \mathbf{v}.$$

Commutative property:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle \\ &= u_1v_1 + u_2v_2 + \dots + u_nv_n \\ &= v_1u_1 + v_2u_2 + \dots + v_nu_n \\ &= \langle v_1, v_2, \dots, v_n \rangle \cdot \langle u_1, u_2, \dots, u_n \rangle \\ &= \mathbf{v} \cdot \mathbf{u}.\end{aligned}$$

Distributive property:

$$\begin{aligned}
 \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1 + w_1, v_2 + w_2, \dots, v_n + w_n \rangle \\
 &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \dots + u_n(v_n + w_n) \\
 &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \dots + u_nv_n + u_nw_n \\
 &= u_1v_1 + u_2v_2 + \dots + u_nv_n + u_1w_1 + u_2w_2 + \dots + u_nw_n \\
 &= \langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle + \langle u_1, u_2, \dots, u_n \rangle \cdot \langle w_1, w_2, \dots, w_n \rangle \\
 &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
 \end{aligned}$$

Linearity property:

$$\begin{aligned}
 a(\mathbf{u} \cdot \mathbf{v}) &= a(\langle u_1, u_2, \dots, u_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle) \\
 &= a(u_1v_1 + u_2v_2 + \dots + u_nv_n) \\
 &= (au_1)v_1 + (au_2)v_2 + \dots + (au_n)v_n \\
 &= \langle au_1, au_2, \dots, au_n \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle \\
 &= (a \langle u_1, u_2, \dots, u_n \rangle) \cdot \langle v_1, v_2, \dots, v_n \rangle \\
 &= (a\mathbf{u}) \cdot \mathbf{v}.
 \end{aligned}$$

Zero-vector property:

$$\mathbf{0} \cdot \mathbf{v} = \langle 0, 0, \dots, 0 \rangle \cdot \langle v_1, v_2, \dots, v_n \rangle = 0v_1 + 0v_2 + \dots + 0v_n = 0.$$

Challenge Problems

84. The angle θ_u between $\mathbf{u}$ and $\mathbf{w}$ satisfies

$$\begin{aligned}
 \cos \theta_u &= \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\| \|\mathbf{w}\|} \\
 &= \frac{\mathbf{u} \cdot (\|\mathbf{v}\| \mathbf{u} + \|\mathbf{u}\| \mathbf{v})}{\|\mathbf{u}\| \|\mathbf{w}\|} \\
 &= \frac{\|\mathbf{v}\| \mathbf{u} \cdot \mathbf{u} + \|\mathbf{u}\| \mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{w}\|} \\
 &= \frac{\|\mathbf{v}\| \|\mathbf{u}\|^2 + \|\mathbf{u}\| \mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{w}\|} \\
 &= \frac{\|\mathbf{v}\| \|\mathbf{u}\| + \mathbf{u} \cdot \mathbf{v}}{\|\mathbf{w}\|}.
 \end{aligned}$$

The angle θ_v between $\mathbf{v}$ and $\mathbf{w}$ satisfies

$$\begin{aligned}
 \cos \theta_v &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \\
 &= \frac{\mathbf{v} \cdot (\|\mathbf{v}\| \mathbf{u} + \|\mathbf{u}\| \mathbf{v})}{\|\mathbf{v}\| \|\mathbf{w}\|} \\
 &= \frac{\|\mathbf{v}\| \mathbf{v} \cdot \mathbf{u} + \|\mathbf{u}\| \mathbf{v} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{w}\|} \\
 &= \frac{\|\mathbf{v}\| \mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\| \|\mathbf{v}\|^2}{\|\mathbf{v}\| \|\mathbf{w}\|} \\
 &= \frac{\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\| \|\mathbf{v}\|}{\|\mathbf{w}\|}.
 \end{aligned}$$

The two expressions are equal, so that $\cos \theta_u = \cos \theta_v$ and therefore $\theta_u = \theta_v$. Since the two angle are equal, $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

85. Computing the left-hand side gives

$$\begin{aligned}
 \|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\
 &= (\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}) \\
 &= 2\mathbf{u} \cdot \mathbf{v} - (-2\mathbf{u} \cdot \mathbf{v}) \\
 &= 4(\mathbf{u} \cdot \mathbf{v}).
 \end{aligned}$$

86. (a) Compute the dot product:

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.$$

Since $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude, this difference is zero, so the two vectors are orthogonal.

(b) In the diagram, the left-hand black diagonal line is $-(-\mathbf{v}) + \mathbf{u} = \mathbf{u} + \mathbf{v}$, and the right-hand black diagonal line is $-\mathbf{v} + \mathbf{u} = \mathbf{u} - \mathbf{v}$. Since those two are orthogonal by part (a), the angle at the terminal point of $\mathbf{u}$ is a right angle.

87. (a) By the theorem in objective 5 of Section 10.3, if $\mathbf{u}$ is any vector in the plane making an angle of θ with the positive x -axis, then $\mathbf{u} = \|\mathbf{u}\|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$. Applying this to $\mathbf{v}$ and $\mathbf{w}$, both of which are assumed to be unit vectors, gives

$$\mathbf{v} = \|\mathbf{v}\|(\cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}) = \cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}, \quad \mathbf{w} = \|\mathbf{w}\|(\cos \beta \mathbf{i} + \sin \beta \mathbf{j}) = \cos \beta \mathbf{i} + \sin \beta \mathbf{j}.$$

(b) The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\alpha - \beta$; by the angle formula, since both vectors are unit vectors,

$$\cos(\alpha - \beta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = (\cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}) \cdot (\cos \beta \mathbf{i} + \sin \beta \mathbf{j}) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

Next consider the vector $\mathbf{w}' = \cos \beta \mathbf{i} - \sin \beta \mathbf{j}$; this is the reflection of $\mathbf{w}$ through the x -axis, so it makes an angle of $-\beta$ with the positive x -axis. So the angle between $\mathbf{v}$ and $\mathbf{w}'$ is $\alpha - (-\beta) = \alpha + \beta$. Again use the angle formula, remembering that both vectors are unit vectors:

$$\cos(\alpha + \beta) = \frac{\mathbf{v} \cdot \mathbf{w}'}{\|\mathbf{v}\| \|\mathbf{w}'\|} = (\cos \alpha \mathbf{i} + \sin \alpha \mathbf{j}) \cdot (\cos \beta \mathbf{i} - \sin \beta \mathbf{j}) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

88. Each of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ is a unit vector, since each is equal to a vector divided by its norm. It remains to show that they are mutually orthogonal.

$$\begin{aligned}
 \mathbf{w}_1 \cdot \mathbf{v}_2 &= \mathbf{w}_1 \cdot (\mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{w}_1)\mathbf{w}_1) \\
 &= \mathbf{w}_1 \cdot \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{w}_1)(\mathbf{w}_1 \cdot \mathbf{w}_1) \\
 &= \mathbf{w}_1 \cdot \mathbf{u}_2 - \mathbf{u}_2 \cdot \mathbf{w}_1 \\
 &= 0.
 \end{aligned}$$

Since $\mathbf{w}_2$ is a scalar multiple of $\mathbf{v}_2$, we see that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal.

$$\begin{aligned}
 \mathbf{w}_1 \cdot \mathbf{v}_3 &= \mathbf{w}_1 \cdot (\mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_3 \cdot \mathbf{w}_2)\mathbf{w}_2) \\
 &= \mathbf{w}_1 \cdot \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)(\mathbf{w}_1 \cdot \mathbf{w}_1) - (\mathbf{u}_3 \cdot \mathbf{w}_2)(\mathbf{w}_1 \cdot \mathbf{w}_2) \\
 &= \mathbf{w}_1 \cdot \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1) \cdot 1 - (\mathbf{u}_3 \cdot \mathbf{w}_2) \cdot 0 \\
 &= \mathbf{w}_1 \cdot \mathbf{u}_3 - \mathbf{u}_3 \cdot \mathbf{w}_1 \\
 &= 0.
 \end{aligned}$$

Since $\mathbf{w}_3$ is a scalar multiple of $\mathbf{v}_3$, we see that $\mathbf{w}_1$ and $\mathbf{w}_3$ are orthogonal.

$$\begin{aligned}
 \mathbf{w}_2 \cdot \mathbf{v}_3 &= \mathbf{w}_2 \cdot (\mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_3 \cdot \mathbf{w}_2)\mathbf{w}_2) \\
 &= \mathbf{w}_2 \cdot \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)(\mathbf{w}_2 \cdot \mathbf{w}_1) - (\mathbf{u}_3 \cdot \mathbf{w}_2)(\mathbf{w}_2 \cdot \mathbf{w}_2) \\
 &= \mathbf{w}_2 \cdot \mathbf{u}_3 - \mathbf{u}_3 \cdot \mathbf{w}_2 \\
 &= 0.
 \end{aligned}$$

Since $\mathbf{w}_3$ is a scalar multiple of $\mathbf{v}_3$, we see that $\mathbf{w}_2$ and $\mathbf{w}_3$ are orthogonal.

89. With $\mathbf{u}_1 = \langle -1, 1, 0 \rangle$, $\mathbf{u}_2 = \langle 2, 0, 1 \rangle$, and $\mathbf{u}_3 = \langle 3, -1, 2 \rangle$, we first get

$$\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{\langle -1, 1, 0 \rangle}{\sqrt{2}} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

Next,

$$\mathbf{v}_2 = \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{w}_1)\mathbf{w}_1 = \langle 2, 0, 1 \rangle + \sqrt{2} \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle = \langle 2 - 1, 0 + 1, 1 + 0 \rangle = \langle 1, 1, 1 \rangle$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle.$$

Finally,

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - (\mathbf{u}_3 \cdot \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_3 \cdot \mathbf{w}_2)\mathbf{w}_2 \\ &= \langle 3, -1, 2 \rangle - (-2\sqrt{2})\mathbf{w}_1 - \frac{4}{\sqrt{3}}\mathbf{w}_2 \\ &= \langle 3, -1, 2 \rangle + \langle -2, 2, 0 \rangle - \left\langle \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right\rangle \\ &= \left\langle -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \\ \mathbf{w}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left\langle -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}}} = \left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle. \end{aligned}$$

90. In general,

$$\begin{aligned} \mathbf{v}_k &= \mathbf{u}_k - (\mathbf{u}_k \cdot \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_k \cdot \mathbf{w}_2)\mathbf{w}_2 - \cdots - (\mathbf{u}_k \cdot \mathbf{w}_{k-1})\mathbf{w}_{k-1} \\ \mathbf{w}_k &= \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|}. \end{aligned}$$

Since $(\mathbf{u}_k \cdot \mathbf{w}_i)\mathbf{w}_i$ is just the projection of the vector $\mathbf{u}_k$ on the unit vector $\mathbf{w}_i$, at each step we subtract off the portion of $\mathbf{u}_k$ parallel to each of the preceding $\mathbf{w}_i$, so what is left is orthogonal to all of them.

91. Computing using the Magnitude Property gives

$$g(t) = \|\mathbf{u} - t\mathbf{v}\|^2 = (\mathbf{u} - t\mathbf{v}) \cdot (\mathbf{u} - t\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - t\mathbf{v} \cdot \mathbf{u} - t\mathbf{u} \cdot \mathbf{v} + t^2\mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - 2t\mathbf{u} \cdot \mathbf{v} + t^2\|\mathbf{v}\|^2.$$

This is a parabola with a positive leading coefficient, so it opens upwards and therefore has a minimum value. That minimum value occurs when the t -derivative of the quadratic is zero, so when $-2\mathbf{u} \cdot \mathbf{v} + 2t\|\mathbf{v}\|^2 = 0$. Solving for t gives

$$t = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}.$$

The minimum value is then

$$\begin{aligned} g(t) &= \|\mathbf{u}\|^2 - 2\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{u} \cdot \mathbf{v} + \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)^2 \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 - 2\frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} + \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} \\ &= \|\mathbf{u}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2}. \end{aligned}$$

Since $g(t) = \|\mathbf{u} - t\mathbf{v}\|^2$, we see that $g(t) \geq 0$ everywhere, so that the minimum value must be nonnegative. That is,

$$\|\mathbf{u}\|^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} \geq 0 \quad \text{so that} \quad \|\mathbf{u}\|^2 \geq \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{v}\|^2} \quad \text{and then} \quad \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \geq (\mathbf{u} \cdot \mathbf{v})^2.$$

Taking square roots gives $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$, as desired.

92. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors in question, and $\mathbf{w}_1$ and $\mathbf{w}_2$ the fixed mutually orthogonal vectors. First suppose that $\mathbf{u}$ and $\mathbf{v}$ are parallel. Then there is some scalar c such that $\mathbf{v} = c\mathbf{u}$. Then let $\mathbf{x}$ be any vector:

$$\text{proj}_{\mathbf{x}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} = \frac{c\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} = c \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} = c \text{proj}_{\mathbf{x}} \mathbf{u}.$$

Therefore the projections of $\mathbf{u}$ and $\mathbf{v}$ on *any* vector have ratio c , so in particular are proportional for $\mathbf{w}_1$ and $\mathbf{w}_2$.

Now suppose that the projections of $\mathbf{u}$ and $\mathbf{v}$ on $\mathbf{w}_1$ and $\mathbf{w}_2$ are proportional; this means that there is some constant c such that

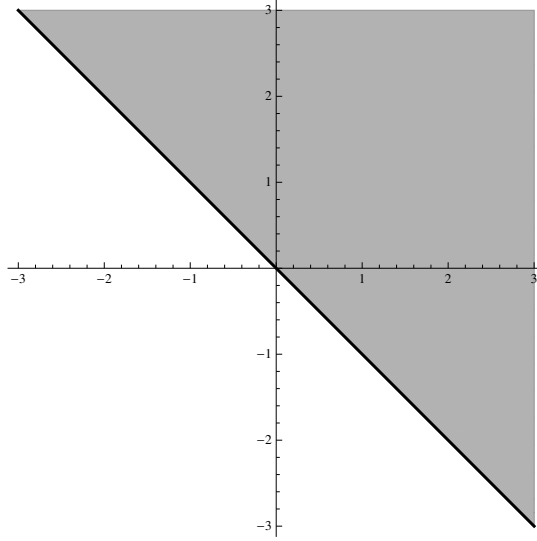
$$\text{proj}_{\mathbf{w}_1} \mathbf{v} = c \text{proj}_{\mathbf{w}_1} \mathbf{u} \quad \text{and} \quad \text{proj}_{\mathbf{w}_2} \mathbf{v} = c \text{proj}_{\mathbf{w}_2} \mathbf{u}.$$

But the projection of $\mathbf{v}$ on $\mathbf{w}_2$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{w}_1$, since $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal. Similarly, the projection of $\mathbf{u}$ on $\mathbf{w}_2$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{w}_1$. Then we have

$$\begin{aligned} \mathbf{v} &= \text{proj}_{\mathbf{w}_1} \mathbf{v} + (\mathbf{v} - \text{proj}_{\mathbf{w}_1} \mathbf{v}) = \text{proj}_{\mathbf{w}_1} \mathbf{v} + \text{proj}_{\mathbf{w}_2} \mathbf{v} = c \text{proj}_{\mathbf{w}_1} \mathbf{u} + c \text{proj}_{\mathbf{w}_2} \mathbf{u} \\ &= c (\text{proj}_{\mathbf{w}_1} \mathbf{u} + \text{proj}_{\mathbf{w}_2} \mathbf{u}) = c\mathbf{u}, \end{aligned}$$

so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

93. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Then $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j}) = x + y$, so that $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j}) \geq 0$ when $x + y \geq 0$. So the region is the set of points in the plane such that $x + y \geq 0$:



94. Let $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then

$$(\mathbf{r} - \mathbf{b}) \cdot (\mathbf{r} + \mathbf{b}) = \langle x - b_1, y - b_2, z - b_3 \rangle \cdot \langle x + b_1, y + b_2, z + b_3 \rangle = x^2 - b_1^2 + y^2 - b_2^2 + z^2 - b_3^2 = 0.$$

So this is the surface $x^2 + y^2 + z^2 = b_1^2 + b_2^2 + b_3^2 = \|\mathbf{b}\|^2$, which is a sphere centered at the origin with radius $\|\mathbf{b}\|$. So all points with that magnitude lie on the sphere; in particular, $\mathbf{b}$ and $-\mathbf{b}$ do. Therefore the surface is as claimed. Next, let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$; then

$$\begin{aligned} (\mathbf{r} - \mathbf{r}_0 - \mathbf{b}) \cdot (\mathbf{r} - \mathbf{r}_0 + \mathbf{b}) &= \langle (x - x_0) - b_1, (y - y_0) - b_2, (z - z_0) - b_3 \rangle \cdot \langle (x - x_0) + b_1, (y - y_0) + b_2, (z - z_0) + b_3 \rangle \\ &= (x - x_0)^2 - b_1^2 + (y - y_0)^2 - b_2^2 + (z - z_0)^2 - b_3^2 = 0. \end{aligned}$$

So this is the surface $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = b_1^2 + b_2^2 + b_3^2 = \|\mathbf{b}\|^2$, which is a sphere with center $(x_0, y_0, z_0) = \mathbf{r}_0$ and radius $\|\mathbf{b}\|$.

10.5: The Cross Product

Concepts and Vocabulary

1. True. See the discussion in objective 3.
2. False. The cross product of two vectors is zero if and only if the vectors are *parallel*. See item (9) in objective 3.
3. **(b)** is correct; $\mathbf{v} \times \mathbf{w}$ is always orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. See item (6) in objective 3.
4. False. See item (2) in objective 2.
5. **(d)** is correct; see item (9) in objective 3.
6. True, since $\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w})$. See item (2) in objective 2.
7. **(c)** is correct; see item (8) in objective 3.
8. False; in fact $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta$. See item (7) in objective 3.

Skill Building

$$9. \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 4 \cdot 1 = 6 - 4 = \boxed{2}.$$

$$10. \begin{vmatrix} -2 & 4 \\ 2 & -1 \end{vmatrix} = -2 \cdot (-1) - 2 \cdot 4 = 2 - 8 = \boxed{-6}.$$

11.

$$\begin{vmatrix} 1 & -2 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1(2 \cdot 4 - 1 \cdot 1) + 2(1 \cdot 4 - 1 \cdot 2) + 2(1 \cdot 1 - 2 \cdot 2) = \boxed{5}.$$

12.

$$\begin{vmatrix} 7 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 7 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 7(2 \cdot 3 - 3 \cdot 1) + 1(0 \cdot 1 - 2 \cdot 0) = \boxed{21}.$$

13. **(a)**

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= (1 \cdot 1 - (-1) \cdot (-1))\mathbf{i} - (2 \cdot 1 - (-1) \cdot 1)\mathbf{j} + (2 \cdot (-1) - 1 \cdot 1)\mathbf{k} \\ &= \boxed{-3\mathbf{j} - 3\mathbf{k}}. \end{aligned}$$

(b) As a check,

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (-3\mathbf{j} - 3\mathbf{k}) = 2 \cdot 0 + 1 \cdot (-3) - 1 \cdot (-3) = 0 \\ \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (-3\mathbf{j} - 3\mathbf{k}) = 1 \cdot 0 - 1 \cdot (-3) + 1 \cdot (-3) = 0. \end{aligned}$$

14. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} \\
&= (-1 \cdot 1 - 2 \cdot 1)\mathbf{i} - (4 \cdot 1 - 2 \cdot 2)\mathbf{j} + (4 \cdot 1 - (-1) \cdot 2)\mathbf{k} \\
&= \boxed{-3\mathbf{i} + 6\mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (-3\mathbf{i} + 6\mathbf{k}) = 4 \cdot (-3) - 1 \cdot 0 + 2 \cdot 6 = 0 \\
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (-3\mathbf{i} + 6\mathbf{k}) = 2 \cdot (-3) + 1 \cdot 0 + 1 \cdot 6 = 0.
\end{aligned}$$

15. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
&= (1 \cdot (-1) - 0 \cdot 0)\mathbf{i} - (1 \cdot (-1) - 0 \cdot 1)\mathbf{j} + (1 \cdot 0 - 1 \cdot 1)\mathbf{k} \\
&= \boxed{-\mathbf{i} + \mathbf{j} - \mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} + \mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j} - \mathbf{k}) = 1 \cdot (-1) + 1 \cdot 0 + 0 \cdot (-1) = 0 \\
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} - \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} - \mathbf{k}) = 1 \cdot (-1) + 0 \cdot 1 - 1 \cdot (-1) = 0.
\end{aligned}$$

16. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\
&= (1 \cdot 0 - (-1) \cdot (-1))\mathbf{i} - (0 \cdot 0 - (-1) \cdot 1)\mathbf{j} + (0 \cdot (-1) - 1 \cdot 1)\mathbf{k} \\
&= \boxed{-\mathbf{i} - \mathbf{j} - \mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} - \mathbf{j} - \mathbf{k}) = 0 \cdot (-1) + 1 \cdot (-1) - 1 \cdot (-1) = 0 \\
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} - \mathbf{j}) \cdot (-\mathbf{i} - \mathbf{j} - \mathbf{k}) = 1 \cdot (-1) - 1 \cdot (-1) + 0 \cdot (-1) = 0.
\end{aligned}$$

17. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
&= (-2 \cdot 0 - 1 \cdot 1)\mathbf{i} - (3 \cdot 0 - 1 \cdot 1)\mathbf{j} + (3 \cdot 1 - (-2) \cdot 1)\mathbf{k} \\
&= \boxed{-\mathbf{i} + \mathbf{j} + 5\mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 3 \cdot (-1) - 2 \cdot 1 + 1 \cdot 5 = 0 \\ \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} + \mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = 1 \cdot (-1) + 1 \cdot 1 + 0 \cdot 5 = 0.\end{aligned}$$

18. (a)

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 1 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 0 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (-1 \cdot (-3) - 0 \cdot 1) \mathbf{i} - (2 \cdot (-3) - 0 \cdot 1) \mathbf{j} + (2 \cdot 1 - (-1) \cdot 1) \mathbf{k} \\ &= \boxed{3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}}.\end{aligned}$$

(b) As a check,

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (2\mathbf{i} - \mathbf{j}) \cdot (3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) = 2 \cdot 3 - 1 \cdot 6 + 0 \cdot 3 = 0 \\ \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) = 1 \cdot 3 + 1 \cdot 6 - 3 \cdot 3 = 0.\end{aligned}$$

19. (a)

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 8 & 3 \\ 7 & 2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 8 & 3 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 7 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 8 \\ 7 & 2 \end{vmatrix} \mathbf{k} \\ &= (8 \cdot 0 - 3 \cdot 2) \mathbf{i} - (-1 \cdot 0 - 3 \cdot 7) \mathbf{j} + (-1 \cdot 2 - 8 \cdot 7) \mathbf{k} \\ &= \boxed{-6\mathbf{i} + 21\mathbf{j} - 58\mathbf{k}}.\end{aligned}$$

(b) As a check,

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (-\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}) \cdot (-6\mathbf{i} + 21\mathbf{j} - 58\mathbf{k}) = -1 \cdot (-6) + 8 \cdot 21 + 3 \cdot (-58) = 0 \\ \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (7\mathbf{i} + 2\mathbf{j}) \cdot (-6\mathbf{i} + 21\mathbf{j} - 58\mathbf{k}) = 7 \cdot (-6) + 2 \cdot 21 = 0.\end{aligned}$$

20. (a)

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot 1 - (-1) \cdot 1) \mathbf{i} - (0 \cdot 1 - (-1) \cdot (-3)) \mathbf{j} + (0 \cdot 1 - 2 \cdot (-3)) \mathbf{k} \\ &= \boxed{3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}.\end{aligned}$$

(b) As a check,

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (2\mathbf{j} - \mathbf{k}) \cdot (3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = 0 \cdot 3 + 2 \cdot 3 - 1 \cdot 6 = 0 \\ \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (-3\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = -3 \cdot 3 + 1 \cdot 3 + 1 \cdot 6 = 0.\end{aligned}$$

21. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ -1 & 1 & -4 \end{vmatrix} \\
&= \begin{vmatrix} 3 & -4 \\ 1 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -4 \\ -1 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k} \\
&= (3 \cdot (-4) - (-4) \cdot 1) \mathbf{i} - (2 \cdot (-4) - (-4) \cdot (-1)) \mathbf{j} + (2 \cdot 1 - 3 \cdot (-1)) \mathbf{k} \\
&= \boxed{-8\mathbf{i} + 12\mathbf{j} + 5\mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot (-8\mathbf{i} + 12\mathbf{j} + 5\mathbf{k}) = 2 \cdot (-8) + 3 \cdot 12 - 4 \cdot 5 = 0 \\
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= (-\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \cdot (-8\mathbf{i} + 12\mathbf{j} + 5\mathbf{k}) = -1 \cdot (-8) + 1 \cdot 12 - 4 \cdot 5 = 0.
\end{aligned}$$

22. (a)

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \end{vmatrix} \\
&= \begin{vmatrix} -\sin \theta & 0 \\ \cos \theta & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \cos \theta & 0 \\ \sin \theta & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \mathbf{k} \\
&= (-\sin \theta \cdot 0 - 0 \cdot \cos \theta) \mathbf{i} - (\cos \theta \cdot 0 - 0 \cdot \sin \theta) \mathbf{j} + (\cos \theta \cos \theta - (-\sin \theta) \sin \theta) \mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + (\cos^2 \theta + \sin^2 \theta) \mathbf{k} \\
&= \boxed{\mathbf{k}}.
\end{aligned}$$

(b) As a check,

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= ((\cos \theta) \mathbf{i} - (\sin \theta) \mathbf{j}) \cdot \mathbf{k} = \cos \theta \cdot 0 - \sin \theta \cdot 0 + 0 \cdot 1 = 0 \\
\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= ((\sin \theta) \mathbf{i} + (\cos \theta) \mathbf{j}) \cdot \mathbf{k} = \sin \theta \cdot 0 + \cos \theta \cdot 0 + 0 \cdot 1 = 0.
\end{aligned}$$

23. Since $\mathbf{u}$ is parallel to itself, $\mathbf{u} \times \mathbf{u} = \boxed{\mathbf{0}}$ by property (9) in objective 3.24. Since $\mathbf{w}$ is parallel to itself, $\mathbf{w} \times \mathbf{w} = \boxed{\mathbf{0}}$ by property (9) in objective 3.

25.

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -4 \\ 1 & 1 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -4 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -4 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
&= (1 \cdot (-3) - (-4) \cdot 1) \mathbf{i} - (-3 \cdot (-3) - (-4) \cdot 1) \mathbf{j} + (-3 \cdot 1 - 1 \cdot 1) \mathbf{k} \\
&= \boxed{\mathbf{i} - 13\mathbf{j} - 4\mathbf{k}}.
\end{aligned}$$

26.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ -3 & 1 & -4 \end{vmatrix} \\
&= \begin{vmatrix} 3 & -4 \\ 1 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -4 \\ -3 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} \mathbf{k} \\
&= (3 \cdot (-4) - (-4) \cdot 1) \mathbf{i} - (2 \cdot (-4) - (-4) \cdot (-3)) \mathbf{j} + (2 \cdot 1 - 3 \cdot (-3)) \mathbf{k} \\
&= \boxed{-8\mathbf{i} + 20\mathbf{j} + 11\mathbf{k}}.
\end{aligned}$$

27. By property (3) in objective 2, $(3\mathbf{v}) \times \mathbf{w} = 3(\mathbf{v} \times \mathbf{w})$. Then using Problem 25, we get

$$(3\mathbf{v}) \times \mathbf{w} = 3(\mathbf{v} \times \mathbf{w}) = 3(\mathbf{i} - 13\mathbf{j} - 4\mathbf{k}) = \boxed{3\mathbf{i} - 39\mathbf{j} - 12\mathbf{k}}.$$

28. By property (3) in objective 2, $\mathbf{u} \times (-\mathbf{v}) = -1(\mathbf{u} \times \mathbf{v})$. Then using Problem 26, we get

$$\mathbf{u} \times (-\mathbf{v}) = -1(\mathbf{u} \times \mathbf{v}) = -1(-8\mathbf{i} + 20\mathbf{j} + 11\mathbf{k}) = \boxed{8\mathbf{i} - 20\mathbf{j} - 11\mathbf{k}}.$$

29. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. From Problem 25, this vector is $\boxed{\mathbf{i} - 13\mathbf{j} - 4\mathbf{k}}$.

30. We must compute $\mathbf{w} \times \mathbf{u}$:

$$\begin{aligned} \mathbf{w} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 2 & 3 & -4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot (-4) - (-3) \cdot 3)\mathbf{i} - (1 \cdot (-4) - (-3) \cdot 2)\mathbf{j} + (1 \cdot 3 - 1 \cdot 2)\mathbf{k} \\ &= \boxed{5\mathbf{i} - 2\mathbf{j} + \mathbf{k}}. \end{aligned}$$

The vector $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is orthogonal to both $\mathbf{w}$ and $\mathbf{u}$.

31. We must compute $\mathbf{w} \times (\mathbf{i} + \mathbf{j} - \mathbf{k})$:

$$\begin{aligned} \mathbf{w} \times (\mathbf{i} + \mathbf{j} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot (-1) - (-3) \cdot 1)\mathbf{i} - (1 \cdot (-1) - (-3) \cdot 1)\mathbf{j} + (1 \cdot 1 - 1 \cdot 1)\mathbf{k} \\ &= \boxed{2\mathbf{i} - 2\mathbf{j}}. \end{aligned}$$

The vector $2\mathbf{i} - 2\mathbf{j}$ is orthogonal to both $\mathbf{w}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

32. We must compute $\mathbf{u} \times \mathbf{k}$:

$$\begin{aligned} \mathbf{u} \times \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -4 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} \mathbf{k} \\ &= (3 \cdot 1 - (-4) \cdot 0)\mathbf{i} - (2 \cdot 1 - (-4) \cdot 0)\mathbf{j} + (2 \cdot 0 - 3 \cdot 0)\mathbf{k} \\ &= \boxed{3\mathbf{i} - 2\mathbf{j}}. \end{aligned}$$

The vector $3\mathbf{i} - 2\mathbf{j}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{k}$.

33. Any vector normal to both $\mathbf{u}$ and $\mathbf{v}$ will be normal to the plane containing those vectors, so $\mathbf{u} \times \mathbf{v}$ is such a vector. From Problem 26, $\mathbf{u} \times \mathbf{v} = -8\mathbf{i} + 20\mathbf{j} + 11\mathbf{k}$. Since we want a unit vector, we must normalize this vector:

$$\|\mathbf{u} \times \mathbf{v}\| = \|-8\mathbf{i} + 20\mathbf{j} + 11\mathbf{k}\| = \sqrt{8^2 + 20^2 + 11^2} = \sqrt{585} = 3\sqrt{65}.$$

Therefore a unit vector normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{1}{3\sqrt{65}}(\mathbf{u} \times \mathbf{v}) = \frac{\sqrt{65}}{195}(-8\mathbf{i} + 20\mathbf{j} + 11\mathbf{k}) = \boxed{-\frac{8}{3\sqrt{65}}\mathbf{i} + \frac{20}{3\sqrt{65}}\mathbf{j} + \frac{11}{3\sqrt{65}}\mathbf{k}}.$$

The vector opposite to this vector, which is $\boxed{\frac{8}{3\sqrt{65}}\mathbf{i} - \frac{20}{3\sqrt{65}}\mathbf{j} - \frac{11}{3\sqrt{65}}\mathbf{k}}$, is also a unit vector normal to the plane.

34. Any vector normal to both $\mathbf{u}$ and $\mathbf{w}$ will be normal to the plane containing those vectors, so $\mathbf{w} \times \mathbf{u}$ is such a vector. From Problem 30, $\mathbf{w} \times \mathbf{u} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Since we want a unit vector, we must normalize this vector:

$$\|\mathbf{w} \times \mathbf{u}\| = \|5\mathbf{i} - 2\mathbf{j} + \mathbf{k}\| = \sqrt{5^2 + 2^2 + 1^2} = \sqrt{30}.$$

Therefore a unit vector normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{1}{\sqrt{30}}(\mathbf{w} \times \mathbf{u}) = \frac{\sqrt{30}}{30}(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \boxed{\frac{\sqrt{30}}{6}\mathbf{i} - \frac{\sqrt{30}}{15}\mathbf{j} + \frac{\sqrt{30}}{30}\mathbf{k}}.$$

The vector opposite to this vector, which is $\boxed{-\frac{\sqrt{30}}{6}\mathbf{i} + \frac{\sqrt{30}}{15}\mathbf{j} - \frac{\sqrt{30}}{30}\mathbf{k}}$, is also a unit vector normal to the plane.

35. Any vector normal to both $2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ will be normal to the plane containing those vectors, so $(2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) \times (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ is such a vector:

$$\begin{aligned} (2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) \times (4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -6 & -3 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -6 \\ 4 & 3 \end{vmatrix} \mathbf{k} \\ &= (-6 \cdot (-1) - (-3) \cdot 3)\mathbf{i} - (2 \cdot (-1) - (-3) \cdot 4)\mathbf{j} + (2 \cdot 3 - (-6) \cdot 4)\mathbf{k} \\ &= 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}. \end{aligned}$$

Since we want a unit vector, we must normalize the cross product:

$$\|15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}\| = \sqrt{15^2 + 10^2 + 30^2} = 35.$$

Therefore a unit vector normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{1}{35}(15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}) = \boxed{\frac{3}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}}.$$

The vector opposite to this vector, which is $\boxed{-\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}}$, is also a unit vector normal to the plane.

36. Any vector normal to both $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ will be normal to the plane containing those vectors, so $(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ is such a vector:

$$\begin{aligned} (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot (-1) - (-2) \cdot 2)\mathbf{i} - (1 \cdot (-1) - (-2) \cdot 3)\mathbf{j} + (1 \cdot 2 - 1 \cdot 3)\mathbf{k} \\ &= 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}. \end{aligned}$$

Since we want a unit vector, we must normalize the cross product:

$$\|3\mathbf{i} - 5\mathbf{j} - \mathbf{k}\| = \sqrt{3^2 + 5^2 + 1^2} = \sqrt{35}.$$

Therefore a unit vector normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{1}{\sqrt{35}}(3\mathbf{i} - 5\mathbf{j} - \mathbf{k}) = \boxed{\frac{3\sqrt{35}}{35}\mathbf{i} - \frac{\sqrt{35}}{7}\mathbf{j} - \frac{\sqrt{35}}{35}\mathbf{k}}.$$

The vector opposite to this vector, which is $\boxed{-\frac{3\sqrt{35}}{35}\mathbf{i} + \frac{\sqrt{35}}{7}\mathbf{j} + \frac{\sqrt{35}}{35}\mathbf{k}}$, is also a unit vector normal to the plane.

37. Two sides are

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} = \langle 2, 1, 1 \rangle - \langle 1, -3, 7 \rangle = \langle 1, 4, -6 \rangle \\ \mathbf{w} &= \overrightarrow{PR} = \langle 6, -1, 2 \rangle - \langle 1, -3, 7 \rangle = \langle 5, 2, -5 \rangle.\end{aligned}$$

The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -6 \\ 5 & 2 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 4 & -6 \\ 2 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -6 \\ 5 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} \mathbf{k} \\ &= (4 \cdot (-5) - (-6) \cdot 2)\mathbf{i} - (1 \cdot (-5) - (-6) \cdot 5)\mathbf{j} + (1 \cdot 2 - 4 \cdot 5)\mathbf{k} \\ &= -8\mathbf{i} - 25\mathbf{j} - 18\mathbf{k}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|-8\mathbf{i} - 25\mathbf{j} - 18\mathbf{k}\| = \sqrt{8^2 + 25^2 + 18^2} = \boxed{\sqrt{1013}}.$$

38. Two sides are

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} = \langle 2, 0, -4 \rangle - \langle 0, 1, 1 \rangle = \langle 2, -1, -5 \rangle \\ \mathbf{w} &= \overrightarrow{PR} = \langle -3, -2, 1 \rangle - \langle 0, 1, 1 \rangle = \langle -3, -3, 0 \rangle.\end{aligned}$$

The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -5 \\ -3 & -3 & 0 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -5 \\ -3 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -5 \\ -3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -3 & -3 \end{vmatrix} \mathbf{k} \\ &= (-1 \cdot 0 - (-5) \cdot (-3))\mathbf{i} - (2 \cdot 0 - (-5) \cdot (-3))\mathbf{j} + (2 \cdot (-3) - (-1) \cdot (-3))\mathbf{k} \\ &= -15\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|-15\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}\| = \sqrt{15^2 + 15^2 + 9^2} = \sqrt{531} = \boxed{3\sqrt{59}}.$$

39. Two sides are

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} = \langle 2, 1, -7 \rangle - \langle -2, 1, 6 \rangle = \langle 4, 0, -13 \rangle \\ \mathbf{w} &= \overrightarrow{PR} = \langle 4, 1, 1 \rangle - \langle -2, 1, 6 \rangle = \langle 6, 0, -5 \rangle.\end{aligned}$$

The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -13 \\ 6 & 0 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -13 \\ 0 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -13 \\ 6 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix} \mathbf{k} \\ &= (0 \cdot (-5) - (-13) \cdot 0) \mathbf{i} - (4 \cdot (-5) - (-13) \cdot 6) \mathbf{j} + (4 \cdot 0 - 0 \cdot 6) \mathbf{k} \\ &= -58\mathbf{j}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|58\mathbf{j}\| = \sqrt{58^2} = \boxed{58}.$$

40. Two sides are

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} = \langle 2, -5, 3 \rangle - \langle 0, 0, 3 \rangle = \langle 2, -5, 0 \rangle \\ \mathbf{w} &= \overrightarrow{PR} = \langle 1, 1, -2 \rangle - \langle 0, 0, 3 \rangle = \langle 1, 1, -5 \rangle.\end{aligned}$$

The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 0 \\ 1 & 1 & -5 \end{vmatrix} \\ &= \begin{vmatrix} -5 & 0 \\ 1 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -5 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (-5 \cdot (-5) - 0 \cdot 1) \mathbf{i} - (2 \cdot (-5) - 0 \cdot 1) \mathbf{j} + (2 \cdot 1 - (-5) \cdot 1) \mathbf{k} \\ &= 25\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|25\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}\| = \sqrt{25^2 + 10^2 + 7^2} = \sqrt{774} = \boxed{3\sqrt{86}}.$$

41. Two sides are

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} = \langle 5, -3, 0 \rangle - \langle 1, 1, -6 \rangle = \langle 4, -4, 6 \rangle \\ \mathbf{w} &= \overrightarrow{PR} = \langle -2, 4, 1 \rangle - \langle 1, 1, -6 \rangle = \langle -3, 3, 7 \rangle.\end{aligned}$$

The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 6 \\ -3 & 3 & 7 \end{vmatrix} \\ &= \begin{vmatrix} -4 & 6 \\ 3 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 6 \\ -3 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -4 \\ -3 & 3 \end{vmatrix} \mathbf{k} \\ &= (-4 \cdot 7 - 6 \cdot 3) \mathbf{i} - (4 \cdot 7 - 6 \cdot (-3)) \mathbf{j} + (4 \cdot 3 - (-4) \cdot (-3)) \mathbf{k} \\ &= -46\mathbf{i} - 46\mathbf{j}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\| -46\mathbf{i} - 46\mathbf{j} \| = \sqrt{46^2 + 46^2} = \boxed{46\sqrt{2}}.$$

42. Two sides are

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1, 1, -5 \rangle - \langle -4, 6, 3 \rangle = \langle 5, -5, -8 \rangle$$

$$\mathbf{w} = \overrightarrow{PR} = \langle 2, 2, 2 \rangle - \langle -4, 6, 3 \rangle = \langle 6, -4, -1 \rangle.$$

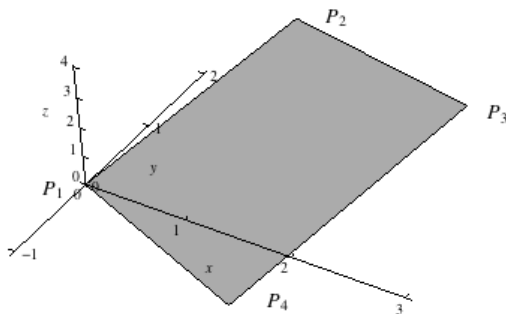
The area of the parallelogram with one corner at P and sides PQ and PR is then $\|\mathbf{v} \times \mathbf{w}\|$. Computing $\mathbf{v} \times \mathbf{w}$ gives

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -5 & -8 \\ 6 & -4 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -5 & -8 \\ -4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 5 & -8 \\ 6 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 5 & -5 \\ 6 & -4 \end{vmatrix} \mathbf{k} \\ &= (-5 \cdot (-1) - (-8) \cdot (-4))\mathbf{i} - (5 \cdot (-1) - (-8) \cdot 6)\mathbf{j} + (5 \cdot (-4) - (-5) \cdot 6)\mathbf{k} \\ &= -27\mathbf{i} - 43\mathbf{j} + 10\mathbf{k}. \end{aligned}$$

Therefore the area of the parallelogram is

$$\| -27\mathbf{i} - 43\mathbf{j} + 10\mathbf{k} \| = \sqrt{27^2 + 43^2 + 10^2} = \boxed{\sqrt{2678}}.$$

43. First plot the points to determine which pairs of points form sides:



We will use

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle 1, 2, 3 \rangle - \langle 0, 0, 0 \rangle = \langle 1, 2, 3 \rangle$$

$$\mathbf{w} = \overrightarrow{P_1P_4} = \langle 2, -1, 1 \rangle - \langle 0, 0, 0 \rangle = \langle 2, -1, 1 \rangle$$

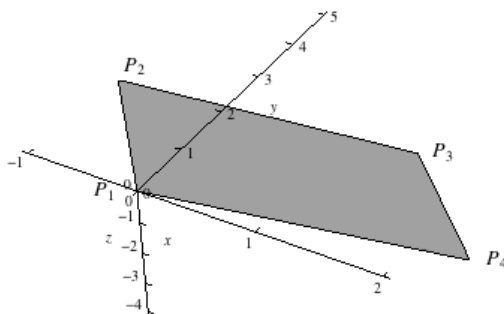
as the two sides. Then the area of the parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$.

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot 1 - 3 \cdot (-1))\mathbf{i} - (1 \cdot 1 - 3 \cdot 2)\mathbf{j} + (1 \cdot (-1) - 2 \cdot 2)\mathbf{k} \\ &= 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}. \end{aligned}$$

Therefore the area of the parallelogram is

$$\|5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}\| = \sqrt{5^2 + 5^2 + 5^2} = \boxed{5\sqrt{3}}.$$

44. First plot the points to determine which pairs of points form sides:



We will use

$$\begin{aligned}\mathbf{v} &= \overrightarrow{P_1P_2} = \langle -1, 2, 0 \rangle - \langle 0, 0, 0 \rangle = \langle -1, 2, 0 \rangle \\ \mathbf{w} &= \overrightarrow{P_1P_4} = \langle 2, 3, -4 \rangle - \langle 0, 0, 0 \rangle = \langle 2, 3, -4 \rangle\end{aligned}$$

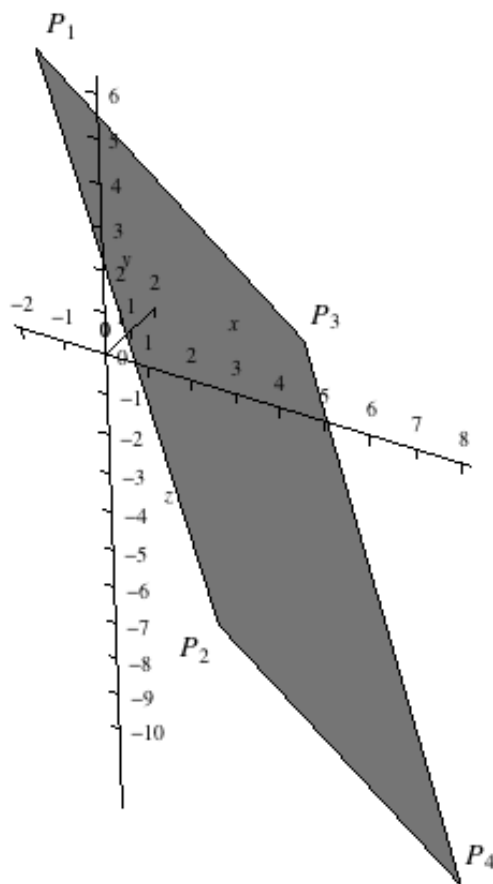
as the two sides. Then the area of the parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$.

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 2 & 3 & -4 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ 2 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot (-4) - 0 \cdot 3)\mathbf{i} - (-1 \cdot (-4) - 0 \cdot 2)\mathbf{j} + (-1 \cdot 3 - 2 \cdot 2)\mathbf{k} \\ &= -8\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|-8\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}\| = \sqrt{8^2 + 4^2 + 7^2} = \boxed{\sqrt{129}}.$$

45. First plot the points to determine which pairs of points form sides:



We will use

$$\begin{aligned}\mathbf{v} &= \overrightarrow{P_1P_2} = \langle 2, 1, -7 \rangle - \langle -2, 1, 6 \rangle = \langle 4, 0, -13 \rangle \\ \mathbf{w} &= \overrightarrow{P_1P_3} = \langle 4, 1, 1 \rangle - \langle -2, 1, 6 \rangle = \langle 6, 0, -5 \rangle\end{aligned}$$

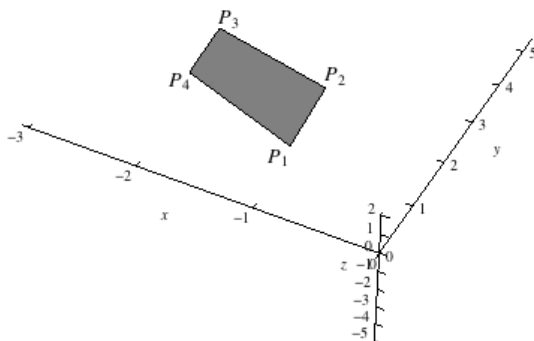
as the two sides. Then the area of the parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$.

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -13 \\ 6 & 0 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -13 \\ 0 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -13 \\ 6 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix} \mathbf{k} \\ &= (0 \cdot (-5) - (-13) \cdot 0) \mathbf{i} - (4 \cdot (-5) - (-13) \cdot 6) \mathbf{j} + (4 \cdot 0 - 0 \cdot 6) \mathbf{k} \\ &= -58\mathbf{j}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|-58\mathbf{j}\| = \sqrt{58^2} = \boxed{58}.$$

46. First plot the points to determine which pairs of points form sides:



We will use

$$\begin{aligned}\mathbf{v} &= \overrightarrow{P_1P_2} = \langle -1, 2, 2 \rangle - \langle -1, 1, 1 \rangle = \langle 0, 1, 1 \rangle \\ \mathbf{w} &= \overrightarrow{P_1P_4} = \langle -3, 4, -5 \rangle - \langle -1, 1, 1 \rangle = \langle -2, 3, -6 \rangle\end{aligned}$$

as the two sides. Then the area of the parallelogram is $\|\mathbf{v} \times \mathbf{w}\|$.

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 3 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot (-6) - 1 \cdot 3) \mathbf{i} - (0 \cdot (-6) - 1 \cdot (-2)) \mathbf{j} + (0 \cdot 3 - 1 \cdot (-2)) \mathbf{k} \\ &= -9\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

Therefore the area of the parallelogram is

$$\|-9\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\| = \sqrt{9^2 + 2^2 + 2^2} = \boxed{\sqrt{89}}.$$

Applications and Extensions

47. Since $\boldsymbol{\omega}$ points in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$, we have $\boldsymbol{\omega} = k\mathbf{i} + k\mathbf{j} + k\mathbf{k}$. Then

$$30 = \omega = \|\boldsymbol{\omega}\| = \sqrt{k^2 + k^2 + k^2} = k\sqrt{3},$$

so that $k = \frac{30}{\sqrt{3}} = 10\sqrt{3}$. Therefore $\boldsymbol{\omega} = 10\sqrt{3}\mathbf{i} + 10\sqrt{3}\mathbf{j} + 10\sqrt{3}\mathbf{k}$. The position of the object is $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, so the velocity of the object is

$$\begin{aligned}\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ &= (10\sqrt{3}\mathbf{i} + 10\sqrt{3}\mathbf{j} + 10\sqrt{3}\mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10\sqrt{3} & 10\sqrt{3} & 10\sqrt{3} \\ -1 & 2 & 3 \end{vmatrix} \\ &= 10\sqrt{3}\mathbf{i} - 40\sqrt{3}\mathbf{j} + 30\sqrt{3}\mathbf{k}.\end{aligned}$$

The object's speed is then

$$\|\mathbf{v}\| = \sqrt{(10\sqrt{3})^2 + (40\sqrt{3})^2 + (30\sqrt{3})^2} = \boxed{10\sqrt{78} \text{ m/s}}.$$

48. ω points in the direction of $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; a unit vector in this direction is

$$\frac{1}{\sqrt{3^2 + 1^2 + 2^2}}(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{3}{\sqrt{14}}\mathbf{i} + \frac{1}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k}.$$

Since the angular speed ω is constant, we have

$$\boldsymbol{\omega} = \omega \left(\frac{3}{\sqrt{14}}\mathbf{i} + \frac{1}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k} \right).$$

(a) The position of the object is $4\mathbf{i} + 4\mathbf{j}$, so the velocity is

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ &= \omega \left(\frac{3}{\sqrt{14}}\mathbf{i} + \frac{1}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k} \right) \times (4\mathbf{i} + 4\mathbf{j}) \\ &= \omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} \\ 4 & 4 & 0 \end{vmatrix} \\ &= \left(\frac{8}{\sqrt{14}}\mathbf{i} - \frac{8}{\sqrt{14}}\mathbf{j} + \frac{8}{\sqrt{14}}\mathbf{k} \right) \omega. \end{aligned}$$

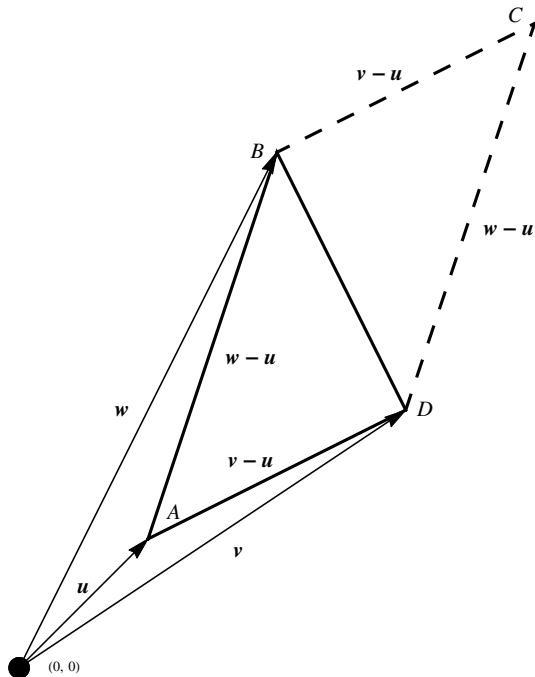
The object's speed is then

$$\|\mathbf{v}\| = \omega \sqrt{\left(\frac{8}{\sqrt{14}}\right)^2 + \left(\frac{8}{\sqrt{14}}\right)^2 + \left(\frac{8}{\sqrt{14}}\right)^2} = 8\omega \sqrt{\frac{3}{14}} = \frac{8\sqrt{42}}{14} \omega = \boxed{\frac{4\sqrt{42}}{7} \omega}.$$

(b) If $\|\mathbf{v}\| = 8\sqrt{14}$, then

$$8\sqrt{14} = \frac{4\sqrt{42}}{7} \omega, \quad \text{so that} \quad \omega = \frac{56\sqrt{14}}{4\sqrt{42}} = \frac{14}{\sqrt{3}} = \boxed{\frac{14\sqrt{3}}{3} \text{ m/s}}.$$

49. Consider the diagram below:



The triangle whose area we wish to determine is $\triangle ABD$. Draw two additional sides, BC and DC . Then $ABCD$ is a parallelogram: it has two pairs of equal sides, and then $\triangle ABD$ and $\triangle CDB$ have all three pairs of sides equal, so they are congruent. It follows that $\angle BAD = \angle DCB$ and therefore $\angle CBA = \angle ADC$. So $ABCD$ is a parallelogram with sides $\mathbf{v} - \mathbf{u}$ and $\mathbf{w} - \mathbf{u}$, and therefore its area is $\|(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})\|$. Since the two triangles are congruent, each has half the area of the parallelogram, so the area of the triangle in question is

$$\frac{1}{2} \|(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})\|.$$

50. Using Problem 49, the area of the triangle is

$$\begin{aligned} A &= \frac{1}{2} \|(\langle 2, 3, -2 \rangle - \langle 0, 0, 0 \rangle) \times (\langle -1, 1, 4 \rangle - \langle 0, 0, 0 \rangle)\| \\ &= \frac{1}{2} \|\langle 2, 3, -2 \rangle \times \langle -1, 1, 4 \rangle\| \\ &= \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ -1 & 1 & 4 \end{vmatrix} \right\| \\ &= \frac{1}{2} \left\| \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k} \right\| \\ &= \frac{1}{2} \|14\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}\| \\ &= \frac{1}{2} \sqrt{14^2 + 6^2 + 5^2} \\ &= \boxed{\frac{1}{2} \sqrt{257}}. \end{aligned}$$

51. (a) Answers will vary.

(b) We have

$$\mathbf{F}_1 = 250\mathbf{j}, \quad \mathbf{F}_2 = 125\sqrt{2}\mathbf{i} + 125\sqrt{2}\mathbf{j}, \quad \mathbf{F}_3 = 250\mathbf{i}.$$

The point Q is $(0.25, 0, 0)$ since the wrench is 0.25 m long. Let $\boldsymbol{\tau}_i$ be the torque due to $\mathbf{F}_i$. Then

$$\boldsymbol{\tau}_1 = (0.25, 0, 0) \times \mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.25 & 0 & 0 \\ 0 & 250 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 250 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0.25 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0.25 & 0 \\ 0 & 250 \end{vmatrix} \mathbf{k} = \boxed{62.5\mathbf{k} \text{ N}\cdot\text{m}}$$

$$\boldsymbol{\tau}_2 = (0.25, 0, 0) \times \mathbf{F}_2$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.25 & 0 & 0 \\ 125\sqrt{2} & 125\sqrt{2} & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 \\ 125\sqrt{2} & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0.25 & 0 \\ 125\sqrt{2} & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0.25 & 0 \\ 125\sqrt{2} & 125\sqrt{2} \end{vmatrix} \mathbf{k} \\ &= \boxed{31.25\sqrt{2}\mathbf{k} \approx 44.19 \text{ N}\cdot\text{m}} \end{aligned}$$

$$\boldsymbol{\tau}_3 = (0.25, 0, 0) \times \mathbf{F}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.25 & 0 & 0 \\ 250 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0.25 & 0 \\ 250 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0.25 & 0 \\ 250 & 0 \end{vmatrix} \mathbf{k} = \boxed{\mathbf{0} \text{ N}\cdot\text{m}}.$$

(We would expect $\boldsymbol{\tau}_3$ to be zero, since the force vector $\mathbf{F}_3$ and $\mathbf{r}$ are parallel, so no turning force is imparted on the wrench.)

(c) Answers will vary.

52. The force vector $\mathbf{F}$ is $-75\mathbf{j}$, since the force of gravity acts downwards. The radial vector through which the force acts has magnitude 0.65 m, so that vector is

$$\mathbf{r} = 0.65(\cos 40^\circ \mathbf{i} + \sin 40^\circ \mathbf{j}).$$

Then the torque is

$$\begin{aligned} \boldsymbol{\tau} &= 0.65(\cos 40^\circ \mathbf{i} + \sin 40^\circ \mathbf{j}) \times (-75\mathbf{j}) \\ &= -75 \cdot 0.65 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos 40^\circ & \sin 40^\circ & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= -48.75 \left(\begin{vmatrix} \sin 40^\circ & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \cos 40^\circ & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \cos 40^\circ & \sin 40^\circ \\ 0 & 1 \end{vmatrix} \mathbf{k} \right) \\ &= -48.75 \cdot \cos 40^\circ \boxed{\approx -37.3447 \text{ N}\cdot\text{m}}. \end{aligned}$$

53. We first compute $\mathbf{v} \times \mathbf{B}$:

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.0050 & 0.0035 & 0 \\ 0.47 & 0.50 & -0.25 \end{vmatrix} = -0.000875\mathbf{i} + 0.00125\mathbf{j} + 0.000855\mathbf{k}.$$

Then the Lorentz force is

$$\begin{aligned} \mathbf{F} &= q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \\ &= 2.5[(0.0064\mathbf{i} - 0.0075\mathbf{j} - 0.0023\mathbf{k}) + (-0.000875\mathbf{i} + 0.00125\mathbf{j} + 0.000855\mathbf{k})] \\ &= 2.5(0.005525\mathbf{i} - 0.00625\mathbf{j} - 0.001445\mathbf{k}) \\ &= \boxed{0.0138125\mathbf{i} - 0.015625\mathbf{j} - 0.0036125\mathbf{k} \text{ N}}. \end{aligned}$$

54. From the text, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$, where θ is the angle between the vectors. If the vectors are orthogonal, $\theta = 90^\circ$, so that $\sin \theta = 1$, and the formula above becomes $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\|$.
55. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then by Problem 54, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\|$. But $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, so that $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. It follows that $\|\mathbf{u} \times \mathbf{v}\| = 1$ as well, so that $\mathbf{u} \times \mathbf{v}$ is a unit vector.
56. Let $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned}
 (a\mathbf{v}) \times \mathbf{w} &= (av_1\mathbf{i} + av_2\mathbf{j} + av_3\mathbf{k}) \times (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ av_1 & av_2 & av_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
 &= \begin{vmatrix} av_2 & av_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} av_1 & av_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} av_1 & av_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\
 &= (av_2w_3 - av_3w_2)\mathbf{i} - (av_1w_3 + av_3w_1)\mathbf{j} + (av_1w_2 + av_2w_1)\mathbf{k} \\
 &= a((v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 + v_3w_1)\mathbf{j} + (v_1w_2 + v_2w_1)\mathbf{k}) \\
 &= a\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k}\right) \\
 &= a(\mathbf{v} \times \mathbf{w}).
 \end{aligned}$$

For the second part, we show that $a(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (a\mathbf{w})$ using property (2) and the first equality:

$$\mathbf{v} \times (a\mathbf{w}) = -((a\mathbf{w}) \times \mathbf{v}) = -(a(\mathbf{w} \times \mathbf{v})) = a(-(\mathbf{w} \times \mathbf{v})) = a(\mathbf{v} \times \mathbf{w}).$$

57. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned}
 \mathbf{v} \times (\mathbf{w} + \mathbf{u}) &= (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \times ((w_1 + u_1)\mathbf{i} + (w_2 + u_2)\mathbf{j} + (w_3 + u_3)\mathbf{k}) \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 + u_1 & w_2 + u_2 & w_3 + u_3 \end{vmatrix} \\
 &= \begin{vmatrix} v_2 & v_3 \\ w_2 + u_2 & w_3 + u_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 + u_1 & w_3 + u_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 + u_1 & w_2 + u_2 \end{vmatrix} \mathbf{k} \\
 &= (v_2(w_3 + u_3) - v_3(w_2 + u_2))\mathbf{i} - (v_1(w_3 + u_3) - v_3(w_1 + u_1))\mathbf{j} \\
 &\quad + (v_1(w_2 + u_2) - v_2(w_1 + u_1))\mathbf{k} \\
 &= ((v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\
 &\quad + ((v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}) \\
 &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\
 &\quad + \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} \mathbf{k} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\
 &= \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u}.
 \end{aligned}$$

58. Answers will vary.

59. Answers will vary.

60. From Problem 25, $\mathbf{v} \times \mathbf{w} = \mathbf{i} - 13\mathbf{j} - 4\mathbf{k}$, so that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot (\mathbf{i} - 13\mathbf{j} - 4\mathbf{k}) = 2 \cdot 1 + 3 \cdot (-13) - 4 \cdot (-4) = \boxed{-21}.$$

61. From Problem 30, $\mathbf{w} \times \mathbf{u} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then

$$(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (-3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 5 \cdot (-3) - 2 \cdot 1 + 1 \cdot (-4) = \boxed{-21}.$$

62. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot ((v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\ &= (u_1v_2w_3 - u_1v_3w_2) - (u_2v_1w_3 - u_2v_3w_1) + (u_3v_1w_2 - u_3v_2w_1) \\ l(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \mathbf{w} \\ &= ((u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}) \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\ &= (u_2v_3w_1 - u_3v_2w_1) - (u_1v_3w_2 - u_3v_1w_2) + (u_1v_2w_3 - u_2v_1w_3). \end{aligned}$$

Rearranging and comparing terms, we see that the two triple scalar products are equal.

63. By Problem 62 and the fact that the dot product is commutative,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}),$$

so that the first and third formulas are equal. For the second, apply Problem 64 to $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ to get

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

64. If $\mathbf{x}$ is a nonzero vector, then all vectors orthogonal to $\mathbf{x}$ lie in a single plane. But $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and to $\mathbf{w}$, so if in addition $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$, then $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{u}$ as well, and therefore $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in the same plane.

65. Since $\mathbf{v}'$ is orthogonal to both $\mathbf{w}$ and $\mathbf{u}$, it is parallel to $\mathbf{w} \times \mathbf{u}$, so it is a scalar multiple of that vector, say $\mathbf{v}' = k_v(\mathbf{w} \times \mathbf{u})$. Similarly, $\mathbf{w}' = k_w(\mathbf{u} \times \mathbf{v})$ and $\mathbf{u}' = k_u(\mathbf{v} \times \mathbf{w})$. Then

$$\begin{aligned} \mathbf{v}' \cdot \mathbf{v} &= k_v(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = 1 \quad \text{so that} \quad k_v = \frac{1}{\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})} \quad \text{and therefore} \quad \mathbf{v}' = \frac{\mathbf{w} \times \mathbf{u}}{\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})} \\ \mathbf{w}' \cdot \mathbf{w} &= k_w(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 1 \quad \text{so that} \quad k_w = \frac{1}{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})} \quad \text{and therefore} \quad \mathbf{w}' = \frac{\mathbf{u} \times \mathbf{v}}{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})} \\ \mathbf{u}' \cdot \mathbf{u} &= k_u(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = 1 \quad \text{so that} \quad k_u = \frac{1}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})} \quad \text{and therefore} \quad \mathbf{u}' = \frac{\mathbf{v} \times \mathbf{w}}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}. \end{aligned}$$

66. We first compute $\mathbf{v} \times \mathbf{w}$:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 3 & 6 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 4 \\ 6 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 3 & 6 \end{vmatrix} \mathbf{k} \\ &= (-2 \cdot (-2) - 4 \cdot 6)\mathbf{i} - (3 \cdot (-2) - 4 \cdot 3)\mathbf{j} + (3 \cdot 6 - (-2) \cdot 3)\mathbf{k} \\ &= -20\mathbf{i} + 18\mathbf{j} + 24\mathbf{k}. \end{aligned}$$

Then the volume of the parallelepiped is

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (-20\mathbf{i} + 18\mathbf{j} + 24\mathbf{k})| = |2 \cdot (-20) + 1 \cdot 18 - 2 \cdot 24| = \boxed{70}.$$

67. We first compute $\mathbf{v} \times \mathbf{w}$:

$$\begin{aligned}
 \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -8 \\ 1 & 0 & 6 \end{vmatrix} \\
 &= \begin{vmatrix} 3 & -8 \\ 0 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -8 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
 &= (3 \cdot 6 - (-8) \cdot 0) \mathbf{i} - (2 \cdot 6 - (-8) \cdot 1) \mathbf{j} + (2 \cdot 0 - 3 \cdot 1) \mathbf{k} \\
 &= 18\mathbf{i} - 20\mathbf{j} - 3\mathbf{k}
 \end{aligned}$$

Then the volume of the parallelepiped is

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |(8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}) \cdot (18\mathbf{i} - 20\mathbf{j} - 3\mathbf{k})| = |8 \cdot 18 - 6 \cdot (-20) + 5 \cdot (-3)| = \boxed{249}.$$

68. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned}
 \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
 &= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k} \\
 &= (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}.
 \end{aligned}$$

Next,

$$\begin{aligned}
 \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} \\
 &= \begin{vmatrix} u_2 & u_3 \\ v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_2w_3 - v_3w_2 & v_1w_2 - v_2w_1 \end{vmatrix} \mathbf{j} \\
 &\quad + \begin{vmatrix} u_1 & u_2 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 \end{vmatrix} \mathbf{k} \\
 &= (u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3)\mathbf{i} \\
 &\quad - (u_1v_1w_2 - u_1v_2w_1 - u_3v_2w_3 + u_3v_3w_2)\mathbf{j} \\
 &\quad + (u_1v_3w_1 - u_1v_1w_3 - u_2v_2w_3 + u_2v_3w_2)\mathbf{k} \\
 &= (u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3)\mathbf{i} \\
 &\quad + (u_1v_2w_1 - u_1v_1w_2 + u_3v_2w_3 - u_3v_3w_2)\mathbf{j} \\
 &\quad + (u_1v_3w_1 - u_1v_1w_3 - u_2v_2w_3 + u_2v_3w_2)\mathbf{k}
 \end{aligned}$$

The other side of the formula is

$$\begin{aligned}
 (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= ((u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}))(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
 &\quad - ((u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}))(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\
 &= (u_1w_1 + u_2w_2 + u_3w_3)(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) - (u_1v_1 + u_2v_2 + u_3v_3)(w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}) \\
 &= (u_1v_1w_1 - u_1v_1w_1 + u_2v_1w_2 - u_2v_2w_1 + u_3v_1w_3 - u_3v_3w_1)\mathbf{i} \\
 &\quad + (u_1v_2w_1 - u_1v_1w_2 + u_2v_2w_2 - u_2v_2w_2 + u_3v_2w_3 - u_3v_3w_2)\mathbf{j} \\
 &\quad + (u_1v_3w_1 - u_1v_1w_3 + u_2v_3w_2 - u_2v_2w_3 + u_3v_3w_3 - u_3v_3w_3)\mathbf{k} \\
 &= (u_2v_1w_2 - u_2v_2w_1 + u_3v_1w_3 - u_3v_3w_1)\mathbf{i} \\
 &\quad + (u_1v_2w_1 - u_1v_1w_2 + u_3v_2w_3 - u_3v_3w_2)\mathbf{j} \\
 &\quad + (u_1v_3w_1 - u_1v_1w_3 + u_2v_3w_2 - u_2v_2w_3)\mathbf{k}.
 \end{aligned}$$

The two expressions are the same.

69. Using Problem 68, we have

$$\begin{aligned}
 \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \\
 &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} \\
 &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \\
 &= \mathbf{0},
 \end{aligned}$$

since the dot product is commutative.

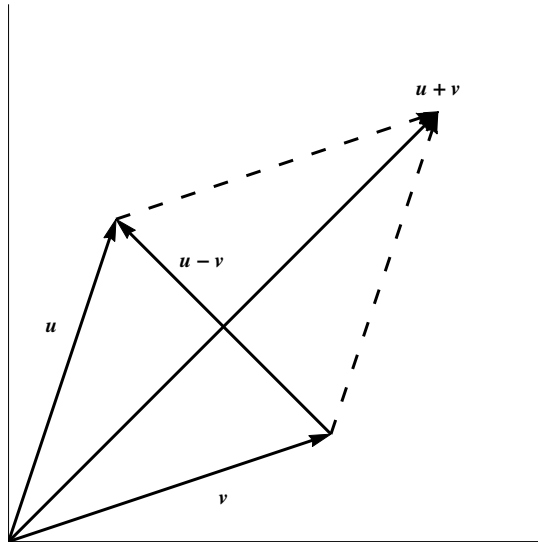
70. We have

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \cdot \mathbf{d} && \text{(Problem 62)} \\
 &= -(\mathbf{c} \times (\mathbf{a} \times \mathbf{b})) \cdot \mathbf{d} && \text{(Cross product is anticommutative)} \\
 &= -((\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}) \cdot \mathbf{d} && \text{(Problem 68)} \\
 &= ((\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}) \cdot \mathbf{d} && \text{(Algebraic manipulation)} \\
 &= (\mathbf{c} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{c} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{d}) && \text{(Properties of dot product)} \\
 &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) && \text{(Dot product and multiplication are commutative)}
 \end{aligned}$$

71. We have

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} && \text{(Problem 68)} \\
 &= [(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}))]\mathbf{c} - [(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))]\mathbf{d} && \text{(Problem 62)}.
 \end{aligned}$$

72. Consider the following parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$:

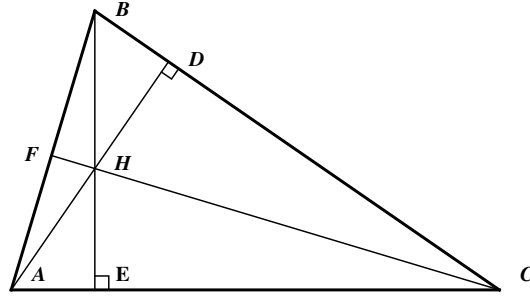


The diagonals are perpendicular when $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$. But

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$$

since the dot product is commutative. Therefore the diagonals are perpendicular if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$, which happens exactly when $\|\mathbf{u}\| = \|\mathbf{v}\|$; that is, when $\mathbf{u}$ and $\mathbf{v}$ have equal lengths. Therefore the diagonals are perpendicular precisely when the parallelogram is a rhombus.

73. For the altitudes, consider the diagram below:



Given $\triangle ABC$, we draw altitudes BE and AD meeting in H , and then extend CH to meet AB at F . We know that $\overrightarrow{AD}$ is orthogonal to $\overrightarrow{BC}$. But $\overrightarrow{AD}$ and $\overrightarrow{AH}$ are parallel, so that $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$ as well; that is, $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$. Also $\overrightarrow{BE}$ is orthogonal to $\overrightarrow{AC}$. But $\overrightarrow{BE}$ and $\overrightarrow{BH}$ are parallel, so that $\overrightarrow{BH}$ is orthogonal to $\overrightarrow{AC}$ as well; that is, $\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$. We must show that $\overrightarrow{CF} \cdot \overrightarrow{AB} = 0$. Use the notation that the vector $\overrightarrow{OX}$ (that is, the point X) may be written $\mathbf{x}$; then

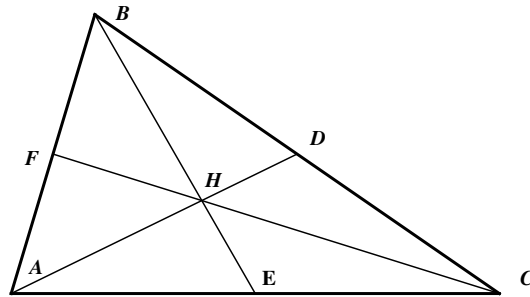
$$\begin{aligned}\overrightarrow{AH} \cdot \overrightarrow{BC} = 0 & \text{ so that } (\mathbf{h} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) = 0 \text{ and therefore } \mathbf{h} \cdot \mathbf{b} - \mathbf{h} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0 \\ \overrightarrow{BH} \cdot \overrightarrow{AC} = 0 & \text{ so that } (\mathbf{h} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \text{ and therefore } \mathbf{h} \cdot \mathbf{c} - \mathbf{h} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0.\end{aligned}$$

Adding these two equations together and canceling like terms gives

$$0 = \mathbf{h} \cdot \mathbf{b} - \mathbf{h} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{a} = (\mathbf{h} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = \overrightarrow{CH} \cdot \overrightarrow{AB},$$

so that $\overrightarrow{CH}$ and $\overrightarrow{AB}$ are orthogonal. Since $\overrightarrow{CH}$ and $\overrightarrow{CF}$ are parallel, it follows that $\overrightarrow{CF}$ and $\overrightarrow{AB}$ are orthogonal as well. Therefore CF is the third altitudes, so that all the altitudes intersect at the orthocenter H .

For the medians, consider the diagram below:



Here we have drawn all three medians, so that $\overrightarrow{AF} = \overrightarrow{BF}$, $\overrightarrow{BD} = \overrightarrow{DC}$, and $\overrightarrow{AD} = \overrightarrow{EC}$. We have shown the medians meeting at a single point (because they do!), although that is what we want to prove. To prove that they do, we show that any pair of medians intersect at a point that is two thirds of the way along each median. First take AD and BE . Using again the convention that $\mathbf{a} = \overrightarrow{OA}$, the point that is two thirds of the way from B to E along BE is

$$\overrightarrow{OB} + \frac{2}{3}\overrightarrow{BE} = \mathbf{b} + \frac{2}{3}(\mathbf{e} - \mathbf{b}) = \frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{e}.$$

However, E is the midpoint of AC , so that

$$\mathbf{e} = \frac{1}{2}(\mathbf{a} + \mathbf{c}).$$

Substituting that into the formula for the intersection point gives

$$\frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{e} = \frac{1}{3}\mathbf{b} + \frac{2}{3} \cdot \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

Next take AD and CF . The point that is two thirds of the way from C to F along CF is

$$\overrightarrow{OC} + \frac{2}{3}\overrightarrow{CF} = \mathbf{c} + \frac{2}{3}(\mathbf{f} - \mathbf{c}) = \frac{1}{3}\mathbf{c} + \frac{2}{3}\mathbf{f}.$$

However, F is the midpoint of AB , so that

$$\mathbf{f} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Substituting that into the formula for the intersection point again gives

$$\frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{f} = \frac{1}{3}\mathbf{b} + \frac{2}{3} \cdot \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

So AD , BE , and CF all meet at the point that is the terminal point of the vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

74. Take the dot product of both sides of the given equation with $\mathbf{a}$:

$$(a\mathbf{x} + (\mathbf{x} \times \mathbf{a})) \cdot \mathbf{a} = a\mathbf{x} \cdot \mathbf{a} + (\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}.$$

Now Problem 62 shows that $(\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a} = \mathbf{x} \cdot (\mathbf{a} \times \mathbf{a}) = \mathbf{0}$ since $\mathbf{a}$ is parallel to itself, so the equation becomes

$$a\mathbf{x} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}, \text{ so that } \mathbf{x} \cdot \mathbf{a} = \frac{1}{a}\mathbf{b} \cdot \mathbf{a}.$$

Now take the cross product of the left side of the original equation with $\mathbf{a}$:

$$\begin{aligned} (a\mathbf{x} + (\mathbf{x} \times \mathbf{a})) \times \mathbf{a} &= a(\mathbf{x} \times \mathbf{a}) + (\mathbf{x} \times \mathbf{a}) \times \mathbf{a} \\ &= a(\mathbf{x} \times \mathbf{a}) - \mathbf{a} \times (\mathbf{x} \times \mathbf{a}) \\ &= a(\mathbf{x} \times \mathbf{a}) - ((\mathbf{a} \cdot \mathbf{a})\mathbf{x} - (\mathbf{a} \times \mathbf{x})\mathbf{a}) \\ &= a(\mathbf{x} \times \mathbf{a}) - (\mathbf{a} \cdot \mathbf{a})\mathbf{x} + \frac{1}{a}(\mathbf{b} \cdot \mathbf{a})\mathbf{a}. \end{aligned}$$

It follows that the last formula above is equal to the cross product of the right side of the original equation with $\mathbf{a}$, so that

$$a(\mathbf{x} \times \mathbf{a}) - (\mathbf{a} \cdot \mathbf{a})\mathbf{x} + \frac{1}{a}(\mathbf{b} \cdot \mathbf{a})\mathbf{a} = \mathbf{b} \times \mathbf{a}.$$

Solving for $\mathbf{x} \times \mathbf{a}$ gives

$$\mathbf{x} \times \mathbf{a} = \frac{1}{a}\mathbf{b} \times \mathbf{a} + \frac{1}{a}(\mathbf{a} \cdot \mathbf{a})\mathbf{x} - \frac{1}{a^2}(\mathbf{b} \cdot \mathbf{a})\mathbf{a}.$$

We can substitute this expression for $\mathbf{x} \times \mathbf{a}$ in the original equation, giving

$$a\mathbf{x} + \frac{1}{a}\mathbf{b} \times \mathbf{a} + \frac{1}{a}(\mathbf{a} \cdot \mathbf{a})\mathbf{x} - \frac{1}{a^2}(\mathbf{b} \cdot \mathbf{a})\mathbf{a} = \mathbf{b}.$$

Now multiply both sides by a^2 to clear fractions, then solve for $\mathbf{x}$ to get

$$\begin{aligned} \mathbf{x} &= \frac{a^2\mathbf{b} - a(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \cdot \mathbf{a})\mathbf{a}}{a^3 + a(\mathbf{a} \cdot \mathbf{a})} \\ &= \boxed{\frac{a^2\mathbf{b} + a(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{a(a^2 + \mathbf{a} \cdot \mathbf{a})}}. \end{aligned}$$

Challenge Problems

75. From Problem 62, the triple scalar product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (u_1v_2w_3 - u_1v_3w_2) - (u_2v_1w_3 - u_2v_3w_1) + (u_3v_1w_2 - u_3v_2w_1).$$

Computing the determinant, we get

$$\begin{aligned} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= (u_1v_2w_3 - u_1v_3w_2) - (u_2v_1w_3 - u_2v_3w_1) + (u_3v_1w_2 - u_3v_2w_1), \end{aligned}$$

and the two expressions are equal.

76. Consider the diagram preceding Problem 66. The height of the parallelepiped is $\|\mathbf{u}\| \cos \theta$, and the area of the base is $\|\mathbf{v} \times \mathbf{w}\|$. The volume can therefore be computed by an integral from 0 to the height $\|\mathbf{u}\| \cos \theta$ of the area of each cross-section:

$$\int_0^{\|\mathbf{u}\| \cos \theta} \|\mathbf{v} \times \mathbf{w}\| \, dz = [z \|\mathbf{v} \times \mathbf{w}\|]_0^{\|\mathbf{u}\| \cos \theta} = \|\mathbf{u}\| \|\mathbf{v} \times \mathbf{w}\| \cos \theta = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

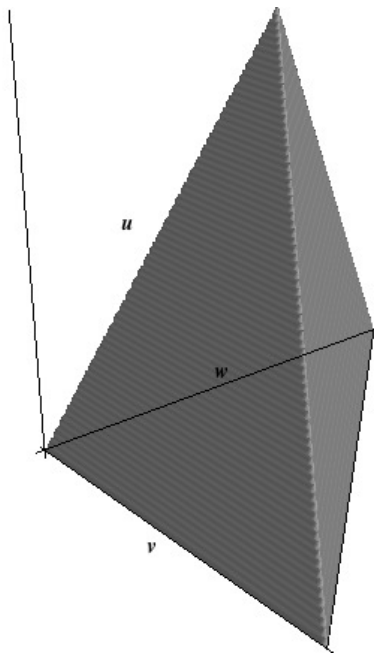
by the discussion of the triple scalar product given preceding Problem 60. See also Problem 22 in Section 6.4.

77. To show that the given vector is normal to the plane, it suffices to show that it is orthogonal to any two nonparallel vectors in the plane, say $\mathbf{u}$ and $\mathbf{v}$, since two nonparallel vectors determine a plane. Now,

$$\begin{aligned} (\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}) \cdot \mathbf{u} &= -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} + (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{u} \\ &= -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{u}) + (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} + \mathbf{w} \cdot (\mathbf{u} \times \mathbf{u}) \\ &= -\mathbf{v} \cdot \mathbf{0} + (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{0} \\ &= (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \\ (\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} - (\mathbf{w} \times \mathbf{v}) \cdot \mathbf{v} + (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) - \mathbf{w} \cdot (\mathbf{v} \times \mathbf{v}) + (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{0} - \mathbf{w} \cdot \mathbf{0} + (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \\ &= (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}. \end{aligned}$$

However, since $\mathbf{v} \times \mathbf{w}$ is orthogonal to the plane containing $\mathbf{v}$ and $\mathbf{w}$, which also contains $\mathbf{u}$, we see that $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = 0$; similarly, since $\mathbf{w} \times \mathbf{u}$ is orthogonal to the plane containing $\mathbf{w}$ and $\mathbf{u}$, which also contains $\mathbf{v}$, we see that $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = 0$. So both of these dot products are zero, and it follows that the given vector is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{w}$ lies in the same plane as $\mathbf{u}$, the given vector is orthogonal to $\mathbf{w}$ as well.

78. A tetrahedron, with one face on the xy -plane, is below:



Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the vectors from $(0,0,0)$ to each of the three vertices, with $\mathbf{u}$ going to the top vertex. Let θ be the angle that $\mathbf{u}$ makes with the positive z -axis (see the diagram preceding Problem 66 as well). Then the tip of the pyramid has z -coordinate $\|\mathbf{u}\| \cos \theta$. We will compute the volume of this pyramid as an integral along the z coordinate, where the area of each cross-section is a triangle. At the base ($z = 0$), the area is the area of the triangle with sides $\mathbf{v}$ and $\mathbf{w}$, which, from Problem 49, is $\frac{1}{2} \|\mathbf{v} \times \mathbf{w}\|$. Using similar triangles on each face, we see that two of the sides of the cross-sectional triangle at a height z are

$$\left(1 - \frac{z}{\|\mathbf{u}\| \cos \theta}\right) \mathbf{v} \text{ and } \left(1 - \frac{z}{\|\mathbf{u}\| \cos \theta}\right) \mathbf{w}.$$

So again from Problem 49, the area of the cross-sectional triangle at height z is

$$\frac{1}{2} \left(1 - \frac{z}{\|\mathbf{u}\| \cos \theta}\right)^2 \|\mathbf{v} \times \mathbf{w}\|.$$

In what follows, write $h = \|\mathbf{u}\| \cos \theta$ for simplicity. Then the volume of the pyramid is

$$\begin{aligned} \int_0^h \frac{1}{2} \left(1 - \frac{z}{h}\right)^2 \|\mathbf{v} \times \mathbf{w}\| \, dz &= \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| \int_0^h \left(1 - \frac{2}{h}z + \frac{1}{h^2}z^2\right) \, dz \\ &= \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| \left[z - \frac{1}{h}z^2 + \frac{1}{3h^2}z^3\right]_0^h \\ &= \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| \left(h - h + \frac{1}{3}h\right) \\ &= \frac{1}{6} h \|\mathbf{v} \times \mathbf{w}\| \\ &= \frac{1}{6} \|\mathbf{u}\| \|\mathbf{v} \times \mathbf{w}\| \cos \theta \\ &= \frac{1}{6} |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

79. (a) The magnitude of the force is $\|\mathbf{F}\| = |q| \|\mathbf{v} \times \mathbf{B}\| = |q| \|\mathbf{v}\| \|\mathbf{B}\| \sin \theta$, from which the magnitude of $\mathbf{v}$ equals

$$\|\mathbf{v}\| = \frac{\|\mathbf{F}\|}{|q| \|\mathbf{B}\| \sin \theta}.$$

For minimum velocity, $\sin \theta$ must be a maximum, so $\sin \theta = 1$ and $\theta = 90^\circ$. That is, the velocity $\mathbf{v}$ is orthogonal to the magnetic field $\mathbf{B}$. Then

$$\|\mathbf{v}\| = \frac{1.63 \times 10^{-13} \text{ N}}{(1.60 \times 10^{-19} \text{ C})(1.20 \text{ T})(1)} = 8.49 \times 10^5 \text{ m/s}.$$

The direction of the force is in the direction of $q\mathbf{v} \times \mathbf{B}$, and since q is positive for a proton, the force $\mathbf{F}$ is in the direction of $\mathbf{v} \times \mathbf{B}$. We know the force $\mathbf{F}$ is horizontal and to the left, and parallel to $-\mathbf{i}$, so $\mathbf{v} \times \mathbf{B}$ is parallel to $-\mathbf{i}$. The field $\mathbf{B}$ points vertically downward and is parallel to $-\mathbf{j}$. Therefore the velocity $\mathbf{v}$ is parallel to $-\mathbf{k}$.

- (b) Answers will vary.

80. Fix axes so that the positive z -axis is pointing to the north and the positive x -axis is pointing to the east; the direction of the positive y -axis is into the earth, determined by the right-hand rule.

- (a) If the electrons move from north to south, their velocity is $-2 \times 10^6 \mathbf{k}$ m/s, and we have

$$\begin{aligned} \mathbf{F}_{\text{mag}} &= q\mathbf{v} \times \mathbf{B} \\ &= 1.6 \times 10^{-19}(-2 \times 10^6 \mathbf{k}) \times (5 \times 10^{-3} \mathbf{k}) \\ &= -1.6 \times 10^{-15} \mathbf{k} \times \mathbf{k} \\ &= \boxed{\mathbf{0}} \end{aligned}$$

since the cross product of parallel vectors is zero.

- (b) If the electrons move from west to east, their velocity is $2 \times 10^6 \mathbf{i}$ m/s, and we have

$$\begin{aligned} \mathbf{F}_{\text{mag}} &= q\mathbf{v} \times \mathbf{B} \\ &= 1.6 \times 10^{-19}(2 \times 10^6 \mathbf{i}) \times (5 \times 10^{-3} \mathbf{k}) \\ &= 1.6 \times 10^{-15} \mathbf{i} \times \mathbf{k} \\ &= \boxed{1.6 \times 10^{-15} \mathbf{j} \text{ N}}. \end{aligned}$$

The resultant force points into the earth.

81. Fix axes so that the positive z -axis is pointing to the north and the positive x -axis is pointing to the east; the direction of the positive y -axis is into the earth, determined by the right-hand rule. The cord lies in the xz plane; suppose it makes an angle θ with the positive x -axis. Its length has magnitude 2, so

$$\mathbf{L} = 2(\cos \theta \mathbf{i} + \sin \theta \mathbf{k}) = 2 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{k}.$$

The magnetic field $\mathbf{B}$ is given by $\mathbf{B} = 5 \times 10^{-3} \mathbf{k}$, and $I = 5$. Then

$$\begin{aligned} \mathbf{F}_{\text{mag}} &= I\mathbf{L} \times \mathbf{B} \\ &= 5(2 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{k}) \times (5 \times 10^{-3} \mathbf{k}) \\ &= 0.05(\cos \theta \mathbf{i} \times \mathbf{k} + \sin \theta \mathbf{k} \times \mathbf{k}) \\ &= -0.05 \cos \theta \mathbf{j}. \end{aligned}$$

- (a) The minimum magnitude force is $\boxed{\text{zero}}$; this happens when $\cos \theta = 0$, which is when the cord is at an angle of 90° with the positive x -axis. This means that it is pointing due north and south.
- (b) The maximum magnitude force occurs when $\cos \theta = \pm 1$; when that happens, the magnitude of the force is $\|\mathbf{F}_{\text{mag}}\| = \|\pm 0.05 \mathbf{j}\| = \boxed{0.05 \text{ N}}$. This occurs when the cord is at an angle of 0° (or 180°) with the positive x -axis, so that it is lying due east and west.

- (c) For the magnitude of the force to be half its maximum value, or 0.025, we need $\cos \theta = \pm \frac{1}{2}$, so that $\theta = \pm 60^\circ$ or $\pm 120^\circ$. The cord should be lying at an angle of 60° from east-west, so it should be lying at an angle of $\boxed{30^\circ \text{ from the direction of } \mathbf{B}}$.

10.6: Equations of Lines and Planes in Space

Concepts and Vocabulary

- False. The direction vector of this line is found by looking at the denominators, so one direction vector is $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
- False. If the lines lie in the same plane, then they are either parallel or they intersect; in either case, they are not skew.
- True. See the discussion in objective 6.
- False. Two planes are parallel if and only if their normal vectors are parallel. However, the normal vector of p_1 is $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and that of p_2 is $6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$; these two vectors are not multiples of each other, so are not parallel.
- False. The two planes are parallel, since they each have normal vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, but they are not one unit apart. To see how far apart they are, take the point $(3, 0, 0)$ on the first plane and compute its distance from the second plane using the formula in the text:

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot 3 + 2 \cdot 0 + 1 \cdot 0 - 4|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{6}}.$$

- Answers will vary.
- Answers will vary.
- Two distinct lines in space that do not intersect and are not parallel are called *skew lines*.

Skill Building

9. (a) Since we are given a direction vector $\mathbf{D} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and a point $\mathbf{r}_0 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ on the line, a vector equation for the line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{D} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = \boxed{(1 + 2t)\mathbf{i} + (2 - t)\mathbf{j} + (3 + t)\mathbf{k}}.$$

- (b) From part (a), a set of parametric equations is

$$\boxed{x = 1 + 2t, \quad y = 2 - t, \quad z = 3 + t}.$$

- (c) Solving the equations in part (b) for t gives the symmetric form

$$\boxed{\frac{x - 1}{2} = \frac{y - 2}{-1} = \frac{z - 3}{1}}.$$

10. (a) Since we are given a direction vector $\mathbf{D} = \mathbf{i} + \mathbf{j}$ and a point $\mathbf{r}_0 = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ on the line, a vector equation for the line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{D} = (4\mathbf{i} - \mathbf{j} + 6\mathbf{k}) + t(\mathbf{i} + \mathbf{j}) = \boxed{(4 + t)\mathbf{i} + (-1 + t)\mathbf{j} + 6\mathbf{k}}.$$

- (b) From part (a), a set of parametric equations is

$$\boxed{x = 4 + t, \quad y = -1 + t, \quad z = 6}.$$

(c) Solving the first two equations in part (b) for t gives the symmetric form

$$\boxed{\frac{x-4}{1} = \frac{y+1}{1}, \quad z=6}.$$

11. Answers may vary depending on the choice of $\mathbf{D}$ and $\mathbf{r}_0$.

(a) Since we are given two points, we can compute a direction vector

$$\mathbf{D} = \overrightarrow{P_0P_1} = \langle 4, 2, 1 \rangle - \langle 1, -1, 3 \rangle = \langle 3, 3, -2 \rangle = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}.$$

Using P_0 for the point on the line, we get the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{D} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = \boxed{(1+3t)\mathbf{i} + (-1+3t)\mathbf{j} + (3-2t)\mathbf{k}}.$$

(b) From part (a), a set of parametric equations is

$$\boxed{x = 1 + 3t, \quad y = -1 + 3t, \quad z = 3 - 2t}.$$

(c) Solving the equations in part (b) for t gives the symmetric form

$$\boxed{\frac{x-1}{3} = \frac{y+1}{3} = \frac{z-3}{-2}}.$$

12. Answers may vary depending on the choice of $\mathbf{D}$ and $\mathbf{r}_0$.

(a) Since we are given two points, we can compute a direction vector

$$\mathbf{D} = \overrightarrow{P_0P_1} = \langle 1, -1, 2 \rangle - \langle -2, 3, 0 \rangle = \langle 3, -4, 2 \rangle = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

Using P_1 for the point on the line, we get the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{D} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \boxed{(1+3t)\mathbf{i} + (-1-4t)\mathbf{j} + (2+2t)\mathbf{k}}.$$

(b) From part (a), a set of parametric equations is

$$\boxed{x = 1 + 3t, \quad y = -1 - 4t, \quad z = 2 + 2t}.$$

(c) Solving the equations in part (b) for t gives the symmetric form

$$\boxed{\frac{x-1}{3} = \frac{y+1}{-4} = \frac{z-2}{2}}.$$

13. The direction vector of the line

$$\frac{x+1}{5} = \frac{y-2}{4} = \frac{z-3}{-3}$$

is $5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$. So we want a line with that direction vector passing through $(-1, 5, 6)$. Such a line has vector form

$$-\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + t(5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}),$$

and therefore parametric form

$$\boxed{x = -1 + 5t, \quad y = 5 + 4t, \quad z = 6 - 3t}.$$

14. The direction vector of the line

$$\frac{x+1}{6} = \frac{y+2}{2} = \frac{z}{-1}$$

is $6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. So we want a line with that direction vector passing through $(1, -2, -3)$. Such a line has vector form

$$\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(6\mathbf{i} + 2\mathbf{j} - \mathbf{k}),$$

and therefore parametric form

$$\boxed{x = 1 + 6t, \quad y = -2 + 2t, \quad z = -3 - t}.$$

15. Answers will vary.

16. Rewriting $x + 1$ as $\frac{x+1}{1}$ and $y + 3$ as $\frac{y+3}{1}$, we get the line

$$\frac{x+1}{1} = \frac{y+3}{1} = \frac{z+4}{2}.$$

A direction vector for this line is the vector whose components are the denominators of the fractions, so it is $\mathbf{D} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$. This is a vector in the direction of the line. One point on the line is $(-1, -3, -4)$, since then all three numerators are zero so the fractions are equal. To find a second point, take for example $x = 1$; then the equations become

$$\frac{1+1}{1} = 2 = \frac{y+3}{1} = \frac{z+4}{2}.$$

Solving for y and z gives $y = -1$ and $z = 0$. So a second point on the line is $(1, -1, 0)$. Answers may vary depending on the choice of x .

17. For $\mathbf{D} = 2\mathbf{i} + \mathbf{k}$, $a = 2$, $b = 0$, and $c = 1$. So the symmetric equations of the line through $P_0 = (4, 2, 1)$ in direction $\mathbf{D}$ are

$$\boxed{\frac{x-4}{2} = \frac{z-1}{1}, \quad y = 2}.$$

18. For $\mathbf{D} = 2\mathbf{k} - \mathbf{j}$, $a = 0$, $b = -1$, and $c = 2$. So the symmetric equations of the line through $P_0 = (-1, 2, 0)$ in direction $\mathbf{D}$ are

$$\boxed{x = -1, \quad \frac{y-2}{-1} = \frac{z}{2}}.$$

19. The direction vectors of the two given lines are $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. So a vector perpendicular to both of these is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 8\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}.$$

We want symmetric equations of a line through $P_0 = (0, 0, 0)$ with direction vector $\mathbf{D} = 8\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$. Then $a = 8$, $b = -10$, and $c = -7$, so the equations are

$$\boxed{\frac{x}{8} = \frac{y}{-10} = \frac{z}{-7}}.$$

The negative of these equations, which is $\boxed{\frac{x}{-8} = \frac{y}{10} = \frac{z}{7}}$, is also a valid answer.

20. The direction vectors of the two given lines are $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. So a vector perpendicular to both of these is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 1 \\ 5 & 3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

We want symmetric equations of a line through $P_0 = (0, 0, 0)$ with direction vector $\mathbf{D} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Then $a = -1$, $b = 1$, and $c = 2$, so the equations are

$$\boxed{\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}}.$$

21. A vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -2 \\ -3 & -2 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 4 \\ -3 & -2 \end{vmatrix} = 4\mathbf{j} + 8\mathbf{k}.$$

We can divide through by 4 and use instead the vector $\mathbf{D} = \mathbf{j} + 2\mathbf{k}$. Then we want symmetric equations of a line through $P_0 = (1, 2, -1)$ with direction vector $\mathbf{D}$, so that $a = 0$, $b = 1$, and $c = 2$ and therefore the equations are

$$\boxed{x = 1, \quad \frac{y - 2}{1} = \frac{z + 1}{2}}.$$

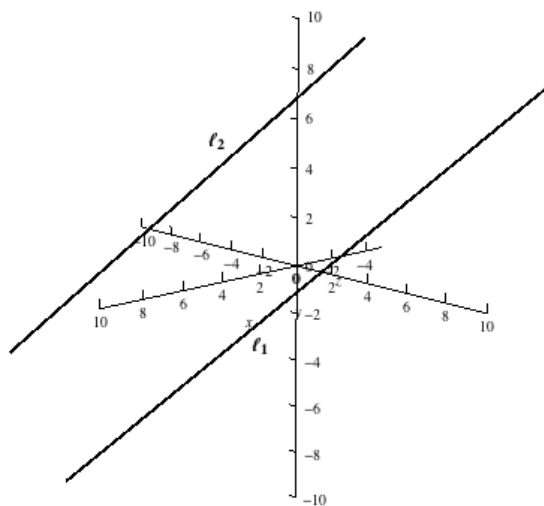
22. A vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -3 \\ 4 & 2 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 2 \\ 4 & 2 \end{vmatrix} = 8\mathbf{i} - 10\mathbf{j} - 12\mathbf{k}.$$

We can divide through by 2 and use instead the vector $\mathbf{D} = 4\mathbf{j} - 5\mathbf{j} - 6\mathbf{k}$. Then we want symmetric equations of a line through $P_0 = (-1, 3, 2)$ with direction vector $\mathbf{D}$, so that $a = 4$, $b = -5$, and $c = -6$ and therefore the equations are

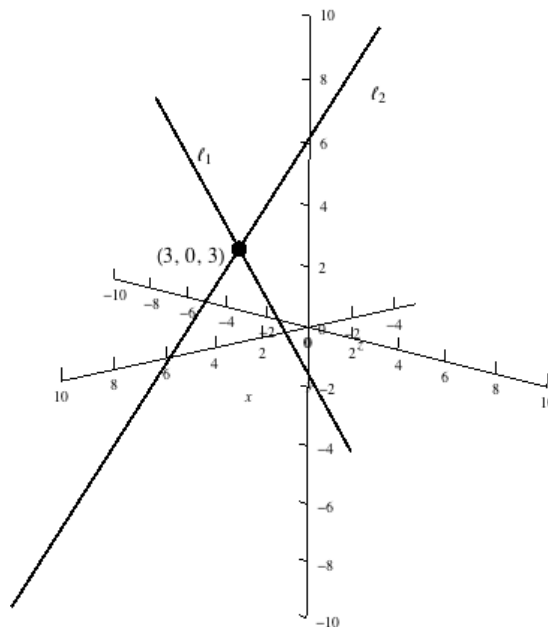
$$\boxed{\frac{x + 1}{4} = \frac{y - 3}{-5} = \frac{z - 2}{-6}}.$$

23. (a) Since the direction vector for ℓ_1 is 3 times the direction vector for ℓ_2 , these lines are parallel.
The graph confirms this result.
- (b)



24. (a) The direction vectors are not multiples of each other, so the lines are not parallel. However, they do intersect since for $t_1 = t_2 = 0$, both lines pass through $(3, 0, 3)$. The graph confirms this result.

(b)



25. (a) The direction vectors are not multiples of each other, so the lines are not parallel. To see if they intersect, we must solve the equation

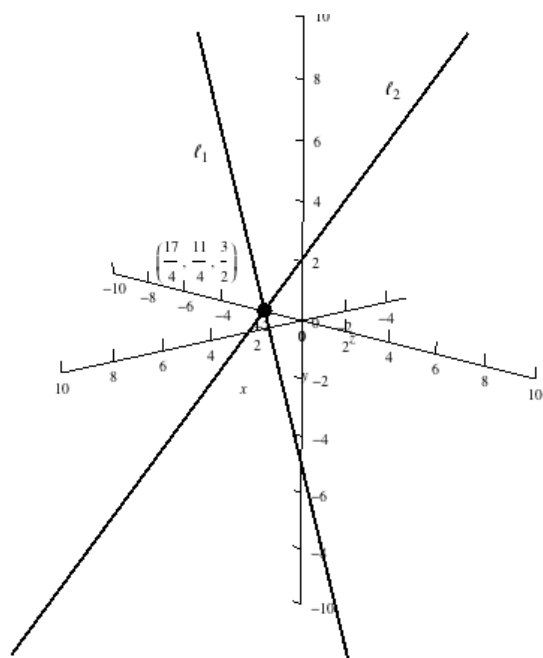
$$4\mathbf{i} + 3\mathbf{j} + t_1(\mathbf{i} - \mathbf{j} + 6\mathbf{k}) = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + t_2(\mathbf{i} - \mathbf{j} - 2\mathbf{k}).$$

Looking at the $\mathbf{i}$ component, we have $4\mathbf{i} + t_1\mathbf{i} = 4\mathbf{i} + t_2\mathbf{i}$, so that $t_1 = t_2$. The second component gives $3\mathbf{j} - t_1\mathbf{j} = 3\mathbf{j} - t_2\mathbf{j}$, so that again $t_1 = t_2$. The third component, with $t_1 = t_2$, gives $6t_1\mathbf{k} = 2\mathbf{k} - 2t_1\mathbf{k}$, so that $t_1 = \frac{1}{4}$. So the two lines intersect when $t_1 = t_2 = \frac{1}{4}$, which is at the point

$$4\mathbf{i} + 3\mathbf{j} + \frac{1}{4}(\mathbf{i} - \mathbf{j} + 6\mathbf{k}) = \frac{17}{4}\mathbf{i} + \frac{11}{4}\mathbf{j} + \frac{3}{2}\mathbf{k}.$$

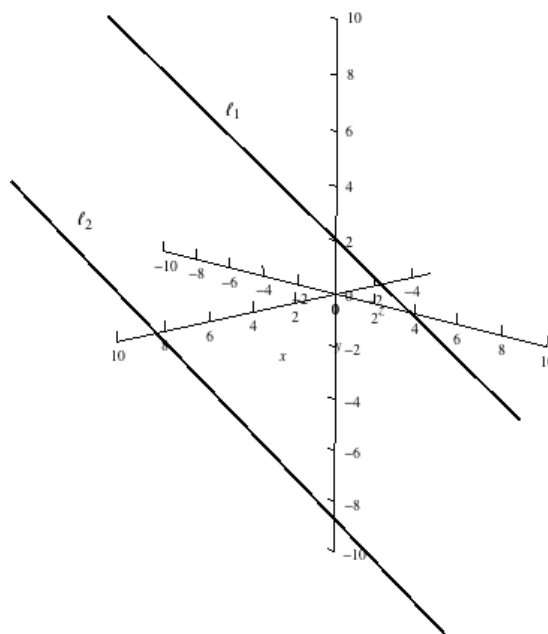
The graph confirms this result.

(b)



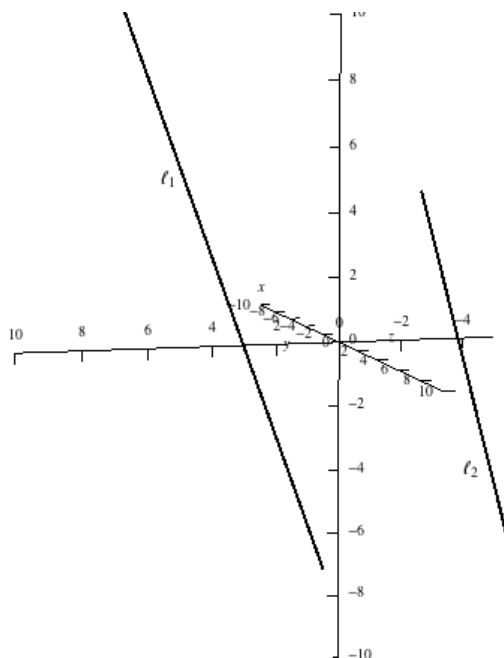
26. (a) Since the direction vector for ℓ_1 is -2 times the direction vector for ℓ_2 , these lines are parallel. The graph confirms this result.

(b)



27. (a) The direction vector for ℓ_1 is $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, while the direction vector for ℓ_2 is $-4\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}$, which is twice that for ℓ_1 . Therefore these lines are parallel. The graph confirms this result.

(b)



28. (a) The direction vector for ℓ_1 is $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, while the direction vector for ℓ_2 is $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Since these vectors are not multiples of one another, the two lines are not parallel. To see if they intersect, we use the symmetric equations to derive the vector form of the two lines:

$$\ell_1 : 2\mathbf{j} - 4\mathbf{k} + t_1(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}), \quad \ell_2 : -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t_2(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}).$$

To see if these lines intersect, we must see if the system

$$\begin{aligned} 3t_1 &= -6 + 3t_2 \\ 2 + 4t_1 &= -2 + 4t_2 \\ -4 + t_1 &= 3 + 2t_2. \end{aligned}$$

The third equation gives $t_1 = 2t_2 + 7$; substituting into the first equation gives

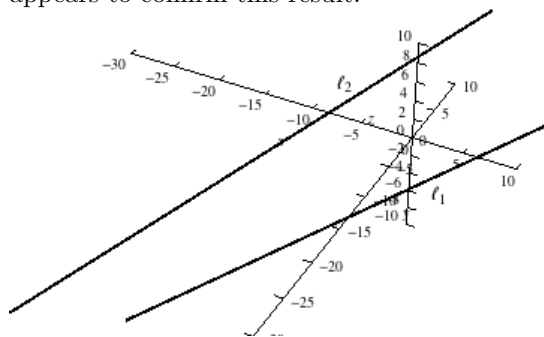
$$3(2t_2 + 7) = -6 + 3t_2, \text{ so that } 3t_2 = -27 \text{ and therefore } t_2 = -9.$$

Then $t_1 = 2 \cdot (-9) + 7 = -11$; substituting these values into the second equation gives

$$2 + 4 \cdot (-11) = -2 + 4 \cdot (-9), \text{ or } -42 = -38,$$

which is false. So this set of equations has no solution, and the lines are skew. The graph appears to confirm this result.

(b)



29. (a) The direction vector for ℓ_1 is $5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, while the direction vector for ℓ_2 is $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Since these vectors are not multiples of one another, the two lines are not parallel. To see if they intersect, we use the symmetric equations to derive the vector form of the two lines:

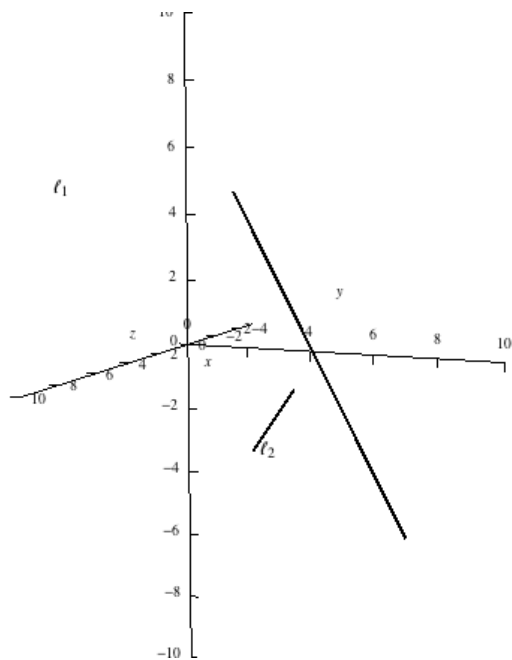
$$\ell_1 : -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t_1(5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}), \quad \ell_2 : -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + t_2(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}).$$

To see if these lines intersect, we must see if the system

$$\begin{aligned} -1 + 5t_1 &= -1 + 6t_2 \\ 2 + 4t_1 &= 2 + 3t_2 \\ 3 - 3t_1 &= -3 + 2t_2. \end{aligned}$$

The first two equations give $5t_1 = 6t_2$ and $4t_1 = 3t_2$, so that $t_1 = t_2 = 0$. But $t_1 = t_2 = 0$ does not satisfy the third equation, so this set of equations has no solution, and the lines are skew. The graph appears to confirm this result.

(b)



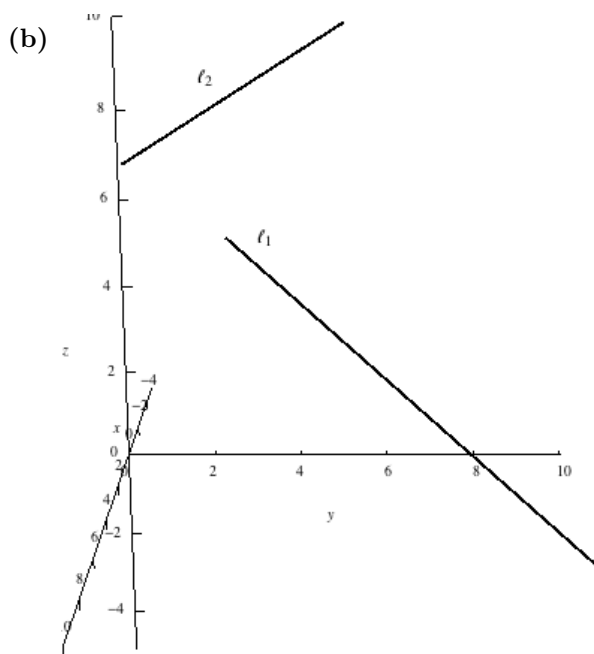
30. (a) The direction vector for ℓ_1 is $6\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, while the direction vector for ℓ_2 is $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Since these vectors are not multiples of one another, the two lines are not parallel. To see if they intersect, we use the symmetric equations to derive the vector form of the two lines:

$$\ell_1 : -5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + t_1(6\mathbf{i} + 3\mathbf{j} - \mathbf{k}), \quad \ell_2 : 2\mathbf{j} + 8\mathbf{k} + t_2(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}).$$

To see if these lines intersect, we must see if the system

$$\begin{aligned} -5 + 6t_1 &= t_2 \\ 2 + 3t_1 &= 2 + 3t_2 \\ 4 - t_1 &= 8 + 2t_2. \end{aligned}$$

The second equation gives $t_1 = t_2$; substituting t_1 for t_2 in the first equation and solving gives $t_1 = 1$, so that $t_2 = 1$. But $t_1 = t_2 = 1$ does not satisfy the third equation. So these lines are skew. The graph appears to confirm this result.



31. The general equation of a plane with normal $\langle a, b, c \rangle$ is $ax + by + cz = d$ for some d . So the equation of this plane is $2x - y + z = d$; substituting $(1, -1, 2)$ for (x, y, z) gives $2 \cdot 1 - 1 \cdot (-1) + 2 = 5 = d$. The equation of the plane is $\boxed{2x - y + z = 5}$.
32. The general equation of a plane with normal $\langle a, b, c \rangle$ is $ax + by + cz = d$ for some d . So the equation of this plane is $x + y - 2z = d$; substituting $(-3, 2, 1)$ for (x, y, z) gives $-3 + 2 - 2 \cdot 1 = -3$. The equation of the plane is $\boxed{x + y - 2z = -3}$.
33. Since the desired plane is parallel to $x + 2y - z = 6$, it is of the form $x + 2y - z = d$. Substituting the given point $(0, 5, -2)$ for (x, y, z) gives $0 + 2 \cdot 5 - (-2) = 12 = d$, so the equation of the plane is $\boxed{x + 2y - z = 12}$.
34. Since the desired plane is parallel to $2x - y + 3z = 10$, it is of the form $2x - y + 3z = d$. Substituting the given point $(1, -2, 0)$ for (x, y, z) gives $2 \cdot 1 - (-2) + 3 \cdot 0 = 4 = d$, so the equation of the plane is $\boxed{2x - y + 3z = 4}$.
35. The direction vector of the given line is determined by looking at the denominators; it is $2\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$. Since the desired plane is normal to the line, this vector is a normal vector to the plane, so the plane has equation $2x + 5y - 2z = d$. Substituting the given point $(2, 3, -1)$ for (x, y, z) gives $2 \cdot 2 + 5 \cdot 3 - 2 \cdot (-1) = 21 = d$, so the equation of the plane is $\boxed{2x + 5y - 2z = 21}$.
36. The direction vector of the given line is determined by looking at the denominators; it is $3\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Since the desired plane is normal to the line, this vector is a normal vector to the plane, so the plane has equation $3x + 4y + 4z = d$. Substituting the given point $(-1, 2, 3)$ for (x, y, z) gives $3 \cdot (-1) + 4 \cdot 2 + 4 \cdot 3 = 17 = d$, so the equation of the plane is $\boxed{3x + 4y + 4z = 17}$.
37. A normal vector to the xy -plane is $\mathbf{k}$, so that also forms a normal to the desired plane. Therefore, the plane has equation $z = d$ for some d . Since it contains $(0, 0, 4)$, we have $4 = d$ and the equation is $\boxed{z = 4}$.
38. A normal vector to the yz -plane is $\mathbf{i}$, so that also forms a normal to the desired plane. Therefore, the plane has equation $x = d$ for some d . Since it contains $(2, 0, 0)$, we have $2 = d$ and the equation is $\boxed{x = 2}$.

39. A normal vector to the xz -plane is $\mathbf{j}$, so that also forms a normal to the desired plane. Therefore, the plane has equation $y = d$ for some d . Since it contains $(1, -2, 3)$, we have $-2 = d$ and the equation is $\boxed{y = -2}$.

40. A normal vector to the xy -plane is $\mathbf{k}$, so that also forms a normal to the desired plane. Therefore, the plane has equation $z = d$ for some d . Since it contains $(1, -3, 4)$, we have $4 = d$ and the equation is $\boxed{z = 4}$.

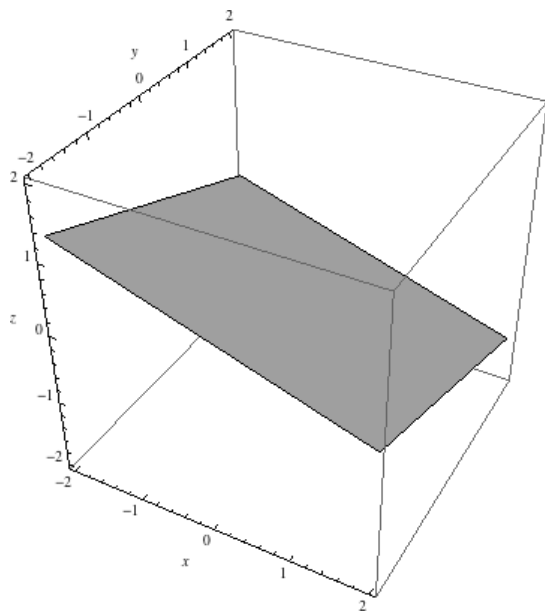
41. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = -\mathbf{i} + \mathbf{j}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

Using the point $(0, 0, 0)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$1(x - 0) + 1(y - 0) + 3(z - 0) = 0, \quad \text{or} \quad \boxed{x + y + 3z = 0}.$$

(b)

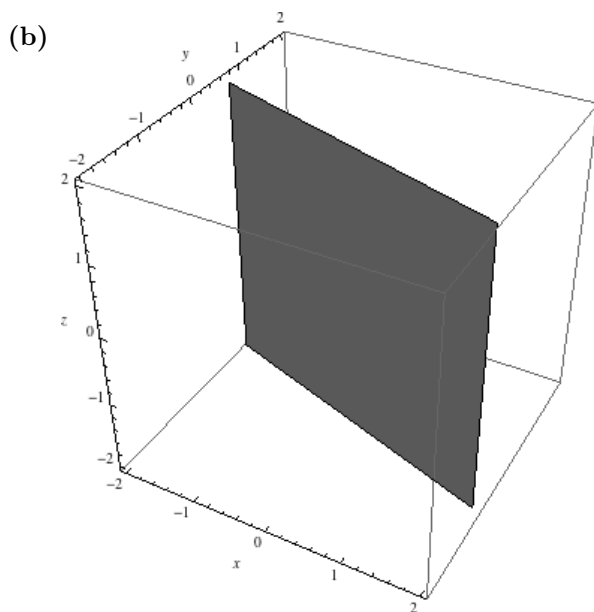


42. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = -3\mathbf{i} + \mathbf{j}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -3 & 1 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & 2 \\ -3 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -1 \\ -3 & 1 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j}.$$

Using the point $(0, 0, 0)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$-2(x - 0) - 6(y - 0) = 0, \quad \text{or} \quad -2x - 6y = 0, \quad \text{or} \quad \boxed{x + 3y = 0}.$$

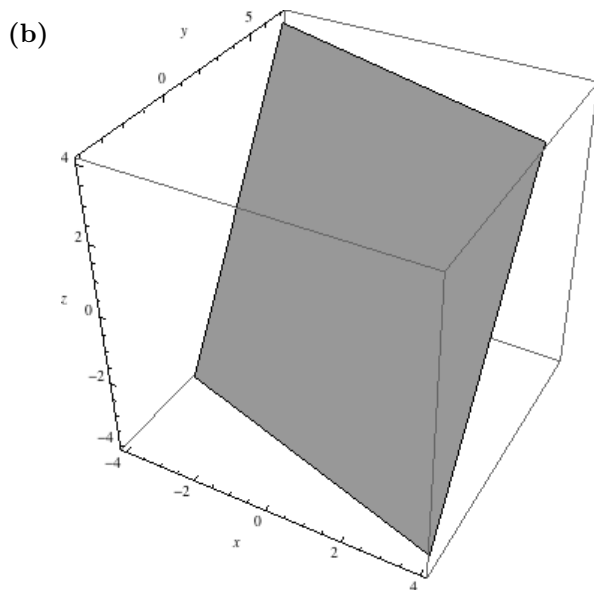


43. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 3 & -3 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 1 \\ -3 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 0 \\ 3 & -3 \end{vmatrix} = 3\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}.$$

Using the point $(1, 2, 1)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$3(x - 1) + 7(y - 2) - 6(z - 1) = 0, \quad \text{or} \quad \boxed{3x + 7y - 6z = 11}.$$



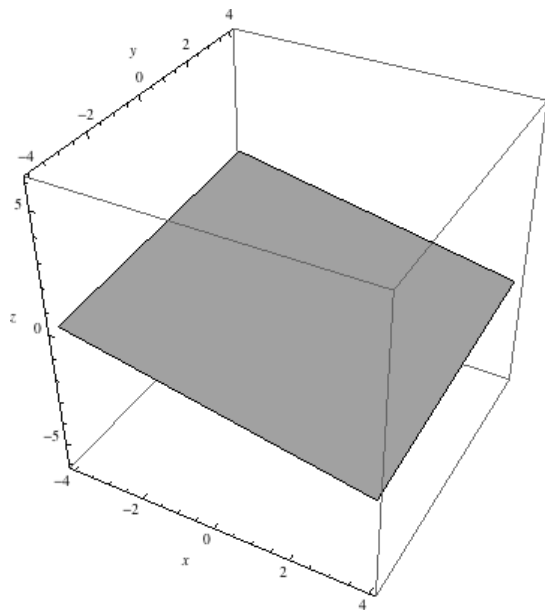
44. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = -\mathbf{i} - 3\mathbf{j}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & -1 \\ -1 & -3 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & -1 \\ -1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 2 \\ -1 & -3 \end{vmatrix} = -3\mathbf{i} + 1\mathbf{j} - 10\mathbf{k}.$$

Using the point $(-1, 2, 0)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$-3(x - (-1)) + (y - 2) - 10(z - 0) = 0, \quad \text{or} \quad -3x + y - 10z = 5, \quad \text{or} \quad \boxed{3x - y + 10z = -5}.$$

(b)



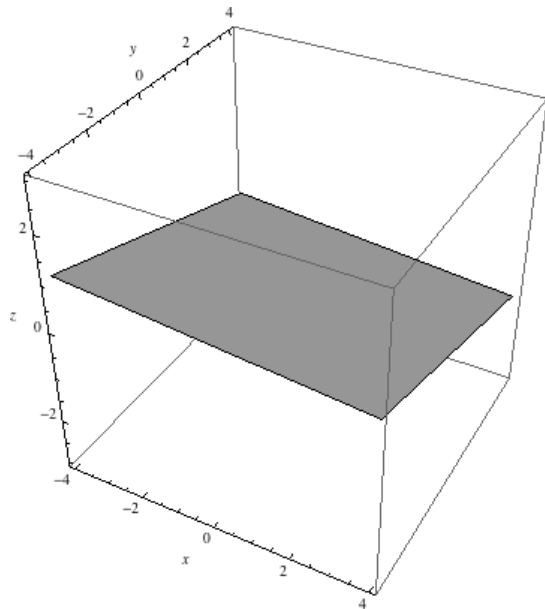
45. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = -2\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = -5\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -9 & 2 \\ -5 & -8 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -9 & 2 \\ -8 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 2 \\ -5 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & -9 \\ -5 & -8 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j} - 29\mathbf{k}.$$

Using the point $(1, 0, 0)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$-2(x - 1) - 6(y - 0) - 29(z - 0) = 0, \quad \text{or} \quad -2x - 6y - 29z = -2, \quad \text{or} \quad \boxed{2x + 6y + 29z = 2}.$$

(b)



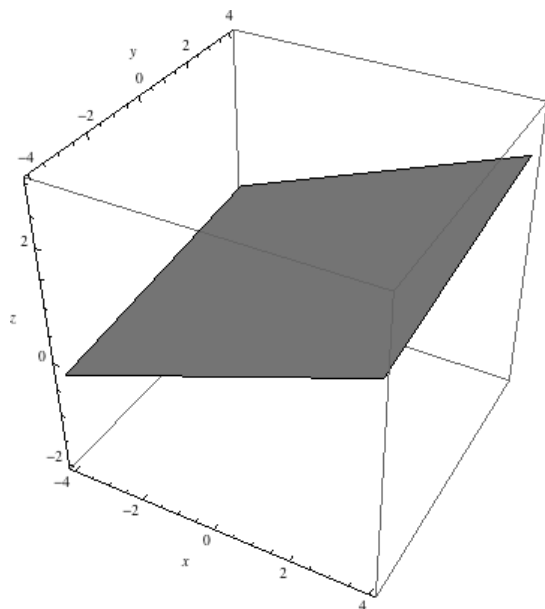
46. (a) The vectors $\mathbf{v} = \overrightarrow{P_1P_2} = 9\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \overrightarrow{P_1P_3} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ both lie in the plane, so that a normal $\mathbf{N}$ to the plane is

$$\mathbf{N} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & -3 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & 2 \\ 4 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 9 & 2 \\ 3 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 9 & -3 \\ 3 & 4 \end{vmatrix} = -11\mathbf{i} - 3\mathbf{j} + 45\mathbf{k}.$$

Using the point $(0, 0, 1)$ and the normal vector $\mathbf{N}$, the general equation of the plane is

$$-11(x - 0) - 3(y - 0) + 45(z - 1) = 0 \quad \text{or} \quad \boxed{-11x - 3y + 45z = 45}.$$

(b)



47. The normals of the two planes are $\mathbf{N}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{N}_2 = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 3 & -4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

To find a point on the line, we need a point that lies on both planes. Set $z = 0$; this gives the equations $x + y = -5$ and $2x + 3y = -1$, which have the solution $x = -14$, $y = 9$. Therefore the point $(-14, 9, 0)$ lies in both planes, so it lies on their line of intersection. So the symmetric equations of the line of intersection are

$$\boxed{\frac{x + 14}{-1} = \frac{y - 9}{2} = \frac{z}{1}}.$$

Answers may vary depending on the specific choice of direction vector and the point on the planes.

48. The normals of the two planes are $\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{N}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

To find a point on the line, we need a point that lies on both planes. Set $x = 0$; this gives the equations $-y + z = 2$ and $y + z = 3$, which have the solution $y = \frac{1}{2}$, $z = \frac{5}{2}$. Therefore the point $(0, \frac{1}{2}, \frac{5}{2})$ lies in

both planes, so it lies on their line of intersection. So the symmetric equations of the line of intersection are

$$\frac{x}{-2} = \frac{y - \frac{1}{2}}{-1} = \frac{z - \frac{5}{2}}{3}.$$

Answers may vary depending on the specific choice of direction vector and the point on the planes.

49. The normals of the two planes are $\mathbf{N}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{N}_2 = \mathbf{j} - \mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

To find a point on the line, we need a point that lies on both planes. Set $z = 0$; this gives the equations $x - y = 2$ and $y = 2$, so that $y = 2$ and $x = 4$. Therefore the point $(4, 2, 0)$ lies in both planes, so it lies on their line of intersection. So the symmetric equations of the line of intersection are

$$\frac{x - 4}{1} = \frac{y - 2}{1} = \frac{z}{1}.$$

Answers may vary depending on the specific choice of direction vector and the point on the planes.

50. The normals of the two planes are $\mathbf{N}_1 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{N}_2 = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 2 & -3 & 4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & 1 \\ -3 & 4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} = -9\mathbf{i} - 6\mathbf{j}.$$

We can multiply this vector by the nonzero constant $-\frac{1}{3}$ to simplify it to $\mathbf{D} = 3\mathbf{i} + 2\mathbf{j}$. To find a point on the line, we need a point that lies on both planes. Set $y = 0$; this gives the equations $2x + z = 1$ and $2x + 4z = 2$, so that $x = z = \frac{1}{3}$. Therefore the point $(\frac{1}{3}, 0, \frac{1}{3})$ lies in both planes, so it lies on their line of intersection. So the symmetric equations of the line of intersection are

$$\frac{x - \frac{1}{3}}{3} = \frac{y}{2}, \quad z = \frac{1}{3}.$$

Answers may vary depending on the specific choice of direction vector and the point on the planes.

51. Using the formula in the text, we have $A = 2$, $B = -1$, $C = 1$, $D = 1$, and $(x_0, y_0, z_0) = (1, 2, -1)$, so the distance from the point to the plane is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|2 \cdot 1 - 1 \cdot 2 + 1 \cdot (-1) - 1|}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}.$$

52. Using the formula in the text, we have $A = 1$, $B = 2$, $C = -3$, $D = 4$, and $(x_0, y_0, z_0) = (-1, 3, -2)$, so the distance from the point to the plane is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot (-1) + 2 \cdot 3 - 3 \cdot (-2) - 4|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}.$$

53. Using the formula in the text, we have $A = -1$, $B = 1$, $C = -3$, $D = 6$, and $(x_0, y_0, z_0) = (2, -1, 1)$, so the distance from the point to the plane is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-1 \cdot 2 + 1 \cdot (-1) - 3 \cdot 1 - 6|}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{12\sqrt{11}}{11}.$$

54. Using the formula in the text, we have $A = -3$, $B = 2$, $C = 1$, $D = 1$, and $(x_0, y_0, z_0) = (-2, 1, 1)$, so the distance from the point to the plane is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-3 \cdot (-2) + 2 \cdot 1 + 1 \cdot 1 - 1|}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{8}{\sqrt{14}} = \boxed{\frac{4\sqrt{14}}{7}}.$$

55. Since both planes have normal vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, they are indeed parallel. To find the distance between them, choose a point on p_1 . To do this, set $x = z = 0$; then $y = -1$ and $(x_0, y_0, z_0) = (0, -1, 0)$ is a point on p_1 . Then using the formula in the text for the distance from a point to a plane, we have from p_2 $A = 2$, $B = 1$, $C = -2$, and $D = 3$, so the distance is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|2 \cdot 0 + 1 \cdot (-1) - 2 \cdot 0 - 3|}{\sqrt{2^2 + 1^2 + 2^2}} = \boxed{\frac{4}{3}}.$$

56. Since both planes have normal vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, they are indeed parallel. To find the distance between them, choose a point on p_1 . To do this, set $y = z = 0$; then $x = -3$ and $(x_0, y_0, z_0) = (-3, 0, 0)$ is a point on p_1 . Then using the formula in the text for the distance from a point to a plane, we have from p_2 $A = 1$, $B = 2$, $C = -2$, and $D = 3$, so the distance is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot (-3) + 2 \cdot 0 - 2 \cdot 0 - 3|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{6}{3} = \boxed{2}.$$

57. Since both planes have normal vector $\mathbf{i} - 2\mathbf{k}$, they are indeed parallel. To find the distance between them, choose a point on p_1 . To do this, set $y = z = 0$; then $x = -1$ and $(x_0, y_0, z_0) = (-1, 0, 0)$ is a point on p_1 . Then using the formula in the text for the distance from a point to a plane, we have for p_2 $A = 1$, $B = 0$, $C = -2$, and $D = 3$, so the distance is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot (-1) + 0 \cdot 0 - 2 \cdot 0 - 3|}{\sqrt{1^2 + 0^2 + 2^2}} = \frac{4}{\sqrt{5}} = \boxed{\frac{4\sqrt{5}}{5}}.$$

58. Since both planes have normal vector $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, they are indeed parallel. To find the distance between them, choose a point on p_1 . To do this, set $y = z = 0$; then $x = 4$ and $(x_0, y_0, z_0) = (4, 0, 0)$ is a point on p_1 . Then using the formula in the text for the distance from a point to a plane, we have for p_2 $A = -1$, $B = 1$, $C = 2$, and $D = -1$, so the distance is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-1 \cdot 4 + 1 \cdot 0 + 2 \cdot 0 - (-1)|}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{6}} = \boxed{\frac{\sqrt{6}}{2}}.$$

59. First convert the symmetric equations for the line to parametric form. The direction vector of the line is $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, and the line passes through the point $(1, -3, 1)$. Therefore the parametric equations are

$$x = 1 + 2t, \quad y = -3 + 4t, \quad z = 1 + t.$$

So to find a point (x, y, z) on the line that also lies on the plane, substitute the equations for x , y , and z from the parametric equations into the equation of the plane and solve for t :

$$2x + y - z = 2(1 + 2t) + (-3 + 4t) - (1 + t) = 7t - 2 = 5,$$

so that $t = 1$. It follows that the line intersects the plane when $t = 1$ in the parametric equations; this is the point $(x, y, z) = (1 + 2 \cdot 1, -3 + 4 \cdot 1, 1 + 1) = \boxed{(3, 1, 2)}$.

60. First convert the symmetric equations for the line to parametric form. The direction vector of the line is $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and the line passes through the point $(-1, 3, 4)$. Therefore the parametric equations are

$$x = -1 + 2t, \quad y = 3 + t, \quad z = 4 - 2t.$$

So to find a point (x, y, z) on the line that also lies on the plane, substitute the equations for x , y , and z from the parametric equations into the equation of the plane and solve for t :

$$x + y - 2z = (-1 + 2t) + (3 + t) - 2(4 - 2t) = 7t - 6 = 8,$$

so that $t = 2$. It follows that the line intersects the plane when $t = 2$ in the parametric equations; this is the point $(x, y, z) = (-1 + 2 \cdot 2, 3 + 2, 4 - 2 \cdot 2) = \boxed{(3, 5, 0)}$.

61. First convert the symmetric equations for the line to parametric form. The direction vector of the line is $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, and the line passes through the point $(3, -4, 1)$. Therefore the parametric equations are

$$x = 3 + t, \quad y = -4 + 2t, \quad z = 1 + 2t.$$

So to find a point (x, y, z) on the line that also lies on the plane, substitute the equations for x , y , and z from the parametric equations into the equation of the plane and solve for t :

$$2x + 3y + z = 2(3 + t) + 3(-4 + 2t) + (1 + 2t) = 10t - 5 = 5,$$

so that $t = 1$. It follows that the line intersects the plane when $t = 1$ in the parametric equations; this is the point $(x, y, z) = (3 + 1, -4 + 2 \cdot 1, 1 + 2 \cdot 1) = \boxed{(4, -2, 3)}$.

62. First convert the symmetric equations for the line to parametric form. The direction vector of the line is $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and the line passes through the point $(-2, 3, 0)$. Therefore the parametric equations are

$$x = -2 + 2t, \quad y = 3 + t, \quad z = 2t.$$

So to find a point (x, y, z) on the line that also lies on the plane, substitute the equations for x , y , and z from the parametric equations into the equation of the plane and solve for t :

$$x + y - z = (-2 + 2t) + (3 + t) - 2t = t + 1 = 3,$$

so that $t = 2$. It follows that the line intersects the plane when $t = 2$ in the parametric equations; this is the point $(x, y, z) = (-2 + 2 \cdot 2, 3 + 2, 2 \cdot 2) = \boxed{(2, 5, 4)}$.

Applications and Extensions

63. To find a point of intersection of ℓ_1 and ℓ_2 , we must find values of t_1 and t_2 so that all three coordinates are equal. This means that we must solve the simultaneous system

$$\begin{aligned} 2 - t_1 &= 4 - t_2 \\ 4 + 2t_1 &= 3 + t_2 \\ -5 + t_1 &= -13 + 3t_2. \end{aligned}$$

The first two equations simplify to $t_1 - t_2 = -2$ and $2t_1 - t_2 = -1$, which has the solution $t_1 = 1$ and $t_2 = 3$. Note that this pair of values satisfies the third equation, so that the lines do in fact intersect. The point of intersection is found by substituting $t_1 = 1$ into the equation for ℓ_1 (or, alternatively, by substituting $t_2 = 3$ into the equation for ℓ_2):

$$(2 - 1)\mathbf{i} + (4 + 2 \cdot 1)\mathbf{j} + (-5 + 1)\mathbf{k}, \text{ so the intersection point is } \boxed{(1, 6, -4)}.$$

To find the angle between the lines, we need the direction vectors for each line; from the vector form these are $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The angle θ between the lines satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-1 \cdot (-1) + 2 \cdot 1 + 1 \cdot 3}{\sqrt{1^2 + 2^2 + 1^2} \sqrt{1^2 + 1^2 + 3^2}} = \frac{6}{\sqrt{6}\sqrt{11}} = \frac{\sqrt{66}}{11},$$

so that

$$\theta = \cos^{-1} \frac{\sqrt{66}}{11} = \sin^{-1} \sqrt{\frac{5}{11}} \approx 0.740 \approx 42.392^\circ.$$

64. To find a point of intersection of ℓ_1 and ℓ_2 , we must find values of t_1 and t_2 so that all three coordinates are equal. This means that we must solve the simultaneous system

$$\begin{aligned} 5 + 2t_1 &= 7 + 3t_2 \\ 6 - t_1 &= 5 - 2t_2 \\ 2t_1 &= 2 - t_2. \end{aligned}$$

The third equation gives $t_2 = 2 - 2t_1$; substitute this into the first equation to get $5 + 2t_1 = 7 + 3(2 - 2t_1) = 13 - 6t_1$, so that $t_1 = 1$ and therefore $t_2 = 0$. Note that this pair of values satisfies the second equation, so that the lines do in fact intersect. The point of intersection is found by substituting $t_2 = 0$ into the equation for ℓ_2 (or, alternatively, by substituting $t_1 = 1$ into the equation for ℓ_1):

$$(7 + 3 \cdot 0)\mathbf{i} + (5 - 2 \cdot 0)\mathbf{j} + (2 - 0)\mathbf{k}, \text{ so the intersection point is } \boxed{(7, 5, 2)}.$$

To find the angle between the lines, we need the direction vectors for each line; from the vector form these are $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$. The angle θ between the lines satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2 \cdot 3 - 1 \cdot (-2) + 2 \cdot (-1)}{\sqrt{2^2 + 1^2 + 2^2} \sqrt{3^2 + 2^2 + 1^2}} = \frac{6}{3\sqrt{14}} = \frac{\sqrt{14}}{7},$$

so that

$$\theta = \cos^{-1} \frac{\sqrt{14}}{7} \boxed{\approx 1.007 \approx 57.688^\circ}.$$

65. To find a point of intersection of ℓ_1 and ℓ_2 , we must find values of t_1 and t_2 so that all three coordinates are equal. This means that we must solve the simultaneous system

$$\begin{aligned} t_1 &= -3 + t_2 \\ 1 + 2t_1 &= 1 - 4t_2 \\ -3 + t_1 &= 2 - 7t_2. \end{aligned}$$

The first equation gives $t_1 = t_2 - 3$; substitute this into the second equation to get $1 + 2(t_2 - 3) = 2t_2 - 5 = 1 - 4t_2$, so that $t_2 = 1$ and therefore $t_1 = -2$. Note that this pair of values satisfies the third equation, so that the lines do in fact intersect. The point of intersection is found by substituting $t_2 = 1$ into the equation for ℓ_2 (or, alternatively, by substituting $t_1 = -2$ into the equation for ℓ_1):

$$(-3 + 1)\mathbf{i} + (1 - 4 \cdot 1)\mathbf{j} + (2 - 7 \cdot 1)\mathbf{k}, \text{ so the intersection point is } \boxed{(-2, -3, -5)}.$$

To find the angle between the lines, we need the direction vectors for each line; from the vector form these are $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$. The angle θ between the lines satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1 \cdot 1 + 2 \cdot (-4) + 1 \cdot (-7)}{\sqrt{1^2 + 2^2 + 1^2} \sqrt{1^2 + 4^2 + 7^2}} = -\frac{14}{\sqrt{6}\sqrt{66}} = -\frac{7}{3\sqrt{11}} = -\frac{7\sqrt{11}}{33}.$$

Since this number is negative, the angle we want actually is such that

$$\cos \theta = \frac{7\sqrt{11}}{33} = \frac{7\sqrt{2}}{\sqrt{198}} \text{ and therefore } \sin \theta = \frac{10}{\sqrt{198}},$$

so that

$$\theta = \sin^{-1} \left(\frac{10}{\sqrt{198}} \right) \approx 0.790 \approx 45.298^\circ.$$

66. To find a point of intersection of ℓ_1 and ℓ_2 , we must find values of t_1 and t_2 so that all three coordinates are equal. This means that we must solve the simultaneous system

$$\begin{aligned} 2 - 3t_1 &= -2t_2 \\ 6t_1 &= 1 + t_2 \\ -2 + 5t_1 &= 2t_2. \end{aligned}$$

The second equation gives $t_2 = 6t_1 - 1$; substitute this into the first equation to get $2 - 3t_1 = -2(6t_1 - 1) = -12t_1 + 2$, so that $t_1 = 0$ and therefore $t_2 = -1$. Note that this pair of values satisfies the third equation, so that the lines do in fact intersect. The point of intersection is found by substituting $t_1 = 0$ into the equation for ℓ_1 (or, alternatively, by substituting $t_2 = -1$ into the equation for ℓ_2):

$$(2 - 3 \cdot 0)\mathbf{i} + (6 \cdot 0)\mathbf{j} + (-2 + 5 \cdot 0)\mathbf{k}, \text{ so the intersection point is } \boxed{(2, 0, -2)}.$$

To find the angle between the lines, we need the direction vectors for each line; from the vector form these are $\mathbf{u} = -3\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The angle θ between the lines satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-3 \cdot (-2) + 6 \cdot 1 + 5 \cdot 2}{\sqrt{3^2 + 6^2 + 5^2} \sqrt{2^2 + 1^2 + 2^2}} = \frac{22}{3\sqrt{70}} = \frac{11\sqrt{70}}{105},$$

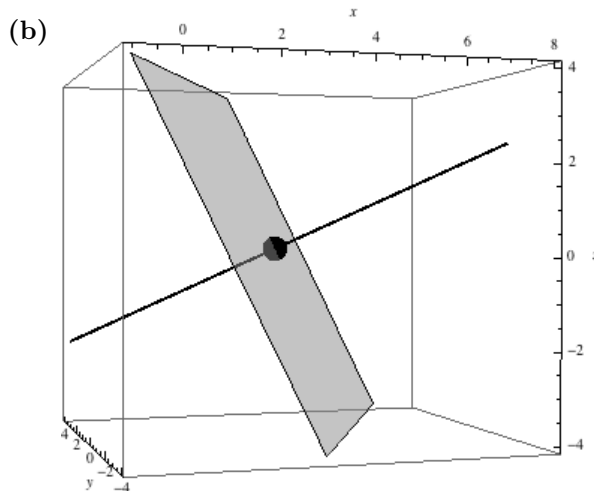
so that

$$\theta = \cos^{-1} \frac{11\sqrt{70}}{105} \approx 0.502 \approx 28.777^\circ.$$

67. (a) Since the line is normal to the plane $2x - y + z - 6 = 0$, its direction vector is parallel to the normal to the plane, which is $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Since the line contains $(1, 2, -1)$, its parametric equations are

$$x = 1 + 2t, \quad y = 2 - t, \quad z = -1 + t.$$

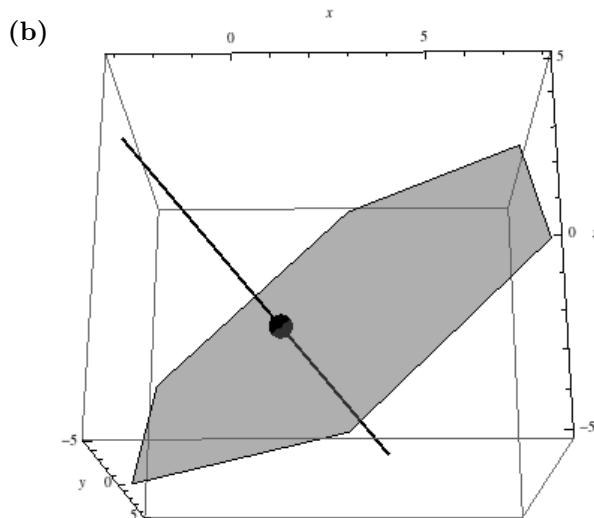
A graph of the plane and the line, with the point of intersection marked, is below.



68. (a) Since the line is normal to the plane $x + y - z - 3 = 0$, its direction vector is parallel to the normal to the plane, which is $\mathbf{i} + \mathbf{j} - \mathbf{k}$. Since the line contains $(2, 3, -1)$, its parametric equations are

$$\boxed{x = 2 + t, \quad y = 3 + t, \quad z = -1 - t}.$$

A graph of the plane and the line, with the point of intersection marked, is shown in part (b).



69. Both lines pass through (x_1, y_1, z_1) . The direction vector of ℓ_1 is $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and the direction vector of ℓ_2 is $\mathbf{w} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$\mathbf{v} \cdot \mathbf{w} = a_1b_1 + a_2b_2 + a_3b_3 = 0,$$

so the direction vectors are orthogonal and therefore the two lines are perpendicular lines through (x_1, y_1, z_1) .

70. The direction vector of ℓ_1 is $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and the direction vector of ℓ_2 is $\mathbf{w} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Suppose that $r = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$. Then $\mathbf{v} = r\mathbf{w}$, so that ℓ_1 and ℓ_2 have parallel direction vectors. Since the lines are assumed distinct, they are parallel lines.

71. p_1 has normal $\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and p_2 has normal $\mathbf{N}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|2 \cdot 1 - 1 \cdot 1 + 1 \cdot 1|}{\sqrt{2^2 + 1^2 + 1^2} \sqrt{1^2 + 1^2 + 1^2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3},$$

so that

$$\theta = \cos^{-1} \frac{\sqrt{2}}{3} \quad \boxed{\approx 1.080 \approx 61.874^\circ}.$$

72. p_1 has normal $\mathbf{N}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and p_2 has normal $\mathbf{N}_2 = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|1 \cdot 2 + 1 \cdot 3 - 1 \cdot (-4)|}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{2^2 + 3^2 + 4^2}} = \frac{9}{\sqrt{3}\sqrt{29}} = \frac{3\sqrt{87}}{29}$$

so that

$$\theta = \cos^{-1} \frac{3\sqrt{87}}{29} \quad \boxed{\approx 0.266 \approx 15.225^\circ}.$$

73. p_1 has normal $\mathbf{N}_1 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, and p_2 has normal $\mathbf{N}_2 = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|2 \cdot 2 - 3 \cdot (-3) + 1 \cdot 4|}{\sqrt{2^2 + 3^2 + 1^2} \sqrt{2^2 + 3^2 + 4^2}} = \frac{17}{\sqrt{14}\sqrt{29}} = \frac{17}{\sqrt{406}}$$

so that

$$\theta = \cos^{-1} \frac{17}{\sqrt{406}} \quad \boxed{\approx 0.567 \approx 32.468^\circ}.$$

74. p_1 has normal $\mathbf{N}_1 = \mathbf{i} - \mathbf{j}$, and p_2 has normal $\mathbf{N}_2 = \mathbf{j} - \mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|1 \cdot 0 - 1 \cdot 1 + 0 \cdot (-1)|}{\sqrt{1^2 + 1^2} \sqrt{1^2 + 1^2}} = \frac{1}{2}.$$

so that

$$\theta = \cos^{-1} \frac{1}{2} = \boxed{\frac{\pi}{3} = 60^\circ}.$$

75. p_1 has normal $\mathbf{N}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, and p_2 has normal $\mathbf{N}_2 = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|2 \cdot 4 - 1 \cdot (-1) + 1 \cdot 6|}{\sqrt{2^2 + 1^2 + 1^2} \sqrt{4^2 + 1^2 + 6^2}} = \frac{15}{\sqrt{6} \sqrt{53}} = \frac{15}{\sqrt{318}},$$

so that

$$\theta = \cos^{-1} \frac{15}{\sqrt{318}} \boxed{\approx 0.571 \approx 32.737^\circ}.$$

76. p_1 has normal $\mathbf{N}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and p_2 has normal $\mathbf{N}_2 = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. So the angle θ between the planes satisfies

$$\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|} = \frac{|1 \cdot 2 + 1 \cdot (-2) - 1 \cdot 2|}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{6} = \frac{1}{3},$$

so that

$$\theta = \cos^{-1} \frac{1}{3} \boxed{\approx 1.231 \approx 70.529^\circ}.$$

77. (a) Both paths are lines; the first is a line with direction vector $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ through $(0, 0, 1)$, while the second is a line with direction vector $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ through $(-3, 0, 4)$.

- (b) The acute angle θ between the paths satisfies

$$\cos \theta = \frac{|\mathbf{v} \cdot \mathbf{w}|}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{|1 \cdot 1 - 1 \cdot 2 - 1 \cdot 4|}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{1^2 + 2^2 + 4^2}} = \frac{5}{\sqrt{63}} = \frac{5}{3\sqrt{7}},$$

so that

$$\theta = \cos^{-1} \frac{5}{3\sqrt{7}} \boxed{\approx 0.889 \approx 50.954^\circ}.$$

- (c) Parametrize the first path using t_1 : $t_1\mathbf{i} - t_1\mathbf{j} + (1 - t_1)\mathbf{k}$, and parametrize the second path using t_2 : $(t_2 - 3)\mathbf{i} + 2t_2\mathbf{j} + (4t_2 - 1)\mathbf{k}$. Then to see if the paths intersect, we must see if there is a solution to the simultaneous system

$$\begin{aligned} t_1 &= -3 + t_2 \\ -t_1 &= 2t_2 \\ 1 - t_1 &= -1 + 4t_2. \end{aligned}$$

The second equation tells us that $t_1 = -2t_2$; substituting into the first equation gives $-2t_2 = -3 + t_2$, so that $t_2 = 1$ and therefore $t_1 = -2$. These values of t_1 and t_2 also satisfy the third equation, so the paths do intersect, at the point

$$(t_1, -t_1, 1 - t_1) = \boxed{(-2, 2, 3)}.$$

- (d) The objects will not collide. Explanations may vary.

78. Complete the squares on each equation:

$$x^2 + y^2 + z^2 - 2x - 4y + 4z = 8$$

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 4z + 4) = 8 + 1 + 4 + 4$$

$$(x - 1)^2 + (y - 2)^2 + (z + 2)^2 = 17$$

$$x^2 + y^2 + z^2 + 2x + 6y + 4z = 20$$

$$(x^2 + 2x + 1) + (y^2 + 6y + 9) + (z^2 + 4z + 4) = 20 + 1 + 9 + 4$$

$$(x + 1)^2 + (y + 3)^2 + (z + 2)^2 = 34.$$

The centers of the spheres are $P_1 = (1, 2, -2)$ and $P_2 = (-1, -3, -2)$. Therefore a direction vector for the line passing through these two points is

$$\mathbf{D} = \overrightarrow{P_1 P_2} = (-1 - 1)\mathbf{i} + (-3 - 2)\mathbf{j} + (-2 - (-2))\mathbf{k} = -2\mathbf{i} - 5\mathbf{j}.$$

Since the line passes through $(1, 2, -2)$, the symmetric equations for the line are

$$\boxed{\frac{x-1}{-2} = \frac{y-2}{-5}, \quad z = -2}.$$

Answers may vary.

79. The direction vector of ℓ_1 is $\mathbf{D}_1 = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and the direction vector of ℓ_2 is $\mathbf{D}_2 = \mathbf{i} - \mathbf{j} + \mathbf{k}$. A vector perpendicular to both of these direction vectors is

$$\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j}.$$

Since the desired line is perpendicular to both of the given lines, it has direction vector $\mathbf{D}$.

The two lines intersect when (parametrizing ℓ_1 with t_1 and ℓ_2 with t_2)

$$\begin{aligned} 1 - t_1 &= 1 + t_2 \\ t_1 &= -t_2 \\ -1 + 2t_1 &= -1 + t_2. \end{aligned}$$

The first two equations both give $t_1 = -t_2$; substituting into the third equation gives $-1 - 2t_2 = -1 + t_2$, so that $t_2 = 0$ and therefore $t_1 = 0$. The two lines intersect when $t_1 = t_2 = 0$, at the point $(1 - 0, 0, 2 \cdot 0 - 1) = (1, 0, -1)$.

Then the line through $(1, 0, -1)$ with direction vector $3\mathbf{i} + 3\mathbf{j}$ has parametric equation

$$\boxed{x = 1 + 3t, \quad y = 3t, \quad z = -1}.$$

Answers may vary.

80. Intuitively, to see that this surface is a plane, start in two dimensions. There, the set of points equidistant from two points is a line. Now rotate that picture through the third dimension; the result is the set of points in space equidistant from the two points, and is a plane.

(a) If the distances are to be the same, then we must have

$$\begin{aligned} \sqrt{(x-1)^2 + (y-3)^2 + (z-0)^2} &= \sqrt{(x+1)^2 + (y-1)^2 + (z-2)^2} \\ x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 &= x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 - 4z + 4 \\ -2x - 6y + 10 &= 2x - 2y - 4z + 6 \\ 4x + 4y - 4z &= 4 \\ \boxed{x + y - z} &= 1. \end{aligned}$$

(b) One point on the plane must be the midpoint of the segment between the two points, which is

$$\left(\frac{1-1}{2}, \frac{3+1}{2}, \frac{0+2}{2} \right) = (0, 2, 1).$$

That segment must be normal to the plane (think about the two-dimensional model), so a normal vector to the plane is $(1 - (-1))\mathbf{i} + (3 - 1)\mathbf{j} + (0 - 2)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Therefore an equation of the plane is

$$2(x - 0) + 2(y - 2) - 2(z - 1) = 0, \quad \text{or} \quad 2x + 2y - 2z = 2, \quad \text{or} \quad \boxed{x + y - z = 1}.$$

81. The normals of the two planes are $\mathbf{N}_1 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{N}_2 = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 2\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}.$$

To find a point on the line, we need a point that lies on both planes. Set $z = 0$; this gives the equations $2x + y = 6$ and $x - y = 4$, which have the solution $x = \frac{10}{3}$, $y = -\frac{2}{3}$. Therefore the point $(\frac{10}{3}, -\frac{2}{3}, 0)$ lies in both planes, so it lies on their line of intersection. So the symmetric equations of the line of intersection are

$$\boxed{\frac{x - \frac{10}{3}}{2} = \frac{y + \frac{2}{3}}{-7} = \frac{z}{-3}}.$$

Answers may vary depending on the specific choice of point.

82. If the line is to be parallel to the given plane, it will be orthogonal to the normal to that plane. Therefore the line is orthogonal to both $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and to $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. So a direction vector for the line is

$$\mathbf{D} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Since the line passes through $(2, 0, -3)$, its symmetric equations are

$$\boxed{\frac{x - 2}{1} = \frac{y}{-1} = \frac{z + 3}{-1}}.$$

83. First determine the equations of the line and the plane. For the line, it has direction vector $(2 - 0)\mathbf{i} + (1 - 2)\mathbf{j} + (-3 - (-2))\mathbf{k} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and passes through $(0, 2, -2)$, so it has parametric equations

$$x = 2t, \quad y = 2 - t, \quad z = -2 - t.$$

To determine an equation for the plane, we first find a normal vector. The three points given allow us to find two vectors that lie in the plane:

$$(1 - 0)\mathbf{i} + (3 - 4)\mathbf{j} + (-2 - (-2))\mathbf{k} = \mathbf{i} - \mathbf{j} \quad \text{and} \quad (2 - 0)\mathbf{i} + (2 - 4)\mathbf{j} + (-3 - (-2))\mathbf{k} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

A normal vector to the plane is then the cross product of these two vectors:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -2 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 0 \\ -2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = \mathbf{i} + \mathbf{j}.$$

Since the plane passes through $(0, 4, -2)$, it has equation

$$1(x - 0) + 1(y - 4) + 0(z + 2) = 0, \quad \text{or} \quad x + y = 4.$$

To find a point (x, y, z) on the line that also lies on the plane, substitute the equations for x , y , and z from the parametric equations into the equation of the plane and solve for t :

$$x + y = 2t + (2 - t) = t + 2 = 4,$$

so that $t = 2$. It follows that the line intersects the plane when $t = 2$ in the parametric equations; this is the point $(x, y, z) = (2 \cdot 2, 2 - 2, -2 - 2) = \boxed{(4, 0, -4)}$.

84. Since the plane contains the two given points, the vector $(2-1)\mathbf{i} + (2-0)\mathbf{j} + (-1-1)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ lies in the plane. Next, since the plane is parallel to the given line, the direction vector for that line, which is $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, also lies in the plane. We take the cross product of these two vectors to get a normal vector to the plane:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -1 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 6\mathbf{i} + 3\mathbf{k}.$$

Since the plane passes through $(1, 0, 1)$, an equation of the plane is

$$6(x-1) + 0(y-0) + 3(z-1) = 0, \quad \text{or} \quad 6x + 3z = 9, \quad \text{or} \quad \boxed{2x + z = 3}.$$

85. Since the desired plane is tangent to the sphere, the radius of the sphere must be the distance from the center to the plane, so a normal vector to the desired plane is the vector from the center of the circle, $(1, -2, 2)$ to the given point $(2, -1, 0)$, which is $(1-2)\mathbf{i} + (-2-(-1))\mathbf{j} + (2-0)\mathbf{k} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The plane passes through the point $(2, -1, 0)$, so its equation is

$$-(x-2) - (y+1) + 2(z-0) = 0, \quad \text{or} \quad -x - y + 2z = -1, \quad \text{or} \quad \boxed{x + y - 2z = 1}.$$

86. Since the desired sphere is tangent to the plane, the radius of the sphere must be the distance from the center of the sphere, $(x_0, y_0, z_0) = (-2, 1, 5)$ to the plane. Using the formula for the distance from a point to a plane, we have $A = 1$, $B = 2$, $C = -2$, and $D = 8$, so the distance, which is the radius of the sphere, is

$$r = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot (-2) + 2 \cdot 1 - 2 \cdot 5 - 8|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{18}{3} = 6.$$

Since the radius of the sphere is 6 and its center is at $(-2, 1, 5)$, its equation is

$$\boxed{(x+2)^2 + (y-1)^2 + (z-5)^2 = 36}.$$

87. From the symmetric equations, parametric equations for the two lines are

$$\begin{aligned} \ell_1: \quad x &= 2 + t_1, & y &= -1 + 2t_1, & z &= 1 + 2t_1 \\ \ell_2: \quad x &= -1 + t_2, & y &= -8 + 3t_2, & z &= -3t_2. \end{aligned}$$

The point of intersection of the lines is found by solving the simultaneous system

$$\begin{aligned} 2 + t_1 &= -1 + t_2 \\ -1 + 2t_1 &= -8 + 3t_2 \\ 1 + 2t_1 &= -3t_2. \end{aligned}$$

The second equation gives $2t_1 = 3t_2 - 7$, while the third gives $2t_1 = -3t_2 - 1$. Therefore $3t_2 - 7 = -3t_2 - 1$, so that $t_2 = 1$ and then $t_1 = -2$. Note that this pair of values satisfies the first equation as well, so that in fact the two lines do intersect. The point of intersection (x_0, y_0, z_0) is found by substituting $t_2 = 1$ into the equation for ℓ_2 (or alternatively, by substituting $t_1 = -2$ into the equation for ℓ_1):

$$x_0 = -1 + 1, \quad y_0 = -8 + 3 \cdot 1, \quad z_0 = -3 \cdot 1 \quad \text{so that} \quad (x_0, y_0, z_0) = (0, -5, -3).$$

A direction vector for the desired line must be of the given lines, which are $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, so we find a direction vector by computing the cross product of these vectors:

$$\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 3 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -12\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

Since the line we are seeking passes through the point of intersection, $(0, -5, -3)$, symmetric equations for the line are

$$\boxed{\frac{x}{-12} = \frac{y+5}{5} = \frac{z+3}{1}}.$$

Challenge Problems

88. Let $ax + by + cz = 0$ be the desired plane $\mathcal{P}$. Then it has normal vector $\mathbf{n} = \langle a, b, c \rangle$; since this plane is orthogonal to a plane with normal vector $\langle 1, -2, -1 \rangle$, these two vectors must be orthogonal, so that $\langle a, b, c \rangle \cdot \langle 1, -2, -1 \rangle = a - 2b - c = 0$ and therefore $a = 2b + c$. Now, we wish $\mathbf{n}$ to form a 60° angle with the positive y -axis, $\mathbf{j}$, so that

$$\frac{\mathbf{n} \cdot \mathbf{j}}{\|\mathbf{n}\| \|\mathbf{j}\|} = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{2}.$$

Simplifying gives $\sqrt{a^2 + b^2 + c^2} = 2b$, so that $4b^2 = a^2 + b^2 + c^2$. Now substitute $2b + c$ for a in this equation, giving

$$4b^2 = (2b + c)^2 + b^2 + c^2 = 4b^2 + 4bc + c^2 + b^2 + c^2, \text{ so that } 2c^2 + 4bc + b^2 = 0.$$

Treating this as a quadratic in c and solving gives

$$c = \frac{-4b \pm \sqrt{16b^2 - 8b^2}}{4} = -b \pm \frac{|b|\sqrt{2}}{2}.$$

Then b must be nonzero; otherwise $c = 0$ and then $a = 2b + c = 0$ as well, so that $\mathcal{P}$ is not defined. Since we can scale all of a , b , and c by any nonzero factor in the equation for $\mathcal{P}$, choose $b = 1$. This gives two solutions:

$$\begin{aligned} c &= -1 + \frac{\sqrt{2}}{2}, & a &= 2b + c = 1 + \frac{\sqrt{2}}{2}, \\ c &= -1 - \frac{\sqrt{2}}{2}, & a &= 2b + c = 1 - \frac{\sqrt{2}}{2}. \end{aligned}$$

So there are two such planes:

$$\left(1 + \frac{\sqrt{2}}{2}\right)x + y + \left(-1 + \frac{\sqrt{2}}{2}\right)z = 0 \text{ and } \left(1 - \frac{\sqrt{2}}{2}\right)x + y + \left(-1 - \frac{\sqrt{2}}{2}\right)z = 0.$$

89. (a) The plane has normal vector $\langle A, B, C \rangle$, so this is a direction vector for the line through $P_0 = (x_0, y_0, z_0)$. Therefore the line has parametric equations

$$x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct.$$

- (b) If (x, y, z) lies on the line, then it satisfies the parametric equations; solving for the term involving t gives $x - x_0 = At$, $y - y_0 = Bt$, and $z - z_0 = Ct$. Since it also lies on the plane, we can find the needed value of t by substituting the parametric equations for x , y , and z into the equation of the plane and solving for t :

$$\begin{aligned} A(x_0 + At) + B(y_0 + Bt) + C(z_0 + Ct) &= D \\ A^2t + B^2t + C^2t &= D - Ax_0 - By_0 - Cz_0 \\ t &= \frac{D - Ax_0 - By_0 - Cz_0}{A^2 + B^2 + C^2} = \frac{-(Ax_0 + By_0 + Cz_0) + D}{A^2 + B^2 + C^2}. \end{aligned}$$

- (c) Since the line through P_0 is normal to the plane, the length of the segment from P_0 to the plane, meeting the plane at P , is in fact the distance to the plane, since if Q is any other point on the plane then $\triangle P_0PQ$ is a right triangle with hypotenuse P_0Q , so that $P_0Q > P_0P$. Therefore this

is the distance from P_0 to P ; call it d . But then

$$\begin{aligned}
 d &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \\
 &= \sqrt{\left(A \left(\frac{-(Ax_0 + By_0 + Cz_0) + D}{A^2 + B^2 + C^2} \right)\right)^2 + \left(B \left(\frac{-(Ax_0 + By_0 + Cz_0) + D}{A^2 + B^2 + C^2} \right)\right)^2 + \left(C \left(\frac{-(Ax_0 + By_0 + Cz_0) + D}{A^2 + B^2 + C^2} \right)\right)^2} \\
 &= \sqrt{(A^2 + B^2 + C^2) \cdot \frac{(-(Ax_0 + By_0 + Cz_0) + D)^2}{(A^2 + B^2 + C^2)^2}} \\
 &= \sqrt{\frac{(-(Ax_0 + By_0 + Cz_0) + D)^2}{A^2 + B^2 + C^2}} \\
 &= \frac{|-(Ax_0 + By_0 + Cz_0) + D|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.
 \end{aligned}$$

90. Since the planes are parallel, the distance between them is the distance from any point on the first plane to the second plane. Let $P = (x_0, y_0, z_0)$ lie on the first plane, so that $Ax_0 + By_0 + Cz_0 = D_1$. Then using the formula for the distance from a point to a plane, the distance from P_0 to the second plane is

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}.$$

91. The plane p_1 containing A with direction vectors $\mathbf{D}_1$ and $\mathbf{D}_2$, and the plane p_2 containing B with the same direction vectors are parallel, since they have the same normal vector $\mathbf{D}_1 \times \mathbf{D}_2$. Also p_1 contains the first skew line, and p_2 the second. So the distance between the skew lines is the distance between these parallel planes. Suppose the two planes have equations

$$rx + sy + tz = u_1, \quad rx + sy + tz = u_2,$$

and that $A = (a_0, a_1, a_2)$ lies on the first plane and $B = (b_0, b_1, b_2)$ on the second. The normal vector $\langle r, s, t \rangle$ to both planes is therefore $\mathbf{D}_1 \times \mathbf{D}_2$. Then the distance between the planes is, from Problem 90,

$$\begin{aligned}
 d &= \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|u_2 - u_1|}{\sqrt{r^2 + s^2 + t^2}} \\
 &= \frac{|(rb_0 + sb_1 + tb_2) - (ra_0 + ra_1 + ra_2)|}{\|\mathbf{D}_1 \times \mathbf{D}_2\|} \\
 &= \frac{|r(b_0 - a_0) + s(b_1 - a_1) + t(b_2 - a_2)|}{\|\mathbf{D}_1 \times \mathbf{D}_2\|} \\
 &= \frac{\overrightarrow{AB} \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{\|\mathbf{D}_1 \times \mathbf{D}_2\|} \\
 &= \frac{\mathbf{w} \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{\|\mathbf{D}_1 \times \mathbf{D}_2\|} \\
 &= \left\| \frac{\mathbf{w} \cdot (\mathbf{D}_1 \times \mathbf{D}_2)}{\|\mathbf{D}_1 \times \mathbf{D}_2\|^2} (\mathbf{D}_1 \times \mathbf{D}_2) \right\| \\
 &= \|\text{proj}_{\mathbf{D}_1 \times \mathbf{D}_2} \mathbf{w}\|
 \end{aligned}$$

where $\mathbf{w} = \overrightarrow{AB}$.

92. Applying Problem 91, we see that the first line has direction vector $\mathbf{D}_1 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and contains the point $(3, 0, 0)$, while the second line has direction vector $\mathbf{D}_2 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and contains the point $(0, -1, 2)$. Then $\mathbf{w} = (0 - 3, -1 - 0, 2 - 0) = (-3, -1, 2)$, and the cross product is

$$\mathbf{D}_1 \times \mathbf{D}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}.$$

Therefore the distance between the lines is

$$\frac{|\mathbf{w} \cdot (\mathbf{D}_1 \times \mathbf{D}_2)|}{\|\mathbf{D}_1 \times \mathbf{D}_2\|} = \frac{|-3 \cdot (-1) - 1 \cdot 3 + 2 \cdot (-7)|}{\sqrt{1^2 + 3^2 + 7^2}} = \boxed{\frac{14}{\sqrt{59}}}.$$

93. To find points on each line at which the distance is a minimum, use the fact that ℓ_1 and ℓ_2 are each perpendicular to the shortest line joining them. The first line ℓ_1 has direction vector $\mathbf{D}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and contains the point $(1, 2, -6)$, while the second line ℓ_2 has direction vector $\mathbf{D}_2 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and contains the point $(1, -2, 10)$. Therefore the vector equations for the two lines are

$$\ell_1 : (1 + t_1, 2 + t_1, -6 + t_1), \quad \ell_2 : (1 + 2t_2, -2 - 3t_2, 10 + t_2).$$

So if P and Q are any points on ℓ_1 and ℓ_2 respectively, then

$$P = (1 + t_1, 2 + t_1, -6 + t_1), \quad Q = (1 + 2t_2, -2 - 3t_2, 10 + t_2), \quad \overrightarrow{PQ} = (2t_2 - t_1, -3t_2 - t_1 - 4, t_2 - t_1 + 16).$$

If P and Q are the closest points on the two lines, then $\overrightarrow{PQ}$ must be perpendicular to both ℓ_1 and ℓ_2 ; that is, it must be perpendicular to the direction vectors of ℓ_1 and ℓ_2 , so that

$$\begin{aligned} \overrightarrow{PQ} \cdot \vec{D}_1 &= (2t_2 - t_1) \cdot 1 + (-3t_2 - t_1 - 4) \cdot 1 + (t_2 - t_1 + 16) \cdot 1 = -3t_1 + 12 = 0 \\ \overrightarrow{PQ} \cdot \vec{D}_2 &= (2t_2 - t_1) \cdot 2 + (-3t_2 - t_1 - 4) \cdot (-3) + (t_2 - t_1 + 16) \cdot 1 = 14t_2 + 28 = 0. \end{aligned}$$

The first of these gives $t_1 = 4$; the second gives $t_2 = -2$. So the points on the two lines whose distance from each other is the shortest are

$$\ell_1 : (1 + 4, 2 + 4, -6 + 4) = \boxed{(5, 6, -2)}, \quad \ell_2 : (1 + 2 \cdot (-2), -2 - 3 \cdot (-2), 10 - 2) = \boxed{(-3, 4, 8)}.$$

It follows that the shortest distance between the lines is the distance between these two points, which is

$$\sqrt{(5 - (-3))^2 + (6 - 4)^2 + (-2 - 8)^2} = \sqrt{8^2 + 2^2 + 10^2} = \sqrt{168} = \boxed{2\sqrt{42}}.$$

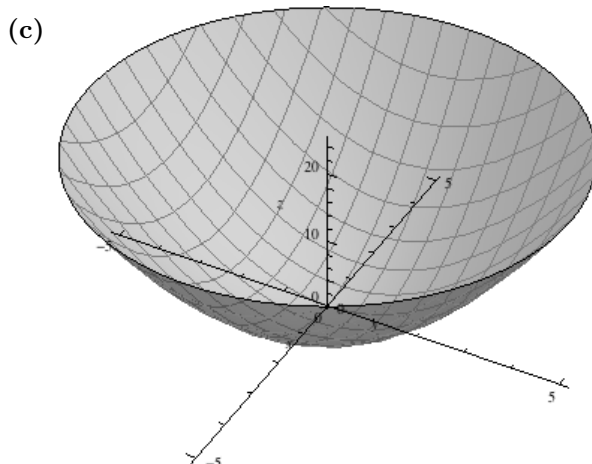
10.7: Quadric Surfaces

Concepts and Vocabulary

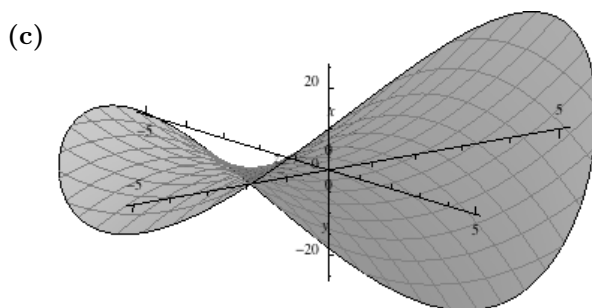
- (c) is correct. The trace in the xy -plane is found by setting $z = 0$, which gives $\frac{x^2}{2} - \frac{y^2}{3} = 1$.
- The intercepts are $(\pm 4, 0, 0)$ and $(0, 0, -4)$.
- True. See the discussion in the text.
- (b) is correct; this is an elliptic cone. See the definition in the text.
- The quadric surface $y^2 - x^2 = 4$ is called a *hyperbolic cylinder*. See the discussion in the text.
- The trace in the xz -plane is $z = -\frac{x^2}{5^2}$, and the trace in the yz -plane is $z = \frac{y^2}{2^2}$. Therefore $(0, 0, 0)$ is a local maximum for the first trace, and a local minimum for the second trace, so it is a *saddle point*.

Skill Building

7. (a) This is an elliptic paraboloid; in fact, since $a = b = 1$, it is a paraboloid of revolution.
- (b) The only intercept is the origin. The trace in the xy -plane has equation $x^2 + y^2 = 0$, so it consists only of the origin; the trace in the xz -plane is the parabola $z = x^2$; the trace in the yz -plane is the parabola $z = y^2$. The z -axis is the axis of the surface.

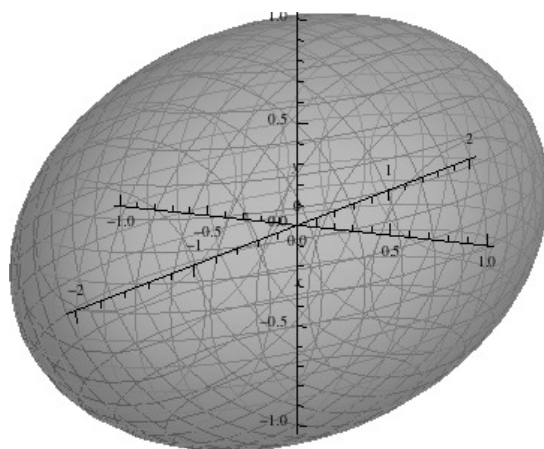


8. (a) This is a hyperbolic paraboloid.
- (b) The only intercept is the origin. The trace in the xy -plane has equation $x^2 - y^2 = (x - y)(x + y) = 0$, so it consists of the two lines $y = x$ and $y = -x$. The trace in the xz -plane is the parabola $z = x^2$; the trace in the yz -plane is the parabola $z = -y^2$.



9. (a) Divide through by 4 to put the equation into standard form: $\frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{1^2} = 1$. This is an ellipsoid of revolution (since two of the denominators are the same).
- (b) The intercepts are $(\pm 1, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, \pm 1)$. The trace in the xy -plane has equation $x^2 + \frac{y^2}{4} = 1$, which is an ellipse. The trace in the xz -plane is $x^2 + z^2 = 1$, which is a circle of radius 1. The trace in the yz -plane is $\frac{y^2}{4} + z^2 = 1$, which is also an ellipse.

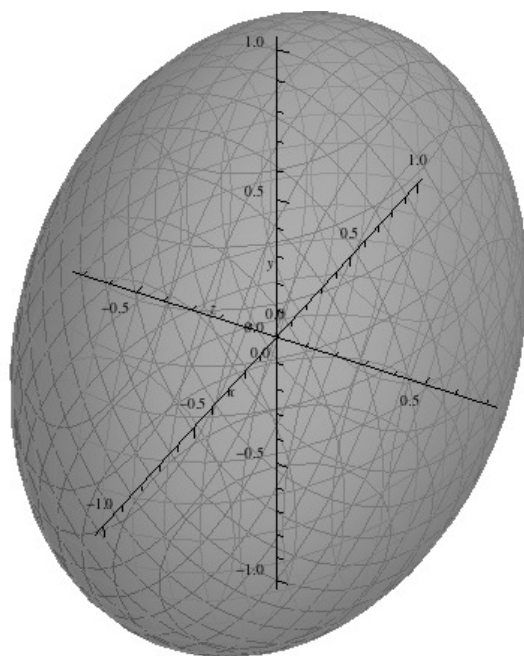
(c)



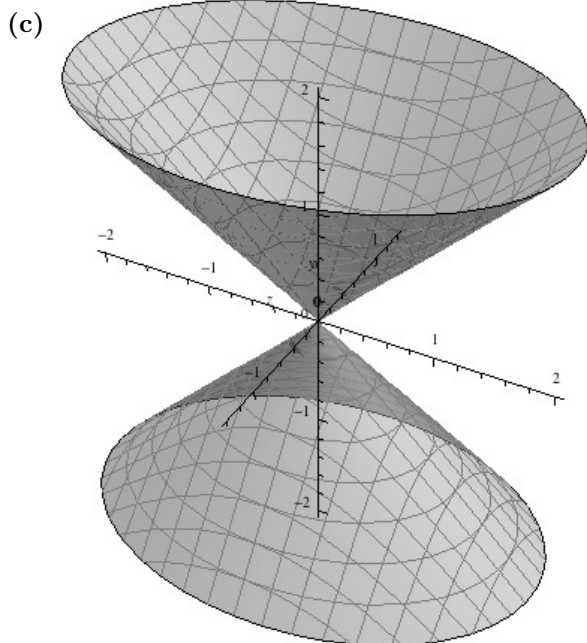
10. (a) Putting this equation into standard form gives $\frac{x^2}{(1/\sqrt{2})^2} + y^2 + z^2 = 1$. This is an ellipsoid of revolution (since two of the denominators are the same).

(b) The intercepts are $(\pm \frac{1}{\sqrt{2}}, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. The trace in the xy -plane has equation $\frac{x^2}{(1/\sqrt{2})^2} + y^2 = 1$, which is an ellipse. The trace in the xz -plane is $\frac{x^2}{(1/\sqrt{2})^2} + z^2 = 1$, which is also an ellipse. The trace in the yz -plane is $y^2 + z^2 = 1$, which is a circle of radius 1.

(c)

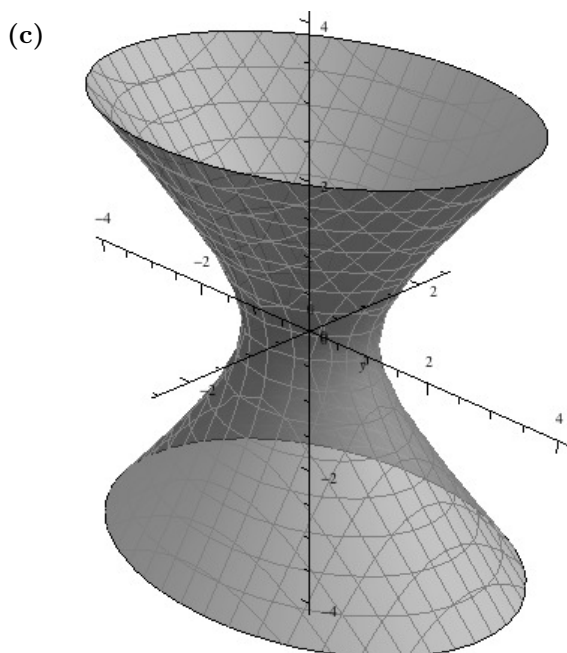


11. (a) Putting this equation into standard form gives $z^2 = x^2 + \frac{y^2}{(1/\sqrt{2})^2}$. This is an elliptic cone.
- (b) The only intercept is the origin. The trace in the xy -plane has equation $x^2 + \frac{y^2}{(1/\sqrt{2})^2} = 0$, which has only the solution $(0, 0)$, so the trace in the xy -plane is the origin. The trace in the xz -plane is $z^2 = x^2$, or $(z - x)(z + x) = 0$, so it consists of the pair of lines $z = x$ and $z = -x$. The trace in the yz -plane is $z^2 = \frac{y^2}{(1/\sqrt{2})^2}$; after simplification this becomes $z^2 = 2y^2$, so $(z - \sqrt{2}y)(z + \sqrt{2}y) = 0$, so it consists of the pair of lines $z = \pm \sqrt{2}y$. The z -axis is the axis of the cone.

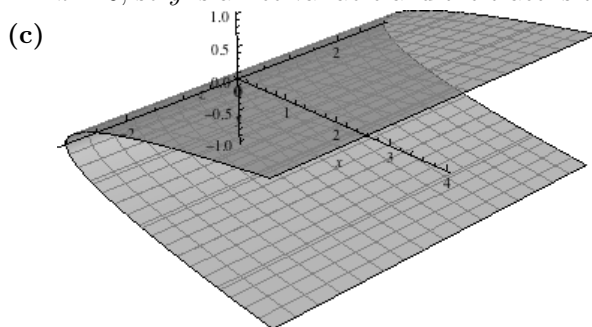


12. (a) Putting this equation into its standard form by dividing through by 2 gives $x^2 + \frac{y^2}{(1/\sqrt{2})^2} - z^2 = 1$. This is a hyperboloid of one sheet.

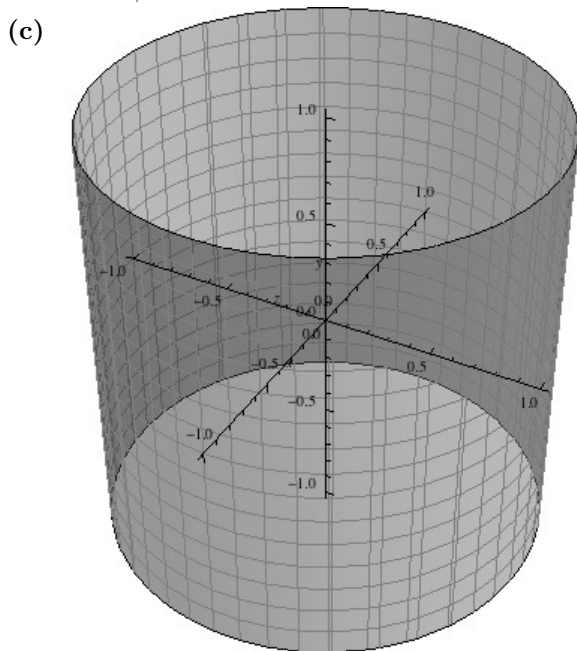
- (b) The x -intercepts are ± 1 . The y -intercepts are solutions to $\frac{y^2}{(1/\sqrt{2})^2} = 1$, which are $y = \pm \frac{1}{\sqrt{2}}$. There are no z -intercepts, since the equation becomes $-z^2 = 1$ upon setting x and y to zero. Therefore the intercepts are $(\pm 1, 0, 0)$ and $(0, \pm \frac{1}{\sqrt{2}}, 0)$. The trace in the xy -plane has equation $x^2 + \frac{y^2}{(1/\sqrt{2})^2} = 1$, which is an ellipse. The trace in the xz -plane is $x^2 - z^2 = (x - z)(x + z) = 0$, so it consists of the pair of lines $z = x$ and $z = -x$. The trace in the yz -plane is $\frac{y^2}{(1/\sqrt{2})^2} - z^2 = 1$, which is a hyperbola. The z -axis is the axis of the hyperboloid.



13. (a) Putting this equation into standard form gives $z^2 = \frac{1}{4}x$. This is a parabolic cylinder.
- (b) The only x -intercept is 0. The y -intercept is found by setting $x = z = 0$, which gives the equation $0 = 0$, so every value of y is a y -intercept. The only z -intercept is 0. Therefore the intercepts are $(0, y, 0)$, where y is a free variable. The trace in the xy -plane has equation $x = 0$, so it is the y -axis. The trace in the xz -plane is $z^2 = \frac{1}{4}x$, which is a parabola. The trace in the yz -plane is $z = 0$, so y is a free variable and the trace is the y -axis. y is the free variable in the surface.



14. (a) This is an elliptic cylinder; in fact, it is a circular cylinder since $a = b = 1$.
- (b) The intercepts are $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$. (Setting $x = y = 0$ in the equation gives $0 = 1$, so there are no z -intercepts.) The trace in the xy -plane is the circle $x^2 + y^2 = 1$. The trace in the xz -plane is $y^2 = 1$, so it consists of the two lines $y = 1$ and $y = -1$. The trace in the yz -plane is $x^2 = 1$, so it consists of the two lines $x = 1$ and $x = -1$. The z -axis is the axis of the cylinder.



15. (a) Divide through by 4 to put this equation into standard form, giving

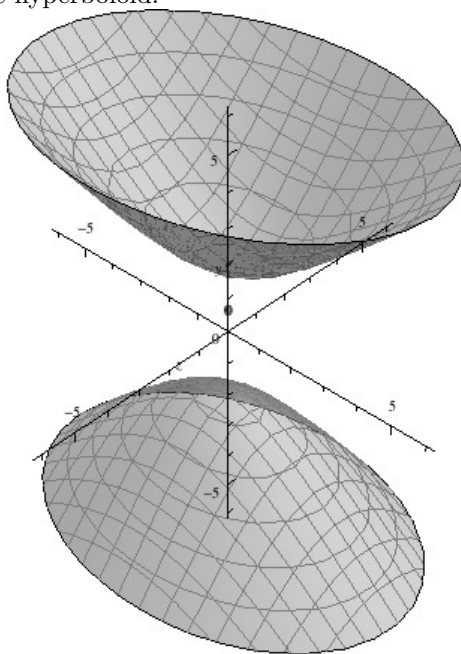
$$\frac{x^2}{2^2} + \frac{y^2}{(1/\sqrt{2})^2} - \frac{z^2}{2^2} = -1.$$

This is a hyperboloid of two sheets.

- (b) The x -intercepts are the solutions to $x^2 = -4$, which do not exist, so there are no x -intercepts. The y -intercepts are the solutions to $2y^2 = -4$, which do not exist, so there are no y -intercepts either. The z -intercept is found by setting $x = y = 0$ in the equation, giving $-z^2 = -4$, so the z -intercepts are ± 2 . Therefore the intercepts are $(0, 0, \pm 2)$. The trace in the xy -plane is the

equation $x^2 + 2y^2 = -4$, which has no solutions, so the trace is empty. However, if $z = k$ with $|k| > 2$, then the trace in the plane $z = k$ parallel to the xy -plane is the ellipse $x^2 + 2y^2 = k^2$. The trace in the xz -plane is $x^2 - z^2 = -4$, or $z^2 - x^2 = 4$, which is a hyperbola. The trace in the yz -plane is $2y^2 - z^2 = -4$, or $z^2 - 2y^2 = 4$, which is also a hyperbola. The z -axis is the axis of the hyperboloid.

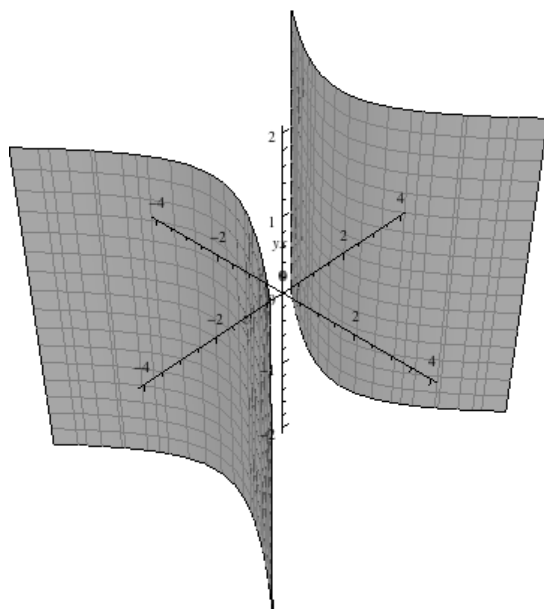
(c)



16. (a) This is a hyperbolic cylinder; in standard form is it $\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1$.

(b) The x -intercepts are the solutions to $-x^2 = 4$, which do not exist, so there are no x -intercepts. The y -intercepts are the solutions to $y^2 = 4$, so the y -intercepts are ± 2 . The z -intercepts are found by setting $x = y = 0$ in the equation, giving $0 = 4$, which has no solutions. So there are no z -intercepts. Therefore the only intercepts are $(0, \pm 2, 0)$. The trace in the xy -plane is the equation $y^2 - x^2 = 4$, which is a hyperbola. The trace in the xz -plane is $-x^2 = 4$, which is empty. The trace in the yz -plane is $y^2 = 4$, so it consists of the two lines $y = 2$ and $y = -2$. z is the free variable in the surface.

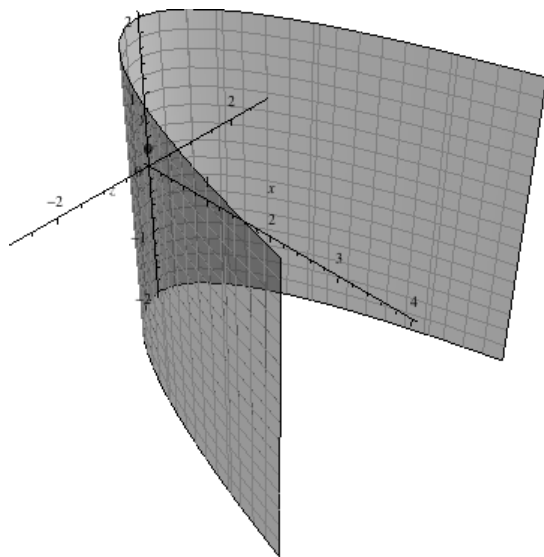
(c)



17. (a) This is a parabolic cylinder.

(b) The only x -intercept is 0, as is the only y -intercept. The z -intercepts are found by setting $x = y = 0$ in the equation, giving $0 = 0$, so any value of z is a z -intercept. Therefore the intercepts are $(0, 0, z)$ where z is a free variable. The trace in the xy -plane is the equation $2x = y^2$, which is a parabola. The trace in the xz -plane is $2x = 0$, so $x = 0$ and the trace is the z -axis. The trace in the yz -plane is $y^2 = 0$, so $y = 0$ and the trace is again the z -axis. z is the free variable in the surface.

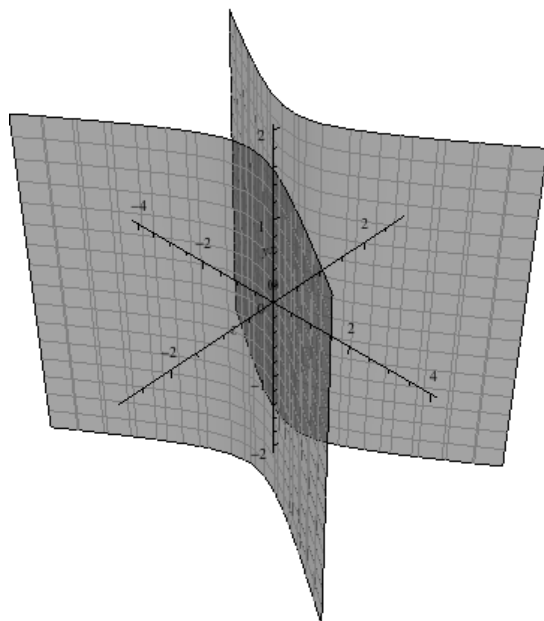
(c)



18. (a) This is a hyperbolic cylinder; in standard form it is $\frac{y^2}{(1/2)^2} - x^2 = 1$.

(b) The x -intercept consists of solutions to $-x^2 = 1$; since there are none, there are no x -intercepts. The y -intercept is solutions to $4y^2 = 1$, so it is $\pm \frac{1}{2}$. The z -intercepts are found by setting $x = y = 0$ in the equation, giving $0 = 1$, so there are no z -intercepts. Therefore the intercepts are $(0, \pm \frac{1}{2}, 0)$. The trace in the xy -plane is the equation $4y^2 - x^2 = 1$, which is a hyperbola. The trace in the xz -plane is $-x^2 = 1$, which is empty. The trace in the yz -plane is $4y^2 = 1$, so it consists of the lines $y = \pm \frac{1}{2}$.

(c)

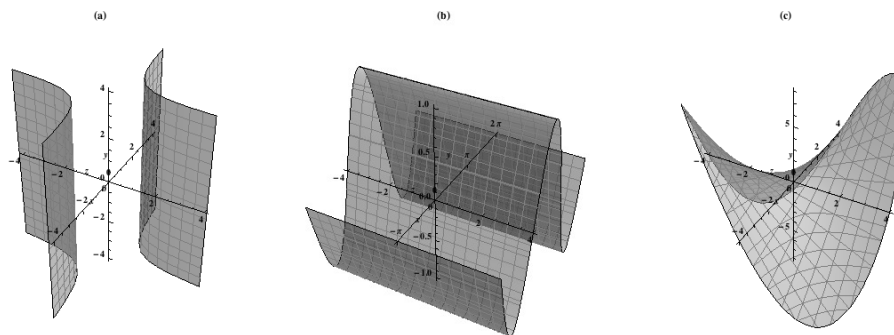


19. This is the equation of a hyperbolic paraboloid, so it is either **A** or **C**. The xy trace of this equation is the equation $4y^2 - x^2 = (2y - x)(2y + x) = 0$, so it consists of the two lines $x = 2y$ and $x = -2y$. This matches **C** but not **A**.
20. This is the equation of an elliptic paraboloid, so it must be **F**.
21. This is the equation of a hyperboloid of one sheet, so it must be **B**.
22. This is the equation of an ellipsoid, so it must be **D**.
23. This is the equation of a parabolic cylinder, so it must be **E**.
24. This is the equation of a hyperbolic paraboloid, so it is either **A** or **C**. The xz trace of this equation is $x^2 - z^2 = (x - z)(x + z) = 0$, so it consists of the two lines $z = x$ and $z = -x$. This matches **A** but not **C**.
25. This is the equation of an elliptic paraboloid, $z = -3x^2 - 4y^2$ (this paraboloid opens down rather than up). It must correspond to graph **L**.
26. Completing the square gives $3x^2 + (2y + 1)^2 = 1$. This is an elliptic cylinder with axis along the z -axis, centered at $(0, -\frac{1}{2})$, which matches graph **I**.
27. Subtracting 1 from both sides gives $3x^2 + 2y^2 - (z - 2)^2 = -1$, which is a hyperboloid of two sheets that has been shifted up by 2 units. This corresponds to graph **K**.
28. This is the equation of a hyperbolic paraboloid, so it must correspond to graph **H**.
29. Completing the square gives $x^2 + 2y^2 - (z - 2)^2 = 0$, or $(z - 2)^2 = x^2 + 2y^2$. This is an elliptic cone with axis along the z -axis, shifted up by 2 units, which matches graph **J**.
30. This is the equation of an ellipsoid of revolution, which matches graph **G**.

Applications and Extensions

31. Answers may vary.
32. As defined in the text, a cylinder is a quadric surface in which one of the variables is missing, so it looks like the planar curve defined by the equation, extended through all possible values of the missing variable. Here x is missing, so this is a cylinder oriented along the x -axis, whose trace in the yz -plane is a sine curve.

33.



Challenge Problem

34. Following the hint, if we rewrite the equation as shown, then it factors as

$$\left(\frac{x}{a} - \frac{z}{c}\right) \left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right) \left(1 + \frac{y}{b}\right), \quad \text{so that} \quad \frac{\frac{x}{a} + \frac{z}{c}}{1 - \frac{y}{b}} = \frac{1 + \frac{y}{b}}{\frac{x}{a} - \frac{z}{c}}.$$

If $P = (x_0, y_0, z_0)$ lies on the one-sheeted hyperboloid, then x_0 , y_0 , and z_0 , satisfy the equation above. Call that common ratio h_P . Now consider the two planes

$$\frac{x}{a} + \frac{z}{c} = h_P \left(1 - \frac{y}{b}\right) \quad \text{and} \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{h_P} \left(1 + \frac{y}{b}\right).$$

The point P lies on both of these planes by construction, so that the planes are not parallel and therefore intersect in a line. Additionally, if (x_1, y_1, z_1) is any point on that line of intersection, so that it lies on both planes, then multiplying the two equations gives

$$\left(\frac{x_1}{a} - \frac{z_1}{c}\right) \left(\frac{x_1}{a} + \frac{z_1}{c}\right) = \left(1 - \frac{y_1}{b}\right) \left(1 + \frac{y_1}{b}\right), \quad \text{so that} \quad \frac{x_1^2}{a^2} - \frac{z_1^2}{c^2} = 1 - \frac{y_1^2}{b^2},$$

and therefore (x_1, y_1, z_1) lies on the surface. So the line of intersection of these two planes lies completely in the surface and contains P .

Now, the equations can be regrouped differently:

$$\left(\frac{x}{a} - \frac{z}{c}\right) \left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right) \left(1 + \frac{y}{b}\right), \quad \text{so that} \quad \frac{\frac{x}{a} + \frac{z}{c}}{1 + \frac{y}{b}} = \frac{1 - \frac{y}{b}}{\frac{x}{a} - \frac{z}{c}}.$$

Again, if $P = (x_0, y_0, z_0)$ lies on the one-sheeted hyperboloid, then x_0 , y_0 , and z_0 , satisfy the equation above. Call that common ratio k_P . Now consider the two planes

$$\frac{x}{a} + \frac{z}{c} = k_P \left(1 + \frac{y}{b}\right) \quad \text{and} \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{k_P} \left(1 - \frac{y}{b}\right).$$

Again the point P lies on both of these planes by construction, so that the planes are not parallel and therefore intersect in a line. Additionally, if (x_1, y_1, z_1) is any point on that line of intersection, so that it lies on both planes, then multiplying the two equations gives the same product as before, so that again (x_1, y_1, z_1) lies on the surface, so that the entire line of intersection lies on the surface.

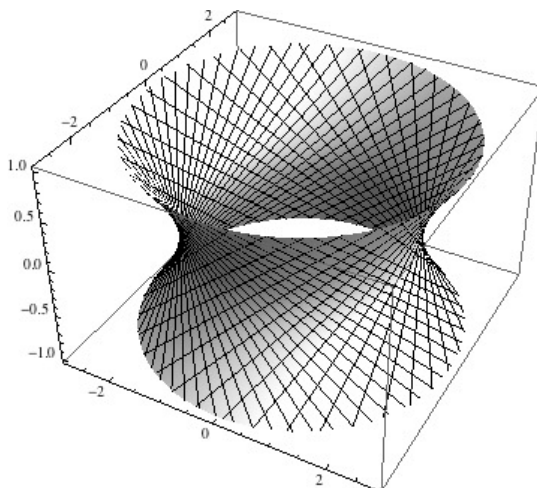
The only remaining point is to show that these two lines are in fact distinct lines. To do that it suffices to show that they have distinct direction vectors. The direction vector to each line is the cross product of the normal vectors of the two planes, so we get for direction vectors

$$\begin{aligned} \mathbf{d}_1 &= \left\langle \frac{1}{a}, \frac{h_P}{b}, \frac{1}{c} \right\rangle \times \left\langle \frac{1}{a}, -\frac{h_P}{b}, -\frac{1}{c} \right\rangle = \left\langle 0, \frac{2}{ac}, -\frac{2h_P}{ab} \right\rangle \\ \mathbf{d}_2 &= \left\langle \frac{1}{a}, -\frac{k_P}{b}, \frac{1}{c} \right\rangle \times \left\langle \frac{1}{a}, \frac{k_P}{b}, -\frac{1}{c} \right\rangle = \left\langle 0, \frac{2}{ac}, \frac{2k_P}{ab} \right\rangle. \end{aligned}$$

Since the second components are identical, these two vectors are parallel if and only if $-\frac{2h_P}{ab} = \frac{2k_P}{ab}$, so if and only if

$$\frac{\frac{x_0}{a} + \frac{z_0}{c}}{1 - \frac{y_0}{b}} = h_P = -k_P = -\frac{\frac{x_0}{a} + \frac{z_0}{c}}{1 + \frac{y_0}{b}}.$$

The numerators are identical, so these two are equal if and only if $1 - \frac{y_0}{b} = -(1 + \frac{y_0}{b}) = -1 - \frac{y_0}{b}$, which is impossible. So the direction vectors are different and the lines are distinct.



Chapter Review

- The other vertices of the box are the other points whose x -coordinates are 1 or 2, whose y -coordinates are 0 or 3, and whose z -coordinates are 2 or 4:

$$(1, 0, 4), (1, 3, 2), (1, 3, 4), (2, 0, 2), (2, 3, 2), (2, 0, 4).$$

- The set of points satisfying $y = 3$ is a plane parallel to the xz -plane and lying 3 units above it.

- (a) $\mathbf{v} = \overrightarrow{P_1P_2} = \langle 7 - 5, 1 - 0, 3 - (-2) \rangle = \langle 2, 1, 5 \rangle$.

- (b) $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

- The magnitude is

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

- The distance between the points is

$$\|\langle 7, 5, 1 \rangle - \langle 1, 2, 3 \rangle\| = \sqrt{(7-1)^2 + (5-2)^2 + (1-3)^2} = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7.$$

- $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4^2 = 16$.

- Completing the square, we get

$$\begin{aligned} x^2 + y^2 + z^2 - 4x + 8y &= 5 \\ (x^2 - 4x + 4) + (y^2 + 8y + 16) + z^2 &= 5 + 4 + 16 \\ (x-2)^2 + (y+4)^2 + z^2 &= 25 = 5^2, \end{aligned}$$

so that this sphere has center $(2, -4, 0)$ and radius 5.

- We have

$$\begin{aligned} 3\mathbf{v} - 2\mathbf{w} &= 3\langle -2, -1, 3 \rangle - 2\langle 5, 4, -2 \rangle \\ &= \langle 3 \cdot (-2) - 2 \cdot 5, 3 \cdot (-1) - 2 \cdot 4, 3 \cdot 3 - 2 \cdot (-2) \rangle = \langle -16, -11, 13 \rangle. \end{aligned}$$

- First compute

$$\mathbf{v} + \mathbf{w} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Then

$$\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{i} + \mathbf{j} + 2\mathbf{k}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}.$$

10. The magnitude of
- $\mathbf{v}$
- is

$$\|\mathbf{v}\| = \sqrt{4^2 + 12^2 + 3^2} = \sqrt{169} = 13,$$

so that a unit vector in the direction of $\mathbf{v}$ is

$$\boxed{\frac{1}{13}\mathbf{v} = \frac{4}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} - \frac{3}{13}\mathbf{k}}.$$

11. Since the angle between
- $\mathbf{v}$
- and the positive
- x
- axis is
- 120°
- , we have

$$\mathbf{v} = \|\mathbf{v}\| (\cos 120^\circ \mathbf{i} + \sin 120^\circ \mathbf{j}) = 25 \left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \right) = \boxed{-\frac{25}{2}\mathbf{i} + \frac{25\sqrt{3}}{2}\mathbf{j}}.$$

12. Multiply the second equation by 3 giving
- $6\mathbf{w} + 3\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$
- . Now add it to the first equation, giving
- $7\mathbf{w} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
- , so that
- $\mathbf{w} = \frac{5}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} - \frac{1}{7}\mathbf{k}$
- . Now substitute this back into the first equation, giving

$$\begin{aligned} \mathbf{w} - 3\mathbf{v} &= 2\mathbf{i} + \mathbf{j} - \mathbf{k} \\ \frac{5}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} - \frac{1}{7}\mathbf{k} - 3\mathbf{v} &= 2\mathbf{i} + \mathbf{j} - \mathbf{k} \\ 3\mathbf{v} &= \frac{5}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} - \frac{1}{7}\mathbf{k} - (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ 3\mathbf{v} &= -\frac{9}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \\ \mathbf{v} &= \boxed{-\frac{3}{7}\mathbf{i} - \frac{1}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}}. \end{aligned}$$

13. (a) We get

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{5^2 + 8^2 + 2^2} = \sqrt{93} \\ \cos \alpha &= \frac{v_1}{\|\mathbf{v}\|} = \frac{5}{\sqrt{93}} \\ \cos \beta &= \frac{v_2}{\|\mathbf{v}\|} = \frac{8}{\sqrt{93}} \\ \cos \gamma &= \frac{v_3}{\|\mathbf{v}\|} = -\frac{2}{\sqrt{93}}. \end{aligned}$$

- (b) From part (a), we have

$$\mathbf{v} = \|\mathbf{v}\| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}) = \boxed{\sqrt{93} \left(\frac{5}{\sqrt{93}}\mathbf{i} + \frac{8}{\sqrt{93}}\mathbf{j} - \frac{2}{\sqrt{93}}\mathbf{k} \right)}.$$

14. The vector projection of
- $\mathbf{v}$
- onto
- $\mathbf{w}$
- is

$$\begin{aligned} \text{proj}_{\mathbf{w}} \mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \mathbf{w} = \frac{3 \cdot 5 + 1 \cdot (-1) - 2 \cdot 1}{\sqrt{3^2 + 1^2 + 2^2} \sqrt{5^2 + 1^2 + 1^2}} \mathbf{w} \\ &= \frac{12}{\sqrt{14}\sqrt{27}} \mathbf{w} = \frac{4}{\sqrt{42}} \mathbf{w} = \frac{20}{\sqrt{42}} \mathbf{i} - \frac{4}{\sqrt{42}} \mathbf{j} + \frac{4}{\sqrt{42}} \mathbf{k}. \end{aligned}$$

Since the projection is parallel to $\mathbf{w}$, we let

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}} \mathbf{v} = \frac{20}{\sqrt{42}} \mathbf{i} - \frac{4}{\sqrt{42}} \mathbf{j} + \frac{4}{\sqrt{42}} \mathbf{k},$$

and then

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = \left(3 - \frac{20}{\sqrt{42}} \right) \mathbf{i} + \left(1 + \frac{4}{\sqrt{42}} \right) \mathbf{j} - \left(2 + \frac{4}{\sqrt{42}} \right) \mathbf{k}.$$

15. The work done is the dot product of the force applied and the vector along which the object is moved:

$$\mathbf{F} \cdot \mathbf{u} = 2 \cdot 3 - 1 \cdot 2 - 1 \cdot (-5) = \boxed{9 \text{ J}}.$$

16. The force along the x -axis is $12\mathbf{i}$, along the y -axis is $16\mathbf{j}$, and along the z -axis is $15\mathbf{k}$, so the total force is

$$\boxed{\mathbf{F} = 12\mathbf{i} + 16\mathbf{j} + 15\mathbf{k}, \text{ with a magnitude of } \sqrt{12^2 + 16^2 + 15^2} = 25}.$$

17. The angle θ between the vectors satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1 \cdot 2 + 2 \cdot (-3) + 3 \cdot (-1)}{\sqrt{1^2 + 2^2 + 3^2} \sqrt{2^2 + 3^2 + 1^2}} = \frac{-7}{14} = -\frac{1}{2},$$

so that

$$\theta = \cos^{-1} \left(-\frac{1}{2} \right) = \boxed{\frac{2\pi}{3} = 120^\circ}.$$

18. The sides emanating from A are

$$\overrightarrow{AB} = \mathbf{u} = \langle 2 - 1, -1 - 0, 1 - 1 \rangle = \langle 1, -1, 0 \rangle, \quad \overrightarrow{AC} = \mathbf{v} = \langle -2 - 1, 1 - 0, 0 - 1 \rangle = \langle -3, 1, -1 \rangle,$$

so the angle θ between those sides satisfies

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1 \cdot (-3) - 1 \cdot 1 + 0 \cdot (-1)}{\sqrt{1^2 + 1^2 + 0^2} \sqrt{3^2 + 1^2 + 1^2}} = -\frac{4}{\sqrt{22}},$$

so that

$$\theta = \cos^{-1} \left(-\frac{4}{\sqrt{22}} \right) \approx 2.592 \approx 148.518^\circ.$$

19. To find the distance between the planes, choose a point on p_1 . To do this, set $y = z = 0$; then $x = 1$ and $(x_0, y_0, z_0) = (1, 0, 0)$ is a point on p_1 . Then using the formula in the text for the distance from a point to a plane, we have from p_2 $A = -2$, $B = -2$, $C = 2$, and $D = 2$, so the distance is

$$\frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-2 \cdot 1 - 2 \cdot 0 + 2 \cdot 0 - 2|}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{4}{\sqrt{12}} = \boxed{\frac{2}{\sqrt{3}}}.$$

20. $\boldsymbol{\omega}$ points in the direction of $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$; a unit vector in this direction is

$$\frac{1}{\sqrt{2^2 + 3^2 + 6^2}}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

Since the angular speed ω is constant, we have

$$\boldsymbol{\omega} = \omega \left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right).$$

The position of the object is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, so the velocity is

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ &= \omega \left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= \omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ 1 & 2 & 3 \end{vmatrix} \\ &= \left(-\frac{3}{7}\mathbf{i} + \frac{1}{7}\mathbf{k} \right) \omega. \end{aligned}$$

The object's speed is then

$$\|\mathbf{v}\| = \omega \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{1}{7}\right)^2} = \boxed{\frac{\sqrt{10}}{7}\omega \text{ m/s}}.$$

21. These lines both pass through the point $(1, 0, -1)$ (when $t = 0$ in each case). ℓ has direction vector $\langle -1, 1, 2 \rangle$, and m has direction vector $\langle 1, -1, 1 \rangle$. Therefore we are looking for a line through $(1, 0, -1)$ with direction vector perpendicular to both of these. We find that vector by taking the cross product:

$$\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j}.$$

So the line perpendicular to the given lines has parametric equations

$$\boxed{x = 1 + 3t, \quad y = 3t, \quad z = -1}.$$

Answers may vary depending on specific choices for the direction vector and the point.

22. Using parametric equations, we get

$$\boxed{x = 1 + 2t, \quad y = 2 - t, \quad z = 3 - 4t}.$$

23. The two vectors are orthogonal when their dot product is zero:

$$\mathbf{v} \cdot \mathbf{w} = (2\mathbf{i} - \mathbf{j} - 4\mathbf{k}) \cdot (a\mathbf{i} + a\mathbf{j} + 3\mathbf{k}) = 2a - a - 12 = a - 12.$$

This is zero when $\boxed{a = 12}$, so this is the value of a for which the vectors are orthogonal.

24. Since we are given two endpoints of a diameter, the center of the sphere is the midpoint of the segment between those two points, which is

$$\left(\frac{2-2}{2}, \frac{2+6}{2}, \frac{-3+5}{2} \right) = (0, 4, 1).$$

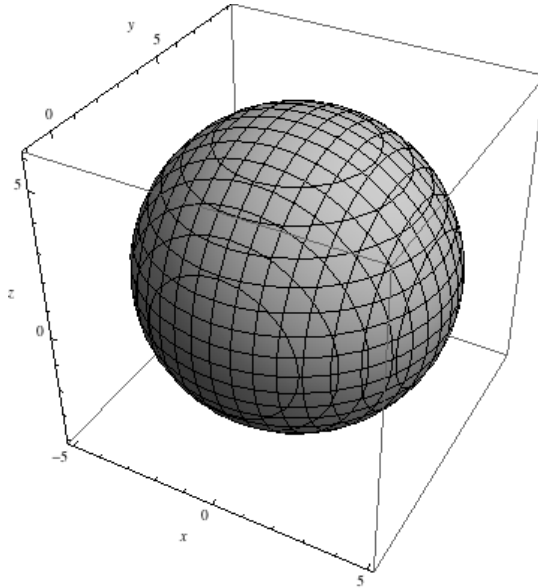
The radius of the sphere is therefore the distance from the center to either of the points; we use the first point:

$$r = \sqrt{(2-0)^2 + (2-4)^2 + (-3-1)^2} = \sqrt{4+4+16} = \sqrt{24},$$

so that the equation of the sphere is

$$\boxed{x^2 + (y-4)^2 + (z-1)^2 = 24}.$$

The sphere is below.



25. The normals of the two planes are $\mathbf{N}_1 = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{N}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; these vectors are not parallel, so the planes intersect in a line. A direction vector to the line lies in both planes, so it is orthogonal to both normals. Therefore a direction vector for the line is

$$\mathbf{D} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 2 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

To find a point on the line, we need a point that lies on both planes. Set $z = 0$; this gives the equations $x - y = 3$ and $2x + 2y = 1$, which have the solution $x = \frac{7}{4}$, $y = -\frac{5}{4}$. Therefore the point $(\frac{7}{4}, -\frac{5}{4}, 0)$ lies in both planes, so it lies on their line of intersection. So the parametric equations of the line of intersection are

$$\boxed{x = \frac{7}{4} - t, \quad y = -\frac{5}{4} + 3t, \quad z = 4t}.$$

Answers may vary depending on specific choices for the direction vector and the point.

26. Since the plane passes through the three given points, the vectors

$$\langle 0, b, 0 \rangle - \langle a, 0, 0 \rangle = \langle -a, b, 0 \rangle, \quad \text{and} \quad \langle 0, 0, c \rangle - \langle a, 0, 0 \rangle = \langle -a, 0, c \rangle$$

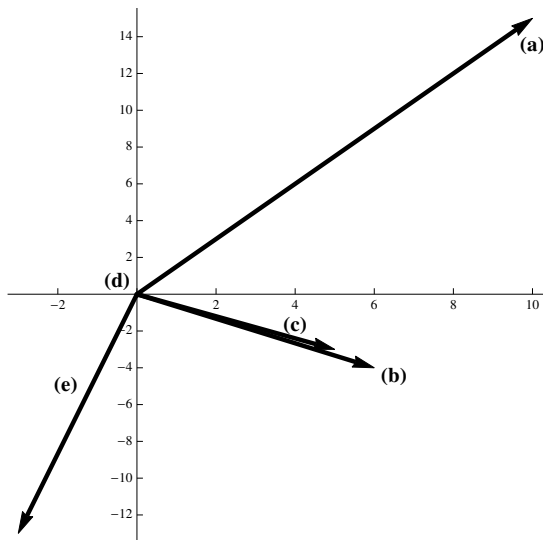
both lie in the plane. Therefore a normal vector to the plane is the cross product of these two vectors:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \mathbf{i} \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - \mathbf{j} \begin{vmatrix} -a & 0 \\ -a & c \end{vmatrix} + \mathbf{k} \begin{vmatrix} -a & b \\ -a & 0 \end{vmatrix} = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}.$$

Since the plane passes through $(a, 0, 0)$, its equation is

$$bc(x - a) + acy + abz = 0, \quad \text{or} \quad \boxed{bcx + acy + abz = abc}.$$

27.



28. From the symmetric equations, both lines have direction vector $\langle 6, 3, 7 \rangle$, so they are either the same line or they are parallel. The parametric equations of ℓ_1 are

$$x = -5 + 6t, \quad y = 2 + 3t, \quad z = -4 + 7t.$$

The two lines are the same if $(-5, -1, 2)$, which lies on ℓ_2 , also lies on ℓ_1 . But $-5 = -5 + 6t$ implies that $t = 0$, so that $y = 2$ and $z = -4$. Therefore this point is not on ℓ_1 , so the lines are parallel.

29. The direction vector of the given line is $\langle 2, 3, 1 \rangle$, so this is the normal $\mathbf{N}$ to the desired plane. Since the plane contains $(2, -1, 4)$, its equation is $2(x - 2) + 3(y - (-1)) + 1(z - 4) = 0$, which simplifies to the general equation

$$\boxed{2x + 3y + z = 5}.$$

30. Since the direction vectors $\mathbf{d}_1 = \langle 1, 3, 1 \rangle$ and $\mathbf{d}_2 = \langle 2, 3, 1 \rangle$ are not multiples of one another, they are not parallel, so the lines are not parallel. To see if they intersect, convert the equation for ℓ_2 to its parametric form, using s as the parameter:

$$\ell_2: \quad x = -3 + 2s, \quad y = 3 + 3s, \quad z = s.$$

If these lines intersect, then the equations

$$t + 1 = 2s - 3$$

$$3t = 3s + 3$$

$$t - 1 = s$$

have a solution. The third equation gives $s = t - 1$; substitute into the first equation to get $t + 1 = 2(t - 1) - 3 = 2t - 5$, so that $t = 6$ and then $s = 5$. These values of s and t also satisfy the second equation, so the lines do intersect. The point of intersection may be found by substituting $s = 5$ into the equations for ℓ_2 to get the point $(7, 18, 5)$.

31. Since the plane contains the two given lines, it contains their direction vectors $\mathbf{d}_1 = \langle 1, 2, 2 \rangle$ and $\mathbf{d}_2 = \langle 1, 3, -3 \rangle$. A line normal to this plane must be normal to both of these direction vectors, so it is parallel to their cross product, which will therefore be a direction vector for the line:

$$\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 3 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -12\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

We need a point on the line, which is the point of intersection of the two given lines. To find their point of intersection, convert each set of symmetric equations to parametric equations:

$$\begin{aligned}\ell_1: \quad x &= 2 + t, & y &= -1 + 2t, & z &= 1 + 2t \\ \ell_2: \quad x &= -1 + s, & y &= -8 + 3s, & z &= -3s.\end{aligned}$$

The intersection point is found by solving the three equations

$$\begin{aligned}2 + t &= -1 + s \\ -1 + 2t &= -8 + 3s \\ 1 + 2t &= -3s.\end{aligned}$$

The first equation gives $t = s - 3$; substitute into the second equation to get $-1 + 2(s - 3) = -8 + 3s$, so that $2s - 7 = 3s - 8$, or $s = 1$. Then $t = -2$, and this pair of values also satisfies the third equation. The point of intersection may then be found by substituting $s = 1$ into ℓ_2 (or substituting $t = -2$ into ℓ_1):

$$x = -1 + 1 = 0, \quad y = -8 + 3 \cdot 1 = -5, \quad z = -3 \cdot 1 = -3,$$

so the point of intersection is $(0, -5, -3)$. Combining this with the direction vector above, the symmetric equations of the desired line are

$$\boxed{\frac{x}{-12} = \frac{y + 5}{5} = \frac{z + 3}{1}}.$$

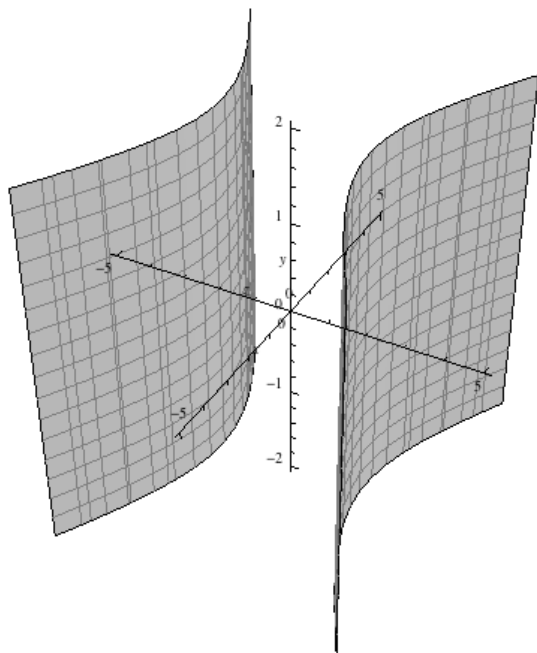
32. A direction vector for the line is $\overrightarrow{P_1P_2} = \langle 5, 3, 6 \rangle - \langle 4, 0, 7 \rangle = \langle 1, 3, -1 \rangle$. Since it passes through the point $(4, 0, 7)$, its symmetric equations are

$$\boxed{\frac{x - 4}{1} = \frac{y}{3} = \frac{z - 7}{-1}}.$$

33. (a) This is a hyperbolic cylinder.

(b) The x -intercepts are solutions to $\frac{x^2}{4} = 1$, so they are ± 2 . The y -intercepts are solutions to $-\frac{y^2}{9} = 1$, which has no solutions, so there are no y -intercepts. The z -intercepts are found by setting $x = y = 0$ in the equation, giving $0 = 1$, so there are no z -intercepts. Therefore the only intercepts are $(\pm 2, 0, 0)$. The trace in the xy -plane is the original equation, which is a hyperbola. The trace in the xz -plane is $\frac{x^2}{4} = 1$, or $x^2 = 4$, so it consists of the two lines $x = \pm 2$. The trace in the yz -plane is $-\frac{y^2}{9} = 1$, so it is empty.

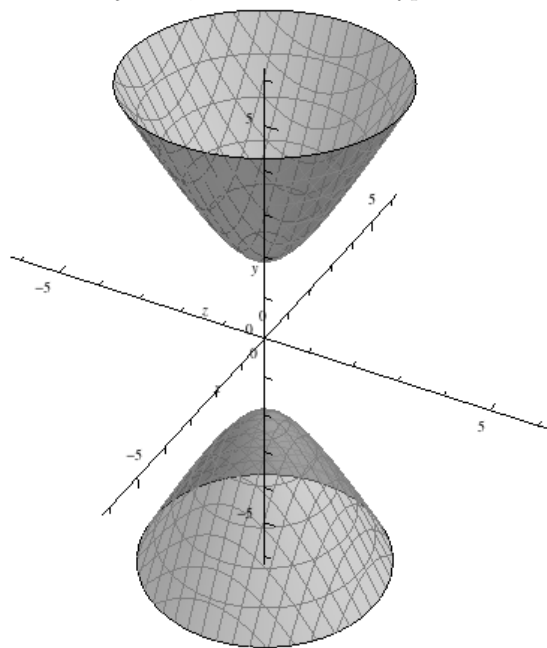
(c)



34. (a) This is a hyperboloid of two sheets.

- (b) The x -intercepts are the solutions to $x^2 = -1$, which do not exist, so there are no x -intercepts. The y -intercepts are the solutions to $y^2 = -1$, which do not exist, so there are no y -intercepts either. The z -intercepts are found by setting $x = y = 0$ in the equation, giving $-\frac{z^2}{4} = -1$, so the z -intercepts are ± 2 . Therefore the only intercepts are $(0, 0, \pm 2)$. The trace in the xy -plane is the equation $x^2 + y^2 = -1$, which has no solutions, so the trace is empty. The trace in the xz -plane is $x^2 - \frac{z^2}{4} = -1$, or $z^2 - 4x^2 = 4$, which is a hyperbola. The trace in the yz -plane is $y^2 - \frac{z^2}{4} = -1$, or $z^2 - 4y^2 = 4$, which is also a hyperbola. The z -axis is the axis of the hyperboloid.

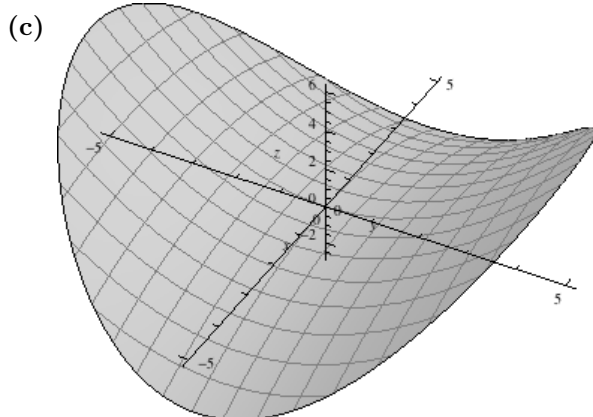
(c)



35. (a) This is a hyperbolic paraboloid.

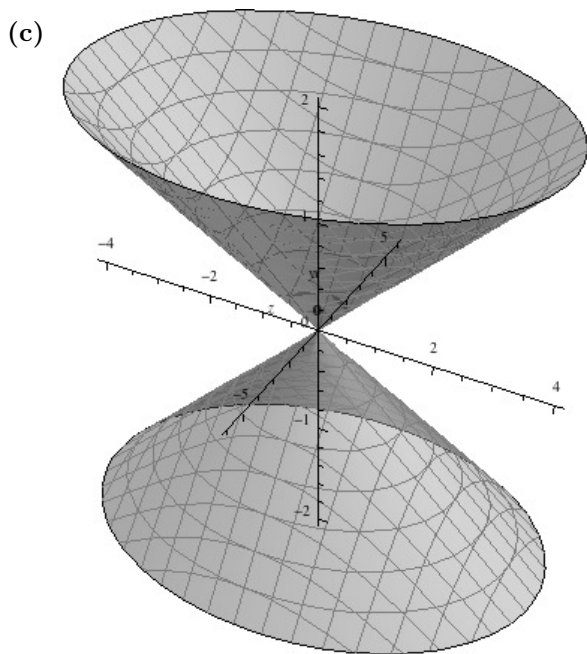
- (b) The only intercept is the origin. The trace in the xy -plane has equation $\frac{x^2}{4} - \frac{y^2}{9} = 0$, so it consists of the two lines $3x = 2y$ and $3x = -2y$. The trace in the xz -plane is the parabola

$z = \frac{1}{4}x^2$; the trace in the yz -plane is the parabola $z = -\frac{1}{9}y^2$.



36. (a) This is an elliptic cone.

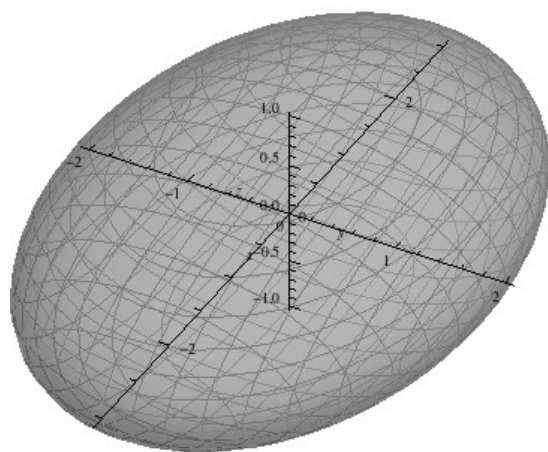
(b) The only intercept is the origin. The trace in the xy -plane has equation $\frac{x^2}{4} + \frac{y^2}{9} = 0$, which has only the solution $(0, 0)$, so the trace in the xy -plane is the origin. The trace in the xz -plane is $z^2 = \frac{x^2}{4}$, or $(2z - x)(2z + x) = 0$, so it consists of the pair of lines $2z = x$ and $2z = -x$. The trace in the yz -plane is $z^2 = \frac{y^2}{9}$; after simplification this becomes $9z^2 = y^2$, so $(3z - y)(3z + y) = 0$, so it consists of the pair of lines $3z = \pm y$. The z -axis is the axis of the cone.



37. (a) This is an ellipsoid.

(b) The intercepts are $(\pm 2, 0, 0)$, $(0, \pm 3, 0)$, and $(0, 0, \pm 1)$. The trace in the xy -plane has equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, which is an ellipse. The trace in the xz -plane is $\frac{x^2}{4} + z^2 = 1$, which is also an ellipse. The trace in the yz -plane is $\frac{y^2}{9} + z^2 = 1$, which is also an ellipse.

(c)



Vector Functions

11.1 Vector Functions and Their Derivatives

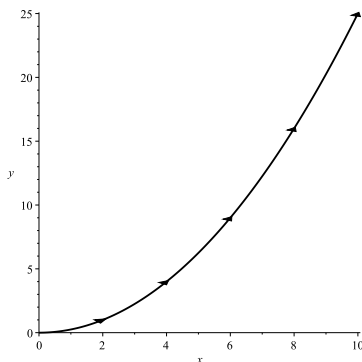
Concepts and Vocabulary

1. The components of $\mathbf{r}(t)$ are $x(t) = 4t + 1$, $y(t) = \sqrt{4 - t}$, and $z(t) = 1$. Since $x(t)$ is defined for all real numbers, $y(t)$ is defined for all real numbers $t \leq 4$, and $z(t)$ is defined for all real numbers, the domain of $\mathbf{r}(t)$ is the intersection of these three domains, or $\boxed{\{t \mid t \leq 4\}}$.
2. False. For a vector function to be continuous at t_0 , all of its component functions must be continuous at t_0 .
3. True. By the theorem on differentiating a vector function, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and if $x(t)$, $y(t)$, and $z(t)$ are each differentiable real functions, then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.
4. True. Since the dot product of two vectors is a scalar function, the derivative of the dot product must also be a scalar function.
5. False. By the scalar multiplication formula, $\frac{d}{dt}[f(t)\mathbf{r}(t)] = \left[\frac{d}{dt}f(t)\right]\mathbf{r}(t) + f(t)\left[\frac{d}{dt}\mathbf{r}(t)\right]$.
6. By the cross product formula, $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$.

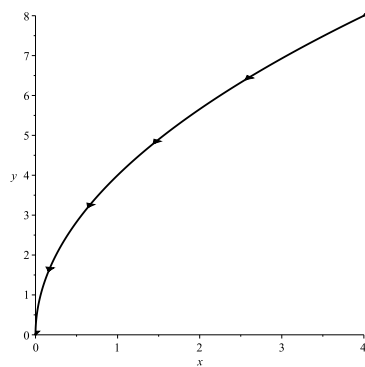
Skill Building

7. $\mathbf{r}(1) = (1)^2\mathbf{i} - (2)(1)\mathbf{j} = \boxed{\mathbf{i} - 2\mathbf{j}}$
8. $\mathbf{r}(2) = (2)^3\mathbf{i} + (2)(2)\mathbf{j} = \boxed{8\mathbf{i} + 4\mathbf{j}}$
9. $\mathbf{v}\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4}\mathbf{i} - \cos \frac{\pi}{4}\mathbf{j} = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}$
10. $\mathbf{v}(0) = \tan(0)\mathbf{i} + \cos(2 \cdot 0)\mathbf{j} = 0\mathbf{i} + 1\mathbf{j} = \boxed{\mathbf{j}}$
11. $\mathbf{u}(0) = e^{(2)(0)}\mathbf{i} + (0)^2\mathbf{j} - 2\mathbf{k} = \boxed{\mathbf{i} - 2\mathbf{k}}$
12. $\mathbf{u}(1) = \ln(1)\mathbf{i} - \mathbf{j} + (1)^3\mathbf{k} = 0\mathbf{i} - \mathbf{j} + \mathbf{k} = \boxed{-\mathbf{j} + \mathbf{k}}$
13. $\mathbf{g}(4) = 4\mathbf{i} - \cos\left(\frac{4\pi}{4}\right)\mathbf{j} + (2)(4)\mathbf{k} = 4\mathbf{i} - (-1)\mathbf{j} + 8\mathbf{k} = \boxed{4\mathbf{i} + \mathbf{j} + 8\mathbf{k}}$
14. $\mathbf{f}(1) = \sin\left(\frac{3\pi(1)}{4}\right)\mathbf{i} + 3\mathbf{j} - 3(1)^2\mathbf{k} = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}}$
15. Since all three component functions, $x(t) = \cos t$, $y(t) = \sin t$, and $z(t) = 2$, are defined for all real numbers, the domain of the vector function $\mathbf{r}(t)$ is $\boxed{\text{all real numbers}}$.

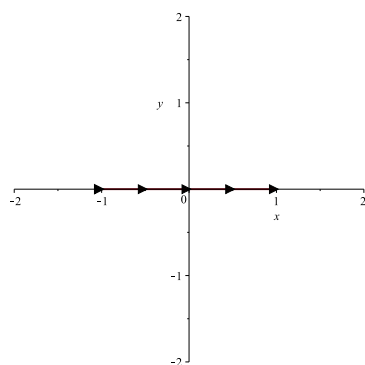
16. Since all three component functions, $x(t) = \cos(2t)$, $y(t) = \sin t$, and $z(t) = 2t$, are defined for all real numbers, the domain of the vector function $\mathbf{r}(t)$ is $\boxed{\text{all real numbers}}$.
17. Both $y(t) = t$ and $z(t) = -1$ are defined for all real numbers, and $x(t) = \sqrt{t}$ is defined for all real numbers $t \geq 0$. The domain of the vector function $\mathbf{f}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t \geq 0\}}$.
18. Both $x(t) = t^2$ and $z(t) = -t$ are defined for all real numbers, and $y(t) = \frac{1}{\sqrt{t}}$ is defined for $t > 0$. The domain of the vector function $\mathbf{r}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t > 0\}}$.
19. Both $x(t) = \frac{(t^2+1)}{t}$ and $z(t) = -\frac{2}{t}$ are defined for $t \neq 0$, and $y(t) = 3t$ is defined for all real numbers. The domain of the vector function $\mathbf{u}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t \neq 0\}}$.
20. Both $x(t) = e^t$ and $y(t) = 5 + 3t$ are defined for all real numbers, and $z(t) = -\frac{2}{t}$ is defined for $t \neq 0$. The domain of the vector function $\mathbf{v}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t \neq 0\}}$.
21. Both $y(t) = t + 1$ and $z(t) = 4t$ are defined for all real numbers, and $x(t) = \ln t$ is defined for $t > 0$. The domain of the vector function $\mathbf{v}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t > 0\}}$.
22. $x(t) = \sqrt{t}$ is defined for $t \geq 0$, $y(t) = \ln t$ is defined for $t > 0$, and $z(t) = t$ is defined for all real numbers. The domain of the vector function $\mathbf{v}(t)$ is the intersection of these three sets, $\boxed{\{t \mid t > 0\}}$.
23. The parametric equations for the curve C are $x(t) = 2t$ and $y(t) = t^2$, $t \geq 0$. We can eliminate the parameter by solving for t in $x = 2t$. Then we substitute $t = \frac{x}{2}$, $x \geq 0$ into y and we get $y = \frac{x^2}{4}$ for $x \geq 0$. This equation represents the right side of a parabola that opens upward with its vertex at the origin. Noting that both $x(t)$ and $y(t)$ increase as t increases gives the orientation.



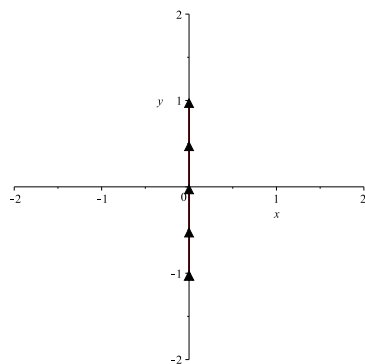
24. The parametric equations for the curve C are $x(t) = t^2$ and $y(t) = -4t$, $t \leq 0$. We can eliminate the parameter by solving for t in $y = -4t$. Then we substitute $t = -\frac{y}{4}$, $y \geq 0$ into x and we get $x = \frac{y^2}{16}$ for $y \geq 0$. This equation represents the top side of a parabola that opens to the right with its vertex at the origin. Noting that for $t \leq 0$, both $x(t)$ and $y(t)$ decrease as t increases gives the orientation.



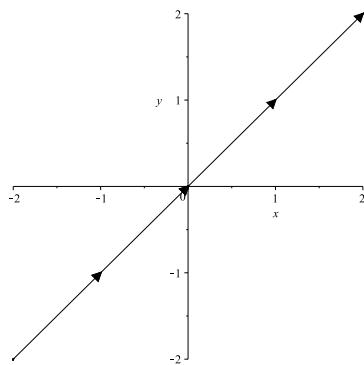
25. Since $x(t) = t$ and $y(t) = 0$ for $-1 \leq t \leq 1$, the graph is the line segment $y = 0$ for $-1 \leq x \leq 1$. The orientation is to the right since x increases as t increases.



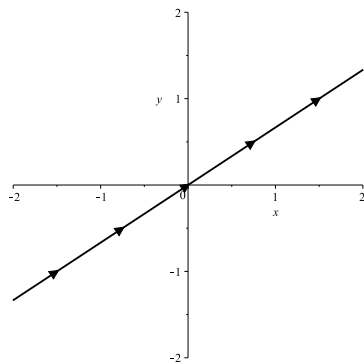
26. Since $x(t) = 0$ and $y(t) = t$ for $-1 \leq t \leq 1$, the graph is the line segment $x = 0$ for $-1 \leq y \leq 1$. The orientation is upward since y increases as t increases.



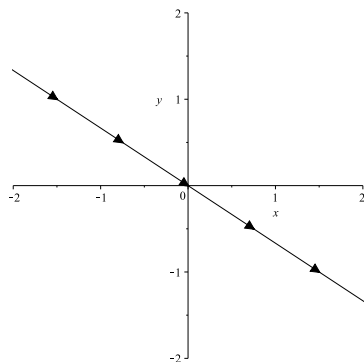
27. Since $x(t) = t$ and $y(t) = t$, C is the line $y = x$. As t increases, both x and y increase.



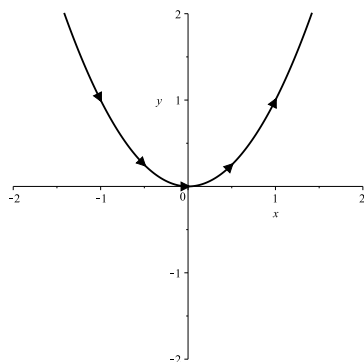
28. Since $x(t) = 3t$ and $y(t) = 2t$, we solve for t in $x = 3t$ to get $t = \frac{x}{3}$. Substituting t into $y = 2t$, we get $y = \frac{2}{3}x$, so C is the line $y = \frac{2}{3}x$. As t increases, both x and y increase.



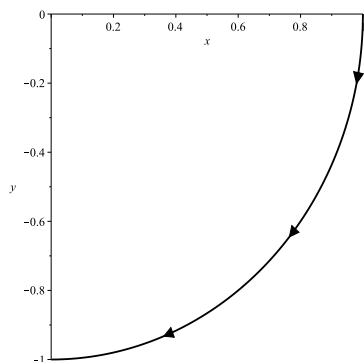
29. Since $x(t) = 3t$ and $y(t) = -2t$, we solve for t in $x = 3t$ to get $t = \frac{x}{3}$. Substituting t into $y = -2t$, we get $y = -\frac{2}{3}x$, so C is the line $y = -\frac{2}{3}x$. As t increases, x increases and y decreases.



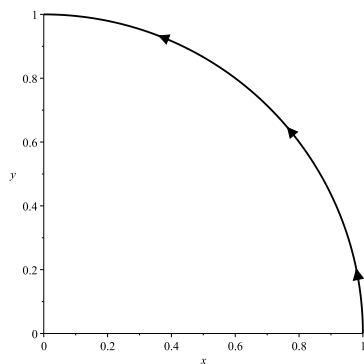
30. The parametric equations for the curve C are $x(t) = t$ and $y(t) = t^2$. We can eliminate the parameter by substituting $t = x$ into y to get $y = x^2$. This equation represents a parabola that opens upward with its vertex at the origin. Noting that $x(t)$ increases as t increases gives the orientation.



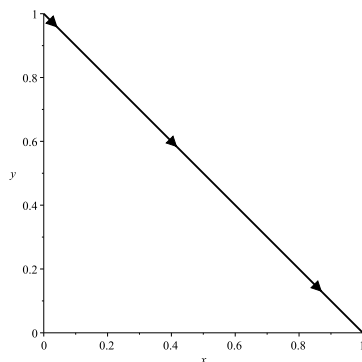
31. The components of $\mathbf{r}(t)$ are $x(t) = \cos t$ and $y(t) = -\sin t$, $0 \leq t \leq \frac{\pi}{2}$. Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the vector function $\mathbf{r}(t)$ traces out part of a circle with its center at the origin and radius 1. For $0 \leq t \leq \frac{\pi}{2}$, $0 \leq \cos t \leq 1$ and $-1 \leq -\sin t \leq 0$, so $0 \leq x \leq 1$ and $-1 \leq y \leq 0$, which means that this curve is in quadrant IV. On $0 \leq t \leq \frac{\pi}{2}$, both x and y decrease as t increases.



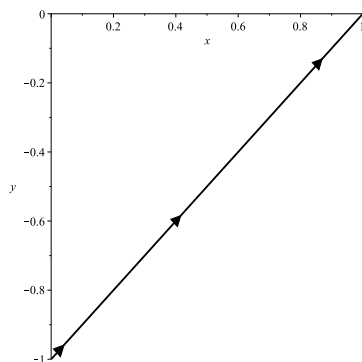
32. The components of $\mathbf{r}(t)$ are $x(t) = \cos t$ and $y(t) = \sin t$, $0 \leq t \leq \frac{\pi}{2}$. For $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the vector function $\mathbf{r}(t)$ traces out part of a circle with its center at the origin and radius 1. Since $0 \leq t \leq \frac{\pi}{2}$, $0 \leq \cos t \leq 1$ and $0 \leq \sin t \leq 1$, so $0 \leq x \leq 1$ and $0 \leq y \leq 1$, which means that this curve is in quadrant I. On $0 \leq t \leq \frac{\pi}{2}$, x decreases and y increases as t increases.



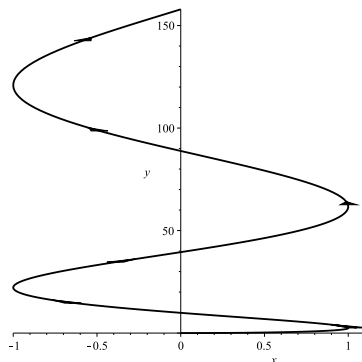
33. The components of $\mathbf{r}(t)$ are $x(t) = \sin^2 t$ and $y(t) = \cos^2 t$, $0 \leq t \leq \frac{\pi}{2}$. Since $x + y = \sin^2 t + \cos^2 t = 1$, or $y = 1 - x$, $0 \leq x \leq 1$, the vector function $\mathbf{r}(t)$ traces out the line segment $y = 1 - x$ from $x = 0$ to $x = 1$. On $0 \leq t \leq \frac{\pi}{2}$, x increases and y decreases as t increases.



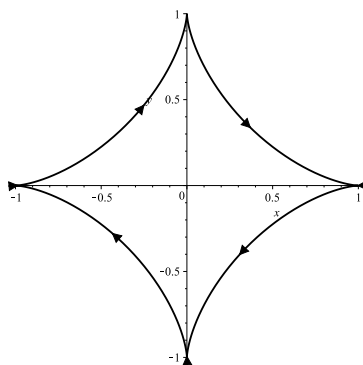
34. The components of $\mathbf{r}(t)$ are $x(t) = \sin^2 t$ and $y(t) = -\cos^2 t$, $0 \leq t \leq \frac{\pi}{2}$. Since $x - y = \sin^2 t + \cos^2 t = 1$, or $y = x - 1$, $0 \leq x \leq 1$, the vector function $\mathbf{r}(t)$ traces out the line segment $y = x - 1$ from $x = 0$ to $x = 1$. On $0 \leq t \leq \frac{\pi}{2}$, both x and y increase as t increases.



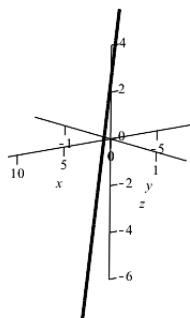
35. For $0 \leq t \leq 4\pi$, $x(t) = \sin t$ oscillates between -1 and 1 , and $y(t) = t^2$ increases from 0 to $16\pi^2$. Using the computer algebra system MapleTM, we get the graph below.



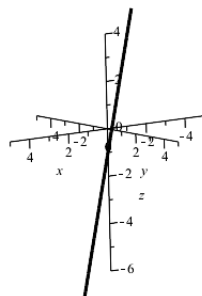
36. The components of $\mathbf{r}(t)$ are $x(t) = \sin^3 t$ and $y(t) = \cos^3 t$, so $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Using the computer algebra system MapleTM, we get the graph of an astroid, seen below.



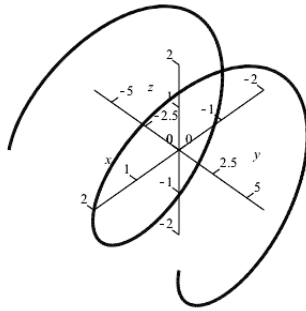
37. For $0 \leq t \leq 2\pi$, $x(t) = t^2$, so $0 \leq x \leq 4\pi^2 \approx 39.5$, and $y(t) = \cos t$ is between -1 and 1 . The graph that meets these criteria is D.
38. For $0 \leq t \leq 2\pi$, $x(t) = \cos t$ is between -1 and 1 , and $y(t) = t^2$, so $0 \leq y \leq 4\pi^2 \approx 39.5$. The graph that meets these criteria is A.
39. For $0 \leq t \leq 2\pi$, $x(t) = \sqrt{t}$, so $0 \leq x \leq \sqrt{2\pi} \approx 2.5$, and $y(t) = \cos t$ is between -1 and 1 . The graph that meets these criteria is C.
40. For $0 \leq t \leq 2\pi$, $x(t) = \cos t$ is between -1 and 1 , and $y(t) = -t$, so $-6.28 \approx -2\pi \leq y \leq 0$. The graph that meets these criteria is B.
41. The graph was generated using the computer algebra system MapleTM.



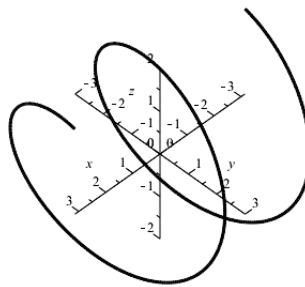
42. The graph was generated using the computer algebra system MapleTM.



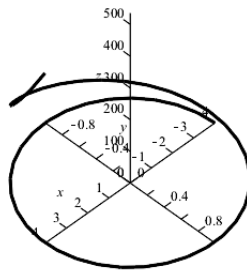
43. The graph was generated using the computer algebra system MapleTM.



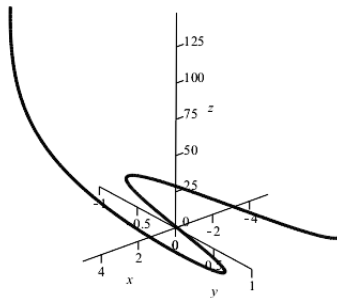
44. The graph was generated using the computer algebra system MapleTM.



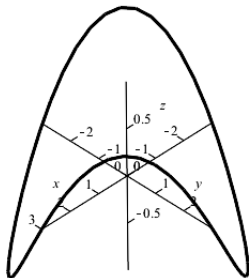
45. The graph was generated using the computer algebra system MapleTM.



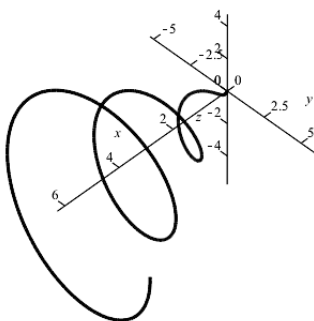
46. The graph was generated using the computer algebra system MapleTM.



47. The graph was generated using the computer algebra system Maple™.



48. The graph was generated using the computer algebra system Maple™.



49. Both $x(t) = t^2$ and $z(t) = \sin t$ are continuous for all real numbers, and $y(t) = \frac{2}{1-t^2}$ is continuous for $t \neq \pm 1$. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these three sets, $\boxed{\{t \mid t \neq \pm 1\}}$.
50. The component function $x(t) = \ln t$ is continuous for $t > 0$, and $y(t) = e^{-t}$ is continuous for all real numbers. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\boxed{\{t \mid t > 0\}}$.
51. The component function $x(t) = \sec t$ is continuous for $t \neq \frac{(2k+1)\pi}{2}$ and $y(t) = -\cos t$ is continuous for all real numbers. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\boxed{\left\{t \mid t \neq \frac{(2k+1)\pi}{2}\right\}}$.
52. The component function $x(t) = \sqrt{t+1}$ is continuous for $t \geq -1$, $y(t) = \tan t$ is continuous for $t \neq \frac{(2k+1)\pi}{2}$, and $z(t) = -\frac{\sin t - 1}{t}$ is continuous for $t \neq 0$. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\boxed{\left\{t \mid t \geq -1, t \neq 0, t \neq \frac{(2k+1)\pi}{2}\right\}}$.
53. The component function $x(t) = \frac{t}{t+1}$ is continuous for $t \neq -1$ and $y(t) = -t$ is continuous for all real numbers. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\boxed{\{t \mid t \neq -1\}}$.
54. Both component functions, $x(t) = \frac{t}{t^2+1}$ and $y(t) = t$, are continuous for all real numbers, so the vector function $\mathbf{r}(t)$ is continuous for $\boxed{\text{all real numbers}}$.
55. The component functions $x(t) = \sin t$ and $z(t) = 1$ are continuous for all real numbers, and $y(t) = -\tan t$ is continuous for $t \neq \frac{(2k+1)\pi}{2}$. The vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\boxed{\left\{t \mid t \neq \frac{(2k+1)\pi}{2}\right\}}$.

56. The component function $x(t) = \sec t$ is continuous for $t \neq \frac{(2k+1)\pi}{2}$, $y(t) = \csc t$ is continuous for $t \neq k\pi$, and $z(t) = 1$ is continuous for all real numbers. Since $x(t)$ is continuous except for odd multiples of $\frac{\pi}{2}$ and $y(t)$ is continuous except for even multiples of $\frac{\pi}{2}$, the vector function $\mathbf{r}(t)$ is continuous at the intersection of these sets, $\left\{ t \mid t \neq \frac{k}{2}\pi \right\}$.

57. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{8\mathbf{i} - 6t^2\mathbf{j}} \\ \mathbf{r}''(t) &= \boxed{8\mathbf{i} - 12t\mathbf{j}}\end{aligned}$$

58. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{8\mathbf{i} + 12t^2\mathbf{j}} \\ \mathbf{r}''(t) &= \boxed{24t\mathbf{j}}\end{aligned}$$

59. Find the derivatives of each component. Since $4\sqrt{t} = 4t^{1/2}$, $\frac{d}{dt}(4\sqrt{t}) = 2t^{-1/2} = \frac{2}{\sqrt{t}} = \frac{2\sqrt{t}}{t}$, and $\frac{d}{dt}(2t^{-1/2}) = -t^{-3/2} = -\frac{1}{t^{3/2}}$,

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{\frac{2\sqrt{t}}{t}\mathbf{i} + 2e^t\mathbf{j}} \\ \mathbf{r}''(t) &= \boxed{-\frac{1}{t^{3/2}}\mathbf{i} + 2e^t\mathbf{j}}\end{aligned}$$

60. Find the derivatives of each component. We can rewrite $\sqrt[3]{t} = t^{1/3}$, so $\frac{d}{dt}(\sqrt[3]{t}) = \frac{1}{3}t^{-2/3} = \frac{1}{3t^{2/3}}$ and $\frac{d}{dt}(\frac{1}{3}t^{-2/3}) = -\frac{2}{9}t^{-5/3} = -\frac{2}{9t^{5/3}}$. By the Chain Rule, $\frac{d}{dt}(e^{3t}) = 3e^{3t}$. Then

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{3e^{3t}\mathbf{i} + \frac{1}{3t^{2/3}}\mathbf{j}} \\ \mathbf{r}''(t) &= \boxed{9e^{3t}\mathbf{i} - \frac{2}{9t^{5/3}}\mathbf{j}}\end{aligned}$$

61. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{\mathbf{j} + 2t\mathbf{k}} \\ \mathbf{r}''(t) &= \boxed{2\mathbf{k}}\end{aligned}$$

62. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{\mathbf{k}} \\ \mathbf{r}''(t) &= \boxed{\mathbf{0}}\end{aligned}$$

63. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{2t\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}} \\ \mathbf{r}''(t) &= \boxed{2\mathbf{i} + 6t\mathbf{j}}\end{aligned}$$

64. Find the derivatives of each component.

$$\mathbf{r}'(t) = \boxed{\mathbf{i} - 6t\mathbf{j} + \mathbf{k}}$$

$$\mathbf{r}''(t) = \boxed{-6\mathbf{j}}$$

65. Find the derivatives of each component. Using the Chain Rule, $\frac{d}{dt}(\sin^2 t) = 2(\sin t)(\cos t) = \sin(2t)$ and $\frac{d}{dt}(-\cos^2 t) = -2(\cos t)(-\sin t) = 2\sin t \cos t = \sin(2t)$. The Chain Rule gives $\frac{d}{dt}(\sin(2t)) = \cos(2t)(2) = 2\cos(2t)$

$$\mathbf{r}'(t) = \boxed{\sin(2t)\mathbf{i} + \sin(2t)\mathbf{j}}$$

$$\mathbf{r}''(t) = \boxed{2\cos(2t)\mathbf{i} + 2\cos(2t)\mathbf{j}}$$

66. Find the derivatives of each component. Using the Chain Rule, $\frac{d}{dt}(\sin(2t)) = 2\cos(2t)$, and $\frac{d}{dt}(-\cos(2t)) = 2\sin(2t)$.

$$\mathbf{r}'(t) = \boxed{2\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j}}$$

$$\mathbf{r}''(t) = \boxed{-4\sin(2t)\mathbf{i} + 4\cos(2t)\mathbf{j}}$$

67. By a property of logarithmic functions, $-\ln(e^{t^2+t}) = -(t^2 + t)$. Using the Chain Rule gives

$$\frac{d}{dt}(5e^{t^2+t}) = 5e^{t^2+t}(2t + 1),$$

and the Product Rule gives

$$\frac{d}{dt}[5e^{t^2+t}(2t + 1)] = 5e^{t^2+t}(2t + 1)(2t + 1) + 5e^{t^2+t}(2).$$

We can now factor out a $5e^{t^2+t}$ and then FOIL to get

$$\frac{d}{dt}[5e^{t^2+t}(2t + 1)] = 5e^{t^2+t}(4t^2 + 4t + 1 + 2) = 5e^{t^2+t}(4t^2 + 4t + 3).$$

Find the derivatives of each component.

$$\mathbf{r}'(t) = \boxed{5e^{t^2+t}(2t + 1)\mathbf{i} - (2t + 1)\mathbf{j}}$$

$$\mathbf{r}''(t) = \boxed{5e^{t^2+t}(4t^2 + 4t + 3)\mathbf{i} - 2\mathbf{j}}$$

68. Find the derivatives of each component. Using the Chain Rule,

$$\begin{aligned}\mathbf{r}'(t) &= \cos(3t^3 - t)(9t^2 - 1)\mathbf{i} - 2\cos(3t^3 - t)(-\sin(3t^3 - t))(9t^2 - 1)\mathbf{j} \\ &= \boxed{(9t^2 - 1)\cos(3t^3 - t)\mathbf{i} + (9t^2 - 1)\sin[2(3t^3 - t)]\mathbf{j}}\end{aligned}$$

Using the Product Rule,

$$\begin{aligned}\frac{d}{dt}[(9t^2 - 1)\cos(3t^3 - t)] &= (9t^2 - 1)(-\sin(3t^3 - t))(9t^2 - 1) + (18t)\cos(3t^3 - t) \\ &= -(9t^2 - 1)^2\sin(3t^3 - t) + 18t\cos(3t^3 - t)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}[(9t^2 - 1)\sin[2(3t^3 - t)]] &= (9t^2 - 1)\cos[2(3t^3 - t)](2)(9t^2 - 1) + (18t)\sin[2(3t^3 - t)] \\ &= 2(9t^2 - 1)^2\cos[2(3t^3 - t)] + 18t\sin[2(3t^3 - t)],\end{aligned}$$

so

$$\mathbf{r}''(t) = \boxed{[-(9t^2 - 1)^2\sin(3t^3 - t) + 18t\cos(3t^3 - t)]\mathbf{i} + [2(9t^2 - 1)^2\cos[2(3t^3 - t)] + 18t\sin[2(3t^3 - t)]]\mathbf{j}}$$

69. Find the derivatives of each component. Using the Product Rule,

$$\begin{aligned}\mathbf{r}'(t) &= (-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j} + \mathbf{k} \\ &= \boxed{e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + \mathbf{k}} \\ \mathbf{r}''(t) &= [e^t(-\sin t - \cos t) + e^t(\cos t - \sin t)]\mathbf{i} + [e^t(-\sin t + \cos t) + e^t(\cos t + \sin t)]\mathbf{j} \\ &= \boxed{-2e^t \sin t \mathbf{i} + 2e^t \cos t \mathbf{j}}\end{aligned}$$

70. Find the derivatives of each component. Using the Product Rule,

$$\begin{aligned}\mathbf{r}'(t) &= (-e^{-t} \sin t - e^{-t} \cos t)\mathbf{i} + (e^{-t} \cos t - e^{-t} \sin t)\mathbf{j} - \mathbf{k} \\ &= \boxed{-e^{-t}(\sin t + \cos t)\mathbf{i} + e^{-t}(\cos t - \sin t)\mathbf{j} - \mathbf{k}}\end{aligned}$$

By the Product Rule, $\frac{d}{dt}(-e^{-t}(\sin t + \cos t)) = -e^{-t}(\cos t - \sin t) + e^{-t}(\sin t + \cos t) = 2e^{-t} \sin t$ and $\frac{d}{dt}(e^{-t}(\cos t - \sin t)) = e^{-t}(-\sin t - \cos t) - e^{-t}(\cos t - \sin t) = -2e^{-t} \cos t$, so

$$\mathbf{r}''(t) = \boxed{2e^{-t} \sin t \mathbf{i} - 2e^{-t} \cos t \mathbf{j}}$$

71. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{(1 - 3t^2)\mathbf{i} + (1 + 3t^2)\mathbf{j} - \mathbf{k}} \\ \mathbf{r}''(t) &= \boxed{-6t\mathbf{i} + 6t\mathbf{j}}\end{aligned}$$

72. Find the derivatives of each component.

$$\begin{aligned}\mathbf{r}'(t) &= \boxed{(2t - 1)\mathbf{i} + (2t + 1)\mathbf{j} + \mathbf{k}} \\ \mathbf{r}''(t) &= \boxed{2\mathbf{i} + 2\mathbf{j}}\end{aligned}$$

73. The derivatives of $\mathbf{u}(t) = t^2\mathbf{i} - t\mathbf{j}$ and $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j}$ are $\mathbf{u}'(t) = 2t\mathbf{i} - \mathbf{j}$ and $\mathbf{v}'(t) = \mathbf{i} + 2t\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned}[\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [2t\mathbf{i} - \mathbf{j}] \cdot [t\mathbf{i} + t^2\mathbf{j}] + [t^2\mathbf{i} - t\mathbf{j}] \cdot [\mathbf{i} + 2t\mathbf{j}] \\ &= 2t^2 - t^2 + t^2 - 2t^2 \\ &= \boxed{0}\end{aligned}$$

74. The derivatives of $\mathbf{u}(t) = t^3\mathbf{i} + t\mathbf{j}$ and $\mathbf{v}(t) = t\mathbf{i} - 2t\mathbf{j}$ are $\mathbf{u}'(t) = 3t^2\mathbf{i} + \mathbf{j}$ and $\mathbf{v}'(t) = \mathbf{i} - 2\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned}[\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [3t^2\mathbf{i} + \mathbf{j}] \cdot [t\mathbf{i} - 2t\mathbf{j}] + [t^3\mathbf{i} + t\mathbf{j}] \cdot [\mathbf{i} - 2\mathbf{j}] \\ &= 3t^3 - 2t + t^3 - 2t \\ &= \boxed{4t^3 - 4t}\end{aligned}$$

75. The derivatives of $\mathbf{u}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$ and $\mathbf{v}(t) = t\mathbf{i} - t^2\mathbf{j}$ are $\mathbf{u}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$ and $\mathbf{v}'(t) = \mathbf{i} - 2t\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned}[\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [e^t\mathbf{i} - e^{-t}\mathbf{j}] \cdot [t\mathbf{i} - t^2\mathbf{j}] + [e^t\mathbf{i} + e^{-t}\mathbf{j}] \cdot [\mathbf{i} - 2t\mathbf{j}] \\ &= \boxed{te^t + t^2e^{-t} + e^t - 2te^{-t}}\end{aligned}$$

76. The derivatives of $\mathbf{u}(t) = t\mathbf{i} + 4\sqrt{t}\mathbf{j}$ and $\mathbf{v}(t) = 2t\mathbf{i} - t^2\mathbf{j}$ are $\mathbf{u}'(t) = \mathbf{i} + \frac{2\sqrt{t}}{t}\mathbf{j}$ and $\mathbf{v}'(t) = 2\mathbf{i} - 2t\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= \left[\mathbf{i} + \frac{2\sqrt{t}}{t}\mathbf{j} \right] \cdot [2t\mathbf{i} - t^2\mathbf{j}] + [t\mathbf{i} + 4\sqrt{t}\mathbf{j}] \cdot [2\mathbf{i} - 2t\mathbf{j}] \\ &= 2t - 2t^{3/2} + 2t - 8t^{3/2} \\ &= \boxed{4t - 10t^{3/2}} \end{aligned}$$

77. The derivatives of $\mathbf{u}(t) = \sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} + \mathbf{j}$ are $\mathbf{u}'(t) = \omega \cos(\omega t)\mathbf{i} - \omega \sin(\omega t)\mathbf{j}$ and $\mathbf{v}'(t) = \mathbf{0}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [\omega \cos(\omega t)\mathbf{i} - \omega \sin(\omega t)\mathbf{j}] \cdot [\mathbf{i} + \mathbf{j}] + [\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}] \cdot [\mathbf{0}] \\ &= \boxed{\omega \cos(\omega t) - \omega \sin(\omega t)} \end{aligned}$$

78. The derivatives of $\mathbf{u}(t) = \sin^2 t\mathbf{i} - \cos^2 t\mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} - \mathbf{j}$ are $\mathbf{u}'(t) = 2\sin t \cos t\mathbf{i} - 2(\cos t)(-\sin t)\mathbf{j} = \sin(2t)\mathbf{i} + \sin(2t)\mathbf{j}$ and $\mathbf{v}'(t) = \mathbf{0}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [\sin(2t)\mathbf{i} + \sin(2t)\mathbf{j}] \cdot [\mathbf{i} - \mathbf{j}] + [\sin^2 t\mathbf{i} - \cos^2 t\mathbf{j}] \cdot [\mathbf{0}] \\ &= \sin(2t) - \sin(2t) \\ &= \boxed{0} \end{aligned}$$

79. The derivatives of $\mathbf{u}(t) = 2t\mathbf{i} + t^2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ are $\mathbf{u}'(t) = 2\mathbf{i} + 2t\mathbf{j}$ and $\mathbf{v}'(t) = 2t\mathbf{i} + 2\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [2\mathbf{i} + 2t\mathbf{j}] \cdot [t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}] + [2t\mathbf{i} + t^2\mathbf{j} - 5\mathbf{k}] \cdot [2t\mathbf{i} + 2\mathbf{j}] \\ &= 2t^2 + 4t^2 + 0 + 4t^2 + 2t^2 + 0 \\ &= \boxed{12t^2} \end{aligned}$$

To find $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$, we use the cross product formula.

$$\begin{aligned} [\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\ &= [2\mathbf{i} + 2t\mathbf{j}] \times [t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}] + [2t\mathbf{i} + t^2\mathbf{j} - 5\mathbf{k}] \times [2t\mathbf{i} + 2\mathbf{j}] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2t & 0 \\ t^2 & 2t & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & t^2 & -5 \\ 2t & 2 & 0 \end{vmatrix} \\ &= [2t\mathbf{i} - 2\mathbf{j} + [(2)(2t) - (t^2)(2t)]\mathbf{k}] + [10\mathbf{i} - 10t\mathbf{j} + [(2t)(2) - (2t)(t^2)]\mathbf{k}] \\ &= \boxed{(2t + 10)\mathbf{i} - (10t + 2)\mathbf{j} + (8t - 4t^3)\mathbf{k}} \end{aligned}$$

80. The derivatives of $\mathbf{u}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$ and $\mathbf{v}(t) = t\mathbf{i} - t^2\mathbf{j} + t^3\mathbf{k}$ are $\mathbf{u}'(t) = 3t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{v}'(t) = \mathbf{i} - 2t\mathbf{j} + 3t^2\mathbf{k}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\ &= [3t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \cdot [t\mathbf{i} - t^2\mathbf{j} + t^3\mathbf{k}] + [t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}] \cdot [\mathbf{i} - 2t\mathbf{j} + 3t^2\mathbf{k}] \\ &= 3t^3 + 2t^3 + t^3 + t^3 + 2t^3 + 3t^3 \\ &= \boxed{12t^3} \end{aligned}$$

To find $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$, we use the cross product formula.

$$\begin{aligned}
 [\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\
 &= [3t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \times [t\mathbf{i} - t^2\mathbf{j} + t^3\mathbf{k}] + [t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}] \times [\mathbf{i} - 2t\mathbf{j} + 3t^2\mathbf{k}] \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & -2t & 1 \\ t & -t^2 & t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t^2 & t \\ 1 & -2t & 3t^2 \end{vmatrix} \\
 &= [(-2t^4 + t^2)\mathbf{i} - (3t^5 - t)\mathbf{j} + (-3t^4 + 2t^2)\mathbf{k}] + [(-3t^4 + 2t^2)\mathbf{i} - (3t^5 - t)\mathbf{j} + (-2t^4 + t^2)\mathbf{k}] \\
 &= (-5t^4 + 3t^2)\mathbf{i} + (-6t^5 + 2t)\mathbf{j} + (-5t^4 + 3t^2)\mathbf{k} \\
 &= \boxed{t^2(-5t^2 + 3)\mathbf{i} + 2t(-3t^4 + 1)\mathbf{j} + t^2(-5t^2 + 3)\mathbf{k}}
 \end{aligned}$$

81. The derivatives of $\mathbf{u}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + \mathbf{k}$ and $\mathbf{v}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ are $\mathbf{u}'(t) = -2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j}$ and $\mathbf{v}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned}
 [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\
 &= [-2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j}] \cdot [\cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}] + [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + \mathbf{k}] \cdot [-\sin t\mathbf{i} + \cos t\mathbf{j}] \\
 &= -2\sin(2t)\cos t + 2\cos(2t)\sin t - \cos(2t)\sin t + \sin(2t)\cos t \\
 &= \boxed{-\sin(2t)\cos t + \cos(2t)\sin t}
 \end{aligned}$$

To find $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$, we use the cross product formula.

$$\begin{aligned}
 [\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\
 &= [-2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j}] \times [\cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}] + [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + \mathbf{k}] \times [-\sin t\mathbf{i} + \cos t\mathbf{j}] \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin(2t) & 2\cos(2t) & 0 \\ \cos t & \sin t & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(2t) & \sin(2t) & 1 \\ -\sin t & \cos t & 0 \end{vmatrix}
 \end{aligned}$$

Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin(2t) & 2\cos(2t) & 0 \\ \cos t & \sin t & 1 \end{vmatrix} = 2\cos(2t)\mathbf{i} + 2\sin(2t)\mathbf{j} + (-2\sin(2t)\sin t - 2\cos(2t)\cos t)\mathbf{k}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(2t) & \sin(2t) & 1 \\ -\sin t & \cos t & 0 \end{vmatrix} = -\cos t\mathbf{i} - \sin t\mathbf{j} + (\cos(2t)\cos t + \sin(2t)\sin t)\mathbf{k},$$

$$[\mathbf{u}(t) \times \mathbf{v}(t)]' = \boxed{(2\cos(2t) - \cos t)\mathbf{i} + (2\sin(2t) - \sin t)\mathbf{j} + (-\sin(2t)\sin t - \cos(2t)\cos t)\mathbf{k}}.$$

82. The derivatives of $\mathbf{u}(t) = e^{2t}\mathbf{i} + e^{-2t}\mathbf{j} + t\mathbf{k}$ and $\mathbf{v}(t) = e^{-t}\mathbf{i} + e^{-2t}\mathbf{j} - t\mathbf{k}$ are $\mathbf{u}'(t) = 2e^{2t}\mathbf{i} - 2e^{-2t}\mathbf{j} + \mathbf{k}$ and $\mathbf{v}'(t) = -e^{-t}\mathbf{i} - 2e^{-2t}\mathbf{j} - \mathbf{k}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned}
 [\mathbf{u}(t) \cdot \mathbf{v}(t)]' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\
 &= [2e^{2t}\mathbf{i} - 2e^{-2t}\mathbf{j} + \mathbf{k}] \cdot [e^{-t}\mathbf{i} + e^{-2t}\mathbf{j} - t\mathbf{k}] + [e^{2t}\mathbf{i} + e^{-2t}\mathbf{j} + t\mathbf{k}] \cdot [-e^{-t}\mathbf{i} - 2e^{-2t}\mathbf{j} - \mathbf{k}] \\
 &= 2e^t - 2e^{-4t} - t - e^t - 2e^{-4t} - t \\
 &= \boxed{e^t - 4e^{-4t} - 2t}
 \end{aligned}$$

To find $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$, we use the cross product formula.

$$\begin{aligned} [\mathbf{u}(t) \times \mathbf{v}(t)]' &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\ &= [2e^{2t}\mathbf{i} - 2e^{-2t}\mathbf{j} + \mathbf{k}] \times [e^{-t}\mathbf{i} + e^{-2t}\mathbf{j} - t\mathbf{k}] + [e^{2t}\mathbf{i} + e^{-2t}\mathbf{j} + t\mathbf{k}] \times [-e^{-t}\mathbf{i} - 2e^{-2t}\mathbf{j} - \mathbf{k}] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2e^{2t} & -2e^{-2t} & 1 \\ e^{-t} & e^{-2t} & -t \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^{2t} & e^{-2t} & t \\ -e^{-t} & -2e^{-2t} & -1 \end{vmatrix} \end{aligned}$$

Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2e^{2t} & -2e^{-2t} & 1 \\ e^{-t} & e^{-2t} & -t \end{vmatrix} = (2te^{-2t} - e^{-2t})\mathbf{i} - (-2te^{2t} - e^{-t})\mathbf{j} + (2 + 2e^{-3t})\mathbf{k}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^{2t} & e^{-2t} & t \\ -e^{-t} & -2e^{-2t} & -1 \end{vmatrix} = (-e^{-2t} + 2te^{-2t})\mathbf{i} - (-e^{2t} + te^{-t})\mathbf{j} + (-2 + e^{-3t})\mathbf{k},$$

$$[\mathbf{u}(t) \times \mathbf{v}(t)]' = \boxed{(4te^{-2t} - 2e^{-2t})\mathbf{i} + (2te^{2t} + e^{-t} + e^{2t} - te^{-t})\mathbf{j} + 3e^{-3t}\mathbf{k}}.$$

Applications and Extensions

83. (a) Find the derivative of each component. Since $\frac{d}{dt} \cos(\omega t) = -\omega \sin(\omega t)$ and $\frac{d}{dt} \sin(\omega t) = \omega \cos(\omega t)$,

$$\frac{d\mathbf{u}}{dt} = \boxed{-\omega \sin(\omega t)\mathbf{i} + \omega \cos(\omega t)\mathbf{j}}$$

(b)

$$\begin{aligned} \left\| \frac{d\mathbf{u}}{dt} \right\| &= \sqrt{(-\omega \sin(\omega t))^2 + (\omega \cos(\omega t))^2} = \sqrt{\omega^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t)} \\ &= \sqrt{\omega^2 (\sin^2(\omega t) + \cos^2(\omega t))} = \sqrt{\omega^2} = \boxed{|\omega|} \end{aligned}$$

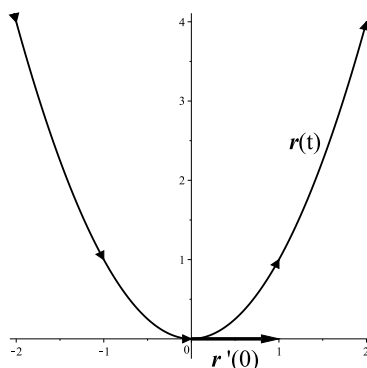
84. Differentiating $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ gives $\frac{d\mathbf{v}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$ and $\frac{d^2\mathbf{v}}{dt^2} = 2\mathbf{j} + 6t\mathbf{k}$.

$$\left\| \frac{d^2\mathbf{v}}{dt^2} \right\| = \sqrt{2^2 + (6t)^2} = \sqrt{4 + 36t^2} = \boxed{2\sqrt{1 + 9t^2}}$$

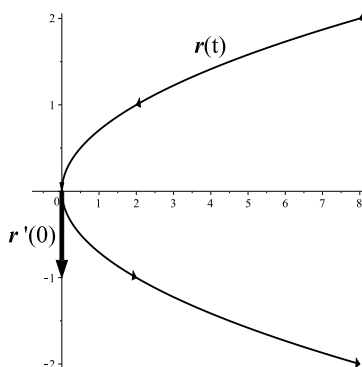
85. Since $\mathbf{f}(t) = \sin t\mathbf{i} - \cos t\mathbf{j}$, $\mathbf{f}'(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ and $\mathbf{f}''(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. The vectors $\mathbf{f}(t)$ and $\mathbf{f}''(t)$ are parallel since $\mathbf{f}(t) = -\mathbf{f}''(t)$.

86. Since $\mathbf{f}(t) = e^{3t}\mathbf{i} + e^{-3t}\mathbf{j}$, $\mathbf{f}'(t) = 3e^{3t}\mathbf{i} - 3e^{-3t}\mathbf{j}$ and $\mathbf{f}''(t) = 9e^{3t}\mathbf{i} + 9e^{-3t}\mathbf{j}$. The vectors $\mathbf{f}(t)$ and $\mathbf{f}''(t)$ are parallel since $\mathbf{f}(t) = \frac{1}{9}\mathbf{f}''(t)$.

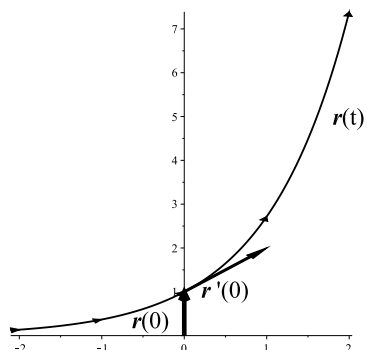
87. The component functions for $\mathbf{r}(t)$ are $x(t) = t$ and $y(t) = t^2$. Substituting $t = x$ into y gives $y = x^2$. This equation represents a parabola that opens upward with its vertex at the origin. Noting that x increases as t increases gives the orientation. For $t = 0$, $\mathbf{r}(0) = \mathbf{0}$, and since $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$, $\mathbf{r}'(0) = \mathbf{i}$.



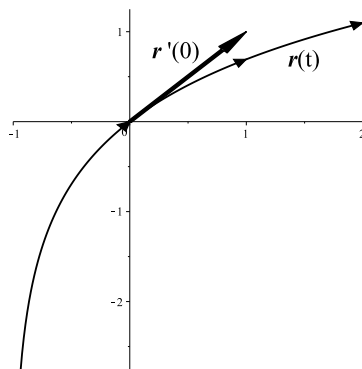
88. The component functions for $\mathbf{r}(t)$ are $x(t) = 2t^2$ and $y(t) = -t$. We can eliminate the parameter by solving for t in $y = -t$. Then we substitute $t = -y$ into x to get $x = 2y^2$. This equation represents a parabola that opens to the right with its vertex at the origin. Noting that y decreases as t increases gives the orientation. For $t = 0$, $\mathbf{r}(0) = \mathbf{0}$, and since $\mathbf{r}'(t) = 4t\mathbf{i} - \mathbf{j}$, $\mathbf{r}'(0) = -\mathbf{j}$.



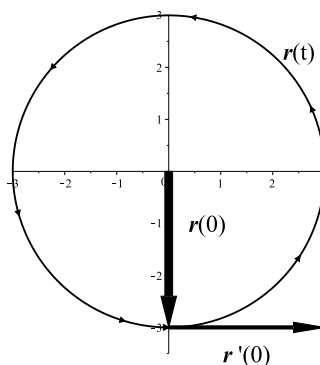
89. The component functions for $\mathbf{r}(t)$ are $x(t) = t$ and $y(t) = e^t$. Substituting $t = x$ into y gives $y = e^x$. This equation represents the natural exponential function. Noting that both x and y increase as t increases gives the orientation. For $t = 0$, $\mathbf{r}(0) = \mathbf{j}$, and since $\mathbf{r}'(t) = \mathbf{i} + e^t\mathbf{j}$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$.



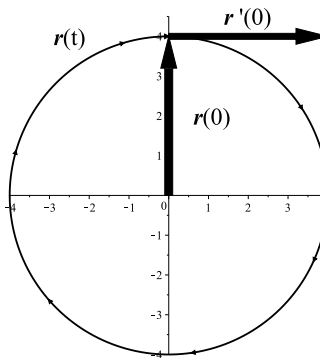
90. The component functions for $\mathbf{r}(t)$ are $x(t) = t$ and $y(t) = \ln(1 + t)$. Substituting $t = x$ into y gives $y = \ln(1 + x)$. This equation represents the natural logarithmic function shifted one unit to the left. Noting that both x and y increase as t increases gives the orientation. For $t = 0$, $\mathbf{r}(0) = \mathbf{0}$, and since $\mathbf{r}'(t) = \mathbf{i} + \frac{1}{1+t}\mathbf{j}$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$.



91. The components of $\mathbf{r}(t)$ are $x(t) = 3 \sin t$ and $y(t) = -3 \cos t$. Since $x^2 + y^2 = 9 \sin^2 t + 9 \cos^2 t = 9$, the vector function $\mathbf{r}(t)$ traces out a circle with its center at the origin and radius 3. Note that $\mathbf{r}(0) = -3\mathbf{j}$, $\mathbf{r}(\frac{\pi}{2}) = 3\mathbf{i}$, and $\mathbf{r}(\pi) = 3\mathbf{j}$, so the orientation is counterclockwise. For $t = 0$, $\mathbf{r}(0) = -3\mathbf{j}$, and since $\mathbf{r}'(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$, $\mathbf{r}'(0) = 3\mathbf{i}$.

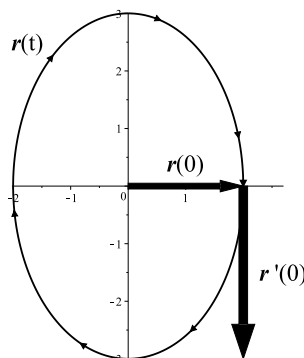


92. The components of $\mathbf{r}(t)$ are $x(t) = 4 \sin t$ and $y(t) = 4 \cos t$. Since $x^2 + y^2 = 16 \sin^2 t + 16 \cos^2 t = 16$, the vector function $\mathbf{r}(t)$ traces out a circle with its center at the origin and radius 4. Note that $\mathbf{r}(0) = 4\mathbf{j}$, $\mathbf{r}(\frac{\pi}{2}) = 4\mathbf{i}$, and $\mathbf{r}(\pi) = -4\mathbf{j}$, so the orientation is clockwise. For $t = 0$, $\mathbf{r}(0) = 4\mathbf{j}$, and since $\mathbf{r}'(t) = 4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$, $\mathbf{r}'(0) = 4\mathbf{i}$.

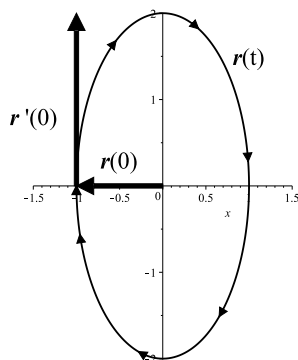


93. The components of $\mathbf{r}(t)$ are $x(t) = 2 \cos t$ and $y(t) = -3 \sin t$, which means $\frac{x}{2} = \cos t$ and $-\frac{y}{3} = \sin t$. Since $(\frac{x}{2})^2 + (-\frac{y}{3})^2 = 1$, the vector function $\mathbf{r}(t)$ traces out an ellipse with its center at the origin, its major axis along the y -axis, and its intercepts are $(-2, 0)$, $(2, 0)$, $(0, -3)$, and $(0, 3)$. Note that

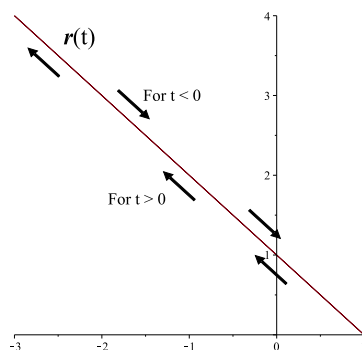
$\mathbf{r}(0) = 2\mathbf{i}$, $\mathbf{r}(\frac{\pi}{2}) = -3\mathbf{j}$, and $\mathbf{r}(\pi) = -2\mathbf{i}$, so the orientation is clockwise. For $t = 0$, $\mathbf{r}(0) = 2\mathbf{i}$, and since $\mathbf{r}'(t) = -2\sin t\mathbf{i} - 3\cos t\mathbf{j}$, $\mathbf{r}'(0) = -3\mathbf{j}$.



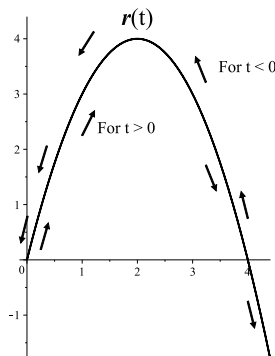
94. The components of $\mathbf{r}(t)$ are $x(t) = -\cos t$ and $y(t) = 2\sin t$, which means $-x = \cos t$ and $\frac{y}{2} = \sin t$. Since $(-x)^2 + (\frac{y}{2})^2 = 1$, the vector function $\mathbf{r}(t)$ traces out an ellipse with its center at the origin, its major axis along the y -axis, and its intercepts are $(-1, 0)$, $(1, 0)$, $(0, -2)$, and $(0, 2)$. Note that $\mathbf{r}(0) = -\mathbf{i}$, $\mathbf{r}(\frac{\pi}{2}) = 2\mathbf{j}$, and $\mathbf{r}(\pi) = \mathbf{i}$, so the orientation is clockwise. For $t = 0$, $\mathbf{r}(0) = -\mathbf{i}$, and since $\mathbf{r}'(t) = \sin t\mathbf{i} + 2\cos t\mathbf{j}$, $\mathbf{r}'(0) = 2\mathbf{j}$.



95. (a) The component functions for $\mathbf{r}(t)$ are $x(t) = (1-t^2)$ and $y(t) = t^2$. We can eliminate the parameter substituting $t^2 = y$ into x to get $x = 1 - y$ or $y = 1 - x$ for $y \geq 0$. This equation represents a ray with its vertex at $(1, 0)$, a slope of -1 and an intercept of 1 . For $t < 0$, x increases while y decreases, so the orientation is downward and right. For $t > 0$, x decreases while y increases, so the orientation is upward and left.
- (b) The graph of $\mathbf{r}(t)$ is below.



96. (a) The component functions for $\mathbf{r}(t)$ are $x(t) = t^2$ and $y(t) = 4t^2 - t^4$. We can eliminate the parameter substituting $x = t^2$ into y to get $y = 4x - x^2$ for $x \geq 0$. This equation represents a parabola that opens downward. Its axis of symmetry is the vertical line $x = -\frac{b}{2a} = -\frac{4}{(2)(-1)} = 2$. For $t < 0$, x decreases as t increases, so the orientation is along the parabola toward the origin. For $t > 0$, x increases as t increases, so the orientation is along the parabola away from the origin.
- (b) The graph of $\mathbf{r}(t)$ is below.

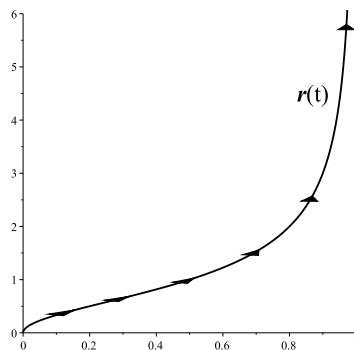


97. (a) The component functions for $\mathbf{r}(t)$ are $x(t) = \sin^2 t$ and $y(t) = \tan t$ for $0 < t < \frac{\pi}{2}$, so $y^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t}$. Since $\sin^2 t + \cos^2 t = 1$, $\cos^2 t = 1 - \sin^2 t$, so we can rewrite y , eliminate the parameter, and solve for y .

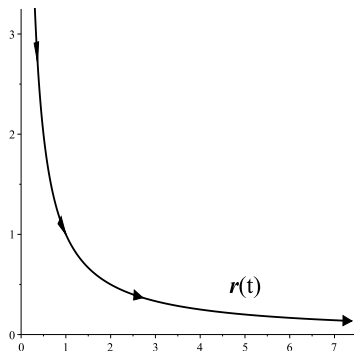
$$\begin{aligned} y^2 &= \frac{\sin^2 t}{\cos^2 t} = \frac{\sin^2 t}{1 - \sin^2 t} \\ &= \frac{x}{1 - x} \quad \text{for } 0 < x < 1, \text{ so} \\ y &= \sqrt{\frac{x}{1 - x}} \quad \text{for } 0 < x < 1 \end{aligned}$$

On the interval $(0, \frac{\pi}{2})$, as t increases, y increases.

- (b) The graph of $\mathbf{r}(t)$ is below.



98. (a) The component functions for $\mathbf{r}(t)$ are $x(t) = e^t$ and $y(t) = e^{-t} = \frac{1}{e^t}$. We can eliminate the parameter substituting $x = e^t$ into y to get $y = \frac{1}{x}$ for $x > 0$. This equation represents the graph of the reciprocal function for $x > 0$. As t increases, x increases and y decreases.
- (b) The graph of $\mathbf{r}(t)$ is below.



99. Let $\mathbf{u}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{v}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then

$$\begin{aligned}\mathbf{u}(t) + \mathbf{v}(t) &= (x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}) + (x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}) \\ &= (x_1(t) + x_2(t))\mathbf{i} + (y_1(t) + y_2(t))\mathbf{j} + (z_1(t) + z_2(t))\mathbf{k}\end{aligned}$$

We use the Sum Rule to get

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] &= \frac{d}{dt}[(x_1(t) + x_2(t))\mathbf{i} + (y_1(t) + y_2(t))\mathbf{j} + (z_1(t) + z_2(t))\mathbf{k}] \\ &= (x_1'(t) + x_2'(t))\mathbf{i} + (y_1'(t) + y_2'(t))\mathbf{j} + (z_1'(t) + z_2'(t))\mathbf{k}.\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}&(x_1'(t) + x_2'(t))\mathbf{i} + (y_1'(t) + y_2'(t))\mathbf{j} + (z_1'(t) + z_2'(t))\mathbf{k} \\ &= (x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}) + (x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}) \\ &= \mathbf{u}'(t) + \mathbf{v}'(t)\end{aligned}$$

■

100. Let $\mathbf{u}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{v}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then

$$\begin{aligned}\mathbf{u}(t) \cdot \mathbf{v}(t) &= (x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}) \cdot (x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}) \\ &= x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t).\end{aligned}$$

We use the Product Rule to get

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d}{dt}[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\ &= x_1'(t)x_2(t) + x_1(t)x_2'(t) + y_1'(t)y_2(t) + y_1(t)y_2'(t) + z_1'(t)z_2(t) + z_1(t)z_2'(t).\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}& x_1'(t)x_2(t) + x_1(t)x_2'(t) + y_1'(t)y_2(t) + y_1(t)y_2'(t) + z_1'(t)z_2(t) + z_1(t)z_2'(t) \\ &= [x_1'(t)x_2(t) + y_1'(t)y_2(t) + z_1'(t)z_2(t)] + [x_1(t)x_2'(t) + y_1(t)y_2'(t) + z_1(t)z_2'(t)]\end{aligned}$$

Since

$$x_1'(t)x_2(t) + y_1'(t)y_2(t) + z_1'(t)z_2(t) = \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

and

$$x_1(t)x_2'(t) + y_1(t)y_2'(t) + z_1(t)z_2'(t) = \mathbf{u}(t) \cdot \mathbf{v}'(t),$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

■

101. Let $\mathbf{u}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{v}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then

$$\begin{aligned}\mathbf{u}(t) \times \mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \end{vmatrix} \\ &= [y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k}\end{aligned}$$

and

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \frac{d}{dt}[(y_1(t)z_2(t) - y_2(t)z_1(t))\mathbf{i} - (x_1(t)z_2(t) - x_2(t)z_1(t))\mathbf{j} + (x_1(t)y_2(t) - x_2(t)y_1(t))\mathbf{k}]$$

We use the Product Rule for each component and rearrange terms.

$$\begin{aligned}\frac{d}{dt}[y_1(t)z_2(t) - y_2(t)z_1(t)] &= y_1'(t)z_2(t) + y_1(t)z_2'(t) - y_2'(t)z_1(t) - y_2(t)z_1'(t) \\ &= [y_1'(t)z_2(t) - y_2'(t)z_1(t)] + [y_1(t)z_2'(t) - y_2(t)z_1'(t)] \\ \frac{d}{dt}[x_1(t)z_2(t) - x_2(t)z_1(t)] &= x_1'(t)z_2(t) + x_1(t)z_2'(t) - x_2'(t)z_1(t) - x_2(t)z_1'(t) \\ &= [x_1'(t)z_2(t) - x_2'(t)z_1(t)] + [x_1(t)z_2'(t) - x_2(t)z_1'(t)] \\ \frac{d}{dt}[x_1(t)y_2(t) - x_2(t)y_1(t)] &= x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2'(t)y_1(t) - x_2(t)y_1'(t) \\ &= [x_1'(t)y_2(t) - x_2'(t)y_1(t)] + [x_1(t)y_2'(t) - x_2(t)y_1'(t)].\end{aligned}$$

Notice that

$$[y_1'(t)z_2(t) - y_2'(t)z_1(t)]\mathbf{i} - [x_1'(t)z_2(t) - x_2'(t)z_1(t)]\mathbf{j} + [x_1'(t)y_2(t) - x_2'(t)y_1(t)]\mathbf{k} = \mathbf{u}'(t) \times \mathbf{v}(t)$$

and

$$[y_1(t)z_2'(t) - y_2(t)z_1'(t)]\mathbf{i} - [x_1(t)z_2'(t) - x_2(t)z_1'(t)]\mathbf{j} + [x_1(t)y_2'(t) - x_2(t)y_1'(t)]\mathbf{k} = \mathbf{u}(t) \times \mathbf{v}'(t)$$

so

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

■

102. Let $\mathbf{v}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{v}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$, so

$$\frac{d}{dt}[\mathbf{v}(f(t))] = \frac{d}{dt}[x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}]$$

By the Chain Rule, $\frac{d}{dt}[x(f(t))] = x'(f(t))f'(t)$, $\frac{d}{dt}[y(f(t))] = y'(f(t))f'(t)$, and $\frac{d}{dt}[z(f(t))] = z'(f(t))f'(t)$. Then

$$\begin{aligned}\frac{d}{dt}[\mathbf{v}(f(t))] &= x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \\ &= f'(t)[x'(f(t))\mathbf{i} + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] \\ &= f'(t)\mathbf{v}'(f(t)) = \mathbf{v}'(f(t))f'(t)\end{aligned}$$

■

103. Answers will vary. Suppose $\mathbf{u}(t) = 3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$ and $\mathbf{v}(t) = t\mathbf{i} + 2\mathbf{j}$. Then $\mathbf{u}'(t) = 3\mathbf{i} + 2t\mathbf{k}$ and $\mathbf{v}'(t) = \mathbf{i}$. By the Cross Product Formula,

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\ &= [3\mathbf{i} + 2t\mathbf{k}] \times [t\mathbf{i} + 2\mathbf{j}] + [3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}] \times [\mathbf{i}] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 2t \\ t & 2 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t & -1 & t^2 \\ 1 & 0 & 0 \end{vmatrix} \\ &= [-4t\mathbf{i} + 2t^2\mathbf{j} + 6\mathbf{k}] + [t^2\mathbf{j} + \mathbf{k}] \\ &= -4t\mathbf{i} + 3t^2\mathbf{j} + 7\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}[\mathbf{v}(t) \times \mathbf{u}(t)] &= \mathbf{v}'(t) \times \mathbf{u}(t) + \mathbf{v}(t) \times \mathbf{u}'(t) \\ &= [\mathbf{i}] \times [3t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}] + [t\mathbf{i} + 2\mathbf{j}] \times [3\mathbf{i} + 2t\mathbf{k}] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 3t & -1 & t^2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2 & 0 \\ 3 & 0 & 2t \end{vmatrix} \\ &= [-t^2\mathbf{j} - \mathbf{k}] + [4t\mathbf{i} - 2t^2\mathbf{j} - 6\mathbf{k}] \\ &= 4t\mathbf{i} - 3t^2\mathbf{j} - 7\mathbf{k}.\end{aligned}$$

Since $-4t\mathbf{i} + 3t^2\mathbf{j} + 7\mathbf{k} \neq 4t\mathbf{i} - 3t^2\mathbf{j} - 7\mathbf{k}$, $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] \neq \frac{d}{dt}[\mathbf{v}(t) \times \mathbf{u}(t)]$, which means the derivative of the cross product of these two vectors is not commutative. In fact, $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = -\frac{d}{dt}[\mathbf{v}(t) \times \mathbf{u}(t)]$.

104. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ and $\mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$. By the Cross Product Formula,

$$\begin{aligned}[\mathbf{r}(t) \times \mathbf{r}'(t)]' &= \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t) & z'(t) \\ x'(t) & y'(t) & z'(t) \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x(t) & y(t) & z(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}\end{aligned}$$

The determinants can be rewritten as

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t) & z'(t) \\ x'(t) & y'(t) & z'(t) \end{vmatrix} &= [y'(t)z'(t) - y'(t)z'(t)]\mathbf{i} - [x'(t)z'(t) - x'(t)z'(t)]\mathbf{j} + [x'(t)y'(t) - x'(t)y'(t)]\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x(t) & y(t) & z(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = [y(t)z''(t) - y''(t)z(t)]\mathbf{i} - [x(t)z''(t) - x''(t)z(t)]\mathbf{j} + [x(t)y''(t) - x''(t)y(t)]\mathbf{k} \\ = \mathbf{r}(t) \times \mathbf{r}''(t),$$

which means that $[\mathbf{r}(t) \times \mathbf{r}'(t)]' = \mathbf{r}(t) \times \mathbf{r}''(t)$. ■

105. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ and $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$.

Suppose $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$. Then $\lim_{t \rightarrow t_0} x(t) = L_1$, $\lim_{t \rightarrow t_0} y(t) = L_2$, and $\lim_{t \rightarrow t_0} z(t) = L_3$, which means that for any $\epsilon > 0$, there are numbers δ_1 , δ_2 , and δ_3 so that

$$\begin{aligned} \text{whenever } 0 < |t - t_0| < \delta_1 \text{ then } |x(t) - L_1| &< \frac{\epsilon}{\sqrt{3}}, \\ \text{whenever } 0 < |t - t_0| < \delta_2 \text{ then } |y(t) - L_2| &< \frac{\epsilon}{\sqrt{3}}, \text{ and} \\ \text{whenever } 0 < |t - t_0| < \delta_3 \text{ then } |z(t) - L_3| &< \frac{\epsilon}{\sqrt{3}}. \end{aligned}$$

Equivalently,

$$\begin{aligned} \text{whenever } 0 < |t - t_0| < \delta_1 \text{ then } (x(t) - L_1)^2 &< \frac{\epsilon^2}{3}, \\ \text{whenever } 0 < |t - t_0| < \delta_2 \text{ then } (y(t) - L_2)^2 &< \frac{\epsilon^2}{3}, \text{ and} \\ \text{whenever } 0 < |t - t_0| < \delta_3 \text{ then } (z(t) - L_3)^2 &< \frac{\epsilon^2}{3}. \end{aligned}$$

Adding terms we have

$$(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2 < \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} = \epsilon^2$$

or

$$\sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2} < \epsilon.$$

Let $\epsilon > 0$ be given, and let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |t - t_0| < \delta$,

$$\sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2} = \|\mathbf{r}(t) - \mathbf{L}\| < \epsilon.$$

Therefore, if $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$, $\lim_{t \rightarrow t_0} \|\mathbf{r}(t) - \mathbf{L}\| = 0$. ■

Conversely, suppose $\lim_{t \rightarrow t_0} \|\mathbf{r}(t) - \mathbf{L}\| = 0$. Then for any $\epsilon > 0$, there is a number $\delta > 0$ such that whenever $0 < |t - t_0| < \delta$, $\|\mathbf{r}(t) - \mathbf{L}\| < \epsilon$. Since by definition

$$\|\mathbf{r}(t) - \mathbf{L}\| = \sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2} < \epsilon,$$

and since

$$\begin{aligned} |x(t) - L_1| &\leq \sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2} \\ |y(t) - L_2| &\leq \sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2}, \text{ and} \\ |z(t) - L_3| &\leq \sqrt{(x(t) - L_1)^2 + (y(t) - L_2)^2 + (z(t) - L_3)^2}, \end{aligned}$$

$\lim_{t \rightarrow t_0} x(t) = L_1$, $\lim_{t \rightarrow t_0} y(t) = L_2$, and $\lim_{t \rightarrow t_0} z(t) = L_3$. Therefore, if $\lim_{t \rightarrow t_0} \|\mathbf{r}(t) - \mathbf{L}\| = 0$, $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$. ■

106. Since $\|\mathbf{r}(t)\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$, the Chain Rule gives

$$\begin{aligned} \frac{d}{dt} \|\mathbf{r}(t)\| &= \frac{1}{2} [(x(t))^2 + (y(t))^2 + (z(t))^2]^{-1/2} (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \\ &= \frac{x(t)x'(t) + y(t)y'(t) + z(t)z'(t)}{\sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}} \\ &= \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|} \end{aligned}$$

if $\mathbf{r}(t) \neq \mathbf{0}$. Now if $\mathbf{r}(t)$ is constant, $\frac{d}{dt} \|\mathbf{r}(t)\| = 0$ and so $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for $\mathbf{r}(t) \neq \mathbf{0}$. Conversely, if $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, $\frac{d}{dt} \|\mathbf{r}(t)\| = 0$, which implies that $\|\mathbf{r}(t)\|$ is constant.

Challenge Problems

107. By the Quotient Rule,

$$\frac{d}{dt} \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} = \frac{\|\mathbf{r}(t)\| \mathbf{r}'(t) - \mathbf{r}(t) \frac{d}{dt} \|\mathbf{r}(t)\|}{\|\mathbf{r}(t)\|^2}$$

for $\mathbf{r}(t) \neq \mathbf{0}$. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then

$$\|\mathbf{r}(t)\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

and

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

Using the Chain Rule,

$$\begin{aligned} \frac{d}{dt} \|\mathbf{r}(t)\| &= \frac{1}{2} ((x(t))^2 + (y(t))^2 + (z(t))^2)^{-1/2} (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \\ &= ((x(t))^2 + (y(t))^2 + (z(t))^2)^{-1/2} (x(t)x'(t) + y(t)y'(t) + z(t)z'(t)). \end{aligned}$$

Note that $x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r}(t) \cdot \mathbf{r}'(t)$, so

$$\frac{d}{dt} \|\mathbf{r}(t)\| = \|\mathbf{r}(t)\|^{-1} (\mathbf{r}(t) \cdot \mathbf{r}'(t)) = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}.$$

Substituting $\frac{d}{dt} \|\mathbf{r}(t)\|$ into the equation from the Quotient Rule,

$$\begin{aligned} \frac{d}{dt} \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} &= \frac{\|\mathbf{r}(t)\| \mathbf{r}'(t) - \mathbf{r}(t) \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}}{\|\mathbf{r}(t)\|^2} \\ &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}(t)\|} - \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|^3} \mathbf{r}(t) \end{aligned}$$

108. (a) Answers may vary. As $t \rightarrow -\infty$, the curve C approaches the origin since $\lim_{t \rightarrow -\infty} e^t = 0$, $-1 \leq \sin t \leq 1$, and $-1 \leq \cos t \leq 1$. The curve spirals upward exponentially as $t \rightarrow \infty$.

(b) We know $\sin^2 t + \cos^2 t = 1$, so $e^{2t} \sin^2 t + e^{2t} \cos^2 t = e^{2t}$, or $x^2 + y^2 = z^2$. The curve lies on a cone given by $x^2 + y^2 = z^2$.

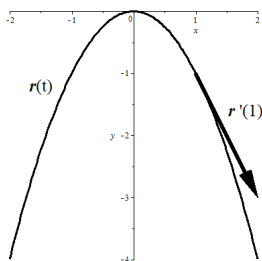
11.2 Unit Tangent and Principal Unit Normal Vectors; Arc Length

Concepts and Vocabulary

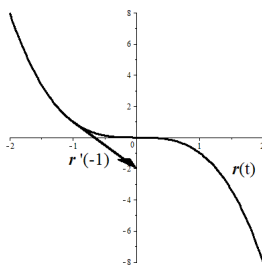
1. False. The tangent line to the curve at t_0 contains the point P_0 but has direction $\mathbf{r}'(t_0)$.
2. False. A curve traced out by $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t)$ exists and is continuous on (a, b) and if $\mathbf{r}'(t) \neq \mathbf{0}$ on (a, b) .
3. False. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$
4. False. $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$
5. True by the theorem about arc length of a vector function.
6. True by definition of the principal unit normal vector.

Skill Building

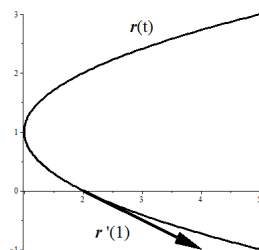
7. (a) $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$ so $\mathbf{r}'(1) = \boxed{\mathbf{i} - 2\mathbf{j}}$
 (b) and (c)



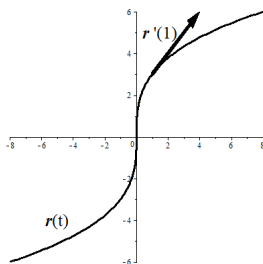
8. (a) $\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$ so $\mathbf{r}'(-1) = \boxed{\mathbf{i} - 3\mathbf{j}}$
 (b) and (c)



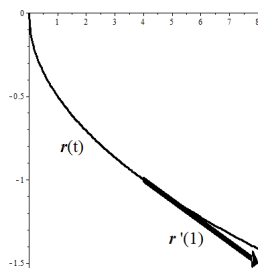
9. (a) $\mathbf{r}'(t) = 2t\mathbf{i} - \mathbf{j}$ so $\mathbf{r}'(1) = \boxed{2\mathbf{i} - \mathbf{j}}$
 (b) and (c)



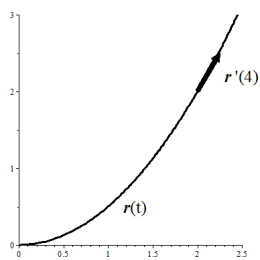
10. (a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + 3\mathbf{j}$ so $\mathbf{r}'(1) = \boxed{3\mathbf{i} + 3\mathbf{j}}$
 (b) and (c)



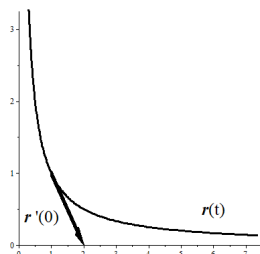
11. (a) $\mathbf{r}'(t) = 4\mathbf{i} - \frac{1}{2\sqrt{t}}\mathbf{j}$ so $\mathbf{r}'(1) = \boxed{4\mathbf{i} - \frac{1}{2}\mathbf{j}}$
 (b) and (c)



12. (a) $\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + \frac{1}{2}\mathbf{j}$ so $\mathbf{r}'(4) = \boxed{\frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}}$
 (b) and (c)

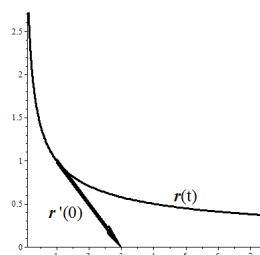


13. (a) $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} - \mathbf{j}}$
 (b) and (c)



14. (a) $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} - e^{-t}\mathbf{j}$ so $\mathbf{r}'(0) = \boxed{2\mathbf{i} - \mathbf{j}}$

(b) and (c)



15. $\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ so $\mathbf{r}'(0) = \boxed{-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}}$

16. $\mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} - \mathbf{j} + 3\mathbf{k}}$

17. $\mathbf{r}'(t) = -2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j}$ so $\mathbf{r}'(\frac{\pi}{4}) = \boxed{-2\mathbf{i}}$

18. $\mathbf{r}'(t) = -\sin t\mathbf{j} + \cos t\mathbf{k}$ so $\mathbf{r}'(\frac{\pi}{6}) = \boxed{-\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}}$

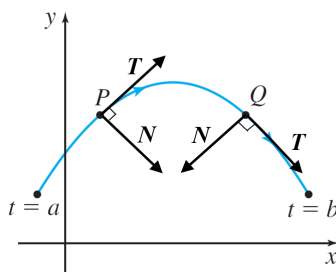
19. $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{k}$ so $\mathbf{r}'(\frac{\pi}{6}) = \boxed{-\mathbf{i} + \sqrt{3}\mathbf{k}}$

20. $\mathbf{r}'(t) = -4\sin(2t)\mathbf{i} + 4\cos(2t)\mathbf{j}$ so $\mathbf{r}'(\frac{\pi}{2}) = \boxed{-4\mathbf{j}}$

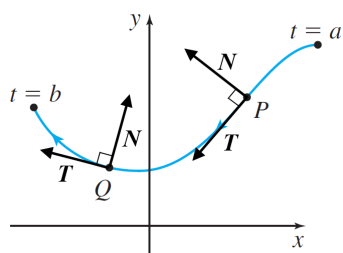
21. $\mathbf{r}'(t) = (-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j} + e^t\mathbf{k}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} + \mathbf{j} + \mathbf{k}}$

22. $\mathbf{r}'(t) = (-e^{-t} \sin t - e^{-t} \cos t)\mathbf{i} + (e^{-t} \cos t - e^{-t} \sin t)\mathbf{j} + e^{-t}\mathbf{k}$ so $\mathbf{r}'(0) = \boxed{-\mathbf{i} + \mathbf{j} + \mathbf{k}}$

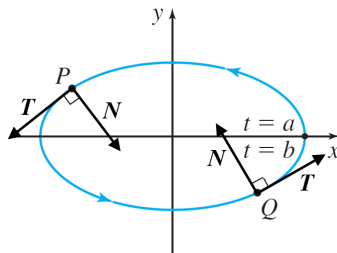
23. The tangent vectors are tangent to C and follow the orientation of C , and the normal vectors are orthogonal to the tangent vectors and point into the curve. See the figure below, which is not drawn to scale.



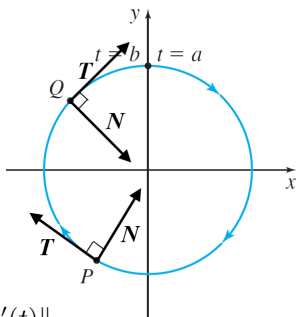
24. The tangent vectors are tangent to C and follow the orientation of C , and the normal vectors are orthogonal to the tangent vectors and point into the curve. See the figure below, which is not drawn to scale.



25. The tangent vectors are tangent to C and follow the orientation of C , and the normal vectors are orthogonal to the tangent vectors and point into the curve. See the figure below, which is not drawn to scale.



26. The tangent vectors are tangent to C and follow the orientation of C , and the normal vectors are orthogonal to the tangent vectors and point into the curve. See the figure below, which is not drawn to scale.



27. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}} \quad \text{and} \quad \mathbf{T}(1) = \boxed{\frac{\sqrt{5}}{5}\mathbf{i} + \frac{2\sqrt{5}}{5}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{(2\mathbf{j})(\sqrt{1 + 4t^2}) - (\mathbf{i} + 2t\mathbf{j})(\frac{1}{2})(1 + 4t^2)^{-1/2}(8t)}{(1 + 4t^2)}.$$

We can multiply the numerator and the denominator by $\sqrt{1 + 4t^2}$ to simplify.

$$\mathbf{T}'(t) = \frac{(2\mathbf{j})(1 + 4t^2) - (4t)(\mathbf{i} + 2t\mathbf{j})}{(1 + 4t^2)^{3/2}} = \frac{(2 + 8t^2)\mathbf{j} - 4t\mathbf{i} - 8t^2\mathbf{j}}{(1 + 4t^2)^{3/2}} = \frac{-4t\mathbf{i} + 2\mathbf{j}}{(1 + 4t^2)^{3/2}}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{16t^2 + 4}{(1 + 4t^2)^3}} = \sqrt{\frac{4(4t^2 + 1)}{(1 + 4t^2)^3}} = \frac{2}{1 + 4t^2}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{-4t\mathbf{i} + 2\mathbf{j}}{(1 + 4t^2)^{3/2}}}{\frac{2}{1 + 4t^2}} = \frac{-2t\mathbf{i} + \mathbf{j}}{(1 + 4t^2)^{1/2}}, \quad \text{and so} \quad \mathbf{N}(1) = \boxed{-\frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}}.$$

28. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 9t^4}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{\mathbf{i} - 3t^2\mathbf{j}}{\sqrt{1 + 9t^4}} \quad \text{and} \quad \mathbf{T}(-1) = \boxed{\frac{\sqrt{10}}{10}\mathbf{i} - \frac{3\sqrt{10}}{10}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{-6t\mathbf{j}\sqrt{1 + 9t^4} - (\mathbf{i} - 3t^2\mathbf{j})(\frac{1}{2})(1 + 9t^4)^{-1/2}(36t^3)}{(1 + 9t^4)}.$$

We can multiply the numerator and the denominator by $\sqrt{1 + 9t^4}$ to simplify.

$$\mathbf{T}'(t) = \frac{-6t\mathbf{j}(1 + 9t^4) - (\mathbf{i} - 3t^2\mathbf{j})(18t^3)}{(1 + 9t^4)^{3/2}} = \frac{(-6t)[(1 + 9t^4)\mathbf{j} + 3t^2\mathbf{i} - 9t^4\mathbf{j}]}{(1 + 9t^4)^{3/2}} = \frac{-(6t)(3t^2\mathbf{i} + \mathbf{j})}{(1 + 9t^4)^{3/2}}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{(-6t)^2(9t^4 + 1)}{(1 + 9t^4)^3}} = \frac{6t}{1 + 9t^4}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{-(6t)(3t^2\mathbf{i} + \mathbf{j})}{(1 + 9t^4)^{3/2}}}{\frac{6t}{1 + 9t^4}} = \frac{-(3t^2\mathbf{i} + \mathbf{j})}{\sqrt{1 + 9t^4}} \quad \text{and} \quad \mathbf{N}(-1) = \boxed{-\frac{3\sqrt{10}}{10}\mathbf{i} - \frac{\sqrt{10}}{10}\mathbf{j}}.$$

29. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2t\mathbf{i} - \mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{2t\mathbf{i} - \mathbf{j}}{\sqrt{4t^2 + 1}} \quad \text{and} \quad \mathbf{T}(1) = \boxed{\frac{2\sqrt{5}}{5}\mathbf{i} - \frac{\sqrt{5}}{5}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{[2\sqrt{4t^2 + 1} - (2t)(\frac{1}{2})(4t^2 + 1)^{-1/2}(8t)]\mathbf{i} + \frac{1}{2}(4t^2 + 1)^{-1/2}(8t)\mathbf{j}}{4t^2 + 1}.$$

We can multiply the numerator and the denominator by $\sqrt{4t^2 + 1}$ to simplify.

$$\mathbf{T}'(t) = \frac{[2(4t^2 + 1) - 8t^2]\mathbf{i} + 4t\mathbf{j}}{(4t^2 + 1)^{3/2}} = \frac{2\mathbf{i} + 4t\mathbf{j}}{(4t^2 + 1)^{3/2}}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{4 + 16t^2}{(4t^2 + 1)^3}} = \sqrt{\frac{4(1 + 4t^2)}{(4t^2 + 1)^3}} = \frac{2}{4t^2 + 1}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{2\mathbf{i} + 4t\mathbf{j}}{(4t^2 + 1)^{3/2}}}{\frac{2}{4t^2 + 1}} = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{4t^2 + 1}} \quad \text{and} \quad \mathbf{N}(1) = \boxed{\frac{\sqrt{5}}{5}\mathbf{i} + \frac{2\sqrt{5}}{5}\mathbf{j}}.$$

30. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 3\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{9t^4 + 9} = 3\sqrt{t^4 + 1}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{3t^2\mathbf{i} + 3\mathbf{j}}{3\sqrt{t^4 + 1}} = \frac{t^2\mathbf{i} + \mathbf{j}}{\sqrt{t^4 + 1}} \quad \text{and} \quad \mathbf{T}(1) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{[2t\sqrt{t^4 + 1} - t^2(\frac{1}{2})(t^4 + 1)^{-1/2}(4t^3)]\mathbf{i} - (\frac{1}{2})(t^4 + 1)^{-1/2}(4t^3)\mathbf{j}}{t^4 + 1}.$$

We can multiply the numerator and the denominator by $\sqrt{t^4 + 1}$ to simplify.

$$\mathbf{T}'(t) = \frac{[(2t)(t^4 + 1) - 2t^5]\mathbf{i} - 2t^3\mathbf{j}}{(t^4 + 1)^{3/2}} = \frac{2t\mathbf{i} - 2t^3\mathbf{j}}{(t^4 + 1)^{3/2}} = \frac{(2t)(\mathbf{i} - t^2\mathbf{j})}{(t^4 + 1)^{3/2}}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{(2t)^2(1 + t^4)}{(t^4 + 1)^3}} = \frac{2t}{t^4 + 1}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{(2t)(\mathbf{i} - t^2\mathbf{j})}{(t^4 + 1)^{3/2}}}{\frac{2t}{t^4 + 1}} = \frac{\mathbf{i} - t^2\mathbf{j}}{\sqrt{t^4 + 1}} \quad \text{and} \quad \mathbf{N}(1) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.$$

31. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$. Using the Quotient Rule to find the derivative of $\mathbf{r}(t) = \frac{2t}{t+1}\mathbf{i} - \frac{t^2}{t+1}\mathbf{j}$,

$$\mathbf{r}'(t) = \frac{2(t+1) - 2t}{(t+1)^2}\mathbf{i} - \frac{2t(t+1) - t^2}{(t+1)^2}\mathbf{j} = \frac{2\mathbf{i} - t(2+t)\mathbf{j}}{(t+1)^2}$$

so $\|\mathbf{r}'(t)\| = \sqrt{\frac{4+t^2(2+t)^2}{(t+1)^4}} = \frac{\sqrt{4+4t^2+4t^3+t^4}}{(t+1)^2}$. Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{\frac{2\mathbf{i} - t(2+t)\mathbf{j}}{(t+1)^2}}{\frac{\sqrt{4+4t^2+4t^3+t^4}}{(t+1)^2}} = \frac{2\mathbf{i} - t(2+t)\mathbf{j}}{\sqrt{4+4t^2+4t^3+t^4}} \quad \text{and} \quad \mathbf{T}(1) = \boxed{\frac{2\sqrt{13}}{13}\mathbf{i} - \frac{3\sqrt{13}}{13}\mathbf{j}}.$$

We find the derivative of the $x(t) = \frac{2}{\sqrt{4+4t^2+4t^3+t^4}}$ component of $\mathbf{T}(t)$.

$$\frac{dx}{dt} = \frac{-(8t + 12t^2 + 4t^3)}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} = \frac{-4t(2 + 3t + t^2)}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} = \frac{-4t(t+1)(t+2)}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}}$$

To find the derivative of the $y(t) = \frac{-t(2+t)}{\sqrt{4+4t^2+4t^3+t^4}}$ component of $\mathbf{T}(t)$, we use the Quotient and Chain Rules.

$$\frac{dy}{dt} = \frac{(-2 - 2t)\sqrt{4 + 4t^2 + 4t^3 + t^4} + t(2 + t)(\frac{1}{2})(4 + 4t^2 + 4t^3 + t^4)^{-1/2}(8t + 12t^2 + 4t^3)}{4 + 4t^2 + 4t^3 + t^4}$$

We can multiply the numerator and the denominator by $\sqrt{4 + 4t^2 + 4t^3 + t^4}$ to simplify.

$$\begin{aligned} \frac{dy}{dt} &= \frac{(-2 - 2t)(4 + 4t^2 + 4t^3 + t^4) + t(2 + t)(4t + 6t^2 + 2t^3)}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} \\ &= \frac{-8 - 8t - 8t^2 - 16t^3 - 10t^4 - 2t^5 + 8t^2 + 16t^3 + 10t^4 + 2t^5}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} \\ &= \frac{-8 - 8t}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} = \frac{-8(1 + t)}{(4 + 4t^2 + 4t^3 + t^4)^{3/2}} \end{aligned}$$

Since $\mathbf{T}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$,

$$\mathbf{T}'(t) = \frac{-4t(t+1)(t+2)\mathbf{i} - 8(1+t)\mathbf{j}}{(4+4t^2+4t^3+t^4)^{3/2}}.$$

Then

$$\begin{aligned}\|\mathbf{T}'(t)\| &= \sqrt{\frac{16t^2(t+1)^2(t+2)^2 + 64(1+t)^2}{(4+4t^2+4t^3+t^4)^3}} \\ &= \frac{4(t+1)\sqrt{t^2(t+2)^2+4}}{(4+4t^2+4t^3+t^4)^{3/2}} \\ &= \frac{4(t+1)\sqrt{t^4+4t^3+4t^2+4}}{(4+4t^2+4t^3+t^4)^{3/2}} \\ &= \frac{4(t+1)}{4+4t^2+4t^3+t^4}.\end{aligned}$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{-4t(t+1)(t+2)\mathbf{i} - 8(1+t)\mathbf{j}}{(4+4t^2+4t^3+t^4)^{3/2}}}{\frac{4(t+1)}{4+4t^2+4t^3+t^4}} = \frac{-t(t+2)\mathbf{i} - 2\mathbf{j}}{\sqrt{t^4+4t^3+4t^2+4}} \quad \text{and} \quad \mathbf{N}(1) = \boxed{-\frac{3\sqrt{13}}{13}\mathbf{i} - \frac{2\sqrt{13}}{13}\mathbf{j}}.$$

32. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + \frac{1}{2}\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{4t} + \frac{1}{4}} = \sqrt{\frac{1+t}{4t}} = \frac{1}{2}\sqrt{\frac{1+t}{t}}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{\frac{1}{2\sqrt{t}}\mathbf{i} + \frac{1}{2}\mathbf{j}}{\frac{1}{2}\sqrt{\frac{1+t}{t}}} = \frac{\mathbf{i} + \sqrt{t}\mathbf{j}}{\sqrt{1+t}} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \sqrt{\frac{t}{1+t}}\mathbf{j} \quad \text{and} \quad \mathbf{T}(4) = \boxed{\frac{\sqrt{5}}{5}\mathbf{i} + \frac{2\sqrt{5}}{5}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = -\frac{1}{2(1+t)^{3/2}}\mathbf{i} + \frac{1}{2}\sqrt{\frac{1+t}{t}} \left[\frac{1+t-t}{(1+t)^2} \right] \mathbf{j} = -\frac{1}{2(1+t)^{3/2}}\mathbf{i} + \frac{1}{2\sqrt{t}(1+t)^{3/2}}\mathbf{j}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{1}{4(1+t)^3} + \frac{1}{4t(1+t)^3}} = \sqrt{\frac{t+1}{4t(1+t)^3}} = \frac{1}{2\sqrt{t}(1+t)}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{-\frac{1}{2(1+t)^{3/2}}\mathbf{i} + \frac{1}{2\sqrt{t}(1+t)^{3/2}}\mathbf{j}}{\frac{1}{2\sqrt{t}(1+t)}} = -\sqrt{\frac{t}{1+t}}\mathbf{i} + \frac{1}{\sqrt{1+t}}\mathbf{j} \quad \text{and} \quad \mathbf{N}(4) = \boxed{-\frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}}.$$

33. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{e^{2t} + e^{-2t}}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}} \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{(e^t \mathbf{i} + e^{-t} \mathbf{j})\sqrt{e^{2t} + e^{-2t}} - (e^t \mathbf{i} - e^{-t} \mathbf{j})(\frac{1}{2})(e^{2t} + e^{-2t})^{-1/2}(2e^{2t} - 2e^{-2t})}{e^{2t} + e^{-2t}}.$$

We can multiply the numerator and the denominator by $\sqrt{e^{2t} + e^{-2t}}$ to simplify.

$$\begin{aligned} \mathbf{T}'(t) &= \frac{(e^t \mathbf{i} + e^{-t} \mathbf{j})(e^{2t} + e^{-2t}) - (e^t \mathbf{i} - e^{-t} \mathbf{j})(e^{2t} - e^{-2t})}{(e^{2t} + e^{-2t})^{3/2}} \\ &= \frac{(e^{3t} + e^{-t})\mathbf{i} + (e^t + e^{-3t})\mathbf{j} - (e^{3t} - e^{-t})\mathbf{i} + (e^t - e^{-3t})\mathbf{j}}{(e^{2t} + e^{-2t})^{3/2}} \\ &= \frac{2e^{-t}\mathbf{i} + 2e^t\mathbf{j}}{(e^{2t} + e^{-2t})^{3/2}} \end{aligned}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{4e^{-2t} + 4e^{2t}}{(e^{2t} + e^{-2t})^3}} = \sqrt{\frac{4(e^{-2t} + e^{2t})}{(e^{2t} + e^{-2t})^3}} = \frac{2}{e^{2t} + e^{-2t}}$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{2e^{-t}\mathbf{i} + 2e^t\mathbf{j}}{(e^{2t} + e^{-2t})^{3/2}}}{\frac{2}{e^{2t} + e^{-2t}}} = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}} \quad \text{and} \quad \mathbf{N}(0) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}.$$

34. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} - e^{-t}\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4e^{4t} + e^{-2t}}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{2e^{2t}\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{4e^{4t} + e^{-2t}}} \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{2\sqrt{5}}{5}\mathbf{i} - \frac{\sqrt{5}}{5}\mathbf{j}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{(4e^{2t}\mathbf{i} + e^{-t}\mathbf{j})(\sqrt{4e^{4t} + e^{-2t}}) - (2e^{2t}\mathbf{i} - e^{-t}\mathbf{j})(\frac{1}{2})(4e^{4t} + e^{-2t})^{-1/2}(16e^{4t} - 2e^{-2t})}{4e^{4t} + e^{-2t}}.$$

We can multiply the numerator and the denominator by $\sqrt{4e^{4t} + e^{-2t}}$ to simplify.

$$\begin{aligned} \mathbf{T}'(t) &= \frac{(4e^{2t}\mathbf{i} + e^{-t}\mathbf{j})(4e^{4t} + e^{-2t}) - (2e^{2t}\mathbf{i} - e^{-t}\mathbf{j})(8e^{4t} - e^{-2t})}{(4e^{4t} + e^{-2t})^{3/2}} \\ &= \frac{16e^{6t}\mathbf{i} + 4\mathbf{i} + 4e^{3t}\mathbf{j} + e^{-3t}\mathbf{j} - (16e^{6t}\mathbf{i} - 2\mathbf{i} - 8e^{3t}\mathbf{j} + e^{-3t}\mathbf{j})}{(4e^{4t} + e^{-2t})^{3/2}} \\ &= \frac{6\mathbf{i} + 12e^{3t}\mathbf{j}}{(4e^{4t} + e^{-2t})^{3/2}} = \frac{6(\mathbf{i} + 2e^{3t}\mathbf{j})}{(4e^{4t} + e^{-2t})^{3/2}} \end{aligned}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{36(1 + 4e^{6t})}{(4e^{4t} + e^{-2t})^3}} = 6\sqrt{\frac{1 + 4e^{6t}}{(4e^{4t} + e^{-2t})^3}}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{6(\mathbf{i} + 2e^{3t}\mathbf{j})}{(4e^{4t} + e^{-2t})^{3/2}}}{6\sqrt{\frac{1 + 4e^{6t}}{(4e^{4t} + e^{-2t})^3}}} = \frac{\mathbf{i} + 2e^{3t}\mathbf{j}}{\sqrt{1 + 4e^{6t}}} \quad \text{and} \quad \mathbf{N}(0) = \boxed{\frac{\sqrt{5}}{5}\mathbf{i} + \frac{2\sqrt{5}}{5}\mathbf{j}}.$$

35. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{9 + 4 + 1} = \sqrt{14}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{14}} \quad \text{and} \quad \mathbf{T}(0) = \boxed{-\frac{3\sqrt{14}}{14}\mathbf{i} + \frac{\sqrt{14}}{7}\mathbf{j} - \frac{\sqrt{14}}{14}\mathbf{k}}.$$

Since $\mathbf{T}(t)$ is a constant vector, $\mathbf{T}'(t) = \mathbf{0}$, which means that $\|\mathbf{T}'(t)\| = 0$, so $\mathbf{N}(t)$ is not defined.

$$\mathbf{N}(0) \text{ is } \boxed{\text{undefined}}.$$

36. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 3\mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 1 + 9} = \sqrt{11}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{\mathbf{i} - \mathbf{j} + 3\mathbf{k}}{\sqrt{11}} \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{11}}{11}\mathbf{i} - \frac{\sqrt{11}}{11}\mathbf{j} + \frac{3\sqrt{11}}{11}\mathbf{k}}.$$

Since $\mathbf{T}(t)$ is a constant vector, $\mathbf{T}'(t) = \mathbf{0}$, which means that $\|\mathbf{T}'(t)\| = 0$, so $\mathbf{N}(t)$ is not defined.

$$\mathbf{N}(0) \text{ is } \boxed{\text{undefined}}.$$

37. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4(\sin^2 t + \cos^2 t)} = 2.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{k} \quad \text{and} \quad \mathbf{T}\left(\frac{\pi}{6}\right) = \boxed{-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{k}}.$$

Then $\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{k}$ and $\|\mathbf{T}'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$. Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{k} \quad \text{and} \quad \mathbf{N}\left(\frac{\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{k}}.$$

38. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -\sin t\mathbf{j} + \cos t\mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = -\sin t\mathbf{j} + \cos t\mathbf{k} \quad \text{and} \quad \mathbf{T}\left(\frac{\pi}{6}\right) = \boxed{-\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}}.$$

Then $\mathbf{T}'(t) = -\cos t\mathbf{j} - \sin t\mathbf{k}$ and $\|\mathbf{T}'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$. Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = -\cos t\mathbf{j} - \sin t\mathbf{k} \quad \text{and} \quad \mathbf{N}\left(\frac{\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}}.$$

39. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{4[\sin^2(2t) + \cos^2(2t)]} = 2.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = -\sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} \quad \text{and} \quad \mathbf{T}\left(\frac{\pi}{4}\right) = \boxed{-\mathbf{i}}.$$

$$\text{Then } \mathbf{T}'(t) = -2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} \text{ and } \|\mathbf{T}'(t)\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2. \text{ Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = -\cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} = \quad \text{and} \quad \mathbf{N}\left(\frac{\pi}{4}\right) = \boxed{-\mathbf{j}}.$$

40. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -4\sin(2t)\mathbf{i} + 4\cos(2t)\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{16\sin^2(2t) + 16\cos^2(2t)} = 4.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = -\sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} \quad \text{and} \quad \mathbf{T}\left(\frac{\pi}{2}\right) = \boxed{-\mathbf{j}}.$$

$$\text{Then } \mathbf{T}'(t) = -2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} \text{ and } \|\mathbf{T}'(t)\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2. \text{ Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = -\cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} \quad \text{and} \quad \mathbf{N}\left(\frac{\pi}{2}\right) = \boxed{\mathbf{i}}.$$

41. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$. $\mathbf{r}'(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$ so

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t}} \\ &= \sqrt{e^{2t}(\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t + 1)} = e^t \sqrt{3}. \end{aligned}$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{\sqrt{3}}{3}[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + \mathbf{k}] \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})}.$$

Then

$$\mathbf{T}'(t) = \frac{\sqrt{3}}{3}[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}]$$

and

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{1}{3}(\sin^2 t + 2\sin t \cos t + \cos^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t)} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{\sqrt{3}}{3}[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}]}{\frac{\sqrt{6}}{3}} = \frac{\sqrt{2}}{2}[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}]$$

and

$$\mathbf{N}(0) = \boxed{-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}.$$

42. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= (-e^{-t} \cos t - e^{-t} \sin t)\mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t)\mathbf{j} + e^{-t}\mathbf{k} \\ &= e^{-t}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]\end{aligned}$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{e^{-2t}(\cos^2 t + 2 \sin t \cos t + \sin^2 t + \sin^2 t - 2 \sin t \cos t + \cos^2 t + 1)} = e^{-t}\sqrt{3}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{\sqrt{3}}{3}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}] \quad \text{and} \quad \mathbf{T}(0) = \boxed{-\frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k}}.$$

Then

$$\mathbf{T}'(t) = \frac{\sqrt{3}}{3}[(\sin t - \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

and

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{1}{3}(\sin^2 t - 2 \sin t \cos t + \cos^2 t + \cos^2 t + 2 \sin t \cos t + \sin^2 t)} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{\sqrt{3}}{3}[(\sin t - \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]}{\frac{\sqrt{6}}{3}} = \frac{\sqrt{2}}{2}[(\sin t - \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

and

$$\mathbf{N}(0) = \boxed{\frac{-\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.$$

43. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + \frac{9}{4}t} = \sqrt{\frac{4+9t}{4}} = \frac{1}{2}\sqrt{4+9t}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_1^8 \frac{1}{2}\sqrt{4+9t} dt$$

Let $u = 4 + 9t$. Then $du = 9dt$ so $dt = \frac{du}{9}$. Now we change the limits of integration. When $t = 1$, $u = 13$, and when $t = 8$, $u = 76$. Therefore,

$$\frac{1}{2} \int_1^8 \sqrt{4+9t} dt = \frac{1}{2} \int_{13}^{76} u^{1/2} \frac{du}{9} = \left[\frac{1}{18} \cdot \frac{2}{3} u^{3/2} \right]_{13}^{76} = \boxed{\frac{1}{27}(76^{3/2} - 13^{3/2})}.$$

44. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + \frac{9}{4}t} = \sqrt{\frac{4+9t}{4}} = \frac{1}{2}\sqrt{4+9t}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^4 \frac{1}{2}\sqrt{4+9t} dt$$

Let $u = 4 + 9t$. Then $du = 9dt$ so $dt = \frac{du}{9}$. Now we change the limits of integration. When $t = 0$, $u = 4$, and when $t = 4$, $u = 40$. Therefore,

$$\frac{1}{2} \int_0^4 \sqrt{4+9t} dt = \frac{1}{2} \int_4^{40} u^{1/2} \frac{du}{9} = \left[\frac{1}{18} \cdot \frac{2}{3} u^{3/2} \right]_4^{40} = \frac{1}{27} (40^{3/2} - 4^{3/2}) = \boxed{\frac{1}{27} (80\sqrt{10} - 8)}.$$

45. We can rewrite $\frac{t^6+2}{t^2}$ as $t^4 + 2t^{-2}$ to simplify differentiating the y -component of $\mathbf{r}(t)$. We find $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 8\mathbf{i} + (4t^3 - 4t^{-3})\mathbf{j} \text{ so}$$

$$\|\mathbf{r}'(t)\| = \sqrt{64 + 16t^6 - 32 + 16t^{-6}} = 4\sqrt{t^6 + 2 + t^{-6}} = 4\sqrt{(t^3 + t^{-3})^2} = 4(t^3 + t^{-3}).$$

Now we use the formula for arc length.

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_1^2 4(t^3 + t^{-3}) dt = \left[4 \left(\frac{1}{4} t^4 - \frac{1}{2} t^{-2} \right) \right]_1^2 \\ &= [t^4 - 2t^{-2}]_1^2 = (2^4 - (2)(2)^{-2}) - (1^4 - (2)(1)^{-2}) = \left(16 - \frac{1}{2} \right) - (1 - 2) = \frac{31}{2} + 1 = \boxed{\frac{33}{2}} \end{aligned}$$

46. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}(1 - t^{2/3})^{1/2} \left(-\frac{2}{3} \right) t^{-1/3} \mathbf{j} = \mathbf{i} - \frac{\sqrt{1 - t^{2/3}}}{t^{1/3}} \mathbf{j} \text{ so}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + \frac{1 - t^{2/3}}{t^{2/3}}} = \sqrt{\frac{t^{2/3} + 1 - t^{2/3}}{t^{2/3}}} = \sqrt{\frac{1}{t^{2/3}}} = t^{-1/3}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_{\frac{1}{8}}^1 t^{-1/3} dt = \left[\frac{3}{2} t^{2/3} \right]_{\frac{1}{8}}^1 = \frac{3}{2} \left[1 - \left(\frac{1}{8} \right)^{2/3} \right] = \frac{3}{2} \left(1 - \frac{1}{4} \right) = \boxed{\frac{9}{8}}$$

47. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 4 + 1} = \sqrt{6}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^1 \sqrt{6} dt = [\sqrt{6}t]_0^1 = \boxed{\sqrt{6}}$$

48. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4 + 1 + 9} = \sqrt{14}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_1^3 \sqrt{14} dt = [\sqrt{14}t]_1^3 = \boxed{2\sqrt{14}}$$

49. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 1} = \sqrt{5}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^\pi \sqrt{5} dt = [\sqrt{5}t]_0^\pi = \boxed{\sqrt{5}\pi}$$

50. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + b \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + b^2} = \sqrt{1 + b^2}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{1 + b^2} dt = \left[t \sqrt{1 + b^2} \right]_0^{2\pi} = \boxed{2\pi \sqrt{1 + b^2}}$$

51. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = e^t \mathbf{i} - e^{-t} \mathbf{j} + \sqrt{2} \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{e^{2t} + e^{-2t} + 2} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^1 (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^1 = (e - e^{-1}) - (1 - 1) = \boxed{e - \frac{1}{e}}$$

52. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j} \quad \text{so}$$

$$\|\mathbf{r}'(t)\| = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} = \sqrt{9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} = 3 \cos t \sin t.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{\pi/2} 3 \sin t \cos t dt$$

Let $u = \sin t$. Then $du = \cos t dt$. Now we change the limits of integration. When $t = 0$, $u = 0$, and when $t = \frac{\pi}{2}$, $u = 1$.

$$\int_0^{\pi/2} 3 \sin t \cos t dt = \int_0^1 3u du = \left[\frac{3}{2} u^2 \right]_0^1 = \frac{3}{2} (1 - 0) = \boxed{\frac{3}{2}}$$

53. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j} + e^t \mathbf{k} \quad \text{so}$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} \cos^2 t - 2e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t}} \\ &= \sqrt{e^{2t} (2 \cos^2 t + 2 \sin^2 t + 1)} = e^t \sqrt{3}. \end{aligned}$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} e^t \sqrt{3} dt = \left[e^t \sqrt{3} \right]_0^{2\pi} = \sqrt{3} e^{2\pi} - \sqrt{3} = \boxed{\sqrt{3} (e^{2\pi} - 1)}$$

54. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2t \mathbf{i} - 2\sqrt{2} \mathbf{j} + 2t \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 8 + 4t^2} = \sqrt{8t^2 + 8} = (2\sqrt{2}) \sqrt{t^2 + 1}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^1 (2\sqrt{2}) \sqrt{t^2 + 1} dt$$

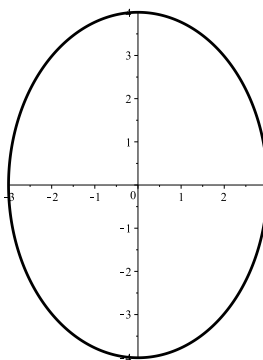
Let $\tan u = t$. Then $\sec^2 u \, du = dt$, and $t^2 + 1 = \tan^2 u + 1 = \sec^2 u$. Now we change the limits of integration. When $t = 0$, $u = 0$, and when $t = 1$, $u = \frac{\pi}{4}$, and

$$\int_0^1 (2\sqrt{2})\sqrt{t^2+1} \, dt = \int_0^{\pi/4} 2\sqrt{2} \cdot \sqrt{\sec^2 u} \sec^2 u \, du = \int_0^{\pi/4} 2\sqrt{2} \sec^3 u \, du$$

Using the secant reduction formula with $n = 3$,

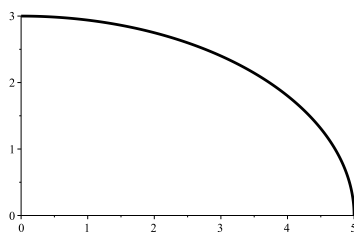
$$\begin{aligned} \int_0^{\pi/4} 2\sqrt{2} \sec^3 u \, du &= 2\sqrt{2} \left[\frac{\sec u \tan u}{2} + \frac{1}{2} \ln |\sec u + \tan u| \right]_0^{\pi/4} \\ &= 2\sqrt{2} \left[\left(\frac{(\sqrt{2})(1)}{2} + \frac{\ln(\sqrt{2}+1)}{2} \right) - \left(0 + \frac{1}{2} \ln 1 \right) \right] \\ &= 2\sqrt{2} \left(\frac{(\sqrt{2})(1)}{2} + \frac{\ln(\sqrt{2}+1)}{2} \right) = \boxed{2 + \sqrt{2} \ln(\sqrt{2}+1)}. \end{aligned}$$

55. (a) The curve traced out by the vector function $\mathbf{r}(t)$ is below.



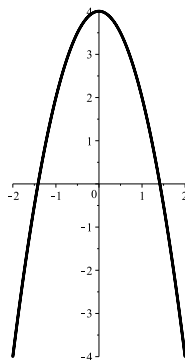
- (b) We find $\mathbf{r}'(t) = 3 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$ so $\|\mathbf{r}'(t)\| = \sqrt{9 \cos^2 t + 16 \sin^2 t}$ and $s = \int_0^\pi \sqrt{9 \cos^2 t + 16 \sin^2 t} \, dt$.
Using Maple™, $s \approx \boxed{11.052}$.

56. (a) The curve traced out by the vector function $\mathbf{r}(t)$ is below.



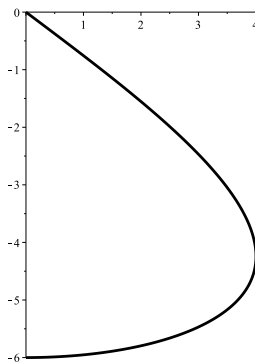
- (b) We find $\mathbf{r}'(t) = -5 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$ so $\|\mathbf{r}'(t)\| = \sqrt{25 \sin^2 t + 9 \cos^2 t}$ and $s = \int_0^{\pi/2} \sqrt{25 \sin^2 t + 9 \cos^2 t} dt$.
Using MapleTM, $s \approx \boxed{6.382}$.

57. (a) The curve traced out by the vector function $\mathbf{r}(t)$ is below.



- (b) We find $\mathbf{r}'(t) = 2 \cos t \mathbf{i} - 8 \sin(2t) \mathbf{j}$ so $\|\mathbf{r}'(t)\| = \sqrt{4 \cos^2 t + 64 \sin^2(2t)}$ and $s = \int_0^{2\pi} \sqrt{4 \cos^2 t + 64 \sin^2(2t)} dt$.
Using MapleTM, $s \approx \boxed{33.637}$.

58. (a) The curve traced out by the vector function $\mathbf{r}(t)$ is below.



- (b) We find $\mathbf{r}'(t) = 8 \cos(2t) \mathbf{i} + 6 \sin t \mathbf{j}$ so $\|\mathbf{r}'(t)\| = \sqrt{64 \cos^2(2t) + 36 \sin^2 t}$ and $s = \int_0^{\pi/2} \sqrt{64 \cos^2(2t) + 36 \sin^2 t} dt$.
Using MapleTM, $s \approx \boxed{10.716}$.

Applications and Extensions

59. The vector function and its tangent vector will be orthogonal when their dot product is zero. Since $\mathbf{r}(t) = t^2 \mathbf{i} + (t^2 - 1) \mathbf{j} - t \mathbf{k}$, $\mathbf{r}'(t) = 2t \mathbf{i} + 2t \mathbf{j} - \mathbf{k}$.

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = (t^2)(2t) + (t^2 - 1)(2t) + (-t)(-1) = 4t^3 - t = t(4t^2 - 1)$$

Then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal when $t(4t^2 - 1) = 0$, or $\{t|t = -\frac{1}{2}, 0, \frac{1}{2}\}$. The points at which $\mathbf{r}(t)$ and its tangent vector are horizontal are $\boxed{\mathbf{r}(0) = -\mathbf{j}, \mathbf{r}\left(\frac{1}{2}\right) = \frac{1}{4}\mathbf{i} - \frac{3}{4}\mathbf{j} - \frac{1}{2}\mathbf{k}, \mathbf{r}\left(-\frac{1}{2}\right) = \frac{1}{4}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}}$.

60. A tangent vector at any point on the helix is given by $\mathbf{r}'(t) = -\sin t \mathbf{i} + \mathbf{j} + \cos t \mathbf{k}$, and the direction of the y -axis can be represented by the vector $c\mathbf{j}$ for $c \neq 0$. Then $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \sqrt{2}$ and $\|c\mathbf{j}\| = c$. The cosine of the angle θ between $\mathbf{r}'(t)$ and the y -axis is

$$\cos \theta = \frac{\mathbf{r}'(t) \cdot c\mathbf{j}}{\|\mathbf{r}'(t)\| \|c\mathbf{j}\|} = \frac{c}{(\sqrt{2})(c)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\text{so } \theta = \boxed{\frac{\pi}{4}}.$$

61. A tangent vector at any point on the helix is given by $\mathbf{r}'(t) = 3 \cos t \mathbf{i} - 3 \sin t \mathbf{j} + 4\mathbf{k}$, so $\|\mathbf{r}'(t)\| = \sqrt{9 \cos^2 t + 9 \sin^2 t + 16} = 5$. The cosine of the angle θ between $\mathbf{r}'(t)$ and $\mathbf{k}$ is

$$\cos \theta = \frac{\mathbf{r}'(t) \cdot \mathbf{k}}{\|\mathbf{r}'(t)\| \|\mathbf{k}\|} = \frac{4}{5},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{4}{5}\right) \approx \boxed{37^\circ}.$$

62. A tangent vector at any point on the helix is given by $\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$, so $\|\mathbf{r}'(t)\| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}$. The cosine of the angle θ between $\mathbf{r}'(t)$ and $\mathbf{k}$ is

$$\cos \theta = \frac{\mathbf{r}'(t) \cdot \mathbf{k}}{\|\mathbf{r}'(t)\| \|\mathbf{k}\|} = \frac{1}{\sqrt{5}},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx \boxed{63^\circ}.$$

63. For $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2) = (1, 0, 1)$, $t_1 = 1$ and $t_2 = 0$. We begin by finding $\mathbf{r}'_1(t_1)$, $\|\mathbf{r}'_1(t_1)\|$, $\mathbf{r}'_2(t_2)$, and $\|\mathbf{r}'_2(t_2)\|$. Since $\mathbf{r}'_1(t_1) = \mathbf{i} + \pi \cos(\pi t_1)\mathbf{j}$ and $\mathbf{r}'_2(t_2) = \mathbf{j} + \mathbf{k}$,

$$\begin{aligned} \mathbf{r}'_1(1) &= \mathbf{i} - \pi\mathbf{j} & \text{and} & \quad \|\mathbf{r}'_1(1)\| = \sqrt{1 + \pi^2} \\ \mathbf{r}'_2(0) &= \mathbf{j} + \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_2(0)\| = \sqrt{2} \end{aligned}$$

The cosine of the angle θ between $\mathbf{r}'_1(1)$ and $\mathbf{r}'_2(0)$ is

$$\cos \theta = \frac{\mathbf{r}'_1(1) \cdot \mathbf{r}'_2(0)}{\|\mathbf{r}'_1(1)\| \|\mathbf{r}'_2(0)\|} = \frac{-\pi}{\sqrt{2 + 2\pi^2}},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{-\pi}{\sqrt{2 + 2\pi^2}}\right) \approx 2.310 \text{ rad} \approx \boxed{132.361^\circ}.$$

64. For $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2) = (0, 0, 0)$, $t_1 = 0$ and $t_2 = 0$. We begin by finding $\mathbf{r}'_1(t_1)$, $\|\mathbf{r}'_1(t_1)\|$, $\mathbf{r}'_2(t_2)$, and $\|\mathbf{r}'_2(t_2)\|$. Since $\mathbf{r}'_1(t_1) = 2 \cos(2t_1)\mathbf{i} + \cos t_1\mathbf{j} + \mathbf{k}$ and $\mathbf{r}'_2(t_2) = 2t_2\mathbf{i} + \mathbf{j} + 3(t_2)^2\mathbf{k}$,

$$\begin{aligned} \mathbf{r}'_1(0) &= 2\mathbf{i} + \mathbf{j} + \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_1(0)\| = \sqrt{4 + 1 + 1} = \sqrt{6} \\ \mathbf{r}'_2(0) &= \mathbf{j} & \text{and} & \quad \|\mathbf{r}'_2(0)\| = \sqrt{1} = 1. \end{aligned}$$

The cosine of the angle θ between $\mathbf{r}'_1(0)$ and $\mathbf{r}'_2(0)$ is

$$\cos \theta = \frac{\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(0)}{\|\mathbf{r}'_1(0)\| \|\mathbf{r}'_2(0)\|} = \frac{1}{\sqrt{6}},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{\sqrt{6}}{6}\right) \approx \boxed{65.905^\circ}.$$

65. For $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2) = (0, -1, 0)$, $t_1 = 0$ and $t_2 = 1$. We begin by finding $\mathbf{r}'_1(t_1)$, $\|\mathbf{r}'_1(t_1)\|$, $\mathbf{r}'_2(t_2)$, and $\|\mathbf{r}'_2(t_2)\|$. Since $\mathbf{r}'_1(t_1) = e^{t_1}\mathbf{i} + \pi \sin(\pi t_1)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}'_2(t_2) = -\mathbf{i} + \mathbf{k}$,

$$\begin{aligned}\mathbf{r}'_1(0) &= \mathbf{i} + \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_1(0)\| = \sqrt{2} \\ \mathbf{r}'_2(1) &= -\mathbf{i} + \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_2(1)\| = \sqrt{2}.\end{aligned}$$

The cosine of the angle θ between $\mathbf{r}'_1(0)$ and $\mathbf{r}'_2(1)$ is

$$\cos \theta = \frac{\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(1)}{\|\mathbf{r}'_1(0)\| \|\mathbf{r}'_2(1)\|} = 0,$$

$$\text{so } \theta = \cos^{-1}(0) = \boxed{90^\circ}.$$

66. For $\mathbf{r}_1(t_1) = \mathbf{r}_2(t_2) = (4, 1, 1)$, $t_1 = 2$ and $t_2 = 1$. We begin by finding $\mathbf{r}'_1(t_1)$, $\|\mathbf{r}'_1(t_1)\|$, $\mathbf{r}'_2(t_2)$, and $\|\mathbf{r}'_2(t_2)\|$. Since $\mathbf{r}'_1(t_1) = 2t_1\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{r}'_2(t_2) = 2t_2\mathbf{i} - \mathbf{j} + \mathbf{k}$,

$$\begin{aligned}\mathbf{r}'_1(2) &= 4\mathbf{i} - \mathbf{j} - \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_1(2)\| = \sqrt{16 + 1 + 1} = \sqrt{18} = 3\sqrt{2} \\ \mathbf{r}'_2(1) &= 2\mathbf{i} - \mathbf{j} + \mathbf{k} & \text{and} & \quad \|\mathbf{r}'_2(1)\| = \sqrt{4 + 1 + 1} = \sqrt{6}.\end{aligned}$$

The cosine of the angle θ between $\mathbf{r}'_1(2)$ and $\mathbf{r}'_2(1)$ is

$$\cos \theta = \frac{\mathbf{r}'_1(2) \cdot \mathbf{r}'_2(1)}{\|\mathbf{r}'_1(2)\| \|\mathbf{r}'_2(1)\|} = \frac{8}{(3\sqrt{2})(\sqrt{6})} = \frac{8}{6\sqrt{3}} = \frac{4\sqrt{3}}{9},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{4\sqrt{3}}{9}\right) \approx \boxed{39.664^\circ}.$$

67. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2t\mathbf{i} - \mathbf{j} - \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1 + 1} = \sqrt{4t^2 + 2}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{2t\mathbf{i} - \mathbf{j} - \mathbf{k}}{\sqrt{4t^2 + 2}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{2i\sqrt{4t^2 + 2} - (2t\mathbf{i} - \mathbf{j} - \mathbf{k})\left(\frac{1}{2}\right)(4t^2 + 2)^{-1/2}(8t)}{4t^2 + 2}.$$

We can multiply the numerator and the denominator by $\sqrt{4t^2 + 2}$ to simplify.

$$\begin{aligned}\mathbf{T}'(t) &= \frac{(2i)(4t^2 + 2) - (4t)(2t\mathbf{i} - \mathbf{j} - \mathbf{k})}{(4t^2 + 2)^{3/2}} = \frac{8t^2\mathbf{i} + 4\mathbf{i} - 8t^2\mathbf{i} + 4t\mathbf{j} + 4t\mathbf{k}}{(4t^2 + 2)^{3/2}} \\ &= \frac{4\mathbf{i} + 4t\mathbf{j} + 4t\mathbf{k}}{(4t^2 + 2)^{3/2}}\end{aligned}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{16 + 16t^2 + 16t^2}{(4t^2 + 2)^3}} = \sqrt{\frac{8(2 + 4t^2)}{(4t^2 + 2)^3}} = \frac{2\sqrt{2}}{4t^2 + 2}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{4\mathbf{i} + 4t\mathbf{j} + 4t\mathbf{k}}{(4t^2 + 2)^{3/2}}}{\frac{2\sqrt{2}}{4t^2 + 2}} = \frac{\frac{2}{\sqrt{2}}\mathbf{i} + \frac{2}{\sqrt{2}}t\mathbf{j} + \frac{2}{\sqrt{2}}t\mathbf{k}}{\sqrt{4t^2 + 2}} = \frac{\sqrt{2}(\mathbf{i} + t\mathbf{j} + t\mathbf{k})}{\sqrt{4t^2 + 2}} = \boxed{\frac{\mathbf{i} + t\mathbf{j} + t\mathbf{k}}{\sqrt{2t^2 + 1}}}$$

The binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, so

$$\begin{aligned}\mathbf{B}(t) &= \left[\frac{2t\mathbf{i} - \mathbf{j} - \mathbf{k}}{\sqrt{4t^2 + 2}} \right] \times \left[\frac{\sqrt{2}(\mathbf{i} + t\mathbf{j} + t\mathbf{k})}{\sqrt{4t^2 + 2}} \right] = \frac{\sqrt{2}}{4t^2 + 2} ([2t\mathbf{i} - \mathbf{j} - \mathbf{k}] \times [\mathbf{i} + t\mathbf{j} + t\mathbf{k}]) \\ &= \frac{\sqrt{2}}{4t^2 + 2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & -1 & -1 \\ 1 & t & t \end{vmatrix} = \frac{\sqrt{2}}{2(2t^2 + 1)} [(-t + t)\mathbf{i} - (2t^2 + 1)\mathbf{j} + (2t^2 + 1)\mathbf{k}] \\ &= \boxed{-\frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}}.\end{aligned}$$

68. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 2t\mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 1 + 4t^2} = \sqrt{2 + 4t^2}.$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \boxed{\frac{\mathbf{i} - \mathbf{j} + 2t\mathbf{k}}{\sqrt{4t^2 + 2}}}.$$

Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{2\mathbf{k}\sqrt{4t^2 + 2} - (\mathbf{i} - \mathbf{j} + 2t\mathbf{k})\left(\frac{1}{2}\right)(4t^2 + 2)^{-1/2}(8t)}{4t^2 + 2}.$$

We can multiply the numerator and the denominator by $\sqrt{4t^2 + 2}$ to simplify.

$$\mathbf{T}'(t) = \frac{(2\mathbf{k})(4t^2 + 2) - (4t)(\mathbf{i} - \mathbf{j} + 2t\mathbf{k})}{(4t^2 + 2)^{3/2}} = \frac{-4t\mathbf{i} + 4t\mathbf{j} + 4\mathbf{k}}{(4t^2 + 2)^{3/2}}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{16t^2 + 16t^2 + 16}{(4t^2 + 2)^3}} = \sqrt{\frac{8(4t^2 + 2)}{(4t^2 + 2)^3}} = \frac{2\sqrt{2}}{4t^2 + 2}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{-4t\mathbf{i} + 4t\mathbf{j} + 4\mathbf{k}}{(4t^2 + 2)^{3/2}}}{\frac{2\sqrt{2}}{4t^2 + 2}} = \frac{-\frac{2}{\sqrt{2}}t\mathbf{i} + \frac{2}{\sqrt{2}}t\mathbf{j} + \frac{2}{\sqrt{2}}\mathbf{k}}{\sqrt{4t^2 + 2}} = \boxed{\frac{\sqrt{2}(-t\mathbf{i} + t\mathbf{j} + \mathbf{k})}{\sqrt{4t^2 + 2}}}$$

The binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, so

$$\begin{aligned}\mathbf{B}(t) &= \left[\frac{\mathbf{i} - \mathbf{j} + 2t\mathbf{k}}{\sqrt{4t^2 + 2}} \right] \times \left[\frac{\sqrt{2}(-t\mathbf{i} + t\mathbf{j} + \mathbf{k})}{\sqrt{4t^2 + 2}} \right] = \frac{\sqrt{2}}{4t^2 + 2} ([\mathbf{i} - \mathbf{j} + 2t\mathbf{k}] \times [-t\mathbf{i} + t\mathbf{j} + \mathbf{k}]) \\ &= \frac{\sqrt{2}}{4t^2 + 2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2t \\ -t & t & 1 \end{vmatrix} = \frac{\sqrt{2}}{2(2t^2 + 1)} [(-1 - 2t^2)\mathbf{i} - (1 + 2t^2)\mathbf{j} + (t - t)\mathbf{k}] \\ &= \boxed{-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.\end{aligned}$$

69. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}(-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k})}.$$

Then

$$\mathbf{T}'(t) = \frac{-\cos t \mathbf{i} - \sin t \mathbf{j}}{\sqrt{2}}$$

and

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{\cos^2 t + \sin^2 t}{2}} = \frac{1}{\sqrt{2}}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{-\cos t \mathbf{i} - \sin t \mathbf{j}}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \boxed{-\cos t \mathbf{i} - \sin t \mathbf{j}}$$

The binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, so

$$\begin{aligned} \mathbf{B}(t) &= \left[\frac{-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}}{\sqrt{2}} \right] \times [-\cos t \mathbf{i} - \sin t \mathbf{j}] = \frac{1}{\sqrt{2}}([-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}] \times [-\cos t \mathbf{i} - \sin t \mathbf{j}]) \\ &= \frac{1}{\sqrt{2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{\sqrt{2}}{2}[\sin t \mathbf{i} - \cos t \mathbf{j} + (\sin^2 t + \cos^2 t)\mathbf{k}] \\ &= \boxed{\frac{\sqrt{2}}{2}(\sin t \mathbf{i} - \cos t \mathbf{j} + \mathbf{k})}. \end{aligned}$$

70. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} + 3 \mathbf{k} \quad \text{so} \quad \|\mathbf{r}'(t)\| = \sqrt{4 \cos^2 t + 4 \sin^2 t + 9} = \sqrt{13}.$$

Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \boxed{\frac{2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} + 3 \mathbf{k}}{\sqrt{13}}}.$$

Then

$$\mathbf{T}'(t) = \frac{-2 \sin t \mathbf{i} - 2 \cos t \mathbf{j}}{\sqrt{13}}$$

and

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{4 \sin^2 t + 4 \cos^2 t}{13}} = \frac{2}{\sqrt{13}}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{-2 \sin t \mathbf{i} - 2 \cos t \mathbf{j}}{\sqrt{13}}}{\frac{2}{\sqrt{13}}} = \boxed{-\sin t \mathbf{i} - \cos t \mathbf{j}}$$

The binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, so

$$\begin{aligned}\mathbf{B}(t) &= \left[\frac{2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} + 3 \mathbf{k}}{\sqrt{13}} \right] \times [-\sin t \mathbf{i} - \cos t \mathbf{j}] = -\frac{1}{\sqrt{13}} ([2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} + 3 \mathbf{k}] \times [\sin t \mathbf{i} + \cos t \mathbf{j}]) \\ &= -\frac{1}{\sqrt{13}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos t & -2 \sin t & 3 \\ \sin t & \cos t & 0 \end{vmatrix} = -\frac{1}{\sqrt{13}} [-3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + (2 \cos^2 t + 2 \sin^2 t) \mathbf{k}] \\ &= \left[\frac{3\sqrt{13}}{13} \cos t \mathbf{i} - \frac{3\sqrt{13}}{13} \sin t \mathbf{j} - \frac{2\sqrt{13}}{13} \mathbf{k} \right].\end{aligned}$$

71. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. The tangent vector at t_0 is $\mathbf{r}'(t_0) = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$. Recall that a nonzero vector $\mathbf{r}'(t_0)$ in space can be described with its three direction angles and its magnitude, $\|\mathbf{r}'(t_0)\|$. The direction angles are defined as

$$\begin{aligned}\alpha &= \text{the angle between } \mathbf{r}'(t_0) \text{ and } \mathbf{i}, 0 \leq \alpha \leq \pi, \\ \beta &= \text{the angle between } \mathbf{r}'(t_0) \text{ and } \mathbf{j}, 0 \leq \beta \leq \pi, \\ \gamma &= \text{the angle between } \mathbf{r}'(t_0) \text{ and } \mathbf{k}, 0 \leq \gamma \leq \pi.\end{aligned}$$

From the Theorem about the Angle Between Vectors,

$$\cos \alpha = \frac{\mathbf{r}'(t_0) \cdot \mathbf{i}}{\|\mathbf{r}'(t_0)\| \|\mathbf{i}\|}, \quad \cos \beta = \frac{\mathbf{r}'(t_0) \cdot \mathbf{j}}{\|\mathbf{r}'(t_0)\| \|\mathbf{j}\|}, \quad \text{and} \quad \cos \gamma = \frac{\mathbf{r}'(t_0) \cdot \mathbf{k}}{\|\mathbf{r}'(t_0)\| \|\mathbf{k}\|},$$

so the direction cosines are
$$\cos \alpha = \frac{x'(t_0)}{\|\mathbf{r}'(t_0)\|}, \cos \beta = \frac{y'(t_0)}{\|\mathbf{r}'(t_0)\|}, \text{ and } \cos \gamma = \frac{z'(t_0)}{\|\mathbf{r}'(t_0)\|}.$$

72. The tangent vector to the helix is $\mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}$, and the direction of the z -axis can be represented by the vector $c \mathbf{k}$ for $c \neq 0$. Then $\|\mathbf{r}'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$ and $\|c \mathbf{k}\| = c$. The cosine of the angle θ between $\mathbf{r}'(t)$ and the z -axis is

$$\cos \theta = \frac{\mathbf{r}'(t) \cdot c \mathbf{k}}{\|\mathbf{r}'(t)\| \|c \mathbf{k}\|} = \frac{(b)(c)}{\sqrt{a^2 + b^2}(c)} = \frac{b}{\sqrt{a^2 + b^2}},$$

so $\theta = \cos^{-1} \left(\frac{b}{\sqrt{a^2 + b^2}} \right)$. Since a and b are real numbers, θ is a constant.

Challenge Problems

73. We begin by finding $\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j} + 2 \mathbf{k}$ and $\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 4t^2 + 4}$. We wish to approximate $\int_0^2 \|\mathbf{r}'(t)\| dt = \int_0^2 \sqrt{9t^4 + 4t^2 + 4}$ using $n = 4$, $\Delta t = \frac{b-a}{n} = \frac{1}{2}$. Then using Simpson's Rule,

$$\begin{aligned}\int_0^2 \|\mathbf{r}'(t)\| dt &\approx \frac{2-0}{3 \cdot 4} \left[\|\mathbf{r}'(0)\| + 4 \left\| \mathbf{r}'\left(\frac{1}{2}\right) \right\| + 2 \|\mathbf{r}'(1)\| + 4 \left\| \mathbf{r}'\left(\frac{3}{2}\right) \right\| + \|\mathbf{r}'(2)\| \right] \\ &= \frac{1}{6} \left[2 + 4 \left(\frac{\sqrt{89}}{4} \right) + 2(\sqrt{17}) + 4 \left(\frac{\sqrt{937}}{4} \right) + 2\sqrt{41} \right] \approx \boxed{10.516}\end{aligned}$$

74. We begin by finding $\mathbf{r}'(t) = 3 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$ and $\|\mathbf{r}'(t)\| = \sqrt{9 \cos^2 t + 16 \sin^2 t}$. We wish to approximate $\int_0^\pi \|\mathbf{r}'(t)\| dt = \int_0^\pi \sqrt{9 \cos^2 t + 16 \sin^2 t} dt$.

(a) Using $n = 4$, $\Delta t = \frac{b-a}{n} = \frac{\pi}{4}$. Then Simpson's Rule gives

$$\begin{aligned}\int_0^\pi \|\mathbf{r}'(t)\| dt &\approx \frac{\pi-0}{3 \cdot 4} \left[\|\mathbf{r}'(0)\| + 4 \left\| \mathbf{r}'\left(\frac{\pi}{4}\right) \right\| + 2 \left\| \mathbf{r}'\left(\frac{\pi}{2}\right) \right\| + 4 \left\| \mathbf{r}'\left(\frac{3\pi}{4}\right) \right\| + \|\mathbf{r}'(\pi)\| \right] \\ &= \frac{\pi}{12} \left[3 + (4) \left(\frac{5\sqrt{2}}{2} \right) + (2)(4) + (4) \left(\frac{5\sqrt{2}}{2} \right) + 3 \right] \approx \boxed{11.070}\end{aligned}$$

(b) Using $n = 6$, $\Delta t = \frac{b-a}{n} = \frac{\pi}{6}$. Then the Trapezoidal Rule gives

$$\begin{aligned} \int_0^\pi \|\mathbf{r}'(t)\| dt &\approx \frac{\pi-0}{2 \cdot 6} \left[\|\mathbf{r}'(0)\| + 2 \left\| \mathbf{r}'\left(\frac{\pi}{6}\right) \right\| + 2 \left\| \mathbf{r}'\left(\frac{\pi}{3}\right) \right\| + \dots + 2 \left\| \mathbf{r}'\left(\frac{5\pi}{6}\right) \right\| + \|\mathbf{r}'(\pi)\| \right] \\ &= \frac{\pi}{12} \left[3 + (2) \left(\frac{\sqrt{43}}{2} \right) + (2) \left(\frac{\sqrt{57}}{2} \right) + (2)(4) + (2) \left(\frac{\sqrt{57}}{2} \right) + (2) \left(\frac{\sqrt{43}}{2} \right) + 3 \right] \\ &\approx \boxed{11.052} \end{aligned}$$

(c) Both answers are close to the result given by technology of 11.052. The Trapezoidal Rule with $n = 6$ gives a better estimate than Simpson's Rule with $n = 4$.

75. We begin by finding $\mathbf{r}'(t) = 2 \cos t \mathbf{i} - 8 \sin(2t) \mathbf{j}$ and $\|\mathbf{r}'(t)\| = \sqrt{4 \cos^2 t + 64 \sin^2(2t)} = 2\sqrt{\cos^2 t + 16 \sin^2(2t)}$.

We wish to approximate $\int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} 2\sqrt{\cos^2 t + 16 \sin^2(2t)} dt$.

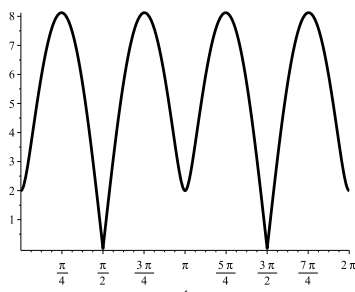
(a) Using $n = 4$, $\Delta t = \frac{b-a}{n} = \frac{\pi}{2}$. Then Simpson's Rule gives

$$\begin{aligned} \int_0^{2\pi} \|\mathbf{r}'(t)\| dt &\approx \frac{2\pi-0}{3 \cdot 4} \left[\|\mathbf{r}'(0)\| + 4 \left\| \mathbf{r}'\left(\frac{\pi}{2}\right) \right\| + 2 \|\mathbf{r}'(\pi)\| + 4 \left\| \mathbf{r}'\left(\frac{3\pi}{2}\right) \right\| + \|\mathbf{r}'(2\pi)\| \right] \\ &= \frac{\pi}{6} [2 + (4)(0) + (2)(2) + (4)(0) + 2] = \frac{4\pi}{3} \approx \boxed{4.189} \end{aligned}$$

(b) Using $n = 6$, $\Delta t = \frac{b-a}{n} = \frac{\pi}{6}$. Then the Trapezoidal Rule gives

$$\begin{aligned} \int_0^\pi \|\mathbf{r}'(t)\| dt &\approx \frac{2\pi-0}{2 \cdot 6} \left[\|\mathbf{r}'(0)\| + 2 \left\| \mathbf{r}'\left(\frac{\pi}{3}\right) \right\| + 2 \left\| \mathbf{r}'\left(\frac{2\pi}{3}\right) \right\| + \dots + 2 \left\| \mathbf{r}'\left(\frac{5\pi}{3}\right) \right\| + \|\mathbf{r}'(2\pi)\| \right] \\ &= \frac{\pi}{6} [2 + (2)(7) + (2)(7) + (2)(2) + (2)(7) + (2)(7) + 2] \\ &= \frac{32\pi}{3} \approx \boxed{33.510} \end{aligned}$$

(c) The result from the Trapezoidal Rule with $n = 6$ is close to the result given by technology of 33.637 while Simpson's Rule with $n = 4$ gives a bad approximation. Looking at the graph of $\|\mathbf{r}'(t)\|$, we see that at two of the points used in the Simpson's Rule approximation, $\|\mathbf{r}'(t)\| = 0$, so the parabolic arcs used to approximate $\|\mathbf{r}'(t)\|$ give rise to an underestimate of the integral.



The graph of
 $\|\mathbf{r}'(t)\| = 2\sqrt{\cos^2 t + 16 \sin^2(2t)}$.

76. We begin by finding $\mathbf{r}'(t) = t^2\mathbf{i} + \mathbf{j}$ and $\|\mathbf{r}'(t)\| = \sqrt{t^4 + 1}$. We wish to approximate $s = \int_0^{1/2} \|\mathbf{r}'(t)\| dt = \int_0^{1/2} \sqrt{t^4 + 1} dt$ by expanding the integrand, $\sqrt{t^4 + 1}$, in a binomial series with $x = t^4$ and $m = \frac{1}{2}$, which gives

$$(1 + t^4)^{1/2} = 1 + \frac{1}{2}t^4 - \frac{1}{8}t^8 + \cdots$$

Using the integration property of power series,

$$\begin{aligned} \int_0^{1/2} \sqrt{t^4 + 1} dt &= \int_0^{1/2} \left(1 + \frac{1}{2}t^4 - \frac{1}{8}t^8 + \cdots \right) dx \\ &= \left[t + \frac{1}{2(5)}t^5 - \frac{1}{8(9)}t^9 + \cdots \right]_0^{1/2} \\ &= \frac{1}{2} + \frac{1}{10(2)^5} - \frac{1}{72(2)^9} + \cdots \\ &\approx 0.5 + 0.003125 - 0.0000271 + \cdots \end{aligned}$$

Since this series satisfies the two conditions of the Alternating Series Test, the error due to the first two terms as an approximation is less than the third term, $\frac{1}{72(2)^9} \approx 2.7127 \times 10^{-5}$, so

$$\int_0^{1/2} \sqrt{t^4 + 1} dt \approx \frac{1}{2} + \frac{1}{10(2)^5} \approx \boxed{0.5031}.$$

11.3 Arc Length as Parameter; Curvature

Concepts and Vocabulary

1. d by the Theorem on Criterion for Arc Length as Parameter
2. False. The curvature of a line is 0.
3. False. The curvature of a circle of radius R is $\frac{1}{R}$, the reciprocal of its radius.
4. True. $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$
5. False. $\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$
6. If $y = f(x)$ is a twice differentiable function, $\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$.
7. Since the equation $x^2 + (y - 2)^2 = 9$ represents a circle of radius 3 with center at $(0, 2)$, the curvature of the circle is the reciprocal of its radius, or $\kappa = \frac{1}{3}$.
8. False. The radius ρ of the osculating circle is $\frac{1}{\kappa}$, provided $\kappa \neq 0$.

Skill Building

9. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} - 4 \cos t \mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4 \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

10. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 3 \cos(3t) \mathbf{i} - 3 \sin(3t) \mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{9 \cos^2(3t) + 9 \sin^2(3t)} = 3 \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

11. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2t \mathbf{i} + \mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

12. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{i} + 3t^2 \mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + 9t^4} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

13. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{4 + 9} = \sqrt{13} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

14. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{1} = 1 \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| = 1$ for all t , yes, the parameter t is arc length as measured along C .

15. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{\frac{4}{13} + \frac{9}{13}} = 1 \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| = 1$ for all t , yes, the parameter t is arc length as measured along C .

16. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2t\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2} = 2t \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

17. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

18. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = a \cos t \mathbf{i} - a \sin t \mathbf{j} + \sqrt{1 - a^2} \mathbf{k} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t + 1 - a^2} = 1 \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| = 1$ for all t , yes, the parameter t is arc length as measured along C .

19. Curvature is a measure of how sharply a curve bends. The curve bends the most sharply at P and the least sharply at Q , so the rank is $P R Q$.

20. Curvature is a measure of how sharply a curve bends. The curve bends the most sharply at Q and the least sharply at R , so the rank is $Q P R$.

21. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= 2t\mathbf{i} - \frac{2}{t^2}\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{4t^2 + \frac{4}{t^4}} = 2\sqrt{\frac{t^6 + 1}{t^4}} = \frac{2\sqrt{t^6 + 1}}{t^2} \\ \mathbf{T}(t) &= \frac{2t\mathbf{i} - \frac{2}{t^2}\mathbf{j}}{\frac{2\sqrt{t^6+1}}{t^2}} = \frac{t^3}{\sqrt{t^6 + 1}}\mathbf{i} - \frac{1}{\sqrt{t^6 + 1}}\mathbf{j} \\ \mathbf{T}'(t) &= \frac{3t^2\sqrt{t^6 + 1} - t^3(\frac{1}{2})(t^6 + 1)^{-1/2}(6t^5)}{t^6 + 1}\mathbf{i} + \frac{6t^5}{2(t^6 + 1)^{3/2}}\mathbf{j} \end{aligned}$$

We simplify $\mathbf{T}(t)$ by multiplying the numerator and the denominator of the $\mathbf{i}$ -component by $\sqrt{t^6 + 1}$ to get

$$\begin{aligned} \mathbf{T}'(t) &= \frac{3t^2(t^6 + 1) - 3t^8}{(t^6 + 1)^{3/2}}\mathbf{i} + \frac{3t^5}{(t^6 + 1)^{3/2}}\mathbf{j} = \frac{3t^8 + 3t^2 - 3t^8}{(t^6 + 1)^{3/2}}\mathbf{i} + \frac{3t^5}{(t^6 + 1)^{3/2}}\mathbf{j} \\ &= \frac{3t^2}{(t^6 + 1)^{3/2}}\mathbf{i} + \frac{3t^5}{(t^6 + 1)^{3/2}}\mathbf{j}, \end{aligned}$$

so

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{9t^4}{(t^6 + 1)^3} + \frac{9t^{10}}{(t^6 + 1)^3}} = 3t^2 \sqrt{\frac{1 + t^6}{(t^6 + 1)^3}} = \frac{3t^2}{t^6 + 1}.$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{3t^2}{t^6+1}}{\frac{2\sqrt{t^6+1}}{t^2}} = \frac{3t^4}{2(t^6 + 1)^{3/2}}.$$

22. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= 2\mathbf{i} + 3t^2\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{4 + 9t^4} \\ \mathbf{T}(t) &= \frac{2\mathbf{i} + 3t^2\mathbf{j}}{\sqrt{4 + 9t^4}} \\ \mathbf{T}'(t) &= \frac{-36t^3}{(4 + 9t^4)^{3/2}}\mathbf{i} + \frac{6t\sqrt{4 + 9t^4} - 3t^2(\frac{1}{2})(4 + 9t^4)^{-1/2}(36t^3)}{(4 + 9t^4)^{3/2}}\mathbf{j}\end{aligned}$$

We simplify $\mathbf{T}(t)$ by multiplying the numerator and the denominator of the $\mathbf{j}$ -component by $\sqrt{4 + 9t^4}$ to get

$$\mathbf{T}(t) = \frac{-36t^3}{(4 + 9t^4)^{3/2}}\mathbf{i} + \frac{6t(4 + 9t^4) - 54t^5}{(4 + 9t^4)^{3/2}}\mathbf{j} = \frac{-36t^3}{(4 + 9t^4)^{3/2}}\mathbf{i} + \frac{24t}{(4 + 9t^4)^{3/2}}\mathbf{j},$$

so

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{(-36t^3)^2}{(4 + 9t^4)^3} + \frac{(24t)^2}{(4 + 9t^4)^3}} = \sqrt{\frac{(12t)^2[(-3t^2)^2 + 2^2]}{(4 + 9t^4)^3}} = 12t\sqrt{\frac{9t^4 + 4}{(4 + 9t^4)^3}} = \frac{12t}{4 + 9t^4}.$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{12t}{4 + 9t^4}}{\sqrt{4 + 9t^4}} = \boxed{\frac{12t}{(4 + 9t^4)^{3/2}}}.$$

23. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= 2\cos t\mathbf{i} - 2\sin t\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{4\cos^2 t + 4\sin^2 t} = 2 \\ \mathbf{T}(t) &= \cos t\mathbf{i} - \sin t\mathbf{j} \\ \mathbf{T}'(t) &= -\sin t\mathbf{i} - \cos t\mathbf{j} \\ \|\mathbf{T}'(t)\| &= \sqrt{\sin^2 t + \cos^2 t} = 1\end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \boxed{\frac{1}{2}}.$$

24. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -\sin t\mathbf{i} + 2\cos t\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{\sin^2 t + \cos^2 t + 3\cos^2 t} = \sqrt{1 + 3\cos^2 t} \\ \mathbf{T}(t) &= \frac{-\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{1 + 3\cos^2 t}}\end{aligned}$$

We differentiate the components of $\mathbf{T}(t)$ individually. For the $\mathbf{i}$ -component,

$$\frac{d}{dt} \frac{-\sin t}{\sqrt{1 + 3\cos^2 t}} = \frac{-\cos t\sqrt{1 + 3\cos^2 t} + \sin t(\frac{1}{2})(1 + 3\cos^2 t)^{-1/2}(-6\cos t\sin t)}{1 + 3\cos^2 t}.$$

We multiply the numerator and denominator by $\sqrt{1 + 3 \cos^2 t}$ to get

$$\begin{aligned} \frac{d}{dt} \frac{-\sin t}{\sqrt{1 + 3 \cos^2 t}} &= \frac{-\cos t(1 + 3 \cos^2 t) - 3 \sin^2 t \cos t}{(1 + 3 \cos^2 t)^{3/2}} \\ &= \frac{-\cos t - 3 \cos^3 t - 3 \sin^2 t \cos t}{(1 + 3 \cos^2 t)^{3/2}} = \frac{-\cos t - 3 \cos t(\cos^2 t + \sin^2 t)}{(1 + 3 \cos^2 t)^{3/2}} \\ &= \frac{-4 \cos t}{(1 + 3 \cos^2 t)^{3/2}}. \end{aligned}$$

Similarly, for the $\mathbf{j}$ -component,

$$\frac{d}{dt} \frac{2 \cos t}{\sqrt{1 + 3 \cos^2 t}} = \frac{-2 \sin t \sqrt{1 + 3 \cos^2 t} - 2 \cos t(\frac{1}{2})(1 + 3 \cos^2 t)^{-1/2}(-6 \cos t \sin t)}{1 + 3 \cos^2 t}.$$

We multiply the numerator and denominator by $\sqrt{1 + 3 \cos^2 t}$ to get

$$\begin{aligned} \frac{d}{dt} \frac{2 \cos t}{\sqrt{1 + 3 \cos^2 t}} &= \frac{-2 \sin t(1 + 3 \cos^2 t) + 6 \cos^2 t \sin t}{(1 + 3 \cos^2 t)^{3/2}} \\ &= \frac{-2 \sin t - 6 \sin t \cos^2 t + 6 \cos^2 t \sin t}{(1 + 3 \cos^2 t)^{3/2}} = \frac{-2 \sin t}{(1 + 3 \cos^2 t)^{3/2}}, \end{aligned}$$

so combining the components, we have

$$\mathbf{T}'(t) = -\frac{4 \cos t}{(1 + 3 \cos^2 t)^{3/2}} \mathbf{i} - \frac{2 \sin t}{(1 + 3 \cos^2 t)^{3/2}} \mathbf{j}.$$

$$\begin{aligned} \|\mathbf{T}'(t)\| &= \sqrt{\frac{16 \cos^2 t + 4 \sin^2 t}{(1 + 3 \cos^2 t)^3}} = 2 \sqrt{\frac{3 \cos^2 t + \cos^2 t + \sin^2 t}{(1 + 3 \cos^2 t)^3}} \\ &= 2 \sqrt{\frac{3 \cos^2 t + 1}{(1 + 3 \cos^2 t)^3}} = \frac{2}{1 + 3 \cos^2 t}. \end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2}{1 + 3 \cos^2 t}}{\sqrt{1 + 3 \cos^2 t}} = \boxed{\frac{2}{(1 + 3 \cos^2 t)^{3/2}}}.$$

25. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= (3 - 3t^2)\mathbf{i} + 6t\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{9 - 18t^2 + 9t^4 + 36t^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{9(1 + t^2)^2} = 3(1 + t^2) \\ \mathbf{T}(t) &= \frac{(3 - 3t^2)\mathbf{i} + 6t\mathbf{j}}{3(1 + t^2)} = \frac{(1 - t^2)\mathbf{i} + 2t\mathbf{j}}{1 + t^2} \\ \mathbf{T}'(t) &= \frac{-2t(1 + t^2) - (1 - t^2)2t}{(1 + t^2)^2} \mathbf{i} + \frac{2(1 + t^2) - 2t(2t)}{(1 + t^2)^2} \mathbf{j} \\ &= \frac{-2t - 2t^3 - 2t + 2t^3}{(1 + t^2)^2} \mathbf{i} + \frac{2 + 2t^2 - 4t^2}{(1 + t^2)^2} \mathbf{j} = \frac{-4t}{(1 + t^2)^2} \mathbf{i} + \frac{2 - 2t^2}{(1 + t^2)^2} \mathbf{j} \\ \|\mathbf{T}'(t)\| &= \sqrt{\frac{16t^2 + 4 - 8t^2 + 4t^4}{(1 + t^2)^4}} = \sqrt{\frac{4t^4 + 8t^2 + 4}{(1 + t^2)^4}} = 2 \sqrt{\frac{(t^2 + 1)^2}{(1 + t^2)^4}} = \frac{2}{1 + t^2} \end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2}{1 + t^2}}{3(1 + t^2)} = \boxed{\frac{2}{3(1 + t^2)^2}}.$$

26. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= (3 - 3t^2)\mathbf{i} + (3 + 3t^2)\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{9 - 18t^2 + 9t^4 + 9 + 18t^2 + 9t^4} = \sqrt{18 + 18t^4} = 3\sqrt{2 + 2t^4} \\ \mathbf{T}(t) &= \frac{(3 - 3t^2)\mathbf{i} + (3 + 3t^2)\mathbf{j}}{3\sqrt{2 + 2t^4}} = \frac{(1 - t^2)\mathbf{i} + (1 + t^2)\mathbf{j}}{\sqrt{2 + 2t^4}}\end{aligned}$$

We differentiate the components of $\mathbf{T}(t)$ individually. For the $\mathbf{i}$ -component,

$$\frac{d}{dt} \left[\frac{(1 - t^2)\mathbf{i}}{\sqrt{2 + 2t^4}} \right] = \frac{-2t\sqrt{2 + 2t^4} - (1 - t^2)(\frac{1}{2})(2 + 2t^4)^{-1/2}(8t^3)}{2 + 2t^4}$$

We multiply the numerator and denominator by $\sqrt{2 + 2t^4}$ to get

$$\frac{d}{dt} \left[\frac{(1 - t^2)\mathbf{i}}{\sqrt{2 + 2t^4}} \right] = \frac{-2t(2 + 2t^4) - (1 - t^2)4t^3}{(2 + 2t^4)^{3/2}} = \frac{-4t - 4t^5 - 4t^3 + 4t^5}{(2 + 2t^4)^{3/2}} = \frac{-4t(1 + t^2)}{(2 + 2t^4)^{3/2}}.$$

Similarly, for the $\mathbf{j}$ -component,

$$\frac{d}{dt} \left[\frac{(1 + t^2)\mathbf{j}}{\sqrt{2 + 2t^4}} \right] = \frac{2t\sqrt{2 + 2t^4} - (1 + t^2)(\frac{1}{2})(2 + 2t^4)^{-1/2}(8t^3)}{2 + 2t^4}$$

We multiply the numerator and denominator by $\sqrt{2 + 2t^4}$ to get

$$\frac{d}{dt} \left[\frac{(1 + t^2)\mathbf{j}}{\sqrt{2 + 2t^4}} \right] = \frac{2t(2 + 2t^4) - (1 + t^2)4t^3}{(2 + 2t^4)^{3/2}} = \frac{4t + 4t^5 - 4t^3 - 4t^5}{(2 + 2t^4)^{3/2}} = \frac{4t(1 - t^2)}{(2 + 2t^4)^{3/2}},$$

so combining terms gives us

$$\mathbf{T}'(t) = \frac{-4t(1 + t^2)}{(2 + 2t^4)^{3/2}}\mathbf{i} + \frac{4t(1 - t^2)}{(2 + 2t^4)^{3/2}}\mathbf{j}.$$

$$\begin{aligned}\|\mathbf{T}'(t)\| &= \sqrt{\frac{16t^2(1 + t^2)^2 + 16t^2(1 - t^2)^2}{(2 + 2t^4)^3}} = 4t\sqrt{\frac{1 + 2t^2 + t^4 + 1 - 2t^2 + t^4}{(2 + 2t^4)^3}} \\ &= 4t\sqrt{\frac{2 + 2t^4}{(2 + 2t^4)^3}} = \frac{4t}{2 + 2t^4} = \frac{2t}{1 + t^4}\end{aligned}$$

Then since $\|\mathbf{r}'(t)\| = 3\sqrt{2 + 2t^4} = 3(\sqrt{2})\sqrt{1 + t^4}$,

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2t}{1 + t^4}}{3(\sqrt{2})\sqrt{1 + t^4}} = \frac{2t}{3\sqrt{2}(1 + t^4)^{3/2}} = \boxed{\frac{\sqrt{2}t}{3(1 + t^4)^{3/2}}}.$$

27. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{i} + 2\mathbf{j} + \mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{1 + 4 + 1} = \sqrt{6} \\ \mathbf{r}''(t) &= \mathbf{0}\end{aligned}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{r}'(t) \times \mathbf{0} = \mathbf{0}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \boxed{0}$$

as we would expect for a line.

28. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{4 + 1 + 9} = \sqrt{14} \\ \mathbf{r}''(t) &= \mathbf{0}\end{aligned}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{r}'(t) \times \mathbf{0} = \mathbf{0}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \boxed{0}$$

as we would expect for a line.

29. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + \mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 1} = \sqrt{5} \\ \mathbf{r}''(t) &= -4\sin(2t)\mathbf{i} - 4\cos(2t)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos(2t) & -2\sin(2t) & 1 \\ -4\sin(2t) & -4\cos(2t) & 0 \end{vmatrix} \\ &= 4\cos(2t)\mathbf{i} - 4\sin(2t)\mathbf{j} + [-8\cos^2(2t) - 8\sin^2(2t)]\mathbf{k} \\ &= 4\cos(2t)\mathbf{i} - 4\sin(2t)\mathbf{j} - 8\mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{16\cos^2(2t) + 16\sin^2(2t) + 64} = \sqrt{80} = 4\sqrt{5}\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{4\sqrt{5}}{(\sqrt{5})^3} = \boxed{\frac{4}{5}}.$$

30. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= \cos t\mathbf{i} - \sin t\mathbf{j} + b\mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{\cos^2 t + \sin^2 t + b^2} = \sqrt{1 + b^2} \\ \mathbf{r}''(t) &= -\sin t\mathbf{i} - \cos t\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sin t & b \\ -\sin t & -\cos t & 0 \end{vmatrix} = b\cos t\mathbf{i} - b\sin t\mathbf{j} + (-\cos^2 t - \sin^2 t)\mathbf{k} \\ &= b\cos t\mathbf{i} - b\sin t\mathbf{j} - \mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{b^2\cos^2 t + b^2\sin^2 t + 1} = \sqrt{b^2 + 1}\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{b^2 + 1}}{(\sqrt{1 + b^2})^3} = \boxed{\frac{1}{b^2 + 1}}.$$

31. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= e^t\mathbf{i} - e^{-t}\mathbf{j} + \sqrt{2}\mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t} \\ \mathbf{r}''(t) &= e^t\mathbf{i} + e^{-t}\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix} = -\sqrt{2}e^{-t}\mathbf{i} + \sqrt{2}e^t\mathbf{j} + 2\mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2(e^t + e^{-t})^2} = \sqrt{2}(e^t + e^{-t})\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{2}(e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2} = \boxed{\frac{\sqrt{2}e^{2t}}{(e^{2t} + 1)^2}}.$$

32. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= e^t\mathbf{i} + 2e^{2t}\mathbf{j} - e^{-t}\mathbf{k} \\ \|\mathbf{r}'(t)\| &= \sqrt{e^{2t} + 4e^{4t} + e^{-2t}} = \sqrt{\frac{4e^{6t} + e^{4t} + 1}{e^{2t}}} = \frac{\sqrt{4e^{6t} + e^{4t} + 1}}{e^t} \\ \mathbf{r}''(t) &= e^t\mathbf{i} + 4e^{2t}\mathbf{j} + e^{-t}\mathbf{k} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & 2e^{2t} & -e^{-t} \\ e^t & 4e^{2t} & e^{-t} \end{vmatrix} = (2e^t + 4e^t)\mathbf{i} - (1 + 1)\mathbf{j} + (4e^{3t} - 2e^{3t})\mathbf{k} \\ &= 6e^t\mathbf{i} - 2\mathbf{j} + 2e^{3t}\mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{36e^{2t} + 4 + 4e^{6t}} = 2\sqrt{e^{6t} + 9e^{2t} + 1}\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{e^{6t} + 9e^{2t} + 1}}{\left(\frac{\sqrt{4e^{6t} + e^{4t} + 1}}{e^t}\right)^3} = \boxed{\frac{2e^{3t}\sqrt{e^{6t} + 9e^{2t} + 1}}{(4e^{6t} + e^{4t} + 1)^{3/2}}}.$$

33. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -3\cos^2 t \sin t \mathbf{i} + 3\sin^2 t \cos t \mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = \sqrt{9\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} = 3\sin t \cos t \\ \mathbf{r}''(t) &= (6\cos t \sin^2 t - 3\cos^3 t)\mathbf{i} + (6\sin t \cos^2 t - 3\sin^3 t)\mathbf{j} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos^2 t \sin t & 3\sin^2 t \cos t & 0 \\ 6\cos t \sin^2 t - 3\cos^3 t & 6\sin t \cos^2 t - 3\sin^3 t & 0 \end{vmatrix} \\ &= (-18\sin^2 t \cos^4 t + 9\sin^4 t \cos^2 t - 18\sin^4 t \cos^2 t + 9\sin^2 t \cos^4 t)\mathbf{k} \\ &= (-9\sin^2 t \cos^4 t - 9\sin^4 t \cos^2 t)\mathbf{k} \\ &= -9\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)\mathbf{k} = -9\sin^2 t \cos^2 t \mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(-9\sin^2 t \cos^2 t)^2} = 9\sin^2 t \cos^2 t\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{9\sin^2 t \cos^2 t}{(3\sin t \cos t)^3} = \boxed{\frac{1}{|3\sin t \cos t|}}.$$

34. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -12 \cos^2 t \sin t \mathbf{i} + 12 \sin^2 t \cos t \mathbf{k} \\ \|\mathbf{r}'(t)\| &= \sqrt{144 \cos^4 t \sin^2 t + 144 \sin^4 t \cos^2 t} = \sqrt{144 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} = 12 \sin t \cos t \\ \mathbf{r}''(t) &= (24 \cos t \sin^2 t - 12 \cos^3 t) \mathbf{i} + (24 \sin t \cos^2 t - 12 \sin^3 t) \mathbf{k} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -12 \cos^2 t \sin t & 0 & 12 \sin^2 t \cos t \\ 24 \cos t \sin^2 t - 12 \cos^3 t & 0 & 24 \sin t \cos^2 t - 12 \sin^3 t \end{vmatrix} \\ &= 288 \sin^4 t \cos^2 t - 144 \sin^2 t \cos^4 t - (-288 \sin^2 t \cos^4 t + 144 \sin^4 t \cos^2 t) \\ &= 144 \sin^4 t \cos^2 t + 144 \sin^2 t \cos^4 t = 144 \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t) \\ &= 144 \sin^2 t \cos^2 t \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(144 \sin^2 t \cos^2 t)^2} = 144 \sin^2 t \cos^2 t\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{144 \sin^2 t \cos^2 t}{(12 \sin t \cos t)^3} = \boxed{\frac{1}{|12 \sin t \cos t|}}.$$

35. To find the curvature of $f(x)$ at $(1, 1)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$\begin{aligned}f'(x) &= 2x & f'(1) &= 2 \\ f''(x) &= 2 & f''(1) &= 2 & |f''(1)| &= 2 \\ \kappa &= \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{2}{(1 + 2^2)^{3/2}} = \frac{2}{5\sqrt{5}} = \boxed{\frac{2\sqrt{5}}{25}}.\end{aligned}$$

36. To find the curvature of $f(x)$ at $(1, 1)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$\begin{aligned}f'(x) &= 2 - 2x & f'(1) &= 0 \\ f''(x) &= -2 & f''(1) &= -2 & |f''(1)| &= 2 \\ \kappa &= \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{2}{(1 + 0^2)^{3/2}} = \boxed{2}.\end{aligned}$$

37. To find the curvature of $f(x)$ at $(1, 0)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$\begin{aligned}f'(x) &= 2x - 3x^2 & f'(1) &= -1 \\ f''(x) &= 2 - 6x & f''(1) &= -4 & |f''(1)| &= 4 \\ \kappa &= \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{4}{(1 + [-1]^2)^{3/2}} = \frac{4}{2\sqrt{2}} = \boxed{\sqrt{2}}.\end{aligned}$$

38. To find the curvature of $f(x)$ at $(1, 1)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$\begin{aligned}f'(x) &= -\frac{3}{2}x^{-5/2} & f'(1) &= -\frac{3}{2} \\ f''(x) &= \frac{15}{4}x^{-7/2} & f''(1) &= \frac{15}{4} & |f''(1)| &= \frac{15}{4} \\ \kappa &= \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{\frac{15}{4}}{\left(1 + \left[-\frac{3}{2}\right]^2\right)^{3/2}} = \frac{\frac{15}{4}}{\left(\frac{4+9}{4}\right)^{3/2}} = \frac{15}{4} \cdot \frac{8}{13\sqrt{13}} = \frac{30}{13\sqrt{13}} = \boxed{\frac{30\sqrt{13}}{169}}.\end{aligned}$$

39. To find the curvature of $f(x)$ at $(2, \sqrt{2})$, we need to find $f'(x)$, $f'(2)$, $f''(x)$, $f''(2)$, and $|f''(2)|$.

$$f'(x) = \frac{1}{2}x^{-1/2} \quad f'(2) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \quad f''(2) = -\frac{1}{8\sqrt{2}} = -\frac{\sqrt{2}}{16} \quad |f''(2)| = \frac{\sqrt{2}}{16}$$

$$\kappa = \frac{|f''(2)|}{(1 + [f'(2)]^2)^{3/2}} = \frac{\frac{\sqrt{2}}{16}}{\left(1 + \left(\frac{\sqrt{2}}{4}\right)^2\right)^{3/2}} = \frac{\frac{\sqrt{2}}{16}}{\left[\frac{9}{8}\right]^{3/2}} = \frac{\sqrt{2}}{16} \cdot \frac{8\sqrt{8}}{27} = \boxed{\frac{2}{27}}.$$

40. To find the curvature of $f(x)$ at $(1, 1)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$f'(x) = -\frac{1}{2}x^{-3/2} \quad f'(1) = -\frac{1}{2}$$

$$f''(x) = \frac{3}{4}x^{-5/2} \quad f''(1) = \frac{3}{4} \quad |f''(1)| = \frac{3}{4}$$

$$\kappa = \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{\frac{3}{4}}{\left(1 + \left[-\frac{1}{2}\right]^2\right)^{3/2}} = \frac{\frac{3}{4}}{\left(\frac{5}{4}\right)^{3/2}} = \frac{6}{5\sqrt{5}} = \boxed{\frac{6\sqrt{5}}{25}}.$$

41. To find the curvature of $f(x)$ at $(0, 2)$, we need to find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$. We use implicit differentiation.

$$\frac{d}{dx}(4x^2 + 9y^2) = \frac{d}{dx}(36)$$

$$8x + 18y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}$$

At $(0, 2)$, $\frac{dy}{dx} = -\frac{(4)(0)}{(9)(2)} = 0$. To get $f''(x)$, we use implicit differentiation again.

$$\frac{d}{dx}\left(8x + 18y \frac{dy}{dx}\right) = \frac{d}{dx}(0)$$

$$8 + 18\left(\frac{dy}{dx}\right)^2 + 18y \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{4 + 9\left(\frac{dy}{dx}\right)^2}{9y}$$

At $(0, 2)$, $\frac{d^2y}{dx^2} = -\frac{4+9(0^2)}{9(2)} = -\frac{2}{9}$, so $|f''(0)| = \frac{2}{9}$. Then

$$\kappa = \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{\frac{2}{9}}{(1 + [0]^2)^{3/2}} = \boxed{\frac{2}{9}}.$$

42. To find the curvature of $f(x)$ at $(\frac{\pi}{4}, \sqrt{2} - 1)$, we need to find $f'(x)$, $f'(\frac{\pi}{4})$, $f''(x)$, $f''(\frac{\pi}{4})$, and $|f''(\frac{\pi}{4})|$.

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f''(x) = \sec x \tan^2 x + \sec^3 x \quad f''\left(\frac{\pi}{4}\right) = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2} \quad \left|f''\left(\frac{\pi}{4}\right)\right| = 3\sqrt{2}$$

$$\kappa = \frac{|f''(\frac{\pi}{4})|}{(1 + [f'(\frac{\pi}{4})]^2)^{3/2}} = \frac{3\sqrt{2}}{(1 + [\sqrt{2}]^2)^{3/2}} = \boxed{\frac{\sqrt{6}}{3}}.$$

43. To find the curvature of $f(x)$ at $(0, 1)$, we need to find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$.

$$f'(x) = e^x \quad f'(0) = 1$$

$$f''(x) = e^x \quad f''(0) = 1 \quad |f''(0)| = 1$$

$$\kappa = \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}.$$

44. To find the curvature of $f(x)$ at $(2, \ln 3)$, we need to find $f'(x)$, $f'(2)$, $f''(x)$, $f''(2)$, and $|f''(2)|$.

$$f'(x) = \frac{1}{x+1} \quad f'(2) = \frac{1}{3}$$

$$f''(x) = -\frac{1}{(x+1)^2} \quad f''(2) = -\frac{1}{9} \quad |f''(2)| = \frac{1}{9}$$

$$\kappa = \frac{|f''(2)|}{(1 + [f'(2)]^2)^{3/2}} = \frac{\frac{1}{9}}{\left(1 + \left[\frac{1}{3}\right]^2\right)^{3/2}} = \frac{\frac{1}{9}}{\left(\frac{10}{9}\right)^{3/2}} = \boxed{\frac{3\sqrt{10}}{100}}.$$

45. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$f'(x) = 3x^2 - 6 \quad f'(1) = -3$$

$$f''(x) = 6x \quad f''(1) = 6 \quad |f''(1)| = 6$$

$$\kappa = \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{6}{(1 + [-3]^2)^{3/2}} = \frac{6}{10\sqrt{10}} = \frac{3}{5\sqrt{10}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{\frac{5\sqrt{10}}{3}}.$$

46. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(-1)$, $f''(x)$, $f''(-1)$, and $|f''(-1)|$.

$$f'(x) = -\frac{2}{x^3} \quad f'(-1) = -2$$

$$f''(x) = \frac{6}{x^4} \quad f''(-1) = 6 \quad |f''(-1)| = 6$$

$$\kappa = \frac{|f''(-1)|}{(1 + [f'(-1)]^2)^{3/2}} = \frac{6}{(1 + [-2]^2)^{3/2}} = \frac{6}{5\sqrt{5}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{\frac{5\sqrt{5}}{6}}.$$

47. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(\frac{\pi}{2})$, $f''(x)$, $f''(\frac{\pi}{2})$, and $|f''(\frac{\pi}{2})|$.

$$\begin{aligned} f'(x) &= \cos x & f'\left(\frac{\pi}{2}\right) &= 0 \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{2}\right) &= -1 & \left|f''\left(\frac{\pi}{2}\right)\right| &= 1 \end{aligned}$$

$$\kappa = \frac{|f''(\frac{\pi}{2})|}{(1 + [f'(\frac{\pi}{2})]^2)^{3/2}} = \frac{1}{(1 + [0]^2)^{3/2}} = 1.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{1}.$$

48. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$.

$$\begin{aligned} f'(x) &= -e^{-x} & f'(0) &= -1 \\ f''(x) &= e^{-x} & f''(0) &= 1 & |f''(0)| &= 1 \end{aligned}$$

$$\kappa = \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{1}{(1 + [-1]^2)^{3/2}} = \frac{1}{2\sqrt{2}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{2\sqrt{2}}.$$

49. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$. We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}(x^2 + xy + y^2) &= \frac{d}{dx}(3) \\ 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(x + 2y) &= -2x - y \\ \frac{dy}{dx} &= \frac{-2x - y}{x + 2y} \end{aligned}$$

At $(1, 1)$, $\frac{dy}{dx} = \frac{(-2)(1)-1}{1+2(1)} = -1$. To get $f''(x)$, we use implicit differentiation again.

$$\begin{aligned} \frac{d}{dx}\left(2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx}\right) &= \frac{d}{dx}(0) \\ 2 + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} &= 0 \end{aligned}$$

$$\frac{d^2y}{dx^2}(x+2y) = -2 - 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} = \frac{-2 - 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2}{x+2y}$$

At $(1, 1)$, $\frac{d^2y}{dx^2} = \frac{-2-2(-1)-2(-1)^2}{1+2(1)} = -\frac{2}{3}$, so $|f''(1)| = \frac{2}{3}$. Then

$$\kappa = \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{\frac{2}{3}}{(1 + [-1]^2)^{3/2}} = \frac{\frac{2}{3}}{2\sqrt{2}} = \frac{2}{6\sqrt{2}} = \frac{1}{3\sqrt{2}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{3\sqrt{2}}.$$

50. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$. We use implicit differentiation.

$$\frac{d}{dx}(y^2 - y + x) = \frac{d}{dx}(0)$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} + 1 = 0$$

$$(2y - 1)\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{2y - 1}$$

At $(0, 0)$, $\frac{dy}{dx} = 1$. To get $f''(x)$, we use implicit differentiation again.

$$\frac{d}{dx}\left((2y - 1)\frac{dy}{dx}\right) = \frac{d}{dx}(-1)$$

$$2\left(\frac{dy}{dx}\right)^2 + (2y - 1)\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-2\left(\frac{dy}{dx}\right)^2}{2y - 1}$$

At $(0, 0)$, $\frac{d^2y}{dx^2} = 2$, so $|f''(0)| = 2$. Then

$$\kappa = \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{2}{(1 + [-1]^2)^{3/2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{\sqrt{2}}.$$

51. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(\frac{\pi}{4})$, $f''(x)$, $f''(\frac{\pi}{4})$, and $|f''(\frac{\pi}{4})|$.

$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \quad f'\left(\frac{\pi}{4}\right) = 1$$

$$f''(x) = \sec^2 x \quad f''\left(\frac{\pi}{4}\right) = 2 \quad \left|f''\left(\frac{\pi}{4}\right)\right| = 2$$

$$\kappa = \frac{|f''(\frac{\pi}{4})|}{(1 + [f'(\frac{\pi}{4})]^2)^{3/2}} = \frac{2}{(1 + [1]^2)^{3/2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{\sqrt{2}}.$$

52. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$.

$$\begin{aligned} f'(x) &= \sinh x & f'(0) &= 0 \\ f''(x) &= \cosh x & f''(0) &= 1 & |f''(0)| &= 1 \end{aligned}$$

$$\kappa = \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{1}{(1 + [0]^2)^{3/2}} = 1.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{1}.$$

53. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= 6t\mathbf{i} + (3 - 3t^2)\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{36t^2 + 9 - 18t^2 + 9t^4} = 3\sqrt{t^4 + 2t^2 + 1} = 3\sqrt{(t^2 + 1)^2} = 3(t^2 + 1) \\ \mathbf{r}''(t) &= 6\mathbf{i} - 6t\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t & 3 - 3t^2 & 0 \\ 6 & -6t & 0 \end{vmatrix} \\ &= (-36t^2 - 18 + 18t^2)\mathbf{k} \\ &= -18(t^2 + 1)\mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{[-18(t^2 + 1)]^2} = 18(t^2 + 1) \end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{18(t^2 + 1)}{[3(t^2 + 1)]^3} = \frac{2}{3(t^2 + 1)^2}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{3}{2}(t^2 + 1)^2,$$

so the radius at $t = 1$ is $\rho = \frac{3}{2}(2^2) = \boxed{6}$.

54. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{1 + 4t^2 + 9t^4} \\ \mathbf{r}''(t) &= 2\mathbf{j} + 6t\mathbf{k} \end{aligned}$$

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} \\
&= (12t^2 - 6t^2)\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} \\
&= 2(3t^2\mathbf{i} - 3t\mathbf{j} + \mathbf{k}) \\
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= 2\sqrt{9t^4 + 9t^2 + 1}
\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{(1 + 4t^2 + 9t^4)^{3/2}}{2\sqrt{9t^4 + 9t^2 + 1}},$$

so the radius at $t = 1$ is $\rho = \frac{14\sqrt{14}}{2\sqrt{19}} = \boxed{\frac{7\sqrt{266}}{19}}$.

55. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}
\mathbf{r}'(t) &= \cos t \mathbf{i} - 2 \sin(2t) \mathbf{j} & \|\mathbf{r}'(t)\| &= \sqrt{\cos^2 t + 4 \sin^2(2t)} \\
\mathbf{r}''(t) &= -\sin t \mathbf{i} - 4 \cos(2t) \mathbf{j}
\end{aligned}$$

Using double-angle formulas,

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -2 \sin(2t) & 0 \\ -\sin t & -4 \cos(2t) & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -4 \sin t \cos t & 0 \\ -\sin t & 4 \sin^2 t - 4 \cos^2 t & 0 \end{vmatrix} \\
&= (4 \sin^2 t \cos t - 4 \cos^3 t - 4 \sin^2 t \cos t) \mathbf{k} = -4 \cos^3 t \mathbf{k} \\
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(-4 \cos^3 t)^2} = |4 \cos^3 t|
\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{|4 \cos^3 t|}{[\cos^2 t + 4 \sin^2(2t)]^{3/2}}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{[\cos^2 t + 4 \sin^2(2t)]^{3/2}}{|4 \cos^3 t|},$$

so the radius at $t = \frac{\pi}{4}$ is

$$\rho = \frac{[\frac{1}{2} + (4)(1)]^{3/2}}{4(\frac{1}{\sqrt{2}})^3} = \frac{\frac{27}{2\sqrt{2}}}{\sqrt{2}} = \boxed{\frac{27}{4}}.$$

56. See Problem 30 where it was shown that $\kappa = \frac{1}{1+b^2}$. Then the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = 1 + b^2,$$

so the radius at $t = \frac{\pi}{4}$ is $\rho = \boxed{1 + b^2}$.

57. See Problem 29 where it was shown that $\kappa = \frac{4}{5}$. Then the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{5}{4},$$

so the radius at $t = \frac{\pi}{4}$ is $\rho = \boxed{\frac{5}{4}}$.

58. See Problem 33 where it was shown that $\kappa = \frac{1}{3 \sin t \cos t}$. Then the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = 3 \sin t \cos t,$$

so the radius at $t = \frac{\pi}{3}$ is $\rho = 3(\frac{\sqrt{3}}{2})(\frac{1}{2}) = \boxed{\frac{3\sqrt{3}}{4}}$.

59. See Problem 31 where it was shown that $\kappa = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$. Then the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{(e^t + e^{-t})^2}{\sqrt{2}},$$

so the radius at $t = 0$ is $\rho = \frac{4}{\sqrt{2}} = \boxed{2\sqrt{2}}$.

60. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= a(3 - 3t^2)\mathbf{i} + 6at\mathbf{j} + a(3 + 3t^2)\mathbf{k} \\ \|\mathbf{r}'(t)\| &= a\sqrt{9 - 18t^2 + 9t^4 + 36t^2 + 9 + 18t^2 + 9t^4} \\ &= 3\sqrt{2}a\sqrt{1 + 2t^2 + t^4} = 3\sqrt{2}a\sqrt{(1 + t^2)^2} = 3\sqrt{2}a(1 + t^2) \\ \mathbf{r}''(t) &= -6ta\mathbf{i} + 6a\mathbf{j} + 6at\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a(3 - 3t^2) & 6at & a(3 + 3t^2) \\ -6ta & 6a & 6at \end{vmatrix} = 18a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - t^2 & 2t & 1 + t^2 \\ -t & 1 & t \end{vmatrix} \\ &= 18a^2[(2t^2 - 1 - t^2)\mathbf{i} - (t - t^3 + t + t^3)\mathbf{j} + (1 - t^2 + 2t^2)\mathbf{k}] \\ &= 18a^2[(t^2 - 1)\mathbf{i} - 2t\mathbf{j} + (t^2 + 1)\mathbf{k}] \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= 18a^2\sqrt{t^4 - 2t^2 + 1 + 4t^2 + t^4 + 2t^2 + 1} = 18a^2\sqrt{2t^4 + 4t^2 + 2} \\ &= 18\sqrt{2}a^2\sqrt{(t^2 + 1)^2} = 18\sqrt{2}a^2(t^2 + 1)\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{18\sqrt{2}a^2(t^2 + 1)}{[3\sqrt{2}a(1 + t^2)]^3} = \frac{1}{3a(1 + t^2)^2}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = 3a(1 + t^2)^2,$$

so the radius at $t = 1$ is $\rho = \boxed{12a}$.

61. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$. Differentiating and simplifying using the double-angle formulas,

$$\begin{aligned}\mathbf{r}'(t) &= -12a \cos^2 t \sin t \mathbf{i} + 12a \sin^2 t \cos t \mathbf{j} - 6a \sin(2t) \mathbf{k} \\ &= -12a \cos^2 t \sin t \mathbf{i} + 12a \sin^2 t \cos t \mathbf{j} - 12a \sin t \cos t \mathbf{k} \\ \|\mathbf{r}'(t)\| &= \sqrt{144a^2 \cos^4 t \sin^2 t + 144a^2 \sin^4 t \cos^2 t + 144a^2 \sin^2 t \cos^2 t} \\ &= \sqrt{144a^2 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) + 144a^2 \sin^2 t \cos^2 t} = 12a\sqrt{2} \sin t \cos t \\ \mathbf{r}''(t) &= (24a \cos t \sin^2 t - 12a \cos^3 t) \mathbf{i} + (24a \sin t \cos^2 t - 12a \sin^3 t) \mathbf{j} - (12a \cos^2 t - 12a \sin^2 t) \mathbf{k}\end{aligned}$$

Using a property of cross products, we can factor out common terms in both $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ to get

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = 144a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos^2 t \sin t & \sin^2 t \cos t & -\sin t \cos t \\ 2 \cos t \sin^2 t - \cos^3 t & 2 \sin t \cos^2 t - \sin^3 t & \sin^2 t - \cos^2 t \end{vmatrix}$$

We consider the sub-determinants separately. For the $\mathbf{i}$ sub-determinant,

$$\begin{aligned}\begin{vmatrix} \sin^2 t \cos t & -\sin t \cos t \\ 2 \sin t \cos^2 t - \sin^3 t & \sin^2 t - \cos^2 t \end{vmatrix} \mathbf{i} &= (\sin^4 t \cos t - \sin^2 t \cos^3 t + 2 \sin^2 t \cos^3 t - \sin^4 t \cos t) \mathbf{i} \\ &= \sin^2 t \cos^3 t \mathbf{i}\end{aligned}$$

For the $\mathbf{j}$ sub-determinant,

$$\begin{aligned}-\begin{vmatrix} -\cos^2 t \sin t & -\sin t \cos t \\ 2 \cos t \sin^2 t - \cos^3 t & \sin^2 t - \cos^2 t \end{vmatrix} \mathbf{j} &= -(\cos^4 t \sin t - \cos^2 t \sin^3 t + 2 \sin^3 t \cos^2 t - \sin t \cos^4 t) \mathbf{j} \\ &= -\sin^3 t \cos^2 t \mathbf{j}\end{aligned}$$

For the $\mathbf{k}$ sub-determinant,

$$\begin{aligned}\begin{vmatrix} -\cos^2 t \sin t & \sin^2 t \cos t \\ 2 \cos t \sin^2 t - \cos^3 t & 2 \sin t \cos^2 t - \sin^3 t \end{vmatrix} \mathbf{k} &= (\sin^4 t \cos^2 t - 2 \sin^2 t \cos^4 t - 2 \sin^4 t \cos^2 t + \sin^2 t \cos^4 t) \mathbf{k} \\ &= (-\sin^4 t \cos^2 t - \sin^2 t \cos^4 t) \mathbf{k} \\ &= -\sin^2 t \cos^2 t (\sin^2 t + \cos^2 t) \mathbf{k} = -\sin^2 t \cos^2 t \mathbf{k}\end{aligned}$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = 144a^2 [\sin^2 t \cos^3 t \mathbf{i} - \sin^3 t \cos^2 t \mathbf{j} + \sin^2 t \cos^2 t \mathbf{k}]$$

and

$$\begin{aligned}\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= 144a^2 \sqrt{\sin^4 t \cos^6 t + \sin^6 t \cos^4 t + \sin^4 t \cos^4 t} \\ &= 144a^2 \sqrt{\sin^4 t \cos^4 t (\sin^2 t + \cos^2 t + 1)} = 144\sqrt{2}a^2 \sin^2 t \cos^2 t\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{144\sqrt{2}a^2 \sin^2 t \cos^2 t}{(12a\sqrt{2} \sin t \cos t)^3} = \frac{1}{24a \sin t \cos t}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = 24a \sin t \cos t,$$

so the radius at $t = \frac{\pi}{4}$ is $\rho = \boxed{12a}$.

62. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{i} + 2\mathbf{j} - \frac{5t}{\sqrt{1-5t^2}}\mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{1+4+\frac{25t^2}{1-5t^2}} = \sqrt{\frac{5}{1-5t^2}} \\ \mathbf{r}''(t) &= \frac{-5\sqrt{1-5t^2} + 5t(\frac{1}{2})(1-5t^2)^{-1/2}(-10t)}{1-5t^2}\end{aligned}$$

We can multiply the numerator and the denominator by $\sqrt{1-5t^2}$ to simplify.

$$\mathbf{r}''(t) = \frac{-5(1-5t^2) - 25t^2}{(1-5t^2)^{3/2}}\mathbf{k} = \frac{-5}{(1-5t^2)^{3/2}}\mathbf{k}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & \frac{-5t}{\sqrt{1-5t^2}} \\ 0 & 0 & \frac{-5}{(1-5t^2)^{3/2}} \end{vmatrix} = \frac{-10}{(1-5t^2)^{3/2}}\mathbf{i} + \frac{5}{(1-5t^2)^{3/2}}\mathbf{j} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{\frac{100}{(1-5t^2)^3} + \frac{25}{(1-5t^2)^3}} = \sqrt{\frac{125}{(1-5t^2)^3}} = \frac{5\sqrt{5}}{(1-5t^2)^{3/2}}\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\frac{5\sqrt{5}}{(1-5t^2)^{3/2}}}{\left(\frac{5}{1-5t^2}\right)^{3/2}} = 1$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = 1,$$

so the radius at $t = 0$ is $\rho = \boxed{1}$.

Applications and Extensions

63. If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Then

$$\kappa(x) = \frac{|2a|}{(1 + [2ax + b]^2)^{3/2}} = \frac{|2a|}{(4a^2x^2 + 4abx + b^2 + 1)^{3/2}}$$

and

$$\rho(x) = \frac{1}{\kappa(x)} = \frac{(4a^2x^2 + 4abx + b^2 + 1)^{3/2}}{|2a|}.$$

To find the minimum radius of curvature, we use $\rho'(x)$ to find the critical point(s).

$$\rho'(x) = \frac{3(4a^2x^2 + 4abx + b^2 + 1)^{1/2}(8a^2x + 4ab)}{2|2a|} = \frac{3a}{|a|}(4a^2x^2 + 4abx + b^2 + 1)^{1/2}(2ax + b)$$

$\rho'(x) = 0$ when $2ax + b = 0$ or at $x = -\frac{b}{2a}$ which is the vertex of the parabola. To show that there is a minimum at $x = -\frac{b}{2a}$, we use the Second Derivative Test.

$$\begin{aligned}\rho''(x) &= \frac{3a}{|a|} \left(\frac{1}{2} \right) (4a^2x^2 + 4abx + b^2 + 1)^{-1/2} (8a^2x + 4ab)(2ax + b) + \frac{3a}{|a|} (4a^2x^2 + 4abx + b^2 + 1)^{1/2} (2a) \\ &= \frac{6a^2[(2ax + b)^2 + 4a^2x^2 + 4abx + b^2 + 1]}{|a|\sqrt{4a^2x^2 + 4abx + b^2 + 1}} = \frac{6a^2[8a^2x^2 + 8abx + 2b^2 + 1]}{|a|\sqrt{4a^2x^2 + 4abx + b^2 + 1}}\end{aligned}$$

Then

$$\rho''\left(-\frac{b}{2a}\right) = \frac{6a^2[8a^2\left(\frac{b^2}{4a^2}\right) + 8ab\left(-\frac{b}{2a}\right) + 2b^2 + 1]}{|a|\sqrt{4a^2\left(\frac{b^2}{4a^2}\right) + 4ab\left(-\frac{b}{2a}\right) + b^2 + 1}} = \frac{6a^2}{|a|}.$$

Since $\rho''\left(-\frac{b}{2a}\right) > 0$, by the Second Derivative Test, $\rho\left(-\frac{b}{2a}\right)$ is a minimum. The radius of curvature of the parabola $y = ax^2 + bx + c$ is a minimum at its vertex. ■

64. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ intersects the axes at $(\pm a, 0)$ and $(0, \pm b)$. To find the radius of curvature, we first need to find the curvature, so we need $f'(x)$, $f''(x)$, and $|f'''(x)|$. We use implicit differentiation.

$$\frac{d}{dx}(b^2x^2 + a^2y^2) = \frac{d}{dx}(a^2b^2)$$

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

To get $f''(x)$, we use implicit differentiation again.

$$\frac{d}{dx}\left(2b^2x + 2a^2y\frac{dy}{dx}\right) = \frac{d}{dx}(0)$$

$$2b^2 + 2a^2\left(\frac{dy}{dx}\right)^2 + 2a^2y\frac{d^2y}{dx^2} = 0$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-b^2 - a^2\left(\frac{dy}{dx}\right)^2}{a^2y} = \frac{-b^2 - a^2\left(-\frac{b^2x}{a^2y}\right)^2}{a^2y} = \frac{-a^2b^2y^2 - b^4x^2}{a^4y^3} \\ &= \frac{-b^2(a^2y^2 + b^2x^2)}{a^4y^3} = \frac{-b^2(a^2b^2)}{a^4y^3} = \frac{-b^4}{a^2y^3}\end{aligned}$$

Then

$$\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{\left|\frac{-b^4}{a^2y^3}\right|}{\left(1 + \left[-\frac{b^2x}{a^2y}\right]^2\right)^{3/2}} = \frac{\frac{b^4}{a^2y^3}}{\left(\frac{a^4y^2 + b^4x^2}{a^4y^2}\right)^{3/2}} = \frac{a^4b^4}{(a^4y^2 + b^4x^2)^{3/2}}.$$

The radius ρ of the osculating circle is

$$\rho = \frac{1}{\kappa} = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4},$$

so at $(\pm a, 0)$,

$$\rho = \frac{(b^4a^2)^{3/2}}{a^4b^4} = \frac{b^2}{a}$$

and at $(0, \pm b)$,

$$\rho = \frac{(a^4b^2)^{3/2}}{a^4b^4} = \frac{a^2}{b}.$$

■

65. To find the curvature, we will need $f'(x)$, $f''(x)$, and $|f''(x)|$.

$$f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad |f''(x)| = \frac{1}{x^2}$$

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{\frac{1}{x^2}}{\left(1 + \left[\frac{1}{x}\right]^2\right)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}.$$

The maximum curvature occurs at a critical point.

$$\begin{aligned} \kappa'(x) &= \frac{(x^2 + 1)^{3/2} - x(\frac{3}{2})(x^2 + 1)^{1/2}(2x)}{(x^2 + 1)^3} = \frac{(x^2 + 1) - 3x^2}{(x^2 + 1)^{5/2}} \\ &= \frac{1 - 2x^2}{(x^2 + 1)^{5/2}} \end{aligned}$$

The critical points of κ occur at $x = \pm \frac{\sqrt{2}}{2}$, but $x = -\frac{\sqrt{2}}{2}$ is not in the domain of $\ln x$. Since $\kappa'(x) > 0$ for $x < \frac{\sqrt{2}}{2}$ and $\kappa'(x) < 0$ for $x > \frac{\sqrt{2}}{2}$, the maximum curvature occurs at $\left(\frac{\sqrt{2}}{2}, \ln \frac{\sqrt{2}}{2}\right)$.

66. To find the curvature, we will need $f'(x)$, $f''(x)$, and $|f''(x)|$.

$$f'(x) = e^x \quad f''(x) = e^x \quad |f''(x)| = e^x$$

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{e^x}{(1 + [e^x]^2)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}.$$

The maximum curvature occurs at a critical point.

$$\kappa'(x) = \frac{e^x(1 + e^{2x})^{3/2} - e^x(\frac{3}{2})(1 + e^{2x})^{1/2}(2e^{2x})}{(1 + e^{2x})^3} = \frac{e^x(1 + e^{2x} - 3e^{2x})}{(1 + e^{2x})^{5/2}} = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$$

The critical points of κ occur when $1 - 2e^{2x} = 0$, or when $x = \frac{1}{2} \ln \frac{1}{2}$. Since $\kappa'(x) > 0$ for $x < \frac{1}{2} \ln \frac{1}{2}$ and $\kappa'(x) < 0$ for $x > \frac{1}{2} \ln \frac{1}{2}$, the maximum curvature occurs at $\left(\frac{1}{2} \ln \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

67. To find the curvature, we will need $f'(x)$, $f''(x)$, and $|f''(x)|$.

$$f'(x) = x^2 \quad f''(x) = 2x \quad |f''(x)| = |2x|$$

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{|2x|}{(1 + [x^2]^2)^{3/2}} = \frac{|2x|}{(1 + x^4)^{3/2}}.$$

The maximum curvature occurs at a critical point.

Assuming $x > 0$, $\kappa(x) = \frac{|2x|}{(1+x^4)^{3/2}} = \frac{2x}{(1+x^4)^{3/2}}$, and

$$\kappa'(x) = \frac{2(1 + x^4)^{3/2} - 2x(\frac{3}{2})(1 + x^4)^{1/2}(4x^3)}{(1 + x^4)^3} = \frac{2 + 2x^4 - 12x^4}{(1 + x^4)^{5/2}} = \frac{2 - 10x^4}{(1 + x^4)^{5/2}}.$$

The critical point of $\kappa(x)$ for $x > 0$ occurs at $x = (\frac{1}{5})^{1/4}$. Since $\kappa'(x) > 0$ for $0 < x < (\frac{1}{5})^{1/4}$ and $\kappa'(x) < 0$ for $x > (\frac{1}{5})^{1/4}$, a maximum κ occurs at $x = (\frac{1}{5})^{1/4}$.

Similarly, assuming $x < 0$, $\kappa(x) = \frac{|2x|}{(1+x^4)^{3/2}} = \frac{-2x}{(1+x^4)^{3/2}}$, and

$$\kappa'(x) = \frac{-2(1+x^4)^{3/2} + 2x(\frac{3}{2})(1+x^4)^{1/2}(4x^3)}{(1+x^4)^3} = \frac{-2 - 2x^4 + 12x^4}{(1+x^4)^{5/2}} = \frac{-2 + 10x^4}{(1+x^4)^{5/2}}.$$

The critical point of $\kappa(x)$ for $x < 0$ occurs at $x = -(\frac{1}{5})^{1/4}$. Since $\kappa'(x) > 0$ for $x < -(\frac{1}{5})^{1/4}$, and $\kappa'(x) < 0$ for $-(\frac{1}{5})^{1/4} < x < 0$, a maximum κ occurs at $x = -(\frac{1}{5})^{1/4}$.

Note that

$$\kappa\left(-\left(\frac{1}{5}\right)^{1/4}\right) = \kappa\left(\left(\frac{1}{5}\right)^{1/4}\right) = \frac{2\left(\left(\frac{1}{5}\right)^{1/4}\right)}{\left(1 + \left[\left(\frac{1}{5}\right)^{1/4}\right]^4\right)^{3/2}} = \frac{5\sqrt[4]{5}\sqrt{6}}{18},$$

so the maximum curvature occurs at $\left(\pm\left(\frac{1}{5}\right)^{1/4}, f\left(\pm\left(\frac{1}{5}\right)^{1/4}\right)\right) = \left(\pm\left(\frac{1}{5}\right)^{1/4}, \frac{1}{3}\left[\pm\left(\frac{1}{5}\right)^{1/4}\right]^3\right)$

$$= \left[\left(\pm\left(\frac{1}{5}\right)^{1/4}, \pm\frac{1}{3}\left(\frac{1}{5}\right)^{3/4}\right)\right].$$

68. To find the curvature, we will need $f'(x)$, $f''(x)$, and $|f''(x)|$.

$$f'(x) = \cos x \quad f''(x) = -\sin x \quad |f''(x)| = |\sin x|$$

We can restrict the domain to $0 \leq x \leq \pi$ to ensure that $\sin x$ is positive and then use the periodicity of the sine curve to find other maxima.

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{\sin x}{(1 + \cos^2 x)^{3/2}}.$$

The maximum curvature occurs at a critical point.

$$\begin{aligned} \kappa'(x) &= \frac{\cos x(1 + \cos^2 x)^{3/2} - \sin x(\frac{3}{2})(1 + \cos^2 x)^{1/2}(-2 \sin x \cos x)}{(1 + \cos^2 x)^3} \\ &= \frac{\cos x + \cos^3 x + 3 \sin^2 x \cos x}{(1 + \cos^2 x)^{5/2}} = \frac{\cos x(1 + \cos^2 x + 3 \sin^2 x)}{(1 + \cos^2 x)^{5/2}} \end{aligned}$$

The critical points of κ occur when $\cos x = 0$ or when $x = \frac{(2k+1)\pi}{2}$. The maximum curvature occurs at

$$\left[\left(\frac{[2k+1]\pi}{2}, \pm 1\right)\right].$$

69. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= -\alpha \sin t \mathbf{i} + \alpha \cos t \mathbf{j} + \mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{\alpha^2 \sin^2 t + \alpha^2 \cos^2 t + 1} = \sqrt{\alpha^2 + 1} \\ \mathbf{r}''(t) &= -\alpha \cos t \mathbf{i} - \alpha \sin t \mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\alpha \sin t & \alpha \cos t & 1 \\ -\alpha \cos t & -\alpha \sin t & 0 \end{vmatrix} = \alpha \sin t \mathbf{i} - \alpha \cos t \mathbf{j} + (\alpha^2 \sin^2 t + \alpha^2 \cos^2 t) \mathbf{k} \\ &= \alpha \sin t \mathbf{i} - \alpha \cos t \mathbf{j} + \alpha^2 \mathbf{k} \end{aligned}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{\alpha^2 \sin^2 t + \alpha^2 \cos^2 t + \alpha^4} = \alpha \sqrt{1 + \alpha^2}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\alpha\sqrt{1+\alpha^2}}{(\sqrt{\alpha^2+1})^3} = \frac{\alpha}{\alpha^2+1},$$

and to find the α that maximizes the curvature, we find $\kappa'(\alpha) = \frac{\alpha^2+1-\alpha(2\alpha)}{(\alpha^2+1)^2} = \frac{1-\alpha^2}{(\alpha^2+1)^2}$. Since $\alpha > 0$, the critical point is $\alpha = 1$. Since $\kappa'(\alpha) > 0$ for $\alpha < 1$ and $\kappa'(\alpha) < 0$ for $\alpha > 1$, the maximum curvature occurs for $\boxed{\alpha = 1}$.

70. The curvature of a plane curve is given by $\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{(\|\mathbf{r}'(t)\|)^3}$ or $\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$. At a point of inflection, $\mathbf{r}''(t) = \mathbf{0}$ or $f''(x) = 0$, which means $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 0$ or $|f''(x)| = 0$. Then at a point of inflection of a plane curve the curvature is $\boxed{\kappa = 0}$.

71. To find the curvature of $f(x)$, we need to find $f'(x)$, $f''(x)$, and $|f''(x)|$.

$$f'(x) = a \sinh \frac{x}{a} \left(\frac{1}{a} \right) = \sinh \frac{x}{a} \quad f''(x) = \frac{1}{a} \cosh \frac{x}{a} \quad |f''(x)| = \frac{1}{a} \cosh \frac{x}{a}$$

$$\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}} = \frac{\frac{1}{a} \cosh \frac{x}{a}}{(1+\sinh^2 \frac{x}{a})^{3/2}} = \frac{\frac{1}{a} \cosh \frac{x}{a}}{(\cosh^2 \frac{x}{a})^{3/2}} = \frac{1}{a \cosh^2 \frac{x}{a}} = \frac{a}{a^2 \cosh^2 \frac{x}{a}} = \frac{a}{y^2}.$$

■

72. To find the curvature of $f(x)$ at $(1, 1)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$. We use implicit differentiation.

$$\frac{d}{dx} [y^2(2-x)] = \frac{d}{dx} (x^3)$$

$$2y \frac{dy}{dx} (2-x) - y^2 = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$$

At $(1, 1)$, $\frac{dy}{dx} = \frac{4}{(2)(1)} = 2$. To get $f''(x)$, we use implicit differentiation again.

$$\frac{d}{dx} \left[2y \frac{dy}{dx} (2-x) - y^2 \right] = \frac{d}{dx} (3x^2)$$

$$2 \left(\frac{dy}{dx} \right)^2 (2-x) + 2y \frac{d^2y}{dx^2} (2-x) - 2y \frac{dy}{dx} - 2y \frac{dy}{dx} = 6x$$

We now substitute $(1, 1)$ into the above equation to solve for $\frac{d^2y}{dx^2}$.

$$(2)(2)^2(2-1) + 2(1) \frac{d^2y}{dx^2} (2-1) - 2(1)(2) - (2)(1)(2) = 6(1)$$

$$2 \frac{d^2y}{dx^2} = 6$$

so at $(1, 1)$, $\frac{d^2y}{dx^2} = 3$. Then

$$\kappa = \frac{|f''(1)|}{(1+[f'(1)]^2)^{3/2}} = \frac{3}{(1+[2]^2)^{3/2}} = \frac{3}{5\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{25}}.$$

73. Since $\mathbf{r}(t)(\theta) = (\theta - \sin \theta)\mathbf{i} + (1 - \cos \theta)\mathbf{j}$,

$$\begin{aligned}\mathbf{r}'(\theta) &= (1 - \cos \theta)\mathbf{i} + \sin \theta\mathbf{j} & \|\mathbf{r}'(t)\| &= \sqrt{1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta} = \sqrt{2 - 2\cos \theta} \\ \mathbf{r}''(t) &= \sin \theta\mathbf{i} + \cos \theta\mathbf{j}\end{aligned}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - \cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \end{vmatrix} = (\cos \theta - \cos^2 \theta - \sin^2 \theta)\mathbf{k} = (\cos \theta - 1)\mathbf{k}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = |\cos \theta - 1|$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{|\cos \theta - 1|}{[\sqrt{2(1 - \cos \theta)}]^3} = \frac{|1 - \cos \theta|}{2\sqrt{2}(1 - \cos \theta)^{3/2}} = \frac{1}{2\sqrt{2}(1 - \cos \theta)^{1/2}}.$$

The cycloid reaches its maximum when $y(\theta) = 1 - \cos(\theta)$ is maximized, and since $-1 \leq \cos \theta \leq 1$, the maximum y value of 2 occurs at $\theta = (2n + 1)\pi$. The curvature of the cycloid at the highest point of

an arch is $\kappa = \frac{1}{2\sqrt{2}(1 - (-1))^{1/2}} = \boxed{\frac{1}{4}}$.

74. (a) We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -3t^2\mathbf{i} + 2t\mathbf{j} & \|\mathbf{r}'(t)\| &= \sqrt{9t^4 + 4t^2} = t\sqrt{9t^2 + 4} \\ \mathbf{r}''(t) &= -6t\mathbf{i} + 2\mathbf{j}\end{aligned}$$

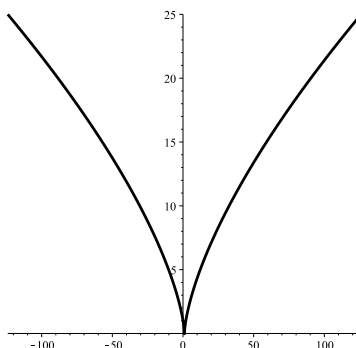
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3t^2 & 2t & 0 \\ -6t & 2 & 0 \end{vmatrix} = (-6t^2 + 12t^2)\mathbf{k} = 6t^2\mathbf{k}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 6t^2$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6t^2}{t^3(9t^2 + 4)^{3/2}} = \boxed{\frac{6}{t(9t^2 + 4)^{3/2}}, t \neq 0}.$$

(b) From the figure below generated by Maple™, we can see that at $t = 0$ the curve has a cusp. The curve is not smooth at $t = 0$, so the curvature is undefined.



75. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= (-e^{-t} \cos t - e^{-t} \sin t)\mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t)\mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{e^{-2t} \cos^2 t + 2e^{-2t} \sin t \cos t + e^{-2t} \sin^2 t + e^{-2t} \sin^2 t - 2e^{-t} \sin t \cos t + e^{-2t} \cos^2 t} \\ &= e^{-t} \sqrt{2} \\ \mathbf{T}(t) &= \frac{(-e^{-t} \cos t - e^{-t} \sin t)\mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t)\mathbf{j}}{e^{-t} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}(-\cos t - \sin t)\mathbf{i} + \frac{1}{\sqrt{2}}(-\sin t + \cos t)\mathbf{j} \\ \mathbf{T}'(t) &= \frac{1}{\sqrt{2}}(\sin t - \cos t)\mathbf{i} + \frac{1}{\sqrt{2}}(-\cos t - \sin t)\mathbf{j} \\ \|\mathbf{T}'(t)\| &= \sqrt{\frac{1}{2}(\sin^2 t - 2 \sin t \cos t + \cos^2 t + \cos^2 t + 2 \sin t \cos t + \sin^2 t)} = 1\end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{e^{-t} \sqrt{2}} = \boxed{\frac{e^t \sqrt{2}}{2}}.$$

As $t \rightarrow \infty$, $\kappa \rightarrow \infty$. We see that as $t \rightarrow \infty$, the spiral wraps more and more tightly around the origin.

76. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -2a \sin t \mathbf{i} + 2a \cos t \mathbf{j} + 2bt \mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{4a^2 \sin^2 t + 4a^2 \cos^2 t + 4b^2 t^2} = 2\sqrt{a^2 + b^2 t^2} \\ \mathbf{r}''(t) &= -2a \cos t \mathbf{i} - 2a \sin t \mathbf{j} + 2b \mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2a \sin t & 2a \cos t & 2bt \\ -2a \cos t & -2a \sin t & 2b \end{vmatrix} \\ &= (4ab \cos t + 4abt \sin t)\mathbf{i} - (-4ab \sin t + 4abt \cos t)\mathbf{j} + (4a^2 \sin^2 t + 4a^2 \cos^2 t)\mathbf{k} \\ &= 4ab(\cos t + t \sin t)\mathbf{i} - 4ab(-\sin t + t \cos t)\mathbf{j} + 4a^2 \mathbf{k}\end{aligned}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 4a\sqrt{b^2(\cos^2 t + t^2 \sin^2 t + \sin^2 t + t^2 \cos^2 t) + a^2} = 4a\sqrt{b^2(1 + t^2) + a^2}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{4a\sqrt{b^2(1 + t^2) + a^2}}{8(a^2 + b^2 t^2)^{3/2}} = \boxed{\frac{a\sqrt{b^2(1 + t^2) + a^2}}{2(a^2 + b^2 t^2)^{3/2}}}$$

77. We can express equation for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the vector function $\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -a \sin t \mathbf{i} + b \cos t \mathbf{j} & \|\mathbf{r}'(t)\| &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \\ \mathbf{r}''(t) &= -a \cos t \mathbf{i} - b \sin t \mathbf{j}\end{aligned}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & b \cos t & 0 \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = (ab \sin^2 t + ab \cos^2 t)\mathbf{k} = ab \mathbf{k}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = ab$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}},$$

and it has its minimum value where the denominator is largest. Since $a > b$, the denominator is largest when $\sin t = 1$ and $\cos t = 0$, or at $t = \frac{\pi}{2}, \frac{3\pi}{2}$, which corresponds to the points $(0, \pm b)$. Similarly, the curvature has its maximum value where the denominator is smallest. Since $a > b$, the denominator is smallest when $\sin t = 0$ and $\cos t = 1$, or at $t = 0, \pi$, which corresponds to the points $(\pm a, 0)$. ■

78. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -a \sin t \mathbf{i} + a \cos t \mathbf{j} + \mathbf{k} & \|\mathbf{r}'(t)\| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + 1} = \sqrt{a^2 + 1} \\ \mathbf{r}''(t) &= -a \cos t \mathbf{i} - a \sin t \mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & 1 \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = a \sin t \mathbf{i} - a \cos t \mathbf{j} + (a^2 \sin^2 t + a^2 \cos^2 t) \mathbf{k} \\ &= a \sin t \mathbf{i} - a \cos t \mathbf{j} + a^2 \mathbf{k}\end{aligned}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + a^4} = a\sqrt{1 + a^2}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a\sqrt{1 + a^2}}{(a^2 + 1)^{3/2}} = \frac{a}{a^2 + 1},$$

which is a constant. ■

79. We need to find $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$ so that we can determine the arc length function.

$$\mathbf{r}'(t) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \|\mathbf{r}'(t)\| = \sqrt{4 + 4 + 1} = 3.$$

Since the arc length function from $t = 0$ to an arbitrary t is given by

$$s(t) = \int_0^t \|\mathbf{r}'(u)\| du,$$

$$s(t) = \int_0^t 3 du = [3u]_0^t = 3t.$$

Then $t = \frac{s}{3}$ for $0 \leq s \leq 6$, and the vector equation for C using the arc length s as the parameter is

$$\boxed{\mathbf{r}(s) = \frac{2s}{3}\mathbf{i} + \frac{2s-3}{3}\mathbf{j} + \frac{s}{3}\mathbf{k}, \quad 0 \leq s \leq 6.}$$

80. The vector equations in Problems 13 and 15 represent the same line, but in one of them the parameter used is arc length and the other it is not. Since

$$\mathbf{r}'(t) = a\mathbf{i} + c\mathbf{j} \quad \|\mathbf{r}'(t)\| = \sqrt{a^2 + c^2},$$

$$s(t) = \int_0^t \sqrt{a^2 + c^2} du = \sqrt{a^2 + c^2} t$$

and $t = \frac{s}{\sqrt{a^2 + c^2}}$. The line whose parameter is arc length is $\boxed{\mathbf{r}(s) = \left(\frac{as}{\sqrt{a^2 + c^2}} + b\right)\mathbf{i} + \left(\frac{cs}{\sqrt{a^2 + c^2}} + d\right)\mathbf{j}}.$

81. Every point P on the curve C is traced out by the vector function $\mathbf{r}(t)$. The vector connecting P and $\mathbf{C}(t)$ at any t must be perpendicular to the unit tangent vector $\mathbf{T}(t)$, so it must be a scalar multiple of the principal unit normal vector. Since the radius of the osculating circle is $\rho = \frac{1}{\kappa}$, $\mathbf{C}(t) = \mathbf{r}(t) + \rho\mathbf{N}(t)$. ■
82. To find the radius of the osculating circle, we need $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= a \cos t \mathbf{i} - a \sin t \mathbf{j} + a^2 \mathbf{k} \\ \|\mathbf{r}'(t)\| &= \sqrt{a^2 \cos^2 t + a^2 \sin^2 t + a^4} = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2} \\ \mathbf{T}(t) &= \frac{a \cos t \mathbf{i} - a \sin t \mathbf{j} + a^2 \mathbf{k}}{a\sqrt{1 + a^2}} = \frac{1}{\sqrt{1 + a^2}} (\cos t \mathbf{i} - \sin t \mathbf{j} + a \mathbf{k}) \\ \mathbf{T}'(t) &= \frac{1}{\sqrt{1 + a^2}} (-\sin t \mathbf{i} - \cos t \mathbf{j}) \quad \|\mathbf{T}'(t)\| = \frac{1}{\sqrt{1 + a^2}} \sqrt{\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{1 + a^2}}\end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{1}{\sqrt{1+a^2}}}{a\sqrt{1+a^2}} = \frac{1}{a(1+a^2)},$$

so the radius of the osculating circle for the helix at any t is $\rho = \boxed{a(1+a^2)}$. To find the center of the osculating circle, we use the result from Problem 81.

$$\mathbf{C}(t) = \mathbf{r}(t) + \rho\mathbf{N}(t)$$

We just found ρ , so we need only to find $\mathbf{N}(t)$.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{\sqrt{1+a^2}}(-\sin t \mathbf{i} - \cos t \mathbf{j})}{\frac{1}{\sqrt{1+a^2}}} = -\sin t \mathbf{i} - \cos t \mathbf{j}$$

- (a) The center of the osculating circle at $t = \frac{\pi}{2}$ is

$$\begin{aligned}\mathbf{C}\left(\frac{\pi}{2}\right) &= \mathbf{r}\left(\frac{\pi}{2}\right) + a(1+a^2)\mathbf{N}\left(\frac{\pi}{2}\right) \\ &= a\mathbf{i} + a^2\frac{\pi}{2}\mathbf{k} + (a+a^3)(-\mathbf{i}) \\ &= \boxed{-a^3\mathbf{i} + a^2\frac{\pi}{2}\mathbf{k}}.\end{aligned}$$

- (b) The center of the osculating circle at $t = \pi$ is

$$\begin{aligned}\mathbf{C}(\pi) &= \mathbf{r}(\pi) + a(1+a^2)\mathbf{N}(\pi) \\ &= -a\mathbf{j} + a^2\pi\mathbf{k} + (a+a^3)(\mathbf{j}) \\ &= \boxed{a^3\mathbf{j} + a^2\pi\mathbf{k}}.\end{aligned}$$

83. $\mathbf{T}(s)$ is the unit tangent vector, which means that $\mathbf{T}(s) \cdot \mathbf{T}(s) = \|\mathbf{T}(s)\|^2 = 1$. Differentiating this equation,

$$\frac{d}{ds}[\mathbf{T}(s) \cdot \mathbf{T}(s)] = \frac{d}{ds}(1) = 0$$

Using the dot product formula for derivatives,

$$\frac{d}{ds}[\mathbf{T}(s) \cdot \mathbf{T}(s)] = \mathbf{T}'(s) \cdot \mathbf{T}(s) + \mathbf{T}(s) \cdot \mathbf{T}'(s) = 2\mathbf{T}'(s) \cdot \mathbf{T}(s)$$

Then $\mathbf{T}'(s) \cdot \mathbf{T}(s) = 0$, so $\mathbf{T}'(s)$ and $\mathbf{T}(s)$ are orthogonal. The principal unit normal vector $\mathbf{N}(s)$ is a scalar multiple of $\mathbf{T}'(s)$, so it must also be orthogonal to $\mathbf{T}(s)$. Since one of the properties of the cross product is that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ must be orthogonal to both $\mathbf{T}(s)$ and $\mathbf{N}(s)$, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are mutually orthogonal. ■

84. By the definition of curvature, $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$. Since $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt}$, and $\frac{ds}{dt} = \|\mathbf{r}'(t)\| \neq 0$,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} \right\| = \frac{\left\| \frac{d\mathbf{T}}{dt} \right\|}{\frac{ds}{dt}} = \frac{\|\mathbf{T}'(t)\|}{\frac{ds}{dt}}.$$

Using $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\kappa \mathbf{N} = \left(\frac{\|\mathbf{T}'(t)\|}{\frac{ds}{dt}} \right) \left(\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \right) = \frac{\mathbf{T}'(t)}{\frac{ds}{dt}} = \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{T}}{ds}$$

by the Chain Rule. ■

85. In Problem 83 we established that $\mathbf{T}$ is orthogonal to $\mathbf{N}$, so the angle between them must be $\theta = \frac{\pi}{2}$. Using one of the properties of the cross product,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \quad \text{so} \quad \|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\| \|\mathbf{N}\| \sin \frac{\pi}{2} = \|\mathbf{T}\| \|\mathbf{N}\| = 1.$$

Then $\mathbf{B}$ is a unit vector, which means that $\mathbf{B} \cdot \mathbf{B} = \|\mathbf{B}\|^2 = 1$. Differentiating this equation,

$$\frac{d}{ds} [\mathbf{B} \cdot \mathbf{B}] = \frac{d}{ds} (1) = 0$$

Using the dot product formula for derivatives,

$$\frac{d}{ds} [\mathbf{B} \cdot \mathbf{B}] = \mathbf{B}' \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B}' = 2\mathbf{B}' \cdot \mathbf{B}$$

Then $\mathbf{B}' \cdot \mathbf{B} = 0$, so $\mathbf{B}'$ and $\mathbf{B}$ are orthogonal.

Using the Chain Rule and the cross product formula for derivatives,

$$\begin{aligned} \frac{d}{ds} (\mathbf{B}) &= \frac{d}{ds} (\mathbf{T} \times \mathbf{N}) = \frac{\frac{d}{dt} (\mathbf{T} \times \mathbf{N})}{\frac{ds}{dt}} = \frac{\frac{d}{dt} (\mathbf{T} \times \mathbf{N})}{\|\mathbf{r}'(t)\|} \\ &= \frac{1}{\|\mathbf{r}'(t)\|} [(\mathbf{T}' \times \mathbf{N}) + (\mathbf{T} \times \mathbf{N}')] \\ &= \frac{1}{\|\mathbf{r}'(t)\|} \left[\left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \right) + (\mathbf{T} \times \mathbf{N}') \right] \\ &= \frac{1}{\|\mathbf{r}'(t)\|} [\mathbf{0} + (\mathbf{T} \times \mathbf{N}')] = \frac{\mathbf{T} \times \mathbf{N}'}{\|\mathbf{r}'(t)\|}. \end{aligned}$$

By one of the properties of the cross product, since $\frac{d\mathbf{B}}{ds} = \frac{\mathbf{T} \times \mathbf{N}'}{\|\mathbf{r}'(t)\|}$, $\frac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$. ■

86. In Problem 83 we showed that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a collection of mutually orthogonal unit vectors, so since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, we must also have $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ as we do with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then

$$\begin{aligned} \frac{d}{ds} (\mathbf{N}) &= \frac{d}{ds} (\mathbf{B} \times \mathbf{T}) = (\mathbf{B}' \times \mathbf{T}) + (\mathbf{B} \times \mathbf{T}') \\ &= [(-\tau \mathbf{N}) \times \mathbf{T}] + [\mathbf{B} \times (\kappa \mathbf{N})] = -\tau (\mathbf{N} \times \mathbf{T}) + \kappa (\mathbf{B} \times \mathbf{N}) \\ &= -\tau (-\mathbf{B}) + \kappa (-\mathbf{T}) = \tau \mathbf{B} - \kappa \mathbf{T} \end{aligned}$$
■

87. We need $\mathbf{r}'(s)$ and $\|\mathbf{r}'(s)\|$.

$$\mathbf{r}'(s) = \frac{\sqrt{2}}{2} [\cos s \mathbf{i} - \sin s \mathbf{j} + \mathbf{k}] \quad \|\mathbf{r}'(s)\| = \frac{\sqrt{2}}{2} \sqrt{\cos^2 s + \sin^2 s + 1} = \frac{\sqrt{2}}{2} \sqrt{2} = 1$$

Then

$$\mathbf{T}(s) = \boxed{\frac{\sqrt{2}}{2} \cos s \mathbf{i} - \frac{\sqrt{2}}{2} \sin s \mathbf{j} + \frac{\sqrt{2}}{2} \mathbf{k}}$$

and

$$\mathbf{T}'(s) = \frac{\sqrt{2}}{2} [-\sin s \mathbf{i} - \cos s \mathbf{j}] \quad \|\mathbf{T}'(s)\| = \frac{\sqrt{2}}{2}$$

so

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \boxed{-\sin s \mathbf{i} - \cos s \mathbf{j}}$$

and

$$\kappa = \frac{\|\mathbf{T}'(s)\|}{\|\mathbf{r}'(s)\|} = \boxed{\frac{\sqrt{2}}{2}}.$$

The binormal vector $\mathbf{B}$ is defined as $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$, so

$$\begin{aligned} \mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{2}}{2} \cos s & -\frac{\sqrt{2}}{2} \sin s & \frac{\sqrt{2}}{2} \\ -\sin s & -\cos s & 0 \end{vmatrix} \\ &= \frac{\sqrt{2}}{2} \cos s \mathbf{i} - \frac{\sqrt{2}}{2} \sin s \mathbf{j} - \left(\frac{\sqrt{2}}{2} \cos^2 s + \frac{\sqrt{2}}{2} \sin^2 s \right) \mathbf{k} \\ &= \boxed{\frac{\sqrt{2}}{2} \cos s \mathbf{i} - \frac{\sqrt{2}}{2} \sin s \mathbf{j} - \frac{\sqrt{2}}{2} \mathbf{k}}. \end{aligned}$$

Challenge Problems

88. We represent the polar curve $r = f(\theta)$ as

$$\mathbf{r}(\theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} = f(\theta) \cos \theta \mathbf{i} + f(\theta) \sin \theta \mathbf{j}.$$

Note that $\mathbf{r}(\theta)$ is a vector function and $r = f(\theta)$ is a polar equation with $\frac{dr}{d\theta} = f'(\theta)$. We need $\mathbf{r}'(\theta)$, $\|\mathbf{r}'(\theta)\|$, $\mathbf{r}''(\theta)$, $\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)$, and $\|\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)\|$.

$$\mathbf{r}'(\theta) = [f'(\theta) \cos \theta - f(\theta) \sin \theta] \mathbf{i} + [f'(\theta) \sin \theta + f(\theta) \cos \theta] \mathbf{j}$$

Since $\|\mathbf{r}'(\theta)\|^2 = [f'(\theta) \cos \theta - f(\theta) \sin \theta]^2 + [f'(\theta) \sin \theta + f(\theta) \cos \theta]^2$,

$$[f'(\theta) \cos \theta - f(\theta) \sin \theta]^2 = [f'(\theta)]^2 \cos^2 \theta - 2f(\theta)f'(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \sin^2 \theta,$$

and

$$[f'(\theta) \sin \theta + f(\theta) \cos \theta]^2 = [f'(\theta)]^2 \sin^2 \theta + 2f(\theta)f'(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \cos^2 \theta,$$

$$\|\mathbf{r}'(\theta)\| = \sqrt{[f'(\theta)]^2 + [f(\theta)]^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Using the Product Rule and combining like terms,

$$\mathbf{r}''(\theta) = [f''(\theta) \cos \theta - 2f'(\theta) \sin \theta - f(\theta) \cos \theta] \mathbf{i} + [f''(\theta) \sin \theta + 2f'(\theta) \cos \theta - f(\theta) \sin \theta] \mathbf{j}$$

so

$$\mathbf{r}'(\theta) \times \mathbf{r}''(\theta) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'(\theta) \cos \theta - f(\theta) \sin \theta & f'(\theta) \sin \theta + f(\theta) \cos \theta & 0 \\ f''(\theta) \cos \theta - 2f'(\theta) \sin \theta - f(\theta) \cos \theta & f''(\theta) \sin \theta + 2f'(\theta) \cos \theta - f(\theta) \sin \theta & 0 \end{vmatrix}$$

The **i** and **j** components become **0**. To find the **k** terms, we distribute both

$$[f'(\theta) \cos \theta - f(\theta) \sin \theta][f''(\theta) \sin \theta + 2f'(\theta) \cos \theta - f(\theta) \sin \theta]$$

and

$$-[f'(\theta) \sin \theta + f(\theta) \cos \theta][f''(\theta) \cos \theta - 2f'(\theta) \sin \theta - f(\theta) \cos \theta].$$

$$\begin{aligned} f'(\theta) \cos \theta [f''(\theta) \sin \theta + 2f'(\theta) \cos \theta - f(\theta) \sin \theta] &= f'(\theta) f''(\theta) \sin \theta \cos \theta + 2[f'(\theta)]^2 \cos^2 \theta - f(\theta) f'(\theta) \sin \theta \cos \theta \\ -f(\theta) \sin \theta [f''(\theta) \sin \theta + 2f'(\theta) \cos \theta - f(\theta) \sin \theta] &= -f(\theta) f''(\theta) \sin^2 \theta - 2f(\theta) f'(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \sin^2 \theta \\ -f'(\theta) \sin \theta [f''(\theta) \cos \theta - 2f'(\theta) \sin \theta - f(\theta) \cos \theta] &= -f'(\theta) f''(\theta) \sin \theta \cos \theta + 2[f'(\theta)]^2 \sin^2 \theta + f(\theta) f'(\theta) \sin \theta \cos \theta \\ -f(\theta) \cos \theta [f''(\theta) \cos \theta - 2f'(\theta) \sin \theta - f(\theta) \cos \theta] &= -f(\theta) f''(\theta) \cos^2 \theta + 2f(\theta) f'(\theta) \sin \theta \cos \theta + [f(\theta)]^2 \cos^2 \theta \end{aligned}$$

Adding these terms together, we find

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = ([f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta))\mathbf{k}$$

and

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = |[f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta)| = \left| r^2 + 2 \left(\frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2} \right|.$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)\|}{\|\mathbf{r}'(\theta)\|^3} = \frac{\left| r^2 + 2 \left(\frac{dr}{d\theta} \right)^2 - r \frac{d^2r}{d\theta^2} \right|}{\left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right]^{3/2}}.$$

■

89. Differentiating r gives $\frac{dr}{d\theta} = -4 \sin(2\theta)$ and $\frac{d^2r}{d\theta^2} = -8 \cos(2\theta)$. At $\theta = \frac{\pi}{12}$, $r\left(\frac{\pi}{12}\right) = \sqrt{3}$, $r'\left(\frac{\pi}{12}\right) = -2$, and $r''\left(\frac{\pi}{12}\right) = -4\sqrt{3}$. We use the formula given in Problem 88 and evaluate κ .

$$\kappa = \frac{|(\sqrt{3})^2 + 2(-2)^2 - \sqrt{3}(-4\sqrt{3})|}{[(\sqrt{3})^2 + (-2)^2]^{3/2}} = \frac{23}{7\sqrt{7}} = \boxed{\frac{23}{7^{3/2}}}$$

90. Differentiating r gives $\frac{dr}{d\theta} = ae^{a\theta}$ and $\frac{d^2r}{d\theta^2} = a^2e^{a\theta}$. At $\theta = \frac{\pi}{2}$, $r\left(\frac{\pi}{2}\right) = e^{a\pi/2}$, $r'\left(\frac{\pi}{2}\right) = ae^{a\pi/2}$, and $r''\left(\frac{\pi}{2}\right) = a^2e^{a\pi/2}$. We use the formula given in Problem 88 and evaluate κ .

$$\kappa = \frac{|(e^{a\pi/2})^2 + 2(ae^{a\pi/2})^2 - e^{a\pi/2}a^2e^{a\pi/2}|}{[(e^{a\pi/2})^2 + (ae^{a\pi/2})^2]^{3/2}} = \frac{|e^{a\pi} + 2a^2e^{a\pi} - a^2e^{a\pi}|}{(e^{a\pi} + a^2e^{a\pi})^{3/2}} = \boxed{\frac{1}{\sqrt{e^{a\pi}(1+a^2)}}}$$

91. Differentiating r gives $\frac{dr}{d\theta} = a$ and $\frac{d^2r}{d\theta^2} = 0$. At $\theta = 1$, $r(1) = a$, $r'(1) = a$, and $r''(1) = 0$. We use the formula given in Problem 88 and evaluate κ .

$$\kappa = \frac{|a^2 + 2(a)^2 - a(0)|}{[(a)^2 + (a)^2]^{3/2}} = \frac{3}{2\sqrt{2}a} = \boxed{\frac{3\sqrt{2}}{4a}}.$$

92. We represent the polar curve $r = 1 - \cos \theta$ as

$$\mathbf{r}(\theta) = (1 - \cos \theta) \cos \theta \mathbf{i} + (1 - \cos \theta) \sin \theta \mathbf{j}.$$

Then

$$\mathbf{r}'(\theta) = [\sin \theta \cos \theta - (1 - \cos \theta) \sin \theta] \mathbf{i} + [\sin^2 \theta + (1 - \cos \theta) \cos \theta] \mathbf{j}$$

and

$$\mathbf{r}'(0) = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}.$$

Since $\mathbf{r}'(\theta) = \mathbf{0}$ at $\theta = 0$, $\mathbf{r}(\theta)$ is not smooth and $\boxed{\kappa \text{ does not exist}}$ for $\theta = 0$. Equivalently, recognizing the polar curve as a cardioid with a cusp at $\theta = 0$ is sufficient to conclude that the curvature does not exist there.

93. Differentiating r gives $\frac{dr}{d\theta} = -2\cos\theta$ and $\frac{d^2r}{d\theta^2} = 2\sin\theta$. At $\theta = \frac{\pi}{6}$, $r\left(\frac{\pi}{6}\right) = 2$, $r'\left(\frac{\pi}{6}\right) = -\sqrt{3}$, and $r''\left(\frac{\pi}{6}\right) = 1$. We use the formula given in Problem 88 and evaluate κ .

$$\kappa = \frac{|2^2 + 2(-\sqrt{3})^2 - 2(1)|}{[2^2 + (-\sqrt{3})^2]^{3/2}} = \boxed{\frac{8\sqrt{7}}{49}}$$

94. Differentiating r gives $\frac{dr}{d\theta} = -3\sin\theta$ and $\frac{d^2r}{d\theta^2} = -3\cos\theta$. At $\theta = \frac{\pi}{3}$, $r\left(\frac{\pi}{3}\right) = \frac{7}{2}$, $r'\left(\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$, and $r''\left(\frac{\pi}{3}\right) = -\frac{3}{2}$. We use the formula given in Problem 88 and evaluate κ .

$$\kappa = \frac{\left|\left(\frac{7}{2}\right)^2 + 2\left(-\frac{3\sqrt{3}}{2}\right)^2 - \left(\frac{7}{2}\right)\left(-\frac{3}{2}\right)\right|}{\left[\left(\frac{7}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2\right]^{3/2}} = \frac{31}{(19)^{3/2}} = \boxed{\frac{31\sqrt{19}}{361}}$$

95. From the figure we see that h is less than x by $\rho\sin\phi$ and k is greater than y by $\rho\cos\phi$, so

$$h = x - \rho\sin\phi \quad k = y + \rho\cos\phi.$$

The point P is on the principal unit normal vector, which has a slope of $\frac{-1}{y'}$. We know slope represents the change in y over the change in x . The change in y has a magnitude of 1, so the side of the triangle adjacent to ϕ has length 1. The change in x has a magnitude of y' , so the side of the triangle opposite ϕ has a magnitude of y' . By the Pythagorean Theorem, the hypotenuse of the triangle has a magnitude of $\sqrt{1 + (y')^2}$. Using the relations $\sin\phi = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\phi = \frac{\text{adjacent}}{\text{hypotenuse}}$,

$$\sin\phi = \frac{y'}{\sqrt{1 + (y')^2}} \quad \cos\phi = \frac{1}{\sqrt{1 + (y')^2}}.$$

Since $\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|}$,

$$\rho\sin\phi = \left(\frac{[1 + (y')^2]^{3/2}}{|y''|}\right) \left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = \frac{y'[1 + (y')^2]}{y''}$$

and

$$\rho\cos\phi = \left(\frac{[1 + (y')^2]^{3/2}}{|y''|}\right) \left(\frac{1}{\sqrt{1 + (y')^2}}\right) = \frac{1 + (y')^2}{y''}.$$

Then

$$h = x - \rho\sin\phi = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \rho\cos\phi = y + \frac{1 + (y')^2}{y''}$$

96. We need y' , $y'(1)$, y'' , and $y''(1)$.

$$y' = 2x \quad y'(1) = 2 \quad y'' = 2 \quad y''(1) = 2$$

From Problem 95, we know

$$h = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \frac{1 + (y')^2}{y''},$$

so

$$h = 1 - \frac{2[1 + (2)^2]}{2} = -4 \quad k = 1 + \frac{1 + (2)^2}{2} = \frac{7}{2}.$$

The coordinates of the center of curvature are $\boxed{\left(-4, \frac{7}{2}\right)}$.

97. We need y' , $y'(\frac{\pi}{2})$, y'' , and $y''(\frac{\pi}{2})$.

$$y' = \cos x \quad y'(\frac{\pi}{2}) = 0 \quad y'' = -\sin x \quad y''(\frac{\pi}{2}) = -1$$

From Problem 95, we know

$$h = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \frac{1 + (y')^2}{y''},$$

so

$$h = \frac{\pi}{2} - \frac{0[1 + (0)^2]}{-1} = \frac{\pi}{2} \quad k = 1 + \frac{1 + (0)^2}{-1} = 0.$$

The coordinates of the center of curvature are $\boxed{(\frac{\pi}{2}, 0)}$.

98. We need y' , $y'(0)$, y'' , and $y''(0)$.

$$y' = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} \quad y'(0) = 1 \quad y'' = -2(x+1)^{-3} = \frac{-2}{(x+1)^3} \quad y''(0) = -2$$

From Problem 95, we know

$$h = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \frac{1 + (y')^2}{y''},$$

so

$$h = 0 - \frac{[1 + (1)^2]}{-2} = 1 \quad k = 0 + \frac{1 + (1)^2}{-2} = -1.$$

The coordinates of the center of curvature are $\boxed{(1, -1)}$.

99. We need y' , $y'(2)$, y'' , and $y''(2)$. We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx} [x^3 + y^3] &= \frac{d}{dx} (4xy) \\ 3x^2 + 3y^2 \frac{dy}{dx} &= 4y + 4x \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{3x^2 - 4y}{4x - 3y^2} \end{aligned}$$

At $(2, 2)$, $\frac{dy}{dx} = \frac{4}{-4} = -1$. To get y'' , we use implicit differentiation again.

$$\begin{aligned} \frac{d}{dx} \left[3x^2 + 3y^2 \frac{dy}{dx} \right] &= \frac{d}{dx} \left(4y + 4x \frac{dy}{dx} \right) \\ 6x + 6y \left(\frac{dy}{dx} \right)^2 + 3y^2 \frac{d^2y}{dx^2} &= 4 \frac{dy}{dx} + 4 \frac{dy}{dx} + 4x \frac{d^2y}{dx^2} \end{aligned}$$

We now substitute $(2, 2)$ into the above equation to solve for $\frac{d^2y}{dx^2}$.

$$\begin{aligned} (6)(2) + (6)(2)(-1)^2 + 3(2)^2 \frac{d^2y}{dx^2} &= 8(-1) + 4(2) \frac{d^2y}{dx^2} \\ 24 + 12 \frac{d^2y}{dx^2} &= -8 + 8 \frac{d^2y}{dx^2} \end{aligned}$$

so at $(2, 2)$, $\frac{d^2y}{dx^2} = -8$. From Problem 95, we know

$$h = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \frac{1 + (y')^2}{y''},$$

so

$$h = 2 - \frac{(-1)[1 + (-1)^2]}{-8} = \frac{7}{4} \quad k = 2 + \frac{1 + (-1)^2}{-8} = \frac{7}{4}.$$

The coordinates of the center of curvature are $\boxed{\left(\frac{7}{4}, \frac{7}{4}\right)}$.

100. We need y' , $y'(2)$, y'' , and $y''(2)$. We use implicit differentiation.

$$\frac{d}{dx}[xy] = \frac{d}{dx}(4)$$

$$y + x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At $(2, 2)$, $\frac{dy}{dx} = -\frac{2}{2} = -1$. To get y'' , we use implicit differentiation again.

$$\frac{d}{dx}\left[y + x \frac{dy}{dx}\right] = \frac{d}{dx}(0)$$

$$\frac{dy}{dx} + \frac{dy}{dx} + x \frac{d^2y}{dx^2} = 0$$

We now substitute $(2, 2)$ into the above equation to solve for $\frac{d^2y}{dx^2}$.

$$(2)(-1) + 2 \frac{d^2y}{dx^2} = 0$$

so at $(2, 2)$, $\frac{d^2y}{dx^2} = 1$.

From Problem 95, we know

$$h = x - \frac{y'[1 + (y')^2]}{y''} \quad k = y + \frac{1 + (y')^2}{y''},$$

so

$$h = 2 - \frac{(-1)[1 + (-1)^2]}{1} = 4 \quad k = 2 + \frac{1 + (-1)^2}{1} = 4.$$

The coordinates of the center of curvature are $\boxed{(4, 4)}$.

101. To use the result from Problem 95, we need y' and y'' .

$$y' = x \quad y'' = 1$$

Then

$$h = x - x \left(\frac{1 + x^2}{1} \right) = -x^3 \quad k = \frac{1}{2}x^2 + \frac{1 + x^2}{1} = \frac{3}{2}x^2 + 1$$

Solve for x in the equation for k .

$$x = \sqrt{\frac{2}{3}(k - 1)}$$

Substitute x into the equation for h .

$$h = - \left[\sqrt{\frac{2}{3}(k - 1)} \right]^3 = - \left[\frac{2}{3}(k - 1) \right]^{3/2}$$

Square both sides.

$$h^2 = \left[\frac{2}{3}(k - 1) \right]^3 = \frac{8}{27}(k - 1)^3$$

102. We use the result from Problem 81 to find the position vector for the evolute of the ellipse. The components of $\mathbf{C}(t) = \mathbf{r}(t) + \rho \mathbf{N}(t)$ will be the parametric equations for this evolute. We need to find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, $\|\mathbf{T}'(t)\|$, and $\mathbf{N}(t)$.

$$\mathbf{r}'(t) = -a \sin t \mathbf{i} + b \cos t \mathbf{j} \quad \|\mathbf{r}'(t)\| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a \sin t \mathbf{i} + b \cos t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

We differentiate the components individually.

$$\begin{aligned} \frac{d}{dt} \left(\frac{-a \sin t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \right) &= \frac{-a \cos t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} + \frac{a \sin t (2a^2 \sin t \cos t - 2b^2 \sin t \cos t)}{2(a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}}}{a^2 \sin^2 t + b^2 \cos^2 t} \\ &= \frac{-a \cos t (a^2 \sin^2 t + b^2 \cos^2 t) + a \sin t (a^2 \sin t \cos t - b^2 \sin t \cos t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-ab^2 \cos^3 t - ab^2 \sin^2 t \cos t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-ab^2 \cos t (\cos^2 t + \sin^2 t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-ab^2 \cos t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \left(\frac{b \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \right) &= \frac{-b \sin t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} - \frac{b \cos t (2a^2 \sin t \cos t - 2b^2 \sin t \cos t)}{2(a^2 \sin^2 t + b^2 \cos^2 t)^{1/2}}}{a^2 \sin^2 t + b^2 \cos^2 t} \\ &= \frac{-b \sin t (a^2 \sin^2 t + b^2 \cos^2 t) - b \cos t (a^2 \sin t \cos t - b^2 \sin t \cos t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-a^2 b \sin^3 t - a^2 b \sin t \cos^2 t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-a^2 b \sin t (\sin^2 t + \cos^2 t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \\ &= \frac{-a^2 b \sin t}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \end{aligned}$$

so

$$\mathbf{T}'(t) = \frac{-ab^2 \cos t \mathbf{i} - a^2 b \sin t \mathbf{j}}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} = \frac{-ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} (b \cos t \mathbf{i} + a \sin t \mathbf{j}).$$

Then

$$\|\mathbf{T}'(t)\| = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \sqrt{b^2 \cos^2 t + a^2 \sin^2 t} = \frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t}.$$

We know $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ and $\rho = \frac{1}{\kappa}$, so

$$\rho = \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{T}'(t)\|} = \frac{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}{\frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t}} = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab}.$$

Since $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$,

$$\mathbf{N}(t) = \frac{\frac{-ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} (b \cos t \mathbf{i} + a \sin t \mathbf{j})}{\frac{ab}{a^2 \sin^2 t + b^2 \cos^2 t}} = \frac{-b \cos t \mathbf{i} - a \sin t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}.$$

We next find and simplify $\rho\mathbf{N}(t)$.

$$\begin{aligned}\rho\mathbf{N}(t) &= \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab} \left(\frac{-b \cos t \mathbf{i} - a \sin t \mathbf{j}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \right) = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)(-b \cos t \mathbf{i} - a \sin t \mathbf{j})}{ab} \\ &= -\frac{\cos t(a^2 \sin^2 t + b^2 \cos^2 t)}{a} \mathbf{i} - \frac{\sin t(a^2 \sin^2 t + b^2 \cos^2 t)}{b} \mathbf{j}\end{aligned}$$

Then

$$\begin{aligned}\mathbf{C}(t) &= \mathbf{r}(t) + \rho\mathbf{N}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j} - \frac{\cos t(a^2 \sin^2 t + b^2 \cos^2 t)}{a} \mathbf{i} - \frac{\sin t(a^2 \sin^2 t + b^2 \cos^2 t)}{b} \mathbf{j} \\ &= \left[a \cos t - \frac{\cos t(a^2 \sin^2 t + b^2 \cos^2 t)}{a} \right] \mathbf{i} + \left[b \sin t - \frac{\sin t(a^2 \sin^2 t + b^2 \cos^2 t)}{b} \right] \mathbf{j} \\ &= \left[\frac{a^2 \cos t - \cos t(a^2 \sin^2 t + b^2 \cos^2 t)}{a} \right] \mathbf{i} + \left[\frac{b^2 \sin t - \sin t(a^2 \sin^2 t + b^2 \cos^2 t)}{b} \right] \mathbf{j} \\ &= \frac{\cos t(a^2 - a^2 \sin^2 t - b^2 \cos^2 t)}{a} \mathbf{i} + \frac{\sin t(b^2 - a^2 \sin^2 t - b^2 \cos^2 t)}{b} \mathbf{j} \\ &= \frac{\cos t(a^2 \cos^2 t - b^2 \cos^2 t)}{a} \mathbf{i} + \frac{\sin t(-a^2 \sin^2 t + b^2 \sin^2 t)}{b} \mathbf{j} = \frac{\cos^3 t(a^2 - b^2)}{a} \mathbf{i} - \frac{\sin^3 t(a^2 - b^2)}{b} \mathbf{j}\end{aligned}$$

so the parametric equations of the evolute of the ellipse are $\boxed{h = \frac{a^2 - b^2}{a} \cos^3 t \text{ and } k = -\frac{a^2 - b^2}{b} \sin^3 t}.$

To eliminate the parameter, we solve the h and k equations for $\cos t$ and $\sin t$, respectively.

$$\cos t = \left(\frac{ah}{a^2 - b^2} \right)^{1/3} \quad \sin t = \left(\frac{-bk}{a^2 - b^2} \right)^{1/3}$$

or

$$\cos^2 t = \left(\frac{ah}{a^2 - b^2} \right)^{2/3} \quad \sin^2 t = \left(\frac{bk}{a^2 - b^2} \right)^{2/3}$$

and the rectangular equation for the evolute is

$$\left(\frac{ah}{a^2 - b^2} \right)^{2/3} + \left(\frac{bk}{a^2 - b^2} \right)^{2/3} = 1$$

or

$$\boxed{(ah)^{2/3} + (bk)^{2/3} = (a^2 - b^2)^{2/3}}.$$

11.4 Motion Along a Curve

Applications and Extensions

Concepts and Vocabulary

1. True by the definition at the beginning of subsection 1.
2. False. We know the acceleration vector $\mathbf{a}(t)$ can be written in terms of the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector as $\mathbf{N}(t)$ as $\mathbf{a}(t) = v'(t)\mathbf{T}(t) + [v(t)]^2\kappa\mathbf{N}(t)$. For a particle moving with constant speed, $v'(t) = 0$, so the tangential component is zero. The acceleration of the particle is directed along the unit normal vector, $\mathbf{N}$.
3. True. The vector in the direction of the velocity is the tangential component, and the vector perpendicular to the direction of the velocity is the normal component.
4. $a_{\mathbf{T}} = v'(t)$ is called the tangential component and $a_{\mathbf{N}} = [v(t)]^2\kappa$ is called the normal component.

Skill Building

5. (a) The velocity, acceleration, and speed are

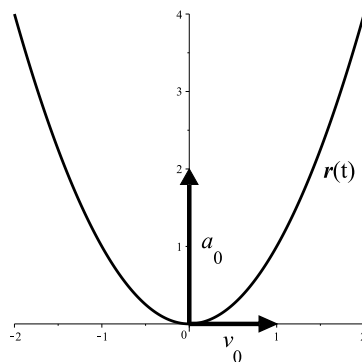
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}t\mathbf{i} + \frac{d}{dt}t^2\mathbf{j} = \boxed{\mathbf{i} + 2t\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}\mathbf{i} + \frac{d}{dt}2t\mathbf{j} = \boxed{2\mathbf{j}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{1^2 + (2t)^2} = \boxed{\sqrt{4t^2 + 1}}.$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = \mathbf{i}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = 2\mathbf{j}$.

- (b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.



6. (a) The velocity, acceleration, and speed are

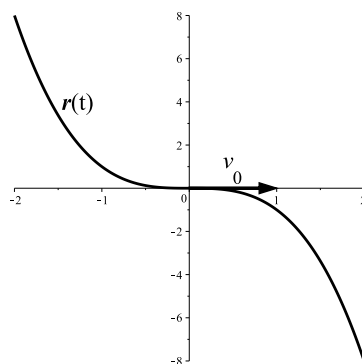
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}t\mathbf{i} - \frac{d}{dt}t^3\mathbf{j} = \boxed{\mathbf{i} - 3t^2\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(\mathbf{i} - 3t^2\mathbf{j}) = \boxed{-6t\mathbf{j}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{1^2 + (-3t^2)^2} = \boxed{\sqrt{9t^4 + 1}}.$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = \mathbf{i}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = \mathbf{0}$.

(b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.

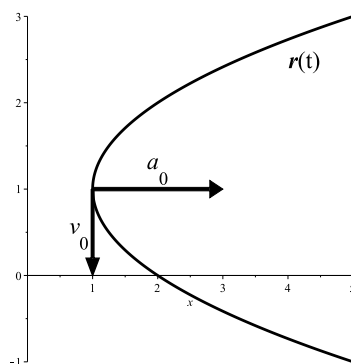


7. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(t^2 + 1)\mathbf{i} - \frac{d}{dt}(1 - t)\mathbf{j} = \boxed{2t\mathbf{i} - \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(2t\mathbf{i} - \mathbf{j}) = \boxed{2\mathbf{i}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + (-1)^2} = \boxed{\sqrt{4t^2 + 1}}.\end{aligned}$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = -\mathbf{j}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = 2\mathbf{i}$.

(b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.



8. (a) The velocity, acceleration, and speed are

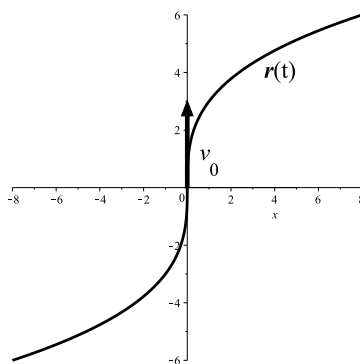
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}t^3\mathbf{i} + \frac{d}{dt}3t\mathbf{j} = \boxed{3t^2\mathbf{i} + 3\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(3t^2\mathbf{i} + 3\mathbf{j}) = \boxed{6t\mathbf{i}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{(3t^2)^2 + 3^2} = \boxed{3\sqrt{t^4 + 1}}.$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = 3\mathbf{j}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = \mathbf{0}$.

- (b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.



9. (a) The velocity, acceleration, and speed are

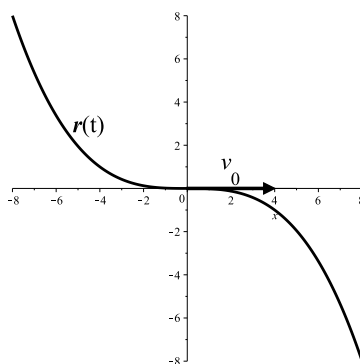
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}4t\mathbf{i} - \frac{d}{dt}t^3\mathbf{j} = \boxed{4\mathbf{i} - 3t^2\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(4\mathbf{i} - 3t^2\mathbf{j}) = \boxed{-6t\mathbf{j}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{4^2 + (-3t^2)^2} = \boxed{\sqrt{9t^4 + 16}}.$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = 4\mathbf{i}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = \mathbf{0}$.

- (b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.



10. (a) The velocity, acceleration, and speed are

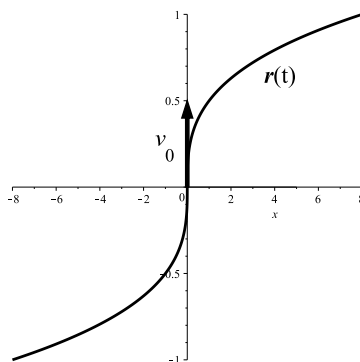
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}t^3\mathbf{i} + \frac{d}{dt}\left(\frac{1}{2}t\right)\mathbf{j} = \boxed{3t^2\mathbf{i} + \frac{1}{2}\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}\left(3t^2\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \boxed{6t\mathbf{i}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{(3t^2)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{36t^4 + 1}{4}} = \boxed{\frac{1}{2}\sqrt{36t^4 + 1}}.$$

At $t = 0$, the velocity is $\mathbf{v}(0) = \mathbf{r}'(0) = \frac{1}{2}\mathbf{j}$ and the acceleration is $\mathbf{a}(0) = \mathbf{r}''(0) = \mathbf{0}$.

- (b) The motion of the particle and the vectors $\mathbf{v}(0)$ and $\mathbf{a}(0)$ are graphed below.



11. The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}t^2\mathbf{i} + \frac{d}{dt}t\mathbf{j} - \frac{d}{dt}3t^3\mathbf{k} = \boxed{2t\mathbf{i} + \mathbf{j} - 9t^2\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(2t\mathbf{i} + \mathbf{j} - 9t^2\mathbf{k}) = \boxed{2\mathbf{i} - 18t\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + 1^2 + (-9t^2)^2} = \boxed{\sqrt{81t^4 + 4t^2 + 1}}.\end{aligned}$$

12. The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(t+1)\mathbf{i} + \frac{d}{dt}6t\mathbf{j} + \frac{d}{dt}t^2\mathbf{k} = \boxed{\mathbf{i} + 6\mathbf{j} + 2t\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(\mathbf{i} + 6\mathbf{j} + 2t\mathbf{k}) = \boxed{2\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{1^2 + 6^2 + (2t)^2} = \boxed{\sqrt{4t^2 + 37}}.\end{aligned}$$

13. The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}\sqrt{16-t^2}\mathbf{i} + \frac{d}{dt}t^2\mathbf{j} + \frac{d}{dt}t\mathbf{k} = \frac{1}{2}(16-t^2)^{-1/2}(-2t)\mathbf{i} + 2t\mathbf{j} + \mathbf{k} = \boxed{-\frac{t}{\sqrt{16-t^2}}\mathbf{i} + 2t\mathbf{j} + \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}\left(\frac{-t}{\sqrt{16-t^2}}\mathbf{i} + 2t\mathbf{j} + \mathbf{k}\right) = \frac{-\sqrt{16-t^2} + t(\frac{1}{2})(16-t^2)^{-1/2}(-2t)}{16-t^2}\mathbf{i} + 2\mathbf{j} \\ &= \frac{t^2 - 16 - t^2}{(16-t^2)^{3/2}}\mathbf{i} + 2\mathbf{j} = \boxed{-\frac{16}{(16-t^2)^{3/2}}\mathbf{i} + 2\mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\left(\frac{-t}{\sqrt{16-t^2}}\right)^2 + (2t)^2 + 1^2} = \boxed{\sqrt{\frac{t^2}{16-t^2} + 4t^2 + 1}}.\end{aligned}$$

14. The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}\left(\frac{1}{t^2}\right)\mathbf{i} + \frac{d}{dt}\left(\frac{t+1}{t^2}\right)\mathbf{j} + \frac{d}{dt}t\mathbf{k} = \frac{d}{dt}t^{-2}\mathbf{i} + \frac{d}{dt}(t^{-1} + t^{-2})\mathbf{j} + \frac{d}{dt}t\mathbf{k} \\ &= -2t^{-3}\mathbf{i} + (-t^{-2} - 2t^{-3})\mathbf{j} + \mathbf{k} = \boxed{-\frac{2}{t^3}\mathbf{i} - \frac{t+2}{t^3}\mathbf{j} + \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}[-2t^{-3}\mathbf{i} + (-t^{-2} - 2t^{-3})\mathbf{j} + \mathbf{k}] = 6t^{-4}\mathbf{i} + (2t^{-3} + 6t^{-4})\mathbf{j} = \boxed{\frac{6}{t^4}\mathbf{i} + \frac{2(t+3)}{t^4}\mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\left(-\frac{2}{t^3}\right)^2 + \left(\frac{t+2}{t^3}\right)^2 + 1^2} = \sqrt{\frac{4}{t^6} + \frac{t^2 + 4t + 4}{t^6} + 1} = \boxed{\frac{\sqrt{t^6 + t^2 + 4t + 8}}{t^3}}.\end{aligned}$$

15. The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}2\cos t\mathbf{i} + \frac{d}{dt}\sin t\mathbf{j} + \frac{d}{dt}t\mathbf{k} = \boxed{-2\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(-2\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) = \boxed{-2\cos t\mathbf{i} - \sin t\mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{(2\sin t)^2 + \cos^2 t + 1} = \sqrt{4\sin^2 t + \cos^2 t + 1} = \sqrt{3\sin^2 t + \sin^2 t + \cos^2 t + 1} \\ &= \boxed{\sqrt{3\sin^2 t + 2}}.\end{aligned}$$

16. The velocity, acceleration, and speed are

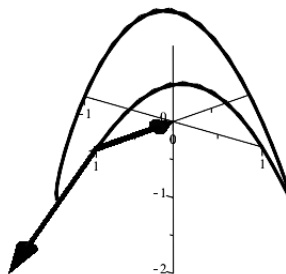
$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}t\mathbf{i} + \frac{d}{dt}\sin(4t)\mathbf{j} + \frac{d}{dt}\cos(4t)\mathbf{k} = \boxed{\mathbf{i} + 4\cos(4t)\mathbf{j} - 4\sin(4t)\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(\mathbf{i} + 4\cos(4t)\mathbf{j} - 4\sin(4t)\mathbf{k}) = \boxed{-16\sin(4t)\mathbf{j} - 16\cos(4t)\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{1^2 + [4\cos(4t)]^2 + [-4\sin(4t)]^2} = \sqrt{1 + 16\cos^2(4t) + 16\sin^2(4t)} = \boxed{\sqrt{17}}.\end{aligned}$$

17. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}\sin t\mathbf{i} + \frac{d}{dt}\cos t\mathbf{j} + \frac{d}{dt}\sin(2t)\mathbf{k} = \boxed{\cos t\mathbf{i} - \sin t\mathbf{j} + 2\cos(2t)\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(\cos t\mathbf{i} - \sin t\mathbf{j} + 2\cos(2t)\mathbf{k}) = \boxed{-\sin t\mathbf{i} - \cos t\mathbf{j} - 4\sin(2t)\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + 4\cos^2(2t)} = \boxed{\sqrt{4\cos^2(2t) + 1}}.\end{aligned}$$

(b) At $t = \frac{\pi}{2}$, the velocity is $\mathbf{v}(\frac{\pi}{2}) = \mathbf{r}'(\frac{\pi}{2}) = \boxed{-\mathbf{j} - 2\mathbf{k}}$, the acceleration is $\mathbf{a}(\frac{\pi}{2}) = \mathbf{r}''(\frac{\pi}{2}) = \boxed{-\mathbf{i}}$, and the speed is $v(\frac{\pi}{2}) = \|\mathbf{v}(\frac{\pi}{2})\| = \boxed{\sqrt{5}}$.

(c) The motion of the particle and the vectors $\mathbf{v}(\frac{\pi}{2})$ and $\mathbf{a}(\frac{\pi}{2})$ are graphed below.

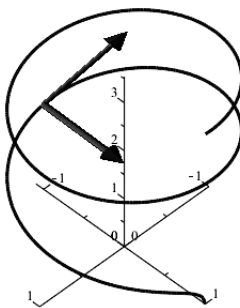


18. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(\sin t)\mathbf{i} + \frac{d}{dt}(\cos t)\mathbf{j} + \frac{d}{dt}(\sqrt{t})\mathbf{k} = \boxed{\cos t\mathbf{i} - \sin t\mathbf{j} + \frac{\sqrt{t}}{2t}\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}\left(\cos t\mathbf{i} - \sin t\mathbf{j} + \frac{\sqrt{t}}{2t}\mathbf{k}\right) = \boxed{-\sin t\mathbf{i} - \cos t\mathbf{j} - \frac{\sqrt{t}}{4t^2}\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + \frac{1}{4t}} = \sqrt{1 + \frac{1}{4t}} = \boxed{\sqrt{\frac{4t+1}{4t}}}.\end{aligned}$$

(b) At $t = \pi$, the velocity is $\mathbf{v}(\pi) = \mathbf{r}'(\pi) = \boxed{-\mathbf{i} + \frac{\sqrt{\pi}}{2\pi}\mathbf{k}}$, the acceleration is $\mathbf{a}(\pi) = \mathbf{r}''(\pi) = \boxed{\mathbf{j} - \frac{\sqrt{\pi}}{4\pi^2}\mathbf{k}}$, and the speed is $v(\pi) = \|\mathbf{v}(\pi)\| = \boxed{\sqrt{\frac{4\pi+1}{4\pi}}}$.

(c) The motion of the particle and the vectors $\mathbf{v}(\pi)$ and $\mathbf{a}(\pi)$ are graphed below.



19. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= e^t \mathbf{i} + e^{-t} \mathbf{j} \\ \mathbf{v}(t) &= \mathbf{r}'(t) = e^t \mathbf{i} - e^{-t} \mathbf{j} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = e^t \mathbf{i} + e^{-t} \mathbf{j} = \mathbf{r}(t)\end{aligned}$$

- (a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{m\mathbf{r}(t)}.$$

- (b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \boxed{\sqrt{e^{2t} + e^{-2t}}}.$$

20. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= e^{2t} \mathbf{i} + e^{-t} \mathbf{j} \\ \mathbf{v}(t) &= \mathbf{r}'(t) = 2e^{2t} \mathbf{i} - e^{-t} \mathbf{j} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = 4e^{2t} \mathbf{i} + e^{-t} \mathbf{j}\end{aligned}$$

- (a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{4me^{2t} \mathbf{i} + me^{-t} \mathbf{j}}.$$

- (b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \boxed{\sqrt{16e^{4t} + e^{-2t}}}.$$

21. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= t\mathbf{i} + e^t \mathbf{j} \\ \mathbf{v}(t) &= \mathbf{r}'(t) = \mathbf{i} + e^t \mathbf{j} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = e^t \mathbf{j}\end{aligned}$$

- (a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{me^t \mathbf{j}}.$$

(b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \sqrt{e^{2t}} = \boxed{e^t}.$$

22. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= t\mathbf{i} + \ln(1+t)\mathbf{j} \\ \mathbf{v}(t) = \mathbf{r}'(t) &= \mathbf{i} + \frac{1}{1+t}\mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= -\frac{1}{(1+t)^2}\mathbf{j}\end{aligned}$$

(a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{-\frac{m}{(1+t)^2}\mathbf{j}}.$$

(b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \boxed{\frac{1}{(1+t)^2}}.$$

23. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= 3\sin(2t)\mathbf{i} + 3\cos(2t)\mathbf{j} \\ \mathbf{v}(t) = \mathbf{r}'(t) &= 6\cos(2t)\mathbf{i} - 6\sin(2t)\mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= -12\sin(2t)\mathbf{i} - 12\cos(2t)\mathbf{j} = -4\mathbf{r}(t)\end{aligned}$$

(a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{-4m\mathbf{r}(t)}.$$

(b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \sqrt{144\sin^2(2t) + 144\cos^2(2t)} = \boxed{12}.$$

24. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= 4\sin t\mathbf{i} - 4\cos t\mathbf{j} \\ \mathbf{v}(t) = \mathbf{r}'(t) &= 4\cos t\mathbf{i} + 4\sin t\mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= -4\sin t\mathbf{i} + 4\cos t\mathbf{j} = -\mathbf{r}(t)\end{aligned}$$

(a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{-m\mathbf{r}(t)}.$$

(b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = \boxed{4}.$$

25. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= 2 \cos t \mathbf{i} - 3 \sin t \mathbf{j} \\ \mathbf{v}(t) = \mathbf{r}'(t) &= -2 \sin t \mathbf{i} - 3 \cos t \mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= -2 \cos t \mathbf{i} + 3 \sin t \mathbf{j} = -\mathbf{r}(t)\end{aligned}$$

- (a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{-m\mathbf{r}(t)}.$$

- (b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = \sqrt{4 \cos^2 t + 9 \sin^2 t} = \sqrt{4 \cos^2 t + 4 \sin^2 t + 5 \sin^2 t} = \boxed{\sqrt{4 + 5 \sin^2 t}}.$$

26. To find the force $\mathbf{F}$ acting on the particle at any time t , we Use Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. We begin by finding the acceleration $\mathbf{a}$ of the particle.

$$\begin{aligned}\mathbf{r}(t) &= -\cos(3t)\mathbf{i} + 2 \sin(3t)\mathbf{j} \\ \mathbf{v}(t) = \mathbf{r}'(t) &= 3 \sin(3t)\mathbf{i} + 6 \cos(3t)\mathbf{j} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= 9 \cos(3t)\mathbf{i} - 18 \sin(3t)\mathbf{j} = -9\mathbf{r}(t)\end{aligned}$$

- (a) By Newton's Law,

$$\mathbf{F}(t) = m\mathbf{a}(t) = \boxed{-9m\mathbf{r}(t)}.$$

- (b) The magnitude of the acceleration is

$$\|\mathbf{a}(t)\| = 9\sqrt{\cos^2(3t) + 4 \sin^2(3t)} = 9\sqrt{\cos^2(3t) + \sin^2(3t) + 3 \sin^2(3t)} = \boxed{9\sqrt{1 + 3 \sin^2(3t)}}.$$

27. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \frac{d}{dt} (2t\mathbf{i} + (t+1)\mathbf{j}) = \boxed{2\mathbf{i} + \mathbf{j}} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (2\mathbf{i} + \mathbf{j}) = \boxed{\mathbf{0}} \\ v(t) = \|\mathbf{r}'(t)\| &= \boxed{\sqrt{5}}.\end{aligned}$$

- (b) Since the speed is constant, $a_{\mathbf{T}} = v'(t) = \boxed{0}$, and since $\mathbf{r}''(t) = \mathbf{0}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{0}$ which implies $\kappa = 0$ and $a_{\mathbf{N}} = \boxed{0}$.

28. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \frac{d}{dt} [(1-3t)\mathbf{i} + 2t\mathbf{j}] = \boxed{-3\mathbf{i} + 2\mathbf{j}} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-3\mathbf{i} + 2\mathbf{j}) = \boxed{\mathbf{0}} \\ v(t) = \|\mathbf{r}'(t)\| &= \sqrt{9+4} = \boxed{\sqrt{13}}.\end{aligned}$$

- (b) Since the speed is constant, $a_{\mathbf{T}} = v'(t) = \boxed{0}$, and since $\mathbf{r}''(t) = \mathbf{0}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{0}$ which implies $\kappa = 0$ and $a_{\mathbf{N}} = \boxed{0}$.

29. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(e^t \mathbf{i} + e^{2t} \mathbf{j}) = \boxed{e^t \mathbf{i} + 2e^{2t} \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt}(e^t \mathbf{i} + 2e^{2t} \mathbf{j}) = \boxed{e^t \mathbf{i} + 4e^{2t} \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{e^{2t} + 4e^{4t}} = \boxed{e^t \sqrt{1 + 4e^{2t}}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(e^{2t} + 4e^{4t})^{-1/2}(2e^{2t} + 16e^{4t}) = \frac{2e^{2t}(1 + 8e^{2t})}{2e^t \sqrt{1 + 4e^{2t}}} = \boxed{\frac{e^t(1 + 8e^{2t})}{\sqrt{1 + 4e^{2t}}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & 2e^{2t} & 0 \\ e^t & 4e^{2t} & 0 \end{vmatrix} = (4e^{3t} - 2e^{3t})\mathbf{k} = 2e^{3t}\mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|2e^{3t}\mathbf{k}\|}{e^t \sqrt{1 + 4e^{2t}}} = \frac{2e^{3t}}{e^t \sqrt{1 + 4e^{2t}}} = \boxed{\frac{2e^{2t}}{\sqrt{1 + 4e^{2t}}}}.$$

30. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(e^{-t} \mathbf{i} + e^{-2t} \mathbf{j}) = \boxed{-e^{-t} \mathbf{i} - 2e^{-2t} \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt}(-e^{-t} \mathbf{i} - 2e^{-2t} \mathbf{j}) = \boxed{e^{-t} \mathbf{i} + 4e^{-2t} \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{e^{-2t} + 4e^{-4t}} = \boxed{e^{-t} \sqrt{1 + 4e^{-2t}}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(e^{-2t} + 4e^{-4t})^{-1/2}(-2e^{-2t} - 16e^{-4t}) = \frac{-e^{-2t}(1 + 8e^{-2t})}{e^{-t} \sqrt{1 + 4e^{-2t}}} = \boxed{\frac{-e^{-t}(1 + 8e^{-2t})}{\sqrt{1 + 4e^{-2t}}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -e^{-t} & -2e^{-2t} & 0 \\ e^{-t} & 4e^{-2t} & 0 \end{vmatrix} = (-4e^{-3t} + 2e^{-3t})\mathbf{k} = -2e^{-3t}\mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\| -2e^{-3t}\mathbf{k} \|}{e^{-t}\sqrt{1+4e^{-2t}}} = \frac{2e^{-3t}}{e^{-t}\sqrt{1+4e^{-2t}}} = \boxed{\frac{2e^{-2t}}{\sqrt{1+4e^{-2t}}}}.$$

31. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(2\sin t\mathbf{i} + \cos t\mathbf{j}) = \boxed{2\cos t\mathbf{i} - \sin t\mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(2\cos t\mathbf{i} - \sin t\mathbf{j}) = \boxed{-2\sin t\mathbf{i} - \cos t\mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{4\cos^2 t + \sin^2 t} = \sqrt{3\cos^2 t + \cos^2 t + \sin^2 t} = \boxed{\sqrt{3\cos^2 t + 1}}.\end{aligned}$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(1 + 3\cos^2 t)^{-1/2}6\cos t(-\sin t) = \boxed{-\frac{3\sin t \cos t}{\sqrt{1 + 3\cos^2 t}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2\kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos t & -\sin t & 0 \\ -2\sin t & -\cos t & 0 \end{vmatrix} = (-2\cos^2 t - 2\sin^2 t)\mathbf{k} = -2\mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\| -2\mathbf{k} \|}{\sqrt{3\cos^2 t + 1}} = \boxed{\frac{2}{\sqrt{3\cos^2 t + 1}}}.$$

32. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt}(2\cos t\mathbf{i} + 3\sin t\mathbf{j}) = \boxed{-2\sin t\mathbf{i} + 3\cos t\mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}(-2\sin t\mathbf{i} + 3\cos t\mathbf{j}) = \boxed{-2\cos t\mathbf{i} - 3\sin t\mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4\sin^2 t + 4\cos^2 t + 5\cos^2 t} = \boxed{\sqrt{4 + 5\cos^2 t}}.\end{aligned}$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(4 + 5\cos^2 t)^{-1/2}5\cos t(-2\sin t) = \boxed{-\frac{5\sin t \cos t}{\sqrt{4 + 5\cos^2 t}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2\kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin t & 3 \cos t & 0 \\ -2 \cos t & -3 \sin t & 0 \end{vmatrix} = (6 \sin^2 t + 6 \cos^2 t) \mathbf{k} = 6 \mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|6\mathbf{k}\|}{\sqrt{4+5\cos^2 t}} = \boxed{\frac{6}{\sqrt{4+5\cos^2 t}}}.$$

33. (a) The velocity, acceleration, and speed are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt} [(1-3t)\mathbf{i} + 2t\mathbf{j} - (5+t)\mathbf{k}] = \boxed{-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \boxed{\mathbf{0}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{9+4+1} = \boxed{\sqrt{14}}.$$

- (b) Since the speed is constant, $a_{\mathbf{T}} = v'(t) = \boxed{0}$, and since $\mathbf{r}''(t) = \mathbf{0}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{0}$ which implies $\kappa = 0$ and $a_{\mathbf{N}} = \boxed{0}$.

34. (a) The velocity, acceleration, and speed are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt} [(2+t)\mathbf{i} + (2-t)\mathbf{j} + 3t\mathbf{k}] = \boxed{\mathbf{i} - \mathbf{j} + 3\mathbf{k}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \boxed{\mathbf{0}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{1+1+9} = \boxed{\sqrt{11}}.$$

- (b) Since the speed is constant, $a_{\mathbf{T}} = v'(t) = \boxed{0}$, and since $\mathbf{r}''(t) = \mathbf{0}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{0}$ which implies $\kappa = 0$ and $a_{\mathbf{N}} = \boxed{0}$.

35. (a) The velocity, acceleration, and speed are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt} (t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \boxed{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) = \boxed{2\mathbf{j} + 6t\mathbf{k}}$$

$$v(t) = \|\mathbf{r}'(t)\| = \boxed{\sqrt{1+4t^2+9t^4}}.$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{8t + 36t^3}{2\sqrt{1+4t^2+9t^4}} = \boxed{\frac{18t^3 + 4t}{\sqrt{1+4t^2+9t^4}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (12t^2 - 6t^2)\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k} = 2(3t^2\mathbf{i} - 3t\mathbf{j} + \mathbf{k})$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|2(3t^2\mathbf{i} - 3t\mathbf{j} + \mathbf{k})\|}{\sqrt{1+4t^2+9t^4}} = \boxed{\frac{2\sqrt{9t^4+9t^2+1}}{\sqrt{1+4t^2+9t^4}}}.$$

36. (a) The velocity, acceleration, and speed are

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} \left(\frac{t^2}{t+1}\mathbf{i} + \frac{1}{t+1}\mathbf{j} + \frac{t}{t+1}\mathbf{k} \right) = \frac{2t(t+1)-t^2}{(t+1)^2}\mathbf{i} - \frac{1}{(t+1)^2}\mathbf{j} + \frac{t+1-t}{(t+1)^2}\mathbf{k} \\ &= \boxed{\frac{t(t+2)}{(t+1)^2}\mathbf{i} - \frac{1}{(t+1)^2}\mathbf{j} + \frac{1}{(t+1)^2}\mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt} \left[\frac{t(t+2)}{(t+1)^2}\mathbf{i} - \frac{1}{(t+1)^2}\mathbf{j} + \frac{1}{(t+1)^2}\mathbf{k} \right] \\ &= \frac{(2t+2)(t+1)^2 - t(t+2)(2)(t+1)}{(t+1)^4}\mathbf{i} + \frac{2}{(t+1)^3}\mathbf{j} - \frac{2}{(t+1)^3}\mathbf{k} \\ &= \boxed{\frac{2}{(t+1)^3}\mathbf{i} + \frac{2}{(t+1)^3}\mathbf{j} - \frac{2}{(t+1)^3}\mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\frac{t^4+4t^3+4t^2+1+1}{(t+1)^4}} = \boxed{\frac{\sqrt{t^4+4t^3+4t^2+2}}{(t+1)^2}}. \end{aligned}$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{\frac{1}{2}(t^4+4t^3+4t^2+2)^{-1/2}(4t^3+12t^2+8t)(t+1)^2 - (t^4+4t^3+4t^2+2)^{1/2}(2)(t+1)}{(t+1)^4}$$

To simplify, we multiply the numerator and denominator by $(t^4+4t^3+4t^2+2)^{1/2}$ and factor out a $t+1$ term.

$$\begin{aligned} a_{\mathbf{T}} &= \frac{(2t^3+6t^2+4t)(t+1) - 2(t^4+4t^3+4t^2+2)}{(t+1)^3\sqrt{t^4+4t^3+4t^2+2}} = \frac{2t^4+8t^3+10t^2+4t-2t^4-8t^3-8t^2-4}{(t+1)^3\sqrt{t^4+4t^3+4t^2+2}} \\ &= \boxed{\frac{2(t^2+2t-2)}{(t+1)^3\sqrt{t^4+4t^3+4t^2+2}}}. \end{aligned}$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2\kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t^2+2t}{(t+1)^2} & \frac{-1}{(t+1)^2} & \frac{1}{(t+1)^2} \\ \frac{2}{(t+1)^3} & \frac{2}{(t+1)^3} & \frac{-2}{(t+1)^3} \end{vmatrix} = \frac{2}{(t+1)^5} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2+2t & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \frac{2}{(t+1)^5} [(t^2+2t+1)\mathbf{j} + (t^2+2t+1)\mathbf{k}] = \frac{2}{(t+1)^3}\mathbf{j} + \frac{2}{(t+1)^3}\mathbf{k} \end{aligned}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\left\| \frac{2}{(t+1)^3} \mathbf{j} + \frac{2}{(t+1)^3} \mathbf{k} \right\|}{\frac{\sqrt{t^4+4t^3+4t^2+2}}{(t+1)^2}} = \frac{\frac{2\sqrt{2}}{(t+1)^3}}{\frac{\sqrt{t^4+4t^3+4t^2+2}}{(t+1)^2}} = \boxed{\frac{2\sqrt{2}}{(t+1)\sqrt{t^4+4t^3+4t^2+2}}}.$$

37. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (3\mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k}) = \boxed{-\sin t \mathbf{j} + \cos t \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-\sin t \mathbf{j} + \cos t \mathbf{k}) = \boxed{-\cos t \mathbf{j} - \sin t \mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = \boxed{1}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \boxed{0}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t) \mathbf{i} = \mathbf{i}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{i}\|}{1} = \boxed{1}.$$

38. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}) = \boxed{\mathbf{i} + \cos t \mathbf{j} - \sin t \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (\mathbf{i} + \cos t \mathbf{j} - \sin t \mathbf{k}) = \boxed{-\sin t \mathbf{j} - \cos t \mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{1^2 + \cos^2 t + \sin^2 t} = \boxed{\sqrt{2}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \boxed{0}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -\mathbf{i} + \cos t \mathbf{j} - \sin t \mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|-\mathbf{i} + \cos t \mathbf{j} - \sin t \mathbf{k}\|}{\sqrt{2}} = \frac{\sqrt{1^2 + \cos^2 t + \sin^2 t}}{\sqrt{2}} = \boxed{1}.$$

39. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (\ln t \mathbf{i} + \sqrt{t} \mathbf{j} + t^{3/2} \mathbf{k}) = \boxed{\frac{1}{t} \mathbf{i} + \frac{1}{2\sqrt{t}} \mathbf{j} + \frac{3}{2} \sqrt{t} \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} \left(\frac{1}{t} \mathbf{i} + \frac{1}{2\sqrt{t}} \mathbf{j} + \frac{3}{2} \sqrt{t} \mathbf{k} \right) = \boxed{-\frac{1}{t^2} \mathbf{i} - \frac{1}{4t^{3/2}} \mathbf{j} + \frac{3}{4\sqrt{t}} \mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t^2} + \frac{1}{4t} + \frac{9}{4}t} = \sqrt{\frac{4+t+9t^3}{4t^2}} = \boxed{\frac{\sqrt{9t^3+t+4}}{2t}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{2t(\frac{1}{2})(9t^3+t+4)^{-1/2}(27t^2+1) - (9t^3+t+4)^{1/2}(2)}{4t^2}$$

We simplify by multiplying the numerator and the denominator by $(9t^3+t+4)^{1/2}$.

$$a_{\mathbf{T}} = \frac{27t^3+t-18t^3-2t-8}{4t^2\sqrt{9t^3+t+4}} = \boxed{\frac{9t^3-t-8}{4t^2\sqrt{9t^3+t+4}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{t} & \frac{1}{2\sqrt{t}} & \frac{3}{2}\sqrt{t} \\ -\frac{1}{t^2} & -\frac{1}{4t^{3/2}} & \frac{3}{4\sqrt{t}} \end{vmatrix} = \frac{3}{4t} \mathbf{i} - \frac{9}{4t^{3/2}} \mathbf{j} + \frac{1}{4t^{5/2}} \mathbf{k}$$

so

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{\frac{9}{16t^2} + \frac{81}{16t^3} + \frac{1}{16t^5}} = \sqrt{\frac{9t^3+81t^2+1}{16t^5}}$$

and

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{\sqrt{9t^3+81t^2+1}}{4t^{5/2}}}{\frac{\sqrt{9t^3+t+4}}{2t}} = \boxed{\frac{\sqrt{9t^3+81t^2+1}}{2t^{3/2}\sqrt{9t^3+t+4}}}.$$

40. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (\cos(2t) \mathbf{i} + \sin(2t) \mathbf{j} - 5\mathbf{k}) = \boxed{-2\sin(2t) \mathbf{i} + 2\cos(2t) \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-2\sin(2t) \mathbf{i} + 2\cos(2t) \mathbf{j}) = \boxed{-4\cos(2t) \mathbf{i} - 4\sin(2t) \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = \boxed{2}.\end{aligned}$$

- (b) Since the speed is constant, $a_{\mathbf{T}} = v'(t) = \boxed{0}$. To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin(2t) & 2\cos(2t) & 0 \\ -4\cos(2t) & -4\sin(2t) & 0 \end{vmatrix} = [8\sin^2(2t) + 8\cos^2(2t)]\mathbf{k} = 8\mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{8}{2} = \boxed{4}.$$

41. (a) The velocity, acceleration, and speed are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}(e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \mathbf{k}) = \boxed{(e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k}}$$

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \frac{d}{dt}[(e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k}] \\ &= (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j} + e^t \mathbf{k} \\ &= \boxed{-2e^t \sin t \mathbf{i} + 2e^t \cos t \mathbf{j} + e^t \mathbf{k}} \end{aligned}$$

$$\begin{aligned} v(t) &= \|\mathbf{r}'(t)\| \\ &= \sqrt{e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t}} \\ &= \sqrt{e^{2t}(\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t + 1)} = \boxed{e^t \sqrt{3}}. \end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \boxed{e^t \sqrt{3}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{vmatrix} \\ &= (e^{2t} \sin t - e^{2t} \cos t)\mathbf{i} - (e^{2t} \cos t + e^{2t} \sin t)\mathbf{j} + (2e^{2t} \cos^2 t + 2e^{2t} \sin^2 t)\mathbf{k} \\ &= [e^{2t}(\sin t - \cos t)]\mathbf{i} - [e^{2t}(\cos t + \sin t)]\mathbf{j} + 2e^{2t}\mathbf{k} \end{aligned}$$

so

$$\begin{aligned} a_{\mathbf{N}} &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|[e^{2t}(\sin t - \cos t)]\mathbf{i} - [e^{2t}(\cos t + \sin t)]\mathbf{j} + 2e^{2t}\mathbf{k}\|}{e^t \sqrt{3}} \\ &= \frac{e^{2t} \sqrt{\sin^2 t - 2 \sin t \cos t + \cos^2 t + \cos^2 t + 2 \sin t \cos t + \sin^2 t + 4}}{e^t \sqrt{3}} \\ &= \frac{e^{2t} \sqrt{6}}{e^t \sqrt{3}} = \boxed{e^t \sqrt{2}}. \end{aligned}$$

42. (a) The velocity, acceleration, and speed are

$$\begin{aligned}
 \mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (e^{-t} \cos t \mathbf{i} + e^{-t} \sin t \mathbf{j} - e^{-t} \mathbf{k}) \\
 &= \boxed{(-e^{-t} \cos t - e^{-t} \sin t) \mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t) \mathbf{j} + e^{-t} \mathbf{k}} \\
 \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} [(-e^{-t} \cos t - e^{-t} \sin t) \mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t) \mathbf{j} + e^{-t} \mathbf{k}] \\
 &= (e^{-t} \cos t + e^{-t} \sin t + e^{-t} \sin t - e^{-t} \cos t) \mathbf{i} + (e^{-t} \sin t - e^{-t} \cos t - e^{-t} \cos t - e^{-t} \sin t) \mathbf{j} - e^{-t} \mathbf{k} \\
 &= \boxed{2e^{-t} \sin t \mathbf{i} - 2e^{-t} \cos t \mathbf{j} - e^{-t} \mathbf{k}} \\
 v(t) &= \|\mathbf{r}'(t)\| \\
 &= \sqrt{e^{-2t} \cos^2 t + 2e^{-2t} \sin t \cos t + e^{-2t} \sin^2 t + e^{-2t} \sin^2 t - 2e^{-2t} \sin t \cos t + e^{-2t} \cos^2 t + e^{-2t}} \\
 &= \sqrt{e^{-2t} (\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t + 1)} = \boxed{e^{-t} \sqrt{2}}.
 \end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \boxed{-e^{-t} \sqrt{2}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\begin{aligned}
 \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -e^{-t} \cos t - e^{-t} \sin t & -e^{-t} \sin t + e^{-t} \cos t & e^{-t} \\ 2e^{-t} \sin t & -2e^{-t} \cos t & -e^{-t} \end{vmatrix} \\
 &= (e^{-2t} \sin t + e^{-2t} \cos t) \mathbf{i} - (e^{-2t} \cos t - e^{-2t} \sin t) \mathbf{j} + (2e^{-2t} \sin^2 t - 2e^{-2t} \cos^2 t) \mathbf{k} \\
 &= [e^{-2t} (\sin t + \cos t)] \mathbf{i} - [e^{-2t} (\cos t - \sin t)] \mathbf{j} + 2e^{-2t} \mathbf{k}
 \end{aligned}$$

so

$$\begin{aligned}
 a_{\mathbf{N}} &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|[e^{-2t} (\sin t + \cos t)] \mathbf{i} - [e^{-2t} (\cos t - \sin t)] \mathbf{j} + 2e^{-2t} \mathbf{k}\|}{e^{-t} \sqrt{2}} \\
 &= \frac{e^{-2t} \sqrt{\sin^2 t + 2 \sin t \cos t + \cos^2 t + \cos^2 t - 2 \sin t \cos t + \sin^2 t + 4}}{e^{-t} \sqrt{2}} \\
 &= \frac{e^{-2t} \sqrt{6}}{e^{-t} \sqrt{2}} = \boxed{e^{-t} \sqrt{3}}.
 \end{aligned}$$

43. (a) The velocity, acceleration, and speed are

$$\begin{aligned}
 \mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (a \cos t \mathbf{i} + b \sin t + ct \mathbf{k}) = \boxed{-a \sin t \mathbf{i} + b \cos t \mathbf{j} + c \mathbf{k}} \\
 \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-a \sin t \mathbf{i} + b \cos t \mathbf{j} + c \mathbf{k}) = \boxed{-a \cos t \mathbf{i} - b \sin t \mathbf{j}} \\
 v(t) &= \|\mathbf{r}'(t)\| = \boxed{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2}}.
 \end{aligned}$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{1}{2}(a^2 \sin^2 t + b^2 \cos^2 t + c^2)^{-1/2} (2a^2 \sin t \cos t - 2b^2 \sin t \cos t)$$

$$= \frac{(a^2 - b^2) \sin t \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & b \cos t & c \\ -a \cos t & -b \sin t & 0 \end{vmatrix} = bc \sin t \mathbf{i} - ac \cos t \mathbf{j} + (ab \sin^2 t + ab \cos^2 t) \mathbf{k}$$

$$= bc \sin t \mathbf{i} - ac \cos t \mathbf{j} + ab \mathbf{k}$$

and

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{b^2 c^2 \sin^2 t + a^2 c^2 \cos^2 t + a^2 b^2}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \sqrt{\frac{b^2 c^2 \sin^2 t + a^2 c^2 \cos^2 t + a^2 b^2}{a^2 \sin^2 t + b^2 \cos^2 t + c^2}}.$$

44. (a) The velocity, acceleration, and speed are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt} (\cosh t \mathbf{i} + \sinh t \mathbf{j} + t \mathbf{k}) = \sinh t \mathbf{i} + \cosh t \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (\sinh t \mathbf{i} + \cosh t \mathbf{j} + \mathbf{k}) = \cosh t \mathbf{i} + \sinh t \mathbf{j}$$

Since $\cosh^2 t - \sinh^2 t = 1$, $\sinh^2 t + 1 = \cosh^2 t$, so

$$v(t) = \|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2 \cosh^2 t} = \cosh t \sqrt{2}.$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \sinh t \sqrt{2}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix} = -\sinh t \mathbf{i} + \cosh t \mathbf{j} + (\sinh^2 t - \cosh^2 t) \mathbf{k} = -\sinh t \mathbf{i} + \cosh t \mathbf{j} - \mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|-\sinh t \mathbf{i} + \cosh t \mathbf{j} - \mathbf{k}\|}{\cosh t \sqrt{2}} = \frac{\sqrt{\sinh^2 t + \cosh^2 t + 1}}{\cosh t \sqrt{2}} = \frac{\sqrt{2 \cosh^2 t}}{\cosh t \sqrt{2}} = 1.$$

Applications and Extensions

45. The velocity and acceleration of a particle moving on the cycloid are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \boxed{(\pi - \pi \cos(\pi t))\mathbf{i} + \pi \sin(\pi t)\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \boxed{\pi^2 \sin(\pi t)\mathbf{i} + \pi^2 \cos(\pi t)\mathbf{j}}.$$

46. (a) The velocity and acceleration of a particle moving on the parabola are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \boxed{(2t - 2)\mathbf{i} + \mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \boxed{2\mathbf{i}}.$$

- (b) The speed of the particle is $v(t) = \|\mathbf{r}'(t)\| = \sqrt{4t^2 - 8t + 4 + 1} = \sqrt{4t^2 - 8t + 5}$. The speed $v(t) = \sqrt{4t^2 - 8t + 5} = 0$ when $4t^2 - 8t + 5 = 0$. Using the quadratic formula,

$$t = \frac{8 \pm \sqrt{64 - 4(4)(5)}}{2(4)} = \frac{8 \pm \sqrt{-16}}{8},$$

so the speed of the particle is never 0.

47. We parametrize the equation $y = 3x^2 - x^3$. We choose $x(t) = \frac{1}{3}t$ because we know the horizontal component of the velocity is equal to $\frac{1}{3}$. Then

$$\mathbf{r}(t) = \frac{1}{3}t\mathbf{i} + \left(\frac{1}{3}t^2 - \frac{1}{27}t^3\right)\mathbf{j}.$$

The velocity and acceleration vectors are

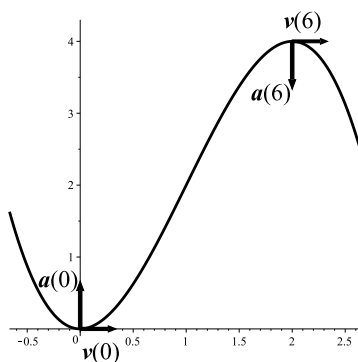
$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{1}{3}\mathbf{i} + \left(\frac{2}{3}t - \frac{1}{9}t^2\right)\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = \left(\frac{2}{3} - \frac{2}{9}t\right)\mathbf{j}$$

The velocity $\mathbf{v}$ is horizontal when $\frac{2}{3}t - \frac{1}{9}t^2 = \frac{1}{9}t(6 - t) = 0$ or at $t = 0$ and $t = 6$. We find

$$\mathbf{v}(0) = \frac{1}{3}\mathbf{i} \quad \boxed{\mathbf{a}(0) = \frac{2}{3}\mathbf{j}} \quad \mathbf{v}(6) = \frac{1}{3}\mathbf{i} \quad \boxed{\mathbf{a}(6) = -\frac{2}{3}\mathbf{j}}.$$

The graph below shows the motion of the particle in addition to $\mathbf{v}(0)$, $\mathbf{a}(0)$, $\mathbf{v}(0)$, and $\mathbf{v}(6)$.



48. Let the speed of the particle be $\|\mathbf{v}(t)\| = c$. Then

$$\mathbf{v}(t) \cdot \mathbf{v}(t) = \|\mathbf{v}(t)\|^2 = c^2.$$

We can differentiate this equation.

$$\begin{aligned}\frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{v}(t)) &= \frac{d}{dt}(c^2) \\ \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) &= 0 \\ 2\mathbf{v}(t) \cdot \mathbf{v}'(t) &= 0\end{aligned}$$

which is equivalent to

$$2\mathbf{v}(t) \cdot \mathbf{a}(t) = 0.$$

Since $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, the velocity and acceleration vectors are orthogonal. ■

49. By definition, $a_{\mathbf{N}} = [v(t)]^2 \kappa$ and $\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$. At a point of inflection, $f''(x) = 0$, so $\kappa = 0$ which makes $a_{\mathbf{N}} = 0$. ■
50. (a) If $\mathbf{r}(t) = e^t \mathbf{i} + e^{-t} \mathbf{j}$, then $\mathbf{v}(t) = e^t \mathbf{i} - e^{-t} \mathbf{j}$ and $\mathbf{a}(t) = e^t \mathbf{i} + e^{-t} \mathbf{j}$. Then $\mathbf{a}(t) = \mathbf{r}(t)$, and since $\mathbf{r}(t)$ is a set of vectors that begin at the origin and point away from it, $\mathbf{a}(t)$ must always point away from the origin, and consequently $\mathbf{F} = m\mathbf{a}$ must also be directed away from the origin.
- (b) We found $\mathbf{v}(t) = e^t \mathbf{i} - e^{-t} \mathbf{j}$, so $v(t) = \|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$. The minimum speed of the particle must occur at a critical point.

$$v'(t) = \frac{1}{2}(e^{2t} + e^{-2t})^{-1/2}(2e^{2t} - 2e^{-2t}) = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$$

The minimum speed of the particle occurs when $v'(t) = 0$ or at $\boxed{t=0}$, and the speed at that time $v(0) = \sqrt{1+1} = \boxed{\sqrt{2}}$.

- (c) The tangential component of acceleration $a_{\mathbf{T}} = v'(t) = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$, so $a_{\mathbf{T}}(0) = \boxed{0}$. To find the normal component of acceleration, we find $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & 0 \\ e^t & e^{-t} & 0 \end{vmatrix} = (1+1)\mathbf{k} = 2\mathbf{k}$$

Then $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 2$ and

$$a_{\mathbf{N}} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}} \quad \text{so} \quad a_{\mathbf{N}}(0) = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}.$$

- (d) We were considering the time at which the minimum speed occurred, so $a_{\mathbf{T}} = v'(t) = 0$. At $t = 0$, $\mathbf{a}(0) = \mathbf{i} + \mathbf{j}$, so $\|\mathbf{a}(0)\| = \sqrt{2}$. If the tangential component of the acceleration is 0, then $\|\mathbf{a}(t)\| = a_{\mathbf{N}} = \sqrt{2}$.
51. If $y = \frac{1}{2}x^2$, then $y' = x$ and $y'' = 1$. Since $v(t) = v_0 = 2$, $v'(t) = a_{\mathbf{T}} = \boxed{0}$. To find $a_{\mathbf{N}}$, we can use $a_{\mathbf{N}} = [v(t)]^2 \kappa$ where $\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$.

$$\kappa = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}} \quad \text{and} \quad [v(t)]^2 = 2^2 = 4$$

so

$$a_{\mathbf{N}} = \boxed{\frac{4}{(1+x^2)^{3/2}}}.$$

52. Given $\mathbf{r}(t) = \alpha \cosh t \mathbf{i} + \beta \sinh t \mathbf{j}$, $\mathbf{v}(t) = \alpha \sinh t \mathbf{i} + \beta \cosh t \mathbf{j}$ and $\mathbf{a}(t) = \alpha \sinh t \mathbf{i} + \beta \cosh t \mathbf{j} = \mathbf{r}(t)$. Then $\mathbf{F} = m\mathbf{a} = m\mathbf{r}$. Since $\mathbf{F}$ is a scalar multiple of $\mathbf{r}$, the force acting on the particle is in the direction of $\mathbf{r}$. ■

53. Using Newton's Second Law of Motion with circular motion, $\|F\| = m \left(\frac{v_0^2}{R} \right)$.

- (a) The speed of the motorcycle is increased by 10% from 120 km/h to 132 km/h. At a speed of 120 km/h, $\|\mathbf{F}\| \approx 1667$ N. With $v_0 = 132$ km/h,

$$\|\mathbf{F}\| = 150 \text{ kg} \left[\frac{(132 \text{ km/h})^2}{100 \text{ m}} \right] \left(\frac{1000 \text{ m}}{1 \text{ km}} \right)^2 \left(\frac{1 \text{ hr}}{3600 \text{ s}} \right)^2 \approx 2017 \text{ N}.$$

The frictional force needs to increase by $2017 \text{ N} - 1667 \text{ N} = \boxed{350 \text{ N}}$.

- (b) If the radius of the track is halved from 100 m to 50 m and we wish to keep the frictional force at 1667 N, the speed v can be found using

$$150 \text{ kg} \left[\frac{(v \text{ km/h})^2}{50 \text{ m}} \right] \left(\frac{1000 \text{ m}}{1 \text{ km}} \right)^2 \left(\frac{1 \text{ hr}}{3600 \text{ s}} \right)^2 = 1667$$

or

$$\frac{25}{108} v^2 = 1667$$

which means $v \approx 85$ km/h. The motorcycle should be driven

$$120 \text{ km/h} - 85 \text{ km/h} = \boxed{35 \text{ km/h more slowly}}.$$

54. Using Newton's Second Law of Motion with circular motion, $\|F\| = m \left(\frac{v_0^2}{R} \right)$.

$$\|\mathbf{F}\| = 1000 \text{ kg} \left[\frac{(200 \text{ km/h})^2}{75 \text{ m}} \right] \left(\frac{1000 \text{ m}}{1 \text{ km}} \right)^2 \left(\frac{1 \text{ hr}}{3600 \text{ s}} \right)^2 \approx \boxed{41152 \text{ N}}.$$

55. Using Newton's Second Law of Motion with circular motion, $\|F\| = m \left(\frac{v_0^2}{R} \right)$. If the circumference of the Large Hadron Collider is 26,659 m, then its radius is

$$R = \frac{26659 \text{ m}}{2\pi} \approx 4243 \text{ m}$$

$$\|\mathbf{F}\| = m \left[\frac{(299792455.3 \text{ m/s})^2}{4243 \text{ m}} \right] \approx \boxed{2.118 \times 10^{13} m \text{ N}}$$

where m is the mass of a proton in kg.

56. Using Newton's Second Law of Motion with circular motion, $\|F\| = m \left(\frac{v_0^2}{R} \right)$. If the radius of the route is 1 km = 1000 m, then

$$\|\mathbf{F}\| = m \left[\frac{(280000 \text{ km/s})^2}{1000 \text{ m}} \right] \left(\frac{1000 \text{ m}}{1 \text{ km}} \right)^2 \approx \boxed{7.84 \times 10^{13} m \text{ N}}$$

where m is the mass of a proton in kg.

57. If $\mathbf{r}(t) = 4 \cos t \mathbf{i} - 2 \cos(2t) \mathbf{j}$, then the velocity, acceleration, and magnitude of the acceleration are

$$\mathbf{v}(t) = -4 \sin t \mathbf{i} + 4 \sin(2t) \mathbf{j}$$

$$\mathbf{a}(t) = -4 \cos t \mathbf{i} + 8 \cos(2t) \mathbf{j}$$

$$\|\mathbf{a}(t)\| = \sqrt{16 \cos^2 t + 64 \cos^2(2t)} = 4\sqrt{\cos^2 t + 4 \cos^2(2t)}$$

so $\|\mathbf{F}(t)\| = m\|\mathbf{a}(t)\| = 4m\sqrt{\cos^2 t + 4\cos^2(2t)}$. Using the identity $\cos(2t) = 2\cos^2 t - 1$, $4\cos^2(2t) = 4(2\cos^2 t - 1)^2 = 4(4\cos^4 t - 4\cos^2 t + 1)$, so letting $F = \|\mathbf{F}\|$,

$$\|\mathbf{F}\| = F = 4m\sqrt{\cos^2 t + 16\cos^4 t - 16\cos^2 t + 4} = 4m\sqrt{16\cos^4 t - 15\cos^2 t + 4}$$

The maximum force will occur at a critical point.

$$F'(t) = 2m[16\cos^4 t - 15\cos^2 t + 4]^{-1/2}[-64\cos^3 t \sin t + 30\cos t \sin t] = \frac{4m \sin t \cos t (-32\cos^2 t + 15)}{\sqrt{16\cos^4 t - 15\cos^2 t + 4}}$$

$F'(t) = 0$ at $t = n\pi, \frac{(2n+1)\pi}{2}, \cos^{-1}\left(\frac{\sqrt{30}}{8}\right) + n\pi, \cos^{-1}\left(-\frac{\sqrt{30}}{8}\right) + n\pi$, but $F'(t)$ only changes from positive to negative at $t = n\pi$ and $t = \frac{(2n+1)\pi}{2}$.

$$F(n\pi) = 4m\sqrt{1+4} = 4m\sqrt{5} \quad \text{and} \quad F\left[\frac{(2n+1)\pi}{2}\right] = 4m\sqrt{4} = 8m.$$

The maximum magnitude of force is $F = \boxed{4\sqrt{5}m}$.

58. For $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - \frac{1}{2}gt^2]\mathbf{j}$, the velocity, acceleration, and speed of the golf ball are

$$\mathbf{v}(t) = v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - gt)\mathbf{j}$$

$$\mathbf{a}(t) = -g\mathbf{j}$$

$$v(t) = \sqrt{v_0^2 - 2v_0 \sin \theta gt + g^2 t^2}$$

(a) The highest point on the ball's trajectory occurs when $v_0 \sin \theta - gt = 0$ or at $t = \frac{v_0 \sin \theta}{g}$. To find the curvature κ , we need $\mathbf{r}' \times \mathbf{r}'' = \mathbf{v} \times \mathbf{a}$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_0 \cos \theta & v_0 \sin \theta - gt & 0 \\ 0 & -g & 0 \end{vmatrix} = -gv_0 \cos \theta \mathbf{k}$$

Then

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{gv_0 \cos \theta}{(v_0^2 - 2v_0 \sin \theta gt + g^2 t^2)^{3/2}}$$

and

$$\begin{aligned} \kappa\left(\frac{v_0 \sin \theta}{g}\right) &= \frac{gv_0 \cos \theta}{\left[v_0^2 - 2v_0 \sin \theta g\left(\frac{v_0 \sin \theta}{g}\right) + g^2\left(\frac{v_0 \sin \theta}{g}\right)^2\right]^{3/2}} \\ &= \frac{gv_0 \cos \theta}{(v_0^2 - v_0^2 \sin^2 \theta)^{3/2}} = \frac{gv_0 \cos \theta}{(v_0^2 \cos^2 \theta)^{3/2}} = \boxed{\frac{g}{v_0^2 \cos^2 \theta}}. \end{aligned}$$

(b) Since

$$a_{\mathbf{T}} = v'(t) = \frac{-v_0 g \sin \theta + g^2 t}{\sqrt{v_0^2 - 2v_0 \sin \theta gt + g^2 t^2}},$$

$$a_{\mathbf{T}}\left(\frac{v_0 \sin \theta}{g}\right) = \frac{-v_0 g \sin \theta + g^2\left(\frac{v_0 \sin \theta}{g}\right)}{\sqrt{v_0^2 - 2v_0 \sin \theta g\left(\frac{v_0 \sin \theta}{g}\right) + g^2\left(\frac{v_0 \sin \theta}{g}\right)^2}} = \boxed{0}.$$

Then

$$\begin{aligned} a_{\mathbf{N}} &= [v(t)]^2 \kappa = (v_0^2 - 2v_0 \sin \theta gt + g^2 t^2) \left[\frac{gv_0 \cos \theta}{(v_0^2 - 2v_0 \sin \theta gt + g^2 t^2)^{3/2}} \right] \\ &= \frac{gv_0 \cos \theta}{(v_0^2 - 2v_0 \sin \theta gt + g^2 t^2)^{1/2}} \end{aligned}$$

and

$$\begin{aligned} a_{\mathbf{N}} \left(\frac{v_0 \sin \theta}{g} \right) &= \frac{gv_0 \cos \theta}{\left[v_0^2 - 2v_0 \sin \theta g \left(\frac{v_0 \sin \theta}{g} \right) + g^2 \left(\frac{v_0 \sin \theta}{g} \right)^2 \right]^{1/2}} \\ &= \frac{gv_0 \cos \theta}{(v_0^2 - v_0^2 \sin^2 \theta)^{1/2}} = \frac{gv_0 \cos \theta}{v_0 \cos \theta} = \boxed{g}. \end{aligned}$$

These components are physically reasonable because at the height of the trajectory, the rate of change of speed is 0, so $a_{\mathbf{T}} = 0$. All acceleration then points in the direction of $a_{\mathbf{N}}$, meaning $a_{\mathbf{N}} = g$.

59. We are given $\mathbf{r}(t) = \frac{1-t^2}{1+t^2}\mathbf{i} + \frac{2t}{1+t^2}\mathbf{j}$.

(a) Then

$$\begin{aligned} \mathbf{v}(t) = \mathbf{r}'(t) &= \frac{-2t(1+t^2) - (1-t^2)(2t)}{(1+t^2)^2}\mathbf{i} + \frac{2(1+t^2) - 2t(2t)}{(1+t^2)^2}\mathbf{j} \\ &= \boxed{-\frac{4t}{(1+t^2)^2}\mathbf{i} + \frac{2(1-t^2)}{(1+t^2)^2}\mathbf{j}}. \end{aligned}$$

(b) $\boxed{\text{No}}$, the particle is never at rest. For the particle to be at rest, both components of $\mathbf{v}$ must be zero. The x -component of the velocity vector is zero for $t = 0$, and the y -component of the velocity vector is zero for $t = \pm 1$. Since there is no t such that both components are zero at the same time, the particle is never at rest.

(c) As $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} \left(\frac{1-t^2}{1+t^2}\mathbf{i} + \frac{2t}{1+t^2}\mathbf{j} \right) = \lim_{t \rightarrow \infty} \left(\frac{1-t^2}{1+t^2} \right)\mathbf{i} + \lim_{t \rightarrow \infty} \left(\frac{2t}{1+t^2} \right)\mathbf{j} = -\mathbf{i},$$

so the particle approaches the point $\boxed{(-1, 0)}$ as t increases without bound.

60. (a) We can express $\mathbf{r}(t)$ in parametric form with $x(t) = e^{-t} \cos t$ and $y(t) = e^{-t} \sin t$, so

$$\frac{x}{e^{-t}} = \cos t \quad \frac{y}{e^{-t}} = \sin t,$$

$$\left(\frac{x}{e^{-t}} \right)^2 + \left(\frac{y}{e^{-t}} \right)^2 = 1$$

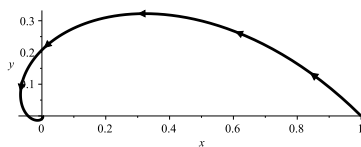
or

$$x^2 + y^2 = e^{-2t}.$$

Since $x^2 + y^2 = r^2$,

$$r^2 = e^{-2t} \quad \text{or} \quad r = e^{-t}.$$

(b) The path of the particle is shown below.



(c) First we find $\mathbf{v}$.

$$\mathbf{v} = \frac{d}{dt}(e^{-t} \cos t \mathbf{i} + e^{-t} \sin t \mathbf{j}) = (-e^{-t} \cos t - e^{-t} \sin t) \mathbf{i} + (-e^{-t} \sin t + e^{-t} \cos t) \mathbf{j}$$

To find the angle between $\mathbf{r}$ and $\mathbf{v}$, we use

$$\cos^{-1} \left(\frac{\mathbf{r} \cdot \mathbf{v}}{\|\mathbf{r}\| \|\mathbf{v}\|} \right).$$

Since

$$\mathbf{r} \cdot \mathbf{v} = -e^{-2t} \cos^2 t - e^{-2t} \sin t \cos t - e^{-2t} \sin^2 t + e^{-2t} \sin t \cos t = -e^{-2t}$$

and

$$\|\mathbf{r}\| = \sqrt{e^{-2t} \cos^2 t + e^{-2t} \sin^2 t} = e^{-t}$$

$$\|\mathbf{v}\| = \sqrt{e^{-2t} \cos^2 t + 2e^{-2t} \sin t \cos t + e^{-2t} \sin^2 t + e^{-2t} \sin^2 t - 2e^{-2t} \sin t \cos t + e^{-2t} \cos^2 t} = e^{-t} \sqrt{2},$$

$$\cos^{-1} \left[\frac{-e^{-2t}}{(e^{-t})(e^{-t} \sqrt{2})} \right] = \cos^{-1} \left(\frac{-1}{\sqrt{2}} \right) = \cos^{-1} \left(\frac{-\sqrt{2}}{2} \right) = \frac{3\pi}{4} = \boxed{135^\circ}.$$

61. (a) If $\mathbf{r}(t) = 4 \cos t \mathbf{i} - 2 \cos(2t) \mathbf{j}$, then $x = 4 \cos t$ and $y = -2 \cos(2t)$ with $-4 \leq x \leq 4$ and $-2 \leq y \leq 2$. Using the double-angle identity $\cos(2t) = 2 \cos^2 t - 1$,

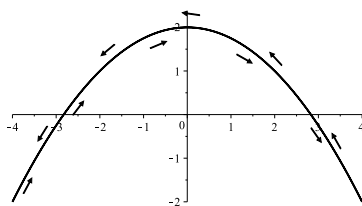
$$y = -2 \cos(2t) = -2(2 \cos^2 t - 1) = 2 - 4 \cos^2 t.$$

We can eliminate the parameter using $\cos t = \frac{x}{4}$ so that

$$y = 2 - 4 \left(\frac{x}{4} \right)^2 = 2 - \frac{x^2}{4}, \quad -4 \leq x \leq 4.$$

The motion of the particle oscillates on the arc of a parabola.

- (b) The path of the particle is shown below.



- (c) The velocity and acceleration of the particle are

$$\begin{aligned} \mathbf{v}(t) &= -4 \sin t \mathbf{i} + 4 \sin(2t) \mathbf{j} \\ \mathbf{a}(t) &= -4 \cos t \mathbf{i} + 8 \cos(2t) \mathbf{j}. \end{aligned}$$

The velocity is 0 at $t = n\pi$ where n is an integer, and the acceleration is $\mathbf{a}(n\pi) = \boxed{\pm 4\mathbf{i} + 8\mathbf{j} \text{ where } n \text{ is an integer}}$.

62. If $\mathbf{r}'(t) = \mathbf{b} \times \mathbf{r}(t)$ and $\mathbf{b}$ is a constant vector, then $\mathbf{b}' = \mathbf{0}$ and

$$\mathbf{a}(t) = \mathbf{r}''(t) = \frac{d}{dt}[\mathbf{b} \times \mathbf{r}(t)] = \mathbf{b}' \times \mathbf{r}(t) + \mathbf{b} \times \mathbf{r}'(t) = \mathbf{b} \times \mathbf{r}'(t).$$

Since $\mathbf{a}(t) = \mathbf{b} \times \mathbf{r}'(t)$ and the cross product of two vectors is orthogonal to both vectors, $\mathbf{a}(t)$ is perpendicular to both $\mathbf{b}$ and $\mathbf{r}'(t)$. ■

If $\mathbf{a}(t) = \mathbf{r}''(t)$ and $\mathbf{r}'(t)$ are perpendicular, then $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$. Because

$$2\mathbf{r}''(t) \cdot \mathbf{r}'(t) = \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)] = 0,$$

$$\mathbf{r}'(t) \cdot \mathbf{r}'(t) = \|\mathbf{r}'(t)\|^2 = c,$$

which implies that $\|\mathbf{r}'(t)\| = v(t)$ is constant. ■

63. Since the *Odyssey* is in an orbit at a constant height of 400 km above Mars' surface, it is in a circular orbit. From Equation (2) and Example 4, the magnitude of the acceleration of a satellite in circular orbit is $\|\mathbf{a}(t)\| = \frac{v_0^2}{R}$. Since $R = 3390 + 400 \text{ km} = 3790000 \text{ m}$ and $\|\mathbf{a}(t)\| = 3.71 \text{ m/s}^2$ on Mars,

$$v_0 = \sqrt{\|\mathbf{a}(t)\|R} = \sqrt{(3.71)(3790000)} \approx \boxed{3750 \text{ m/s}}.$$

64. Power is $\mathbf{F} \cdot \mathbf{v}$ and from Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a} = m\mathbf{v}'$, so

$$\text{Power} = \mathbf{F} \cdot \mathbf{v} = m\mathbf{v}' \cdot \mathbf{v} = \frac{m}{2} \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d}{dt} \left(\frac{m}{2} \|\mathbf{v}\|^2 \right).$$

Since kinetic energy $= \frac{1}{2}m\|\mathbf{v}(t)\|^2$, power is the rate of change of its kinetic energy. ■

65. Momentum is $\mathbf{p}(t) = m\mathbf{v}(t)$, so

$$\frac{d\mathbf{p}}{dt} = m \frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F}.$$

66. Since the angular momentum $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$,

$$\frac{d\mathbf{L}}{dt} = (\mathbf{r}' \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{v}') = m(\mathbf{v} \times \mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}.$$

67. We know that for a central field of force $\mathbf{F}(t) = u(t)\mathbf{r}(t)$ and torque is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Then

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times [u\mathbf{r}] = u(\mathbf{r} \times \mathbf{r}) = \mathbf{0}.$$

A central field of force produces zero torque. ■

68. From Equation (2) in Example 3, $\|\mathbf{a}\| = \frac{v^2}{r}$ for a circular orbit. Then

$$F_{\text{elec}} = \frac{ke^2}{r^2} = \frac{mv^2}{r}$$

and

$$v = \sqrt{\frac{ke^2}{mr}}.$$

69. (a) The push against the seat is in the direction of the motion, so this force is the tangential force, $m \frac{dv}{dt}$.
 (b) The push against the door is centripetal force or normal force, $mv^2\kappa$.
 (c) The normal force varies linearly with the curvature of the road.
 (d) The normal force varies quadratically with the speed of the car. ■

70. The normal component of acceleration is $a_N = [v(t)]^2 \kappa$, so if $a_N = 0$ for all t , $\kappa = 0$, which means the motion of the particle is in a straight line. ■

71. If $\mathbf{r}$ lies on a sphere, then $\|\mathbf{r}\|$ is constant, which means $\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2 = c$. Then

$$\begin{aligned}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) &= \frac{d}{dt}(c) \\ (\mathbf{r}' \cdot \mathbf{r}) + (\mathbf{r} \cdot \mathbf{r}') &= 0 \\ 2\mathbf{r} \cdot \mathbf{r}' &= 0,\end{aligned}$$

so the tangent vector $\mathbf{r}'$ is always orthogonal to $\mathbf{r}$. ■

Challenge Problems

72. (a) Since $\mathbf{r}(t) = t\mathbf{R}(t)$,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{R}(t) + t\mathbf{R}'(t) = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j} + t\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$. ■

(b) From part (a), $\mathbf{v}(t) = \mathbf{R}(t) + t\mathbf{R}'(t)$,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \mathbf{R}'(t) + \mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{v}_d + t\mathbf{a}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ and $\mathbf{a}_d = \mathbf{R}''(t)$. ■

73. (a) Since $\mathbf{r}(t) = t^2\mathbf{R}(t)$,

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{R}(t) + t^2\mathbf{R}'(t) = \boxed{2t\cos(\omega t)\mathbf{i} + 2t\sin(\omega t)\mathbf{j} + t^2\mathbf{v}_d}$$

and

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{v}'(t) = \mathbf{r}''(t) = 2\mathbf{R}(t) + 2t\mathbf{R}'(t) + 2t\mathbf{R}'(t) + t^2\mathbf{R}''(t) = 2\mathbf{R}(t) + 4t\mathbf{R}'(t) + t^2\mathbf{R}''(t) \\ &= \boxed{2\cos(\omega t)\mathbf{i} + 2\sin(\omega t)\mathbf{j} + 4t\mathbf{v}_d + t^2\mathbf{a}_d}\end{aligned}$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ and $\mathbf{a}_d = \mathbf{R}''(t)$. ■

(b) From Problem 72(b), the Coriolis acceleration is the extra term(s) other than the acceleration of the rotating disk $\mathbf{a}_d$. In this case, it is $\boxed{2\cos(\omega t)\mathbf{i} + 2\sin(\omega t)\mathbf{j} + 4t\mathbf{v}_d}$.

74. Both $\boxed{\mathbf{T} \times \mathbf{r}}$ and $\boxed{\mathbf{T} \times \mathbf{N}}$ are orthogonal to $\mathbf{T}$ since the cross product of two vectors is orthogonal to both vectors.

75. We are given $\mathbf{u}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ and $\mathbf{u}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$.

(a) Then $\|\mathbf{u}_r\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$ and $\|\mathbf{u}_\theta\| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$, so $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are unit vectors.

$$\mathbf{u}_r \cdot \mathbf{u}_\theta = -\sin \theta \cos \theta + \sin \theta \cos \theta = 0$$

Since the dot product of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ is zero, the vectors are orthogonal.

(b) The derivatives of the $\mathbf{u}$ vectors with respect to θ are

$$\begin{aligned}\frac{d\mathbf{u}_r}{d\theta} &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} = \mathbf{u}_\theta \\ \frac{d\mathbf{u}_\theta}{d\theta} &= -\cos \theta \mathbf{i} - \sin \theta \mathbf{j} = -\mathbf{u}_r.\end{aligned}$$

■

(c) Since $\mathbf{r}(t)$ has polar coordinates $(r(t), \theta(t))$,

$$\begin{aligned}\mathbf{r}(t) &= r(t) \cos(\theta(t))\mathbf{i} + r(t) \sin(\theta(t))\mathbf{j} \\ \|\mathbf{r}(t)\| &= \sqrt{[r(t)]^2 \cos^2(\theta(t)) + [r(t)]^2 \sin^2(\theta(t))} = r(t)\end{aligned}$$

and $\mathbf{r}(t) = r(t) \cos(\theta(t))\mathbf{i} + r(t) \sin(\theta(t))\mathbf{j} = \|\mathbf{r}(t)\|[\cos(\theta(t))\mathbf{i} + \sin(\theta(t))\mathbf{j}] = \|\mathbf{r}(t)\|\mathbf{u}_r$. ■

(d) From part (c), $\mathbf{r}(t) = \|\mathbf{r}(t)\|\mathbf{u}_r = r(t)\mathbf{u}_r$ and from part (b), $\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta$. Then

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \frac{d}{dt}\mathbf{r}(t) = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} \\ &= \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta\end{aligned}$$

(e) From part (b), $\frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r$ and $\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta$, so

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= \frac{d}{dt}\mathbf{v}(t) = \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt} \\ &= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\left(\frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}\right) + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\left(\frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt}\right) \\ &= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\mathbf{u}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\left(\frac{d\theta}{dt}\right)^2(-\mathbf{u}_r) \\ &= [r'' - r(\theta')^2]\mathbf{u}_r + [r\theta'' + 2r'\theta']\mathbf{u}_\theta\end{aligned}$$

76. If $r = 2 + \cos t$ and $\theta = 2t$, then $r' = -\sin t$, $r'' = -\cos t$, $\theta' = 2$, and $\theta'' = 0$, so from Problem 75 parts (d) and (e),

$$\begin{aligned}\mathbf{v}(t) &= r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta = \boxed{-\sin t\mathbf{u}_r + 2(\cos t + 2)\mathbf{u}_\theta} \\ \mathbf{a}(t) &= [r'' - r(\theta')^2]\mathbf{u}_r + [r\theta'' + 2r'\theta']\mathbf{u}_\theta \\ &= [(-\cos t) - (2 + \cos t)(2)^2]\mathbf{u}_r + [(2 + \cos t)(0) + 2(-\sin t)(2)]\mathbf{u}_\theta \\ &= \boxed{-(5\cos t + 8)\mathbf{u}_r - 4\sin t\mathbf{u}_\theta}.\end{aligned}$$

77. To convert rectangular equations into polar equations, we use $x = r \cos \theta$ and $y = r \sin \theta$, so $\mathbf{r}(t) = e^{2t} \cos t \mathbf{i} + e^{2t} \sin t \mathbf{j}$ implies $\boxed{r(t) = e^{2t} \text{ where } \theta = t}$. Then $r' = 2e^{2t}$, $r'' = 4e^{2t}$, $\theta' = 1$, and $\theta'' = 0$, so from Problem 75 parts (d) and (e),

$$\begin{aligned}\mathbf{v}(t) &= r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta = 2e^{2t}\mathbf{u}_r + e^{2t}(1)\mathbf{u}_\theta = \boxed{2r\mathbf{u}_r + r\mathbf{u}_\theta} \\ \mathbf{a}(t) &= [r'' - r(\theta')^2]\mathbf{u}_r + [r\theta'' + 2r'\theta']\mathbf{u}_\theta = [4e^{2t} - e^{2t}(1)^2]\mathbf{u}_r + [e^{2t}(0) + 2(2e^{2t})(1)]\mathbf{u}_\theta \\ &= \boxed{3r\mathbf{u}_r + 4r\mathbf{u}_\theta}\end{aligned}$$

78. (a) Using the results from Problem 75,

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = (r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta) \cdot (r\mathbf{u}_r) = rr'\mathbf{u}_r \cdot \mathbf{u}_r + r^2\theta'\mathbf{u}_\theta \cdot \mathbf{u}_r = rr'\mathbf{u}_r \cdot \mathbf{u}_r = rr'\|\mathbf{u}_r\|^2 = rr'$$

Since $\mathbf{r}'(t) \cdot \mathbf{r}(t) = rr' = 0$, either $r = 0$ or $r' = 0$. If $r = 0$, then $\mathbf{r}(t)$ is a single point at the pole. If $r' = 0$, then r is a constant, meaning $\mathbf{r}(t) = r\mathbf{u}_r$ is a sphere centered at the origin. ■

(b) If the acceleration and velocity vectors are orthogonal for all t , the velocity will change direction without changing its magnitude, so the motion is circular.

(c) If velocity and position vectors are parallel or anti-parallel for all t , the motion is in a straight line radially toward or away from the origin.

11.5 Integrals of Vector Functions; Projectile Motion

Concepts and Vocabulary

1. True by the statement of the integral of a vector function in subsection 1.
2. False. We must insert a constant of integration into each component.

Skill Building

3. We integrate each component individually.

$$\begin{aligned}
 \int \sin t \mathbf{i} dt - \int \cos t \mathbf{j} dt + \int t \mathbf{k} dt &= (-\cos t + c_1) \mathbf{i} + (-\sin t + c_2) \mathbf{j} + \left(\frac{1}{2}t^2 + c_3\right) \mathbf{k} \\
 &= -\cos t \mathbf{i} - \sin t \mathbf{j} + \frac{1}{2}t^2 \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\
 &= \boxed{-\cos t \mathbf{i} - \sin t \mathbf{j} + \frac{1}{2}t^2 \mathbf{k} + \mathbf{c}}
 \end{aligned}$$

4. We integrate each component individually.

$$\begin{aligned}
 \int \cos t \mathbf{i} dt + \int \sin t \mathbf{j} dt - \int \mathbf{k} dt &= (\sin t + c_1) \mathbf{i} + (-\cos t + c_2) \mathbf{j} + (-t + c_3) \mathbf{k} \\
 &= \boxed{\sin t \mathbf{i} - \cos t \mathbf{j} - t \mathbf{k} + \mathbf{c}}
 \end{aligned}$$

5. We integrate each component individually.

$$\begin{aligned}
 \int t^2 \mathbf{i} dt - \int t \mathbf{j} dt + \int e^t \mathbf{k} dt &= \left(\frac{1}{3}t^3 + c_1\right) \mathbf{i} + \left(-\frac{1}{2}t^2 + c_2\right) \mathbf{j} + (e^t + c_3) \mathbf{k} \\
 &= \frac{1}{3}t^3 \mathbf{i} - \frac{1}{2}t^2 \mathbf{j} + e^t \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\
 &= \boxed{\frac{1}{3}t^3 \mathbf{i} - \frac{1}{2}t^2 \mathbf{j} + e^t \mathbf{k} + \mathbf{c}}
 \end{aligned}$$

6. We integrate each component individually.

$$\begin{aligned}
 \int e^t \mathbf{i} dt - \int \sqrt{t} \mathbf{j} dt + \int t^2 \mathbf{k} dt &= (e^t + c_1) \mathbf{i} + \left(-\frac{2}{3}t^{3/2} + c_2\right) \mathbf{j} + \left(\frac{1}{3}t^3 + c_3\right) \mathbf{k} \\
 &= e^t \mathbf{i} - \frac{2}{3}t^{3/2} \mathbf{j} + \frac{1}{3}t^3 \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\
 &= \boxed{e^t \mathbf{i} - \frac{2}{3}t^{3/2} \mathbf{j} + \frac{1}{3}t^3 \mathbf{k} + \mathbf{c}}
 \end{aligned}$$

7. We integrate each component individually. We use integration by parts for the integrals of the first two components. For $\int \ln t dt$, we choose

$$u = \ln t \quad \text{and} \quad dv = dt.$$

Then

$$du = \frac{1}{t} dt \quad \text{and} \quad v = \int dt = t$$

and

$$\int \ln t dt = t \ln t - \int t \left(\frac{1}{t}\right) dt = t \ln t - \int dt = t \ln t - t.$$

For $\int t \ln t \, dt$, we choose

$$u = \ln t \quad \text{and} \quad dv = t \, dt.$$

Then

$$du = \frac{1}{t} \, dt \quad \text{and} \quad v = \int t \, dt = \frac{1}{2} t^2$$

and

$$\int t \ln t \, dt = \frac{1}{2} t^2 \ln t - \int \frac{1}{2} t^2 \left(\frac{1}{t} \right) dt = \frac{1}{2} t^2 \ln t - \frac{1}{2} \int t \, dt = \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2.$$

$$\begin{aligned} \int \ln t \, \mathbf{i} \, dt - \int t \ln t \, \mathbf{j} \, dt - \int 2\mathbf{k} \, dt &= (t \ln t - t + c_1) \mathbf{i} + \left(-\frac{t^2}{2} \ln t + \frac{t^2}{4} + c_2 \right) \mathbf{j} + (-2t + c_3) \mathbf{k} \\ &= (t \ln t - t) \mathbf{i} + \left(\frac{t^2}{4} - \frac{t^2}{2} \ln t \right) \mathbf{j} - 2t \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\ &= \boxed{(t \ln t - t) \mathbf{i} + \left(\frac{t^2}{4} - \frac{t^2}{2} \ln t \right) \mathbf{j} - 2t \mathbf{k} + \mathbf{c}} \end{aligned}$$

8. We integrate each component individually. We use integration by parts for the integrals of the second component. For $\int \ln t \, dt$, we choose

$$u = \ln t \quad \text{and} \quad dv = dt.$$

Then

$$du = \frac{1}{t} \, dt \quad \text{and} \quad v = \int dt = t$$

and

$$\int \ln t \, dt = t \ln t - \int t \left(\frac{1}{t} \right) dt = t \ln t - \int dt = t \ln t - t.$$

$$\begin{aligned} \int \mathbf{i} \, dt + \int \ln t \, \mathbf{j} \, dt + \int \frac{1}{t} \mathbf{k} \, dt &= (t + c_1) \mathbf{i} + (t \ln t - t + c_2) \mathbf{j} + (\ln t + c_3) \mathbf{k} \\ &= t \mathbf{i} + (t \ln t - t) \mathbf{j} + \ln t \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\ &= \boxed{t \mathbf{i} + (t \ln t - t) \mathbf{j} + \ln t \mathbf{k} + \mathbf{c}} \end{aligned}$$

9. We integrate each component individually. We use the substitution $u = t - 2$, $du = dt$ for the first two components.

$$\begin{aligned} \int (t - 2) \mathbf{i} \, dt - \int (t - 2)^2 \mathbf{j} \, dt + \int \mathbf{k} \, dt &= \left[\frac{1}{2} (t - 2)^2 + c_1 \right] \mathbf{i} - \left[\frac{1}{3} (t - 2)^3 + c_2 \right] \mathbf{j} + (t + c_3) \mathbf{k} \\ &= \frac{1}{2} (t - 2)^2 \mathbf{i} - \frac{1}{3} (t - 2)^3 \mathbf{j} + t \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\ &= \boxed{\frac{1}{2} (t - 2)^2 \mathbf{i} - \frac{1}{3} (t - 2)^3 \mathbf{j} + t \mathbf{k} + \mathbf{c}} \end{aligned}$$

Note that if we do not use substitution to integrate the first term, the answer is

$$\left(\frac{1}{2} t^2 - 2t \right) \mathbf{i} - \frac{1}{3} (t - 2)^3 \mathbf{j} + t \mathbf{k} + \mathbf{c},$$

which varies from the answer above by only a constant. Both answers are correct.

10. We integrate each component individually. We use the substitution $u = 3t + 1$, $du = 3dt$, $\frac{du}{3} = dt$ for each component.

$$\begin{aligned} \int (3t+1)\mathbf{i} dt + \int (3t+1)^2\mathbf{j} dt + \int (3t+1)^{-1}\mathbf{k} dt &= \left[\frac{(3t+1)^2}{6} + c_1 \right] \mathbf{i} + \left[\frac{(3t+1)^3}{9} + c_2 \right] \mathbf{j} + \left[\frac{\ln(3t+1)}{3} + c_3 \right] \mathbf{k} \\ &= \frac{(3t+1)^2}{6} \mathbf{i} + \frac{(3t+1)^3}{9} \mathbf{j} + \frac{\ln(3t+1)}{3} \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\ &= \boxed{\frac{(3t+1)^2}{6} \mathbf{i} + \frac{(3t+1)^3}{9} \mathbf{j} + \frac{\ln(3t+1)}{3} \mathbf{k} + \mathbf{c}} \end{aligned}$$

11. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (e^t \mathbf{i} - \ln t \mathbf{j} + 2t \mathbf{k}) dt = (e^t + c_1) \mathbf{i} + (-t \ln t + t + c_2) \mathbf{j} + (t^2 + c_3) \mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(1) = \mathbf{j} + \mathbf{k}$

$$\mathbf{r}(1) = (e + c_1) \mathbf{i} + (1 + c_2) \mathbf{j} + (1 + c_3) \mathbf{k} = \mathbf{j} + \mathbf{k}$$

from which we find

$$c_1 = -e \quad c_2 = 0 \quad c_3 = 0.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{(e^t - e) \mathbf{i} + (t - t \ln t) \mathbf{j} + t^2 \mathbf{k}}$.

12. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int \left(t \mathbf{i} + e^{-t} \mathbf{j} - \frac{1}{t} \mathbf{k} \right) dt = \left(\frac{1}{2} t^2 + c_1 \right) \mathbf{i} + (-e^{-t} + c_2) \mathbf{j} + (-\ln t + c_3) \mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(1) = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}(1) = \left(\frac{1}{2} + c_1 \right) \mathbf{i} + \left(-\frac{1}{e} + c_2 \right) \mathbf{j} + c_3 \mathbf{k}$$

from which we find

$$c_1 = \frac{1}{2} \quad c_2 = \frac{1}{e} - 1 \quad c_3 = 2.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{\left(\frac{1}{2} t^2 + \frac{1}{2} \right) \mathbf{i} + \left(\frac{1}{e} - 1 - e^{-t} \right) \mathbf{j} + (2 - \ln t) \mathbf{k}}$.

13. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (2 \sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}) dt = (-2 \cos t + c_1) \mathbf{i} + (\sin t + c_2) \mathbf{j} + (t + c_3) \mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(0) = \mathbf{i} - \mathbf{j}$

$$\mathbf{r}(0) = (-2 + c_1) \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$$

from which we find

$$c_1 = 3 \quad c_2 = -1 \quad c_3 = 0.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{(3 - 2 \cos t) \mathbf{i} + (\sin t - 1) \mathbf{j} + t \mathbf{k}}$.

14. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int [\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + 2\mathbf{k}] dt = \left(\frac{1}{2}\sin(2t) + c_1\right)\mathbf{i} + \left(-\frac{1}{2}\cos(2t) + c_2\right)\mathbf{j} + (2t + c_3)\mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$

$$\mathbf{r}(0) = c_1\mathbf{i} + \left(-\frac{1}{2} + c_2\right)\mathbf{j} + c_3\mathbf{k}$$

from which we find

$$c_1 = 1 \quad c_2 = \frac{1}{2} \quad c_3 = 1.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{\left(\frac{1}{2}\sin(2t) + 1\right)\mathbf{i} + \left[-\frac{1}{2}\cos(2t) + \frac{1}{2}\right]\mathbf{j} + (2t + 1)\mathbf{k}}.$

15. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (t^{-1}\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}) dt = (\ln t + c_1)\mathbf{i} + \left(\frac{1}{2}t^2 + c_2\right)\mathbf{j} + \left(\frac{1}{3}t^3 + c_3\right)\mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{r}(1) = c_1\mathbf{i} + \left(\frac{1}{2} + c_2\right)\mathbf{j} + \left(\frac{1}{3} + c_3\right)\mathbf{k}$$

from which we find

$$c_1 = 1 \quad c_2 = \frac{1}{2} \quad c_3 = \frac{2}{3}.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{(\ln t + 1)\mathbf{i} + \frac{1}{2}(t^2 + 1)\mathbf{j} + \frac{1}{3}(t^3 + 2)\mathbf{k}}.$

16. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int \left(t^3\mathbf{i} + \frac{1}{t+1}\mathbf{j} + \mathbf{k}\right) dt = \left(\frac{1}{4}t^4 + c_1\right)\mathbf{i} + (\ln(t+1) + c_2)\mathbf{j} + (t + c_3)\mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{r}(0) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

from which we find

$$c_1 = 1 \quad c_2 = 1 \quad c_3 = 1.$$

The vector function $\mathbf{r} = \mathbf{r}(t) = \boxed{\left(\frac{1}{4}t^4 + 1\right)\mathbf{i} + (\ln(t+1) + 1)\mathbf{j} + (t + 1)\mathbf{k}}.$

17. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -32\mathbf{k} dt = c_1\mathbf{i} + c_2\mathbf{j} + (-32t + c_3)\mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{0}$, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, so the velocity vector is

$$\mathbf{v}(t) = \boxed{-32t\mathbf{k}}$$

and the speed is

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{(-32t)^2} = \boxed{32t}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int -32t\mathbf{k} dt = d_1\mathbf{i} + d_2\mathbf{j} + (-16t^2 + d_3)\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $d_1 = 0$, $d_2 = 0$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{-16t^2\mathbf{k}}.$$

18. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -32\mathbf{k} dt = c_1\mathbf{i} + c_2\mathbf{j} + (-32t + c_3)\mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{\mathbf{i} + \mathbf{j} - 32t\mathbf{k}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{1 + 1 + 1024t} = \boxed{\sqrt{1024t + 2}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (\mathbf{i} + \mathbf{j} - 32t\mathbf{k}) dt = (t + d_1)\mathbf{i} + (t + d_2)\mathbf{j} + (-16t^2 + d_3)\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $d_1 = 0$, $d_2 = 0$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{t\mathbf{i} + t\mathbf{j} - 16t^2\mathbf{k}}.$$

19. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (\cos t\mathbf{i} + \sin t\mathbf{j}) dt = (\sin t + c_1)\mathbf{i} + (-\cos t + c_2)\mathbf{j} + c_3\mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i}$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{(\sin t + 1)\mathbf{i} + (1 - \cos t)\mathbf{j}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{(\sin t + 1)^2 + (1 - \cos t)^2} = \sqrt{\sin^2 t + 2\sin t + 1 + 1 - 2\cos t + \cos^2 t} = \boxed{\sqrt{2\sin t - 2\cos t + 3}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [(\sin t + 1)\mathbf{i} + (1 - \cos t)\mathbf{j}] dt = (-\cos t + t + d_1)\mathbf{i} + (t - \sin t + d_2)\mathbf{j} + d_3\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{j}$, $d_1 = 1$, $d_2 = 1$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{(t - \cos t + 1)\mathbf{i} + (t - \sin t + 1)\mathbf{j}}.$$

20. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (\cos t \mathbf{i} + \sin t \mathbf{j}) dt = (\sin t + c_1) \mathbf{i} + (-\cos t + c_2) \mathbf{j} + c_3 \mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{j}$, $c_1 = 0$, $c_2 = 2$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{\sin t \mathbf{i} + (2 - \cos t) \mathbf{j}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{\sin^2 t + (2 - \cos t)^2} = \sqrt{\sin^2 t + 4 - 4 \cos t + \cos^2 t} = \boxed{\sqrt{5 - 4 \cos t}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [\sin t \mathbf{i} + (2 - \cos t) \mathbf{j}] dt = (-\cos t + d_1) \mathbf{i} + (2t - \sin t + d_2) \mathbf{j} + d_3 \mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{i}$, $d_1 = 2$, $d_2 = 0$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{(2 - \cos t) \mathbf{i} + (2t - \sin t) \mathbf{j}}.$$

21. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -9.8 \mathbf{k} dt = c_1 \mathbf{i} + c_2 \mathbf{j} + (-9.8t + c_3) \mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i}$, $c_1 = 1$, $c_2 = 0$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{\mathbf{i} - 9.8t \mathbf{k}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{1 + (-9.8t)^2} = \boxed{\sqrt{96.04t^2 + 1}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (\mathbf{i} - 9.8t \mathbf{k}) dt = (t + d_1) \mathbf{i} + d_2 \mathbf{j} + (-4.9t^2 + d_3) \mathbf{k}.$$

Since $\mathbf{r}(0) = 5 \mathbf{k}$, $d_1 = 0$, $d_2 = 0$, and $d_3 = 5$, the position vector is given by

$$\mathbf{r}(t) = \boxed{t \mathbf{i} + (-4.9t^2 + 5) \mathbf{k}}.$$

22. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -9.8 \mathbf{k} dt = c_1 \mathbf{i} + c_2 \mathbf{j} + (-9.8t + c_3) \mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{\mathbf{i} + \mathbf{j} - 9.8t \mathbf{k}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{1 + 1 + (-9.8t)^2} = \boxed{\sqrt{96.04t^2 + 2}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (\mathbf{i} + \mathbf{j} - 9.8t \mathbf{k}) dt = (t + d_1) \mathbf{i} + (t + d_2) \mathbf{j} + (-4.9t^2 + d_3) \mathbf{k}.$$

Since $\mathbf{r}(0) = 2 \mathbf{i}$, $d_1 = 2$, $d_2 = 0$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{(t + 2) \mathbf{i} + t \mathbf{j} - 4.9t^2 \mathbf{k}}.$$

23. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (e^{-t}\mathbf{i} + \mathbf{j}) dt = (-e^{-t} + c_1)\mathbf{i} + (t + c_2)\mathbf{j} + c_3\mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, $c_1 = 2$, $c_2 = 1$, and $c_3 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{(-e^{-t} + 2)\mathbf{i} + (t + 1)\mathbf{j}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{(-e^{-t} + 2)^2 + (t + 1)^2} = \sqrt{e^{-2t} - 4e^{-t} + 4 + t^2 + 2t + 1} = \boxed{\sqrt{e^{-2t} - 4e^{-t} + t^2 + 2t + 5}}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [(-e^{-t} + 2)\mathbf{i} + (t + 1)\mathbf{j}] dt = (e^{-t} + 2t + d_1)\mathbf{i} + \left(\frac{1}{2}t^2 + t + d_2\right)\mathbf{j} + d_3\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{i} - \mathbf{j}$, $d_1 = 0$, $d_2 = -1$, and $d_3 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{(e^{-t} + 2t)\mathbf{i} + \left(\frac{1}{2}t^2 + t - 1\right)\mathbf{j}}.$$

24. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (t^2\mathbf{i} - e^{-t}\mathbf{k}) dt = \left(\frac{1}{3}t^3 + c_1\right)\mathbf{i} + c_2\mathbf{j} + (e^{-t} + c_3)\mathbf{k}.$$

Since $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$, $c_1 = 1$, $c_2 = -1$, and $c_3 = -1$, the velocity vector is

$$\mathbf{v}(t) = \boxed{\left(\frac{1}{3}t^3 + 1\right)\mathbf{i} - \mathbf{j} + (e^{-t} - 1)\mathbf{k}}$$

and the speed is

$$\begin{aligned} v = \|\mathbf{v}(t)\| &= \sqrt{\left(\frac{1}{3}t^3 + 1\right)^2 + 1 + (e^{-t} - 1)^2} = \sqrt{\frac{1}{9}t^6 + \frac{2}{3}t^3 + 1 + 1 + e^{-2t} - 2e^{-t} + 1} \\ &= \boxed{\sqrt{\frac{1}{9}t^6 + \frac{2}{3}t^3 + e^{-2t} - 2e^{-t} + 3}}. \end{aligned}$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \left[\left(\frac{1}{3}t^3 + 1\right)\mathbf{i} - \mathbf{j} + (e^{-t} - 1)\mathbf{k}\right] dt = \left(\frac{1}{12}t^4 + t + d_1\right)\mathbf{i} + (-t + d_2)\mathbf{j} + (-e^{-t} - t + d_3)\mathbf{k}.$$

Since $\mathbf{r}(0) = \mathbf{k}$, $d_1 = 0$, $d_2 = 0$, and $d_3 = 2$, the position vector is given by

$$\mathbf{r}(t) = \boxed{\left(\frac{1}{12}t^4 + t\right)\mathbf{i} - t\mathbf{j} + (-e^{-t} - t + 2)\mathbf{k}}.$$

Applications and Extensions

25. Since $\mathbf{r}(0) = \mathbf{0}$ and the projectile has a constant initial velocity, we can use the formulas derived in the section. We are given that $v_0 = 520\text{m/s}$ and that $\theta = 30^\circ = \frac{\pi}{6}$ radians. The range is

$$x = \frac{v_0^2 \sin(2\theta)}{g} = \frac{520^2 \sin\left(\frac{\pi}{3}\right)}{9.8} \approx \boxed{23,895 \text{ m}},$$

the time in flight is

$$t = \frac{2v_0 \sin \theta}{g} = \frac{(2)(520) \sin\left(\frac{\pi}{6}\right)}{9.8} \approx \boxed{53 \text{ s}},$$

and the greatest height reached is

$$y = \frac{v_0^2 \sin^2 \theta}{2g} = \frac{(520)^2 \sin^2\left(\frac{\pi}{6}\right)}{(2)(9.8)} \approx \boxed{3449 \text{ m}}.$$

26. Since $\mathbf{r}(0) = \mathbf{0}$ and the projectile has a constant initial velocity, we can use the formulas derived in the section. We are given that $v_0 = 200\text{m/s}$ and that $\theta = 60^\circ = \frac{\pi}{3}$ radians. The range is

$$x = \frac{v_0^2 \sin(2\theta)}{g} = \frac{200^2 \sin\left(\frac{2\pi}{3}\right)}{9.8} \approx \boxed{3535 \text{ m}},$$

the time in flight is

$$t = \frac{2v_0 \sin \theta}{g} = \frac{(2)(200) \sin\left(\frac{\pi}{3}\right)}{9.8} \approx \boxed{35 \text{ s}},$$

and the greatest height reached is

$$y = \frac{v_0^2 \sin^2 \theta}{2g} = \frac{(200)^2 \sin^2\left(\frac{\pi}{3}\right)}{(2)(9.8)} \approx \boxed{1531 \text{ m}}.$$

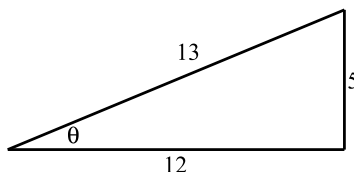
27. Since $\mathbf{r}(0) = \mathbf{0}$ and the projectile has a constant initial velocity, we can use the formulas derived in the section. We are given that $v_0 = 100\text{m/s}$ and that $\theta = \tan^{-1}\left(\frac{5}{12}\right)$.

(a) The parametric equations of the motion of the projectile are

$$\begin{aligned} x &= (v_0 \cos \theta)t = 100 \cos \left[\tan^{-1} \left(\frac{5}{12} \right) \right] t \\ y &= -\frac{1}{2}gt^2 + (v_0 \sin \theta)t = -4.9t^2 + 100 \sin \left[\tan^{-1} \left(\frac{5}{12} \right) \right] t \end{aligned}$$

To simplify the angles, refer to the figure below. The angle $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ is labeled on the triangle. Using the relations $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$,

$$\begin{aligned} x &= 100 \left(\frac{12}{13} \right) t = \frac{1200}{13} t \\ y &= -4.9t^2 + 100 \left(\frac{5}{13} \right) t = -4.9t^2 + \frac{500}{13} t \end{aligned}$$



The parametric equations of the path of the projectile are $x = \frac{1200}{13}t, y = -4.9t^2 + \frac{500}{13}t$.

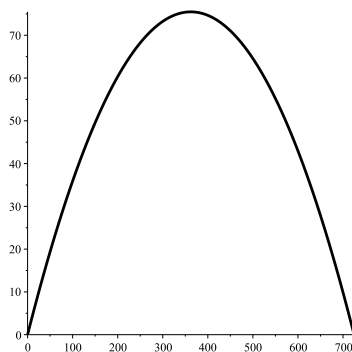
(b) Using the identity $\sin(2\theta) = 2 \sin \theta \cos \theta$, we find the range to be

$$\begin{aligned} x &= \frac{v_0^2 \sin(2\theta)}{g} = \frac{100^2(2) \sin \left[\tan^{-1} \left(\frac{5}{12} \right) \right] \cos \left[\tan^{-1} \left(\frac{5}{12} \right) \right]}{9.8} \\ &= \frac{100^2(2) \left(\frac{5}{13} \right) \left(\frac{12}{13} \right)}{9.8} \approx \boxed{724.6 \text{ m}}. \end{aligned}$$

(c) The projectile is in the air for

$$t = \frac{2v_0 \sin \theta}{g} = \frac{(2)(100) \sin \left[\tan^{-1} \left(\frac{5}{12} \right) \right]}{9.8} = \frac{(2)(100) \left(\frac{5}{13} \right)}{9.8} \approx \boxed{7.8 \text{ s}}.$$

(d) The trajectory of the projectile is shown below.



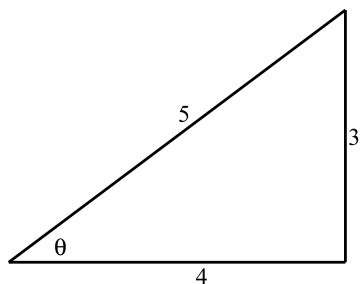
28. Since $\mathbf{r}(0) = \mathbf{0}$ and the projectile has a constant initial velocity, we can use the formulas derived in the section. We are given that $v_0 = 120\text{m/s}$ and that $\theta = \tan^{-1} \left(\frac{3}{4} \right)$.

(a) The parametric equations of the motion of the projectile are

$$\begin{aligned} x &= (v_0 \cos \theta)t = 120 \cos \left[\tan^{-1} \left(\frac{3}{4} \right) \right] t \\ y &= -\frac{1}{2}gt^2 + (v_0 \sin \theta)t = -4.9t^2 + 120 \sin \left[\tan^{-1} \left(\frac{3}{4} \right) \right] t \end{aligned}$$

To simplify the angles, refer to the figure below. The angle $\theta = \tan^{-1} \left(\frac{3}{4} \right)$ is labeled on the triangle. Using the relations $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$,

$$\begin{aligned} x &= 120 \left(\frac{4}{5} \right) t = 96t \\ y &= -4.9t^2 + 120 \left(\frac{3}{5} \right) t = -4.9t^2 + 72t \end{aligned}$$



The parametric equations of the path of the projectile are $x = 96t, y = -4.9t^2 + 72t$.

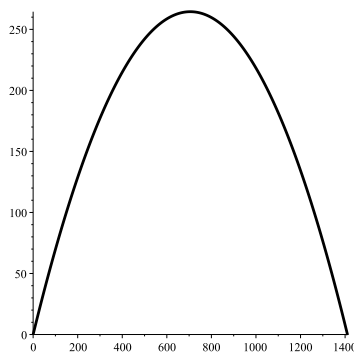
(b) Using the identity $\sin(2\theta) = 2 \sin \theta \cos \theta$, we find the range to be

$$\begin{aligned} x &= \frac{v_0^2 \sin(2\theta)}{g} = \frac{120^2 (2) \sin \left[\tan^{-1} \left(\frac{3}{4} \right) \right] \cos \left[\tan^{-1} \left(\frac{3}{4} \right) \right]}{9.8} \\ &= \frac{120^2 (2) \left(\frac{3}{5} \right) \left(\frac{4}{5} \right)}{9.8} \approx \boxed{1410.6 \text{ m}}. \end{aligned}$$

(c) The projectile is in the air for

$$t = \frac{2v_0 \sin \theta}{g} = \frac{(2)(120) \sin \left[\tan^{-1} \left(\frac{3}{4} \right) \right]}{9.8} = \frac{(2)(120) \left(\frac{3}{5} \right)}{9.8} \approx \boxed{14.7 \text{ s}}.$$

(d) The trajectory of the projectile is shown below.



29. The hill makes a $30^\circ = \frac{\pi}{6}$ radians angle to the horizontal, meaning that $\tan \left(\frac{\pi}{6} \right) = \frac{y}{x}$, or $y = x \tan \left(\frac{\pi}{6} \right) = \frac{\sqrt{3}}{3}x$. The equation for the hill is

$$y_{\text{hill}} = \frac{\sqrt{3}}{3}x.$$

Since $\mathbf{r}(0) = \mathbf{0}$ and the projectile has a constant initial velocity, we can use the formulas derived in the section.

The position vector of the projectile is given by

$$\begin{aligned}\mathbf{r}(t) &= (v_0 \cos \theta)t\mathbf{i} + \left(v_0 \sin \theta t - \frac{1}{2}gt^2\right)\mathbf{j} = 100\left(\frac{\sqrt{2}}{2}\right)\mathbf{i} + \left[100\left(\frac{\sqrt{2}}{2}\right) - 16t^2\right]\mathbf{j} \\ &= 50\sqrt{2}t\mathbf{i} + (-16t^2 + 50\sqrt{2}t)\mathbf{j}.\end{aligned}$$

The hill and the projectile intersect where $y_{\text{hill}} = y_{\text{projectile}}$, and both the equation for the hill and the equation for the projectile must be satisfied. We substitute the equations for the projectile into the equation for the hill.

$$\begin{aligned}-16t^2 + 50\sqrt{2}t &= \left(\frac{\sqrt{3}}{3}\right)(50\sqrt{2}t) \\ -48t^2 + 150\sqrt{2}t &= 50\sqrt{6}t \\ -24t^2 + 75\sqrt{2}t - 25\sqrt{6}t &= 0 \\ t(-24t + 75\sqrt{2} - 25\sqrt{6}) &= 0\end{aligned}$$

The projectile and the hill intersect at $t = 0$ and $t = \frac{25\sqrt{2}}{8} - \frac{25\sqrt{6}}{24} \approx 1.868$ s.

(a) To find the distance between the projectile's starting point and landing point, we evaluate

$$x(1.868) = 50\sqrt{2}(1.868) \approx 132.078 \text{ ft} \quad y(1.868) = -16(1.868)^2 + 50\sqrt{2}(1.868) \approx 76.255 \text{ ft}.$$

Using the Pythagorean Theorem, we find the distance traveled by the projectile to be

$$\sqrt{x^2 + y^2} \approx \sqrt{132.078^2 + 76.255^2} \approx \boxed{152.5 \text{ ft}}.$$

(b) The projectile is in the air for $\boxed{1.9 \text{ s}}$.

30. We know that $y_0 = 3$ m and that when $x = 30$ m, we want $y = 1$ m. Assuming the only other force that acts on the projectile is gravity, $\mathbf{a}(t) = -9.8\mathbf{j}$. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -9.8\mathbf{j} dt = v_x\mathbf{i} + (-9.8t + v_y)\mathbf{j}$$

Because the projectile is being propelled horizontally, $\mathbf{v}(0) = v_0$, so $v_x = v_0$ and $v_y = 0$. The velocity vector is

$$\mathbf{v}(t) = v_0\mathbf{i} - 9.8t\mathbf{j}.$$

The position vector of the projectile is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (v_0\mathbf{i} - 9.8t\mathbf{j}) dt = (v_0t + r_x)\mathbf{i} + (-4.9t^2 + r_y)\mathbf{j}.$$

Since $\mathbf{r}(0) = 3\mathbf{j}$, $r_x = 0$ and $r_y = 3$, the position vector is given by

$$\mathbf{r}(t) = v_0t\mathbf{i} + (-4.9t^2 + 3)\mathbf{j}.$$

At the time that the y -component is 1 m, we want the x -component to be 30 m. We now find the time at which that occurs. Solving $y = -4.9t^2 + 3 = 1$ gives $t \approx 0.639$ s. Substituting this time into the x -component allows us to solve for v_0 .

$$x = (v_0)(0.639) = 30 \quad \text{so} \quad v_0 \approx 47.0$$

The initial velocity of the projectile should be approximately $\boxed{47.0 \text{ m/s}}$.

31. We know $\mathbf{F} = m\mathbf{a}$, so $\|\mathbf{F}\| = \|m\mathbf{a}\| = m\|\mathbf{a}\|$, so $5 = (1)\|\mathbf{a}\|$ and $\|\mathbf{a}\| = 5$. The force acts in the direction of the positive x -axis, so $\theta = 0$, and the acceleration from the applied force is $5\mathbf{i}$. The acceleration is then $\mathbf{a}(t) = 5\mathbf{i} - 9.8\mathbf{j}$.

- (a) The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (5\mathbf{i} - 9.8\mathbf{j}) dt = (5t + c_1)\mathbf{i} + (-9.8t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = 3\mathbf{j}$, $c_1 = 0$ and $c_2 = 3$, the velocity vector is

$$\mathbf{v}(t) = \boxed{5t\mathbf{i} + (-9.8t + 3)\mathbf{j}}$$

and the speed is

$$v = \|\mathbf{v}(t)\| = \sqrt{(5t)^2 + (-9.8t + 3)^2} = \sqrt{25t^2 + 96.04t^2 - 58.8t + 9} = \boxed{\sqrt{121.04t^2 - 58.8t + 9}}.$$

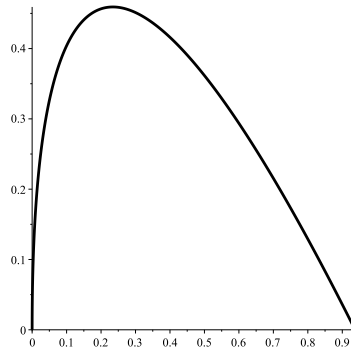
- (b) The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (5t\mathbf{i} + (-9.8t + 3)\mathbf{j}) dt = \left(\frac{5}{2}t^2 + d_1\right)\mathbf{i} + (-4.9t^2 + 3t + d_2)\mathbf{j}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $d_1 = 0$ and $d_2 = 0$, the position vector is given by

$$\mathbf{r}(t) = \boxed{\frac{5}{2}t^2\mathbf{i} + (-4.9t^2 + 3t)\mathbf{j}}.$$

- (c) The projectile is in the air until its y -component is zero, or when $-4.9t^2 + 3t = 0$. The projectile is in the air for $t \approx \boxed{0.612 \text{ s}}$.
- (d) The projectile has a parabolic shape. Because of the acceleration in the positive x -direction, the path of the projectile is stretched to the right.



32. Since $\mathbf{F} = \frac{\sqrt{2}m}{2}(-\mathbf{i} - \mathbf{j})$ and $\mathbf{F} = m\mathbf{a}$, $\mathbf{a} = \frac{\sqrt{2}m}{2}(-\mathbf{i} - \mathbf{j})$. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int \frac{\sqrt{2}m}{2}(-\mathbf{i} - \mathbf{j}) dt = \left(-\frac{\sqrt{2}}{2}t + c_1\right)\mathbf{i} + \left(-\frac{\sqrt{2}}{2}t + c_2\right)\mathbf{j}.$$

Since $\mathbf{v}(0) = 3\mathbf{i} + 4\mathbf{j}$, $c_1 = 3$ and $c_2 = 4$, the velocity vector is

$$\mathbf{v}(t) = \left(-\frac{\sqrt{2}}{2}t + 3\right)\mathbf{i} + \left(-\frac{\sqrt{2}}{2}t + 4\right)\mathbf{j}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \left[\left(-\frac{\sqrt{2}}{2}t + 3 \right) \mathbf{i} + \left(-\frac{\sqrt{2}}{2}t + 4 \right) \mathbf{j} \right] dt = \left(-\frac{\sqrt{2}}{4}t^2 + 3t + d_1 \right) \mathbf{i} + \left(-\frac{\sqrt{2}}{4}t^2 + 4t + d_2 \right) \mathbf{j}.$$

Since $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, $d_1 = 1$ and $d_2 = 2$, the position vector is given by

$$\mathbf{r}(t) = \left[\left(-\frac{\sqrt{2}}{4}t^2 + 3t + 1 \right) \mathbf{i} + \left(-\frac{\sqrt{2}}{4}t^2 + 4t + 2 \right) \mathbf{j} \right].$$

33. We begin with the acceleration due to gravity, $\mathbf{a}(t) = -32\mathbf{j}$, and integrate to first find the velocity vector and then the position vector. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -32\mathbf{j} dt = c_1\mathbf{i} + (-32t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = v_0 \cos(45^\circ)\mathbf{i} + v_0 \sin(45^\circ)\mathbf{j}$, $c_1 = \frac{\sqrt{2}}{2}v_0$ and $c_2 = \frac{\sqrt{2}}{2}v_0$, the velocity vector is

$$\mathbf{v}(t) = \frac{\sqrt{2}}{2}v_0\mathbf{i} + \left(-32t + \frac{\sqrt{2}}{2}v_0 \right) \mathbf{j}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \left[\frac{\sqrt{2}}{2}v_0\mathbf{i} + \left(-32t + \frac{\sqrt{2}}{2}v_0 \right) \mathbf{j} \right] dt = \left(\frac{\sqrt{2}}{2}v_0t + d_1 \right) \mathbf{i} + \left(-16t^2 + \frac{\sqrt{2}}{2}v_0t + d_2 \right) \mathbf{j}.$$

Since $\mathbf{r}(0) = 6.5\mathbf{j}$, $d_1 = 0$ and $d_2 = 6.5$, the position vector is given by

$$\mathbf{r}(t) = \left(\frac{\sqrt{2}}{2}v_0t \right) \mathbf{i} + \left(-16t^2 + \frac{\sqrt{2}}{2}v_0t + 6.5 \right) \mathbf{j}.$$

We will find the time t^* at which the ball lands in the basket. At t^* , $x(t^*) = 89.25$ ft and $y(t^*) = 10$ ft.

$$x(t^*) = \frac{\sqrt{2}}{2}v_0t^* = 89.25$$

$$y(t^*) = -16(t^*)^2 + \frac{\sqrt{2}}{2}v_0t^* + 6.5 = 10$$

so

$$t^* = \frac{89.25\sqrt{2}}{v_0}$$

and

$$y(t^*) = -16 \left[\frac{89.25\sqrt{2}}{v_0} \right]^2 + \frac{\sqrt{2}}{2}v_0 \left(\frac{89.25\sqrt{2}}{v_0} \right) + 6.5 = 10$$

$$v_0^2 = \frac{-16(89.25)^2(2)}{10 - 6.5 - 89.25}$$

$$v_0 \approx 54.521$$

The ball was tossed with initial velocity $\boxed{54.521 \text{ ft/s}}$.

34. We begin with the acceleration due to gravity, $\mathbf{a}(t) = -127008\mathbf{j}$, and integrate to first find the velocity vector and then the position vector. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -127008\mathbf{j} dt = c_1\mathbf{i} + (-127008t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = 400\mathbf{i}$, $c_1 = 400$ and $c_2 = 0$, the velocity vector is

$$\mathbf{v}(t) = 400\mathbf{i} - 127008t\mathbf{j}.$$

The position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int (400\mathbf{i} - 127008t\mathbf{j}) dt = (400t + d_1)\mathbf{i} + (-63504t^2 + d_2)\mathbf{j}.$$

Since $\mathbf{r}(0) = 4\mathbf{j}$, $d_1 = 0$ and $d_2 = 4$, the position vector is given by

$$\mathbf{r}(t) = 400t\mathbf{i} + (-63504t^2 + 4)\mathbf{j}.$$

We need $x(t^*) = T$ or $400t^* = T$. Then $\tan \alpha = \frac{4}{T} = \frac{4}{400t^*} = \frac{1}{100t^*}$. We also need $y(t^*) = 0$ or $4 - 63504(t^*)^2 = 0$, which means $t^* = \frac{1}{126}$ hr and

$$\tan \alpha = \frac{1}{100\left(\frac{1}{126}\right)}$$

or

$$\alpha = \tan^{-1}\left(\frac{63}{50}\right) \approx 0.8999 \text{ rad} \approx 51.6^\circ.$$

The angle of sight $\alpha \approx \boxed{51.6^\circ}$.

35. We begin with the acceleration due to gravity, $\mathbf{a}(t) = -32\mathbf{j}$, and integrate to first find the velocity vector and then the position vector. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -32\mathbf{j} dt = c_1\mathbf{i} + (-32t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = v_0 \cos 45^\circ \mathbf{i} + v_0 \sin 45^\circ \mathbf{j} = \frac{\sqrt{2}}{2}v_0\mathbf{i} + \frac{\sqrt{2}}{2}v_0\mathbf{j}$, $c_1 = \frac{\sqrt{2}}{2}v_0$ and $c_2 = \frac{\sqrt{2}}{2}v_0$, the velocity vector is

$$\mathbf{v}(t) = \frac{\sqrt{2}}{2}v_0\mathbf{i} + \left(-32t + \frac{\sqrt{2}}{2}v_0\right)\mathbf{j}.$$

The position vector of the baseball is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int \left[\frac{\sqrt{2}}{2}v_0\mathbf{i} + \left(-32t + \frac{\sqrt{2}}{2}v_0\right)\mathbf{j} \right] dt = \left(\frac{\sqrt{2}}{2}v_0t + d_1\right)\mathbf{i} + \left(-16t^2 + \frac{\sqrt{2}}{2}v_0t + d_2\right)\mathbf{j}.$$

Since $\mathbf{r}(0) = 3\mathbf{j}$, $d_1 = 0$ and $d_2 = 3$, the position vector is given by

$$\mathbf{r}(t) = \frac{\sqrt{2}}{2}v_0t\mathbf{i} + \left(-16t^2 + \frac{\sqrt{2}}{2}v_0t + 3\right)\mathbf{j}.$$

At the time t^* that the ball reaches the vines, $x(t^*) = 400$ and $y(t^*) = 10$. Then $t^* = \frac{400\sqrt{2}}{v_0}$ and

$$-16\left(\frac{400\sqrt{2}}{v_0}\right)^2 + \frac{\sqrt{2}}{2}v_0\left(\frac{400\sqrt{2}}{v_0}\right) + 3 = 10$$

$$v_0 \approx 114.14.$$

The initial speed of the baseball was $\boxed{114 \text{ ft/s}}$. Substituting $v_0 = 114.14$ into the equation for t^* gives $t^* \approx 4.956$ s. It takes the ball $\boxed{5 \text{ s}}$ to reach the vines.

36. We begin with the acceleration due to gravity, $\mathbf{a}(t) = -32\mathbf{j}$, and integrate to first find the velocity vector and then the position vector. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -32\mathbf{j} dt = c_1\mathbf{i} + (-32t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = 100 \cos 45^\circ \mathbf{i} + 100 \sin 45^\circ \mathbf{j} = 50\sqrt{2}\mathbf{i} + 50\sqrt{2}\mathbf{j}$, $c_1 = 50\sqrt{2}$ and $c_2 = 50\sqrt{2}$, the velocity vector is

$$\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + (-32t + 50\sqrt{2})\mathbf{j}.$$

The position vector of the baseball is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [50\sqrt{2}\mathbf{i} + (-32t + 50\sqrt{2})\mathbf{j}] dt = (50\sqrt{2}t + d_1)\mathbf{i} + (-16t^2 + 50\sqrt{2}t + d_2)\mathbf{j}.$$

Since $\mathbf{r}(0) = 6\mathbf{j}$, $d_1 = 0$ and $d_2 = 6$, the position vector is given by

$$\mathbf{r}(t) = 50\sqrt{2}t\mathbf{i} + (-16t^2 + 50\sqrt{2}t + 6)\mathbf{j}.$$

- (a) The ball reaches home plate when $y(t^*) = -16(t^*)^2 + 50\sqrt{2}t^* + 6 = 0$ or $t^* \approx 4.5$. The maximum distance occurs at $x(t^*) = 50\sqrt{2}(4.5) \approx 318.4$. The furthest an outfielder can be from home plate to ensure that the ball reaches home plate on the fly is about $\boxed{318 \text{ ft}}$.
- (b) The ball is in flight for $\boxed{t^* \approx 4.5 \text{ s}}$.
- (c) If the distance from third place to home plate is 90 ft, the player must run at a speed of $\frac{90}{4.5} \approx \boxed{20 \text{ ft/s}}$.
37. Since $\mathbf{r}(0) = \mathbf{0}$ and the football has a constant initial velocity, we can use the formulas derived in the section.

- (a) The range is

$$x = \frac{v_0^2 \sin(2\theta)}{g} = \frac{(65^2) \sin(60^\circ)}{32} = \boxed{114.342 \text{ ft}}.$$

- (b) The position vector is

$$\begin{aligned} \mathbf{r}(t) &= (v_0 \cos \theta)t\mathbf{i} + \left[(v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = 65 \cos(30^\circ)t\mathbf{i} + [65 \sin(30^\circ)t - 16t^2]\mathbf{j} \\ &= \frac{65\sqrt{3}}{2}t\mathbf{i} + \left(\frac{65}{2}t - 16t^2\right)\mathbf{j}. \end{aligned}$$

We want to find $x(t^*)$ for $y(t^*) = 10$.

$$\frac{65}{2}t^* - 16(t^*)^2 = 10$$

which means that $t^* \approx 0.378$ or $t^* \approx 1.65$. Since $x(t)$ is an increasing function of t for $t > 0$, $t^* \approx 1.65$ gives the maximum distance. The furthest from the goal post the kick can originate and score a field goal is

$$\frac{65\sqrt{3}}{2}(1.65) \approx \boxed{93 \text{ ft}}.$$

38. We will show that when the y -component of the projectile – its height – is the same as the y -component of the target, the x -component of the projectile is the same as the x -component of the target. Since the bullet begins at the origin and has a constant initial velocity, we can use the formulas derived in the section.

$$\mathbf{r}_{\text{projectile}} = (v_0 \cos \theta)t\mathbf{i} + \left(-\frac{1}{2}gt^2 + (v_0 \sin \theta)t\right)\mathbf{j}$$

The target has a constant x position and an initial height of T . We define the distance from the origin to the target which is the hypotenuse of the triangle in the figure as h . This makes the x -coordinate of the position $x_T = h \cos \theta_0$ and the y -coordinate of the position $y_T = h \sin \theta_0$. We begin with the acceleration due to gravity, $\mathbf{a}(t) = -g\mathbf{j}$, and integrate to first find the velocity vector and then the position vector. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -g\mathbf{j} dt = c_1\mathbf{i} + (-gt + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = \mathbf{0}$, the velocity vector is

$$\mathbf{v}(t) = -gt\mathbf{j}.$$

The position vector of the target is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int -gt dt = d_1\mathbf{i} + \left(-\frac{1}{2}gt^2 + d_2\right)\mathbf{j}.$$

Since $\mathbf{r}(0) = x_T\mathbf{i} + y_T\mathbf{j}$, $d_1 = x_T = h \cos \theta_0$ and $d_2 = y_T = h \sin \theta_0$, the position vector is given by

$$\mathbf{r}(t) = h \cos \theta_0 \mathbf{i} + \left(-\frac{1}{2}gt^2 + h \sin \theta_0\right)\mathbf{j}.$$

The time t^* when $y_{\text{projectile}} = y_T$ can be found.

$$\begin{aligned} y(t^*) &= -\frac{1}{2}g(t^*)^2 + h \sin \theta_0 = -\frac{1}{2}g(t^*)^2 + (v_0 \sin \theta_0)t^* \\ t^* &= \frac{h \sin \theta_0}{v_0 \sin \theta_0} = \frac{h}{v_0} \end{aligned}$$

We can now determine the x -position of the projectile at t^* .

$$x(t^*) = v_0 \cos \theta_0 \left(\frac{h}{v_0}\right) = h \cos \theta_0 = x_T$$

Since $x_{\text{projectile}}(t^*) = x_T$ when $y_{\text{projectile}}(t^*) = y_T$ and none of these expressions depend on v_0 , the bullet will always hit the falling target, no matter what the initial speed v_0 is. ■

39. If the position vector is $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - \frac{1}{2}gt^2]\mathbf{j}$, then $\mathbf{r}(0) = \mathbf{0}$, and the range of the projectile is $\frac{(v_0)^2 \sin(2\theta)}{g}$.

- (a) This range will be maximized when $\sin(2\theta)$ is maximized or when $\sin(2\theta) = 1$. We know $\sin(2\theta) = 1$ when $2\theta = \frac{\pi}{2}$ or when $\theta = \frac{\pi}{4}$. Equivalently, we could see that the range is maximized by taking the derivative of the range function.

$$\frac{d}{d\theta}x(\theta) = \frac{d}{d\theta} \left[\frac{v_0^2 \sin(2\theta)}{g} \right] = \frac{v_0^2 \cos(2\theta)2}{g}$$

The critical points for the range are $\theta = 0$ and $\theta = \frac{\pi}{4}$. Since $x'(\theta)$ is positive for $0 < \theta < \frac{\pi}{4}$ and is negative for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, the maximum range occurs at $\theta = \frac{\pi}{4}$. ■

- (b) The maximum range occurs when $\theta = \frac{\pi}{4}$, so the maximum range is

$$\frac{v_0^2 \sin(2\theta)}{g} = \frac{v_0^2 \sin\left(\frac{\pi}{2}\right)}{g} = \frac{v_0^2}{g}.$$

■

40. If the position vector is $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - \frac{1}{2}gt^2]\mathbf{j}$, then $\mathbf{r}(0) = \mathbf{0}$, so the maximum height of the projectile occurs at $t = \frac{v_0 \sin \theta}{g}$ and the velocity vector of the projectile is $\mathbf{v}(t) = v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - gt)\mathbf{j}$. The speed is then

$$v = \sqrt{v_0^2 \cos^2 \theta + v_0^2 \sin^2 \theta - 2v_0 \sin \theta gt + g^2 t^2} = \sqrt{v_0^2 - 2v_0 \sin \theta gt + g^2 t^2}.$$

To find the extrema of the speed, we look at the first derivative.

$$\frac{d}{dt}v = \frac{d}{dt}\sqrt{v_0^2 - 2v_0 \sin \theta gt + g^2 t^2} = \frac{1}{2}(v_0^2 - 2v_0 \sin \theta gt + g^2 t^2)^{-1/2}(-2v_0 \sin \theta g + 2g^2 t)$$

The critical point occurs when $-2v_0 \sin \theta g + 2g^2 t = 0$, or at $t = \frac{v_0 \sin \theta}{g}$. The derivative is negative for $0 < t < \frac{v_0 \sin \theta}{g}$ and positive for $t > \frac{v_0 \sin \theta}{g}$, so the minimum speed occurs at $t = \frac{v_0 \sin \theta}{g}$, which is when the maximum height of the projectile occurs. ■

41. If no outside forces are applied to the particle, then $\mathbf{F} = m\mathbf{a} = \mathbf{0}$, so $\mathbf{a} = \mathbf{0}$. Finding the velocity and position vectors,

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = c_1 \mathbf{i} + c_2 \mathbf{j}$$

and

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = (c_1 t + d_1)\mathbf{i} + (c_2 t + d_2)\mathbf{j}.$$

If $c_1 = c_2 = 0$, the particle is stationary. If either c_1 or c_2 is not zero, the position vector is the vector equation of a line, and its speed is $\|\mathbf{v}(t)\| = \sqrt{c_1^2 + c_2^2}$, which is a constant. Therefore, the particle is either stationary or moves with constant speed along a straight line. ■

42. If $\mathbf{r}'(t) = \mathbf{0}$ for all t on some interval I , then

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = c_1 \mathbf{i} + c_2 \mathbf{j} = \mathbf{c}$$

for all t in I . ■

43. Let $\mathbf{f}'(t) = f'_1(t)\mathbf{i} + f'_2(t)\mathbf{j} + f'_3(t)\mathbf{k}$ and $\mathbf{g}'(t) = g'_1(t)\mathbf{i} + g'_2(t)\mathbf{j} + g'_3(t)\mathbf{k}$. If $\mathbf{f}'(t) = \mathbf{g}'(t)$ for all t in some interval I , then $f'_1(t) = g'_1(t)$, $f'_2(t) = g'_2(t)$, and $f'_3(t) = g'_3(t)$. From Section 4.8 we know that if $f'(x) = g'(x)$ for all numbers x in an interval (a, b) , then there exists a number C for which $f(x) = g(x) + C$ on (a, b) . Applied to this situation, that means that $f_1(t) = g_1(t) + c_1$, $f_2(t) = g_2(t) + c_2$, and $f_3(t) = g_3(t) + c_3$, or $\mathbf{f}(t) = \mathbf{g}(t) + \mathbf{c}$. ■

Challenge Problems

44. Let $\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then

$$\begin{aligned} \int_a^b \mathbf{c} \cdot \mathbf{r}(t) dt &= \int_a^b [c_1 x(t) + c_2 y(t) + c_3 z(t)] dt = \int_a^b c_1 x(t) dt + \int_a^b c_2 y(t) dt + \int_a^b c_3 z(t) dt \\ &= c_1 \int_a^b x(t) dt + c_2 \int_a^b y(t) dt + c_3 \int_a^b z(t) dt = \mathbf{c} \cdot \int_a^b \mathbf{r}(t) dt. \end{aligned}$$

45. Let $\mathbf{c} = \int_a^b \mathbf{r}(t) dt$. Then

$$\|\mathbf{c}\|^2 = \mathbf{c} \cdot \mathbf{c} = \left| \int_a^b \mathbf{r}(s) ds \cdot \int_a^b \mathbf{r}(t) dt \right|.$$

Because $\int_a^b \mathbf{r}(s) ds$ is a constant with respect to the variable t , we can use the result from Problem 44 to bring it into the integral with respect to t .

$$\|\mathbf{c}\|^2 = \left| \int_a^b \left(\int_a^b \mathbf{r}(s) ds \right) \cdot \mathbf{r}(t) dt \right|$$

We can change the order of the terms in the dot product by commutativity, and then, since $\mathbf{r}(t)$ is a constant with respect to the variable s , we can bring $\mathbf{r}(t)$ into the integral with respect to s by using the result from Problem 44 again.

$$\|\mathbf{c}\|^2 = \left| \int_a^b \mathbf{r}(t) \cdot \left(\int_a^b \mathbf{r}(s) ds \right) dt \right| = \left| \int_a^b \left(\int_a^b \mathbf{r}(t) \cdot \mathbf{r}(s) ds \right) dt \right|$$

From problem 115 from section 5.4, we know that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

We use this result twice to obtain the following:

$$\begin{aligned} \|\mathbf{c}\|^2 &= \left| \int_a^b \left(\int_a^b \mathbf{r}(t) \cdot \mathbf{r}(s) ds \right) dt \right| \\ &\leq \int_a^b \left| \int_a^b \mathbf{r}(t) \cdot \mathbf{r}(s) ds \right| dt \\ &\leq \int_a^b \left(\int_a^b |\mathbf{r}(t) \cdot \mathbf{r}(s)| ds \right) dt \end{aligned}$$

We can use properties of the dot product to see that

$$\mathbf{r}(t) \cdot \mathbf{r}(s) = \|\mathbf{r}(t)\| \|\mathbf{r}(s)\| \cos \theta \leq \|\mathbf{r}(t)\| \|\mathbf{r}(s)\|,$$

which implies that

$$\begin{aligned} \|\mathbf{c}\|^2 &\leq \int_a^b \left(\int_a^b |\mathbf{r}(t) \cdot \mathbf{r}(s)| ds \right) dt \\ &\leq \int_a^b \left(\int_a^b \|\mathbf{r}(t)\| \|\mathbf{r}(s)\| ds \right) dt. \end{aligned}$$

Since $\|\mathbf{r}(t)\|$ is a constant with respect to the variable s , we can pull it out of the inner integral, rearrange the order, and simplify, obtaining

$$\begin{aligned} \|\mathbf{c}\|^2 &\leq \int_a^b \left(\int_a^b \|\mathbf{r}(t)\| \|\mathbf{r}(s)\| ds \right) dt \\ &= \int_a^b \|\mathbf{r}(t)\| \left(\int_a^b \|\mathbf{r}(s)\| ds \right) dt \\ &= \int_a^b \left(\int_a^b \|\mathbf{r}(s)\| ds \right) \|\mathbf{r}(t)\| dt. \end{aligned}$$

Finally, since $\int_a^b \|r(s)\| ds$ is a constant with respect to the variable t , we have

$$\begin{aligned}\|\mathbf{c}\|^2 &\leq \int_a^b \left(\int_a^b \|\mathbf{r}(s)\| ds \right) \|\mathbf{r}(t)\| dt \\ &= \left(\int_a^b \|\mathbf{r}(s)\| ds \right) \left(\int_a^b \|\mathbf{r}(t)\| dt \right) \\ &= \left(\int_a^b \|\mathbf{r}(t)\| dt \right) \left(\int_a^b \|\mathbf{r}(t)\| dt \right) \\ &= \left(\int_a^b \|\mathbf{r}(t)\| dt \right)^2.\end{aligned}$$

Since $\|\mathbf{c}\|^2 = \left\| \int_a^b \mathbf{r}(t) dt \right\|^2$, we have

$$\left\| \int_a^b \mathbf{r}(t) dt \right\|^2 \leq \left(\int_a^b \|\mathbf{r}(t)\| dt \right)^2$$

which, in turn, implies

$$\left\| \int_a^b \mathbf{r}(t) dt \right\| \leq \int_a^b \|\mathbf{r}(t)\| dt. \blacksquare$$

46. Let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then

$$\int_a^b \mathbf{c} \times \mathbf{r}(t) dt = \int_a^b ([c_2z(t) - c_3y(t)]\mathbf{i} - [c_1z(t) - c_3x(t)]\mathbf{j} + [c_1y(t) - c_2x(t)]\mathbf{k}) dt$$

We can rewrite the terms using properties of integrals.

$$\begin{aligned}\int_a^b [c_2z(t) - c_3y(t)]\mathbf{i} dt &= \left(c_2 \int_a^b z(t) dt - c_3 \int_a^b y(t) dt \right) \mathbf{i}, \\ \int_a^b -[c_1z(t) - c_3x(t)]\mathbf{j} dt &= - \left(c_1 \int_a^b z(t) dt - c_3 \int_a^b x(t) dt \right) \mathbf{j}, \\ \int_a^b [c_1y(t) - c_2x(t)]\mathbf{k} dt &= \left(c_1 \int_a^b y(t) dt - c_2 \int_a^b x(t) dt \right) \mathbf{k}\end{aligned}$$

Then

$$\int_a^b \mathbf{c} \times \mathbf{r}(t) dt = \mathbf{c} \times \int_a^b \mathbf{r}(t) dt. \quad \blacksquare$$

11.6 Application: Kepler's Laws of Planetary Motion

Applications and Extensions

- Kepler's Third Law says that the square of the period of revolution of a planet is proportional to the cube of the length of the semimajor axis of the planet's elliptical orbit. For a circular orbit, the semimajor axis is the radius, so for this planetary system, $(28.1 \text{ days})^2 = k(1.8 \times 10^{10} \text{ m})^3$, where k is the constant of proportionality. Solving for k , we find that for this system, $k \approx 1.354 \times 10^{-28} \frac{\text{days}^2}{\text{m}^3}$. The second planet is twice as far from the central star, so the length of its radius is $3.6 \times 10^{10} \text{ m}$, which means that the square of the period of revolution of the second planet is approximately $1.354 \times 10^{-28} \frac{\text{days}^2}{\text{m}^3} \cdot (3.6 \times 10^{10} \text{ m})^3 = 6316.8800 \text{ days}^2$. The period of revolution of the other planet is $\boxed{79.479 \text{ days}}$.

- For this system, $(1.77 \text{ days})^2 = k(422000 \text{ km})^3$, so $k \approx 4.169 \times 10^{-17} \frac{\text{days}^2}{\text{km}^3}$.

- Let T_1 be the period of Europa and $d_1 = 671000 \text{ km}$ be the distance from Europa to Jupiter. Then

$$T_1^2 = k(671000 \text{ km})^3 \quad \text{so} \quad T_1^2 \approx 12.594 \text{ days}^2 \quad \text{and} \quad T_1 \approx \boxed{3.55 \text{ days}}.$$

- Let $T_2 = 7.16 \text{ days}$ be the period of Ganymede and d_2 be its distance from Jupiter. Then

$$(7.16 \text{ days})^2 = kd_2^3 \quad \text{so} \quad d_2^3 \approx 1.230 \times 10^{18} \text{ km}^3 \quad \text{and} \quad d_2 \approx \boxed{1.07 \times 10^6 \text{ km}}.$$

- Since $\mathbf{r} \times \mathbf{v}$ is a constant and perpendicular to both $\mathbf{r}$ and $\mathbf{v}$, $\mathbf{r}$ and $\mathbf{v}$ must be in a plane.
- Since $\frac{dA}{dt}$ is constant, a change in area is proportional to a change of time without regard to the actual time.

Challenge Problems

- (a) Since $\frac{dA}{dt} = \frac{1}{2}\|\mathbf{D}\|$ which is a constant,

$$\int_0^T \frac{dA}{dt} dt = \int_0^T \frac{1}{2}\|\mathbf{D}\| dt$$

and

$$A = \frac{1}{2}\|\mathbf{D}\|T.$$

For a circular orbit, $A = \pi\|\mathbf{r}\|^2$, so $T = \frac{2\pi\|\mathbf{r}\|^2}{\|\mathbf{D}\|}$. Given that $\|\mathbf{D}\|^2 = \|\mathbf{r}\|GM(1 + e \cos \theta)$ in the proof of Kepler's first law and that $e = 0$ for a circle,

$$T = \frac{2\pi\|\mathbf{r}\|^2}{\sqrt{\|\mathbf{r}\|GM}} \quad \text{and} \quad T^2 = \frac{4\pi^2\|\mathbf{r}\|^3}{GM},$$

which shows that the square of the period is proportional to the cube of the radius, which is the length of the semimajor axis for a circular orbit. ■

- Since the gravitational constant G is measured in $\text{N}\cdot\text{m}^2/\text{kg}^2$ and $\text{N} = \text{kg}\cdot\text{m}/\text{s}^2$, we must convert the period of the Earth from days into seconds.

$$365 \frac{\text{days}}{\text{year}} \cdot 24 \frac{\text{hours}}{\text{day}} \cdot 60 \frac{\text{min}}{\text{hr}} \cdot 60 \frac{\text{sec}}{\text{min}} = 3.1536 \times 10^7 \frac{\text{sec}}{\text{year}}$$

From part (a), $M = \frac{4\pi^2\|\mathbf{r}\|^3}{GT^2}$, so

$$M = \frac{4\pi^2(1.5 \times 10^{11})^3}{(6.67 \times 10^{-11})(3.1536 \times 10^7)^2} \approx \boxed{2.0 \times 10^{30} \text{ kg}}.$$

6. We are given

$$\|\mathbf{r}\| = \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1 + e \cos \theta} \right).$$

The length of the semimajor axis corresponds to half the sum of the $\|\mathbf{r}\|$ values at $\theta = 0$ and $\theta = \pi$.

$$\text{Length of the semimajor axis} = \frac{1}{2} \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1+e} + \frac{1}{1-e} \right) = \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1-e^2} \right).$$

The length of the semiminor axis corresponds to the height of a right triangle that has as its base the line segment going from the center of the ellipse to the focus at the origin and as its hypotenuse the line segment going from the top minor axis vertex to the origin. The length of the base is equal to $\|\mathbf{r}\|$ at $\theta = 0$ subtracted from the length of the semimajor axis. The length of the hypotenuse is equal to the length of the semimajor axis by definition of the ellipse. Using the Pythagorean Theorem we get

$$\text{Length of the semiminor axis} = \sqrt{\left[\frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1-e^2} \right) \right]^2 - \left[\frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1-e^2} \right) - \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1+e} \right) \right]^2} = \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{\sqrt{1-e^2}} \right).$$

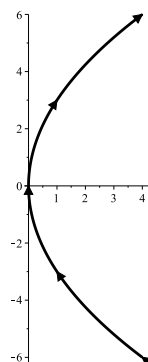
The area of the ellipse is πab where a is the semimajor axis and b is the semiminor axis.

$$\text{Area of the ellipse} = \pi \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{1-e^2} \right) \frac{\|\mathbf{D}\|^2}{GM} \left(\frac{1}{\sqrt{1-e^2}} \right) = \frac{\pi \|\mathbf{D}\|^4}{G^2 M^2} \left[\frac{1}{(1-e^2)^{3/2}} \right].$$

7. (a) Problem 75 (e) in section 11.4 showed that the angular component of acceleration is $r\theta'' + 2r'\theta'$. Given that the force is directed toward the origin, the angular component must be equal to zero. Therefore, $r\theta'' + 2r'\theta' = 0$.
- (b) Multiply by r to get $r^2\theta'' + 2rr'\theta' = 0$. Note that this is equivalent to $\frac{d(r^2\theta')}{dt} = 0$. Integrating both sides with respect to t gives $r^2\theta' = c$ where c is a constant of integration.
- (c) From section 9.6, $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$. We can replace $r^2 d\theta$ in this integral with $c dt$ from part (b). Carrying through the integration gives $A = \frac{1}{2} c_1 t + c_2$ where the c from above has been replaced by c_1 and c_2 is the constant from this integration. This result shows that A is linearly dependent on t so that equal areas are swept out in equal times.

Review Exercises

- Since $x(t) = t^2$ and $y(t) = 3t$ are defined for all real numbers, the domain of $\mathbf{r}(t)$ is all real numbers.
 - The parametric equations for the curve C are $x(t) = t^2$ and $y(t) = 3t$. We can eliminate the parameter by solving for t in $y = 3t$. We then substitute $t = \frac{y}{3}$ into x , and we get $x = \frac{y^2}{9}$. This equation represents a parabola that opens to the right with its vertex at the origin. As t increases, y increases, which gives the orientation. The curve C is shown below.

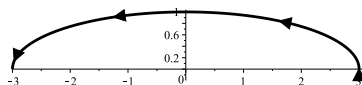


- Since

$$\mathbf{r}'(t) = \frac{d}{dt}(t^2\mathbf{i} + 3t\mathbf{j}) = \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(3t)\mathbf{j} = 2t\mathbf{i} + 3\mathbf{j},$$

$$\mathbf{r}'(2) = \boxed{4\mathbf{i} + 3\mathbf{j}}.$$

- Since the domain is restricted to $0 \leq t \leq \pi$ and $x(t) = 3 \cos t$ and $y(t) = \sin t$ are defined for all real numbers, the domain of $\mathbf{r}(t)$ is $\{t \mid 0 \leq t \leq \pi\}$.
 - The components of $\mathbf{r}(t)$ satisfy the equation $(\frac{x}{3})^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so $\mathbf{r}(t)$ traces out the top half of an ellipse with its semimajor axis along the x . The curve begins at $\mathbf{r}(0) = 3\mathbf{i}$, contains the vector $\mathbf{r}(\frac{\pi}{2}) = \mathbf{j}$, and ends at $\mathbf{r}(\pi) = -3\mathbf{i}$, so the orientation is counterclockwise. The curve C is shown below.

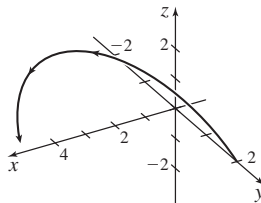


- Since

$$\mathbf{r}'(t) = \frac{d}{dt}(3 \cos t \mathbf{i} + \sin t \mathbf{j}) = \frac{d}{dt}(3 \cos t)\mathbf{i} + \frac{d}{dt}(\sin t)\mathbf{j} = -3 \sin t \mathbf{i} + \cos t \mathbf{j},$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = \boxed{-\frac{3\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}}.$$

- Since $x(t) = t$, $y(t) = 2 \cos t$, and $z(t) = 2 \sin t$ are defined for all real numbers, the domain of $\mathbf{r}(t)$ is all real numbers.
 - The parametric equations satisfy $y^2 + z^2 = 4 \cos^2 t + 4 \sin^2 t = 4$, so any point (x, y, z) on C will lie on the right circular cylinder $y^2 + z^2 = 4$. If $t = 0$, then $\mathbf{r}(0) = 2\mathbf{j}$, so the point $(0, 2, 0)$ is on the curve. As t increases, the x -component increases, so the curve winds around the circular cylinder in the positive x direction. The curve C is shown below.

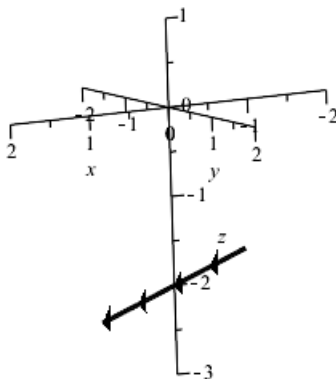


(c) Since

$$\mathbf{r}'(t) = \frac{d}{dt}(t\mathbf{i} + 2\cos t\mathbf{j} + 2\sin t\mathbf{k}) = \frac{d}{dt}t\mathbf{i} + \frac{d}{dt}(2\cos t)\mathbf{j} + \frac{d}{dt}(2\sin t)\mathbf{k} = \mathbf{i} - 2\sin t\mathbf{j} + 2\cos t\mathbf{k},$$

$$\boxed{\mathbf{r}'(0) = \mathbf{i} + 2\mathbf{k}}.$$

4. (a) Since $x(t) = t$, $y(t) = t$, and $z(t) = -2$ are defined for all real numbers, the domain of $\mathbf{r}(t)$ is all real numbers.
- (b) The components of $\mathbf{r}(t)$ are the parametric equations of a line in space containing the point $(0, 0, -2)$ in the direction of the vector $\mathbf{i} + \mathbf{j}$. As t increases, both x and y increase, which gives the orientation. The curve C is shown below.



(c) Since

$$\mathbf{r}'(t) = \frac{d}{dt}(t\mathbf{i} + t\mathbf{j} - 2\mathbf{k}) = \frac{d}{dt}t\mathbf{i} + \frac{d}{dt}t\mathbf{j} + \frac{d}{dt}(-2)\mathbf{k} = \mathbf{i} + \mathbf{j},$$

$$\mathbf{r}'(1) = \boxed{\mathbf{i} + \mathbf{j}}.$$

5.

$$\begin{aligned} \lim_{t \rightarrow 3} \left[(t-3)\mathbf{i} + \frac{t^2-3t}{t^2-9}\mathbf{j} - \frac{4}{t}\mathbf{k} \right] &= \lim_{t \rightarrow 3} (t-3)\mathbf{i} + \lim_{t \rightarrow 3} \left[\frac{t(t-3)}{(t+3)(t-3)} \right] \mathbf{j} - \lim_{t \rightarrow 3} \left(\frac{4}{t} \right) \mathbf{k} \\ &= \lim_{t \rightarrow 3} \left[\frac{t}{t+3} \right] \mathbf{j} - \frac{4}{3}\mathbf{k} = \boxed{\frac{1}{2}\mathbf{j} - \frac{4}{3}\mathbf{k}} \end{aligned}$$

6. The component function $x(t) = \frac{t}{t-2}$ is continuous for $t \neq 2$ and the component function $y(t) = \sin t$ is continuous for all real numbers. The vector function $\mathbf{r}(t)$ is not continuous at each number in the interval $(0, 2\pi)$. It has a discontinuity at $t = 2$.
7. The component function $x(t) = \frac{t}{t+1}$ is continuous for $t \neq -1$, which is not in the interval $[1, 3]$, the component function $y(t) = \sqrt{t-1}$ is continuous for all real numbers in $[1, 3]$, and the component function $z(t) = 3t$ is continuous for all real numbers, so $\mathbf{r}(t)$ is continuous.
8. Given $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j} + 2e^t\mathbf{k}$,

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}(3t\mathbf{i} - \ln t\mathbf{j} + 2e^t\mathbf{k}) = \frac{d}{dt}(3t)\mathbf{i} - \frac{d}{dt}(\ln t)\mathbf{j} + \frac{d}{dt}(2e^t)\mathbf{k} = \boxed{3\mathbf{i} - \frac{1}{t}\mathbf{j} + 2e^t\mathbf{k}} \\ \mathbf{r}''(t) &= \frac{d}{dt}\left(3\mathbf{i} - \frac{1}{t}\mathbf{j} + 2e^t\mathbf{k}\right) = \frac{d}{dt}3\mathbf{i} - \frac{d}{dt}\left(\frac{1}{t}\right)\mathbf{j} + \frac{d}{dt}(2e^t)\mathbf{k} = \boxed{\frac{1}{t^2}\mathbf{j} + 2e^t\mathbf{k}}.\end{aligned}$$

9. Given $\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\cos t\mathbf{j} + t\mathbf{k}$,

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}(2\cos t\mathbf{i} + 3\cos t\mathbf{j} + t\mathbf{k}) = \frac{d}{dt}(2\cos t)\mathbf{i} + \frac{d}{dt}(3\cos t)\mathbf{j} + \frac{d}{dt}t\mathbf{k} = \boxed{-2\sin t\mathbf{i} - 3\sin t\mathbf{j} + \mathbf{k}} \\ \mathbf{r}''(t) &= \frac{d}{dt}(-2\sin t\mathbf{i} - 3\sin t\mathbf{j} + \mathbf{k}) = \frac{d}{dt}(-2\sin t)\mathbf{i} - \frac{d}{dt}(3\sin t)\mathbf{j} + \frac{d}{dt}\mathbf{k} = \boxed{-2\cos t\mathbf{i} - 3\cos t\mathbf{j}}.\end{aligned}$$

10. The derivatives of $\mathbf{f}(t) = t\mathbf{i} - \frac{t}{2}\mathbf{j} + \mathbf{k}$ and $\mathbf{g}(t) = \sqrt{1-t}\mathbf{i} + t^3\mathbf{j} + 2(t+1)\mathbf{k}$ are $\mathbf{f}'(t) = \mathbf{i} - \frac{1}{2}\mathbf{j}$ and $\mathbf{g}'(t) = -\frac{1}{2\sqrt{1-t}}\mathbf{i} + 3t^2\mathbf{j} + 2\mathbf{k}$. To find $\frac{d}{dt}[\mathbf{f}(t) \cdot \mathbf{g}(t)]$, we use the dot product formula.

$$\begin{aligned}[\mathbf{f}(t) \cdot \mathbf{g}(t)]' &= \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t) \\ &= \left[\mathbf{i} - \frac{1}{2}\mathbf{j}\right] \cdot [\sqrt{1-t}\mathbf{i} + t^3\mathbf{j} + 2(t+1)\mathbf{k}] + \left[t\mathbf{i} - \frac{t}{2}\mathbf{j} + \mathbf{k}\right] \cdot \left[-\frac{1}{2\sqrt{1-t}}\mathbf{i} + 3t^2\mathbf{j} + 2\mathbf{k}\right] \\ &= \sqrt{1-t} - \frac{1}{2}t^3 - \frac{t}{2\sqrt{1-t}} - \frac{3}{2}t^3 + 2 \\ &= \boxed{\sqrt{1-t} - 2t^3 - \frac{t}{2\sqrt{1-t}} + 2}\end{aligned}$$

To find $\frac{d}{dt}[\mathbf{f}(t) \times \mathbf{g}(t)]$, we use the cross product formula.

$$\begin{aligned}[\mathbf{f}(t) \times \mathbf{g}(t)]' &= \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t) \\ &= \left[\mathbf{i} - \frac{1}{2}\mathbf{j}\right] \times [\sqrt{1-t}\mathbf{i} + t^3\mathbf{j} + 2(t+1)\mathbf{k}] + \left[t\mathbf{i} - \frac{t}{2}\mathbf{j} + \mathbf{k}\right] \times \left[-\frac{1}{2\sqrt{1-t}}\mathbf{i} + 3t^2\mathbf{j} + 2\mathbf{k}\right] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\frac{1}{2} & 0 \\ \sqrt{1-t} & t^3 & 2(t+1) \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -\frac{t}{2} & 1 \\ -\frac{1}{2\sqrt{1-t}} & 3t^2 & 2 \end{vmatrix}\end{aligned}$$

Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\frac{1}{2} & 0 \\ \sqrt{1-t} & t^3 & 2(t+1) \end{vmatrix} = (-t-1)\mathbf{i} - 2(t+1)\mathbf{j} + \left(t^3 + \frac{1}{2}\sqrt{1-t}\right)\mathbf{k}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -\frac{t}{2} & 1 \\ -\frac{1}{2\sqrt{1-t}} & 3t^2 & 2 \end{vmatrix} = (-t-3t^2)\mathbf{i} - \left(2t + \frac{1}{2\sqrt{1-t}}\right)\mathbf{j} + \left(3t^3 - \frac{t}{4\sqrt{1-t}}\right)\mathbf{k},$$

$$[\mathbf{f}(t) \times \mathbf{g}(t)]' = \boxed{-(3t^2 + 2t + 1)\mathbf{i} - \left(4t + \frac{1}{2\sqrt{1-t}} + 2\right)\mathbf{j} + \left(4t^3 + \frac{1}{2}\sqrt{1-t} - \frac{t}{4\sqrt{1-t}}\right)\mathbf{k}}.$$

11. The derivatives of $\mathbf{f}(t) = t\mathbf{i} + \cos(2t)\mathbf{j} - 5\mathbf{k}$ and $\mathbf{g}(t) = 2t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$ are $\mathbf{f}'(t) = \mathbf{i} - 2\sin(2t)\mathbf{j}$ and $\mathbf{g}'(t) = 2\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$. To find $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, we use the dot product formula.

$$\begin{aligned} [\mathbf{f}(t) \cdot \mathbf{g}(t)]' &= \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t) \\ &= [\mathbf{i} - 2\sin(2t)\mathbf{j}] \cdot [2t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}] + [t\mathbf{i} + \cos(2t)\mathbf{j} - 5\mathbf{k}] \cdot [2\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}] \\ &= 2t - 2\sin(2t)\cos t + 2t - \sin t\cos(2t) - 5\cos t \\ &= \boxed{4t - 2\sin(2t)\cos t - \sin t\cos(2t) - 5\cos t} \end{aligned}$$

To find $\frac{d}{dt}[\mathbf{f}(t) \times \mathbf{g}(t)]$, we use the cross product formula.

$$\begin{aligned} [\mathbf{f}(t) \times \mathbf{g}(t)]' &= \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t) \\ &= [\mathbf{i} - 2\sin(2t)\mathbf{j}] \times [2t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}] + [t\mathbf{i} + \cos(2t)\mathbf{j} - 5\mathbf{k}] \times [2\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2\sin(2t) & 0 \\ 2t & \cos t & \sin t \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & \cos(2t) & -5 \\ 2 & -\sin t & \cos t \end{vmatrix} \end{aligned}$$

Since

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2\sin(2t) & 0 \\ 2t & \cos t & \sin t \end{vmatrix} = -2\sin t\sin(2t)\mathbf{i} - \sin t\mathbf{j} + [\cos t + 4t\sin(2t)]\mathbf{k}$$

and

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & \cos(2t) & -5 \\ 2 & -\sin t & \cos t \end{vmatrix} = [\cos t\cos(2t) - 5\sin t]\mathbf{i} - [t\cos t + 10]\mathbf{j} - [t\sin t + 2\cos(2t)]\mathbf{k},$$

$$[\mathbf{f}(t) \times \mathbf{g}(t)]' =$$

$$\boxed{(-2\sin(2t)\sin t + \cos(2t)\cos t - 5\sin t)\mathbf{i} + (-10 - \sin t - t\cos t)\mathbf{j} + (\cos t - t\sin t + 4t\sin(2t) - 2\cos(2t))\mathbf{k}}.$$

12. We are given $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}$.

(a) $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} - \mathbf{j}}$ is tangent to C at $t = 0$.

(b) $\|\mathbf{r}'(t)\| = \sqrt{e^{2t} + e^{-2t}}$. Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}} \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.$$

- (c) Using the Quotient and Chain Rules,

$$\mathbf{T}'(t) = \frac{\sqrt{e^{2t} + e^{-2t}}(e^t\mathbf{i} + e^{-t}\mathbf{j}) - (e^t\mathbf{i} - e^{-t}\mathbf{j})\left(\frac{1}{2}\right)(e^{2t} + e^{-2t})^{-1/2}(2e^{2t} - 2e^{-2t})}{e^{2t} + e^{-2t}}.$$

We can multiply the numerator and the denominator by $\sqrt{e^{2t} + e^{-2t}}$ to simplify.

$$\begin{aligned}\mathbf{T}'(t) &= \frac{(e^{2t} + e^{-2t})(e^t \mathbf{i} + e^{-t} \mathbf{j}) - (e^t \mathbf{i} - e^{-t} \mathbf{j})(e^{2t} - e^{-2t})}{(e^{2t} + e^{-2t})^{3/2}} \\ &= \frac{e^{3t} \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{i} + e^{-3t} \mathbf{j} - (e^{3t} \mathbf{i} - e^{-t} \mathbf{i} - e^t \mathbf{j} + e^{-3t} \mathbf{j})}{(e^{2t} + e^{-2t})^{3/2}} = \frac{2e^{-t} \mathbf{i} + 2e^t \mathbf{j}}{(e^{2t} + e^{-2t})^{3/2}}\end{aligned}$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{4e^{-2t} + 4e^{2t}}{(e^{2t} + e^{-2t})^3}} = \frac{2\sqrt{e^{-2t} + e^{2t}}}{(e^{2t} + e^{-2t})^{3/2}} = \frac{2}{e^{2t} + e^{-2t}}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{2e^{-t} \mathbf{i} + 2e^t \mathbf{j}}{(e^{2t} + e^{-2t})^{3/2}}}{\frac{2}{e^{2t} + e^{-2t}}} = \frac{e^{-t} \mathbf{i} + e^t \mathbf{j}}{\sqrt{e^{-2t} + e^{2t}}} \quad \text{and} \quad \mathbf{N}(0) = \boxed{\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}}.$$

13. We are given $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \sqrt{1 - 5t^2}\mathbf{k}$, $-\frac{\sqrt{5}}{5} < t < \frac{\sqrt{5}}{5}$.

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} - \frac{5t}{\sqrt{1-5t^2}}\mathbf{k}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} + 2\mathbf{j}}$ is tangent to C at $t = 0$.

(b) $\|\mathbf{r}'(t)\| = \sqrt{1 + 4 + \frac{25t^2}{1-5t^2}} = \sqrt{\frac{5-25t^2+25t^2}{1-5t^2}} = \sqrt{\frac{5}{1-5t^2}}$. Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$,

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2\mathbf{j} - \frac{5t}{\sqrt{1-5t^2}}\mathbf{k}}{\sqrt{\frac{5}{1-5t^2}}} = \sqrt{\frac{1-5t^2}{5}}\mathbf{i} + 2\sqrt{\frac{1-5t^2}{5}}\mathbf{j} - \sqrt{5}t\mathbf{k} \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{5}}{5}\mathbf{i} + \frac{2\sqrt{5}}{5}\mathbf{j}}.$$

(c) Using the Constant Multiple and Chain Rules,

$$\mathbf{T}'(t) = \left(\frac{1}{\sqrt{5}}\right) \frac{-10t}{2\sqrt{1-5t^2}}\mathbf{i} + \left(\frac{2}{\sqrt{5}}\right) \frac{-10t}{2\sqrt{1-5t^2}}\mathbf{j} - \sqrt{5}\mathbf{k} = -t\sqrt{\frac{5}{1-5t^2}}\mathbf{i} - 2t\sqrt{\frac{5}{1-5t^2}}\mathbf{j} - \sqrt{5}\mathbf{k}.$$

Then

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{5t^2}{1-5t^2} + \frac{20t^2}{1-5t^2} + 5} = \sqrt{\frac{5t^2 + 20t^2 + 5 - 25t^2}{1-5t^2}} = \sqrt{\frac{5}{1-5t^2}}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{-t\sqrt{\frac{5}{1-5t^2}}\mathbf{i} - 2t\sqrt{\frac{5}{1-5t^2}}\mathbf{j} - \sqrt{5}\mathbf{k}}{\sqrt{\frac{5}{1-5t^2}}} = -t\mathbf{i} - 2t\mathbf{j} - \sqrt{1-5t^2}\mathbf{k} \quad \text{and} \quad \mathbf{N}(0) = \boxed{-\mathbf{k}}.$$

14. We are given $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + e^t \mathbf{k}$.

(a) $\mathbf{r}'(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k}$ so $\mathbf{r}'(0) = \boxed{\mathbf{i} + \mathbf{j} + \mathbf{k}}$ is tangent to C at $t = 0$.

(b)

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \sin^2 t - 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t}} \\ &= \sqrt{e^{2t}(\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t + 1)} = e^t \sqrt{3}.\end{aligned}$$

$$\text{Since } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

$$\mathbf{T}(t) = \frac{\sqrt{3}}{3}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}] \quad \text{and} \quad \mathbf{T}(0) = \boxed{\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})}.$$

(c) Then

$$\mathbf{T}'(t) = \frac{\sqrt{3}}{3}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

and

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{1}{3}(\sin^2 t - 2\sin t \cos t + \cos^2 t + \cos^2 t + 2\sin t \cos t + \sin^2 t)} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}.$$

$$\text{Since } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},$$

$$\mathbf{N}(t) = \frac{\frac{\sqrt{3}}{3}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]}{\frac{\sqrt{6}}{3}} = \frac{\sqrt{2}}{2}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

and

$$\mathbf{N}(0) = \boxed{\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}}.$$

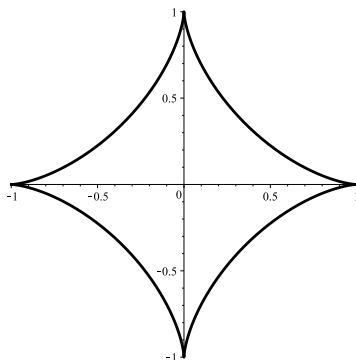
15. To find the angle θ between the tangent vector of $\mathbf{r}(t)$ and $\mathbf{j}$, we use the dot product.

$$\mathbf{r}' \cdot \mathbf{j} = \|\mathbf{r}'\| \|\mathbf{j}\| \cos \theta \quad \text{so} \quad \theta = \cos^{-1} \left(\frac{\mathbf{r}' \cdot \mathbf{j}}{\|\mathbf{r}'\| \|\mathbf{j}\|} \right) = \cos^{-1} \left(\frac{\mathbf{r}' \cdot \mathbf{j}}{\|\mathbf{r}'\|} \right)$$

Given $\mathbf{r}(t) = \sin t \mathbf{i} + 3t \mathbf{j} + \cos t \mathbf{k}$, $\mathbf{r}'(t) = \cos t \mathbf{i} + 3 \mathbf{j} - \sin t \mathbf{k}$ and

$$\mathbf{r}' \cdot \mathbf{j} = 3 \quad \text{and} \quad \|\mathbf{r}'\| = \sqrt{\cos^2 t + 9 + \sin^2 t} = \sqrt{10} \quad \text{so} \quad \theta = \boxed{\cos^{-1} \left(\frac{3\sqrt{10}}{10} \right)}.$$

16. A plot of $\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}$ reveals that the curve C is not smooth on $[0, 2\pi]$. We find the arc length of the curve on $[0, \frac{\pi}{2}]$ and use symmetry to find the total arc length.



$$\mathbf{r}'(t) = -3\cos^2 t \sin t \mathbf{i} + 3\sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = \sqrt{9\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} = |3\sin t \cos t|.$$

Now we use the formula for arc length. Since $3\sin t \cos t$ is positive on $[0, \frac{\pi}{2}]$, $|3\sin t \cos t| = 3\sin t \cos t$. We use symmetry since $\mathbf{r}(t)$ is smooth on $[0, \frac{\pi}{2}]$.

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = 4 \int_0^{\pi/2} 3\sin t \cos t dt$$

Let $\sin t = u$. Then $\cos t \, dt = du$. Now we change the limits of integration. When $t = 0$, $u = 0$, and when $t = \frac{\pi}{2}$, $u = 1$, and

$$4 \int_0^{\pi/2} 3 \sin t \cos t \, dt = 4 \int_0^1 3u \, du = 4 \left[\left(\frac{3}{2} \right) u^2 \right]_0^1 = \boxed{6}.$$

17. We are given $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \sqrt{1 - 5t^2}\mathbf{k}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} - \frac{5t}{\sqrt{1 - 5t^2}}\mathbf{k}$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4 + \frac{25t^2}{1 - 5t^2}} = \sqrt{\frac{5 - 25t^2 + 25t^2}{1 - 5t^2}} = \sqrt{\frac{5}{1 - 5t^2}}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{1/4} \sqrt{\frac{5}{1 - 5t^2}} \, dt$$

Let $u = \sqrt{5}t$. Then $du = \sqrt{5} \, dt$. Now we change the limits of integration. When $t = 0$, $u = 0$, and when $t = \frac{1}{4}$, $u = \frac{\sqrt{5}}{4}$.

$$\int_0^{1/4} \sqrt{\frac{5}{1 - 5t^2}} \, dt = \int_0^{\sqrt{5}/4} \frac{1}{\sqrt{1 - u^2}} \, du$$

We use the Table of Integrals in the subsection *Essential Integrals*.

$$\int_0^{\sqrt{5}/4} \frac{1}{\sqrt{1 - u^2}} \, du = [\sin^{-1} u]_0^{\sqrt{5}/4} = \boxed{\sin^{-1} \left(\frac{\sqrt{5}}{4} \right)}$$

18. We are given $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{t}{5}\mathbf{k}$.

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{5}\mathbf{k}$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{25}} = \sqrt{\frac{25 + 1}{25}} = \frac{\sqrt{26}}{5}.$$

Now we use the formula for arc length.

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{2\pi} \frac{\sqrt{26}}{5} \, dt = \left[\frac{t\sqrt{26}}{5} \right]_0^{2\pi} = \boxed{\frac{2\pi\sqrt{26}}{5}}$$

19. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

20. We begin by finding $\mathbf{r}'(t)$ and $\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 8\mathbf{k} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 64} = \sqrt{73} \quad \text{for all } t.$$

Since $\|\mathbf{r}'(t)\| \neq 1$ for all t , no, the parameter t does not measure arc length along C .

21. To find the curvature, we need $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= -4 \sin t \mathbf{i} - 4 \cos t \mathbf{j} \\ \|\mathbf{r}'(t)\| &= \sqrt{16 \sin^2 t + 16 \cos^2 t} = \sqrt{16} = 4 \\ \mathbf{T}(t) &= \frac{-4 \sin t \mathbf{i} - 4 \cos t \mathbf{j}}{4} = -\sin t \mathbf{i} - \cos t \mathbf{j} \\ \mathbf{T}'(t) &= -\cos t \mathbf{i} + \sin t \mathbf{j} \quad \|\mathbf{T}'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1\end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \boxed{\frac{1}{4}}.$$

22. To find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{i} - \sin t \mathbf{j} + \cos t \mathbf{k} \quad \|\mathbf{r}'(t)\| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2} \\ \mathbf{r}''(t) &= -\cos t \mathbf{j} - \sin t \mathbf{k} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t) \mathbf{i} + \sin t \mathbf{j} - \cos t \mathbf{k} \\ &= \mathbf{i} + \sin t \mathbf{j} - \cos t \mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \boxed{\frac{1}{2}}.$$

23. To find the curvature, we need $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and $\|\mathbf{T}'(t)\|$.

$$\begin{aligned}\mathbf{r}'(t) &= e^t \mathbf{i} + 4 \mathbf{j} - 5e^{-t} \mathbf{k} \quad \|\mathbf{r}'(t)\| = \sqrt{e^{2t} + 16 + 25e^{-2t}} \\ \mathbf{T}(t) &= \frac{e^t \mathbf{i} + 4 \mathbf{j} - 5e^{-t} \mathbf{k}}{\sqrt{e^{2t} + 16 + 25e^{-2t}}}\end{aligned}$$

We differentiate each component separately.

$$\begin{aligned}\frac{d}{dt} \left[\frac{e^t}{\sqrt{e^{2t} + 16 + 25e^{-2t}}} \right] &= \frac{e^t \sqrt{e^{2t} + 16 + 25e^{-2t}} - e^t \left(\frac{1}{2} \right) (e^{2t} + 16 + 25e^{-2t})^{-1/2} (2e^{2t} - 50e^{-2t})}{e^{2t} + 16 + 25e^{-2t}} \\ &= \frac{e^t (e^{2t} + 16 + 25e^{-2t}) - e^t (e^{2t} - 25e^{-2t})}{(e^{2t} + 16 + 25e^{-2t})^{3/2}} = \frac{16e^t + 50e^{-t}}{(e^{2t} + 16 + 25e^{-2t})^{3/2}} \\ \frac{d}{dt} \left[\frac{4}{\sqrt{e^{2t} + 16 + 25e^{-2t}}} \right] &= -2(e^{2t} + 16 + 25e^{-2t})^{-3/2} (2e^{2t} - 50e^{-2t}) = -\frac{4e^{2t} - 100e^{-2t}}{(e^{2t} + 16 + 25e^{-2t})^{3/2}} \\ \frac{d}{dt} \left[\frac{-5e^{-t}}{\sqrt{e^{2t} + 16 + 25e^{-2t}}} \right] &= \frac{5e^{-t} \sqrt{e^{2t} + 16 + 25e^{-2t}} + 5e^{-t} \left(\frac{1}{2} \right) (e^{2t} + 16 + 25e^{-2t})^{-1/2} (2e^{2t} - 50e^{-2t})}{e^{2t} + 16 + 25e^{-2t}} \\ &= \frac{5e^{-t} (e^{2t} + 16 + 25e^{-2t}) + 5e^{-t} (e^{2t} - 25e^{-2t})}{(e^{2t} + 16 + 25e^{-2t})^{3/2}} = \frac{10e^t + 80e^{-t}}{(e^{2t} + 16 + 25e^{-2t})^{3/2}}\end{aligned}$$

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{(16e^t + 50e^{-t})\mathbf{i} - (4e^{2t} - 100e^{-2t})\mathbf{j} + (10e^t + 80e^{-t})\mathbf{k}}{(e^{2t} + 16 + 25e^{-2t})^{3/2}} \\
\|\mathbf{T}'(t)\| &= \sqrt{\frac{(16e^t + 50e^{-t})^2 + (4e^{2t} - 100e^{-2t})^2 + (10e^t + 80e^{-t})^2}{(e^{2t} + 16 + 25e^{-2t})^3}} \\
&= \sqrt{\frac{256e^{2t} + 1600 + 2500e^{-2t} + 16e^{4t} - 800 + 10000e^{-4t} + 100e^{2t} + 1600 + 6400e^{-2t}}{(e^{2t} + 16 + 25e^{-2t})^3}} \\
&= \sqrt{\frac{16e^{4t} + 356e^{2t} + 2400 + 8900e^{-2t} + 10000e^{-4t}}{(e^{2t} + 16 + 25e^{-2t})^3}} \\
&= 2\sqrt{\frac{4e^{4t} + 89e^{2t} + 600 + 2225e^{-2t} + 2500e^{-4t}}{(e^{2t} + 16 + 25e^{-2t})^3}}
\end{aligned}$$

Then

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\sqrt{\frac{4e^{4t} + 89e^{2t} + 600 + 2225e^{-2t} + 2500e^{-4t}}{(e^{2t} + 16 + 25e^{-2t})^3}}}{\sqrt{e^{2t} + 16 + 25e^{-2t}}} = \boxed{\frac{2\sqrt{4e^{4t} + 89e^{2t} + 600 + 2225e^{-2t} + 2500e^{-4t}}}{(e^{2t} + 16 + 25e^{-2t})^2}}.$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \boxed{\frac{2\sqrt{100e^{-2t} + 25 + 4e^{2t}}}{(e^{2t} + 16 + 25e^{-2t})^{3/2}}}.$$

24. To find the curvature of $f(x)$ at $(1, -3)$, we need to find $f'(x)$, $f'(1)$, $f''(x)$, $f''(1)$, and $|f''(1)|$.

$$\begin{aligned}
f'(x) &= 3x^2 & f'(1) &= 3 \\
f''(x) &= 6x & f''(1) &= 6 & |f''(1)| &= 6 \\
\kappa &= \frac{|f''(1)|}{(1 + [f'(1)]^2)^{3/2}} = \frac{6}{(1 + [3]^2)^{3/2}} = \frac{6}{(10)^{3/2}} = \boxed{\frac{3\sqrt{10}}{50}}.
\end{aligned}$$

25. To find the curvature of $f(x)$ at $(0, 4)$, we need to find $f'(x)$, $f'(0)$, $f''(x)$, $f''(0)$, and $|f''(0)|$.

$$\begin{aligned}
f'(x) &= -8e^{-2x} & f'(0) &= -8 \\
f''(x) &= 16e^{-2x} & f''(0) &= 16 & |f''(0)| &= 16 \\
\kappa &= \frac{|f''(0)|}{(1 + [f'(0)]^2)^{3/2}} = \frac{16}{(1 + [-8]^2)^{3/2}} = \boxed{\frac{16}{65^{3/2}}}.
\end{aligned}$$

26. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned}
\mathbf{r}'(t) &= -2t\mathbf{i} - e^{-t}\mathbf{j} & \|\mathbf{r}'(t)\| &= \sqrt{4t^2 + e^{-2t}} \\
\mathbf{r}''(t) &= -2\mathbf{i} + e^{-t}\mathbf{j}
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2t & -e^{-t} & 0 \\ -2 & e^{-t} & 0 \end{vmatrix} = (-2te^{-t} - 2e^{-t})\mathbf{k} = -2e^{-t}(t+1)\mathbf{k} \\
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= |2e^{-t}(t+1)|
\end{aligned}$$

The curvature

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{|2e^{-t}(t+1)|}{(4t^2 + e^{-2t})^{3/2}}$$

and the radius of the osculating circle

$$\rho = \frac{1}{\kappa} = \frac{(4t^2 + e^{-2t})^{3/2}}{|2e^{-t}(t+1)|},$$

so the radius at $t = 0$ is $\rho = \boxed{\frac{1}{2}}$.

27. Since $x = \cos^2 t$ and $y = \sin^2 t$, we can eliminate the parameter using $x + y = \cos^2 t + \sin^2 t = 1$, so $y = 1 - x$, $-1 \leq x \leq 1$. This equation represents a line segment, so the curvature is $\kappa = 0$, and the radius of the osculating circle is

$$\rho = \frac{1}{\kappa} = \boxed{\infty}.$$

28. To find the radius of the osculating circle, we need to find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$ for $-1 < t < 1$.

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{i} + 2\mathbf{j} - \frac{t}{\sqrt{1-t^2}}\mathbf{k} \\ \|\mathbf{r}'(t)\| &= \sqrt{1 + 4 + \frac{t^2}{1-t^2}} = \sqrt{\frac{5-5t^2+t^2}{1-t^2}} = \sqrt{\frac{5-4t^2}{1-t^2}} \\ \mathbf{r}''(t) &= \frac{-\sqrt{1-t^2} + t(\frac{1}{2})(1-t^2)^{-1/2}(-2t)}{(1-t^2)}\mathbf{k}\end{aligned}$$

We can multiply the numerator and the denominator by $\sqrt{1-t^2}$ to simplify.

$$\mathbf{r}''(t) = \frac{-1 + t^2 - t^2}{(1-t^2)^{3/2}} = -\frac{1}{(1-t^2)^{3/2}}\mathbf{k}$$

Then

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -\frac{t}{\sqrt{1-t^2}} \\ 0 & 0 & -\frac{1}{(1-t^2)^{3/2}} \end{vmatrix} = -\frac{2}{(1-t^2)^{3/2}}\mathbf{i} + \frac{1}{(1-t^2)^{3/2}}\mathbf{j} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{\frac{4}{(1-t^2)^3} + \frac{1}{(1-t^2)^3}} = \sqrt{\frac{5}{(1-t^2)^3}} = \frac{\sqrt{5}}{(1-t^2)^{3/2}}\end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\frac{\sqrt{5}}{(1-t^2)^{3/2}}}{\left(\frac{5-4t^2}{1-t^2}\right)^{3/2}} = \frac{\sqrt{5}}{(5-4t^2)^{3/2}}$$

and the radius of the osculating circle is

$$\rho = \frac{1}{\kappa} = \frac{(5-4t^2)^{3/2}}{\sqrt{5}},$$

so the radius at $t = 0$ is $\rho = \frac{5^{3/2}}{\sqrt{5}} = \boxed{5}$.

29. To find the curvature of $y = \sqrt[3]{x}$, $x > 0$, we need y' , y'' , and $|y''|$.

$$y' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \quad y'' = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}} \quad |y''| = \frac{2}{9x^{5/3}}$$

We know that $\frac{2}{9x^{5/3}}$ is positive since $x > 0$. The curvature is

$$\kappa = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} = \frac{\frac{2}{9x^{5/3}}}{(1 + [\frac{1}{3x^{2/3}}]^2)^{3/2}} = \frac{\frac{2}{9x^{5/3}}}{\left(\frac{9x^{4/3}+1}{9x^{4/3}}\right)^{3/2}} = \boxed{\frac{6x^{1/3}}{(9x^{4/3} + 1)^{3/2}}}.$$

30. To find the curvature κ , so we first find $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$, and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$.

$$\begin{aligned} \mathbf{r}'(t) &= -3 \sin t \cos^2 t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j} = 3 \sin t \cos t (-\cos t \mathbf{i} + \sin t \mathbf{j}) \\ \|\mathbf{r}'(t)\| &= 3 \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} = 3 \sin t \cos t \\ \mathbf{r}''(t) &= (-3 \cos^3 t + 6 \sin^2 t \cos t) \mathbf{i} + (6 \sin t \cos^2 t - 3 \sin^3 t) \mathbf{j} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \sin t \cos^2 t & 3 \sin^2 t \cos t & 0 \\ -3 \cos^3 t + 6 \sin^2 t \cos t & 6 \sin t \cos^2 t - 3 \sin^3 t & 0 \end{vmatrix} \\ &= 9 \sin t \cos t \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ -\cos^3 t + 2 \sin^2 t \cos t & 2 \sin t \cos^2 t - \sin^3 t & 0 \end{vmatrix} \\ &= 9 \sin t \cos t (-2 \sin t \cos^3 t + \sin^3 t \cos t + \sin t \cos^3 t - 2 \sin^3 t \cos t) \mathbf{k} \\ &= 9 \sin t \cos t (-\sin t \cos^3 t - \sin^3 t \cos t) \mathbf{k} = -9 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) \mathbf{k} \\ &= -9 \sin^2 t \cos^2 t \mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= 9 \sin^2 t \cos^2 t \end{aligned}$$

The curvature is

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{9 \sin^2 t \cos^2 t}{(3 \sin t \cos t)^3} = \frac{1}{|3 \sin t \cos t|}.$$

The curvature will be minimized when the denominator is maximized. Let

$$f(t) = |3 \sin t \cos t| = \left| \frac{3}{2} \sin(2t) \right|.$$

This function is maximized when $\sin(2t) = \pm 1$, or $t = \frac{(2n+1)\pi}{4}$. Then the minimum curvature is

$$\kappa \left[\frac{(2n+1)\pi}{4} \right] = \frac{1}{\left| 3 \left(\pm \frac{1}{\sqrt{2}} \right) \left(\pm \frac{1}{\sqrt{2}} \right) \right|} = \boxed{\frac{2}{3}}.$$

31. (a) The velocity, acceleration, and speed are

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (2 \cos t \mathbf{i} + \sin t \mathbf{j}) = \boxed{-2 \sin t \mathbf{i} + \cos t \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (-2 \sin t \mathbf{i} + \cos t \mathbf{j}) = \boxed{-2 \cos t \mathbf{i} - \sin t \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{4 \sin^2 t + \cos^2 t} = \sqrt{3 \sin^2 t + \sin^2 t + \cos^2 t} = \boxed{\sqrt{3 \sin^2 t + 1}}. \end{aligned}$$

(b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{6 \sin t \cos t}{2\sqrt{3 \sin^2 t + 1}} = \boxed{\frac{3 \sin t \cos t}{\sqrt{3 \sin^2 t + 1}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin t & \cos t & 0 \\ -2 \cos t & -\sin t & 0 \end{vmatrix} = (2 \sin^2 t + 2 \cos^2 t) \mathbf{k} = 2 \mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|2 \mathbf{k}\|}{\sqrt{3 \sin^2 t + 1}} = \boxed{\frac{2}{\sqrt{3 \sin^2 t + 1}}}.$$

32. (a) The velocity, acceleration, and speed are

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (e^t \sin t \mathbf{i} + e^{-t} \mathbf{j}) = \boxed{(e^t \sin t + e^t \cos t) \mathbf{i} - e^{-t} \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} [(e^t \sin t + e^t \cos t) \mathbf{i} - e^{-t} \mathbf{j}] \\ &= (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \mathbf{i} + e^{-t} \mathbf{j} = \boxed{2e^t \cos t \mathbf{i} + e^{-t} \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{-2t}} \\ &= \boxed{\sqrt{e^{2t} + 2e^{2t} \sin t \cos t + e^{-2t}}}. \end{aligned}$$

(b) The tangential component of the acceleration is

$$\begin{aligned} a_{\mathbf{T}} &= v'(t) = \frac{2e^{2t} + 4e^{2t} \sin t \cos t + 2e^{2t} \cos^2 t - 2e^{2t} \sin^2 t - 2e^{-2t}}{2\sqrt{e^{2t} + 2e^{2t} \sin t \cos t + e^{-2t}}} \\ &= \boxed{\frac{e^{2t} + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t - e^{2t} \sin^2 t - e^{-2t}}{\sqrt{e^{2t} + 2e^{2t} \sin t \cos t + e^{-2t}}}}. \end{aligned}$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t \sin t + e^t \cos t & -e^{-t} & 0 \\ 2e^t \cos t & e^{-t} & 0 \end{vmatrix} = (\sin t + \cos t + 2 \cos t) \mathbf{k} = (\sin t + 3 \cos t) \mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|(\sin t + 3 \cos t) \mathbf{k}\|}{\sqrt{e^{2t} + 2e^{2t} \sin t \cos t + e^{-2t}}} = \boxed{\frac{|\sin t + 3 \cos t|}{\sqrt{e^{2t} + 2e^{2t} \sin t \cos t + e^{-2t}}}}.$$

33. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (e^t \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k}) = \boxed{e^t \mathbf{i} - e^{-t} \mathbf{j}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} (e^t \mathbf{i} - e^{-t} \mathbf{j}) = \boxed{e^t \mathbf{i} + e^{-t} \mathbf{j}} \\ v(t) &= \|\mathbf{r}'(t)\| = \boxed{\sqrt{e^{2t} + e^{-2t}}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \frac{2e^{2t} - 2e^{-2t}}{2\sqrt{e^{2t} + e^{-2t}}} = \boxed{\frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & 0 \\ e^t & e^{-t} & 0 \end{vmatrix} = 2\mathbf{k}$$

so

$$a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|2\mathbf{k}\|}{\sqrt{e^{2t} + e^{-2t}}} = \boxed{\frac{2}{\sqrt{e^{2t} + e^{-2t}}}}.$$

34. (a) The velocity, acceleration, and speed are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \frac{d}{dt} (e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + e^t \mathbf{k}) = \boxed{(e^t \sin t + e^t \cos t) \mathbf{i} + (e^t \cos t - e^t \sin t) \mathbf{j} + e^t \mathbf{k}} \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \frac{d}{dt} \mathbf{r}'(t) = \frac{d}{dt} [e^t \sin t + e^t \cos t] \mathbf{i} + \frac{d}{dt} [e^t \cos t - e^t \sin t] \mathbf{j} + \frac{d}{dt} e^t \mathbf{k} \\ &= (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \mathbf{j} + e^t \mathbf{k} \\ &= \boxed{2e^t \cos t \mathbf{i} - 2e^t \sin t \mathbf{j} + e^t \mathbf{k}} \\ v(t) &= \|\mathbf{r}'(t)\| \\ &= \sqrt{e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t + e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t}} \\ &= \sqrt{e^{2t} (\sin^2 t + \cos^2 t + \cos^2 t + \sin^2 t + 1)} = \boxed{e^t \sqrt{3}}.\end{aligned}$$

- (b) The tangential component of the acceleration is

$$a_{\mathbf{T}} = v'(t) = \boxed{e^t \sqrt{3}}.$$

To find the normal component of the acceleration, note that

$$a_{\mathbf{N}} = [v(t)]^2 \kappa \quad \text{and} \quad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{[v(t)]^3} \quad \text{so} \quad a_{\mathbf{N}} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$

Then

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t & e^t \\ 2e^t \cos t & -2e^t \sin t & e^t \end{vmatrix} \\ &= (e^{2t} \cos t + e^{2t} \sin t)\mathbf{i} - (e^{2t} \sin t - e^{2t} \cos t)\mathbf{j} + (-2e^{2t} \sin^2 t - 2e^{2t} \cos^2 t)\mathbf{k} \\ &= [e^{2t}(\cos t + \sin t)]\mathbf{i} - [e^{2t}(\sin t - \cos t)]\mathbf{j} - 2e^{2t}\mathbf{k}\end{aligned}$$

so

$$\begin{aligned}a_N &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|[e^{2t}(\cos t + \sin t)]\mathbf{i} - [e^{2t}(\sin t - \cos t)]\mathbf{j} - 2e^{2t}\mathbf{k}\|}{e^{3t}\sqrt{3}} \\ &= \frac{e^{2t}\sqrt{\cos^2 t + 2\sin t \cos t + \sin^2 t + \sin^2 t - 2\sin t \cos t + \cos^2 t + 4}}{e^{3t}\sqrt{3}} \\ &= \frac{e^{2t}\sqrt{6}}{e^{3t}\sqrt{3}} = \boxed{e^t\sqrt{2}}.\end{aligned}$$

35. The velocity and acceleration of a particle moving on the cycloid are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \boxed{(\pi - \cos(\pi t))\mathbf{i} + \pi \sin(\pi t)\mathbf{j}} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= \frac{d}{dt}\mathbf{r}'(t) = \boxed{\pi^2 \sin(\pi t)\mathbf{i} + \pi^2 \cos(\pi t)\mathbf{j}}.\end{aligned}$$

36. (a) The velocity and acceleration of a particle moving on the parabola are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \boxed{(2t - 2)\mathbf{i} + \mathbf{j}} \\ \mathbf{a}(t) = \mathbf{r}''(t) &= \frac{d}{dt}\mathbf{r}'(t) = \boxed{2\mathbf{i}}.\end{aligned}$$

(b) The speed of the particle is $v(t) = \|\mathbf{r}'(t)\| = \sqrt{4t^2 - 8t + 4 + 1} = \sqrt{4t^2 - 8t + 5}$. The speed $v(t) = \sqrt{4t^2 - 8t + 5} = 0$ when $4t^2 - 8t + 5 = 0$. Using the quadratic equation,

$$t = \frac{8 \pm \sqrt{64 - 4(4)(5)}}{2(4)} = \frac{8 \pm \sqrt{-16}}{8},$$

so the speed of the particle is $\boxed{\text{never } 0}$.

37. We integrate each component individually. We use the substitution $u = t - 2$ and $du = dt$ in the second component.

$$\begin{aligned}\int (t^2 - 2)\mathbf{i} dt - \int (t - 2)^2 \mathbf{j} dt + \int e^{2t} \mathbf{k} dt &= \left(\frac{1}{3}t^3 - 2t + c_1\right)\mathbf{i} - \left[\frac{1}{3}(t - 2)^3 + c_2\right]\mathbf{j} + \left(\frac{1}{2}e^{2t} + c_3\right)\mathbf{k} \\ &= \left(\frac{1}{3}t^3 - 2t\right)\mathbf{i} - \frac{1}{3}(t - 2)^3 \mathbf{j} + \frac{1}{2}e^{2t}\mathbf{k} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \\ &= \boxed{\left(\frac{1}{3}t^3 - 2t\right)\mathbf{i} - \frac{1}{3}(t - 2)^3 \mathbf{j} + \frac{1}{2}e^{2t}\mathbf{k} + \mathbf{c}}\end{aligned}$$

38. We integrate each component individually. We use the substitution $u = \frac{t}{2}$, $du = \frac{1}{2} dt$, $dt = 2 du$ in the

first component, and we use the identity $\sin^2 t = \frac{1}{2} - \frac{1}{2} \cos(2t)$ in the second component.

$$\begin{aligned}
 \int \cos\left(\frac{t}{2}\right) \mathbf{i} dt + \int \sin^2\left(\frac{t}{2}\right) \mathbf{j} dt + \int 3t \mathbf{k} dt &= \int 2 \cos u \mathbf{i} du + \frac{1}{2} \int (1 - \cos t) \mathbf{j} dt + \int 3t \mathbf{k} dt \\
 &= (2 \sin u + c_1) \mathbf{i} + \frac{1}{2} (t - \sin t + c_2) \mathbf{j} + \left(\frac{3}{2} t^2 + c_3\right) \mathbf{k} \\
 &= \left[2 \sin\left(\frac{t}{2}\right) + c_1\right] \mathbf{i} + \frac{1}{2} (t - \sin t + c_2) \mathbf{j} + \left(\frac{3}{2} t^2 + c_3\right) \mathbf{k} \\
 &= 2 \sin\left(\frac{t}{2}\right) \mathbf{i} + \frac{1}{2} (t - \sin t) \mathbf{j} + \frac{3}{2} t^2 \mathbf{k} + c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k} \\
 &= \boxed{2 \sin\left(\frac{t}{2}\right) \mathbf{i} + \frac{1}{2} (t - \sin t) \mathbf{j} + \frac{3}{2} t^2 \mathbf{k} + \mathbf{c}}
 \end{aligned}$$

39. To find the general solution to the differential equation, we integrate each component. We use the Table of Integrals in the subsection *Integrals Containing Exponential and Logarithmic Functions* or use integration by parts for the integral of the second component.

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (e^{2t} \mathbf{i} + \ln t \mathbf{j} + 2e^t \mathbf{k}) dt = \left(\frac{1}{2} e^{2t} + c_1\right) \mathbf{i} + (t \ln t - t + c_2) \mathbf{j} + (2e^t + c_3) \mathbf{k}$$

Now we use the initial condition $\mathbf{r}(1) = \mathbf{j} + \mathbf{k}$

$$\mathbf{r}(1) = \left(\frac{1}{2} e^2 + c_1\right) \mathbf{i} + (-1 + c_2) \mathbf{j} + (2e + c_3) \mathbf{k} = \mathbf{j} + \mathbf{k}$$

from which we find

$$c_1 = -\frac{1}{2} e^2 \quad c_2 = 2 \quad c_3 = 1 - 2e.$$

The vector function is $\mathbf{r} = \mathbf{r}(t) = \boxed{\left(\frac{1}{2} e^{2t} - \frac{1}{2} e^2\right) \mathbf{i} + (t \ln t - t + 2) \mathbf{j} + (2e^t + 1 - 2e) \mathbf{k}}.$

40. To find the general solution to the differential equation, we integrate each component. For the first component, we use the substitution $u = \sin t$, $du = \cos t dt$.

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (\sin t \cos t \mathbf{i} + \tan t \sec t \mathbf{j} + t \mathbf{k}) dt = \left(\frac{1}{2} \sin^2 t + c_1\right) \mathbf{i} + (\sec t + c_2) \mathbf{j} + \left(\frac{1}{2} t^2 + c_3\right) \mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$

$$\mathbf{r}(0) = c_1 \mathbf{i} + (1 + c_2) \mathbf{j} + c_3 \mathbf{k} = \mathbf{i} + \mathbf{k}$$

from which we find

$$c_1 = 1 \quad c_2 = -1 \quad c_3 = 1.$$

The vector function is $\mathbf{r} = \mathbf{r}(t) = \boxed{\left(\frac{1}{2} \sin^2 t + 1\right) \mathbf{i} + (\sec t - 1) \mathbf{j} + \left(\frac{1}{2} t^2 + 1\right) \mathbf{k}}.$

41. The general solution to the differential equation is

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int \left(\cos t \mathbf{i} - \sin t \mathbf{j} + \frac{t}{2} \mathbf{k}\right) dt = (\sin t + c_1) \mathbf{i} + (\cos t + c_2) \mathbf{j} + \left(\frac{1}{4} t^2 + c_3\right) \mathbf{k}.$$

Now we use the initial condition $\mathbf{r}(0) = \mathbf{i}$

$$\mathbf{r}(0) = c_1\mathbf{i} + (1 + c_2)\mathbf{j} + c_3\mathbf{k} = \mathbf{i}$$

from which we find

$$c_1 = 1 \quad c_2 = -1 \quad c_3 = 0.$$

The vector function is $\mathbf{r} = \mathbf{r}(t) = \boxed{(\sin t + 1)\mathbf{i} + (\cos t - 1)\mathbf{j} + \frac{1}{4}t^2\mathbf{k}}$.

42. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (\tan t \sec t \mathbf{i} + \cos t \mathbf{j}) dt = (\sec t + c_1)\mathbf{i} + (\sin t + c_2)\mathbf{j}.$$

Since $\mathbf{v}(0) = \mathbf{0}$, $c_1 = -1$ and $c_2 = 0$, the velocity vector is

$$\mathbf{v}(t) = \boxed{(\sec t - 1)\mathbf{i} + \sin t \mathbf{j}}$$

and the speed is

$$v(t) = \|\mathbf{v}(t)\| = \boxed{\sqrt{\sec^2 t - 2\sec t + 1 + \sin^2 t}}.$$

To find the position vector, we integrate the velocity vector. We use the Table of Integrals in the subsection *Essential Integrals* for $\int \sec t dt$. Then the position vector of the particle is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [(\sec t - 1)\mathbf{i} + \sin t \mathbf{j}] dt = (\ln |\sec t + \tan t| - t + d_1)\mathbf{i} + (-\cos t + d_2)\mathbf{j}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, $d_1 = 0$ and $d_2 = 1$, the position vector is given by

$$\mathbf{r}(t) = \boxed{(\ln |\sec t + \tan t| - t)\mathbf{i} + (1 - \cos t)\mathbf{j}}.$$

43. The information in the problem gives $v_0 = 500$ m/s, $\theta = 45^\circ$, and $\mathbf{r}(0) = 30\mathbf{j}$. Then $\mathbf{v}(0) = 500 \cos 45^\circ \mathbf{i} + 500 \sin 45^\circ \mathbf{j} = 250\sqrt{2}\mathbf{i} + 250\sqrt{2}\mathbf{j}$. Assuming the only force that acts on the projectile is gravity, $\mathbf{a}(t) = -9.8\mathbf{j}$. The velocity vector $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int -9.8\mathbf{j} dt = c_1\mathbf{i} + (-9.8t + c_2)\mathbf{j}$$

Now we use the initial condition $\mathbf{v}(0) = 250\sqrt{2}\mathbf{i} + 250\sqrt{2}\mathbf{j}$

$$\mathbf{v}(0) = c_1\mathbf{i} + c_2\mathbf{j} = 250\sqrt{2}\mathbf{i} + 250\sqrt{2}\mathbf{j}$$

from which we find

$$c_1 = 250\sqrt{2} \quad c_2 = 250\sqrt{2}.$$

The velocity vector is

$$\mathbf{v}(t) = 250\sqrt{2}\mathbf{i} + (-9.8t + 250\sqrt{2})\mathbf{j}.$$

The position vector of the projectile is given by

$$\mathbf{r}(t) = \int \mathbf{v}(t) dt = \int [250\sqrt{2}\mathbf{i} + (-9.8t + 250\sqrt{2})\mathbf{j}] dt = (250\sqrt{2}t + d_1)\mathbf{i} + (-4.9t^2 + 250\sqrt{2}t + d_2)\mathbf{j}.$$

Since $\mathbf{r}(0) = 30\mathbf{j}$, $d_1 = 0$ and $d_2 = 30$, the position vector is given by

$$\mathbf{r}(t) = 250\sqrt{2}t\mathbf{i} + (-4.9t^2 + 250\sqrt{2}t + 30)\mathbf{j}.$$

We want to find the value of the x -component when the y -component is 0. We find the time at which that occurs. Solving $y = -4.9t^2 + 250\sqrt{2}t + 30 = 0$ gives $t \approx 72.239$ sec. Substituting this time into the x -component gives the position where the projectile strikes the ground.

$$x = 250\sqrt{2}(72.239) = \quad \text{so } x \approx 25540$$

The position where the projectile strikes the ground is approximately 25,540 m away from the point where it was launched.

44. Answers will vary, but Kepler's three Laws of Planetary Motion are

- The orbit of each planet is an ellipse with the Sun at a focus,
- A line joining the Sun to a planet sweeps out equal areas in equal times, and
- The square of the period of revolution of a planet is proportional to the cube of the length of the semimajor axis of the planets elliptical orbit.

Chapter 12: Functions of Several Variables

12.1: Functions of Two or More Variables and Their Graphs

Concepts and Vocabulary

1. Since the ordered pair $(2, 3)$ indicates that $x = 2$ and $y = 3$, $f(2, 3) = \sqrt{16 - 2^2 - 3^2} = \sqrt{16 - 4 - 9} = \boxed{\sqrt{3}}$.
2. True. More specifically, the domain of a function is the nonempty set of points in the xy -plane for which the function is defined. See the definition of “domain” (p. 810).
3. The independent variables are x, y , and z , and the dependent variable is w .
4. True. A level curve is the projection of a contour line onto the xy -plane.

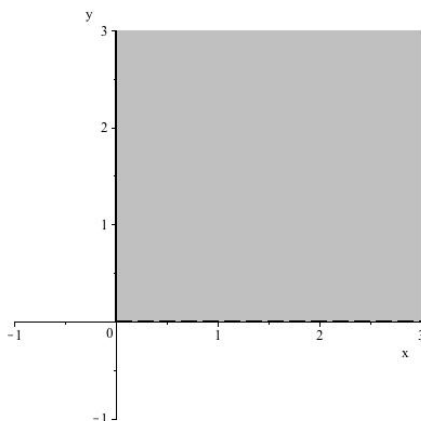
Skill Building

5. (a) $f(0, 1) = 3 \cdot 0 + 2 \cdot 1 + 0 \cdot 1 = 0 + 2 + 0 = \boxed{2}$;
 (b) $f(2, 1) = 3 \cdot 2 + 2 \cdot 1 + 2 \cdot 1 = 6 + 2 + 2 = \boxed{10}$;
 (c) $f(x + \Delta x, y) = 3(x + \Delta x) + 2y + (x + \Delta x)y = \boxed{3x + 3\Delta x + 2y + xy + y\Delta x}$;
 (d) $f(x, y + \Delta y) = 3x + 2(y + \Delta y) + x(y + \Delta y) = \boxed{3x + 2y + 2\Delta y + xy + x\Delta y}$.
6. (a) $f(0, 1) = 0^2 \cdot 1 + 0 + 1 = 0 + 0 + 1 = \boxed{1}$;
 (b) $f(2, 1) = 2^2 \cdot 1 + 2 + 1 = 4 + 2 + 1 = \boxed{7}$;
 (c) $f(x + \Delta x, y) = (x + \Delta x)^2 y + (x + \Delta x) + 1 = (x^2 + 2x\Delta x + (\Delta x)^2)y + x + \Delta x + 1$
 $= \boxed{x^2 y + (2xy + 1)\Delta x + y(\Delta x)^2 + x + 1}$;
 (d) $f(x, y + \Delta y) = x^2(y + \Delta y) + x + 1 = \boxed{x^2 y + x^2 \Delta y + x + 1}$.
7. (a) $f(0, 1) = \sqrt{0 \cdot 1} + 0 = 0 + 0 = \boxed{0}$;
 (b) $f(a^2, t^2) = \sqrt{a^2 t^2} + a^2 = |at| + a^2 = \boxed{at + a^2}$ (because $a > 0$ and $t > 0$);
 (c) $f(x + \Delta x, y) = \sqrt{(x + \Delta x)y} + (x + \Delta x) = \boxed{\sqrt{xy + y\Delta x} + x + \Delta x}$;
 (d) $f(x, y + \Delta y) = \sqrt{x(y + \Delta y)} + x = \boxed{\sqrt{xy + x\Delta y} + x}$.
8. (a) $f(1, -1) = e^{1+(-1)} = e^0 = \boxed{1}$;
 (b) $f(x + \Delta x, y) = e^{(x+\Delta x)+y} = \boxed{e^{x+y+\Delta x}}$;
 (c) $f(x, y + \Delta y) = e^{x+(y+\Delta y)} = \boxed{e^{x+y+\Delta y}}$.
9. (a) $F(0, 0, 1) = \frac{3 \cdot 0 \cdot 0 + 1}{0^2 + 0^2 + 1^2} = \frac{1}{1} = \boxed{1}$;
 (b) $F(0, 1, 0) = \frac{3 \cdot 0 \cdot 1 + 0}{0^2 + 1^2 + 0^2} = \frac{0}{1} = \boxed{0}$;
 (c) $F(\sin t, \cos t, 0) = \frac{3 \sin t \cos t + 0}{\sin^2 t + \cos^2 t + 0^2} = \frac{3 \sin t \cos t}{1} = \boxed{3 \sin t \cos t}$.
10. (a) $F(1, -1, 1) = \frac{1 \cdot (-1) \cdot 1}{1^2 + (-1)^2 + 1^2} = \frac{-1}{3} = \boxed{-\frac{1}{3}}$;

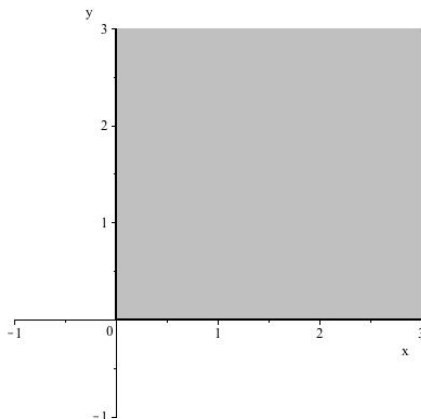
$$(b) F(a, a, a) = \frac{a \cdot a \cdot a}{a^2 + a^2 + a^2} = \frac{a^3}{3a^2} = \boxed{\frac{a}{3}};$$

$$(c) F(\sin t, \cos t, a) = \frac{\sin t \cdot \cos t \cdot a}{\sin^2 t + \cos^2 t + a^2} = \boxed{\frac{a \sin t \cos t}{1 + a^2}}.$$

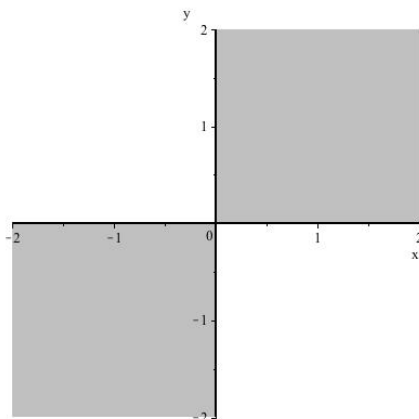
11. (a) $x(0) = \sqrt{0} = 0$ and $y(0) = 0^2 = 0$, so $f(x(0), y(0)) = f(0, 0) = 3 \cdot 0 \cdot 0 - 0^2 = 0 - 0 = \boxed{0}$;
 (b) $x(1) = \sqrt{1} = 1$ and $y(1) = 1^2 = 1$, so $f(x(1), y(1)) = f(1, 1) = 3 \cdot 1 \cdot 1 - 1^2 = 3 - 1 = \boxed{2}$;
 (c) $x(4) = \sqrt{4} = 2$ and $y(4) = 4^2 = 16$, so $f(x(4), y(4)) = f(2, 16) = 3 \cdot 2 \cdot 16 - 2^2 = 96 - 4 = \boxed{92}$.
12. (a) $x(0) = 0$ and $y(0) = 0^2 = 0$, so $f(x(0), y(0)) = f(0, 0) = 0^2 + 0 \cdot 0 + 0^2 = 0 + 0 + 0 = \boxed{0}$;
 (b) $x(1) = 1$ and $y(1) = 1^2 = 1$, so $f(x(1), y(1)) = f(1, 1) = 1^2 + 1 \cdot 1 + 1^2 = 1 + 1 + 1 = \boxed{3}$;
 (c) $x(2) = 2$ and $y(2) = 2^2 = 4$, so $f(x(2), y(2)) = f(2, 4) = 2^2 + 2 \cdot 4 + 4^2 = 4 + 8 + 16 = \boxed{28}$.
13. In order for $f(x, y)$ to be defined, both square roots need to evaluate only nonnegative numbers. So, we require that $x \geq 0$ and $y \geq 0$. However, the denominator $\sqrt{y}$ also cannot be zero, which means y cannot be zero, which restricts our inequality further to $y > 0$. So, the domain is the set of points (x, y) where $x \geq 0, y > 0$, or the set $\boxed{\{(x, y) | x \geq 0, y > 0\}}$.



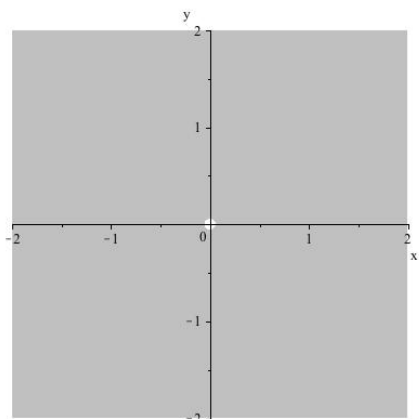
14. In order for $f(x, y)$ to be defined, both square roots need to evaluate only nonnegative numbers. So, we require that $x \geq 0$ and $y \geq 0$. So, the domain is the set of points (x, y) where $x \geq 0, y \geq 0$, or the set $\boxed{\{(x, y) | x \geq 0, y \geq 0\}}$.



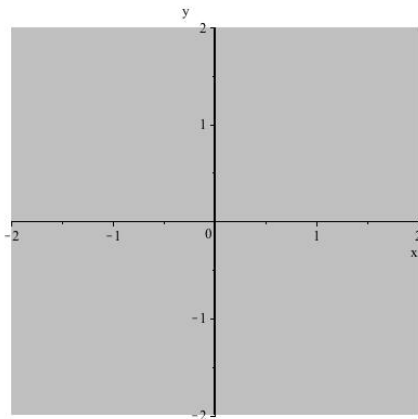
15. In order for $f(x, y)$ to be defined, the square root must evaluate only nonnegative numbers, i.e. we require that $xy \geq 0$. This inequality will be satisfied under either of the following two sets of conditions: $x, y \geq 0$, $x, y \leq 0$. So, the domain is the union of the sets $\{(x, y) | x, y \geq 0\}$ and $\{(x, y) | x, y \leq 0\}$, which can also be written as the set $\{(x, y) | x, y \geq 0 \text{ or } x, y \leq 0\}$.



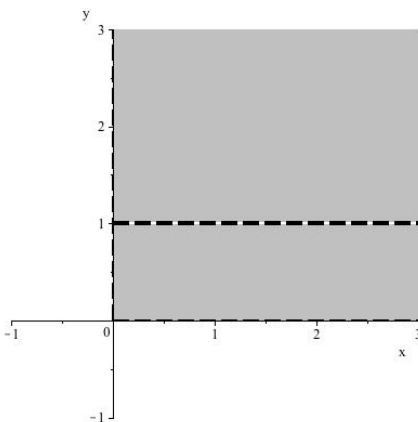
16. In order for $f(x, y)$ to be defined, the denominator $x^2 + y^2$ cannot be zero. So, the domain is the set of points (x, y) where $x^2 + y^2 \neq 0$. Since $x^2 + y^2 = 0$ only when x and y are both zero, the domain is the set of all points (x, y) with the exception of the point $(0, 0)$. So the domain is the set $\{(x, y) | (x, y) \neq (0, 0)\}$.



17. Because both the functions e^x and $\sin y$ are defined for all real numbers x and y , the domain of this function is the entire xy -plane, or $\{(x, y) | x \text{ is any real number, } y \text{ is any real number}\}$.

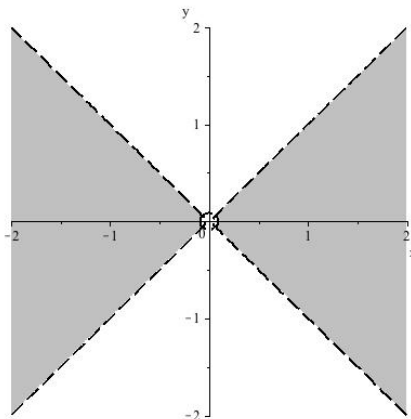


18. In order for $f(x, y)$ to be defined, both $\ln x$ and $\ln y$ must be defined, which means that we must have $x > 0$ and $y > 0$. Furthermore, the denominator cannot be zero, which means that we must have $\ln y \neq 0$, or $y \neq 1$. Therefore, the domain must be the set $\{(x, y) | x > 0, y > 0 \text{ and } y \neq 1\}$.

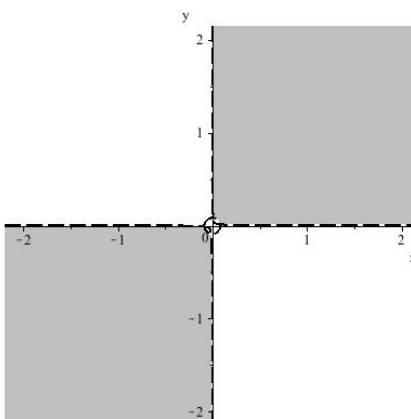


19. In order for $f(x, y)$ to be defined, the expression under the radical, $\frac{x^2 + y^2}{x^2 - y^2}$ must be nonnegative. This can occur only if $x^2 + y^2 \geq 0$ and $x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 0$ and $x^2 - y^2 \leq 0$. The second case will only occur when $(x, y) = (0, 0)$, since $x^2 + y^2 > 0$ for all $(x, y) \neq (0, 0)$. But, this will also make the denominator of the argument of the square root 0, which is not allowed. So, the second case may be ruled out and we consider only the case where $x^2 + y^2 \geq 0$ and $x^2 - y^2 \geq 0$. Before we do so, we note further that in fact, because $x^2 - y^2$ appears in the denominator, we may restrict this inequality further to $x^2 - y^2 > 0$, and also, since we have already pointed out that $(0, 0)$ is not in the domain, we know that we will also have the stricter requirement that $x^2 + y^2 > 0$. This occurs for $(x, y) \neq (0, 0)$, so we only need to further consider the inequality $x^2 - y^2 > 0$. Therefore, the domain can be written as the set $\{(x, y) | x^2 - y^2 > 0\}$.

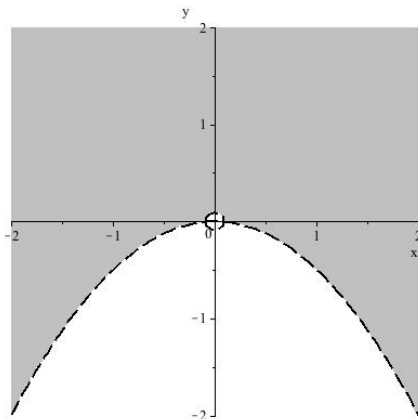
To obtain the graph, we factor to obtain $x^2 - y^2 = (x - y)(x + y) > 0$, which will occur only when $x - y, x + y > 0$ or $x - y, x + y < 0$. We may rewrite these as $x > y, x > -y$, or $x < y, x < -y$. The domain is therefore the union of the sets $\{(x, y) | x > y, x > -y\}$ and $\{(x, y) | x < y, x < -y\}$, which can be graphed as follows:



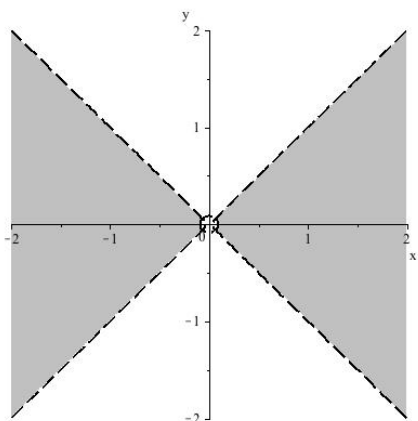
20. In order for $f(x, y)$ to be defined, the quantity under the radical must be nonnegative. However, the numerator of that quantity, $x^2 + y^2$, is always positive or zero. It is zero only when $(x, y) = (0, 0)$; however, when $(x, y) = (0, 0)$, the function is not defined because the denominator is zero. When $x^2 + y^2$ is positive, the denominator must also be positive in order for the quantity under the radical to be positive. The denominator is positive when $xy > 0$; in other words, when $x > 0$ and $y > 0$, as well as when $x < 0$ and $y < 0$. Therefore, the domain is the set $\{(x, y) | xy > 0\}$, which can also be written as the set $\{(x, y) | x > 0, y > 0 \text{ or } x < 0, y < 0\}$.



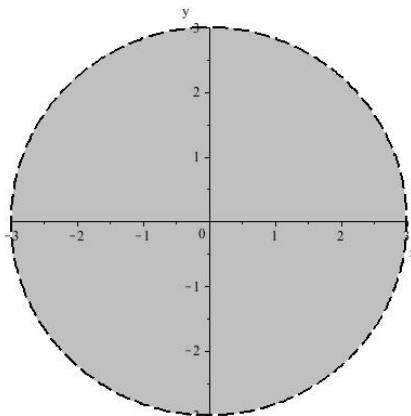
21. In order for this function to be defined, the denominator must be nonzero and the expression under the radical must be nonnegative. Combining these yields the inequality $2y + x^2 > 0$. The domain of the function is therefore the set $\{(x, y) | 2y + x^2 > 0\}$. To obtain the graph, we note that the inequality is equivalent to $y > -\frac{x^2}{2}$, which can be graphed as follows:



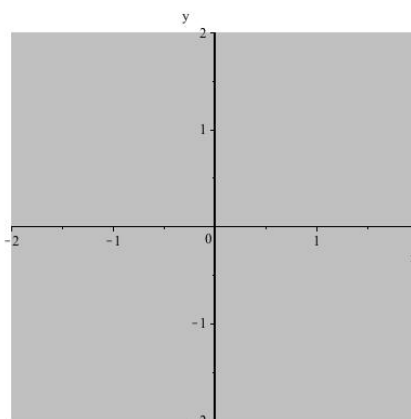
22. In order for this function to be defined, the quantity $x^2 - y^2$, which we need to take the logarithm of, must be positive. So (x, y) is in the domain when $x^2 - y^2 > 0$. Factoring, we obtain $x^2 - y^2 = (x - y)(x + y) > 0$, which will occur only when $x - y, x + y > 0$ or $x - y, x + y < 0$. We may rewrite these as $x > y, x > -y$, or $x < y, x < -y$. The domain is therefore the union of the sets $\{(x, y) | x > y, x > -y\}$ and $\{(x, y) | x < y, x < -y\}$, which can also be written as the set $\{(x, y) | x > y, x > -y \text{ or } x < y, x < -y\}$.



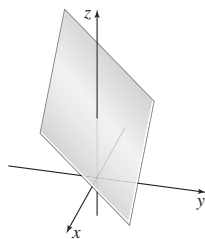
23. In order for this function to be defined, the denominator must be nonzero and the expression under the radical must be nonnegative. Combining these yields the inequality $9 - x^2 - y^2 > 0$. We may rewrite this as $x^2 + y^2 < 9$, which we recognize as the interior of the circle of radius 3 centered at the origin. The domain is therefore the set $\{(x, y) | x^2 + y^2 < 9\}$.



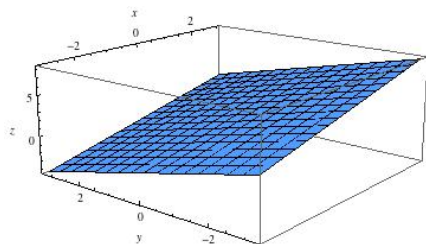
24. Since the inverse tangent function is defined for all real numbers, the domain of this function is the entire xy -plane, or $\{(x, y) | x \text{ is any real number, } y \text{ is any real number}\}$.



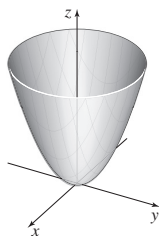
25. Because of the z^2 in the denominator, the function will be defined everywhere, except when $z^2 = 0$, which is equivalent to $z = 0$. So the domain is $\{(x, y, z) | z \neq 0\}$.
26. While e^z is always defined, $\ln(x^2 + y^2)$ is defined only when $x^2 + y^2 > 0$; however, this is always true, except when x and y are both equal to zero. So the domain is $\{(x, y, z) | (x, y) \neq (0, 0)\}$, which can also be written as $\{(x, y, z) | x \neq 0 \text{ and } y \neq 0\}$.
27. Because of the $\cos y$ in the denominator, the function is defined only when $\cos y \neq 0$. We know that $\cos y = 0$ when $y = \frac{(2k+1)\pi}{2}$ for k an integer, so the domain is $\{(x, y, z) | y \neq \frac{(2k+1)\pi}{2} \text{ for } k \text{ an integer}\}$.
28. Because of the $\sqrt{x^2 + y^2 + z^2}$ in the denominator, the function is defined only when $x^2 + y^2 + z^2 > 0$. However, this is always true, except when $(x, y, z) = (0, 0, 0)$. So the domain is $\{(x, y, z) | (x, y, z) \neq (0, 0, 0)\}$.
29. By rewriting the equation as $z = 3 - x - y$, we see that this is a plane with z -intercept 3. We can also find the x - and y -intercepts by setting y, z and x, z equal to 0 respectively. This yields intercepts of $(3, 0, 0)$ and $(0, 3, 0)$. Also, by rearranging the equation into $x + y + z = 3$, we can see that a normal vector for this plane is $\langle 1, 1, 1 \rangle$.



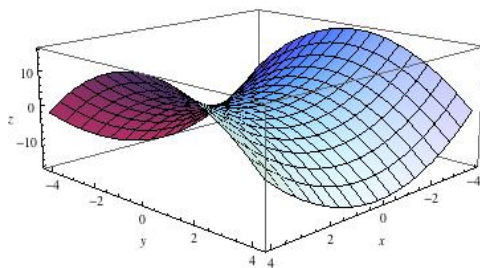
30. By rewriting the equation as $z = 2 + x - y$, we see that this is a plane with z -intercept 2. We can also find the x - and y -intercepts by setting y, z and x, z equal to 0 respectively. This yields intercepts of $(-2, 0, 0)$ and $(0, 2, 0)$. Also, by rearranging the equation into $-x + y + z = 2$, we can see that a normal vector for this plane is $\langle -1, 1, 1 \rangle$.



31. By rewriting the equation as $z = x^2 + y^2$, we recognize this as a paraboloid opening upward about the z -axis, with circular cross-sections.



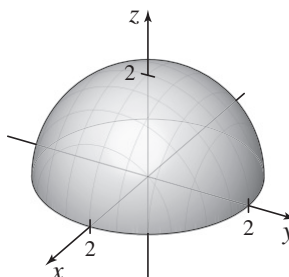
32. By rewriting the equation as $z = x^2 - y^2$, we recognize this as a hyperbolic paraboloid. We can obtain further information about the graph by considering the traces at $x = 0$ and $y = 0$. When $x = 0$, we have the equation $z = -y^2$, which is a downward-facing parabola. When $y = 0$, we have the equation $z = x^2$, which is an upward-facing parabola.



33. We manipulate algebraically the given equation:

$$\begin{aligned} z = f(x, y) &= \sqrt{4 - x^2 - y^2} \\ z^2 &= 4 - x^2 - y^2 \\ x^2 + y^2 + z^2 &= 4 \end{aligned}$$

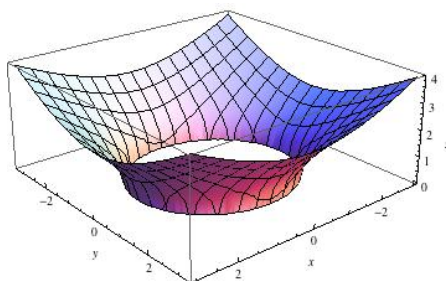
which we recognize as a sphere of radius 2 centered at the origin. However, because the original function is a square root, which produces only nonnegative values, this is the portion of the sphere with only nonnegative z -values, or the top half. Therefore, this is the top half of the sphere of radius 2 centered at the origin.



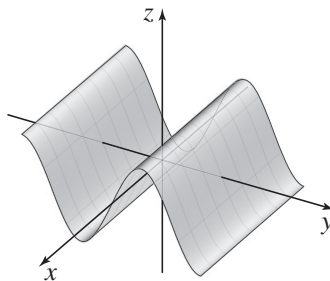
34. We manipulate algebraically the given equation:

$$\begin{aligned} z = f(x, y) &= \sqrt{x^2 + y^2 - 4} \\ z^2 &= x^2 + y^2 - 4 \\ x^2 + y^2 - z^2 &= 4 \end{aligned}$$

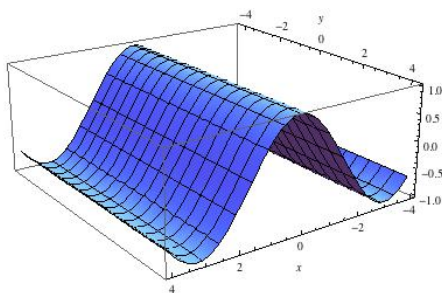
which we recognize as a hyperboloid of one sheet with its center at the origin, opening about the z -axis. However, because the original function is a square root, which produces only nonnegative values, this is the portion of the hyperboloid with only nonnegative z -values, or the top half.



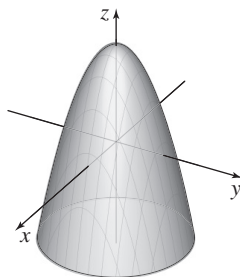
35. We see that the variable x does not appear in the expression for the function, and so it is an unrestricted variable. We therefore first consider the behavior in only y and z , which is the curve $z = \sin y$. This is a graph of the sine function with y as the independent variable and z as the dependent variable. We then extend this curve in the x direction.



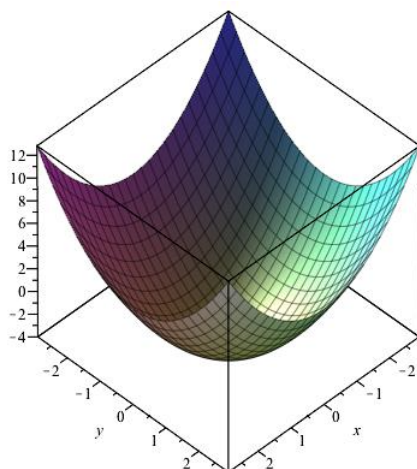
36. We see that the variable y does not appear in the expression for the function, and so it is an unrestricted variable. We therefore first consider the behavior in only x and z , which is the curve $z = \cos x$. This is a graph of the cosine function with x as the independent variable and z as the dependent variable. We then extend this curve in the y direction.



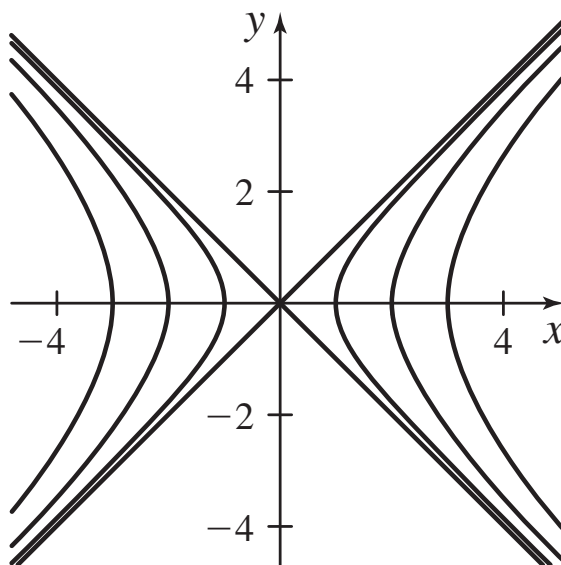
37. By rewriting the equation as $z = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$, we recognize this as a paraboloid opening downward about the z -axis, with circular cross-sections, with vertex at height 4 above the xy -plane.



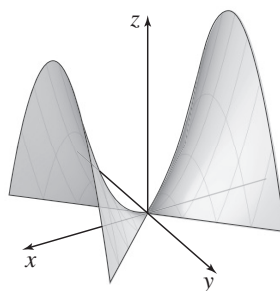
38. By rewriting the equation as $z = x^2 + y^2 - 4$, we recognize this as a paraboloid opening upward about the z -axis, with circular cross-sections, with vertex at height -4 below the xy -plane.



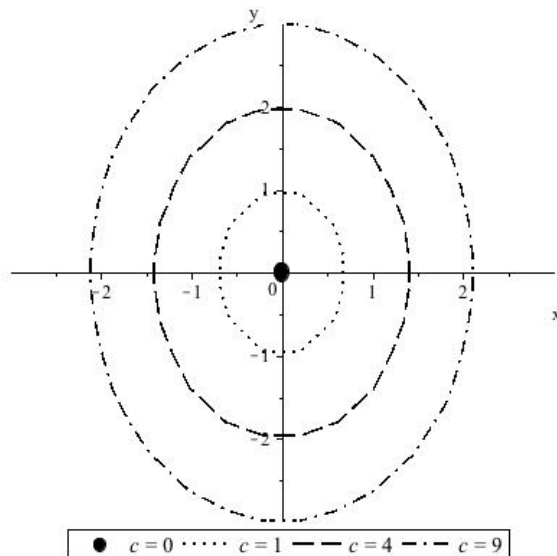
39. (a) We set $z = c$, and we get $c = x^2 - y^2$, which is equivalent to $\frac{x^2}{c} - \frac{y^2}{c} = 1$, provided that $c \neq 0$. We recognize these equations; their graphs are hyperbolas. For $c > 0$, they open about the x -axis. So, the level curves corresponding to $c = 1, 4, 9$ are hyperbolas opening about the x -axis. For $c = 0$, the equation becomes $x^2 = y^2$, or $y = \pm x$, so the level curve consists of two intersecting lines, whose slopes are 1 and -1 .



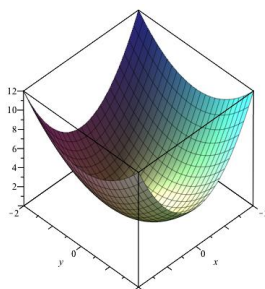
- (b) The graph is:



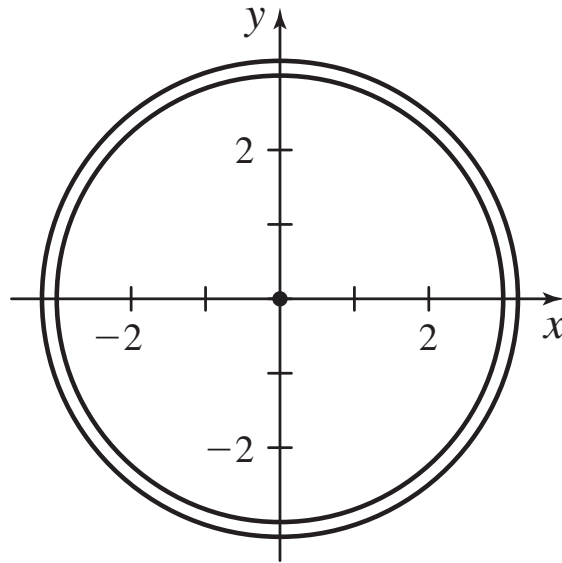
40. (a) We set $z = c$, and we get $c = 2x^2 + y^2$, which is equivalent to $\frac{x^2}{\frac{c}{2}} + \frac{y^2}{c} = 1$, provided that $c \neq 0$. We recognize these equations; their graphs are ellipses centered at the origin. So, the level curves corresponding to $c = 1, 4, 9$ are ellipses centered at the origin. For $c = 0$, the equation becomes $2x^2 + y^2 = 0$, which has only one solution, $x = 0$ and $y = 0$, so the level curve consists of one point, the origin.



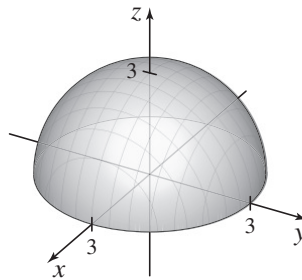
- (b) The graph is:



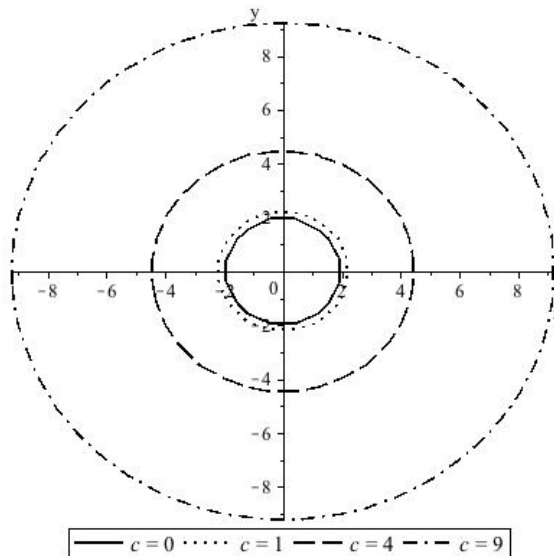
41. (a) We set $z = c$, and we get $c = \sqrt{9 - x^2 - y^2}$, which is equivalent to $c^2 = 9 - x^2 - y^2$, or $x^2 + y^2 = 9 - c^2$. We recognize these as the equations of circles centered at the origin, with radius 3 for $c = 0$ and radius $\sqrt{8}$ for $c = 1$. For $c = 3$, the equation becomes $x^2 + y^2 = 0$, so the level curve consists of one point, the origin.



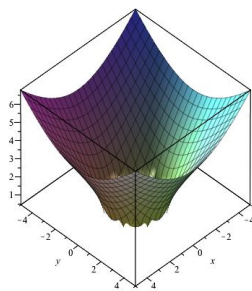
(b) The graph is:



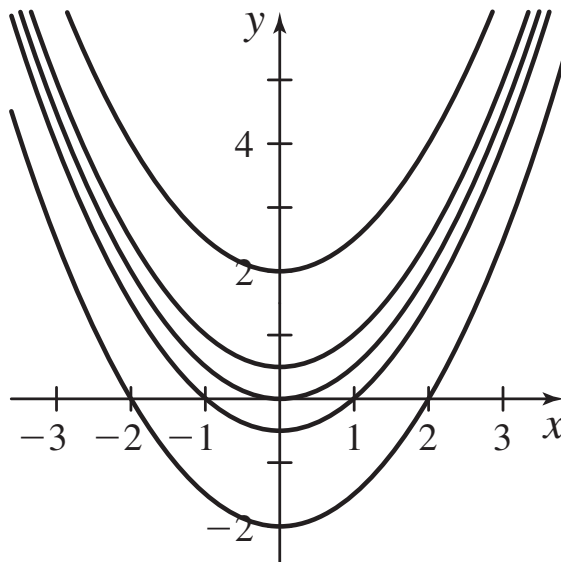
42. (a) We set $z = c$, and we get $c = \sqrt{x^2 + y^2 - 4}$, which is equivalent to $c^2 = x^2 + y^2 - 4$, or $x^2 + y^2 = c^2 + 4$. We recognize these as the equations of circles centered at the origin, with radius 2 for $c = 0$, radius $\sqrt{5}$ for $c = 1$, radius $\sqrt{20}$ for $c = 4$ and radius $\sqrt{85}$ for $c = 9$.



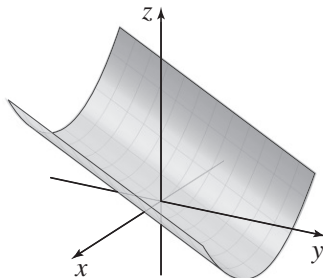
(b) The graph is:



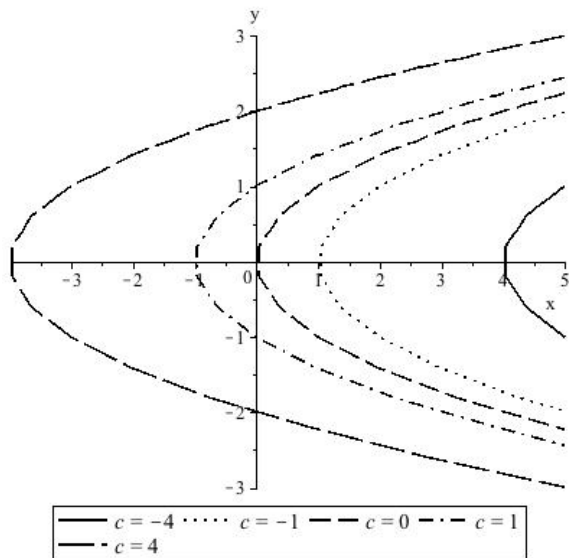
43. (a) We set $z = c$, and we get $c = x^2 - 2y$, which is equivalent to $2y = x^2 - c$, or $y = \frac{x^2}{2} - \frac{c}{2}$. For $c = 0$, this is the equation of a parabola that opens about the positive y -axis, with vertex at the origin. For $c = -4, -1, 1, 4$, this is the equation of the same parabola, translated up 2 units, up $\frac{1}{2}$ unit, down $\frac{1}{2}$ unit, and down 2 units (respectively).



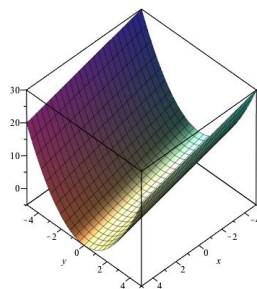
(b) The graph is:



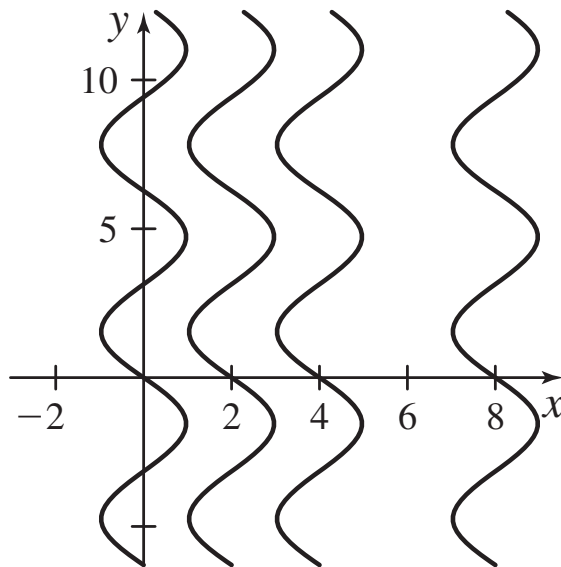
44. (a) We set $z = c$, and we get $c = y^2 - x$, which is equivalent to $x = y^2 - c$. For $c = 0$, this is the equation of a parabola that opens about the positive x -axis, with vertex at the origin. For $c = -4, -1, 1, 4$, this is the equation of the same parabola, translated right 4 units, right 1 unit, left 1 unit, and left 4 units (respectively).



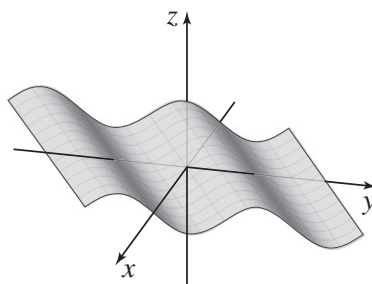
(b) The graph is:



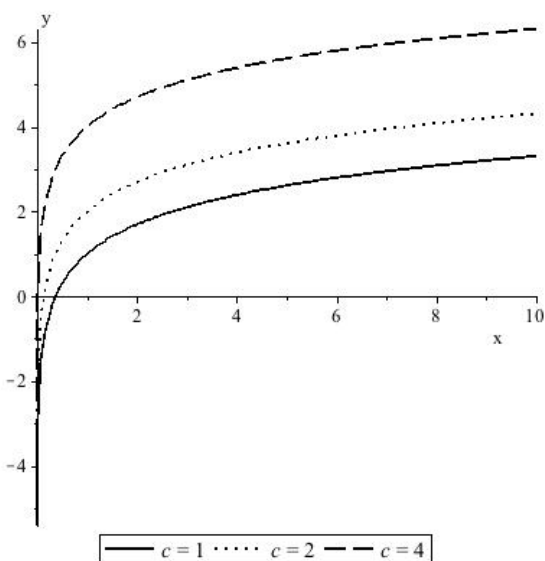
45. (a) We set $z = c$, and we get $c = x + \sin y$, which is equivalent to $x = -\sin y + c$. For $c = 0$, this is the equation of a sinusoidal curve; for $c = 2, 4, 8$, this is the equation of the same curve, translated right 2, 4 and 8 units respectively.



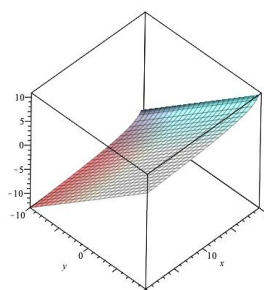
(b) The graph is:



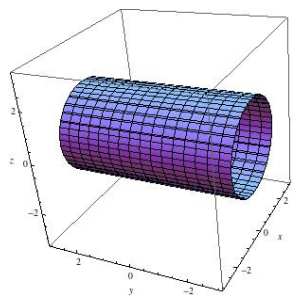
46. (a) We set $z = c$, and we get $c = y - \ln x$, which is equivalent to $y = \ln x + c$. So the level curves are just translations of the graph of $y = \ln x$, up by 1, 2 and 4 units for $c = 1, 2$ and 4 (respectively).



(b) The graph is:



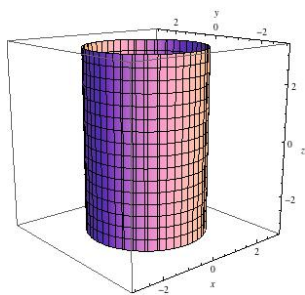
47. (a) We consider surfaces of the form $f(x, y, z) = c = x^2 + z^2$, where c is a constant (which in this case, cannot be negative since $x^2 + z^2 \geq 0$ for all x, z). We observe that the variable y is an unrestricted variable, and $c = x^2 + z^2$ forms a circle in x and z , centered about the y -axis. $c = x^2 + z^2$ is therefore a cylinder about the y -axis, and the level surfaces are such cylinders of increasing radii as c increases.
- (b) Shown is the level surface for $c = 3$.



(c) Yes.

48. (a) We consider surfaces of the form $f(x, y, z) = c = x^2 + y^2$, where c is a constant (which in this case, cannot be negative since $x^2 + y^2 \geq 0$ for all x, y). We observe that the variable z is an unrestricted variable, and $c = x^2 + y^2$ forms a circle in x and y , centered about the z -axis. $c = x^2 + y^2$ is therefore a cylinder about the z -axis, and the level surfaces are such cylinders of increasing radii as c increases.

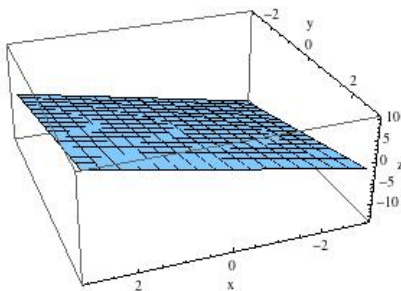
(b) Shown is the level surface for $c = 4$.



(c) Yes.

49. (a) We consider surfaces of the form $f(x, y, z) = c = z - 2x - 2y$, where c is a constant. We observe that this surface forms a plane perpendicular to the vector $\langle -2, -2, 1 \rangle$. We can rewrite this surface as $z = 2x + 2y + c$, from which we can see that increasing values of c increase the z -intercept of the plane (i.e. move it up).

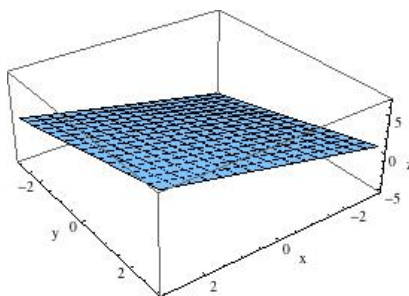
(b) Shown is the level surface for $c = -2$.



(c) Yes.

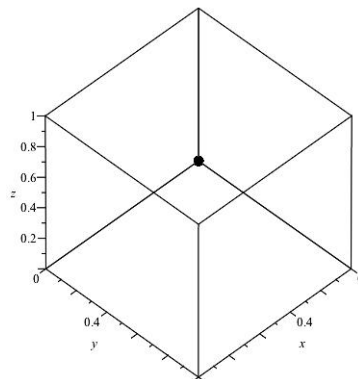
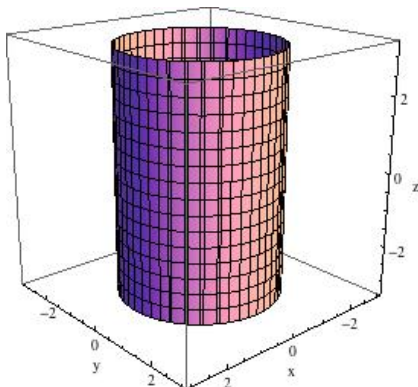
50. (a) We consider surfaces of the form $f(x, y, z) = c = x + y - z$, where c is a constant. We observe that this surface forms a plane perpendicular to the vector $\langle 1, 1, -1 \rangle$. We can rewrite this surface as $z = x + y - c$, from which we can see that increasing values of c decrease the z -intercept of the plane (i.e. move it down).

- (b) Shown is the level surface for $c = -1$.



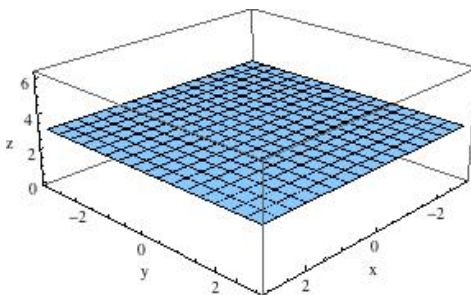
- (c) Yes.

51. (a) We consider surfaces of the form $f(x, y, z) = c = 4 - x^2 - y^2$, where c is a constant. We observe that the variable z is an unrestricted variable, and rearrange our equation to obtain $x^2 + y^2 = 4 - c$. We see that since $x^2 + y^2 \geq 0$ for all x, y , we must have $4 - c \geq 0$ or $c \leq 4$. We see that for such c , this forms a circle in x and y (of radius $\sqrt{4 - c}$), centered about the z -axis. $x^2 + y^2 = 4 - c$ is therefore a cylinder about the z -axis for $c < 4$, and in this case the level surfaces are such cylinders of decreasing radii as c increases. $c = 4$ is a special case, as this yields $x^2 + y^2 = 0$, which is satisfied only by the point $(0, 0)$. So, the level surface is just the origin itself for $c = 4$.
- (b) Shown are the level surface for $c = 0$, and $c = 4$.



- (c) Yes.

52. (a) We consider surfaces of the form $f(x, y, z) = c = z$, where c is a constant. We observe that this surface is a plane parallel to the xy -plane, at height c above (or below) the xy -plane. Increasing values of c increase the height of the plane above the xy -plane (negative values of c will place the plane below the xy -plane).
- (b) Shown is the level surface for $c = \pi$.



(c) Yes.

Applications and Extensions

53. It appears that the contour lines are straight lines. To gain more information, we consider the curves of the form $f(x, y) = c = 2x - y - 2$, and manipulate this equation algebraically.

$$\begin{aligned}c &= 2x - y - 2 \\y &= 2x - 2 - c\end{aligned}$$

We can see that the level curves in the xy -plane are then lines of slope 2, with y -intercept $-2 - c$. This intercept decreases as c increases. This corresponds to graph **F**.

54. It appears that the contour lines are straight lines. To gain more information, we consider the curves of the form $f(x, y) = c = x + y - 3$, and manipulate this equation algebraically.

$$\begin{aligned}c &= x + y - 3 \\y &= -x + 3 + c\end{aligned}$$

We can see that the level curves in the xy -plane are then lines of slope -1 , with y -intercept $3 + c$. This intercept increases as c increases. This corresponds to graph **D**.

55. The contour lines appear to be ellipses centered at the origin; we verify algebraically. We set $f(x, y) = c = 4x^2 + y^2$, and further manipulate the equation:

$$\begin{aligned}c &= 4x^2 + y^2 \\1 &= \frac{4x^2}{c} + \frac{y^2}{c} \quad \text{for } c \neq 0 \\1 &= \frac{x^2}{\frac{c}{4}} + \frac{y^2}{c}\end{aligned}$$

We see then that for $c \neq 0$, the level curves in the xy -plane will indeed be ellipses centered at the origin (for $c = 0$ we have just the point $(0,0)$); as c increases, so will both the semi-major and semi-minor axes. This matches graph **B**.

56. It appears that the horizontal cross-sections, or contour lines of the cone are concentric circles with increasing radii as z increases. However, we verify this algebraically by considering constant values of $z = f(x, y)$, or $c = f(x, y)$:

$$\begin{aligned}c = f(x, y) &= \sqrt{\frac{x^2}{4^2} + \frac{y^2}{4^2}} = \sqrt{\frac{x^2}{16} + \frac{y^2}{16}} \\c^2 &= \frac{x^2}{16} + \frac{y^2}{16} \\16c^2 &= x^2 + y^2\end{aligned}$$

We see that as $|c|$ increases, we will have circles of increasing radii, centered at the origin. This corresponds to graph **C**.

57. The contour lines appear to be hyperbolas opening about the x -axis or y -axis, centered at the origin. We verify algebraically by manipulating the equation $f(x, y) = c = \frac{x^2}{2^2} - \frac{y^2}{1^2}$:

$$\begin{aligned} c &= \frac{x^2}{2^2} - \frac{y^2}{1^2} \\ 1 &= \frac{x^2}{4c} - \frac{y^2}{c} \quad \text{if } c \neq 0 \end{aligned}$$

These are indeed hyperbolas with the x -axis as the transverse axis if $c > 0$ and the y -axis if $c < 0$. As c increases, the eccentricity decreases. If $c = 0$, the level curve is given by $0 = \frac{x^2}{2^2} - \frac{y^2}{1^2}$, which is satisfied by only the origin. This corresponds to graph A.

58. The contour lines appear to be pairs of hyperbolas centered at the origin. We will not be able to easily see this algebraically, but we do see that horizontal planes will cut the surface at hyperbolas with asymptotes given by combinations of the x - and y -axes and the lines $y = \pm x$. This helps us recognize the corresponding graph of level curves as E.
59. (a) Four Corners is about halfway between two isotherms, one with temperature $45^\circ F$ and one with temperature $50^\circ F$, and seems slightly closer to the $45^\circ F$ isotherm. Therefore, an estimate for the temperature at Four Corners would be 47° F.
- (b) When we move south from the Four Corners, we move from a point with approximate temperature $47^\circ F$ in the direction of the isotherm with temperature $50^\circ F$, which means that the temperature will be increasing.
- (c) To move as quickly to the next warmer temperature zone, we would move toward the isotherm with temperature $50^\circ F$. Therefore, we would move toward the southwest. Regarding how the path intersects the isotherm, answers will vary.
- (d) Answers will vary.
60. (a) Examining the level curves, we see that the altitude at point A is almost 1770 feet. At the other three points, the altitude is below 1730 feet. So, she would start to climb at point A to minimize the distance climbed.
- (b) The climb would be initially the steepest at point D, because the level curves are closest at that point.
- (c) The climber would walk along the level curve, since the level curve is by definition the curve along which the altitude is constant.
- (d) Answers will vary.
61. (a) We take $M = 10$ and $C = 12$, so $IQ = 100 \cdot \frac{10}{12} \approx$ 83.
- (b) We take $M = 12$ and $C = 10$, so $IQ = 100 \cdot \frac{12}{10} =$ 120.
- (c) We take $IQ = 139$ and $C = 10$, and we solve for M : $139 = 100 \cdot \frac{M}{10}$, so $139 = 10M$, and $M = \frac{139}{10} \approx$ 14.
62. Since x is the number of megabytes used domestically in excess of 200, and the total number of megabytes used domestically is equal to 210, we see that $x = 210 - 200$, so $x = 10$. Since y is the number of kilobytes used to or from Canada, we see that $y = 250$.

Since z is the number of kilobytes used outside the United States and Canada, we see that $z = 15$. Therefore, the monthly cost (in dollars) is

$$C(10, 250, 15) = 35.00 + 0.1(10) + 0.015(250) + 0.0195(15) = 40.0425,$$

which rounds to $\boxed{\$40.04}$.

63. (a) $A(3, 4) = 9 \left(\frac{3}{4} \right) = \boxed{6.75}$.
 (b) $A(6, 3) = 9 \left(\frac{6}{3} \right) = \boxed{18}$.
 (c) $A(2, 9) = 9 \left(\frac{2}{9} \right) = \boxed{2}$.
 (d) $A(3, 18) = 9 \left(\frac{3}{18} \right) = \boxed{1.5}$.
64. (a) s is the total number of field goals attempted. Since he attempted 488 two-point field goals and zero three-point field goals, we have $s = 488 + 0 = 488$. Since he attempted zero three-point field goals, he made zero three-point field goals, so $y = 0$. Also, we are told that he made 234 two-point field goals, so $x = 234$. Therefore, his adjusted field goal percentage is $f(234, 0, 488) = \frac{234 + 1.5(0)}{488} \approx \boxed{.480}$.
- (b) s is the total number of field goals attempted. Since he attempted 527 two-point field goals and 246 three-point field goals, we have $s = 527 + 246 = 773$. Since he made 234 three-point field goals, we have $y = 234$. Also, we are told that he made 234 two-point field goals, so $x = 234$. Therefore, his adjusted field goal percentage is $f(234, 234, 773) = \frac{234 + 1.5(234)}{773} \approx \boxed{.757}$.
65. (a) We have $r = 50$ and $t = 95$, so the heat index is given by

$$\begin{aligned} H(95, 50) &= -42.379 + (2.04901523)(95) + (10.1333127)(50) - (0.22475541)(95)(50) \\ &\quad - (0.00683783)(95)^2 - (0.05481717)(50)^2 + (0.00122874)(95)^2(50) \\ &\quad + (0.00085282)(95)(50)^2 - (0.00000199)(95)^2(50)^2 \approx 104.715, \end{aligned}$$

so we have a heat index of approximately $\boxed{104.715^\circ F}$.

- (b) We have $H(t, r) = 105$ and $t = 97$, and we need to solve for r . We have:

$$\begin{aligned} 105 &= -42.379 + (2.04901523)(97) + (10.1333127)r - (0.22475541)(97)r \\ &\quad - (0.00683783)(97)^2 - (0.05481717)r^2 + (0.00122874)(97)^2r \\ &\quad + (0.00085282)(97)r^2 - (0.00000199)(97)^2r^2 \\ &= 92.03833484 - 0.10674741r + 0.00918246r^2. \end{aligned}$$

So we have a quadratic equation for r :

$$0.00918246r^2 - 0.10674741r - 12.96166516 = 0.$$

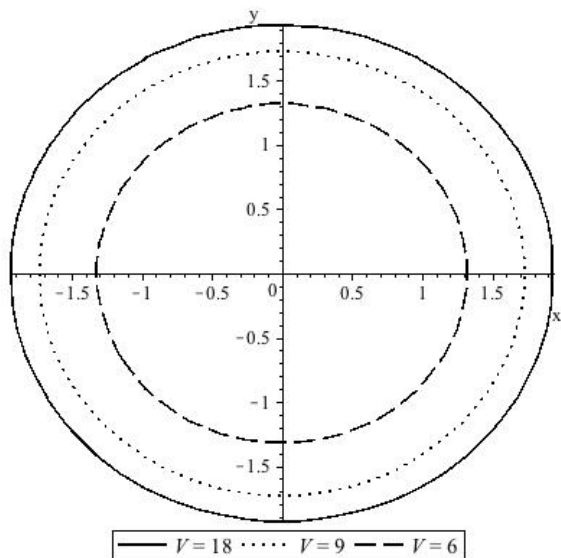
Solving using the quadratic formula, we get

$$\begin{aligned} r &= \frac{0.10674741 \pm \sqrt{(-0.10674741)^2 - 4(0.00918246)(-12.96166516)}}{2(0.00918246)} \\ &\approx \frac{0.10674741 \pm 0.698194025}{0.01836492}. \end{aligned}$$

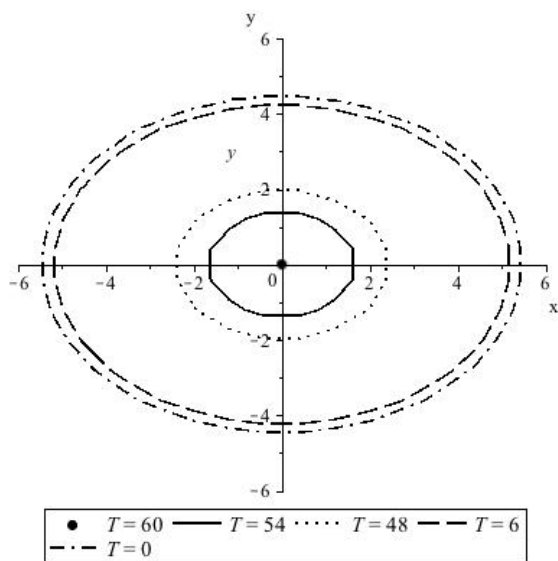
But the negative square root will result in a negative value for r , which makes no sense, so we will take the positive square root.

$r \approx \frac{0.10674741 + 0.698194025}{0.01836492} \approx 43.830$. So, there is only one value for the relative humidity that will result in a heat index of $105^\circ F$, and that value is approximately 43.830%. Since this is the *only* such value, it must be the minimum such value; therefore, the minimum relative humidity that will result in a heat index of $105^\circ F$ is $\boxed{\approx 43.830\%}$.

66. (a) $Q(19, 21) = 400(19)^{0.3}(21)^{0.7} \approx \boxed{8151.538}$.
 (b) $Q(21, 20) = 400(21)^{0.3}(20)^{0.7} \approx \boxed{8117.958}$.
67. The surface of an open box consists of five rectangles. One of them is the base of the box; it is an x by y rectangle, so its area is xy . Two of the sides of the box are x by z rectangles; their area is xz (each). The other two sides of the box are y by z rectangles; their area is yz (each). So we arrive at the total surface area of the open box, which is $\boxed{S(x, y, z) = xy + 2xz + 2yz}$.
68. The bottom and top of the tank are circles of radius R , so their area is πR^2 (each). This means that the cost of the bottom and the top will be $300\pi R^2$ dollars each, or $600\pi R^2$ total. The side of the tank has area equal to the surface area of the cylinder, or $2\pi Rh$, and the cost is $500 \cdot 2\pi Rh$ dollars, or $1000\pi Rh$ dollars. Therefore, the total cost of constructing the tank, in dollars, is $\boxed{C(R, h) = 600\pi R^2 + 1000\pi Rh}$.
69. Denote the length of the box by x , its width by y and its depth by z . Here, x , y and z are in centimeters. The bottom of the box is an x by y rectangle, so its area is xy and its cost is $4xy$. Two of the sides are x by z rectangles, and the other two are y by z rectangles; we will assume that the former are more expensive. So, the more expensive sides have area xz each, so their cost is $3xz$ each, or $6xz$ total. The less expensive sides have area yz each, so their cost is $2yz$ each, or $4yz$ total. So, the total cost of constructing the open box, in dollars, is $\boxed{C(x, y, z) = 4xy + 6xz + 4yz}$.
70. Denote the length of the box by x , its width by y and its depth by z . Here, x , y and z are in centimeters. The bottom of the box is an x by y rectangle, so its area is xy and its cost is $4xy$. The top of the box is also an x by y rectangle, so its area is also xy and its cost is $5xy$. Two of the sides are x by z rectangles, and the other two are y by z rectangles; we will assume that the former are more expensive. So, the more expensive sides have area xz each, so their cost is $3xz$ each, or $6xz$ total. The less expensive sides have area yz each, so their cost is $2yz$ each, or $4yz$ total. So, the total cost of constructing the closed box, in dollars, is $C(x, y, z) = 4xy + 5xy + 6xz + 4yz$, or $\boxed{C(x, y, z) = 9xy + 6xz + 4yz}$.
71. We see the level curves for $V = 18, 9$, and 6 below. The surface described by $z = V(x, y)$ exists only for (x, y) such that (x, y) is in the domain of V , and V has domain consisting of all (x, y) such that $4 - (x^2 + y^2) > 0$, or $x^2 + y^2 < 4$. This is the interior of the circle of radius 2 centered at the origin. So, the surface exists only above this disc. Also, note that a square root always produces a nonnegative output, so $V > 0$ always, and so it exists above the disc as opposed to below. Lastly, note that as distance from the origin ($x^2 + y^2$) increases, the denominator of V decreases, making V increase. So, $V(0, 0) = \frac{9}{2}$ and the z -values increase without bound as we move away from the origin, and asymptotically approach the circle $x^2 + y^2 = 4$.



72. We see the level curves for $T = 60^\circ, 54^\circ, 48^\circ, 6^\circ$, and 0° below. The surface described by $z = T(x, y)$ is an elliptic paraboloid opening downward, shifted up 60 units (degrees) on the z -axis.



73. We consider surfaces of the form $E(x, y, z) = c = \frac{3}{\sqrt{y^2 + z^2}}$. We then perform some algebraic manipulation on this equation:

$$\begin{aligned} c &= \frac{3}{\sqrt{y^2 + z^2}} \\ c^2 &= \frac{9}{y^2 + z^2} \\ y^2 + z^2 &= \frac{9}{c^2} \end{aligned}$$

We see that these are cylinders with the x -axis going through them, of radius $\frac{3}{c}$. (Note that c must be positive to begin with since $\sqrt{y^2 + z^2} \geq 0$ and $\frac{3}{\sqrt{y^2 + z^2}} \neq 0$ for all y, z .) The radii decrease as c increases.

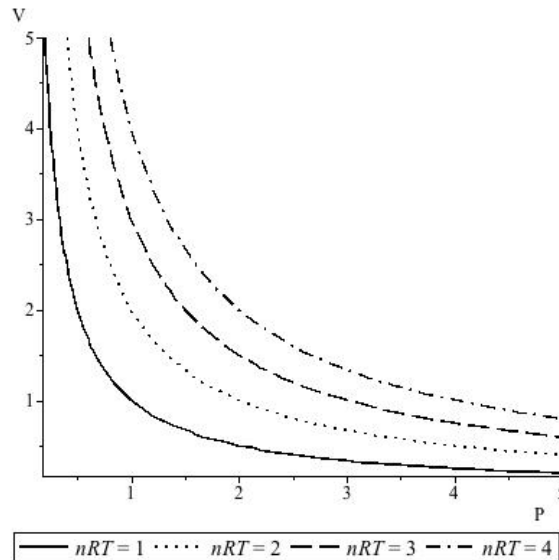
The level surfaces for $E(x, y, z)$ are all cylinders of differing radii with the x -axis going through them.

74. We consider surfaces of the form $F(x, y, z) = c = \frac{GmM}{x^2 + y^2 + z^2}$, and perform some algebraic manipulation on this equation:

$$\begin{aligned} c &= \frac{GmM}{x^2 + y^2 + z^2} \\ x^2 + y^2 + z^2 &= \frac{GmM}{c^2} \end{aligned}$$

We see that these are spheres centered at the origin of radius $\frac{\sqrt{GmM}}{c}$. (Once again note that $c > 0$ since $G, m, M, x^2 + y^2 + z^2 \geq 0$ and $\frac{GmM}{x^2 + y^2 + z^2} \neq 0$ for all x, y, z .) The radii decrease as c increases.

75. (a) Because T is being held constant, we are considering T as our dependent variable, and examining the level curves of the function $T = f(P, V)$. This yields $T = c = f(P, V)$, or $PV = nRc$. If we solve for V , we can think of these curves as existing in the PV -plane. This yields $V = \frac{nRc}{P} = c \left(\frac{nR}{P} \right)$, which are hyperbolas (centered at the origin). Since c is multiplying the function $V = \frac{nR}{P}$, the hyperbolas will be increasing vertical stretches of this function, as c increases (as shown in the graph).



- (b) Because P is being held constant, we are considering P as our dependent variable, and examining the level curves of the function $P = f(T, V)$. This yields $P = c = f(T, V)$, or $cV = nRT$. If

we solve for V , we can think of these curves as existing in the TV -plane. This yields $V = \frac{nRT}{c}$, which are lines with V -intercept 0 and slope $\frac{nR}{c}$. The slopes will decrease as c increases, and the lines will approach vertical lines as c tends to 0.

- (c) Because V is being held constant, we are considering V as our dependent variable, and examining the level curves of the function $V = f(T, P)$. This yields $V = c = f(T, P)$, or $Pc = nRT$. If we solve for P , we can think of these curves as existing in the TP -plane. This yields $P = \frac{nRT}{c}$, which are lines with P -intercept 0 and slope $\frac{nR}{c}$. The slopes will decrease as c increases, and the lines will approach vertical lines as c tends to 0.

76. (a) We consider surfaces of the form $V = c = \frac{kQ}{\sqrt{x^2 + y^2 + z^2}}$. We manipulate this equation algebraically to better understand these surfaces:

$$\begin{aligned} c &= \frac{kQ}{\sqrt{x^2 + y^2 + z^2}} \\ c^2 &= \frac{k^2 Q^2}{x^2 + y^2 + z^2} \\ x^2 + y^2 + z^2 &= \frac{k^2 Q^2}{c^2} \end{aligned}$$

We see that these are spheres centered at the origin of radius $\frac{kQ}{c}$. The radii decrease as c increases.

- (b) We see that considering $V \rightarrow 0$ is equivalent to considering $c \rightarrow 0$ in the above equation of the sphere. As $c \rightarrow 0$, the radii of the spheres increase to infinity.

77. (a) We consider surfaces of the form $B = c = \frac{\mu_0 I}{2\pi\sqrt{x^2 + y^2}}$. We manipulate this equation algebraically to better understand these surfaces:

$$\begin{aligned} c &= \frac{\mu_0 I}{2\pi\sqrt{x^2 + y^2}} \\ c^2 &= \frac{\mu_0^2 I^2}{4\pi^2(x^2 + y^2)} \\ x^2 + y^2 &= \frac{\mu_0^2 I^2}{4\pi^2 c^2} \end{aligned}$$

We see that these are circular cylinders opening about the z -axis, of radius $\frac{\mu_0 I}{2\pi c}$. The radii decrease as c increases.

- (b) We see that considering what happens as $|B|$ increases to ∞ is equivalent to considering $|c| \rightarrow \infty$ in the above equation of the sphere. As $|c| \rightarrow \infty$, the radii of the cylinders decrease toward 0, and tend to the B -axis.

78. (a) We write the equation as $U = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$ for simplicity. We then consider surfaces of the form $U = c = \frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$. We manipulate this equation algebraically to better understand these surfaces:

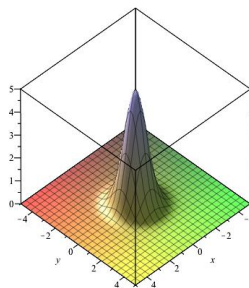
$$\begin{aligned} c &= -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}} \\ c^2 &= \frac{G^2 M^2 m^2}{x^2 + y^2 + z^2} \\ x^2 + y^2 + z^2 &= \frac{G^2 M^2 m^2}{c^2} \end{aligned}$$

We see that these are spheres centered at the origin of radius $\frac{GMm}{|c|}$. (Note that $c < 0$ since

$c = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$.) The radii decrease as $|c|$ increases.

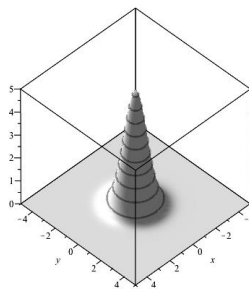
- (b) We see that considering $U \rightarrow 0$ is equivalent to considering $c \rightarrow 0$ in the above equation of the sphere. As $c \rightarrow 0$, the radii of the spheres increase to infinity.

79. (a) The graph is:

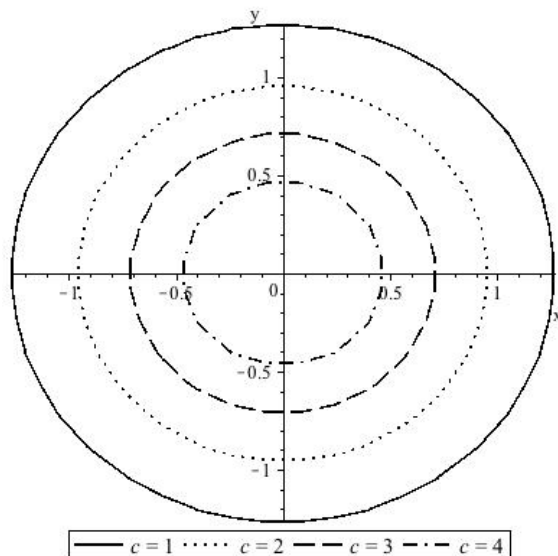


- (b) Answers will vary.

- (c) This is a version the graph that shows contour lines:



- (d) We set $f(x, y) = c$, and we obtain $c = 5e^{-x^2 - y^2}$, or $\frac{c}{5} = e^{-x^2 - y^2}$. Taking the natural logarithm of both sides, we get $\ln\left(\frac{c}{5}\right) = -x^2 - y^2$, or $x^2 + y^2 = \ln\left(\frac{5}{c}\right)$. So the level curve $f(x, y) = c$ is a circle of radius $\sqrt{\ln\left(\frac{5}{c}\right)}$ centered at the origin.



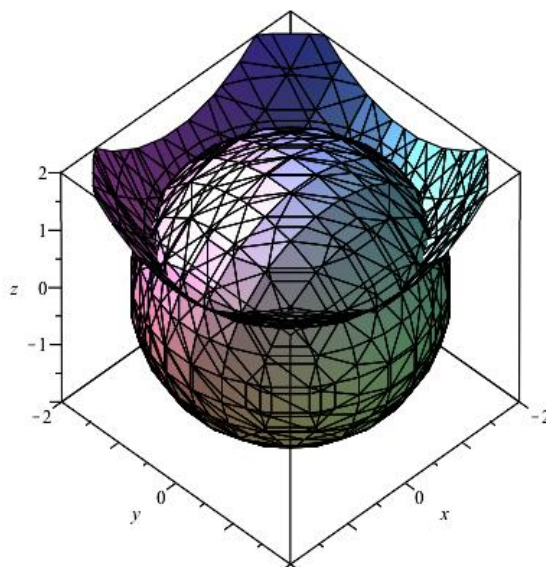
(e) Answers will vary.

80. (a) We manipulate the equation $z = \frac{1}{3}(x^2 + y^2)$ to obtain $x^2 + y^2 = 3z$, and replace $x^2 + y^2$ with $3z$ in the equation $x^2 + y^2 + z^2 = 4$ to obtain $3z + z^2 = 4$. We solve for z :

$$\begin{aligned} z^2 + 3z - 4 &= 0 \\ (z + 4)(z - 1) &= 0 \end{aligned}$$

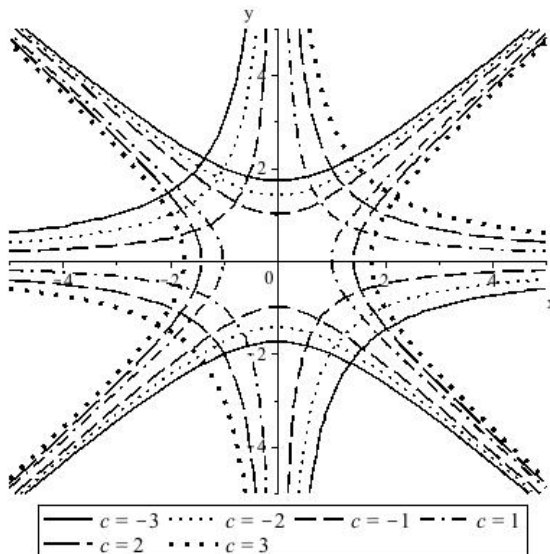
from which we obtain the horizontal planes $z = -4$ and $z = 1$, the latter of which is above the xy -plane. So, the desired curve of intersection is at height $z = 1$, which we substitute into $x^2 + y^2 = 3z$ to obtain $x^2 + y^2 = 3$. This is a circle in the plane $z = 1$ of radius $\sqrt{3}$, centered at $(0, 0, 1)$.

(b) The graphs are:



(c) Yes.

81. The level curves of $f(x, y)$ will be surfaces of the form $f(x, y) = c = x^2 - y^2$, or $1 = \frac{x^2}{c} - \frac{y^2}{c}$. These are symmetric hyperbolas centered about the origin, with asymptotes of $y = \pm x$. The level curves of $g(x, y)$ will be surfaces of the form $g(x, y) = k = xy$, or $y = \frac{k}{x}$. These are also hyperbolas with asymptotes of the x - and y -axes.



82. We consider the level curves of the graphs of f and g through P_0 , which can be represented by $c_1 = x^2 - y^2$ and $c_2 = xy$ respectively, for some constants c_1 and c_2 . We will compare the slopes of these curves at P_0 . We do this by implicitly differentiating each equation with respect to x :

$$\begin{aligned}\frac{d}{dx}(c_1) &= \frac{d}{dx}(x^2 - y^2) \\ 0 &= \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) \\ 0 &= 2x - 2y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{x}{y}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}(c_2) &= \frac{d}{dx}(xy) \\ 0 &= y + x \frac{dy}{dx} \quad \text{product rule} \\ \frac{dy}{dx} &= -\frac{y}{x}\end{aligned}$$

We see that these derivatives hold for all $(x, y) \neq (0, 0)$, and that they are opposite reciprocals of each other. So, the curves are perpendicular at any point $P_0 \neq (0, 0)$.

Challenge Problems

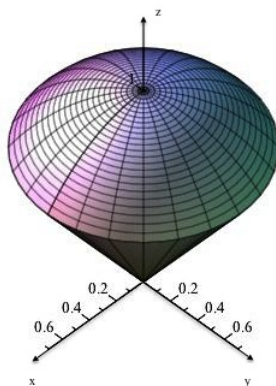
83. We know that the set of points (x, y, z) satisfying the condition $x^2 + y^2 + z^2 < 1$ is the region inside the unit sphere centered at the origin.

We also know that the set of points (x, y, z) satisfying the condition $x^2 + y^2 = z^2$ is a circular cone whose vertex is at the origin.

Therefore, the set of points satisfying the condition $x^2 + y^2 < z^2$, where $z > 0$, is the region located above the upper nappe of that circular cone.

The set of points described in the problem is the region that is both inside the unit sphere and above the upper nappe of the circular cone.

It is the inside of the region portrayed here:



Chapter 12: Functions of Several Variables

12.2: Limits and Continuity

Concepts and Vocabulary

1. **c, a δ -neighborhood**. A δ -neighborhood of P_0 consists of all the points P that lie within a distance δ of P_0 .
2. $\lim_{(x,y) \rightarrow (1,8)} 2y = 2 \left[\lim_{(x,y) \rightarrow (1,8)} y \right] = 2 \lim_{y \rightarrow 8} y = 2(8) = \mathbf{16}$.
3. **False**. For $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ to exist and equal L , it must exist and equal L along *all* curves in the domain of f that contain $(0,0)$, not just the straight-line curves.
4. **True**. This follows from the discussion in the beginning of subsection 3.
5. **c, a boundary**. See definition of “boundary point”.
6. **a, open**. See definition of “open”.
7. **False**. If $f(P_0) = 0$, then $\frac{g}{f}$ is not continuous at P_0 .
8. Since the function is continuous at $(0,0)$, we can evaluate it at $(0,0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (0,0)} e^{x^2+y^2} = e^{0+0} = e^0 = \boxed{1}.$$

Skill Building

9. We use algebraic properties of limits:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} x^2 + xy - y^2 + 8 &= \lim_{(x,y) \rightarrow (1,2)} x^2 + \lim_{(x,y) \rightarrow (1,2)} xy - \lim_{(x,y) \rightarrow (1,2)} y^2 + \lim_{(x,y) \rightarrow (1,2)} 8 \\ &= \lim_{x \rightarrow 1} x^2 + \left(\lim_{x \rightarrow 1} x \right) \left(\lim_{y \rightarrow 2} y \right) - \lim_{y \rightarrow 2} y^2 + 8 \\ &= 1 + 1 \cdot 2 - 4 + 8 = \boxed{7}. \end{aligned}$$

10. We use algebraic properties of limits:

$$\begin{aligned} &\lim_{(x,y) \rightarrow (-1,3)} x^2y + y^2 - 3xy - 2 \\ &= \lim_{(x,y) \rightarrow (-1,3)} x^2y + \lim_{(x,y) \rightarrow (-1,3)} y^2 - \lim_{(x,y) \rightarrow (-1,3)} 3xy - \lim_{(x,y) \rightarrow (-1,3)} 2 \\ &= \left(\lim_{x \rightarrow -1} x^2 \right) \left(\lim_{y \rightarrow 3} y \right) + \lim_{y \rightarrow 3} y^2 - 3 \left(\lim_{x \rightarrow -1} x \right) \left(\lim_{y \rightarrow 3} y \right) - 2 \\ &= 1 \cdot 3 + 9 - 3 \cdot (-1) \cdot 3 - 2 = \boxed{19}. \end{aligned}$$

11. We use algebraic properties of limits:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (1,-1,2)} (3x^2y + y^2z) &= \lim_{(x,y,z) \rightarrow (1,-1,2)} (3x^2y) + \lim_{(x,y,z) \rightarrow (1,-1,2)} (y^2z) \\ &= 3 \left(\lim_{x \rightarrow 1} x^2 \right) \left(\lim_{y \rightarrow -1} y \right) + \left(\lim_{y \rightarrow -1} y^2 \right) \left(\lim_{z \rightarrow 2} z \right) \\ &= 3 \cdot 1 \cdot (-1) + 1 \cdot 2 = \boxed{-1}. \end{aligned}$$

12. We use algebraic properties of limits:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,1,-1)} (x^2 - y^2z^2) &= \lim_{(x,y,z) \rightarrow (0,1,-1)} x^2 - \lim_{(x,y,z) \rightarrow (0,1,-1)} (y^2z^2) \\ &= \lim_{x \rightarrow 0} x^2 - \left(\lim_{y \rightarrow 1} y^2 \right) \left(\lim_{z \rightarrow -1} z^2 \right) \\ &= 0 - 1 \cdot 1 = \boxed{-1}. \end{aligned}$$

13. We use algebraic properties of limits:

$$\lim_{(x,y) \rightarrow (\frac{\pi}{2}, \pi)} \sin x \cos y = \left(\lim_{x \rightarrow \frac{\pi}{2}} \sin x \right) \left(\lim_{y \rightarrow \pi} \cos y \right) = \sin \frac{\pi}{2} \cos \pi = 1 \cdot -1 = \boxed{-1}.$$

14. We use algebraic properties of limits:

$$\lim_{(x,y) \rightarrow (2,e)} x^2y \ln y = \left(\lim_{x \rightarrow 2} x^2 \right) \left(\lim_{y \rightarrow e} y \right) \left(\lim_{y \rightarrow e} \ln y \right) = 4 \cdot e \cdot \ln(e) = 4 \cdot e \cdot 1 = \boxed{4e}.$$

15. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (1,5)} 4y - y^2 = \lim_{(x,y) \rightarrow (1,5)} 4y - \lim_{(x,y) \rightarrow (1,5)} y^2 = 4 \cdot \lim_{y \rightarrow 5} y - \lim_{y \rightarrow 5} y^2 = 4 \cdot 5 - 25 = -5 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (1,5)} 4x - xy + 4 &= \lim_{(x,y) \rightarrow (1,5)} 4x - \lim_{(x,y) \rightarrow (1,5)} xy + \lim_{(x,y) \rightarrow (1,5)} 4 \\
&= 4 \cdot \lim_{x \rightarrow 1} x - \left(\lim_{x \rightarrow 1} x \right) \left(\lim_{y \rightarrow 5} y \right) + 4 \\
&= 4 \cdot 1 - 1 \cdot 5 + 4 = 3.
\end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (1,5)} \frac{4x - xy + 4}{4y - y^2} = \frac{\lim_{(x,y) \rightarrow (1,5)} 4x - xy + 4}{\lim_{(x,y) \rightarrow (1,5)} 4y - y^2} = \frac{3}{-5} = \boxed{-\frac{3}{5}}.$$

16. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (2,2)} x + y - 3 = \lim_{(x,y) \rightarrow (2,2)} x + \lim_{(x,y) \rightarrow (2,2)} y - \lim_{(x,y) \rightarrow (2,2)} 3 = \lim_{x \rightarrow 2} x + \lim_{y \rightarrow 2} y - 3 = 2 + 2 - 3 = 1 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (2,2)} x^2 + 2xy + y^2 - 9 &= \lim_{(x,y) \rightarrow (2,2)} x^2 + \lim_{(x,y) \rightarrow (2,2)} 2xy + \lim_{(x,y) \rightarrow (2,2)} y^2 - \lim_{(x,y) \rightarrow (2,2)} 9 \\
&= \lim_{x \rightarrow 2} x^2 + 2 \left(\lim_{x \rightarrow 2} x \right) \left(\lim_{y \rightarrow 2} y \right) + \lim_{y \rightarrow 2} y^2 - 9 \\
&= 4 + 2 \cdot 2 \cdot 2 + 4 - 9 = 7.
\end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (2,2)} \frac{x^2 + 2xy + y^2 - 9}{x + y - 3} = \frac{\lim_{(x,y) \rightarrow (2,2)} x^2 + 2xy + y^2 - 9}{\lim_{(x,y) \rightarrow (2,2)} x + y - 3} = \frac{7}{1} = \boxed{7}.$$

17. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (\pi,0)} x = \lim_{x \rightarrow \pi} x = \pi \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\lim_{(x,y) \rightarrow (\pi,0)} \cos x \cos y = \left(\lim_{x \rightarrow \pi} \cos x \right) \left(\lim_{y \rightarrow 0} \cos y \right) = \cos \pi \cdot \cos 0 = -1 \cdot 1 = -1.$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (\pi,0)} \frac{\cos x \cos y}{x} = \frac{\lim_{(x,y) \rightarrow (\pi,0)} \cos x \cos y}{\lim_{(x,y) \rightarrow (\pi,0)} x} = \frac{-1}{\pi} = \boxed{-\frac{1}{\pi}}.$$

18. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (\pi,\pi)} xy = \left(\lim_{x \rightarrow \pi} x \right) \left(\lim_{y \rightarrow \pi} y \right) = \pi \cdot \pi = \pi^2 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (\pi,\pi)} \cos y(1 - \cos x) &= \left(\lim_{y \rightarrow \pi} \cos y \right) \left(\lim_{x \rightarrow \pi} (1 - \cos x) \right) \\
 &= \left(\lim_{y \rightarrow \pi} \cos y \right) \left(\lim_{x \rightarrow \pi} 1 - \lim_{x \rightarrow \pi} \cos x \right) \\
 &= \cos \pi (1 - \cos \pi) = (-1) \cdot (1 - (-1)) = -2.
 \end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (\pi,\pi)} \frac{\cos y(1 - \cos x)}{xy} = \frac{\lim_{(x,y) \rightarrow (\pi,\pi)} \cos y(1 - \cos x)}{\lim_{(x,y) \rightarrow (\pi,\pi)} xy} = \frac{-2}{\pi^2} = \boxed{-\frac{2}{\pi^2}}.$$

19. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (0,0)} e^y = \lim_{y \rightarrow 0} e^y = e^0 = 1 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\lim_{(x,y) \rightarrow (0,0)} e^x - 4 = \lim_{(x,y) \rightarrow (0,0)} e^x - \lim_{(x,y) \rightarrow (0,0)} 4 = \lim_{x \rightarrow 0} e^x - 4 = e^0 - 4 = 1 - 4 = -3.$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^x - 4}{e^y} = \frac{\lim_{(x,y) \rightarrow (0,0)} e^x - 4}{\lim_{(x,y) \rightarrow (0,0)} e^y} = \frac{-3}{1} = \boxed{-3}.$$

20. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (0,0)} e^y = e^0 = 1 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (0,0)} e^x \cos y - \cos y &= \lim_{(x,y) \rightarrow (0,0)} e^x \cos y - \lim_{(x,y) \rightarrow (0,0)} \cos y \\
 &= \left(\lim_{x \rightarrow 0} e^x \right) \left(\lim_{y \rightarrow 0} \cos y \right) - \lim_{y \rightarrow 0} \cos y \\
 &= e^0 \cdot \cos 0 - \cos 0 = 1 \cdot 1 - 1 = 0.
 \end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^x \cos y - \cos y}{e^y} = \frac{\lim_{(x,y) \rightarrow (0,0)} e^x \cos y - \cos y}{\lim_{(x,y) \rightarrow (0,0)} e^y} = \frac{0}{1} = \boxed{0}.$$

21. Look at the denominator. Since

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (2,1)} x^2 + 4y^2 &= \lim_{(x,y) \rightarrow (2,1)} x^2 + \lim_{(x,y) \rightarrow (2,1)} 4y^2 \\
 &= \lim_{x \rightarrow 2} x^2 + 4 \cdot \lim_{y \rightarrow 1} y^2 = 4 + 4 \cdot 1 = 8 \neq 0,
 \end{aligned}$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (2,1)} x^2 + xy - 6y^2 &= \lim_{(x,y) \rightarrow (2,1)} x^2 + \lim_{(x,y) \rightarrow (2,1)} xy - \lim_{(x,y) \rightarrow (2,1)} 6y^2 \\
&= \lim_{x \rightarrow 2} x^2 + \left(\lim_{x \rightarrow 2} x \right) \left(\lim_{y \rightarrow 1} y \right) - 6 \cdot \lim_{y \rightarrow 1} y^2 \\
&= 4 + 2 \cdot 1 - 6 \cdot 1 = 0.
\end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 + xy - 6y^2}{x^2 + 4y^2} = \frac{\lim_{(x,y) \rightarrow (2,1)} x^2 + xy - 6y^2}{\lim_{(x,y) \rightarrow (2,1)} x^2 + 4y^2} = \frac{0}{8} = \boxed{0}.$$

22. Look at the denominator. Since

$$\begin{aligned}
\lim_{(x,y) \rightarrow (0,-2)} xy - y + 2x - e^x &= \lim_{(x,y) \rightarrow (0,-2)} xy - \lim_{(x,y) \rightarrow (0,-2)} y + \lim_{(x,y) \rightarrow (0,-2)} 2x - \lim_{(x,y) \rightarrow (0,-2)} e^x \\
&= \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{y \rightarrow -2} y \right) - \left(\lim_{y \rightarrow -2} y \right) + 2 \cdot \left(\lim_{x \rightarrow 0} x \right) - \left(\lim_{x \rightarrow 0} e^x \right) \\
&= 0 \cdot (-2) - (-2) + 2 \cdot 0 - e^0 = 0 + 2 + 0 - 1 = 1 \neq 0,
\end{aligned}$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
&\lim_{(x,y) \rightarrow (0,-2)} y^2 + xy + 4y + e^x \\
&= \lim_{(x,y) \rightarrow (0,-2)} y^2 + \lim_{(x,y) \rightarrow (0,-2)} xy + \lim_{(x,y) \rightarrow (0,-2)} 4y + \lim_{(x,y) \rightarrow (0,-2)} e^x \\
&= \lim_{y \rightarrow -2} y^2 + \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{y \rightarrow -2} y \right) + 4 \cdot \left(\lim_{y \rightarrow -2} y \right) + \left(\lim_{x \rightarrow 0} e^x \right) \\
&= (-2)^2 + 0 \cdot (-2) + 4 \cdot (-2) + e^0 = 4 + 0 - 8 + 1 = -3.
\end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (0,-2)} \frac{y^2 + xy + 4y + e^x}{xy - y + 2x - e^x} = \frac{\lim_{(x,y) \rightarrow (0,-2)} y^2 + xy + 4y + e^x}{\lim_{(x,y) \rightarrow (0,-2)} xy - y + 2x - e^x} = \frac{-3}{1} = \boxed{-3}.$$

23. Look at the denominator. Since

$$\begin{aligned}
\lim_{(x,y) \rightarrow (2,0)} x^3 y + 3xy^2 - 8 &= \lim_{(x,y) \rightarrow (2,0)} x^3 y + \lim_{(x,y) \rightarrow (2,0)} 3xy^2 - \lim_{(x,y) \rightarrow (2,0)} 8 \\
&= \left(\lim_{x \rightarrow 2} x^3 \right) \left(\lim_{y \rightarrow 0} y \right) + 3 \cdot \left(\lim_{x \rightarrow 2} x \right) \left(\lim_{y \rightarrow 0} y^2 \right) - 8 \\
&= 8 \cdot 0 + 3 \cdot 2 \cdot 0 - 8 = -8 \neq 0,
\end{aligned}$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (2,0)} x^2 y + x &= \lim_{(x,y) \rightarrow (2,0)} x^2 y + \lim_{(x,y) \rightarrow (2,0)} x \\
&= \left(\lim_{x \rightarrow 2} x^2 \right) \left(\lim_{y \rightarrow 0} y \right) + \lim_{x \rightarrow 2} x \\
&= 2^2 \cdot 0 + 2 = 0 + 2 = 2.
\end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (2,0)} \frac{x^2y + x}{x^3y + 3xy^2 - 8} = \frac{\lim_{(x,y) \rightarrow (2,0)} x^2y + x}{\lim_{(x,y) \rightarrow (2,0)} x^3y + 3xy^2 - 8} = \frac{2}{-8} = \boxed{-\frac{1}{4}}.$$

24. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (0,1)} xy + 4 = \lim_{(x,y) \rightarrow (0,1)} xy + \lim_{(x,y) \rightarrow (0,1)} 4 = \left(\lim_{x \rightarrow 0} x\right) \left(\lim_{y \rightarrow 1} y\right) + 4 = 0 \cdot 1 + 4 = 4 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,1)} x^3 - 4x^2y + 2 &= \lim_{(x,y) \rightarrow (0,1)} x^3 - \lim_{(x,y) \rightarrow (0,1)} 4x^2y + \lim_{(x,y) \rightarrow (0,1)} 2 \\ &= \lim_{x \rightarrow 0} x^3 - 4 \cdot \left(\lim_{x \rightarrow 0} x^2\right) \left(\lim_{y \rightarrow 1} y\right) + 2 \\ &= 0 - 4 \cdot 0 \cdot 1 + 2 = 2. \end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (0,1)} \frac{x^3 - 4x^2y + 2}{xy + 4} = \frac{\lim_{(x,y) \rightarrow (0,1)} x^3 - 4x^2y + 2}{\lim_{(x,y) \rightarrow (0,1)} xy + 4} = \frac{2}{4} = \boxed{\frac{1}{2}}.$$

25. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x \cdot 0}{2x^2 + 0} = \lim_{x \rightarrow 0} \frac{0}{2x^2} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3 \cdot 0 \cdot y}{2 \cdot 0 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x \cdot x}{2x^2 + x^2} = \lim_{x \rightarrow 0} \frac{3x^2}{3x^2} = \lim_{x \rightarrow 0} 1 = \boxed{1}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x \cdot (3x)}{2x^2 + (3x)^2} = \lim_{x \rightarrow 0} \frac{9x^2}{2x^2 + 9x^2} = \lim_{x \rightarrow 0} \frac{9x^2}{11x^2} = \lim_{x \rightarrow 0} \frac{9}{11} = \boxed{\frac{9}{11}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x \cdot x^2}{2x^2 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{3x^3}{2x^2 + x^4} = \lim_{x \rightarrow 0} \frac{3x^3}{x^2(2 + x^2)} = \lim_{x \rightarrow 0} \frac{3x}{2 + x^2} = \frac{0}{2} = \boxed{0}.$$

Since we did not obtain the same answer for all five parts, we conclude that

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{2x^2 + y^2} \text{ does not exist.}}$$

26. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot 0}{x^2 + 3 \cdot 0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2 \cdot 0 \cdot y}{0 + 3y^2} = \lim_{y \rightarrow 0} \frac{0}{3y^2} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot x}{x^2 + 3x^2} = \lim_{x \rightarrow 0} \frac{2x^2}{4x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \boxed{\frac{1}{2}}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot (3x)}{x^2 + 3(3x)^2} = \lim_{x \rightarrow 0} \frac{6x^2}{x^2 + 27x^2} = \lim_{x \rightarrow 0} \frac{6x^2}{28x^2} = \lim_{x \rightarrow 0} \frac{3}{14} = \boxed{\frac{3}{14}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot x^2}{x^2 + 3(x^2)^2} = \lim_{x \rightarrow 0} \frac{2x^3}{x^2 + 3x^4} = \lim_{x \rightarrow 0} \frac{2x^3}{x^2(1 + 3x^2)} = \lim_{x \rightarrow 0} \frac{2x}{1 + 3x^2} = \frac{0}{1} = \boxed{0}.$$

Since we did not obtain the same answer for all five parts,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + 3y^2} \text{ does not exist.}$$

27. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot 0}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{0 \cdot y^2}{0 + y^3} = \lim_{y \rightarrow 0} \frac{0}{y^3} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot x^2}{x^2 + x^3} = \lim_{x \rightarrow 0} \frac{x^3}{x^2(1 + x)} = \lim_{x \rightarrow 0} \frac{x}{1 + x} = \frac{0}{1} = \boxed{0}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot (3x)^2}{x^2 + (3x)^3} = \lim_{x \rightarrow 0} \frac{9x^3}{x^2 + 27x^3} = \lim_{x \rightarrow 0} \frac{9x^3}{x^2(1 + 27x)} = \lim_{x \rightarrow 0} \frac{9x}{(1 + 27x)} = \frac{0}{1} = \boxed{0}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot (x^2)^2}{x^2 + 3(x^2)^3} = \lim_{x \rightarrow 0} \frac{x^5}{x^2 + 3x^6} = \lim_{x \rightarrow 0} \frac{x^5}{x^2(1 + 3x^4)} = \lim_{x \rightarrow 0} \frac{x^3}{1 + 3x^4} = \frac{0}{1} = \boxed{0}.$$

Although we obtained the same answer for all five parts,

$$\text{no conclusion can be reached about } \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^3}.$$

28. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \cdot 0}{3x^3 + 0} = \lim_{x \rightarrow 0} \frac{0}{3x^3} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2 \cdot 0 \cdot y}{0 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \cdot x}{3x^3 + x^2} = \lim_{x \rightarrow 0} \frac{2x^3}{x^2(3x + 1)} = \lim_{x \rightarrow 0} \frac{2x}{3x + 1} = \frac{0}{1} = \boxed{0}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \cdot (3x)}{3x^3 + (3x)^2} = \lim_{x \rightarrow 0} \frac{6x^3}{3x^3 + 9x^2} = \lim_{x \rightarrow 0} \frac{6x^3}{x^2(3x + 9)} = \lim_{x \rightarrow 0} \frac{6x}{(3x + 9)} = \frac{0}{9} = \boxed{0}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \cdot (x^2)}{3x^3 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{2x^4}{3x^3 + x^4} = \lim_{x \rightarrow 0} \frac{2x^4}{x^3(3 + x)} = \lim_{x \rightarrow 0} \frac{2x}{3 + x} = \frac{0}{3} = \boxed{0}.$$

Although we obtained the same answer for all five parts,

no conclusion can be reached about $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{3x^3 + y^2}.$

29. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 \cdot 0}{x^4 + 0} = \lim_{x \rightarrow 0} \frac{0}{x^4} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3 \cdot 0 \cdot y^2}{0 + y^4} = \lim_{y \rightarrow 0} \frac{0}{y^4} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 \cdot x^2}{x^4 + x^4} = \lim_{x \rightarrow 0} \frac{3x^4}{2x^4} = \lim_{x \rightarrow 0} \frac{3}{2} = \boxed{\frac{3}{2}}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 \cdot (3x)^2}{x^4 + (3x)^4} = \lim_{x \rightarrow 0} \frac{27x^4}{x^4 + 81x^4} = \lim_{x \rightarrow 0} \frac{27x^4}{82x^4} = \lim_{x \rightarrow 0} \frac{27}{82} = \boxed{\frac{27}{82}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 \cdot (x^2)^2}{x^4 + (x^2)^4} = \lim_{x \rightarrow 0} \frac{3x^6}{x^4 + x^8} = \lim_{x \rightarrow 0} \frac{3x^6}{x^4(1 + x^4)} = \lim_{x \rightarrow 0} \frac{3x^2}{1 + x^4} = \frac{0}{1} = \boxed{0}.$$

Since we did not obtain the same answer for all five parts,

$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y^2}{x^4 + y^4}$ does not exist.

30. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \lim_{x \rightarrow 0} 1 = \boxed{1}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{0}{0 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \boxed{\frac{1}{2}}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + (3x)^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2 + 9x^2} = \lim_{x \rightarrow 0} \frac{x^2}{10x^2} = \lim_{x \rightarrow 0} \frac{1}{10} = \boxed{\frac{1}{10}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2 + x^4} = \lim_{x \rightarrow 0} \frac{x^2}{x^2(1 + x^2)} = \lim_{x \rightarrow 0} \frac{1}{1 + x^2} = \frac{1}{1} = \boxed{1}.$$

Since we did not obtain the same answer for all five parts,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} \text{ does not exist.}}$$

31. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + x \cdot 0}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \lim_{x \rightarrow 0} 1 = \boxed{1}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{0 + 0 \cdot y}{0 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} 0 = \boxed{0}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + x \cdot x}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = \lim_{x \rightarrow 0} \frac{2}{2} = \boxed{1}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + x \cdot (3x)}{x^2 + (3x)^2} = \lim_{x \rightarrow 0} \frac{x^2 + 3x^2}{x^2 + 9x^2} = \lim_{x \rightarrow 0} \frac{4x^2}{10x^2} = \lim_{x \rightarrow 0} \frac{4}{10} = \boxed{\frac{2}{5}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + x \cdot x^2}{x^2 + (x^2)^2} = \lim_{x \rightarrow 0} \frac{x^2 + x^3}{x^2 + x^4} = \lim_{x \rightarrow 0} \frac{x^2(1 + x)}{x^2(1 + x^2)} = \lim_{x \rightarrow 0} \frac{1 + x}{1 + x^2} = \frac{1}{1} = \boxed{1}.$$

Since we did not obtain the same answer for all five parts,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + xy}{x^2 + y^2} \text{ does not exist.}}$$

32. (a) Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-0)^2}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \lim_{x \rightarrow 0} 1 = \boxed{1}.$$

(b) Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(0-y)^2}{0+y^2} = \lim_{y \rightarrow 0} \frac{y^2}{y^2} = \lim_{y \rightarrow 0} 1 = \boxed{1}.$$

(c) Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-x)^2}{x^2+x^2} = \lim_{x \rightarrow 0} \frac{0}{2x^2} = \lim_{x \rightarrow 0} 0 = \boxed{0}.$$

(d) Using $y = 3x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-3x)^2}{x^2+(3x)^2} = \lim_{x \rightarrow 0} \frac{(-2x)^2}{x^2+9x^2} = \lim_{x \rightarrow 0} \frac{4x^2}{10x^2} = \lim_{x \rightarrow 0} \frac{4}{10} = \boxed{\frac{2}{5}}.$$

(e) Using $y = x^2$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-x^2)^2}{x^2+(x^2)^2} = \lim_{x \rightarrow 0} \frac{x^2-2x^3+x^4}{x^2+x^4} = \lim_{x \rightarrow 0} \frac{x^2(1-2x+x^2)}{x^2(1+x^2)} = \lim_{x \rightarrow 0} \frac{1-2x+x^2}{1+x^2} = \frac{1}{1} = \boxed{1}.$$

Since we did not obtain the same answer for all five parts,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-y)^2}{x^2+y^2} \text{ does not exist.}$$

33. We investigate the limit using two curves that contain (0,0): the x -axis ($y = 0$) and the y -axis ($x = 0$).
Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2+0}{x^2+0} = \lim_{x \rightarrow 0} \frac{2x^2}{x^2} = \lim_{x \rightarrow 0} 2 = 2.$$

Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2 \cdot 0 + y^2}{0 + y^2} = \lim_{y \rightarrow 0} \frac{y^2}{y^2} = \lim_{y \rightarrow 0} 1 = 1.$$

Since two different answers are obtained, the limit does not exist.

34. We investigate the limit using two curves that contain (0,0): the x -axis ($y = 0$) and the line $y = x$.
Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot 0}{x^2+0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x \cdot x}{x^2+x^2} = \lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = \lim_{x \rightarrow 0} 1 = 1.$$

Since two different answers are obtained, the limit does not exist.

35. We investigate the limit using two curves that contain (0,0): the x -axis ($y = 0$) and the y -axis ($x = 0$).
Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4-0}{x^2+0} = \lim_{x \rightarrow 0} \frac{x^4}{x^2} = \lim_{x \rightarrow 0} x^2 = 0.$$

Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{0-y^2}{0+y^2} = \lim_{y \rightarrow 0} \frac{-y^2}{y^2} = \lim_{y \rightarrow 0} -1 = -1.$$

Since two different answers are obtained, the limit does not exist.

36. We investigate the limit using two curves that contain (0,0): the x -axis ($y = 0$) and the y -axis ($x = 0$). Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 0}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \lim_{x \rightarrow 0} 1 = 1.$$

Using $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{0 + y^4}{0 + y^2} = \lim_{y \rightarrow 0} \frac{y^4}{y^2} = \lim_{y \rightarrow 0} y^2 = 0.$$

Since two different answers are obtained, the limit does not exist.

37. Since f is a polynomial in x and y , it is continuous for all points (x, y) in the plane.
38. Since f is a polynomial in x and y , it is continuous for all points (x, y) in the plane.
39. $f(x, y)$ is a rational function and is therefore continuous on its domain, which is all points (x, y) in the plane at which the denominator is not 0. We see that $x - y = 0$ on the line $y = x$, so this function is continuous at all points (x, y) in the plane except those on the line $y = x$.
40. $f(x, y)$ is a rational function and is therefore continuous on its domain, which is all points (x, y) in the plane at which the denominator is not 0. We see that $1 - xy = 0$ on the hyperbola $xy = 1$, or $y = \frac{1}{x}$, so this function is continuous at all points (x, y) in the plane except those on the hyperbola $y = \frac{1}{x}$.
41. $f(x, y)$ is the composition of two continuous functions, $h(t) = e^t$ and $g(x, y) = x^2 - y^2$, and is therefore continuous itself, for all points (x, y) in the plane.
42. $f(x, y)$ is the composition of the functions $h(t) = \ln t$ and $g(x, y) = x^2 + y^2$. We see that h is continuous on its domain, which is all positive real numbers t . Therefore, f will be continuous on all points (x, y) in the plane such that $x^2 + y^2 > 0$. Since $x^2 + y^2 \geq 0$ for all x, y , we only need eliminate the case $x^2 + y^2 = 0$, which only happens when $(x, y) = (0, 0)$. Therefore, this function is continuous at all points (x, y) in the plane except the point $(0, 0)$.
43. $f(x, y)$ is the composition of two continuous functions, $h(t) = \sin t$ and $g(x, y) = x^2 - y$, and is therefore continuous itself, for all points (x, y) in the plane.
44. $f(x, y)$ is the composition of the functions $h(t) = \cos t$ and $g(x, y) = \sqrt{x^2 - y}$. We see that h is continuous on all real numbers, but g is only continuous on its domain, which is all points (x, y) in the plane except those which will produce a negative quantity under the square root. Therefore, f will be continuous at all points (x, y) in the plane such that $x^2 - y \geq 0$, or such that $y \leq x^2$. Therefore, this function is continuous on all points (x, y) in the plane that satisfy $y \leq x^2$.
45. $f(x, y)$ is the product of two functions. The first the composition of two continuous functions, $k(t) = \sin t$ and $l(x, y) = x + y$, and so is continuous on the entire xy -plane. The second is also the composition of two continuous functions, $k(t) = \cos t$ and $l(x, y) = x - y$, and so is also continuous itself. The product of two continuous functions is continuous, and so f is continuous on all points (x, y) in the plane.
46. $f(x, y)$ is the product of two functions. The first is the function $h(x, y) = h(x) = e^x$, which is continuous on the entire xy -plane. The second is the composition of two continuous functions, $k(t) = \sin t$ and $l(x, y) = xy$, and so is also itself continuous. The product of two continuous functions is continuous, and so f is continuous on all points (x, y) in the plane.

47. $f(x, y)$ is the quotient of two functions. So, we just need to check continuity of both numerator and denominator, as well as find the zeros of the denominator. The numerator is a polynomial so it is continuous on the entire plane. The denominator is an exponential, so it is never equal to 0. We lastly check where the denominator itself is continuous. It is the composition of two continuous functions, $h(t) = e^t$ and $g(x, y) = x^2 - y^2$, and is therefore continuous on the entire xy -plane. Also, note that $e^{x^2-y^2}$ will never be equal to 0. So, f is continuous for all points (x, y) in the plane.

48. $f(x, y)$ is the quotient of two functions. So, we just need to check continuity of both numerator and denominator, as well as find the zeros of the denominator. The numerator is a polynomial so it is continuous on the entire plane. The denominator is the composition of two continuous functions, $h(t) = \ln t$ and $g(x, y) = x^2 + y^2$. We see that h is continuous on its domain, which is all positive real numbers t . Therefore, h will be continuous on all points (x, y) in the plane such that $x^2 + y^2 > 0$. Since $x^2 + y^2 \geq 0$ for all x, y , we only need eliminate the case $x^2 + y^2 = 0$, which only happens when $(x, y) = (0, 0)$. Therefore, the denominator is continuous on all points (x, y) in the plane except the point $(0, 0)$. We still need to determine where the denominator is 0. This will happen when $\ln(x^2 + y^2) = 0$, which occurs when $x^2 + y^2 = 1$. We must eliminate these points. So, f is continuous on all points (x, y) in the plane except $(0, 0)$ and those on the circle $x^2 + y^2 = 1$.

49. Since the function is continuous at $(1, 0)$, we may evaluate it at $(1, 0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - y^2}{x - y} = \frac{1 - 0}{1 - 0} = \boxed{1}.$$

50. Since the function is continuous at $(0, 0)$, we may evaluate it at $(0, 0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y + xy^2}{1 - xy} = \frac{0 + 0}{1 - 0} = \boxed{0}.$$

51. Since the function is continuous at $(1, 1)$, we may evaluate it at $(1, 1)$ to find the limit:

$$\lim_{(x,y) \rightarrow (1,1)} e^{x^2-y^2} = e^{1-1} = e^0 = \boxed{1}.$$

52. Since the function is continuous at $(0, e)$, we may evaluate it at $(0, e)$ to find the limit:

$$\lim_{(x,y) \rightarrow (0,e)} \ln(x^2 + y^2) = \ln(0 + e^2) = \ln(e^2) = \boxed{2}.$$

53. Since the function is continuous at $(\pi/2, \pi)$, we may evaluate it at $(\pi/2, \pi)$ to find the limit:

$$\lim_{(x,y) \rightarrow (\pi/2, \pi)} [\sin(x + y) \cos(x - y)] = \sin\left(\frac{\pi}{2} + \pi\right) \cos\left(\frac{\pi}{2} - \pi\right) = \sin\frac{3\pi}{2} \cos\left(-\frac{\pi}{2}\right) = (-1)(0) = \boxed{0}.$$

54. Since the function is continuous at $(0, \pi/2)$, we may evaluate it at $(0, \pi/2)$ to find the limit:

$$\lim_{(x,y) \rightarrow (0, \pi/2)} e^x \sin(xy) = e^0 \sin\left(0 \cdot \frac{\pi}{2}\right) = 1 \sin 0 = (1)(0) = \boxed{0}.$$

55. Since the function is continuous at $(\pi, 0)$, we may evaluate it at $(\pi, 0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (\pi, 0)} \frac{e^{x^2+y^2} \cos x^2}{\cos y^2} = \frac{e^{\pi^2+0} \cos \pi^2}{\cos 0} = \frac{e^{\pi^2} \cos \pi^2}{1} = \boxed{e^{\pi^2} \cos \pi^2}.$$

56. Since the function is continuous at $(\frac{\pi}{2}, 4)$, we may evaluate it at $(\frac{\pi}{2}, 4)$ to find the limit:

$$\lim_{(x,y) \rightarrow (\pi/2, 4)} \frac{e^{x^2 y}}{\cos(2x)} = \frac{e^{(\frac{\pi}{2})^2 \cdot 4}}{\cos(2 \cdot \frac{\pi}{2})} = \frac{e^{\frac{\pi^2}{4} \cdot 4}}{\cos \pi} = \frac{e^{\pi^2}}{-1} = \boxed{-e^{\pi^2}}.$$

57. Since the function is continuous at $(0, 0)$, we may evaluate it at $(0, 0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \tan^{-1} \left(\frac{e^{x+y}}{y^2 + 1} \right) = \tan^{-1} \left(\frac{e^{0+0}}{0+1} \right) = \tan^{-1} \left(\frac{1}{1} \right) = \tan^{-1}(1) = \boxed{\frac{\pi}{4}}.$$

58. Since the function is continuous at $(\pi, 0)$, we may evaluate it at $(\pi, 0)$ to find the limit:

$$\lim_{(x,y) \rightarrow (\pi,0)} \tan^{-1} [\cos(x+y)] = \tan^{-1} [\cos(\pi+0)] = \tan^{-1}(\cos \pi) = \tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}.$$

Applications and Extensions

59. Note that $(x, y, z) \rightarrow (0, 0, 0)$ corresponds to $t \rightarrow 0$. We replace x, y , and z with their respective expressions in t in the limit:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2yz}{x^4 + y^2 + z^2} &= \lim_{t \rightarrow 0} \frac{2(t)(t)}{t^4 + t^2 + t^2} = \lim_{t \rightarrow 0} \frac{2t^2}{t^4 + 2t^2} = \lim_{t \rightarrow 0} \frac{2t^2}{t^2(t^2 + 2)} \\ &= \lim_{t \rightarrow 0} \frac{2}{t^2 + 2} = \frac{2}{2} = \boxed{1} \end{aligned}$$

60. Note that $(x, y, z) \rightarrow (0, 0, 0)$ corresponds to $t \rightarrow 0$. We replace x, y , and z with their respective expressions in t in the limit:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2yz}{x^4 + y^2 + z^2} &= \lim_{t \rightarrow 0} \frac{2(3t)(4t)}{(2t)^4 + (3t)^2 + (4t)^2} = \lim_{t \rightarrow 0} \frac{24t^2}{16t^4 + 25t^2} \\ &= \lim_{t \rightarrow 0} \frac{24t^2}{t^2(16t^2 + 25)} = \lim_{t \rightarrow 0} \frac{24}{16t^2 + 25} = \boxed{\frac{24}{25}} \end{aligned}$$

61. Note that $(x, y, z) \rightarrow (0, 0, 0)$ corresponds to $t \rightarrow 0$. We replace x, y , and z with their respective expressions in t in the limit:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2yz}{x^4 + y^2 + z^2} &= \lim_{t \rightarrow 0} \frac{2(t^2)(t^2)}{t^4 + (t^2)^2 + (t^2)^2} = \lim_{t \rightarrow 0} \frac{2t^4}{t^4 + t^4 + t^4} = \lim_{t \rightarrow 0} \frac{2t^4}{3t^4} \\ &= \lim_{t \rightarrow 0} \frac{2}{3} = \boxed{\frac{2}{3}} \end{aligned}$$

62. Note that $(x, y, z) \rightarrow (0, 0, 0)$ corresponds to $t \rightarrow 0$. We may then replace x, y , and z with their respective expressions in t to find the limit:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2yz}{x^4 + y^2 + z^2} &= \lim_{t \rightarrow 0} \frac{2(bt)(ct)}{(at)^4 + (bt)^2 + (ct)^2} = \lim_{t \rightarrow 0} \frac{2bct^2}{a^4t^4 + b^2t^2 + c^2t^2} \\ &= \lim_{t \rightarrow 0} \frac{2bct^2}{t^2(a^4t^2 + b^2 + c^2)} = \lim_{t \rightarrow 0} \frac{2bc}{a^4t^2 + b^2 + c^2} = \frac{2bc}{b^2 + c^2} \end{aligned}$$

This limit is valid as long as $b^2 + c^2 > 0$.

63. Given a number $\varepsilon > 0$, we seek a number $\delta > 0$, so that

$$\text{whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \text{ then } \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| < \varepsilon.$$

That is,

$$\text{whenever } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{x^2 y}{x^2 + y^2} \right| < \varepsilon.$$

We need to find a connection between $\frac{x^2 y}{x^2 + y^2}$ and $\sqrt{x^2 + y^2}$. We begin with the observation that since $y^2 \geq 0$,

$$x^2 \leq x^2 + y^2.$$

For all points (x, y) not equal to $(0, 0)$, the quantity $x^2 + y^2$ is greater than zero, so we can divide both sides of the inequality by $x^2 + y^2$. We obtain

$$\frac{x^2}{x^2 + y^2} \leq 1.$$

Now,

$$\left| \frac{x^2 y}{x^2 + y^2} \right| = \frac{x^2 |y|}{x^2 + y^2} = |y| \cdot \frac{x^2}{x^2 + y^2} \leq |y| \cdot 1 = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$$

Given $\varepsilon > 0$, we choose $\delta \leq \varepsilon$. Then, whenever $0 < \sqrt{x^2 + y^2} < \delta$, we have

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \leq \sqrt{x^2 + y^2} < \delta \leq \varepsilon.$$

That is,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

64. Given a number $\varepsilon > 0$, we seek a number $\delta > 0$, so that

$$\text{whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \text{ then } \left| \frac{\sin(x^2 + y^2)}{x^2 + y^2} - 1 \right| < \varepsilon.$$

That is,

$$\text{whenever } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{\sin(x^2 + y^2)}{x^2 + y^2} - 1 \right| < \varepsilon.$$

We need to find a connection between $\frac{\sin(x^2 + y^2)}{x^2 + y^2} - 1$ and $\sqrt{x^2 + y^2}$. We begin by recalling the following limit:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1,$$

which we know from Section 1.4, page 108. Using the ε - δ definition of a limit, it follows that given $\varepsilon > 0$, we can choose $\delta_0 > 0$ such that

$$\text{whenever } 0 < \theta < \delta_0 \text{ we have } \left| \frac{\sin \theta}{\theta} - 1 \right| < \varepsilon.$$

Making the substitution $\theta = x^2 + y^2$, we see that

$$\text{whenever } 0 < x^2 + y^2 < \delta_0 \text{ we have } \left| \frac{\sin(x^2 + y^2)}{x^2 + y^2} - 1 \right| < \varepsilon.$$

But, we also see that the inequality $0 < x^2 + y^2 < \delta_0$ is equivalent to $0 < \sqrt{x^2 + y^2} < \sqrt{\delta_0}$. Therefore, given $\varepsilon > 0$, we choose $\delta = \sqrt{\delta_0} > 0$. Then, whenever $0 < \sqrt{x^2 + y^2} < \delta$, we have $0 < x^2 + y^2 < \delta^2 = \delta_0$, and it follows that

$$\left| \frac{\sin(x^2 + y^2)}{x^2 + y^2} - 1 \right| < \varepsilon.$$

That is,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1.$$

65. (a) f is not continuous at $(0,0)$ since this point gives 0 in the denominator.
 (b) We need to determine whether or not the following limit exists:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2}.$$

We will use the ε - δ definition of a limit to prove that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = 0.$$

Given a number $\varepsilon > 0$, we seek a number $\delta > 0$, so that

$$\text{whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \text{ then } \left| \frac{xy^2}{x^2 + y^2} - 0 \right| < \varepsilon.$$

That is,

$$\text{whenever } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{xy^2}{x^2 + y^2} \right| < \varepsilon.$$

We need to find a connection between $\frac{xy^2}{x^2 + y^2}$ and $\sqrt{x^2 + y^2}$. We begin with the observation that since $x^2 \geq 0$,

$$y^2 \leq x^2 + y^2.$$

For all points (x, y) not equal to $(0, 0)$, the quantity $x^2 + y^2$ is greater than zero, so we can divide both sides of the inequality by $x^2 + y^2$. We obtain

$$\frac{y^2}{x^2 + y^2} \leq 1.$$

Now,

$$\left| \frac{xy^2}{x^2 + y^2} \right| = \frac{|x|y^2}{x^2 + y^2} = |x| \cdot \frac{y^2}{x^2 + y^2} \leq |x| \cdot 1 = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$$

Given $\varepsilon > 0$, we choose $\delta \leq \varepsilon$. Then, whenever $0 < \sqrt{x^2 + y^2} < \delta$, we have

$$\left| \frac{xy^2}{x^2 + y^2} \right| \leq \sqrt{x^2 + y^2} < \delta \leq \varepsilon.$$

That is,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = 0.$$

Because this limit exists, we conclude that yes, it is possible to define $f(0, 0)$ so that f would be continuous at $(0, 0)$.

- (c) Since $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = 0$, we define $f(0,0) = 0$ to have $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = f(0,0)$, making f continuous at $(0,0)$.

66. (a) f is not continuous at $(0,0)$ since this point gives 0 in the denominator.
 (b) We need to determine whether or not the following limit exists:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2}.$$

We have already done this in problem 63, but we review the steps here. We will use the ε - δ definition of a limit to prove that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

Given a number $\varepsilon > 0$, we seek a number $\delta > 0$, so that

$$\text{whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta \text{ then } \left| \frac{x^2y}{x^2+y^2} - 0 \right| < \varepsilon.$$

That is,

$$\text{whenever } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left| \frac{x^2y}{x^2+y^2} \right| < \varepsilon.$$

We need to find a connection between $\frac{x^2y}{x^2+y^2}$ and $\sqrt{x^2+y^2}$. We begin with the observation that since $y^2 \geq 0$,

$$x^2 \leq x^2 + y^2.$$

For all points (x, y) not equal to $(0, 0)$, the quantity $x^2 + y^2$ is greater than zero, so we can divide both sides of the inequality by $x^2 + y^2$. We obtain

$$\frac{x^2}{x^2 + y^2} \leq 1.$$

Now,

$$\left| \frac{x^2y}{x^2+y^2} \right| = \frac{x^2|y|}{x^2+y^2} = |y| \cdot \frac{x^2}{x^2+y^2} \leq |y| \cdot 1 = \sqrt{y^2} \leq \sqrt{x^2+y^2}$$

Given $\varepsilon > 0$, we choose $\delta \leq \varepsilon$. Then, whenever $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\left| \frac{x^2y}{x^2+y^2} \right| \leq \sqrt{x^2+y^2} < \delta \leq \varepsilon.$$

That is,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

Because this limit exists, we conclude that yes, it is possible to define $f(0,0)$ so that f would be continuous at $(0,0)$.

- (c) Since $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2} = 0$, we define $f(0,0) = 0$ to have $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^2+y^2} = f(0,0)$, making f continuous at $(0,0)$.

67. (a) f is not continuous at $(0,0)$ since this point gives 0 in the denominator.

- (b) In order for us to be able to define $f(0,0)$ so that f is continuous at $(0,0)$, we need $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ to exist. We first try the limit along two different curves that contain $(0,0)$ to see if we can show that the limit does *not* exist:

Along $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + y^2}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{2(0^2) + y^2}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{y^2}{y^2} = \lim_{y \rightarrow 0} 1 = 1$$

Along $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + y^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{2x^2 + 0^2}{x^2 + 0^2} = \lim_{x \rightarrow 0} \frac{2x^2}{x^2} = \lim_{x \rightarrow 0} 2 = 2$$

Since we have differing limits along two different curves containing $(0,0)$, the limit does not exist. Therefore, it is not possible to define $f(0,0)$ so that f is continuous at $(0,0)$.

68. (a) f is not continuous at $(0,0)$ since this point gives 0 in the denominator.
 (b) In order for us to be able to define $f(0,0)$ so that f is continuous at $(0,0)$, we need $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ to exist. We first try the limit along two different curves that contain $(0,0)$ to see if we can show that the limit does *not* exist:

Along $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^2}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{0^4 - y^2}{0^2 + y^2} = \lim_{y \rightarrow 0} -\frac{y^2}{y^2} = \lim_{y \rightarrow 0} -1 = -1$$

Along $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^4 - 0^2}{x^2 + 0^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^2} = \lim_{x \rightarrow 0} x^2 = 0$$

Since we have differing limits along two different curves containing $(0,0)$, the limit does not exist. Therefore, it is not possible to define $f(0,0)$ so that f is continuous at $(0,0)$.

69. (a) To determine if f is continuous at $(0,0)$, we must check to see if $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0)$. We are given that $f(0,0) = 0$. Therefore, for f to be continuous at $(0,0)$, we would have to have that $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$. This would mean that the limit must be 0 along every curve containing $(0,0)$. But, we see that along the curve given by $y = x$, we have the following (note that since we are *approaching* $(0,0)$ in the following computation and we are not *at* $(0,0)$, we use the expression $\frac{3xy}{x^2 + y^2}$ for f):

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{3x \cdot x}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{3x^2}{2x^2} = \lim_{x \rightarrow 0} \frac{3}{2} = \frac{3}{2}$$

Since this limit is not 0, the limit of f as (x, y) approaches $(0, 0)$, even if it does exist, cannot be 0, and so this limit, even if it does exist, is not equal to $f(0, 0)$. This means that

f is not continuous at $(0, 0)$.

- (b) In order for us to be able to redefine $f(0, 0)$ so that f is continuous at $(0, 0)$, we need the limit $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ to exist. We already know that this limit along the curve $y = x$ is $\frac{3}{2}$. However, we also have the following limit along the curve $x = 0$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{3xy}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{3(0)y}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{x \rightarrow 0} 0 = 0$$

This is not equal to the limit along $y = x$, so the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ does not exist. Therefore, it is not possible to redefine $f(0, 0)$ so that f is continuous at $(0, 0)$.

70. (a) To determine if f is continuous at $(0, 0)$, we must check to see if $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$. We are given that $f(0, 0) = 1$. Therefore, for f to be continuous at $(0, 0)$, we would have to have that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1$. This would mean that the limit must be 1 along every curve containing $(0, 0)$. But, we see that along the curve given by $x = 0$, we have the following (note that since we are *approaching* $(0, 0)$ in the following computation and we are not *at* $(0, 0)$, we use the expression $\frac{\sin(xy)}{x^2 + y^2}$ for f):

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{\sin((0)y)}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{\sin 0}{y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{x \rightarrow 0} 0 = 0$$

Since this limit is not 1, the limit of f as (x, y) approaches $(0, 0)$, even if it does exist, cannot be 1, and so this limit, even if it does exist, is not equal to $f(0, 0)$. This means that

f is not continuous at $(0, 0)$.

- (b) In order for us to be able to redefine $f(0, 0)$ so that f is continuous at $(0, 0)$, we need the limit $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ to exist. We already know that this limit along the curve $x = 0$ is 0. However, we also have the following limit along the curve $y = x$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(x(x))}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = \frac{1}{2}(1) = \frac{1}{2}$$

This is not equal to the limit along $x = 0$, so the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ does not exist. Therefore, it is not possible to redefine $f(0, 0)$ so that f is continuous at $(0, 0)$.

71. (a) To determine if f is continuous at $(0, 0)$, we must check to see if $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$.

We are given that $f(0, 0) = 1$. Therefore, for f to be continuous at $(0, 0)$, we must have that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1$. By Problem 64, we know that this limit is indeed equal to 1. Therefore,

f is continuous at $(0, 0)$.

72. (a) To determine if f is continuous at $(0, 0)$, we must check to see if $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$. We are given that $f(0, 0) = 1$. Therefore, for f to be continuous at $(0, 0)$, we would have to have that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1$. This would mean that the limit must be 1 along every curve containing

$(0,0)$. But, we see that along the curve given by $x = 0$, we have the following (note that since we are *approaching* $(0,0)$ in the following computation and we are not *at* $(0,0)$, we use the expression $\frac{\sin(x^2 - y^2)}{x^2 + y^2}$ for f):

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 - y^2)}{x^2 + y^2} &= \lim_{y \rightarrow 0} \frac{\sin(0^2 - y^2)}{0^2 + y^2} = \lim_{y \rightarrow 0} \frac{\sin(-y^2)}{y^2} = \lim_{y \rightarrow 0} \frac{-\sin(y^2)}{y^2} = - \lim_{y \rightarrow 0} \frac{\sin(y^2)}{y^2} \\ &= -1 \end{aligned}$$

Since this limit is not 1, the limit of f as (x, y) approaches $(0,0)$, even if it does exist, cannot be 1, and it follows that it is not equal to $f(0,0)$. This means that f is not continuous at $(0,0)$.

- (b) In order for us to be able to redefine $f(0,0)$ so that f is continuous at $(0,0)$, we need the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ to exist. We already know that this limit along the curve $x = 0$ is -1 . However, we also have the following limit along the curve $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 - y^2)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(x^2 - 0^2)}{x^2 + 0^2} = \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = 1$$

This is not equal to the limit along $x = 0$, so the limit of $f(x, y)$ as (x, y) approaches $(0,0)$ does not exist. Therefore, it is not possible to redefine $f(0,0)$ so that f is continuous at $(0,0)$.

Challenge Problems

73. We convert using the formulas $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$. Also, note that if $(x, y) \rightarrow (0, 0)$, this implies that r^2 also goes to 0, as does r . Also, θ may take any value since sending r to 0 automatically takes us toward the origin, and because the function $\frac{(r \cos \theta)^2 (r \sin \theta)}{r^2}$ is defined for any θ . Therefore, we need consider only the limit as $r \rightarrow 0$.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} &= \lim_{(r,\theta) \rightarrow (0,0)} \frac{(r \cos \theta)^2 (r \sin \theta)}{r^2} = \lim_{(r,\theta) \rightarrow (0,0)} \frac{r^3 \cos^2 \theta \sin \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^2 \theta \sin \theta \\ &= 0 \text{ for any value of } \theta. \end{aligned}$$

Therefore, the limit is $\boxed{0}$.

74. We convert using the formulas $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$. Also, note that if $(x, y) \rightarrow (0, 0)$, this implies that r^2 also goes to 0, as does r . Also, θ may take any value since sending r to 0 automatically takes us toward the origin, and because the function $\frac{(r \cos \theta)(r \sin \theta)^2}{r^2}$ is defined for any θ . Therefore, we need consider only the limit as $r \rightarrow 0$.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} &= \lim_{r \rightarrow 0} \frac{(r \cos \theta)(r \sin \theta)^2}{r^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos \theta \sin^2 \theta \\ &= 0 \text{ for any value of } \theta. \end{aligned}$$

Therefore, the limit is $\boxed{0}$.

75. We convert using the formulas $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$. Also, note that if $(x, y) \rightarrow (0, 0)$, this implies that r^2 also goes to 0, as does r . Also, θ may take any value since sending r to 0 automatically takes us toward the origin, and because the function $\frac{\cos(r^2)}{r^2}$ is defined for any θ . Therefore, we need consider only the limit as $r \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\cos(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\cos(r^2)}{r^2}.$$

We see that (whether the value 0 is approached from the left or right) while the numerator goes to 1, the denominator becomes infinitesimally small. This causes the entire quantity to approach ∞ , i.e. the limit does not exist.

76. We convert using the formulas $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$. Also, note that if $(x, y) \rightarrow (0, 0)$, this implies that r^2 also goes to 0, as does r . Also, θ may take any value since sending r to 0 automatically takes us toward the origin, and because the function $\frac{\sin(r^2)}{r^2}$ is defined for any θ . Therefore, we need consider only the limit as $r \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\sin(r^2)}{r^2} = \boxed{1}.$$

77. Since $x - 2y$ is a factor of the numerator, we can use algebra to solve this problem.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - 4x^2y + 4xy^2 + 5x - 10y}{x - 2y} &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - 2x^2y - 2x^2y + 4xy^2 + 5x - 10y}{x - 2y} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2(x - 2y) - 2xy(x - 2y) + 5(x - 2y)}{x - 2y} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - 2xy + 5)(x - 2y)}{x - 2y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x^2 - 2xy + 5 \\ &= \lim_{(x,y) \rightarrow (0,0)} x^2 - 2 \cdot \lim_{x \rightarrow 0} x \cdot \lim_{y \rightarrow 0} y + \lim_{(x,y) \rightarrow (0,0)} 5 \\ &= 0 - 2 \cdot 0 \cdot 0 + 5 = \boxed{5} \end{aligned}$$

Chapter 12: Functions of Several Variables

12.3: Partial Derivatives

Concepts and Vocabulary

- False. It is the other way around; to find the partial derivative $f_y(x, y)$, we need to differentiate f with respect to y while treating x as if it were a constant.
- b, $y = y_0$. See the “Slope of a Tangent Line” theorem, on page 831. Alternatively, since calculating $f_x(x, y)$ involves treating y as if it were a constant, $f_x(x_0, y_0)$ should be the slope of the tangent line to a curve located in a plane of the form $y = \text{constant}$.
- True. This is because of the interpretation of $f_x(x, y)$ as a rate of change, as discussed on page 832.
- mixed. The second-order partial derivatives that are computed by taking the derivative with respect to x first, and then with respect to y , or vice versa, are called the mixed partials.

5. False. This can be checked by computing the partial derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x} (x \cos y) = \cos y \frac{\partial}{\partial x} (x) = \cos y \cdot 1 = \cos y;$$

$$f_y(x, y) = \frac{\partial}{\partial y} (x \cos y) = x \frac{\partial}{\partial y} (\cos y) = x(-\sin y) = -x \sin y.$$

The two are not equal.

6. treat the variables x and z as constants, and differentiate f with respect to y . When we compute partial derivatives, all variables, except for the one that we are computing the partial derivative with respect to, are treated as constants. Since $f_y(x, y, z)$ is the partial derivative with respect to y , x and z must be treated as constants.

Skill Building

7.

$$f_x(x, y) = \frac{\partial}{\partial x} (x^2y + 6y^2) = \frac{\partial}{\partial x} (x^2y) + \frac{\partial}{\partial x} (6y^2) = y \frac{\partial}{\partial x} (x^2) + 0 = y \cdot 2x$$

$$\boxed{f_x(x, y) = 2xy}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} (x^2y + 6y^2) = \frac{\partial}{\partial y} (x^2y) + \frac{\partial}{\partial y} (6y^2) \\ &= x^2 \frac{\partial}{\partial y} (y) + 6 \frac{\partial}{\partial y} (y^2) = x^2 \cdot 1 + 6 \cdot 2y = x^2 + 12y \end{aligned}$$

$$\boxed{f_y(x, y) = x^2 + 12y}$$

8.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (3x^2 + 6xy^3) = \frac{\partial}{\partial x} (3x^2) + \frac{\partial}{\partial x} (6xy^3) \\ &= 3 \frac{\partial}{\partial x} (x^2) + 6y^3 \frac{\partial}{\partial x} (x) = 3 \cdot 2x + 6y^3 \cdot 1 = 6x + 6y^3 \end{aligned}$$

$$\boxed{f_x(x, y) = 6x + 6y^3}$$

$$f_y(x, y) = \frac{\partial}{\partial y} (3x^2 + 6xy^3) = \frac{\partial}{\partial y} (3x^2) + \frac{\partial}{\partial y} (6xy^3) = 0 + 6x \frac{\partial}{\partial y} (y^3) = 6x \cdot 3y^2 = 18xy^2$$

$$\boxed{f_y(x, y) = 18xy^2}$$

9.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left(\frac{x-y}{x+y} \right) = \frac{\left(\frac{\partial}{\partial x} (x-y) \right) (x+y) - (x-y) \left(\frac{\partial}{\partial x} (x+y) \right)}{(x+y)^2} \\ &= \frac{(x+y) \cdot 1 - (x-y) \cdot 1}{(x+y)^2} = \frac{2y}{(x+y)^2} \end{aligned}$$

$$f_x(x, y) = \frac{2y}{(x+y)^2}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left(\frac{x-y}{x+y} \right) = \frac{\left(\frac{\partial}{\partial y} (x-y) \right) (x+y) - (x-y) \left(\frac{\partial}{\partial y} (x+y) \right)}{(x+y)^2} \\ &= \frac{(x+y) \cdot (-1) - (x-y) \cdot 1}{(x+y)^2} = -\frac{2x}{(x+y)^2} \end{aligned}$$

$$f_y(x, y) = -\frac{2x}{(x+y)^2}$$

10.

$$f_x(x, y) = \frac{\partial}{\partial x} \left(\frac{x+y}{y^2} \right) = \frac{1}{y^2} \frac{\partial}{\partial x} (x+y) = \frac{1}{y^2} \cdot 1 = \frac{1}{y^2}$$

$$f_x(x, y) = \frac{1}{y^2}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left(\frac{x+y}{y^2} \right) = \frac{y^2 \left(\frac{\partial}{\partial y} (x+y) \right) - \left(\frac{\partial}{\partial y} (y^2) \right) (x+y)}{(y^2)^2} \\ &= \frac{y^2 \cdot 1 - (2y) \cdot (x+y)}{y^4} = \frac{y^2 - 2xy - 2y^2}{y^4} = \frac{-2xy - y^2}{y^4} = \frac{-y(2x+y)}{y^4} = -\frac{2x+y}{y^3} \end{aligned}$$

$$f_y(x, y) = -\frac{2x+y}{y^3}$$

11.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (e^y \cos x + e^x \sin y) = \frac{\partial}{\partial x} (e^y \cos x) + \frac{\partial}{\partial x} (e^x \sin y) \\ &= e^y \frac{\partial}{\partial x} (\cos x) + \sin y \frac{\partial}{\partial x} (e^x) = e^y (-\sin x) + \sin y \cdot e^x = -e^y \sin x + e^x \sin y \end{aligned}$$

$$f_x(x, y) = -e^y \sin x + e^x \sin y$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} (e^y \cos x + e^x \sin y) = \frac{\partial}{\partial y} (e^y \cos x) + \frac{\partial}{\partial y} (e^x \sin y) \\ &= \cos x \frac{\partial}{\partial y} (e^y) + e^x \frac{\partial}{\partial y} (\sin y) = \cos x \cdot e^y + e^x \cdot \cos y = e^y \cos x + e^x \cos y \end{aligned}$$

$$f_y(x, y) = e^y \cos x + e^x \cos y$$

12.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (x^2 \cos y + y^2 \sin x) = \frac{\partial}{\partial x} (x^2 \cos y) + \frac{\partial}{\partial x} (y^2 \sin x) \\ &= \cos y \frac{\partial}{\partial x} (x^2) + y^2 \frac{\partial}{\partial x} (\sin x) = \cos y \cdot 2x + y^2 \cdot \cos x = 2x \cos y + y^2 \cos x \end{aligned}$$

$$\boxed{f_x(x, y) = 2x \cos y + y^2 \cos x}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} (x^2 \cos y + y^2 \sin x) = \frac{\partial}{\partial y} (x^2 \cos y) + \frac{\partial}{\partial y} (y^2 \sin x) \\ &= x^2 \frac{\partial}{\partial y} (\cos y) + \sin x \frac{\partial}{\partial y} (y^2) = x^2 (-\sin y) + \sin x \cdot 2y = -x^2 \sin y + 2y \sin x \end{aligned}$$

$$\boxed{f_y(x, y) = -x^2 \sin y + 2y \sin x}$$

13.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (x^2 e^{xy}) = \frac{\partial}{\partial x} (x^2) \cdot e^{xy} + x^2 \cdot \frac{\partial}{\partial x} (e^{xy}) \\ &= 2x \cdot e^{xy} + x^2 e^{xy} \cdot \frac{\partial}{\partial x} (xy) = 2x \cdot e^{xy} + x^2 e^{xy} \cdot y = 2x e^{xy} + x^2 y e^{xy} \end{aligned}$$

$$\boxed{f_x(x, y) = 2x e^{xy} + x^2 y e^{xy}}$$

$$f_y(x, y) = \frac{\partial}{\partial y} (x^2 e^{xy}) = x^2 \frac{\partial}{\partial y} (e^{xy}) = x^2 e^{xy} \frac{\partial}{\partial y} (xy) = x^2 e^{xy} \cdot x = x^3 e^{xy}$$

$$\boxed{f_y(x, y) = x^3 e^{xy}}$$

14.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (\cos(x^2 y^3)) = -\sin(x^2 y^3) \cdot \frac{\partial}{\partial x} (x^2 y^3) \\ &= -\sin(x^2 y^3) \cdot y^3 \cdot \frac{\partial}{\partial x} (x^2) = -\sin(x^2 y^3) \cdot y^3 \cdot 2x = -2xy^3 \sin(x^2 y^3) \end{aligned}$$

$$\boxed{f_x(x, y) = -2xy^3 \sin(x^2 y^3)}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} (\cos(x^2 y^3)) = -\sin(x^2 y^3) \cdot \frac{\partial}{\partial y} (x^2 y^3) \\ &= -\sin(x^2 y^3) \cdot x^2 \cdot \frac{\partial}{\partial y} (y^3) = -\sin(x^2 y^3) \cdot x^2 \cdot 3y^2 = -3x^2 y^2 \sin(x^2 y^3) \end{aligned}$$

$$\boxed{f_y(x, y) = -3x^2 y^2 \sin(x^2 y^3)}$$

15.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) \\ &= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot \frac{\partial}{\partial x} (x^{-1}) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot (-x^{-2}) = \frac{x^2}{x^2 + y^2} \cdot \frac{-y}{x^2} = -\frac{y}{x^2 + y^2} \end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2}}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) \\
&= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot \frac{\partial}{\partial y} (y) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot 1 = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2}}$$

16.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (\sin^2(2xy)) = 2 \sin(2xy) \cdot \frac{\partial}{\partial x} (\sin(2xy)) = 2 \sin(2xy) \cdot \cos(2xy) \cdot \frac{\partial}{\partial x} (2xy) \\
&= 2 \sin(2xy) \cdot \cos(2xy) \cdot 2y \cdot \frac{\partial}{\partial x} (x) = 2 \sin(2xy) \cdot \cos(2xy) \cdot 2y \cdot 1 = 4y \sin(2xy) \cos(2xy)
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = 4y \sin(2xy) \cos(2xy)}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (\sin^2(2xy)) = 2 \sin(2xy) \cdot \frac{\partial}{\partial y} (\sin(2xy)) = 2 \sin(2xy) \cdot \cos(2xy) \cdot \frac{\partial}{\partial y} (2xy) \\
&= 2 \sin(2xy) \cdot \cos(2xy) \cdot 2x \cdot \frac{\partial}{\partial y} (y) = 2 \sin(2xy) \cdot \cos(2xy) \cdot 2x \cdot 1 = 4x \sin(2xy) \cos(2xy)
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = 4x \sin(2xy) \cos(2xy)}$$

17.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (\sin(e^{x^2 y})) = \cos(e^{x^2 y}) \cdot \frac{\partial}{\partial x} (e^{x^2 y}) = \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot \frac{\partial}{\partial x} (x^2 y) \\
&= \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot y \cdot \frac{\partial}{\partial x} (x^2) = \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot y \cdot 2x = 2xy e^{x^2 y} \cos(e^{x^2 y})
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = 2xy e^{x^2 y} \cos(e^{x^2 y})}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (\sin(e^{x^2 y})) = \cos(e^{x^2 y}) \cdot \frac{\partial}{\partial y} (e^{x^2 y}) = \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot \frac{\partial}{\partial y} (x^2 y) \\
&= \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot x^2 \cdot \frac{\partial}{\partial y} (y) = \cos(e^{x^2 y}) \cdot e^{x^2 y} \cdot x^2 \cdot 1 = x^2 e^{x^2 y} \cos(e^{x^2 y})
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = x^2 e^{x^2 y} \cos(e^{x^2 y})}$$

18.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (\sin[\ln(x^2 + y^2)]) = \cos[\ln(x^2 + y^2)] \cdot \frac{\partial}{\partial x} (\ln(x^2 + y^2)) \\
&= \cos[\ln(x^2 + y^2)] \cdot \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) = \cos[\ln(x^2 + y^2)] \cdot \frac{1}{x^2 + y^2} \cdot (2x + 0) \\
&= \cos[\ln(x^2 + y^2)] \cdot \frac{2x}{x^2 + y^2}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = \cos[\ln(x^2 + y^2)] \cdot \frac{2x}{x^2 + y^2}}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (\sin[\ln(x^2 + y^2)]) = \cos[\ln(x^2 + y^2)] \cdot \frac{\partial}{\partial y} (\ln(x^2 + y^2)) \\ &= \cos[\ln(x^2 + y^2)] \cdot \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) = \cos[\ln(x^2 + y^2)] \cdot \frac{1}{x^2 + y^2} \cdot (0 + 2y) \\ &= \cos[\ln(x^2 + y^2)] \cdot \frac{2y}{x^2 + y^2}\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = \cos[\ln(x^2 + y^2)] \cdot \frac{2y}{x^2 + y^2}}$$

19.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (e^{(x^2+y^2)^{1.2}}) = e^{(x^2+y^2)^{1.2}} \cdot \frac{\partial}{\partial x} ((x^2 + y^2)^{1.2}) \\ &= e^{(x^2+y^2)^{1.2}} \cdot 1.2 \cdot (x^2 + y^2)^{0.2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) = e^{(x^2+y^2)^{1.2}} \cdot 1.2 \cdot (x^2 + y^2)^{0.2} \cdot (2x + 0) \\ &= 2.4x(x^2 + y^2)^{0.2} e^{(x^2+y^2)^{1.2}}\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = 2.4x(x^2 + y^2)^{0.2} e^{(x^2+y^2)^{1.2}}}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (e^{(x^2+y^2)^{1.2}}) = e^{(x^2+y^2)^{1.2}} \cdot \frac{\partial}{\partial y} ((x^2 + y^2)^{1.2}) \\ &= e^{(x^2+y^2)^{1.2}} \cdot 1.2 \cdot (x^2 + y^2)^{0.2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) = e^{(x^2+y^2)^{1.2}} \cdot 1.2 \cdot (x^2 + y^2)^{0.2} \cdot (0 + 2y) \\ &= 2.4y(x^2 + y^2)^{0.2} e^{(x^2+y^2)^{1.2}}\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = 2.4y(x^2 + y^2)^{0.2} e^{(x^2+y^2)^{1.2}}}$$

20.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (\ln \sqrt{x^2 + y^2}) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} (\sqrt{x^2 + y^2}) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} ((x^2 + y^2)^{1/2}) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot (2x + 0) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{x^2 + y^2}\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{x}{x^2 + y^2}}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \left(\ln \sqrt{x^2 + y^2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y} \left(\sqrt{x^2 + y^2} \right) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y} \left((x^2 + y^2)^{1/2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \frac{\partial}{\partial y} (x^2 + y^2) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot (0 + 2y) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + y^2}} \cdot 2y = \frac{y}{x^2 + y^2}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{y}{x^2 + y^2}}$$

21. We have the following first- and second-order partial derivatives for the function $f(x, y) = 6x^2 - 8xy + 9y^2$:

$$f_x = 12x - 8y$$

$$f_y = -8x + 18y$$

$$\boxed{f_{xx} = 12}$$

$$\boxed{f_{xy} = -8}$$

$$\boxed{f_{yx} = -8}$$

$$\boxed{f_{yy} = 18}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}$.

22. We have the following first- and second-order partial derivatives for the function $f(x, y) = (2x + 3y)(3x - 2y)$, which we obtain by first expanding our expression for $f(x, y)$ to $f(x, y) = 6x^2 + 5xy - 6y^2$:

$$f_x = 12x + 5y$$

$$f_y = 5x - 12y$$

$$\boxed{f_{xx} = 12}$$

$$\boxed{f_{xy} = 5}$$

$$\boxed{f_{yx} = 5}$$

$$\boxed{f_{yy} = -12}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}$.

23. We have the following first- and second-order partial derivatives for the function $f(x, y) = \ln(x^3 + y^2)$:

$$f_x = \frac{1}{x^3 + y^2} (3x^2) = \frac{3x^2}{x^3 + y^2}$$

$$f_{xx} = \frac{6x(x^3 + y^2) - 3x^2(3x^2)}{(x^3 + y^2)^2} = \frac{-3x^4 + 6xy^2}{(x^3 + y^2)^2} = \frac{-3x(x^3 - 2y^2)}{(x^3 + y^2)^2}$$

$$\boxed{f_{xx} = \frac{-3x(x^3 - 2y^2)}{(x^3 + y^2)^2}}$$

$$f_{xy} = 3x^2[(-1)(x^3 + y^2)^{-2}(2y)] = \frac{-6x^2y}{(x^3 + y^2)^2}$$

$$\boxed{f_{xy} = \frac{-6x^2y}{(x^3 + y^2)^2}}$$

$$f_y = \frac{1}{x^3 + y^2} (2y) = \frac{2y}{x^3 + y^2}$$

$$f_{yx} = 2y[(-1)(x^3 + y^2)^{-2}(3x^2)] = \frac{-6x^2y}{(x^3 + y^2)^2}$$

$$\boxed{f_{yx} = \frac{-6x^2y}{(x^3 + y^2)^2}}$$

$$f_{yy} = \frac{2(x^3 + y^2) - 2y(2y)}{(x^3 + y^2)^2} = \frac{2x^3 - 2y^2}{(x^3 + y^2)^2} = \frac{2(x^3 - y^2)}{(x^3 + y^2)^2}$$

$$\boxed{f_{yy} = \frac{2(x^3 - y^2)}{(x^3 + y^2)^2}}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}.$

24. We have the following first- and second-order partial derivatives for the function $f(x, y) = e^{2x+3y}$:

$$f_x = e^{2x+3y} \cdot 2 = 2e^{2x+3y}$$

$$f_y = e^{2x+3y} \cdot 3 = 3e^{2x+3y}$$

$$f_{xx} = 2e^{2x+3y} \cdot 2 = 4e^{2x+3y}$$

$$f_{yx} = 3e^{2x+3y} \cdot 2 = 6e^{2x+3y}$$

$$\boxed{f_{xx} = 4e^{2x+3y}}$$

$$\boxed{f_{yx} = 6e^{2x+3y}}$$

$$f_{xy} = 2e^{2x+3y} \cdot 3 = 6e^{2x+3y}$$

$$f_{yy} = 3e^{2x+3y} \cdot 3 = 9e^{2x+3y}$$

$$\boxed{f_{xy} = 6e^{2x+3y}}$$

$$\boxed{f_{yy} = 9e^{2x+3y}}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}.$

25. We have the following first- and second-order partial derivatives for the function $f(x, y) = \cos(x^2y^3)$:

$$f_x = -\sin(x^2y^3) \cdot 2xy^3 = -2xy^3 \sin(x^2y^3)$$

$$f_{xx} = (-2xy^3)[\cos(x^2y^3) \cdot 2xy^3] + [-2y^3 \sin(x^2y^3)] = -4x^2y^6 \cos(x^2y^3) - 2y^3 \sin(x^2y^3)$$

$$\boxed{f_{xx} = -4x^2y^6 \cos(x^2y^3) - 2y^3 \sin(x^2y^3)}$$

$$f_{xy} = -2xy^3[\cos(x^2y^3) \cdot x^2(3y^2)] + [-2x(3y^2) \sin(x^2y^3)] = -6x^3y^5 \cos(x^2y^3) - 6xy^2 \sin(x^2y^3)$$

$$\boxed{f_{xy} = -6x^3y^5 \cos(x^2y^3) - 6xy^2 \sin(x^2y^3)}$$

$$f_y = -\sin(x^2y^3) \cdot x^2(3y^2) = -3x^2y^2 \sin(x^2y^3)$$

$$f_{yx} = -3x^2y^2[\cos(x^2y^3) \cdot 2xy^3] + [-6xy^2 \sin(x^2y^3)] = -6x^3y^5 \cos(x^2y^3) - 6xy^2 \sin(x^2y^3)$$

$$\boxed{f_{yx} = -6x^3y^5 \cos(x^2y^3) - 6xy^2 \sin(x^2y^3)}$$

$$f_{yy} = -3x^2y^2[\cos(x^2y^3) \cdot x^2(3y^2)] + [-3x^2(2y)\sin(x^2y^3)] = -9x^4y^4\cos(x^2y^3) - 6x^2y\sin(x^2y^3)$$

$$\boxed{f_{yy} = -9x^4y^4\cos(x^2y^3) - 6x^2y\sin(x^2y^3)}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}$.

26. We have the following first- and second-order partial derivatives for the function $f(x, y) = \sin^2(xy)$:

$$f_x = 2\sin(xy)\cos(xy) \cdot y = 2y\sin(xy)\cos(xy)$$

$$f_{xx} = 2y\sin(xy)[- \sin(xy) \cdot y] + [2y\cos(xy) \cdot y]\cos(xy) = -2y^2\sin^2(xy) + 2y^2\cos^2(xy)$$

$$= 2y^2[\cos^2(xy) - \sin^2(xy)] = 2y^2\cos(2xy)$$

$$\boxed{f_{xx} = 2y^2\cos(2xy)}$$

$$f_{xy} = 2y\sin(xy)[- \sin(xy) \cdot x] + [2y\cos(xy) \cdot x + 2\sin(xy)]\cos(xy)$$

$$= -2xy\sin^2(xy) + 2xy\cos^2(xy) + 2\sin(xy)\cos(xy)$$

$$= 2xy[\cos^2(xy) - \sin^2(xy)] + 2\sin(xy)\cos(xy)$$

$$= 2xy\cos(2xy) + \sin(2xy)$$

$$\boxed{f_{xy} = 2xy\cos(2xy) + \sin(2xy)}$$

$$f_y = 2\sin(xy)\cos(xy) \cdot x = 2x\sin(xy)\cos(xy)$$

$$f_{yx} = 2x\sin(xy)[- \sin(xy) \cdot y] + [2x\cos(xy) \cdot y + 2\sin(xy)]\cos(xy)$$

$$= -2xy\sin^2(xy) + 2xy\cos^2(xy) + 2\sin(xy)\cos(xy)$$

$$= 2xy[\cos^2(xy) - \sin^2(xy)] + 2\sin(xy)\cos(xy)$$

$$= 2xy\cos(2xy) + \sin(2xy)$$

$$\boxed{f_{yx} = 2xy\cos(2xy) + \sin(2xy)}$$

$$f_{yy} = 2x\sin(xy)[- \sin(xy) \cdot x] + [2x\cos(xy) \cdot x]\cos(xy) = -2x^2\sin^2(xy) + 2x^2\cos^2(xy)$$

$$= 2x^2[\cos^2(xy) - \sin^2(xy)] = 2x^2\cos(2xy)$$

$$\boxed{f_{yy} = 2x^2\cos(2xy)}$$

We see that $\boxed{\text{it is indeed the case that } f_{xy} = f_{yx}}$.

27. We have the following first-order partial derivatives for the function $f(x, y, z) = xy + yz + xz$:

$$\boxed{f_x = y + z}$$

$$\boxed{f_y = x + z}$$

$$f_z = y + x = x + y$$

$$\boxed{f_z = x + y}$$

28. We have the following first-order partial derivatives for the function $f(x, y, z) = xe^y + ye^z + ze^x$:

$$\boxed{f_x = e^y + ze^x}$$

$$\boxed{f_y = xe^y + e^z}$$

$$\boxed{f_z = ye^z + e^x}$$

29. We have the following first-order partial derivatives for the function $f(x, y, z) = xy \sin z - yz \sin x$:

$$\boxed{f_x = y \sin z - yz \cos x}$$

$$\boxed{f_y = x \sin z - z \sin x}$$

$$\boxed{f_z = xy \cos z - y \sin x}$$

30. We have the following first-order partial derivatives for the function $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$:

$$f_x = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\boxed{f_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}}$$

$$f_y = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y) = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\boxed{f_y = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}}$$

$$f_z = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z) = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\boxed{f_z = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}}$$

31. We have the following first-order partial derivatives for the function $f(x, y, z) = z \tan^{-1} \left(\frac{y}{x} \right)$:

$$f_x = z \left[\frac{1}{1 + \left(\frac{y}{x} \right)^2} \right] (-yx^{-2}) = \frac{-yz}{x^2 \left[1 + \frac{y^2}{x^2} \right]} = \frac{-yz}{x^2 + y^2}$$

$$\boxed{f_x = \frac{-yz}{x^2 + y^2}}$$

$$f_y = z \left[\frac{1}{1 + \left(\frac{y}{x} \right)^2} \right] \left(\frac{1}{x} \right) = \frac{z}{x \left[1 + \frac{y^2}{x^2} \right]} = \frac{z}{x + \frac{y^2}{x}} = \frac{xz}{x^2 + y^2}$$

$$\boxed{f_y = \frac{xz}{x^2 + y^2}}$$

$$\boxed{f_z = \tan^{-1} \left(\frac{y}{x} \right)}$$

32. We have the following first-order partial derivatives for the function $f(x, y, z) = \tan^{-1} \left(\frac{xy}{z} \right)$:

$$f_x = \frac{1}{1 + \left(\frac{xy}{z} \right)^2} \left(\frac{y}{z} \right) = \frac{y}{z \left(1 + \frac{x^2 y^2}{z^2} \right)} = \frac{y}{z + \frac{x^2 y^2}{z}} = \frac{yz}{z^2 + x^2 y^2}$$

$$\boxed{f_x = \frac{yz}{z^2 + x^2 y^2}}$$

$$f_y = \frac{1}{1 + \left(\frac{xy}{z} \right)^2} \left(\frac{x}{z} \right) = \frac{x}{z \left(1 + \frac{x^2 y^2}{z^2} \right)} = \frac{x}{z + \frac{x^2 y^2}{z}} = \frac{xz}{z^2 + x^2 y^2}$$

$$\boxed{f_y = \frac{xz}{z^2 + x^2 y^2}}$$

$$f_z = \frac{1}{1 + \left(\frac{xy}{z} \right)^2} [xy(-z^{-2})] = \frac{-xy}{z^2 \left(1 + \frac{x^2 y^2}{z^2} \right)} = \frac{-xy}{z^2 + x^2 y^2}$$

$$\boxed{f_z = \frac{-xy}{z^2 + x^2 y^2}}$$

33. We have the following first-order partial derivatives for the function $f(x, y, z) = \sin[\ln(x^2 + y^2 + z^2)]$:

$$f_x = \cos[\ln(x^2 + y^2 + z^2)] \left(\frac{1}{x^2 + y^2 + z^2} \right) (2x) = \frac{2x \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}$$

$$\boxed{f_x = \frac{2x \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}}$$

$$f_y = \cos[\ln(x^2 + y^2 + z^2)] \left(\frac{1}{x^2 + y^2 + z^2} \right) (2y) = \frac{2y \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}$$

$$\boxed{f_y = \frac{2y \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}}$$

$$f_z = \cos[\ln(x^2 + y^2 + z^2)] \left(\frac{1}{x^2 + y^2 + z^2} \right) (2z) = \frac{2z \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}$$

$$f_z = \frac{2z \cos[\ln(x^2 + y^2 + z^2)]}{x^2 + y^2 + z^2}$$

34. We have the following first-order partial derivatives for the function $f(x, y, z) = e^{x^2+y^2} \ln z$:

$$f_x = e^{x^2+y^2} (2x) \ln z = 2xe^{x^2+y^2} \ln z$$

$$f_x = 2xe^{x^2+y^2} \ln z$$

$$f_y = e^{x^2+y^2} (2y) \ln z = 2ye^{x^2+y^2} \ln z$$

$$f_y = 2ye^{x^2+y^2} \ln z$$

$$f_z = e^{x^2+y^2} \left(\frac{1}{z} \right) = \frac{e^{x^2+y^2}}{z}$$

$$f_z = \frac{e^{x^2+y^2}}{z}$$

35.

$$\begin{aligned} f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^3 + 0}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^3}{(\Delta x)^2} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 1 = 1 \end{aligned}$$

$$f_x(0, 0) = 1$$

$$\begin{aligned} f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{0 + (\Delta y)^3}{0 + (\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{(\Delta y)^3}{(\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} 1 = 1 \end{aligned}$$

$$f_y(0, 0) = 1$$

36.

$$\begin{aligned} f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^2 \cdot 0}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

$$f_x(0, 0) = 0$$

$$\begin{aligned} f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{0 \cdot (\Delta y)^2}{0 + 4(\Delta y)^3} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0 \end{aligned}$$

$$f_y(0,0) = 0$$

37. The slope of the tangent line to the curve of intersection of $z = x^2 + y^2$ and $y = 2$ at any point is $f_x(x, 2) = 2x$. At the point $(1, 2, 5)$, the slope is 2. This tangent line lies on the plane $y = 2$. Symmetric equations of the tangent line are

$$z - 5 = \frac{x - 1}{1/2} \quad y = 2$$

38. The slope of the tangent line to the curve of intersection of $z = x^2 - y^2$ and $x = 3$ at any point is $f_y(3, y) = -2y$. At the point $(3, 1, 8)$, the slope is -2. This tangent line lies on the plane $x = 3$. Symmetric equations of the tangent line are

$$z - 8 = \frac{y - 1}{-1/2} \quad x = 3$$

39. The slope of the tangent line to the curve of intersection of $z = \sqrt{1 - x^2 - y^2}$ and $x = 0$ at any point can be found as follows:

$$\begin{aligned} f_y(0, y) &= \frac{1}{2}(1 - y^2)^{-1/2}(-2y) \\ &= \frac{-y}{\sqrt{1 - y^2}} \end{aligned}$$

At the point $(0, \frac{1}{2}, \frac{\sqrt{3}}{2})$, the slope is $-\frac{1}{\sqrt{3}}$. This tangent line lies on the plane $x = 0$. Symmetric equations of the tangent line are

$$z - \frac{\sqrt{3}}{2} = \frac{y - \frac{1}{2}}{-\sqrt{3}} \quad x = 0$$

40. The slope of the tangent line to the curve of intersection of $z = \sqrt{16 - x^2 - y^2}$ and $y = 2$ at any point can be found as follows:

$$f(x, 2) = \sqrt{16 - x^2 - 2^2} = \sqrt{12 - x^2}$$

so

$$f_x(x, 2) = \frac{1}{2}(12 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{12 - x^2}}$$

At the point $(\sqrt{3}, 2, 3)$, the slope is $\frac{-\sqrt{3}}{\sqrt{12 - (\sqrt{3})^2}} = \frac{-\sqrt{3}}{\sqrt{9}} = -\frac{\sqrt{3}}{3}$. This tangent line lies on the plane $y = 2$. Symmetric equations of the tangent line are

$$z - 3 = \frac{x - \sqrt{3}}{-\frac{3}{\sqrt{3}}} \quad y = 2$$

Since $\frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$, we can rewrite these as

$$\boxed{z - 3 = \frac{x - \sqrt{3}}{-\sqrt{3}} \quad y = 2}$$

41. The slope of the tangent line to the curve of intersection of $z = \sqrt{x^2 + y^2}$ and $x = 4$ at any point can be found as follows:

$$f(4, y) = \sqrt{4^2 + y^2} = \sqrt{16 + y^2}$$

so

$$f_y(4, y) = \frac{1}{2}(16 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{16 + y^2}}$$

At the point $(4, 2, 2\sqrt{5})$, the slope is $\frac{2}{\sqrt{16 + 2^2}} = \frac{2}{\sqrt{20}} = \frac{2\sqrt{20}}{\sqrt{20}\sqrt{20}} = \frac{2\sqrt{20}}{20} = \frac{\sqrt{20}}{10} = \frac{2\sqrt{5}}{10} = \frac{\sqrt{5}}{5}$. This tangent line lies on the plane $x = 4$. Symmetric equations of the tangent line are

$$z - 2\sqrt{5} = \frac{y - 2}{\frac{5}{\sqrt{5}}} \quad x = 4$$

Since $\frac{5}{\sqrt{5}} = \frac{5\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$, we can rewrite these as

$$\boxed{z - 2\sqrt{5} = \frac{y - 2}{\sqrt{5}} \quad x = 4}$$

42. The slope of the tangent line to the curve of intersection of $z = e^x \ln e$ and $y = e$ at any point is $f_x(x, e) = e^x \ln e = e^x \cdot 1 = e^x$. At the point $(0, e, 1)$, the slope is $e^0 = 1$. This tangent line lies on the plane $y = e$. Symmetric equations of the tangent line are

$$z - 1 = \frac{x}{1} \quad y = e$$

or simplified as

$$\boxed{z - 1 = x \quad y = e}$$

43. (a) The rate of change of $z = \ln \sqrt{x^2 + y^2}$ at the point $(3, 4, \ln 5)$ in the direction of the positive x -axis can be found by evaluating the first partial derivative of z with respect to x at $(3, 4, \ln 5)$. Note that we can simplify the function first by applying a rule of logarithms to obtain $z = \ln(x^2 + y^2)^{1/2} = \frac{1}{2} \ln(x^2 + y^2)$. We then have

$$\frac{\partial z}{\partial x} = \frac{1}{2} \left[\frac{1}{x^2 + y^2} \right] (2x) = \frac{x}{x^2 + y^2}$$

so at the point $(3, 4, \ln 5)$,

$$\frac{\partial z}{\partial x} = \frac{3}{9 + 16} = \frac{3}{25}.$$

So, the rate of change of $z = \ln \sqrt{x^2 + y^2}$ at the point $(3, 4, \ln 5)$ in the direction of the positive

x -axis is $\boxed{\frac{3}{25}}$.

- (b) The rate of change of $z = \ln \sqrt{x^2 + y^2}$ at the point $(3, 4, \ln 5)$ in the direction of the positive y -axis can be found by evaluating the first partial derivative of z with respect to y at $(3, 4, \ln 5)$. Note that we can simplify the function first by applying a rule of logarithms to obtain $z = \ln(x^2 + y^2)^{1/2} = \frac{1}{2} \ln(x^2 + y^2)$. We then have

$$\frac{\partial z}{\partial y} = \frac{1}{2} \left[\frac{1}{x^2 + y^2} \right] (2y) = \frac{y}{x^2 + y^2}$$

so at the point $(3, 4, \ln 5)$,

$$\frac{\partial z}{\partial y} = \frac{4}{9 + 16} = \frac{4}{25}.$$

So, the rate of change of $z = \ln \sqrt{x^2 + y^2}$ at the point $(3, 4, \ln 5)$ in the direction of the positive y -axis is $\boxed{\frac{4}{25}}$.

44. (a) The rate of change of $z = e^y \sin x$ at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$ in the direction of the positive x -axis can be found by evaluating the first partial derivative of z with respect to x at $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$. We have

$$\frac{\partial z}{\partial x} = e^y \cos x$$

so at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$,

$$\frac{\partial z}{\partial x} = (e^0) \cos\left(\frac{\pi}{3}\right) = (1) \left(\frac{1}{2}\right) = \frac{1}{2}.$$

So, the rate of change of $z = e^y \sin x$ at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$ in the direction of the positive x -axis is $\boxed{\frac{1}{2}}$.

- (b) The rate of change of $z = e^y \sin x$ at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$ in the direction of the positive y -axis can be found by evaluating the first partial derivative of z with respect to y at $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$. We have

$$\frac{\partial z}{\partial y} = e^y \sin x$$

so at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$,

$$\frac{\partial z}{\partial y} = (e^0) \sin\left(\frac{\pi}{3}\right) = (1) \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

So, the rate of change of $z = e^y \sin x$ at the point $\left(\frac{\pi}{3}, 0, \frac{\sqrt{3}}{2}\right)$ in the direction of the positive y -axis is $\boxed{\frac{\sqrt{3}}{2}}$.

Applications and Extensions

45. (a) On the circle $x^2 + y^2 = 1$, we have that $T(x, y) = \left(\frac{100}{\ln 2}\right) \ln 1 = \left(\frac{100}{\ln 2}\right) (0) = 0^\circ\text{C}$. On the circle $x^2 + y^2 = 4$, we have that $T(x, y) = \left(\frac{100}{\ln 2}\right) \ln 4 = \left(\frac{100}{\ln 2}\right) \ln 2^2 = \left(\frac{100}{\ln 2}\right) 2 \ln 2 = 200^\circ\text{C}$.

- (b) We can find the rate of change of T in the direction of the positive x -axis at (1,2) and (2,1) by finding the first partial derivative of T with respect to x and then evaluating at (1,2) and (2,1) respectively. We have

$$T_x = \left(\frac{100}{\ln 2}\right) \left(\frac{1}{x^2 + y^2}\right) (2x) = \left(\frac{200}{\ln 2}\right) \left(\frac{x}{x^2 + y^2}\right)$$

Then, $T_x(1, 2) = \left(\frac{200}{\ln 2}\right) \left(\frac{1}{1^2 + 2^2}\right) = \frac{200}{5 \ln 2} = \frac{40}{\ln 2}$, or just $\boxed{T_x(1, 2) = \frac{40}{\ln 2} \approx 57.708}$. For every unit increase in the direction of the positive x -axis from the point (1, 2), there is an increase in the temperature of the plate of approximately 58°C . We also have $T_x(2, 1) = \left(\frac{200}{\ln 2}\right) \left(\frac{2}{2^2 + 1^2}\right) = \frac{400}{5 \ln 2} = \frac{80}{\ln 2}$, or just $\boxed{T_x(2, 1) = \frac{80}{\ln 2} \approx 115.416}$. For every unit increase in the direction of the positive x -axis from the point (2,1), there is an increase in the temperature of the plate of approximately 115°C .

- (c) We can find the rate of change of T in the direction of the positive y -axis at (2,0) and (0,2) by finding the first partial derivative of T with respect to y and then evaluating at (2,0) and (0,2) respectively. We have

$$T_y = \left(\frac{100}{\ln 2}\right) \left(\frac{1}{x^2 + y^2}\right) (2y) = \left(\frac{200}{\ln 2}\right) \left(\frac{y}{x^2 + y^2}\right)$$

Then, $T_y(2, 0) = \left(\frac{200}{\ln 2}\right) \left(\frac{0}{2^2 + 0^2}\right) = 0$, or just $\boxed{T_y(2, 0) = 0}$. Since the rate of change is 0, the temperature is neither increasing nor decreasing for any increase (or decrease) in the direction of the positive y -axis from the point (2,0). We also have $T_y(0, 2) = \left(\frac{200}{\ln 2}\right) \left(\frac{2}{0^2 + 2^2}\right) = \frac{200}{\ln 2} \left(\frac{1}{2}\right) = \frac{100}{\ln 2}$, or just $\boxed{T_y(0, 2) = \frac{100}{\ln 2} \approx 144.27}$. For every unit increase in the direction of the positive y -axis from the point (0,2), there is an increase in the temperature of the plate of approximately 144°C .

46. (a) On the circle $x^2 + y^2 = 1$, we have that $T(x, y) = \frac{100}{\sqrt{1}} = 100^\circ\text{C}$. On the circle $x^2 + y^2 = 4$, we have that $T(x, y) = \frac{100}{\sqrt{4}} = \frac{100}{2} = 50^\circ\text{C}$.

- (b) We can find the rate of change of T in the direction of the positive x -axis at $(1,0)$ and $(0,1)$ by finding the first partial derivative of T with respect to x and then evaluating at $(1,0)$ and $(0,1)$ respectively. We have

$$T_x = 100(-\frac{1}{2})(x^2 + y^2)^{-3/2}(2x) = \frac{-100x}{(x^2 + y^2)^{3/2}}$$

Then, $T_x(1,0) = \frac{-100(1)}{(1^2 + 0^2)^{3/2}} = -100$, or just $\boxed{T_x(1,0) = -100}$. For every unit increase in the direction of the positive x -axis from the point $(1,0)$, there is a decrease in the temperature of the plate of approximately 100°C . We also have $T_x(0,1) = \frac{-100(0)}{(0^2 + 1^2)^{3/2}} = 0$, or just $\boxed{T_x(0,1) = 0}$. Since the rate of change is 0, the temperature is neither increasing nor decreasing for any increase (or decrease) in the direction of the positive x -axis from the point $(0,1)$.

- (c) We can find the rate of change of T in the direction of the positive y -axis at $(2,0)$ and $(0,2)$ by finding the first partial derivative of T with respect to y and then evaluating at $(2,0)$ and $(0,2)$ respectively. We have

$$T_y = 100(-\frac{1}{2})(x^2 + y^2)^{-3/2}(2y) = \frac{-100y}{(x^2 + y^2)^{3/2}}$$

Then, $T_y(2,0) = \frac{-100(0)}{(2^2 + 0^2)^{3/2}} = 0$, or just $\boxed{T_y(2,0) = 0}$. Since the rate of change is 0, the temperature is neither increasing nor decreasing for any increase (or decrease) in the direction of the positive y -axis from the point $(2,0)$. We also have $T_y(0,2) = \frac{-100(2)}{(0^2 + 2^2)^{3/2}} = -\frac{200}{4^{3/2}} = -\frac{200}{8} = -25$, or just $\boxed{T_y(0,2) = -25}$. For every unit increase in the direction of the positive y -axis from the point $(0,2)$, there is a decrease in the temperature of the plate of approximately 25°C .

47. We consider the equation $PV = nrT$, and assume that none of P, V, n, r , or T are 0 since this would result in a trivial situation. We find $\frac{\partial V}{\partial T}$, $\frac{\partial T}{\partial P}$, and $\frac{\partial P}{\partial V}$, and show that their product is -1. To find these partials, we solve for V , T , and then P in the equation $PV = nrT$.

$$V = \frac{nrT}{P}$$

so

$$\frac{\partial V}{\partial T} = \frac{nr}{P}.$$

Also,

$$T = \frac{PV}{nr}$$

so

$$\frac{\partial T}{\partial P} = \frac{V}{nr}.$$

Lastly,

$$P = \frac{nrT}{V}$$

so

$$\frac{\partial P}{\partial V} = nrT(-1 \cdot V^{-2}) = -\frac{nrT}{V^2}.$$

Multiplying gives us $\frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} = \left(\frac{nr}{P}\right) \left(\frac{V}{nr}\right) \left(-\frac{nrT}{V^2}\right) = -\frac{nrT}{PV} = -\frac{PV}{PV} = -1$.

48. (a) $\boxed{\frac{\partial V}{\partial T} = \frac{k}{P}}$ and $\frac{\partial V}{\partial P} = k(T)(-1 \cdot P^{-2}) = -\frac{kT}{P^2}$, so $\boxed{\frac{\partial V}{\partial P} = -\frac{kT}{P^2}}$.

(b) We have that $T \frac{\partial V}{\partial T} + P \frac{\partial V}{\partial P} = T \left(\frac{k}{P}\right) + P \left(-\frac{kT}{P^2}\right) = \frac{kT}{P} - \frac{kT}{P} = 0$.

49. (a) The marginal productivity with respect to capital input K is the partial derivative $\frac{\partial P}{\partial K}$, which is the following:

$$\frac{\partial P}{\partial K} \approx (1.01)(0.25)K^{-0.75}L^{0.75} = 0.2525K^{-0.75}L^{0.75}$$

When $K = 107$ and $L = 105$, we obtain that $\frac{\partial P}{\partial K} \approx 0.2525(107)^{-0.75}(105)^{0.75}$

so

$$\boxed{\frac{\partial P}{\partial K} \approx 0.249}$$

The marginal productivity with respect to labor input L is the partial derivative $\frac{\partial P}{\partial L}$, which is the following:

$$\frac{\partial P}{\partial L} \approx (1.01)(0.75)K^{0.25}L^{-0.25} = 0.7575K^{0.25}L^{-0.25}$$

When $K = 107$ and $L = 105$, we obtain that $\frac{\partial P}{\partial L} \approx 0.7575(107)^{0.25}(105)^{-0.25}$

so

$$\boxed{\frac{\partial P}{\partial L} \approx 0.761}$$

- (b) The marginal productivity with respect to capital input K is the partial derivative $\frac{\partial P}{\partial K}$, which is the following:

$$\frac{\partial P}{\partial K} \approx (1.01)(0.25)K^{-0.75}L^{0.75} = 0.2525K^{-0.75}L^{0.75}$$

When $K = 407$ and $L = 193$, we obtain that $\frac{\partial P}{\partial K} \approx 0.2525(407)^{-0.75}(193)^{0.75}$

so

$$\boxed{\frac{\partial P}{\partial K} \approx 0.144}$$

The marginal productivity with respect to labor input L is the partial derivative $\frac{\partial P}{\partial L}$, which is the following:

$$\frac{\partial P}{\partial L} \approx (1.01)(0.75)K^{0.25}L^{-0.25} = 0.7575K^{0.25}L^{-0.25}$$

When $K = 407$ and $L = 193$, we obtain that $\frac{\partial P}{\partial L} \approx 0.7575(407)^{0.25}(193)^{-0.25}$

so

$$\boxed{\frac{\partial P}{\partial L} \approx 0.913}$$

(c) Answers will vary.

50. (a) $f(0.25, 0.04, 0.05) = \frac{1 + (1 - 0.25)0.04}{1 + 0.05} - 1$

$$\boxed{f(0.25, 0.04, 0.05) \approx -0.019}$$

(b) The rate of change of gain (or loss) of money with respect to the marginal tax rate will be given by the partial derivative $\frac{\partial z}{\partial x}$. We first rewrite the expression for z as follows:

$$z = \frac{1 + (1 - x)y}{1 + r} - 1 = \frac{1 + y - xy}{1 + r} - 1 = \frac{1 + y}{1 + r} - \frac{xy}{1 + r} - 1$$

Then

$$\frac{\partial z}{\partial x} = -\frac{y}{1 + r}$$

So, when the effective yield y is 4%, or 0.04, and the inflation rate r is 5%, or 0.05, we have

$$\frac{\partial z}{\partial x} = -\frac{0.04}{1 + 0.05}, \text{ or } \boxed{\frac{\partial z}{\partial x} \approx -0.038}.$$

(c) The rate of change of gain (or loss) of money with respect to the effective yield will be given by the partial derivative $\frac{\partial z}{\partial y}$. We first rewrite the expression for z as follows:

$$z = \frac{1 + (1 - x)y}{1 + r} - 1 = \frac{1}{1 + r} + \frac{(1 - x)y}{1 + r} - 1$$

Then

$$\frac{\partial z}{\partial y} = \frac{1 - x}{1 + r}$$

So, when the marginal tax rate x is 25%, or 0.25, and the inflation rate r is 5%, or 0.05, we have

$$\frac{\partial z}{\partial y} = \frac{1 - 0.25}{1 + 0.05}, \text{ or } \boxed{\frac{\partial z}{\partial y} \approx 0.714}.$$

- (d) The rate of change of gain (or loss) of money with respect to the inflation rate will be given by the partial derivative $\frac{\partial z}{\partial r}$. We have that

$$\frac{\partial z}{\partial r} = [1 + (1 - x)y] \cdot [-1 \cdot (1 + r)^{-2}] = -\frac{1 + (1 - x)y}{(1 + r)^2}$$

So, when the marginal tax rate x is 25%, or 0.25, the effective yield y is 4%, or 0.04, and the annual inflation rate r is 5%, or 0.05, we have $\frac{\partial z}{\partial r} = -\frac{1 + (1 - 0.25)0.04}{(1.05)^2}$, or $\boxed{\frac{\partial z}{\partial r} \approx -0.934}$.

51. The marginal productivity with respect to x , the marginal productivity with respect to y , and the marginal productivity with respect to z will be given by the partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, and $\frac{\partial w}{\partial z}$ respectively. We have

$$\frac{\partial w}{\partial x} = 2 \cdot \frac{1}{2} x^{-1/2} y^{1/3} z^{1/6}$$

$$\boxed{\frac{\partial w}{\partial x} = \frac{y^{1/3} z^{1/6}}{x^{1/2}}}$$

$$\frac{\partial w}{\partial y} = 2 \cdot \frac{1}{3} x^{1/2} y^{-2/3} z^{1/6}$$

$$\boxed{\frac{\partial w}{\partial y} = \frac{2x^{1/2} z^{1/6}}{3y^{2/3}}}$$

$$\frac{\partial w}{\partial z} = 2 \cdot \frac{1}{6} x^{1/2} y^{1/3} z^{-5/6}$$

$$\boxed{\frac{\partial w}{\partial z} = \frac{x^{1/2} y^{1/3}}{3z^{5/6}}}$$

52. We have that

$$\frac{\partial v}{\partial p} = k \left(\frac{1}{2} \right) \left(\frac{p}{d} \right)^{-1/2} \frac{1}{d}$$

$$\boxed{\frac{\partial v}{\partial p} = \frac{k}{2d} \sqrt{\frac{d}{p}}}$$

and

$$\frac{\partial v}{\partial d} = k \left(\frac{1}{2} \right) \left(\frac{p}{d} \right)^{-1/2} (p \cdot (-d^{-2}))$$

$$\boxed{\frac{\partial v}{\partial d} = -\frac{kp}{2d^2} \sqrt{\frac{d}{p}}}$$

53. We compute $\frac{\partial^2 f}{\partial t^2}$ and $\frac{\partial^2 f}{\partial x^2}$ to show that $\frac{\partial^2 f}{\partial t^2} = 25 \frac{\partial^2 f}{\partial x^2}$. We have

$$\frac{\partial f}{\partial t} = 2(-\sin(5t) \cdot 5) \sin x = -10 \sin(5t) \sin x$$

$$\frac{\partial^2 f}{\partial t^2} = -10(\cos(5t) \cdot 5) \sin x = -50 \cos(5t) \sin x$$

and

$$\frac{\partial f}{\partial x} = 2 \cos(5t) \cos x$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \cos(5t)(-\sin x) = -2 \cos(5t) \sin x$$

so

$$\frac{\partial^2 f}{\partial t^2} = -50 \cos(5t) \sin x = 25(-2 \cos(5t) \sin x) = 25 \frac{\partial^2 f}{\partial x^2} \text{ at all points } (x, t). \quad \checkmark$$

54. (a) We first show that $T_t = -\lambda T$. We have that

$$\begin{aligned} T_t &= 40(-\lambda e^{-\lambda t}) \sin\left(\frac{\pi x}{20}\right) \\ &= -\lambda \left[40e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right) \right] = -\lambda T. \quad \checkmark \end{aligned}$$

Next, we show that $T_{xx} = -\frac{\pi^2}{400}T$. We have that

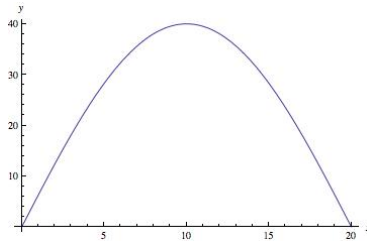
$$\begin{aligned} T_x &= 40e^{-\lambda t} \cos\left(\frac{\pi x}{20}\right) \left(\frac{\pi}{20}\right) = 2\pi e^{-\lambda t} \cos\left(\frac{\pi x}{20}\right) \\ T_{xx} &= 2\pi e^{-\lambda t} \left[-\sin\left(\frac{\pi x}{20}\right) \left(\frac{\pi}{20}\right) \right] = -\frac{\pi^2}{10} e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right) \quad \text{and} \\ -\frac{\pi^2}{400}T &= -\frac{\pi^2}{400} \left[40e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right) \right] = -\frac{\pi^2}{10} e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right) = T_{xx} \quad \checkmark \end{aligned}$$

Lastly, we show that $T_t = \frac{1}{k^2}T_{xx}$ for some constant k . We have already computed that

$$T_t = -40\lambda e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right) \text{ and } T_{xx} = -\frac{\pi^2}{10} e^{-\lambda t} \sin\left(\frac{\pi x}{20}\right), \text{ and so}$$

$$T_t = \frac{40\lambda}{\frac{\pi^2}{10}} T_{xx} = \frac{1}{\frac{\pi^2}{400\lambda}} T_{xx} \text{ (using } \lambda \neq 0 \text{). Letting } k = \frac{\pi}{20\sqrt{\lambda}} \text{ yields } T_t = \frac{1}{k^2} T_{xx} \text{ as desired (and } k \text{ is a constant since } \pi \text{ and } \lambda \text{ are also constants).}$$

(b) The graph of $y = T(0, x) = 40 \sin\left(\frac{\pi x}{20}\right)$ is shown below on $0 \leq x \leq 20$.



55. $\boxed{\frac{\partial x}{\partial r} = \cos \theta}$

$$\frac{\partial x}{\partial \theta} = r(-\sin \theta) \text{ so}$$

$$\boxed{\frac{\partial x}{\partial \theta} = -r \sin \theta}$$

$$\boxed{\frac{\partial y}{\partial r} = \sin \theta}$$

$$\boxed{\frac{\partial y}{\partial \theta} = r \cos \theta}$$

56. $\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot y(-x^{-2}) = \frac{-y}{x^2 \left[1 + \frac{y^2}{x^2}\right]} = \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}$$

$$\boxed{\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}}$$

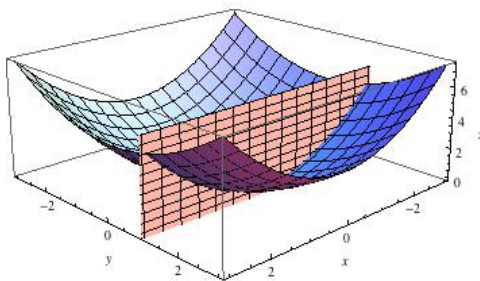
$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{x \left[1 + \frac{y^2}{x^2}\right]} = \frac{1}{x + \frac{y^2}{x}} = \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\boxed{\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}}$$

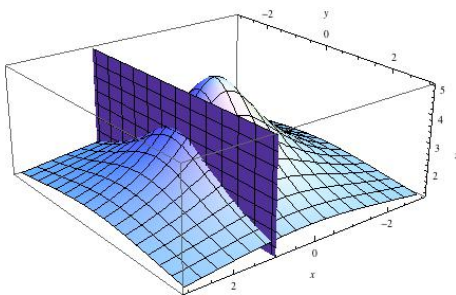
57. (a) Below are the graphs of $f(x, y) = \frac{1}{2}x^2 + \frac{1}{3}y^2$ and $y = 1$.



- (b) The slope of the tangent line to the curve of intersection of $z = \frac{1}{2}x^2 + \frac{1}{3}y^2$ and $y = 1$ at any point is $f_x(x, 1) = x$. At the point $\left(2, 1, \frac{7}{3}\right)$, the slope is 2. This tangent line lies on the plane $y = 1$. Symmetric equations of the tangent line are

$$\boxed{z - \frac{7}{3} = \frac{x - 2}{\frac{1}{2}} \quad y = 1}$$

58. (a) Below are the graphs of $f(x, y) = \frac{5}{\sqrt{x^2 + y^2 + 1}}$ and $x = 1$.



- (b) The slope of the tangent line to the curve of intersection of $z = \frac{5}{\sqrt{x^2 + y^2 + 1}}$ and $x = 1$ at any point is $f_y(1, y) = 5 \cdot \left[-\frac{1}{2}(1^2 + y^2 + 1)^{-3/2}(2y)\right] = \frac{-5y}{(2 + y^2)^{3/2}}$. At the point $\left(1, -2, \frac{5\sqrt{6}}{6}\right)$, the slope is $\frac{-5(-2)}{(2 + (-2)^2)^{3/2}} = \frac{10}{6^{3/2}} = \frac{10}{6\sqrt{6}} = \frac{5}{3\sqrt{6}}$. This tangent line lies on the plane $x = 1$. Symmetric equations of the tangent line are

$$\boxed{z - \frac{5\sqrt{6}}{6} = \frac{y + 2}{\frac{3\sqrt{6}}{5}} \quad x = 1}$$

59. We have that

$$\frac{\partial u}{\partial x} = e^x \cos y \text{ and } \frac{\partial v}{\partial y} = e^x \cos y \text{ so } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \quad \checkmark$$

Also,

$$\frac{\partial u}{\partial y} = e^x(-\sin y) = -e^x \sin y \text{ and } \frac{\partial v}{\partial x} = e^x \sin y \text{ so } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \checkmark$$

60. To find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we first apply a rule of logarithms to rewrite u as $u = \ln(x^2 + y^2)^{1/2} = \frac{1}{2} \ln(x^2 + y^2)$. We then have that

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2x) = \frac{x}{x^2 + y^2} \text{ and}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} (2y) = \frac{y}{x^2 + y^2}.$$

We also have that, for $x \neq 0$, $\frac{\partial v}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{x \left[1 + \frac{y^2}{x^2}\right]} = \frac{1}{x + \frac{y^2}{x}} = \frac{x}{x^2 + y^2} = \frac{\partial u}{\partial x}$, and so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \quad \checkmark$$

Lastly, we have, for $x \neq 0$, $\frac{\partial v}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot y(-x^{-2}) = \frac{-y}{x^2 \left[1 + \frac{y^2}{x^2}\right]} = \frac{-y}{x^2 + y^2} = -\frac{\partial u}{\partial y}$, and so

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \checkmark$$

61. We have that $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 8y$, so

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x(2x) + y(8y) = 2x^2 + 8y^2 = 2(x^2 + 4y^2) = 2u. \quad \checkmark$$

62. We have that $\frac{\partial u}{\partial x} = y^2$ and $\frac{\partial u}{\partial y} = 2xy$, so

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x(y^2) + y(2xy) = xy^2 + 2xy^2 = 3xy^2 = 3u. \quad \checkmark$$

63. We have that $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y - 3z$, and $\frac{\partial w}{\partial z} = -3y$, so

$$\begin{aligned} x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} &= x(2x) + y(2y - 3z) + z(-3y) = 2x^2 + 2y^2 - 3yz - 3yz = 2x^2 + 2y^2 - 6yz \\ &= 2(x^2 + y^2 - 3yz) = 2w. \quad \checkmark \end{aligned}$$

64. We first rewrite w to reduce the computations in finding the partial derivatives. We have

$$w = \frac{xz + y^2}{yz} = \frac{xz}{yz} + \frac{y^2}{yz} = \frac{x}{y} + \frac{y}{z}.$$

This yields $\frac{\partial w}{\partial x} = \frac{1}{y}$, $\frac{\partial w}{\partial y} = x(-y^{-2}) + \frac{1}{z} = -\frac{x}{y^2} + \frac{1}{z}$, and $\frac{\partial w}{\partial z} = y(-z^{-2}) = -\frac{y}{z^2}$, so

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = x \left(\frac{1}{y} \right) + y \left(-\frac{x}{y^2} + \frac{1}{z} \right) + z \left(-\frac{y}{z^2} \right) = \frac{x}{y} - \frac{x}{y} + \frac{y}{z} - \frac{y}{z} = 0. \quad \checkmark$$

65. We have that $\frac{\partial z}{\partial x} = -\sin(x + y) - \sin(x - y)$, and $\frac{\partial^2 z}{\partial x^2} = -\cos(x + y) - \cos(x - y)$. Also,

$$\frac{\partial z}{\partial y} = -\sin(x+y) - \sin(x-y)(-1) = -\sin(x+y) + \sin(x-y),$$

and

$$\frac{\partial^2 z}{\partial y^2} = -\cos(x+y) + \cos(x-y)(-1) = -\cos(x+y) - \cos(x-y).$$

So,

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} &= -\cos(x+y) - \cos(x-y) - [-\cos(x+y) - \cos(x-y)] \\ &= -\cos(x+y) - \cos(x-y) + \cos(x+y) + \cos(x-y) = 0. \quad \checkmark \end{aligned}$$

66. We have that

$$\frac{\partial z}{\partial x} = \cos(x-y) + \frac{1}{x+y}, \text{ and } \frac{\partial^2 z}{\partial x^2} = -\sin(x-y) + [-(x+y)^{-2}] = -\sin(x-y) - \frac{1}{(x+y)^2}.$$

Also,

$$\frac{\partial z}{\partial y} = \cos(x-y)(-1) + \frac{1}{x+y} = -\cos(x-y) + \frac{1}{x+y},$$

and

$$\frac{\partial^2 z}{\partial y^2} = -[-\sin(x-y)(-1)] + [-(x+y)^{-2}] = -\sin(x-y) - \frac{1}{(x+y)^2},$$

So,

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x-y) - \frac{1}{(x+y)^2} = \frac{\partial^2 z}{\partial y^2}. \quad \checkmark$$

67. We have $\frac{\partial u}{\partial t} = e^{-\alpha^2 t}(-\alpha^2)\sin(\alpha x) = -\alpha^2 e^{-\alpha^2 t} \sin(\alpha x)$. Also,

$$\frac{\partial u}{\partial x} = e^{-\alpha^2 t} \cos(\alpha x) \cdot \alpha = \alpha e^{-\alpha^2 t} \cos(\alpha x) \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha e^{-\alpha^2 t} [-\sin(\alpha x) \cdot \alpha] = -\alpha^2 e^{-\alpha^2 t} \sin(\alpha x). \text{ So,}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ for any constant } \alpha. \quad \checkmark$$

68.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \left(\ln \sqrt{x^2 + y^2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} \left(\sqrt{x^2 + y^2} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial x} \left((x^2 + y^2)^{1/2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \frac{\partial}{\partial x} (x^2 + y^2) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot (2x + 0) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{\left(\frac{\partial}{\partial x} (x) \right) (x^2 + y^2) - x \left(\frac{\partial}{\partial x} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\
&= \frac{1 \cdot (x^2 + y^2) - x \cdot (2x + 0)}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \left(\ln \sqrt{x^2 + y^2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y} \left(\sqrt{x^2 + y^2} \right) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial}{\partial y} \left((x^2 + y^2)^{1/2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \frac{\partial}{\partial y} (x^2 + y^2) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot (0 + 2y) = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + y^2}} \cdot 2y = \frac{y}{x^2 + y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \frac{\left(\frac{\partial}{\partial y} (y) \right) (x^2 + y^2) - y \left(\frac{\partial}{\partial y} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\
&= \frac{1 \cdot (x^2 + y^2) - y \cdot (0 + 2y)}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}
\end{aligned}$$

Adding up the two, we get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2 + x^2 - y^2}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2} = 0,$$

which means that the function $z = \ln \sqrt{x^2 + y^2}$ is a harmonic function.

69.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (e^{ax} \sin(ay)) = \sin(ay) \cdot \frac{\partial}{\partial x} (e^{ax}) = \sin(ay) \cdot e^{ax} \cdot \frac{\partial}{\partial x} (ax) = \sin(ay) \cdot e^{ax} \cdot a = ae^{ax} \sin(ay)$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (ae^{ax} \sin(ay)) \\
&= a \sin(ay) \cdot \frac{\partial}{\partial x} (e^{ax}) = a \sin(ay) \cdot e^{ax} \cdot \frac{\partial}{\partial x} (ax) = a \sin(ay) \cdot e^{ax} \cdot a = a^2 e^{ax} \sin(ay)
\end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (e^{ax} \sin(ay)) = e^{ax} \cdot \frac{\partial}{\partial y} (\sin(ay)) = e^{ax} \cdot \cos(ay) \cdot \frac{\partial}{\partial y} (ay) = e^{ax} \cdot \cos(ay) \cdot a = ae^{ax} \cos(ay)$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (ae^{ax} \cos(ay)) \\
&= ae^{ax} \cdot \frac{\partial}{\partial y} (\cos(ay)) = ae^{ax} \cdot (-\sin(ay)) \cdot \frac{\partial}{\partial y} (ay) = ae^{ax} \cdot (-\sin(ay)) \cdot a = -a^2 e^{ax} \sin(ay)
\end{aligned}$$

Adding up the two, we get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = a^2 e^{ax} \sin(ay) - a^2 e^{ax} \sin(ay) = 0,$$

which means that the function $z = e^{ax} \sin(ay)$ is a harmonic function.

70. For $x \neq 0$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) \\ &= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot \frac{\partial}{\partial x} (x^{-1}) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot (-x^{-2}) = \frac{x^2}{x^2 + y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2 + y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{-y}{x^2 + y^2} \right) = -y \cdot \frac{\partial}{\partial x} ((x^2 + y^2)^{-1}) \\ &= -y \cdot (-1(x^2 + y^2)^{-2}) \cdot \frac{\partial}{\partial x} (x^2 + y^2) = y(x^2 + y^2)^{-2} \cdot (2x + 0) = \frac{2xy}{(x^2 + y^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) \\ &= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot \frac{\partial}{\partial y} (y) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot 1 = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = x \cdot \frac{\partial}{\partial y} ((x^2 + y^2)^{-1/2}) \\ &= x \cdot (-1(x^2 + y^2)^{-2}) \cdot \frac{\partial}{\partial y} (x^2 + y^2) = -x(x^2 + y^2)^{-2} \cdot (0 + 2y) = \frac{-2xy}{(x^2 + y^2)^2}\end{aligned}$$

Adding up the two, we get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = \frac{2xy - 2xy}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2} = 0,$$

which means that the function $z = \tan^{-1} \frac{y}{x}$ is a harmonic function.

71.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (e^{ax} \cos(ay)) = \cos(ay) \cdot \frac{\partial}{\partial x} (e^{ax}) = \cos(ay) \cdot e^{ax} \cdot \frac{\partial}{\partial x} (ax) = \cos(ay) \cdot e^{ax} \cdot a = ae^{ax} \cos(ay)$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (ae^{ax} \cos(ay)) \\ &= a \cos(ay) \cdot \frac{\partial}{\partial x} (e^{ax}) = a \cos(ay) \cdot e^{ax} \cdot \frac{\partial}{\partial x} (ax) = a \cos(ay) \cdot e^{ax} \cdot a = a^2 e^{ax} \cos(ay)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (e^{ax} \cos(ay)) \\ &= e^{ax} \cdot \frac{\partial}{\partial y} (\cos(ay)) = e^{ax} \cdot (-\sin(ay)) \cdot \frac{\partial}{\partial y} (ay) = e^{ax} \cdot (-\sin(ay)) \cdot a = -ae^{ax} \sin(ay)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (-ae^{ax} \sin(ay)) \\ &= -ae^{ax} \cdot \frac{\partial}{\partial y} (\sin(ay)) = -ae^{ax} \cdot \cos(ay) \cdot \frac{\partial}{\partial y} (ay) = -ae^{ax} \cdot \cos(ay) \cdot a = -a^2 e^{ax} \cos(ay)\end{aligned}$$

Adding up the two, we get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = a^2 e^{ax} \cos(ay) - a^2 e^{ax} \cos(ay) = 0,$$

which means that the function $z = e^{ax} \cos(ay)$ is a harmonic function.

72. Since $u(x, y)$ and $v(x, y)$ have continuous second-order partial derivatives, we can apply the theorem on the Equality of Mixed Partial (Clairaut's Theorem), and conclude that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. To show that u is a harmonic function, we compute $u_{xx} + u_{yy}$ and show that it is equal to zero:

$$\begin{aligned} u_{xx} + u_{yy} &= \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial x}(v_y) + \frac{\partial}{\partial y}(-v_x) \quad (\text{since } u_x = v_y \text{ and } u_y = -v_x) \\ &= \frac{\partial}{\partial x}(v_y) - \frac{\partial}{\partial y}(v_x) = v_{yx} - v_{xy} = v_{yx} - v_{yx} \quad (\text{by the equality of mixed partials}) \\ &= 0. \end{aligned}$$

Therefore, u is a harmonic function. To show that v is a harmonic function, we compute $v_{xx} + v_{yy}$ and show that it is equal to zero:

$$\begin{aligned} v_{xx} + v_{yy} &= \frac{\partial}{\partial x}(v_x) + \frac{\partial}{\partial y}(v_y) = \frac{\partial}{\partial x}(-u_y) + \frac{\partial}{\partial y}(u_x) \quad (\text{since } u_x = v_y \text{ and } u_y = -v_x) \\ &= -\frac{\partial}{\partial x}(u_y) + \frac{\partial}{\partial y}(u_x) = -u_{yx} + u_{xy} = -u_{yx} + u_{yx} \quad (\text{by the equality of mixed partials}) \\ &= 0. \end{aligned}$$

Therefore, v is a harmonic function.

73. For $y \neq 0$,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} \left(z \tan^{-1} \frac{x}{y} \right) = z \cdot \frac{\partial}{\partial x} \left(z \tan^{-1} \frac{x}{y} \right) = z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{x}{y} \right) \\ &= z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{1}{y} \cdot \frac{\partial}{\partial x}(x) = z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{1}{y} \cdot 1 = z \cdot \frac{y^2}{x^2 + y^2} \cdot \frac{1}{y} = \frac{yz}{x^2 + y^2} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{yz}{x^2 + y^2} \right) = yz \cdot \frac{\partial}{\partial x} ((x^2 + y^2)^{-1}) \\ &= yz \cdot (-1(x^2 + y^2)^{-2}) \cdot \frac{\partial}{\partial x}(x^2 + y^2) = -yz \cdot (x^2 + y^2)^{-2} \cdot (2x + 0) = \frac{-2xyz}{(x^2 + y^2)^2} \\ \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} \left(z \tan^{-1} \frac{x}{y} \right) = z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{x}{y} \right) \\ &= z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot x \cdot \frac{\partial}{\partial y}(y^{-1}) = z \cdot \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot x \cdot (-y^{-2}) = z \cdot \frac{y^2}{x^2 + y^2} \cdot \frac{-x}{y^2} = \frac{-xz}{x^2 + y^2} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{-xz}{x^2 + y^2} \right) = -xz \cdot \frac{\partial}{\partial y} ((x^2 + y^2)^{-1}) \\ &= -xz \cdot (-1(x^2 + y^2)^{-2}) \cdot \frac{\partial}{\partial y}(x^2 + y^2) = xz(x^2 + y^2)^{-2} \cdot (0 + 2y) = \frac{2xyz}{(x^2 + y^2)^2} \end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial}{\partial z} \left(z \tan^{-1} \frac{x}{y} \right) = \tan^{-1} \frac{x}{y} \cdot \frac{\partial}{\partial z} (z) \\ &= \tan^{-1} \frac{x}{y} \cdot 1 = \tan^{-1} \frac{x}{y}\end{aligned}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left(\tan^{-1} \frac{x}{y} \right) = 0$$

The last derivative is zero because, when we compute a partial derivative with respect to z , the variables x and y are treated as constants, which means that the entire expression $\tan^{-1} \frac{x}{y}$ is treated as a constant.

Adding up the three second-order partial derivatives that we computed, we get:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-2xyz}{(x^2 + y^2)^2} + \frac{2xyz}{(x^2 + y^2)^2} + 0 = \frac{-2xyz + 2xyz}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2} = 0,$$

which means that the function $u = z \tan^{-1} \frac{x}{y}$ satisfies the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

74.

$$\begin{aligned}f_x(x, y, z) &= \frac{\partial}{\partial x} \left((x^2 + y^2 + z^2)^{-1/2} \right) = -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot \frac{\partial}{\partial x} (x^2 + y^2 + z^2) \\ &= -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot (2x + 0 + 0) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\end{aligned}$$

$$\begin{aligned}f_{xx}(x, y, z) &= \frac{\partial}{\partial x} (f_x(x, y, z)) = \frac{\partial}{\partial x} \left(-x(x^2 + y^2 + z^2)^{-3/2} \right) \\ &= \frac{\left(\frac{\partial}{\partial x} (-x) \right) (x^2 + y^2 + z^2)^{3/2} - (-x) \left(\frac{\partial}{\partial x} ((x^2 + y^2 + z^2)^{3/2}) \right)}{\left((x^2 + y^2 + z^2)^{3/2} \right)^2} \\ &= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-x) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \left(\frac{\partial}{\partial x} (x^2 + y^2 + z^2) \right)}{(x^2 + y^2 + z^2)^3} \\ &= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-x) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \cdot (2x + 0 + 0)}{(x^2 + y^2 + z^2)^3} \\ &= \frac{-(x^2 + y^2 + z^2)^{3/2} + 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\ &= (x^2 + y^2 + z^2)^{1/2} \cdot \frac{-(x^2 + y^2 + z^2) + 3x^2}{(x^2 + y^2 + z^2)^3} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= \frac{\partial}{\partial y} \left((x^2 + y^2 + z^2)^{-1/2} \right) = -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot \frac{\partial}{\partial y} (x^2 + y^2 + z^2) \\ &= -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot (0 + 2y + 0) = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\end{aligned}$$

$$\begin{aligned}
f_{yy}(x, y, z) &= \frac{\partial}{\partial y} (f_y(x, y, z)) = \frac{\partial}{\partial y} \left(-y(x^2 + y^2 + z^2)^{-3/2} \right) \\
&= \frac{\left(\frac{\partial}{\partial y} (-y) \right) (x^2 + y^2 + z^2)^{3/2} - (-y) \left(\frac{\partial}{\partial y} ((x^2 + y^2 + z^2)^{3/2}) \right)}{\left((x^2 + y^2 + z^2)^{3/2} \right)^2} \\
&= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-y) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \left(\frac{\partial}{\partial y} (x^2 + y^2 + z^2) \right)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-y) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \cdot (0 + 2y + 0)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-(x^2 + y^2 + z^2)^{3/2} + 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\
&= (x^2 + y^2 + z^2)^{1/2} \cdot \frac{-(x^2 + y^2 + z^2) + 3y^2}{(x^2 + y^2 + z^2)^3} = \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

$$\begin{aligned}
f_z(x, y, z) &= \frac{\partial}{\partial z} \left((x^2 + y^2 + z^2)^{-1/2} \right) = -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot \frac{\partial}{\partial z} (x^2 + y^2 + z^2) \\
&= -\frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot (0 + 0 + 2z) = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
f_{zz}(x, y, z) &= \frac{\partial}{\partial z} (f_z(x, y, z)) = \frac{\partial}{\partial z} \left(-z(x^2 + y^2 + z^2)^{-3/2} \right) \\
&= \frac{\left(\frac{\partial}{\partial z} (-z) \right) (x^2 + y^2 + z^2)^{3/2} - (-z) \left(\frac{\partial}{\partial z} ((x^2 + y^2 + z^2)^{3/2}) \right)}{\left((x^2 + y^2 + z^2)^{3/2} \right)^2} \\
&= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-z) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \left(\frac{\partial}{\partial z} (x^2 + y^2 + z^2) \right)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{(-1) \cdot (x^2 + y^2 + z^2)^{3/2} - (-z) \cdot \frac{3}{2} \cdot (x^2 + y^2 + z^2)^{1/2} \cdot (0 + 0 + 2z)}{(x^2 + y^2 + z^2)^3} \\
&= \frac{-(x^2 + y^2 + z^2)^{3/2} + 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\
&= (x^2 + y^2 + z^2)^{1/2} \cdot \frac{-(x^2 + y^2 + z^2) + 3z^2}{(x^2 + y^2 + z^2)^3} = \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

Adding up the three second-order partial derivatives that we computed, we get

$$\begin{aligned}
f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z) &= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{2x^2 - y^2 - z^2 - x^2 + 2y^2 - z^2 - x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{0}{(x^2 + y^2 + z^2)^{5/2}} = 0.
\end{aligned}$$

Therefore, $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies the three-dimensional Laplace equation

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

75. (a) If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left(\frac{xy^3}{x^2 + y^2} \right) = y^3 \cdot \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = y^3 \cdot \frac{\left(\frac{\partial}{\partial x} (x) \right) (x^2 + y^2) - x \left(\frac{\partial}{\partial x} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= y^3 \cdot \frac{1 \cdot (x^2 + y^2) - x \cdot (2x)}{(x^2 + y^2)^2} = y^3 \cdot \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = y^3 \cdot \frac{-x^2 + y^2}{(x^2 + y^2)^2} \end{aligned}$$

and

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left(\frac{xy^3}{x^2 + y^2} \right) = x \cdot \frac{\partial}{\partial y} \left(\frac{y^3}{x^2 + y^2} \right) = x \cdot \frac{\left(\frac{\partial}{\partial y} (y^3) \right) (x^2 + y^2) - y^3 \left(\frac{\partial}{\partial y} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= x \cdot \frac{3y^2 \cdot (x^2 + y^2) - y^3 \cdot (2y)}{(x^2 + y^2)^2} = x \cdot \frac{3x^2y^2 + 3y^4 - 2y^4}{(x^2 + y^2)^2} = x \cdot \frac{3x^2y^2 + y^4}{(x^2 + y^2)^2} \\ &= xy^2 \cdot \frac{3x^2 + y^2}{(x^2 + y^2)^2} \end{aligned}$$

Now we need to compute the partial derivatives at $(0, 0)$. We will use the definition of a partial derivative.

$$\begin{aligned} f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x) \cdot 0}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

$$\begin{aligned} f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{0 \cdot (\Delta y)^3}{0 + (\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0 \end{aligned}$$

$f_x(x, y) = \begin{cases} \frac{-y^3(x^2 - y^2)}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$	$f_y(x, y) = \begin{cases} \frac{xy^2(3x^2 + y^2)}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$
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(b) We will use the definition of a partial derivative to compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

$$\begin{aligned} f_{xy}(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f_x(0, 0 + \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{(\Delta y)^3 \left(\frac{0 + (\Delta y)^2}{(0 + (\Delta y)^2)^2} \right) - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{(\Delta y)^5}{(\Delta y)^4}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} 1 = 1 \end{aligned}$$

$$\begin{aligned} f_{yx}(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f_y(0 + \Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x) \cdot 0 \cdot \frac{3(\Delta x)^2 + 0}{((\Delta x)^2 + 0)^2} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

Since $f_{xy}(0, 0) = 1$ and $f_{yx}(0, 0) = 0$, we conclude that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

76. (a) First, let us compute $f_x(x, y)$. If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = y \cdot \frac{\partial}{\partial x} \left(\frac{x^3 - xy^2}{x^2 + y^2} \right) \\ &= y \cdot \frac{\left(\frac{\partial}{\partial x} (x^3 - xy^2) \right) (x^2 + y^2) - (x^3 - xy^2) \left(\frac{\partial}{\partial x} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= y \cdot \frac{(3x^2 - y^2)(x^2 + y^2) - (x^3 - xy^2)(2x)}{(x^2 + y^2)^2} \\ &= y \cdot \frac{3x^4 + 3x^2y^2 - x^2y^2 - y^4 - 2x^4 + 2x^2y^2}{(x^2 + y^2)^2} = y \cdot \frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2} \end{aligned}$$

In particular, if we let $x = 0$, then for all $y \neq 0$ we have

$$f_x(0, y) = y \cdot \frac{0 + 0 - y^4}{(0 + y^2)^2} = y \cdot \frac{-y^4}{y^4} = -y.$$

So the formula $f_x(0, y) = -y$ is correct for all y , except perhaps $y = 0$. Now we need to compute $f_x(0, 0)$. We will use the definition of a partial derivative. In order for our formula $f_x(0, y) = -y$ to be correct for $y = 0$, $f_x(0, 0)$ should be 0.

$$\begin{aligned} f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x) \cdot 0 \cdot ((\Delta x)^2 - 0)}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

So our formula $f_x(0, y) = -y$ is correct even when $y = 0$.

(b) Now, let us compute $f_y(x, y)$. If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) = x \cdot \frac{\partial}{\partial y} \left(\frac{x^2y - y^3}{x^2 + y^2} \right) \\ &= x \cdot \frac{\left(\frac{\partial}{\partial y} (x^2y - y^3) \right) (x^2 + y^2) - (x^2y - y^3) \left(\frac{\partial}{\partial y} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= x \cdot \frac{(x^2 - 3y^2)(x^2 + y^2) - (x^2y - y^3)(2y)}{(x^2 + y^2)^2} \\ &= x \cdot \frac{x^4 + x^2y^2 - 3x^2y^2 - 3y^4 - 2x^2y^2 + 2y^4}{(x^2 + y^2)^2} = x \cdot \frac{x^4 - 4x^2y^2 + 2xy^3 - y^4}{(x^2 + y^2)^2} \end{aligned}$$

In particular, if we let $y = 0$, then for all $x \neq 0$ we have

$$f_y(x, 0) = x \cdot \frac{x^4 - 0 + 0 - 0}{(x^2 + 0)^2} = x \cdot \frac{x^4}{x^4} = x.$$

So the formula $f_y(x, 0) = x$ is correct for all x , except perhaps $x = 0$. Now we need to compute $f_y(0, 0)$. We will use the definition of a partial derivative. In order for our formula $f_y(x, 0) = x$ to be correct for $x = 0$, $f_y(0, 0)$ should be 0.

$$\begin{aligned} f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{0 \cdot (\Delta y) \cdot (0 - (\Delta y)^2)}{0 + (\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0 \end{aligned}$$

So our formula $f_y(x, 0) = x$ is correct even when $x = 0$.

(c) Using the definition of a partial derivative,

$$\begin{aligned} f_{xy}(0,0) &= \lim_{\Delta y \rightarrow 0} \frac{f_x(0,0+\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{-(\Delta y) - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} -1 = -1. \end{aligned}$$

(d) Using the definition of a partial derivative,

$$\begin{aligned} f_{yx}(0,0) &= \lim_{\Delta x \rightarrow 0} \frac{f_y(0+\Delta x,0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x,0) - f_y(0,0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 1 = 1. \end{aligned}$$

77. If there is such a function f , then it must be the case that:

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} (f_x(x,y)) = \frac{\partial}{\partial y} (3x - y) = \frac{\partial}{\partial y} (3x) - \frac{\partial}{\partial y} (y) = 0 - 1 = -1 \\ f_{yx}(x,y) &= \frac{\partial}{\partial x} (f_y(x,y)) = \frac{\partial}{\partial x} (x - 3y) = \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial x} (3y) = 1 - 0 = 1. \end{aligned}$$

So, the partial derivatives f_x, f_y, f_{xy}, f_{yx} all exist (at every point in $\mathbb{R}^2$), and f_{xy} and f_{yx} are constant functions, so they are continuous at every point in $\mathbb{R}^2$. Therefore, the conditions of the Equality of Mixed Partials Theorem (Clairaut's Theorem) are satisfied, and it must be the case that f_{xy} and f_{yx} are equal at every point in $\mathbb{R}^2$.

However, $f_{xy}(x,y) = -1$, and $f_{yx}(x,y) = 1$, so the two are not equal. This means that such a function f cannot exist. Therefore, no, you should not believe it.

78. If there is such a function $z = f(x,y)$, then it must be the case that:

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} (f_x(x,y)) = \frac{\partial}{\partial y} (2x - y) = \frac{\partial}{\partial y} (2x) - \frac{\partial}{\partial y} (y) = 0 - 1 = -1 \\ f_{yx}(x,y) &= \frac{\partial}{\partial x} (f_y(x,y)) = \frac{\partial}{\partial x} (x - 2y) = \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial x} (2y) = 1 - 0 = 1. \end{aligned}$$

So, the partial derivatives f_x, f_y, f_{xy}, f_{yx} all exist (at every point in $\mathbb{R}^2$), and f_{xy} and f_{yx} are constant functions, so they are continuous at every point in $\mathbb{R}^2$. Therefore, the conditions of the Equality of Mixed Partials Theorem (Clairaut's Theorem) are satisfied, and it must be the case that f_{xy} and f_{yx} are equal at every point in $\mathbb{R}^2$.

However, $f_{xy}(x,y) = -1$, and $f_{yx}(x,y) = 1$, so the two are not equal. This means that such a function f cannot exist.

79. Let us define $f(x,y)$ to be $\sqrt{x^2 + y^2}$, so that $z = f(x,y) = \sqrt{x^2 + y^2}$. Using the definition of a partial derivative,

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{(\Delta x)^2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}. \end{aligned}$$

We investigate the one-sided limits:

$$\begin{aligned}\lim_{\Delta x \rightarrow 0^+} \frac{|\Delta x|}{\Delta x} &= \lim_{\Delta x \rightarrow 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} 1 = 1 \\ \lim_{\Delta x \rightarrow 0^-} \frac{|\Delta x|}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} -1 = -1\end{aligned}$$

Since the one-sided limits are not equal, we conclude that $\lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}$ does not exist. Therefore, $f_x(0, 0)$ does not exist.

$$\begin{aligned}f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\sqrt{0 + (\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\sqrt{(\Delta y)^2}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{|\Delta y|}{\Delta y}.\end{aligned}$$

We investigate the one-sided limits:

$$\begin{aligned}\lim_{\Delta y \rightarrow 0^+} \frac{|\Delta y|}{\Delta y} &= \lim_{\Delta y \rightarrow 0^+} \frac{\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0^+} 1 = 1 \\ \lim_{\Delta y \rightarrow 0^-} \frac{|\Delta y|}{\Delta y} &= \lim_{\Delta y \rightarrow 0^-} \frac{-\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0^-} -1 = -1\end{aligned}$$

Since the one-sided limits are not equal, we conclude that $\lim_{\Delta y \rightarrow 0} \frac{|\Delta y|}{\Delta y}$ does not exist. Therefore, $f_y(0, 0)$ does not exist. We conclude that the function $z = \sqrt{x^2 + y^2}$ does not have partial derivatives at $(0, 0)$. For the geometric reason why this should be so, answers will vary.

80.

$$\begin{aligned}f\left(1, -\frac{1}{3}\right) &= 4 \cdot 1^2 + 9 \cdot \left(-\frac{1}{3}\right)^2 - 12 = -7 \\ f_x(x, y) &= \frac{\partial}{\partial x} (4x^2 + 9y^2 - 12) = 4 \frac{\partial}{\partial x} (x^2) + 9 \frac{\partial}{\partial x} (y^2) - \frac{\partial}{\partial x} (12) = 4 \cdot 2x + 0 - 0 = 8x \\ f_x\left(1, -\frac{1}{3}\right) &= 8 \cdot 1 = 8 \\ f_y(x, y) &= \frac{\partial}{\partial y} (4x^2 + 9y^2 - 12) = 4 \frac{\partial}{\partial y} (x^2) + 9 \frac{\partial}{\partial y} (y^2) - \frac{\partial}{\partial y} (12) = 0 + 9 \cdot 2y - 0 = 18y \\ f_y\left(1, -\frac{1}{3}\right) &= 18 \cdot \left(-\frac{1}{3}\right) = -6\end{aligned}$$

These values can be interpreted as slopes of tangent lines. More specifically,

The slope of the tangent line to the curve of intersection of the surface $z = 4x^2 + 9y^2 - 12$ and the plane $y = -\frac{1}{3}$ at the point $(1, -\frac{1}{3}, -7)$ equals 8.

The slope of the tangent line to the curve of intersection of the surface $z = 4x^2 + 9y^2 - 12$ and the plane $x = 1$ at the point $(1, -\frac{1}{3}, -7)$ equals -6.

81.

$$\begin{aligned}f_x(x, y, z) &= \frac{\partial}{\partial x} (e^{x/y} + e^{y/z} + e^{z/x}) = \frac{\partial}{\partial x} (e^{x/y}) + \frac{\partial}{\partial x} (e^{y/z}) + \frac{\partial}{\partial x} (e^{z/x}) \\ &= e^{x/y} \cdot \frac{\partial}{\partial x} \left(\frac{x}{y}\right) + 0 + e^{z/x} \cdot \frac{\partial}{\partial x} \left(\frac{z}{x}\right) = e^{x/y} \cdot \frac{1}{y} \cdot \frac{\partial}{\partial x} (x) + 0 + e^{z/x} \cdot z \cdot \frac{\partial}{\partial x} (x^{-1}) \\ &= e^{x/y} \cdot \frac{1}{y} \cdot 1 + 0 + e^{z/x} \cdot z \cdot (-x^{-2}) = \frac{1}{y} \cdot e^{x/y} - \frac{z}{x^2} e^{z/x}\end{aligned}$$

$$\begin{aligned}
f_y(x, y, z) &= \frac{\partial}{\partial y} (e^{x/y} + e^{y/z} + e^{z/x}) = \frac{\partial}{\partial y} (e^{x/y}) + \frac{\partial}{\partial y} (e^{y/z}) + \frac{\partial}{\partial y} (e^{z/x}) \\
&= e^{x/y} \cdot \frac{\partial}{\partial y} \left(\frac{x}{y} \right) + e^{y/z} \cdot \frac{\partial}{\partial y} \left(\frac{y}{z} \right) + 0 = e^{x/y} \cdot x \cdot \frac{\partial}{\partial y} (y^{-1}) + e^{y/z} \cdot \frac{1}{z} \cdot \frac{\partial}{\partial y} (y) + 0 \\
&= e^{x/y} \cdot x \cdot (-y^{-2}) + e^{y/z} \cdot \frac{1}{z} \cdot 1 + 0 = -\frac{x}{y^2} \cdot e^{x/y} + \frac{1}{z} e^{y/z}
\end{aligned}$$

$$\begin{aligned}
f_z(x, y, z) &= \frac{\partial}{\partial z} (e^{x/y} + e^{y/z} + e^{z/x}) = \frac{\partial}{\partial z} (e^{x/y}) + \frac{\partial}{\partial z} (e^{y/z}) + \frac{\partial}{\partial z} (e^{z/x}) \\
&= 0 + e^{y/z} \cdot \frac{\partial}{\partial z} \left(\frac{y}{z} \right) + e^{z/x} \cdot \frac{\partial}{\partial z} \left(\frac{z}{x} \right) = 0 + e^{y/z} \cdot y \cdot \frac{\partial}{\partial z} (z^{-1}) + e^{z/x} \cdot \frac{1}{x} \cdot \frac{\partial}{\partial z} (z) \\
&= 0 + e^{y/z} \cdot y \cdot (-z^{-2}) + e^{z/x} \cdot \frac{1}{x} \cdot 1 + 0 = -\frac{y}{z^2} \cdot e^{y/z} + \frac{1}{x} e^{z/x}
\end{aligned}$$

It follows that

$$\begin{aligned}
&xf_x(x, y, z) + yf_y(x, y, z) + zf_z(x, y, z) \\
&= x \left(\frac{1}{y} \cdot e^{x/y} - \frac{z}{x^2} e^{z/x} \right) + y \left(-\frac{x}{y^2} \cdot e^{x/y} + \frac{1}{z} e^{y/z} \right) + z \left(-\frac{y}{z^2} \cdot e^{y/z} + \frac{1}{x} e^{z/x} \right) \\
&= \frac{x}{y} \cdot e^{x/y} - \frac{z}{x} e^{z/x} - \frac{x}{y} e^{x/y} + \frac{y}{z} e^{y/z} - \frac{y}{z} e^{y/z} + \frac{z}{x} e^{z/x} = 0.
\end{aligned}$$

Therefore, $f(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$ satisfies the equation

$$xf_x + yf_y + zf_z = 0.$$

82.

$$\begin{aligned}
f_x(t, x, y) &= \frac{\partial}{\partial x} (e^{at} \sin(bx) \cos(cy)) = e^{at} \cos(cy) \frac{\partial}{\partial x} (\sin(bx)) \\
&= e^{at} \cos(cy) \cos(bx) \cdot \frac{\partial}{\partial x} (bx) = e^{at} \cos(cy) \cos(bx) \cdot b = be^{at} \cos(bx) \cos(cy)
\end{aligned}$$

$$\begin{aligned}
f_{xx}(t, x, y) &= \frac{\partial}{\partial x} (f_x(t, x, y)) = \frac{\partial}{\partial x} (be^{at} \cos(bx) \cos(cy)) = be^{at} \cos(cy) \frac{\partial}{\partial x} (\cos(bx)) \\
&= be^{at} \cos(cy) (-\sin(bx)) \cdot \frac{\partial}{\partial x} (bx) = be^{at} \cos(cy) (-\sin(bx)) \cdot b = -b^2 e^{at} \sin(bx) \cos(cy)
\end{aligned}$$

$$\begin{aligned}
f_y(t, x, y) &= \frac{\partial}{\partial y} (e^{at} \sin(bx) \cos(cy)) = e^{at} \sin(bx) \frac{\partial}{\partial y} (\cos(cy)) \\
&= e^{at} \sin(bx) (-\sin(cy)) \cdot \frac{\partial}{\partial y} (cy) = e^{at} \sin(bx) (-\sin(cy)) \cdot c = -ce^{at} \sin(bx) \sin(cy)
\end{aligned}$$

$$\begin{aligned}
f_{yy}(t, x, y) &= \frac{\partial}{\partial y} (f_y(t, x, y)) = \frac{\partial}{\partial y} (-ce^{at} \sin(bx) \sin(cy)) = -ce^{at} \sin(bx) \frac{\partial}{\partial y} (\sin(cy)) \\
&= -ce^{at} \sin(bx) \cos(cy) \cdot \frac{\partial}{\partial y} (cy) = -ce^{at} \sin(bx) \cos(cy) \cdot c = -c^2 e^{at} \sin(bx) \cos(cy)
\end{aligned}$$

$$\begin{aligned}
f_t(t, x, y) &= \frac{\partial}{\partial t} (e^{at} \sin(bx) \cos(cy)) = \sin(bx) \cos(cy) \frac{\partial}{\partial t} (e^{at}) \\
&= \sin(bx) \cos(cy) e^{at} \cdot \frac{\partial}{\partial t} (at) = \sin(bx) \cos(cy) e^{at} \cdot a = ae^{at} \sin(bx) \cos(cy)
\end{aligned}$$

So, the equation $f_t = f_{xx} + f_{yy}$ will be satisfied when

$$ae^{at} \sin(bx) \cos(cy) = -b^2 e^{at} \sin(bx) \cos(cy) - c^2 e^{at} \sin(bx) \cos(cy),$$

which after factoring becomes

$$ae^{at} \sin(bx) \cos(cy) = (-b^2 - c^2) (e^{at} \sin(bx) \cos(cy)).$$

Dividing both sides by $e^{at} \sin(bx) \cos(cy)$ gives us the desired expression: $\boxed{a = -b^2 - c^2}$. (Note that though there are specific values of x and y that will make the quantity $e^{at} \sin(bx) \cos(cy) = 0$, and which will satisfy the equation before we divide by this quantity, we are seeking a, b , and c values which satisfy $f_t = f_{xx} + f_{yy}$ for *all* values of x, y .)

83.

$$f_x(x, t) = \frac{\partial}{\partial x} (\cos(x + ct)) = -\sin(x + ct) \frac{\partial}{\partial x} (x + ct) = -\sin(x + ct) \cdot 1 = -\sin(x + ct)$$

$$\begin{aligned} f_{xx}(x, t) &= \frac{\partial}{\partial x} (f_x(x, t)) \\ &= \frac{\partial}{\partial x} (-\sin(x + ct)) = -\cos(x + ct) \frac{\partial}{\partial x} (x + ct) = -\cos(x + ct) \cdot 1 = -\cos(x + ct) \end{aligned}$$

$$f_t(x, t) = \frac{\partial}{\partial t} (\cos(x + ct)) = -\sin(x + ct) \frac{\partial}{\partial t} (x + ct) = -\sin(x + ct) \cdot c = -c \sin(x + ct)$$

$$\begin{aligned} f_{tt}(x, t) &= \frac{\partial}{\partial t} (f_t(x, t)) \\ &= \frac{\partial}{\partial t} (-c \sin(x + ct)) = -c \cos(x + ct) \frac{\partial}{\partial t} (x + ct) = -c \cos(x + ct) \cdot c = -c^2 \cos(x + ct) \end{aligned}$$

So,

$$c^2 f_{xx}(x, t) = c^2 (-\cos(x + ct)) = -c^2 \cos(x + ct) = f_{tt}(x, t).$$

Therefore, $f(x, t) = \cos(x + ct)$ does satisfy the equation $f_{tt} = c^2 f_{xx}$.

84.

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left(\int_x^y \ln(\cos \sqrt{t}) dt \right) \\ &= \frac{\partial}{\partial x} \left(- \int_y^x \ln(\cos \sqrt{t}) dt \right) = - \frac{\partial}{\partial x} \left(\int_y^x \ln(\cos \sqrt{t}) dt \right) = -\ln(\cos \sqrt{x}), \end{aligned}$$

by the Fundamental Theorem of Calculus, Part 1 (see Section 5.3, page 362).

So $\boxed{f_x(x, y) = -\ln(\cos \sqrt{x})}$.

$$f_y(x, y) = \frac{\partial}{\partial y} \left(\int_x^y \ln(\cos \sqrt{t}) dt \right) = \ln \cos \sqrt{y},$$

again, by the Fundamental Theorem of Calculus, Part 1 (see Section 5.3, page 362).

So $\boxed{f_y(x, y) = \ln(\cos \sqrt{y})}$.

85. We solve for a :

$$a = \pm \sqrt{b^2 + c^2 - 2bc \cos A}.$$

However, since a is the length of a side of a triangle, it cannot be negative. So, we reject the negative root, and we have

$$a = \sqrt{b^2 + c^2 - 2bc \cos A} = (b^2 + c^2 - 2bc \cos A)^{1/2}.$$

(Note that $b^2 + c^2 - 2bc \cos A \neq 0$ since if it did, side length a would be 0.)

$$\begin{aligned} \frac{\partial a}{\partial b} &= \frac{\partial}{\partial b} \left((b^2 + c^2 - 2bc \cos A)^{1/2} \right) = \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial b} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial b} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \left(\frac{\partial}{\partial b} (b^2) + \frac{\partial}{\partial b} (c^2) - 2c \cos A \frac{\partial}{\partial b} (b) \right) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot (2b + 0 - 2c \cos A \cdot 1) \\ &= \frac{2b - 2c \cos A}{2\sqrt{b^2 + c^2 - 2bc \cos A}} = \frac{b - c \cos A}{\sqrt{b^2 + c^2 - 2bc \cos A}} \end{aligned}$$

$$\boxed{\frac{\partial a}{\partial b} = \frac{b - c \cos A}{\sqrt{b^2 + c^2 - 2bc \cos A}}}$$

$$\begin{aligned} \frac{\partial a}{\partial c} &= \frac{\partial}{\partial c} \left((b^2 + c^2 - 2bc \cos A)^{1/2} \right) = \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial c} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial c} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \left(\frac{\partial}{\partial c} (b^2) + \frac{\partial}{\partial c} (c^2) - 2b \cos A \frac{\partial}{\partial c} (c) \right) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot (0 + 2c - 2b \cos A \cdot 1) \\ &= \frac{2c - 2b \cos A}{2\sqrt{b^2 + c^2 - 2bc \cos A}} = \frac{c - b \cos A}{\sqrt{b^2 + c^2 - 2bc \cos A}} \end{aligned}$$

$$\boxed{\frac{\partial a}{\partial c} = \frac{c - b \cos A}{\sqrt{b^2 + c^2 - 2bc \cos A}}}$$

$$\begin{aligned} \frac{\partial a}{\partial A} &= \frac{\partial}{\partial A} \left((b^2 + c^2 - 2bc \cos A)^{1/2} \right) = \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial A} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \frac{\partial}{\partial A} (b^2 + c^2 - 2bc \cos A) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot \left(\frac{\partial}{\partial A} (b^2) + \frac{\partial}{\partial A} (c^2) - 2bc \frac{\partial}{\partial A} (\cos A) \right) \\ &= \frac{1}{2} \cdot (b^2 + c^2 - 2bc \cos A)^{-1/2} \cdot (0 + 0 - 2bc(-\sin A)) \\ &= \frac{2bc \sin A}{2\sqrt{b^2 + c^2 - 2bc \cos A}} = \frac{bc \sin A}{\sqrt{b^2 + c^2 - 2bc \cos A}} \end{aligned}$$

$$\boxed{\frac{\partial a}{\partial A} = \frac{bc \sin A}{\sqrt{b^2 + c^2 - 2bc \cos A}}}$$

86.

$$\begin{aligned}\frac{\partial x}{\partial r} &= \frac{\partial}{\partial r} (r \cos \theta) = \cos \theta \cdot \frac{\partial}{\partial r} (r) = \cos \theta \cdot 1 = \cos \theta \\ \frac{\partial x}{\partial \theta} &= \frac{\partial}{\partial \theta} (r \cos \theta) = r \cdot \frac{\partial}{\partial \theta} (\cos \theta) = r \cdot (-\sin \theta) = -r \sin \theta \\ \frac{\partial y}{\partial r} &= \frac{\partial}{\partial r} (r \sin \theta) = \sin \theta \cdot \frac{\partial}{\partial r} (r) = \sin \theta \cdot 1 = \sin \theta \\ \frac{\partial y}{\partial \theta} &= \frac{\partial}{\partial \theta} (r \sin \theta) = r \cdot \frac{\partial}{\partial \theta} (\sin \theta) = r \cos \theta\end{aligned}$$

So,

$$\begin{aligned}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} &= \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} \\ &= (\cos \theta)(r \cos \theta) - (-r \sin \theta)(\sin \theta) = r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta) = r \cdot 1 = r.\end{aligned}$$

Next, since $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \left(\frac{y}{x} \right)$, we have (for $x \neq 0$):

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} \left((x^2 + y^2)^{1/2} \right) = \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) \\ &= \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \\ \frac{\partial r}{\partial y} &= \frac{\partial}{\partial y} \left((x^2 + y^2)^{1/2} \right) = \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) \\ &= \frac{1}{2} \cdot (x^2 + y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x} \right) \\ &= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot \frac{\partial}{\partial x} (x^{-1}) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot y \cdot (-x^{-2}) = \frac{x^2}{x^2 + y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} \\ \frac{\partial \theta}{\partial y} &= \frac{\partial}{\partial y} \left(\tan^{-1} \frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x} \right) \\ &= \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot \frac{\partial}{\partial y} (y) = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \cdot \frac{1}{x} \cdot 1 = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}\end{aligned}$$

So,

$$\begin{aligned}\begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} &= \frac{\partial r}{\partial x} \cdot \frac{\partial \theta}{\partial y} - \frac{\partial r}{\partial y} \cdot \frac{\partial \theta}{\partial x} = \left(\frac{x}{r} \right) \left(\frac{x}{r^2} \right) - \left(\frac{y}{r} \right) \left(\frac{-y}{r^2} \right) = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

Challenge Problems

87.

$$f_x(x, y) = \frac{\partial}{\partial x} (x^y), \text{ so } \boxed{f_x(x, y) = x^{y-1}y} \text{ (using the Power Rule).}$$

$$f_y(x, y) = \frac{\partial}{\partial y} (x^y), \text{ so } \boxed{f_y(x, y) = x^y \ln x}.$$

In the last calculation, we used the rule for the derivative of an exponential function, from page 202.

88. For f_x , we use logarithmic differentiation (see page 225). We start by taking the natural logarithm of each side:

$$\ln f(x, y) = \ln x^{2x+3y}$$

$$\ln f(x, y) = (2x + 3y) \cdot \ln x \text{ (using the properties of logarithms)}$$

and then we use implicit differentiation (we take the derivative of each side with respect to x , keeping y constant):

$$\begin{aligned} \frac{\partial}{\partial x} (\ln f(x, y)) &= \frac{\partial}{\partial x} ((2x + 3y) \cdot \ln x) \\ \frac{1}{f(x, y)} \frac{\partial}{\partial x} (f(x, y)) &= \left(\frac{\partial}{\partial x} (2x + 3y) \right) \cdot \ln x + (2x + 3y) \cdot \frac{\partial}{\partial x} (\ln x) \\ \frac{1}{f(x, y)} f_x(x, y) &= 2 \cdot \ln x + (2x + 3y) \cdot \frac{1}{x}. \end{aligned}$$

Multiplying both sides by $f(x, y)$, we get

$$f_x(x, y) = f(x, y) \left(2 \cdot \ln x + (2x + 3y) \cdot \frac{1}{x} \right) = x^{2x+3y} \left(2 \cdot \ln x + (2x + 3y) \cdot \frac{1}{x} \right).$$

$$\boxed{f_x(x, y) = 2 \ln x (x^{2x+3y}) + (2x + 3y) x^{2x+3y-1}}.$$

For f_y , we observe that $f(x, y) = x^{2x+3y} = x^{2x} \cdot x^{3y}$, so

$$f_y(x, y) = \frac{\partial}{\partial y} (x^{2x} \cdot x^{3y}) = x^{2x} \frac{\partial}{\partial y} (x^{3y}) = x^{2x} x^{3y} \cdot \ln x \cdot 3,$$

using the rule for the derivative of an exponential function as well as the chain rule. So $\boxed{f_y(x, y) = 3x^{2x+3y} \cdot \ln x}$.

89.

$$f_x(x, y, z) = \frac{\partial}{\partial x} (x^{y+z}) = (y+z)x^{y+z-1} \text{ (using the Power Rule)}$$

$$\boxed{f_x(x, y, z) = \frac{(y+z)x^{y+z}}{x}}.$$

$$\begin{aligned} f_y(x, y, z) &= \frac{\partial}{\partial y} (x^{y+z}) = \frac{\partial}{\partial y} (x^y \cdot x^z) = x^z \cdot \frac{\partial}{\partial y} (x^y) \\ &= x^z \cdot x^y \cdot \ln x \text{ (using the rule for the derivative of an exponential function, page 202)} \end{aligned}$$

$$\boxed{f_y(x, y, z) = x^{y+z} \ln x}.$$

$$\begin{aligned} f_z(x, y, z) &= \frac{\partial}{\partial z} (x^{y+z}) = \frac{\partial}{\partial z} (x^y \cdot x^z) = x^y \cdot \frac{\partial}{\partial z} (x^z) \\ &= x^y \cdot x^z \cdot \ln x \text{ (using the rule for the derivative of an exponential function, page 202)} \end{aligned}$$

$$\boxed{f_z(x, y, z) = x^{y+z} \ln x}.$$

90.

$$f_x(x, y, z) = \frac{\partial}{\partial x} (x^{yz}) = (yz)x^{yz-1} \text{ (using the Power Rule)}$$

$$\boxed{f_x(x, y, z) = \frac{yzx^{y+z}}{x}}.$$

$$\begin{aligned} f_y(x, y, z) &= \frac{\partial}{\partial y} (x^{yz}) = \frac{\partial}{\partial y} ((x^z)^y) \text{ (using the laws of exponents)} \\ &= (x^z)^y \cdot \ln(x^z) \text{ (using the rule for the derivative of an exponential function, page 202)} \\ &= x^{yz} \cdot z \ln x \end{aligned}$$

$$\boxed{f_y(x, y, z) = zx^{yz} \ln x}.$$

$$\begin{aligned} f_z(x, y, z) &= \frac{\partial}{\partial z} (x^{yz}) = \frac{\partial}{\partial z} ((x^y)^z) \text{ (using the laws of exponents)} \\ &= (x^y)^z \cdot \ln(x^y) \text{ (using the rule for the derivative of an exponential function, page 202)} \\ &= x^{yz} \cdot y \ln x \end{aligned}$$

$$\boxed{f_z(x, y, z) = yx^{yz} \ln x}.$$

91.

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial}{\partial x} ((x+y)^z) = z(x+y)^{z-1} \cdot \frac{\partial}{\partial x} (x+y) \text{ (using the Power Rule and the Chain Rule)} \\ &= z(x+y)^{z-1} \cdot 1 = (x+y)^{z-1} z \end{aligned}$$

$$\boxed{f_x(x, y, z) = (x+y)^{z-1} z}$$

$$\begin{aligned} f_y(x, y, z) &= \frac{\partial}{\partial y} ((x+y)^z) = z(x+y)^{z-1} \cdot \frac{\partial}{\partial y} (x+y) \text{ (using the Power Rule and the Chain Rule)} \\ &= z(x+y)^{z-1} \cdot 1 = (x+y)^{z-1} z \end{aligned}$$

$$\boxed{f_y(x, y, z) = (x+y)^{z-1} z}$$

$$f_z(x, y, z) = \frac{\partial}{\partial z} ((x+y)^z), \text{ so } \boxed{f_z(x, y, z) = (x+y)^z \ln(x+y)}.$$

For the last computation, we used the rule for the derivative of an exponential function, page 202.

92.

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial}{\partial x} ((xy)^z) = z(xy)^{z-1} \cdot \frac{\partial}{\partial x} (xy) \text{ (using the Power Rule and the Chain Rule)} \\ &= z(xy)^{z-1} \cdot y = x^{z-1} y^z z \end{aligned}$$

$$\boxed{f_x(x, y, z) = x^{z-1} y^z z}$$

$$\begin{aligned}
f_y(x, y, z) &= \frac{\partial}{\partial y} ((xy)^z) = z(xy)^{z-1} \cdot \frac{\partial}{\partial y} (xy) \quad (\text{using the Power Rule and the Chain Rule}) \\
&= z(xy)^{z-1} \cdot x = x^z y^{z-1} z
\end{aligned}$$

$$\boxed{f_y(x, y, z) = x^z y^{z-1} z}$$

$$f_z(x, y, z) = \frac{\partial}{\partial z} ((xy)^z), \text{ so } \boxed{f_z(x, y, z) = (xy)^z \ln(xy)}.$$

For the last computation, we used the rule for the derivative of an exponential function, page 202.

93.

$$\begin{aligned}
f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{e^{-\frac{1}{(\Delta x)^2 + 0}} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^{-\frac{1}{(\Delta x)^2}}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{\Delta x}}{e^{\frac{1}{(\Delta x)^2}}} = \lim_{h \rightarrow 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}.
\end{aligned}$$

Here we made the substitution $h = \Delta x$ for simplicity. So, we need to compute the limit

$$\lim_{h \rightarrow 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}.$$

We will do so by computing the two one-sided limits,

$$\lim_{h \rightarrow 0^+} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}} \text{ and } \lim_{h \rightarrow 0^-} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}.$$

Since $\lim_{h \rightarrow 0^+} \frac{1}{h} = \infty$ and $\lim_{h \rightarrow 0^+} e^{\frac{1}{h^2}} = \infty$, the quotient is an indeterminate form at 0^+ of type $\frac{\infty}{\infty}$. Therefore, we will apply L'Hôpital's Rule. Now,

$$\frac{d}{dh} \left(\frac{1}{h} \right) = \frac{d}{dh} (h^{-1}) = -h^{-2}$$

and

$$\frac{d}{dh} \left(e^{\frac{1}{h^2}} \right) = e^{\frac{1}{h^2}} \frac{d}{dh} \left(\frac{1}{h^2} \right) = e^{\frac{1}{h^2}} \frac{d}{dh} (h^{-2}) = e^{\frac{1}{h^2}} (-2h^{-3}).$$

Therefore, by L'Hôpital's Rule,

$$\lim_{h \rightarrow 0^+} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}} = \lim_{h \rightarrow 0^+} \frac{-h^{-2}}{e^{\frac{1}{h^2}} (-2h^{-3})} = \lim_{h \rightarrow 0^+} \frac{h}{2e^{\frac{1}{h^2}}} = 0,$$

since $\lim_{h \rightarrow 0^+} h = 0$ and $\lim_{h \rightarrow 0^+} 2e^{\frac{1}{h^2}} = \infty$.

Since $\lim_{h \rightarrow 0^-} \frac{1}{h} = -\infty$ and $\lim_{h \rightarrow 0^-} e^{\frac{1}{h^2}} = \infty$, the quotient is an indeterminate form at 0^- of type $\frac{\infty}{\infty}$. Therefore, we will apply L'Hôpital's Rule. The derivatives of the numerator and denominator are the same as they were above; therefore, by L'Hôpital's Rule,

$$\lim_{h \rightarrow 0^-} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}} = \lim_{h \rightarrow 0^-} \frac{-h^{-2}}{e^{\frac{1}{h^2}} (-2h^{-3})} = \lim_{h \rightarrow 0^-} \frac{h}{2e^{\frac{1}{h^2}}} = 0,$$

since $\lim_{h \rightarrow 0^-} h = 0$ and $\lim_{h \rightarrow 0^-} 2e^{\frac{1}{h^2}} = \infty$. Since both one-sided limits equal zero, we conclude that $\lim_{h \rightarrow 0} \frac{1}{e^{\frac{1}{h^2}}} = 0$. Therefore, $\boxed{f_x(0,0) = 0}$.

$$\begin{aligned} f_y(0,0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{e^{-\frac{1}{0+(\Delta y)^2}} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{e^{-\frac{1}{(\Delta y)^2}}}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{1}{\Delta y}}{e^{\frac{1}{(\Delta y)^2}}} = \lim_{h \rightarrow 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}. \end{aligned}$$

Here we made the substitution $h = \Delta y$ for simplicity. We then realize that this limit is identical to the one that we computed when we computed $f_x(0,0)$, so it must also equal 0. Therefore, $\boxed{f_y(0,0) = 0}$.

94. We will use logarithmic differentiation (see page 225). We start by taking the natural logarithm of each side:

$$\begin{aligned} \ln f(x,y) &= \ln(xy)^{xy} \\ \ln f(x,y) &= xy \ln(xy) \text{ (using the properties of logarithms)} \end{aligned}$$

To compute f_x , we use implicit differentiation (we take the derivative of each side with respect to x , keeping y constant):

$$\begin{aligned} \frac{\partial}{\partial x} (\ln f(x,y)) &= \frac{\partial}{\partial x} (xy \ln(xy)) \\ \frac{1}{f(x,y)} \frac{\partial}{\partial x} (f(x,y)) &= y \cdot \frac{\partial}{\partial x} (x \ln(xy)) \\ \frac{1}{f(x,y)} \frac{\partial}{\partial x} (f(x,y)) &= y \left(\frac{\partial}{\partial x} (x) \cdot \ln(xy) + x \cdot \frac{\partial}{\partial x} (\ln(xy)) \right) \\ \frac{1}{f(x,y)} f_x(x,y) &= y \left(\ln(xy) + x \cdot \frac{1}{xy} \frac{\partial}{\partial x} (xy) \right) \\ \frac{1}{f(x,y)} f_x(x,y) &= y \left(\ln(xy) + x \frac{y}{xy} \right) \\ \frac{1}{f(x,y)} f_x(x,y) &= y \ln(xy) + y = y(\ln(xy) + 1) \end{aligned}$$

Multiplying both sides by $f(x,y)$, we get

$$f_x(x,y) = y \cdot f(x,y) (\ln(xy) + 1) = y(xy)^{xy} (\ln(xy) + 1).$$

So $\boxed{f_x(x,y) = y(xy)^{xy} (\ln(xy) + 1)}$.

To compute f_y , we use implicit differentiation again (we take the derivative of each side with respect

to y , keeping x constant):

$$\begin{aligned}
\frac{\partial}{\partial y} (\ln f(x, y)) &= \frac{\partial}{\partial y} (xy \ln(xy)) \\
\frac{1}{f(x, y)} \frac{\partial}{\partial y} (f(x, y)) &= x \cdot \frac{\partial}{\partial y} (y \ln(xy)) \\
\frac{1}{f(x, y)} \frac{\partial}{\partial y} (f(x, y)) &= x \left(\frac{\partial}{\partial y} (y) \cdot \ln(xy) + y \cdot \frac{\partial}{\partial y} (\ln(xy)) \right) \\
\frac{1}{f(x, y)} f_y(x, y) &= x \left(\ln(xy) + y \cdot \frac{1}{xy} \frac{\partial}{\partial y} (xy) \right) \\
\frac{1}{f(x, y)} f_y(x, y) &= x \left(\ln(xy) + y \frac{x}{xy} \right) \\
\frac{1}{f(x, y)} f_y(x, y) &= x \ln(xy) + x = x(\ln(xy) + 1)
\end{aligned}$$

Multiplying both sides by $f(x, y)$, we get

$$f_y(x, y) = x \cdot f(x, y) (\ln(xy) + 1) = x(xy)^{xy} (\ln(xy) + 1).$$

So $\boxed{f_y(x, y) = x(xy)^{xy} (\ln(xy) + 1)}$.

Since we already know the first partial derivatives of f , we do not need to use logarithmic differentiation to compute the second partial derivatives. We use the Product Rule.

$$\begin{aligned}
f_{xx}(x, y) &= \frac{\partial}{\partial x} (f_x(x, y)) = \frac{\partial}{\partial x} (y(xy)^{xy} (\ln(xy) + 1)) = y \cdot \frac{\partial}{\partial x} ((xy)^{xy} (\ln(xy) + 1)) \\
&= y \left[\left(\frac{\partial}{\partial x} ((xy)^{xy}) \right) \cdot (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{\partial}{\partial x} (\ln(xy) + 1) \right] \\
&= y \left[(y(xy)^{xy}) (\ln(xy) + 1) (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{\frac{\partial}{\partial x} (xy)}{xy} \right] \\
&= y \left[(y(xy)^{xy}) (\ln(xy) + 1) (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{y}{xy} \right] \\
&= y^2 (xy)^{xy} (\ln(xy) + 1)^2 + \frac{y}{x} \cdot (xy)^{xy} \\
&= (xy)^{xy} \left(y^2 (\ln(xy) + 1)^2 + \frac{y}{x} \right)
\end{aligned}$$

$$\boxed{f_{xx}(x, y) = (xy)^{xy} \left(y^2 (\ln(xy) + 1)^2 + \frac{y}{x} \right)}$$

$$\begin{aligned}
f_{yy}(x, y) &= \frac{\partial}{\partial y} (f_y(x, y)) = \frac{\partial}{\partial y} (x(xy)^{xy} (\ln(xy) + 1)) = x \cdot \frac{\partial}{\partial y} ((xy)^{xy} (\ln(xy) + 1)) \\
&= x \left[\left(\frac{\partial}{\partial y} ((xy)^{xy}) \right) \cdot (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{\partial}{\partial y} (\ln(xy) + 1) \right] \\
&= x \left[(x(xy)^{xy} (\ln(xy) + 1) (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{\frac{\partial}{\partial y} (xy)}{xy} \right] \\
&= x \left[(x(xy)^{xy} (\ln(xy) + 1) (\ln(xy) + 1) + (xy)^{xy} \cdot \frac{x}{xy} \right] \\
&= x^2 (xy)^{xy} (\ln(xy) + 1)^2 + \frac{x}{y} \cdot (xy)^{xy} \\
&= (xy)^{xy} \left(x^2 (\ln(xy) + 1)^2 + \frac{x}{y} \right)
\end{aligned}$$

$$f_{yy}(x, y) = (xy)^{xy} \left(x^2 (\ln(xy) + 1)^2 + \frac{x}{y} \right)$$

$$\begin{aligned}
f_{xy}(x, y) &= \frac{\partial}{\partial y} (f_x(x, y)) = \frac{\partial}{\partial y} (y(xy)^{xy} (\ln(xy) + 1)) = \frac{\partial}{\partial y} ((xy)^{xy} (y \ln(xy) + y)) \\
&= \left(\frac{\partial}{\partial y} ((xy)^{xy}) \right) \cdot (y \ln(xy) + y) + (xy)^{xy} \cdot \frac{\partial}{\partial y} (y \ln(xy) + y) \\
&= x(xy)^{xy} (\ln(xy) + 1) (y \ln(xy) + y) + (xy)^{xy} \left(\frac{\partial}{\partial y} (y) \cdot \ln(xy) + y \cdot \frac{\partial}{\partial y} (\ln(xy)) + \frac{\partial}{\partial y} (y) \right) \\
&= x(xy)^{xy} (\ln(xy) + 1) (y \ln(xy) + y) + (xy)^{xy} \left(\ln(xy) + y \cdot \frac{\frac{\partial}{\partial y} (xy)}{xy} + 1 \right) \\
&= xy(xy)^{xy} (\ln(xy) + 1)^2 + (xy)^{xy} \left(\ln(xy) + y \cdot \frac{x}{xy} + 1 \right) \\
&= (xy)^{xy} (xy (\ln(xy) + 1)^2 + \ln(xy) + 2)
\end{aligned}$$

$$f_{xy}(x, y) = (xy)^{xy} (xy (\ln(xy) + 1)^2 + \ln(xy) + 2)$$

Based on the definition of exponential function (page 202), the domain of f is the set of points (x, y) such that $xy > 0$ and $xy \neq 1$. In other words, the domain of f is $\{(x, y) | xy > 0 \text{ and } xy \neq 1\}$.

95. Since $f(x, y)$ is defined to be $\frac{x^3 - y^3}{x^2 + y^2}$ for all $(x, y) \neq (0, 0)$, and $\frac{x^3 - y^3}{x^2 + y^2}$ is a quotient of polynomials, with a nonzero denominator for all $(x, y) \neq (0, 0)$, we conclude that f has first partial derivatives at (x, y) for all $(x, y) \neq (0, 0)$. So, all that remains to be done is to verify that f has first partial derivatives at $(0, 0)$. We do so by using the definition of a partial derivative:

$$\begin{aligned}
f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^3 - 0}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 1 = 1
\end{aligned}$$

$$\begin{aligned}
f_y(0,0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{0-(\Delta y)^3}{0+(\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} -1 = -1
\end{aligned}$$

So $f_x(0,0) = 1$ and $f_y(0,0) = -1$, and therefore we conclude that f has first partial derivatives at all points in the plane.

96.

$$f_r(r, \theta) = \frac{\partial}{\partial r} (r^n \sin(n\theta)) = \sin(n\theta) \frac{\partial}{\partial r} (r^n) = \sin(n\theta) \cdot nr^{n-1} = nr^{n-1} \sin(n\theta)$$

$$\begin{aligned}
f_{rr}(r, \theta) &= \frac{\partial}{\partial r} (f_r(r, \theta)) = \frac{\partial}{\partial r} (nr^{n-1} \sin(n\theta)) \\
&= n \sin(n\theta) \frac{\partial}{\partial r} (r^{n-1}) = n \sin(n\theta) \cdot (n-1)r^{n-2} = n(n-1)r^{n-2} \sin(n\theta)
\end{aligned}$$

$$f_\theta(r, \theta) = \frac{\partial}{\partial \theta} (r^n \sin(n\theta)) = r^n \frac{\partial}{\partial \theta} (\sin(n\theta)) = r^n \cos(n\theta) \cdot \frac{\partial}{\partial \theta} (n\theta) = r^n \cos(n\theta) \cdot n = nr^n \cos(n\theta)$$

$$\begin{aligned}
f_{\theta\theta}(r, \theta) &= \frac{\partial}{\partial \theta} (f_\theta(r, \theta)) = \frac{\partial}{\partial \theta} (nr^n \cos(n\theta)) \\
&= nr^n \frac{\partial}{\partial \theta} (\cos(n\theta)) = nr^n (-\sin(n\theta)) \cdot \frac{\partial}{\partial \theta} (n\theta) = nr^n (-\sin(n\theta)) \cdot n = -n^2 r^n \sin(n\theta)
\end{aligned}$$

So,

$$\begin{aligned}
&f_{rr}(r, \theta) + \frac{1}{r} f_r(r, \theta) + \frac{1}{r^2} f_{\theta\theta}(r, \theta) \\
&= n(n-1)r^{n-2} \sin(n\theta) + \frac{1}{r} \cdot nr^{n-1} \sin(n\theta) + \frac{1}{r^2} (-n^2 r^n \sin(n\theta)) \\
&= n(n-1)r^{n-2} \sin(n\theta) + nr^{n-2} \sin(n\theta) - n^2 r^{n-2} \sin(n\theta) \\
&= (n(n-1) + n - n^2) r^{n-2} \sin(n\theta) = 0 \cdot r^{n-2} \sin(n\theta) = 0.
\end{aligned}$$

Therefore, $f(r, \theta) = r^n \sin(n\theta)$ satisfies the equation

$$f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0.$$

97. For all m , we have

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r} (r^m \cos(m\theta)) = \cos(m\theta) \frac{\partial}{\partial r} (r^m) = \cos(m\theta) \cdot mr^{m-1} = mr^{m-1} \cos(m\theta)$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \right) = \frac{\partial}{\partial r} (mr^{m-1} \cos(m\theta)) \\
&= m \cos(m\theta) \frac{\partial}{\partial r} (r^{m-1}) = m \cos(m\theta) \cdot (m-1)r^{m-2} = m(m-1)r^{m-2} \cos(m\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial}{\partial \theta} (r^m \cos(m\theta)) \\
&= r^m \frac{\partial}{\partial \theta} (\cos(m\theta)) = r^m (-\sin(m\theta)) \cdot \frac{\partial}{\partial \theta} (m\theta) = r^m (-\sin(m\theta)) \cdot m = -mr^m \sin(m\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial \theta} \right) = \frac{\partial}{\partial \theta} (-mr^m \sin(m\theta)) \\
&= -mr^m \frac{\partial}{\partial \theta} (\sin(m\theta)) = -mr^m (\cos(m\theta)) \cdot \frac{\partial}{\partial \theta} (m\theta) = -mr^m (\cos(m\theta)) \cdot m = -m^2 r^m \cos(m\theta)
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r} \left(\frac{\partial u}{\partial r} \right) \\
&= m(m-1)r^{m-2} \cos(m\theta) + \frac{1}{r^2} (-m^2 r^m \cos(m\theta)) + \frac{1}{r} (mr^{m-1} \cos(m\theta)) \\
&= m(m-1)r^{m-2} \cos(m\theta) - m^2 r^{m-2} \cos(m\theta) + mr^{m-2} \cos(m\theta) \\
&= (m(m-1) - m^2 + m) r^{m-2} \cos(m\theta) = 0 \cdot r^{m-2} \cos(m\theta) = 0.
\end{aligned}$$

Therefore, $u = r^m \cos(m\theta)$ satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r} \left(\frac{\partial u}{\partial r} \right) = 0,$$

for all m .

98. (a) The slope of the tangent line at $(x_0, y_0, f(x_0, y_0))$ to the curve of intersection of $z = f(x, y)$ and $y = y_0$ is the partial derivative $f_x(x_0, y_0)$. One point on the tangent line is $(x_0, y_0, f(x_0, y_0))$, and the tangent line lies on the plane $y = y_0$. Therefore, if $f_x(x_0, y_0) \neq 0$, the symmetric equations of the tangent line are:

$$\frac{z - f(x_0, y_0)}{f_x(x_0, y_0)} = x - x_0, \quad y = y_0.$$

If the partial derivative $f_x(x_0, y_0)$ is 0, then the slope of the curve in the plane $y = y_0$ is 0, and so the curve is the constant curve $z = f(x_0, y_0)$. The symmetric equations are then given by

$$z = f(x_0, y_0), y = y_0$$

The slope of the tangent line at $(x_0, y_0, f(x_0, y_0))$ to the curve of intersection of $z = f(x, y)$ and $x = x_0$ is the partial derivative $f_y(x_0, y_0)$. One point on the tangent line is $(x_0, y_0, f(x_0, y_0))$, and the tangent line lies on the plane $x = x_0$. Therefore, if $f_y(x_0, y_0) \neq 0$, the symmetric equations of the tangent line are:

$$\frac{z - f(x_0, y_0)}{f_y(x_0, y_0)} = y - y_0, \quad x = x_0.$$

If the partial derivative $f_y(x_0, y_0)$ is 0, then the slope of the curve in the plane $x = x_0$ is 0, and so the curve is the constant curve $z = f(x_0, y_0)$. The symmetric equations are then given by

$$z = f(x_0, y_0), x = x_0$$

- (b) From the symmetric equations, we can see that the first tangent line is in the direction of the vector $\mathbf{i} + f_x(x_0, y_0)\mathbf{k}$, and the second tangent line is in the direction of the vector $\mathbf{j} + f_y(x_0, y_0)\mathbf{k}$. Therefore, the vector

$$\begin{aligned}
\mathbf{N} &= (\mathbf{i} + f_x(x_0, y_0)\mathbf{k}) \times (\mathbf{j} + f_y(x_0, y_0)\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} \\
&= \begin{vmatrix} 0 & f_x(x_0, y_0) \\ 1 & f_y(x_0, y_0) \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & f_x(x_0, y_0) \\ 0 & f_y(x_0, y_0) \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}
\end{aligned}$$

is orthogonal to both tangent lines, and is therefore a normal vector to the plane determined by the two lines. Since the plane contains the point $(x_0, y_0, f(x_0, y_0))$, we conclude that an equation of the plane is

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0,$$

which we can also write as $\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$.

(c) Answers will vary.

99. (a)

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (3x^2 + 4y^3) = \frac{\partial}{\partial x} (3x^2) + \frac{\partial}{\partial x} (4y^3) = 6x + 0 = 6x \\ f_x(1, 6) &= 6 \cdot 1 \end{aligned}$$

$$\boxed{f_x(1, 6) = 6}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} (3x^2 + 4y^3) = \frac{\partial}{\partial y} (3x^2) + \frac{\partial}{\partial y} (4y^3) = 0 + 12y^2 = 12y^2 \\ f_y(1, 6) &= 12 \cdot 6^2 \end{aligned}$$

$$\boxed{f_y(1, 6) = 432}$$

(b) Suppose that we would like to determine the $x'y'$ -coordinates of a point (x, y) . Imagine rotating the entire plane 45° clockwise about the origin. Then the x' -axis is mapped to the x -axis, and the y' -axis is mapped to the y -axis. Therefore, the point (x, y) must be mapped to the point (x', y') ; this means that (x', y') is the result of rotating the point (x, y) 45° clockwise about the origin. Or, (x, y) is the result of rotating the point (x', y') 45° counterclockwise about the origin. In other words, if we write $x' = r \cos \theta$ and $y' = r \sin \theta$ (using polar coordinates), then $x = r \cos(\theta + 45^\circ)$ and $y = r \sin(\theta + 45^\circ)$. Using trigonometric identities, we obtain

$$\begin{aligned} x &= r \cos(\theta + 45^\circ) = r (\cos \theta \cos 45^\circ - \sin \theta \sin 45^\circ) = r \cos \theta \cos 45^\circ - r \sin \theta \sin 45^\circ \\ &= \frac{\sqrt{2}}{2} r \cos \theta - \frac{\sqrt{2}}{2} r \sin \theta = \frac{\sqrt{2}}{2} x' - \frac{\sqrt{2}}{2} y' = \frac{\sqrt{2}}{2} (x' - y') \\ y &= r \sin(\theta + 45^\circ) = r (\sin \theta \cos 45^\circ + \cos \theta \sin 45^\circ) = r \sin \theta \cos 45^\circ + r \cos \theta \sin 45^\circ \\ &= \frac{\sqrt{2}}{2} r \sin \theta + \frac{\sqrt{2}}{2} r \cos \theta = \frac{\sqrt{2}}{2} y' + \frac{\sqrt{2}}{2} x' = \frac{\sqrt{2}}{2} (x' + y') \end{aligned}$$

So,

$$f(x, y) = 3x^2 + 4y^3 = 3 \left(\frac{\sqrt{2}}{2} (x' - y') \right)^2 + 4 \left(\frac{\sqrt{2}}{2} (x' + y') \right)^3 = \frac{3}{2} (x' - y')^2 + \sqrt{2} (x' + y')^3$$

Therefore, $\bar{f}(x', y') = \frac{3}{2} (x' - y')^2 + \sqrt{2} (x' + y')^3$. It follows that:

$$\begin{aligned} \bar{f}_{x'}(x', y') &= \frac{3}{2} \cdot \frac{\partial}{\partial x'} ((x' - y')^2) + \sqrt{2} \cdot \frac{\partial}{\partial x'} ((x' + y')^3) \\ &= \frac{3}{2} \cdot 2(x' - y') \frac{\partial}{\partial x'} (x' - y') + \sqrt{2} \cdot 3(x' + y')^2 \frac{\partial}{\partial x'} (x' + y') \\ &= 3(x' - y') + 3\sqrt{2} (x' + y')^2 \\ \bar{f}_{y'}(x', y') &= \frac{3}{2} \cdot \frac{\partial}{\partial y'} ((x' - y')^2) + \sqrt{2} \cdot \frac{\partial}{\partial y'} ((x' + y')^3) \\ &= \frac{3}{2} \cdot 2(x' - y') \frac{\partial}{\partial y'} (x' - y') + \sqrt{2} \cdot 3(x' + y')^2 \frac{\partial}{\partial y'} (x' + y') \\ &= -3(x' - y') + 3\sqrt{2} (x' + y')^2 \end{aligned}$$

Now we need to determine (a, b) . Using the above equations, we have $1 = \frac{\sqrt{2}}{2}(a - b)$ and $6 = \frac{\sqrt{2}}{2}(a + b)$. So, $a - b = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$ and $a + b = \frac{6}{\frac{\sqrt{2}}{2}} = 6\sqrt{2}$. Adding these last two equations gives $2a = 7\sqrt{2}$, while subtracting the first from the second gives $2b = 5\sqrt{2}$. So, $(a, b) = (\frac{7\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})$. Therefore,

$$\bar{f}_{x'}(a, b) = 3(a - b) + 3\sqrt{2}(a + b)^2 = 3\sqrt{2} + 3\sqrt{2} \cdot 72 = 219\sqrt{2}$$

$$\boxed{\bar{f}_{x'}(a, b) = 219\sqrt{2} \approx 309.713}$$

$$\bar{f}_{y'}(a, b) = -3(a - b) + 3\sqrt{2}(a + b)^2 = -3\sqrt{2} + 3\sqrt{2} \cdot 72 = 213\sqrt{2}$$

$$\boxed{\bar{f}_{y'}(a, b) = 213\sqrt{2} \approx 301.227}$$

Chapter 12: Functions of Several Variables

12.4: Differentiability and the Differential

Concepts and Vocabulary

1. True. The change in z , Δz , from $(0.9, 0)$ to $(1, \frac{\pi}{4})$, is defined as

$$f\left(1, \frac{\pi}{4}\right) - f(0.9, 0) = (1) \sin\left(\frac{\pi}{4}\right) - (0.9) \sin 0 = \frac{\sqrt{2}}{2}.$$

2. True. Note that both f_x and f_y exist in a disk centered at $(0, 2)$ (we can choose the disk of radius $\frac{1}{2}$ for example), and f_x and f_y are both continuous at $(0, 2)$, so f is differentiable at $(0, 2)$. Recall that the continuity of the partial derivatives f_x and f_y implies the differentiability of f .
3. True. The differential dz of z at $(0, \frac{\pi}{2})$ is defined as $dz = f_x\left(0, \frac{\pi}{2}\right) dx + f_y\left(0, \frac{\pi}{2}\right) dy$. We have that $f_x = e^x \cos y$ and $f_y = -e^x \sin y$, so $f_x\left(0, \frac{\pi}{2}\right) = e^0 \cos\left(\frac{\pi}{2}\right) = (1)(0) = 0$ and $f_y\left(0, \frac{\pi}{2}\right) = -e^0 \sin\left(\frac{\pi}{2}\right) = -(1)(1) = -1$. This yields:

$$dz = 0 \cdot dx + -1 \cdot dy = -dy.$$

4. True. Note that f_x and f_y exist and are continuous everywhere since they are both polynomial functions, and the continuity of the partial derivatives f_x and f_y implies the continuity of f .
5. True. Note that f_x and f_y exist and are continuous everywhere since they are both polynomial functions, and the continuity of the partial derivatives f_x and f_y implies the differentiability of f .
6. False. The differential dw is defined as $dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$, and we have the following partial derivatives:

$$f_x(x, y, z) = e^{xyz}(yz) = yze^{xyz}$$

$$f_y(x, y, z) = e^{xyz}(xz) = xze^{xyz}$$

$$f_z(x, y, z) = e^{xyz}(xy) = xye^{xyz}$$

This yields $dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz = yze^{xyz}dx + xze^{xyz}dy + xye^{xyz}dz \neq yze^x dx + xze^y dy + xye^z dz$.

Skill Building

7. We compute $\Delta z = f(1.1, 3.2) - f(1, 3)$:

$$f(1.1, 3.2) = 1.1^2 + 3.2^2 = 11.45$$

$$f(1, 3) = 1^2 + 3^2 = 10$$

So, the change in z is $\Delta z = 11.45 - 10$, or $\boxed{\Delta z = 1.45}$.

8. We compute $\Delta z = f(2.1, -1.1) - f(2, -1)$:

$$f(2.1, -1.1) = 2(2.1)^2 + (2.1)(-1.1) - (-1.1)^2 = 5.3$$

$$f(2, -1) = 2(2)^2 + (2)(-1) - (-1)^2 = 5$$

So, the change in z is $\Delta z = 5.3 - 5$, or $\boxed{\Delta z = 0.3}$.

9. We compute $\Delta z = f(0.9, 2.1) - f(1, 2)$:

$$f(0.9, 2.1) = e^{0.9} \ln[(0.9)(2.1)] \approx 1.566$$

$$f(1, 2) = e^1 \ln[(1)(2)] \approx 1.884$$

So, the change in z is $\Delta z \approx 1.566 - 1.884$, or $\boxed{\Delta z \approx -0.318}$.

10. We compute $\Delta z = f(-0.9, 1.9) - f(-1, 2)$:

$$f(-0.9, 1.9) = \frac{(-0.9)(1.9)}{-0.9 + 1.9} = -1.71$$

$$f(-1, 2) = \frac{(-1)(2)}{-1 + 2} = -2$$

So, the change in z is $\Delta z = -1.71 - (-2)$, or $\boxed{\Delta z = 0.29}$.

11. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= 2x \\ f_y(x, y) &= 2y \end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$\boxed{dz = 2x dx + 2y dy}.$$

12. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= 4x + y \\f_y(x, y) &= x - 2y\end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$dz = (4x + y) dx + (x - 2y) dy.$$

13. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= \sin y + y \cos x \\f_y(x, y) &= x \cos y + \sin x\end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$dz = (\sin y + y \cos x) dx + (x \cos y + \sin x) dy.$$

14. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= e^x \cos y + e^x \cdot \frac{\partial}{\partial x}(-x) \cdot \sin y = e^x \cos y - e^x \sin y \\f_y(x, y) &= -e^x \sin y + e^{-x} \cos y\end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$dz = (e^x \cos y - e^x \sin y) dx + (-e^x \sin y + e^{-x} \cos y) dy.$$

15. $f(x, y)$ is defined everywhere in the xy -plane, except when $x = 0$ (because of the denominator x). The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2} \\f_y(x, y) &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{1}{x + \frac{y^2}{x}} = \frac{x}{x^2 + y^2}\end{aligned}$$

Now, f_x and f_y are continuous everywhere in the domain of f (because, if $x \neq 0$, then the denominator of $f_x(x, y)$ and $f_y(x, y)$, which is $x^2 + y^2$, cannot be zero). So, the function $z = f(x, y)$ is differentiable on its domain. The differential dz is

$$dz = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

16. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= \tan^{-1} y \\f_y(x, y) &= \frac{x}{1 + y^2}\end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$dz = \tan^{-1} y dx + \frac{x}{1 + y^2} dy.$$

17. In order for $f(x, y)$ to be defined, the argument of the logarithm, $\frac{y}{x}$, must be positive. This happens when x and y are both positive, or both negative.

So the domain of f is $\{(x, y) | x > 0, y > 0 \text{ or } x < 0, y < 0\}$. The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= \frac{1}{\left(\frac{y}{x}\right)} \cdot \frac{\partial}{\partial x} \left(\frac{y}{x}\right) = \frac{x}{y} \cdot \frac{-y}{x^2} = -\frac{1}{x} \\ f_y(x, y) &= \frac{1}{\left(\frac{y}{x}\right)} \cdot \frac{\partial}{\partial y} \left(\frac{y}{x}\right) = \frac{x}{y} \cdot \frac{1}{x} = \frac{1}{y} \end{aligned}$$

Now, f_x and f_y are continuous everywhere in the domain of f (because, if $x > 0, y > 0$ or $x < 0, y < 0$, then $x \neq 0$ and $y \neq 0$, so the denominators of $f_x(x, y)$ and $f_y(x, y)$ are both nonzero). So, the function $z = f(x, y)$ is differentiable on its domain. The differential dz is

$$\boxed{dz = -\frac{1}{x} dx + \frac{1}{y} dy}.$$

18. We see that the argument of the logarithm, $x^2 + y^2$, is always positive, except when $(x, y) = (0, 0)$. Therefore, f is defined everywhere in the xy -plane, except at the point $(0, 0)$. The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) = \frac{2x}{x^2 + y^2} \\ f_y(x, y) &= \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) = \frac{2y}{x^2 + y^2} \end{aligned}$$

We see that the denominator of $f_x(x, y)$ and $f_y(x, y)$, which is $x^2 + y^2$, is always nonzero when $(x, y) \neq (0, 0)$. So, f_x and f_y are continuous everywhere in the domain of f , which implies that the function $z = f(x, y)$ is differentiable on its domain. The differential dz is

$$\boxed{dz = \frac{2x}{x^2 + y^2} dx + \frac{2y}{x^2 + y^2} dy}.$$

19. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= e^{xy} \cdot \frac{\partial}{\partial x} (xy) = ye^{xy} \\ f_y(x, y) &= e^{xy} \cdot \frac{\partial}{\partial y} (xy) = xe^{xy} \end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$\boxed{dz = ye^{xy} dx + xe^{xy} dy}.$$

20. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned} f_x(x, y) &= e^{x^2+y} \cdot \frac{\partial}{\partial x} (x^2 + y) = 2xe^{x^2+y} \\ f_y(x, y) &= e^{x^2+y} \cdot \frac{\partial}{\partial y} (x^2 + y) = e^{x^2+y} \end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$\boxed{dz = 2xe^{x^2+y} dx + e^{x^2+y} dy}.$$

21. f is defined everywhere in the xy -plane. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= 2xy \\f_y(x, y) &= x^2 + e^{y^2} \cdot \frac{\partial}{\partial y}(y^2) = x^2 + 2ye^{y^2}\end{aligned}$$

Since f_x and f_y are continuous everywhere in the xy -plane, the function $z = f(x, y)$ is differentiable. The differential dz is

$$\boxed{dz = 2xy \, dx + (x^2 + 2ye^{y^2}) \, dy}.$$

22. Since the argument of the logarithm, x^2 , is always positive, except when $x = 0$, we see that $f(x, y)$ is defined everywhere in the xy -plane, except when $x = 0$. The partial derivatives of f are

$$\begin{aligned}f_x(x, y) &= y^3 - \frac{1}{x^2} \cdot \frac{\partial}{\partial x}(x^2) = y^3 - \frac{2x}{x^2} = y^3 - \frac{2}{x} \\f_y(x, y) &= 3xy^2\end{aligned}$$

Now, f_x and f_y are continuous everywhere in the domain of f (because, if $x \neq 0$, then the denominator that appears in $f_x(x, y)$ is nonzero). So, the function $z = f(x, y)$ is differentiable on its domain. The differential dz is

$$\boxed{dz = \left(y^3 - \frac{2}{x}\right) dx + 3xy^2 \, dy}.$$

23. The function f is defined everywhere in space. We begin by computing the partial derivatives of f :

$$f_x(x, y, z) = e^{yz} + ye^{xz}(z) + ze^{xy}(y) = e^{yz} + yze^{xz} + yze^{xy}$$

$$f_y(x, y, z) = xe^{yz}(z) + e^{xz} + ze^{xy}(x) = xze^{yz} + e^{xz} + xze^{xy}$$

$$f_z(x, y, z) = xe^{yz}(y) + ye^{xz}(x) + e^{xy} = xye^{yz} + xye^{xz} + e^{xy}.$$

Since the partial derivatives are continuous everywhere, we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$\boxed{dw = (e^{yz} + yze^{xz} + yze^{xy})dx + (xze^{yz} + e^{xz} + xze^{xy})dy + (xye^{yz} + xye^{xz} + e^{xy})dz}.$$

24. The function f is defined everywhere in space. We begin by computing the partial derivatives of f :

$$f_x(x, y, z) = 2xy + z^2$$

$$f_y(x, y, z) = x^2 + 2yz$$

$$f_z(x, y, z) = y^2 + (2z)x = y^2 + 2xz.$$

Since the partial derivatives are continuous everywhere, we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$\boxed{dw = (2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz}.$$

25. The function f is defined everywhere in space except the origin. We begin by computing the partial derivatives of f :

$$f_x(x, y, z) = \frac{1}{x^2 + y^2 + z^2}(2x) = \frac{2x}{x^2 + y^2 + z^2}$$

$$f_y(x, y, z) = \frac{1}{x^2 + y^2 + z^2}(2y) = \frac{2y}{x^2 + y^2 + z^2}$$

$$f_z(x, y, z) = \frac{1}{x^2 + y^2 + z^2}(2z) = \frac{2z}{x^2 + y^2 + z^2}.$$

Since the partial derivatives are continuous everywhere on the domain of the function f , we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$dw = \frac{2x}{x^2 + y^2 + z^2}dx + \frac{2y}{x^2 + y^2 + z^2}dy + \frac{2z}{x^2 + y^2 + z^2}dz.$$

26. The function f is defined where $xy > 0$, $xz > 0$, and $yz > 0$. More specifically, these three inequalities will hold when either $x, y, z > 0$ or $x, y, z < 0$. We begin by computing the partial derivatives of f :

$$f_x(x, y, z) = \frac{1}{xy} \cdot y + \frac{1}{xz} \cdot z = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$$

$$f_y(x, y, z) = \frac{1}{xy} \cdot x + \frac{1}{yz} \cdot z = \frac{1}{y} + \frac{1}{y} = \frac{2}{y}$$

$$f_z(x, y, z) = \frac{1}{xz} \cdot x + \frac{1}{yz} \cdot y = \frac{1}{z} + \frac{1}{z} = \frac{2}{z}.$$

Since the partial derivatives are continuous everywhere on the domain of the function f , we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$dw = \frac{2}{x}dx + \frac{2}{y}dy + \frac{2}{z}dz.$$

27. The function f is defined everywhere in space. We begin by computing the partial derivatives of f :

$$f_x(x, y, z) = yz$$

$$f_y(x, y, z) = xz$$

$$f_z(x, y, z) = xy.$$

Since the partial derivatives are continuous everywhere, we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$dw = (yz)dx + (xz)dy + (xy)dz.$$

28. The function f is defined everywhere in space except the plane $x + y + z = 0$. We begin by computing the partial derivatives of f :

$$\begin{aligned} f_x(x, y, z) &= \frac{(yz)(x + y + z) - xyz(1)}{(x + y + z)^2} = \frac{xyz + y^2z + yz^2 - xyz}{(x + y + z)^2} = \frac{y^2z + yz^2}{(x + y + z)^2} \\ f_y(x, y, z) &= \frac{(xz)(x + y + z) - xyz(1)}{(x + y + z)^2} = \frac{x^2z + xyz + xz^2 - xyz}{(x + y + z)^2} = \frac{x^2z + xz^2}{(x + y + z)^2} \\ f_z(x, y, z) &= \frac{(xy)(x + y + z) - xyz(1)}{(x + y + z)^2} = \frac{x^2y + xy^2 + xyz - xyz}{(x + y + z)^2} = \frac{x^2y + xy^2}{(x + y + z)^2}. \end{aligned}$$

Since the partial derivatives are continuous everywhere on the domain of the function f , we have

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

$$dw = \frac{y^2z + yz^2}{(x + y + z)^2} dx + \frac{x^2z + xz^2}{(x + y + z)^2} dy + \frac{x^2y + xy^2}{(x + y + z)^2} dz.$$

29. (a) We have that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, so for the function $z = f(x, y) = xy^2 - 2xy$ we have

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= (x + \Delta x)(y + \Delta y)^2 - 2(x + \Delta x)(y + \Delta y) - [xy^2 - 2xy] \\ &= (x + \Delta x)(y^2 + 2y\Delta y + (\Delta y)^2) - 2(xy + x\Delta y + y\Delta x + \Delta x\Delta y) - xy^2 + 2xy \\ &= xy^2 + 2xy\Delta y + x(\Delta y)^2 + y^2\Delta x + 2y\Delta x\Delta y + \Delta x(\Delta y)^2 - 2xy - 2x\Delta y - 2y\Delta x - 2\Delta x\Delta y - xy^2 + 2xy \\ &= 2xy\Delta y + x(\Delta y)^2 + y^2\Delta x + 2y\Delta x\Delta y + \Delta x(\Delta y)^2 - 2x\Delta y - 2y\Delta x - 2\Delta x\Delta y \\ &= (y^2 - 2y)\Delta x + (2xy - 2x)\Delta y + (2y\Delta y - 2\Delta y)\Delta x + (x\Delta y + \Delta x\Delta y)\Delta y \end{aligned}$$

$$\text{So, } \Delta z = (y^2 - 2y)\Delta x + (2xy - 2x)\Delta y + (2y\Delta y - 2\Delta y)\Delta x + (x\Delta y + \Delta x\Delta y)\Delta y.$$

- (b) Note that $f_x(x, y) = y^2 - 2y$ and $f_y(x, y) = 2xy - 2x$. So, we see from our manipulations from part (a) that we may choose η_1 and η_2 as follows:

$$\eta_1 = 2y\Delta y - 2\Delta y$$

$$\eta_2 = x\Delta y + \Delta x\Delta y,$$

which will yield $\Delta z = (y^2 - 2y)\Delta x + (2xy - 2x)\Delta y + \eta_1\Delta x + \eta_2\Delta y$. (Note that had we rearranged Δz differently in part (a), we may have had a different choice of η_1 and η_2 .)

- (c) We now show that $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_1 = 0$ and $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_2 = 0$. We have that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_1 = \lim_{\Delta y \rightarrow 0} [2y\Delta y - 2\Delta y] = 0$$

and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_2 = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} [x\Delta y + \Delta x\Delta y] = 0.$$

30. (a) We have that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, so for the function $z = f(x, y) = 3x^2 + y^2$ we have

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= 3(x + \Delta x)^2 + (y + \Delta y)^2 - [3x^2 + y^2] \end{aligned}$$

$$\begin{aligned}
&= 3(x^2 + 2x\Delta x + (\Delta x)^2) + y^2 + 2y\Delta y + (\Delta y)^2 - 3x^2 - y^2 \\
&= 3x^2 + 6x\Delta x + 3(\Delta x)^2 + y^2 + 2y\Delta y + (\Delta y)^2 - 3x^2 - y^2 \\
&= 6x\Delta x + 3(\Delta x)^2 + 2y\Delta y + (\Delta y)^2 \\
&= 6x\Delta x + 2y\Delta y + [3\Delta x]\Delta x + [\Delta y]\Delta y
\end{aligned}$$

So, $\boxed{\Delta z = 6x\Delta x + 2y\Delta y + [3\Delta x]\Delta x + [\Delta y]\Delta y}$.

- (b) Note that $f_x(x, y) = 6x$ and $f_y(x, y) = 2y$. So, we see from our manipulations from part (a) that we may choose η_1 and η_2 as follows:

$$\boxed{\eta_1 = 3\Delta x}$$

$$\boxed{\eta_2 = \Delta y},$$

which will yield $\Delta z = 6x\Delta x + 2y\Delta y + \eta_1\Delta x + \eta_2\Delta y$. (Note that in this case, there is no way to rearrange Δz differently in part (a), so these are our only correct choices of η_1 and η_2 .)

- (c) We now show that $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = 0$ and $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = 0$. We have that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = \lim_{\Delta x \rightarrow 0} 3\Delta x = 0$$

and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = \lim_{\Delta y \rightarrow 0} \Delta y = 0.$$

31. (a) We have that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, so for the function $z = f(x, y) = \frac{y}{x}$ we have

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\
&= \frac{y + \Delta y}{x + \Delta x} - \frac{y}{x} \\
&= \frac{(y + \Delta y)x - (x + \Delta x)y}{x(x + \Delta x)} \\
&= \frac{xy + x\Delta y - (xy + y\Delta x)}{x(x + \Delta x)} \\
&= \frac{xy + x\Delta y - xy - y\Delta x}{x(x + \Delta x)} \\
&= \frac{x\Delta y - y\Delta x}{x(x + \Delta x)} \\
&= \frac{x\Delta y - y\Delta x}{x(x + \Delta x)} - \frac{y}{x^2}\Delta x + \frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y - \frac{1}{x}\Delta y \\
&= -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \frac{x\Delta y - y\Delta x}{x(x + \Delta x)} + \frac{y}{x^2}\Delta x - \frac{1}{x}\Delta y \\
&= -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \frac{x\Delta y}{x(x + \Delta x)} + \frac{-y\Delta x}{x(x + \Delta x)} + \frac{y}{x^2}\Delta x - \frac{1}{x}\Delta y \\
&= -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \frac{\Delta y}{x + \Delta x} + \frac{-y\Delta x}{x(x + \Delta x)} + \frac{y}{x^2}\Delta x - \frac{1}{x}\Delta y \\
&= -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \frac{y}{x^2}\Delta x + \frac{-y\Delta x}{x(x + \Delta x)} - \frac{1}{x}\Delta y + \frac{\Delta y}{x + \Delta x} \\
&= -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \left[\frac{y}{x^2} - \frac{y}{x(x + \Delta x)} \right] \Delta x + \left[-\frac{1}{x} + \frac{1}{x + \Delta x} \right] \Delta y
\end{aligned}$$

So,
$$\Delta z = -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \left[\frac{y}{x^2} - \frac{y}{x(x+\Delta x)}\right]\Delta x + \left[-\frac{1}{x} + \frac{1}{x+\Delta x}\right]\Delta y.$$

- (b) Note that $f_x(x, y) = -\frac{y}{x^2}$ and $f_y(x, y) = \frac{1}{x}$. So, we see from our manipulations from part (a) that we may choose η_1 and η_2 as follows:

$$\eta_1 = \frac{y}{x^2} - \frac{y}{x(x+\Delta x)}$$

$$\eta_2 = -\frac{1}{x} + \frac{1}{x+\Delta x},$$

which will yield $\Delta z = -\frac{y}{x^2}\Delta x + \frac{1}{x}\Delta y + \eta_1\Delta x + \eta_2\Delta y$. (Note that had we rearranged Δz differently in part (a), we may have had a different choice of η_1 and η_2 .)

- (c) We now show that $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = 0$ and $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = 0$. We have that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = \lim_{\Delta x \rightarrow 0} \left[\frac{y}{x^2} - \frac{y}{x(x+\Delta x)} \right] = \frac{y}{x^2} - \frac{y}{x^2} = 0$$

and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = \lim_{\Delta x \rightarrow 0} \left[-\frac{1}{x} + \frac{1}{x+\Delta x} \right] = -\frac{1}{x} + \frac{1}{x} = 0.$$

32. (a) We have that $\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$, so for the function $z = f(x, y) = \frac{2x}{y}$ we have

$$\begin{aligned} \Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y) \\ &= \frac{2(x+\Delta x)}{y+\Delta y} - \frac{2x}{y} \\ &= \frac{2(x+\Delta x)y - 2x(y+\Delta y)}{y(y+\Delta y)} \\ &= \frac{2xy + 2y\Delta x - 2xy - 2x\Delta y}{y(y+\Delta y)} \\ &= \frac{2y\Delta x - 2x\Delta y}{y(y+\Delta y)} \\ &= \frac{2y\Delta x - 2x\Delta y}{y^2} - \frac{2x}{y^2}\Delta y + \frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x - \frac{2}{y}\Delta x \\ &= -\frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x + \frac{2y\Delta x - 2x\Delta y}{y^2} + \frac{2x}{y^2}\Delta y - \frac{2}{y}\Delta x \\ &= -\frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x + \frac{2y\Delta x}{y^2} - \frac{2x\Delta y}{y^2} + \frac{2x}{y^2}\Delta y - \frac{2}{y}\Delta x \\ &= -\frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x + \frac{2\Delta x}{y} - \frac{2x\Delta y}{y^2} + \frac{2x}{y^2}\Delta y - \frac{2}{y}\Delta x \\ &= -\frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x - \frac{2}{y}\Delta x + \frac{2\Delta x}{y} + \frac{2x}{y^2}\Delta y - \frac{2x\Delta y}{y^2} \\ &= -\frac{2x}{y^2}\Delta y + \frac{2}{y}\Delta x + \left[\frac{-2}{y} + \frac{2}{y+\Delta y} \right] \Delta x + \left[\frac{2x}{y^2} - \frac{2x}{y(y+\Delta y)} \right] \Delta y \end{aligned}$$

So,
$$\Delta z = \frac{2}{y}\Delta x - \frac{2x}{y^2}\Delta y + \left[\frac{-2}{y} + \frac{2}{y+\Delta y} \right] \Delta x + \left[\frac{2x}{y^2} - \frac{2x}{y(y+\Delta y)} \right] \Delta y.$$

- (b) Note that $f_x(x, y) = -\frac{2x}{y^2}$ and $f_y(x, y) = \frac{2}{y}$. So, we see from our manipulations from part (a) that we may choose η_1 and η_2 as follows:

$$\eta_1 = \frac{-2}{y} + \frac{2}{y + \Delta y}$$

$$\eta_2 = \frac{2x}{y^2} - \frac{2x}{y(y + \Delta y)},$$

which will yield $\Delta z = \frac{2}{y}\Delta x - \frac{2x}{y^2}\Delta y + \eta_1\Delta x + \eta_2\Delta y$. (Note that had we rearranged Δz differently in part (a), we may have had a different choice of η_1 and η_2 .)

- (c) We now show that $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_1 = 0$ and $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_2 = 0$. We have that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_1 = \lim_{\Delta y \rightarrow 0} \left[\frac{-2}{y} + \frac{2}{y + \Delta y} \right] = -\frac{2}{y} + \frac{2}{y} = 0$$

and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \eta_2 = \lim_{\Delta y \rightarrow 0} \left[\frac{2x}{y^2} - \frac{2x}{y(y + \Delta y)} \right] = \frac{2x}{y^2} - \frac{2x}{y^2} = 0.$$

33. (a) We compute the differential dz for the function $z = f(x, y) = x^2 + y^2$ by first finding its partial derivatives at the point $(x_0, y_0) = (1, 3)$:

$$\begin{aligned} f_x(x, y) &= 2x \\ f_x(1, 3) &= 2(1) = 2 \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= 2y \\ f_y(1, 3) &= 2(3) = 6 \end{aligned}$$

Also, $dx = \Delta x = 1.1 - 1 = 0.1$ and $dy = \Delta y = 3.2 - 3 = 0.2$. The differential is given by

$$\begin{aligned} dz &= f_x(1, 3)dx + f_y(1, 3)dy \\ &= 2(0.1) + 6(0.2) \\ &= 1.4. \end{aligned}$$

So, the approximate change in z is $\Delta z \approx dz = 1.4$.

- (b) We compare the approximated change, 1.4, with the actual change, computed in problem 7, which is $\Delta z = 1.45$.
34. (a) We compute the differential dz for the function $z = f(x, y) = 2x^2 + xy - y^2$ by first finding its partial derivatives at the point $(x_0, y_0) = (2, -1)$:

$$\begin{aligned} f_x(x, y) &= 4x + y \\ f_x(2, -1) &= 4(2) - 1 = 7 \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= x - 2y \\ f_y(2, -1) &= 2 - 2(-1) = 4 \end{aligned}$$

Also, $dx = \Delta x = 2.1 - 2 = 0.1$ and $dy = \Delta y = -1.1 + 1 = -0.1$. The differential is given by

$$\begin{aligned} dz &= f_x(2, -1)dx + f_y(2, -1)dy \\ &= 7(0.1) + 4(-0.1) \\ &= 0.3. \end{aligned}$$

So, the approximate change in z is $\boxed{dz = 0.3}$.

- (b) We compare our approximated change, 0.3 to the actual change, computed in problem 8, which is $\boxed{\Delta z = 0.3}$.

35. (a) We compute the differential dz for the function $z = f(x, y) = e^x \ln(xy)$ by first finding its partial derivatives at the point $(x_0, y_0) = (1, 2)$:

$$f_x(x, y) = e^x \ln(xy) + e^x \left(\frac{1}{xy} \right) (y) = e^x \ln(xy) + \frac{e^x}{x}$$

$$f_x(1, 2) = e^1 \ln(1 \cdot 2) + \frac{e^1}{1} = e \ln 2 + e$$

$$f_y(x, y) = e^x \left(\frac{1}{xy} \right) (x) = \frac{e^x}{y}$$

$$f_y(1, 2) = \frac{e^1}{2} = \frac{e}{2}$$

Also, $dx = \Delta x = 0.9 - 1 = -0.1$ and $dy = \Delta y = 2.1 - 2 = 0.1$. The differential is given by

$$\begin{aligned} dz &= f_x(1, 2)dx + f_y(1, 2)dy \\ &= (e \ln 2 + e)(-0.1) + \frac{e}{2}(0.1) = -0.1e \ln 2 - 0.1e + 0.05e \\ &= -0.1e \ln 2 - .05e \approx -0.324. \end{aligned}$$

So, the approximate change in z is $\boxed{dz \approx -0.324}$.

- (b) We compare the approximated change, -0.324 , with the actual change, computed in problem 9, which is $\boxed{\Delta z \approx -0.318}$.

36. (a) We compute the differential dz for the function $z = f(x, y) = \frac{xy}{x+y}$ by first finding its partial derivatives at the point $(x_0, y_0) = (-1, 2)$:

$$f_x(x, y) = \frac{y(x+y) - (xy)(1)}{(x+y)^2} = \frac{xy + y^2 - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$f_x(-1, 2) = \frac{2^2}{(-1+2)^2} = 4$$

$$f_y(x, y) = \frac{x(x+y) - (xy)(1)}{(x+y)^2} = \frac{x^2 + xy - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

$$f_y(-1, 2) = \frac{(-1)^2}{(-1+2)^2} = 1$$

Also, $dx = \Delta x = -0.9 + 1 = 0.1$ and $dy = \Delta y = 1.9 - 2 = -0.1$. The differential is given by

$$\begin{aligned} dz &= f_x(-1, 2)dx + f_y(-1, 2)dy \\ &= (4)(0.1) + (1)(-0.1) \\ &= 0.3. \end{aligned}$$

So, the approximate change in z is $\boxed{dz = 0.3}$.

- (b) We compare the approximated change, 0.3, with the actual change, computed in problem 10, which is $\boxed{\Delta z = 0.29}$.

Applications and Extensions

37. We approximate the increase in the area of the triangle using the differential dA , where A is the area of the triangle. The formula for the area of a triangle of base b and height h is $A = \frac{1}{2}bh$. We compute the partial derivatives of A with respect to b and h at the point $(b, h) = (2, 5)$ in order to find the differential:

$$A_b(b, h) = \frac{1}{2}h$$

$$A_b(2, 5) = \frac{1}{2}(5) = \frac{5}{2}$$

$$A_h(b, h) = \frac{1}{2}b$$

$$A_h(2, 5) = \frac{1}{2}(2) = 1.$$

Also, $db = \Delta b = 2.05 - 2 = 0.05$ and $dh = \Delta h = 5.1 - 5 = 0.1$. The differential dA is then given by:

$$\begin{aligned} dA &= A_b(2, 5)db + A_h(2, 5)dh \\ &= \frac{5}{2}(0.05) + 1(0.1) \\ &= 0.225. \end{aligned}$$

So, the area of the triangle increases by $\boxed{\approx 0.225 \text{ cm}^2}$.

38. We approximate the change in the area of the triangle using the differential dA , where A is the area of the triangle. The formula for the area of a triangle of base b and height h is $A = \frac{1}{2}bh$. We compute the partial derivatives of A with respect to b and h at the point $(b, h) = (5, 10)$ in order to find the differential:

$$A_b(b, h) = \frac{1}{2}h$$

$$A_b(5, 10) = \frac{1}{2}(10) = 5$$

$$A_h(b, h) = \frac{1}{2}b$$

$$A_h(5, 10) = \frac{1}{2}(5) = \frac{5}{2}.$$

Also, $db = \Delta b = 5.1 - 5 = 0.1$ and $dh = \Delta h = 9.8 - 10 = -0.2$. The differential dA is then given by:

$$\begin{aligned} dA &= A_b(5, 10)db + A_h(5, 10)dh \\ &= 5(0.1) + \frac{5}{2}(-0.2) \\ &= 0. \end{aligned}$$

So, the area of the triangle changes by $\boxed{\approx 0 \text{ cm}^2}$, or essentially doesn't (or almost doesn't) change at all.

39. (a) We approximate the change in the volume of the right circular cylinder using the differential dV , where V is the volume of the cylinder. The formula for the volume of a right circular cylinder of

radius r and height h is $V = \pi r^2 h$. We compute the partial derivatives of V with respect to r and h at the point $(r, h) = (0.5, 2)$ in order to find the differential:

$$V_r(r, h) = 2\pi r h$$

$$V_r(0.5, 2) = 2\pi(0.5)(2) = 2\pi$$

$$V_h(r, h) = \pi r^2$$

$$V_h(0.5, 2) = \pi(0.5)^2 = \frac{\pi}{4}.$$

Also, $dr = \Delta r = 0.49 - 0.5 = -0.01$ and $dh = \Delta h = 2.1 - 2 = 0.1$. The differential dV is then given by:

$$\begin{aligned} dV &= V_r(0.5, 2)dr + V_h(0.5, 2)dh \\ &= 2\pi(-0.01) + \frac{\pi}{4}(0.1) \\ &= 0.005\pi \approx 0.016. \end{aligned}$$

So, the volume of the cylinder changes by $\boxed{\approx 0.016 \text{ cm}^3}$, which is a positive change, so the volume increases.

- (b) We approximate the change in the surface area of the right circular cylinder using the differential dA , where A is the surface area of the cylinder. The formula for the surface area of a closed right circular cylinder of radius r and height h is $V = 2\pi r h + 2\pi r^2$ (the first term is for the cylinder itself and the second for the two circles on top and bottom). We compute the partial derivatives of A with respect to r and h at the point $(r, h) = (0.5, 2)$ in order to find the differential:

$$A_r(r, h) = 2\pi h + 4\pi r$$

$$A_r(0.5, 2) = 2\pi(2) + 4\pi(0.5) = 6\pi$$

$$A_h(r, h) = 2\pi r$$

$$A_h(0.5, 2) = 2\pi(0.5) = \pi.$$

Also, $dr = \Delta r = 0.49 - 0.5 = -0.01$ and $dh = \Delta h = 2.1 - 2 = 0.1$, same as in part (a). The differential dA is then given by:

$$\begin{aligned} dA &= A_r(0.5, 2)dr + A_h(0.5, 2)dh \\ &= 6\pi(-0.01) + \pi(0.1) \\ &= 0.04\pi \approx 0.126. \end{aligned}$$

So, the surface area of the cylinder changes by $\boxed{\approx 0.126 \text{ cm}^2}$, which is a positive change, so the surface area increases.

40. We approximate the maximum variation of the total resistance R by using the differential dR . We first solve for R :

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_2 + R_1}$$

We will need the partial derivatives R_{R_1} and R_{R_2} :

$$R_{R_1} = \frac{R_2(R_2 + R_1) - R_1 R_2(1)}{(R_2 + R_1)^2} = \frac{R_2^2 + R_1 R_2 - R_1 R_2}{(R_2 + R_1)^2} = \frac{R_2^2}{(R_2 + R_1)^2}$$

and

$$R_{R_2} = \frac{R_1(R_2 + R_1) - R_1 R_2(1)}{(R_2 + R_1)^2} = \frac{R_1 R_2 + R_1^2 - R_1 R_2}{(R_2 + R_1)^2} = \frac{R_1^2}{(R_2 + R_1)^2}$$

Note also that $\frac{|\Delta R_1|}{R_1} = \frac{|dR_1|}{R_1} = 0.012$ and $\frac{|\Delta R_2|}{R_2} = \frac{|dR_2|}{R_2} = 0.01$

We have that $\frac{|\Delta R|}{R} \approx \frac{|dR|}{R} = \frac{\left| \frac{R_2^2}{(R_2 + R_1)^2} dR_1 + \frac{R_1^2}{(R_2 + R_1)^2} dR_2 \right|}{\frac{R_1 R_2}{R_2 + R_1}}$

$$\leq \frac{\left| \frac{R_2^2}{(R_2 + R_1)^2} dR_1 \right|}{\frac{R_1 R_2}{R_2 + R_1}} + \frac{\left| \frac{R_1^2}{(R_2 + R_1)^2} dR_2 \right|}{\frac{R_1 R_2}{R_2 + R_1}}$$

$$= \left| \frac{R_2 dR_1}{R_1(R_2 + R_1)} \right| + \left| \frac{R_1 dR_2}{R_2(R_2 + R_1)} \right|$$

$$\leq \left| \frac{dR_1}{R_1} \right| + \left| \frac{dR_2}{R_2} \right| \quad \text{since } 0 \leq \frac{R_2}{R_2 + R_1} \leq 1 \text{ and } 0 \leq \frac{R_1}{R_2 + R_1} \leq 1, \text{ since } R_1, R_2 \geq 0 \text{ always}$$

$$= 0.012 + 0.01 = 0.022. \text{ So, the maximum variation of the total resistance } R \text{ is } \boxed{\approx 2.2\%}$$

41. We find the differential ds .

$$\begin{aligned} \frac{\partial s}{\partial a} &= \frac{1(a-w) - a(1)}{(a-w)^2} = \frac{-w}{(a-w)^2} \\ \frac{\partial s}{\partial w} &= a[-(a-w)^{-2}(-1)] = \frac{a}{(a-w)^2} \\ ds &= \frac{\partial s}{\partial a} da + \frac{\partial s}{\partial w} dw = \frac{-w}{(a-w)^2} da + \frac{a}{(a-w)^2} dw \end{aligned}$$

We are told that a is subject to a possible error of 1%, and w is subject to a possible error of 2%. This means that

$$\frac{|\Delta a|}{a} = .01 \quad \text{and} \quad \frac{|\Delta w|}{w} = .02$$

We are also told that $s = 6$ and $w = 5$. Therefore, the relative error in the specific gravity s is:

$$\begin{aligned} \frac{|\Delta s|}{s} &\approx \frac{|ds|}{s} = \left| \frac{-w}{s(a-w)^2} da + \frac{a}{s(a-w)^2} dw \right| = \left| \frac{-w}{a(a-w)} da + \frac{1}{a-w} dw \right| \\ &= \left| \frac{-w}{a-w} \cdot \frac{da}{a} + \frac{w}{a-w} \cdot \frac{dw}{w} \right| = \left| \frac{-w}{a-w} \cdot \frac{\Delta a}{a} + \frac{w}{a-w} \cdot \frac{\Delta w}{w} \right| \\ &\leq \left| \frac{-w}{a-w} \right| \cdot \frac{|\Delta a|}{a} + \left| \frac{w}{a-w} \right| \cdot \frac{|\Delta w|}{w} = \frac{5}{1} \cdot (.01) + \frac{5}{1} \cdot (.02) = .15 \end{aligned}$$

So, the approximate maximum variation in the specific gravity s is $\boxed{\approx 15\%}$.

42. We can approximate the maximum variation in the volume of the silo by describing a function representing the volume V of the silo, and computing the differential dV . To find this function, we split the silo into two pieces: the bottom, cylindrical piece and the top, hemispherical piece. If the cylinder has radius r and height h , its volume is $V_{cyl} = \pi r^2 h$. The hemispherical piece must have the same radius as the cylinder, r , and the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$, so the half-spherical portion has volume $V_{sph} = \frac{1}{2}[\frac{4}{3}\pi r^3] = \frac{2}{3}\pi r^3$. The volume of the silo is then the sum of these two: $V = \pi r^2 h + \frac{2}{3}\pi r^3$. The maximum relative error in the radius is $\frac{|\Delta r|}{r} = \frac{|dr|}{r} = 0.01$, as is the maximum relative error in the height, $\frac{|\Delta h|}{h} = \frac{|dh|}{h}$. The maximum relative error in the volume is $\frac{|\Delta V|}{V} \approx \frac{|dV|}{V}$.

We have

$$\begin{aligned} \frac{|\Delta V|}{V} &\approx \frac{|dV|}{V} = \frac{|(2\pi r h + 2\pi r^2)dr + \pi r^2 dh|}{\pi r^2 h + \frac{2}{3}\pi r^3} \\ &\leq \frac{|2\pi r h dr|}{\pi r^2 h + \frac{2}{3}\pi r^3} + \frac{|2\pi r^2 dr|}{\pi r^2 h + \frac{2}{3}\pi r^3} + \frac{|\pi r^2 dh|}{\pi r^2 h + \frac{2}{3}\pi r^3} \\ &\leq \frac{|2\pi r h dr|}{\pi r^2 h} + \frac{|2\pi r^2 dr|}{\frac{2}{3}\pi r^3} + \frac{|\pi r^2 dh|}{\pi r^2 h} \quad \text{since we decreased the denominators} \\ &= 2\frac{|dr|}{r} + 3\frac{|dr|}{r} + \frac{|dh|}{h} \\ &= 2(0.01) + 3(0.01) + 0.01 = 0.06 \end{aligned}$$

So, the maximum variation in the volume of the silo is 0.06, or $\boxed{\approx 6\%}$.

43. We find the differential $d\mu$.

$$\begin{aligned} \frac{\partial \mu}{\partial i} &= \frac{\cos i}{\sin r} \\ \frac{\partial \mu}{\partial r} &= \sin i \cdot \frac{\partial}{\partial r} ((\sin r)^{-1}) = \sin i \cdot (-1) \cdot (\sin r)^{-2} \cdot \frac{\partial}{\partial r} (\sin r) = -\frac{\sin i \cos r}{\sin^2 r} \\ d\mu &= \frac{\partial \mu}{\partial i} di + \frac{\partial \mu}{\partial r} dr = \frac{\cos i}{\sin r} di - \frac{\sin i \cos r}{\sin^2 r} dr \end{aligned}$$

We are told that each measurement is subject to a possible error of 2%. This means that

$$\frac{|\Delta i|}{i} = .02 \quad \text{and} \quad \frac{|\Delta r|}{r} = .02$$

We are also told that $i = 30^\circ$ and $r = 60^\circ$. Converting to radians, we have $i = \frac{\pi}{6}$ and $r = \frac{\pi}{3}$. Therefore,

the relative error in the index of refraction μ is

$$\begin{aligned}
\frac{|\Delta\mu|}{\mu} &\approx \frac{|d\mu|}{\mu} = \frac{\left| \frac{\cos i}{\sin r} di - \frac{\sin i \cos r}{\sin^2 r} dr \right|}{\frac{\sin i}{\sin r}} = \left| \frac{\cos i}{\sin i} di - \frac{\cos r}{\sin r} dr \right| \\
&= \left| \frac{i \cos i}{\sin i} \cdot \frac{di}{i} - \frac{r \cos r}{\sin r} \cdot \frac{dr}{r} \right| = \left| \frac{i \cos i}{\sin i} \cdot \frac{\Delta i}{i} - \frac{r \cos r}{\sin r} \cdot \frac{\Delta r}{r} \right| \\
&\leq \frac{i \cos i}{\sin i} \cdot \frac{|\Delta i|}{i} + \frac{r \cos r}{\sin r} \cdot \frac{|\Delta r|}{r} = \frac{\frac{\pi}{6} \cdot \cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} \cdot (.02) + \frac{\frac{\pi}{3} \cdot \cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} \cdot (.02) \\
&= \frac{\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2}}{\frac{1}{2}} \cdot (.02) + \frac{\frac{\pi}{3} \cdot \frac{1}{2}}{\frac{\sqrt{3}}{2}} \cdot (.02) \approx .0302
\end{aligned}$$

So the approximate maximum relative error for μ is $\boxed{3.02\%}$.

44. For the ideal gas law, we recall (see problem 91 of section 2.4, on page 183) that T must be the *absolute* temperature, so we must convert the degrees Celsius into Kelvin. This is done by adding 273 to the degrees Celsius. So, we have $T = 32 + 273 = 305$. We write $P = \frac{kT}{V} = kTV^{-1}$ and we find the differential dP .

$$\begin{aligned}
\frac{\partial P}{\partial V} &= kT \cdot (-1) \cdot V^{-2} = -\frac{kT}{V^2} \\
\frac{\partial P}{\partial T} &= \frac{k}{V} \\
dP &= \frac{\partial P}{\partial V} dV + \frac{\partial P}{\partial T} dT = -\frac{kT}{V^2} dV + \frac{k}{V} dT
\end{aligned}$$

Now we know that

$$P = 0.1, V = 12, T = 305; \quad \Delta V = 3, \Delta T = -3.$$

We now have the necessary information to compute k :

$$k = \frac{PV}{T} = \frac{(0.1)(12)}{305} \approx 0.0039344.$$

Therefore,

$$\begin{aligned}
\Delta P &\approx dP = -\frac{kT}{V^2} dV + \frac{k}{V} dT = -\frac{kT\Delta V}{V^2} + \frac{k\Delta T}{V} = \frac{k(V\Delta T - T\Delta V)}{V^2} \\
&= \frac{(0.0039344)((12)(-3) - (305)(3))}{(12)^2} \approx -0.026
\end{aligned}$$

So, $\boxed{\Delta P \approx -0.026 \text{ g/mm}^2}$.

45. (a) We know that the volume of a spherical object with radius r is $V = \frac{4\pi r^3}{3}$. Substituting $r = \frac{d}{2}$, we get $V = \frac{\pi d^3}{6}$. The average density of the object is mass per unit volume, so $\rho = \frac{m}{V} = \frac{6m}{\pi d^3} = \frac{6}{\pi}md^{-3}$. Now we need to find the differential $d\rho$.

$$\begin{aligned}
\frac{\partial \rho}{\partial m} &= \frac{6}{\pi d^3} \\
\frac{\partial \rho}{\partial d} &= \frac{6}{\pi} \cdot m(-3d^{-4}) = -\frac{18m}{\pi d^4} \\
d\rho &= \frac{\partial \rho}{\partial m} dm + \frac{\partial \rho}{\partial d} dd = \frac{6}{\pi d^3} dm - \frac{18m}{\pi d^4} dd
\end{aligned}$$

We also know that

$$m = 24, d = 1.5; \quad |\Delta m| = 0.24, |\Delta d| = 0.075.$$

Therefore, the approximate maximum error in the average density ρ is

$$\begin{aligned} |\Delta \rho| &\approx |d\rho| = \left| \frac{6}{\pi d^3} dm - \frac{18m}{\pi d^4} dd \right| = \left| \frac{6\Delta m}{\pi d^3} - \frac{18m\Delta d}{\pi d^4} \right| \\ &\leq \left| \frac{6\Delta m}{\pi d^3} \right| + \left| \frac{18m\Delta d}{\pi d^4} \right| = \frac{6}{\pi d^3} |\Delta m| + \frac{18m}{\pi d^4} |\Delta d| \\ &= \frac{6}{(1.5)^3 \pi} (0.24) + \frac{(18)(24)}{(1.5)^4 \pi} (0.075) \approx 2.173 \end{aligned}$$

So the approximate maximum error for ρ is $\boxed{2.173 \text{ g/cm}^3}$.

(b) We compute the density ρ for the given values, $m = 24.0$ and $d = 1.5$:

$$\rho = \frac{6m}{\pi d^3} = \frac{(6)(24.0)}{(1.5)^3 \pi} \approx 13.581 \text{ g/cm}^3,$$

and therefore

$$\frac{|\Delta \rho|}{\rho} = \frac{|d\rho|}{\rho} \leq \frac{2.173}{13.581} \approx 0.16000.$$

So the approximate maximum percentage error in the computed density is $\boxed{16\%}$.

46. (a) We find the differential dA .

$$\begin{aligned} \frac{\partial A}{\partial b} &= \frac{1}{2} c \sin \alpha \\ \frac{\partial A}{\partial c} &= \frac{1}{2} b \sin \alpha \\ \frac{\partial A}{\partial \alpha} &= \frac{1}{2} bc \cos \alpha \\ dA &= \frac{\partial A}{\partial b} db + \frac{\partial A}{\partial c} dc + \frac{\partial A}{\partial \alpha} d\alpha = \frac{1}{2} (c \sin \alpha db + b \sin \alpha dc + bc \cos \alpha d\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dA}{A} &= \frac{1}{2A} (c \sin \alpha db + b \sin \alpha dc + bc \cos \alpha d\alpha) \\ &= \frac{1}{bc \sin \alpha} (c \sin \alpha db + b \sin \alpha dc + bc \cos \alpha d\alpha) \\ &= \frac{db}{b} + \frac{dc}{c} + \frac{\cos \alpha d\alpha}{\sin \alpha} \\ &= \frac{db}{b} + \frac{dc}{c} + \cot \alpha d\alpha. \end{aligned}$$

(b) We are told that

$$\frac{|\Delta b|}{b} = 0.02, \frac{|\Delta c|}{c} = 0.02, \frac{|\Delta \alpha|}{\alpha} = 0.03, \alpha = \frac{\pi}{4}, \text{ and so } \cot \alpha = 1.$$

Therefore,

$$\begin{aligned} \frac{|\Delta A|}{A} &\approx \frac{|dA|}{A} = \left| \frac{db}{b} + \frac{dc}{c} + \cot \alpha d\alpha \right| \\ &= \left| \frac{\Delta b}{b} + \frac{\Delta c}{c} + \cot \alpha \Delta \alpha \right| \leq \frac{|\Delta b|}{b} + \frac{|\Delta c|}{c} + \cot \alpha |\Delta \alpha| \\ &= \frac{|\Delta b|}{b} + \frac{|\Delta c|}{c} + \alpha \cot \alpha \frac{|\Delta \alpha|}{\alpha} = 0.02 + 0.02 + \left(\frac{\pi}{4} \right) \cdot 1 \cdot (0.03) \approx 0.06356. \end{aligned}$$

So, the approximate maximum relative variation in the computation of A is 6.356%.

47. (a)

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \left((x^2 + y^2 - 2x - 6y + 10)^{1/2} \right) \\ &= \frac{1}{2} (x^2 + y^2 - 2x - 6y + 10)^{-1/2} \cdot \frac{\partial}{\partial x} (x^2 + y^2 - 2x - 6y + 10) \\ &= \frac{x - 1}{\sqrt{x^2 + y^2 - 2x - 6y + 10}} \end{aligned}$$

$$f_x(x, y) = \frac{x - 1}{\sqrt{x^2 + y^2 - 2x - 6y + 10}}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y} \left((x^2 + y^2 - 2x - 6y + 10)^{1/2} \right) \\ &= \frac{1}{2} (x^2 + y^2 - 2x - 6y + 10)^{-1/2} \cdot \frac{\partial}{\partial y} (x^2 + y^2 - 2x - 6y + 10) \\ &= \frac{y - 3}{\sqrt{x^2 + y^2 - 2x - 6y + 10}} \end{aligned}$$

$$f_y(x, y) = \frac{y - 3}{\sqrt{x^2 + y^2 - 2x - 6y + 10}}$$

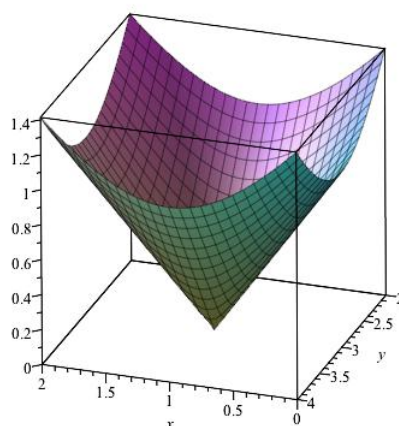
- (b) We look at the expression $x^2 + y^2 - 2x - 6y + 10$, which appears under the radical in the denominator. The partial derivatives will be continuous at all points (x, y) for which this expression is positive. We complete the square:

$$x^2 + y^2 - 2x - 6y + 10 = x^2 - 2x + 1 + y^2 - 6y + 9 = (x - 1)^2 + (y - 3)^2$$

This is a sum of two squares, so it is always positive, except when $x - 1$ and $y - 3$ are both zero. So,

the partial derivatives are continuous everywhere in the xy -plane, except at the point $(1, 3)$.

- (c) The graph is the following:



- (d) The partial derivatives are discontinuous at the point $(1, 3)$ because that is where the vertex of the graph (a circular cone) is located.

48. We compute the differentials dx and dy :

$$\begin{aligned}\frac{\partial x}{\partial r} &= \cos \theta \\ \frac{\partial x}{\partial \theta} &= -r \sin \theta \\ \frac{\partial y}{\partial r} &= \sin \theta \\ \frac{\partial y}{\partial \theta} &= r \cos \theta\end{aligned}$$

$$\begin{aligned}dx &= \cos \theta dr - r \sin \theta d\theta \\ dy &= \sin \theta dr + r \cos \theta d\theta\end{aligned}$$

Therefore,

$$\begin{aligned}x dy - y dx &= x \sin \theta dr + xr \cos \theta d\theta - y \cos \theta dr + yr \sin \theta d\theta \\ &= x \sin \theta dr + x^2 d\theta - y \cos \theta dr + y^2 d\theta \\ &= x \sin \theta dr - y \cos \theta dr + (x^2 + y^2) d\theta \\ &= x \cdot \frac{y}{r} dr - y \cdot \frac{x}{r} dr + r^2 d\theta = \left(\frac{xy}{r} - \frac{xy}{r} \right) dr + r^2 d\theta = r^2 d\theta.\end{aligned}$$

49.

$$\begin{aligned}f_x(0, 0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{2 \cdot \Delta x \cdot 0}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \\ f_y(0, 0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{\frac{2 \cdot 0 \cdot \Delta y}{0 + (\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0\end{aligned}$$

Therefore, f has partial derivatives at $(0, 0)$. To determine whether f is continuous at $(0, 0)$, we need to determine whether $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exists and equals $f(0, 0)$. We investigate the limit using two curves that contain $(0, 0)$: the x -axis ($y = 0$) and the line $y = x$. Using $y = 0$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{2x \cdot 0}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

Using $y = x$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{2x \cdot x}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = \lim_{x \rightarrow 0} 1 = 1.$$

Since two different answers are obtained, the limit $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist. Therefore, f cannot be continuous at $(0, 0)$. But we know that if f is differentiable at $(0, 0)$, then f must be continuous at $(0, 0)$. (“Differentiability is Sufficient for Continuity”, page 845) Since f is not continuous at $(0, 0)$, we conclude that f is **not** differentiable at $(0, 0)$.

50.

$$\begin{aligned}
f_x(0,0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x \cdot 0 \cdot (1+0)}{(\Delta x)^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 = 0 \\
f_y(0,0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, 0 + \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{0 \cdot \Delta y \cdot (1+(\Delta y)^2)}{0+(\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \rightarrow 0} 0 = 0
\end{aligned}$$

Therefore, f has partial derivatives at $(0,0)$. To determine whether f is continuous at $(0,0)$, we need to determine whether $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ exists and equals $f(0,0)$. We investigate the limit using two curves that contain $(0,0)$: the x -axis ($y = 0$) and the line $y = x$. Using $y = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot 0 \cdot (1+0)}{x^2 + 0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

Using $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot x \cdot (1+x^2)}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{x^2 + x^4}{2x^2} = \lim_{x \rightarrow 0} \frac{x^2(1+x^2)}{2x^2} = \lim_{x \rightarrow 0} \frac{1+x^2}{2} = \frac{1}{2}.$$

Since two different answers are obtained, the limit $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist. Therefore, f cannot be continuous at $(0,0)$. But we know that if f is differentiable at $(0,0)$, then f must be continuous at $(0,0)$. (“Differentiability is Sufficient for Continuity”, page 845) Since f is not continuous at $(0,0)$, we conclude that f is **not** differentiable at $(0,0)$.

51. We compute f_x and f_y at $(1,1)$ by using the definition of a partial derivative:

$$\begin{aligned}
f_x(1,1) &= \lim_{\Delta x \rightarrow 0} \frac{f(1 + \Delta x, 1) - f(1,1)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{(1+\Delta x)(1)-1}{(1+\Delta x)^2 + 1^2 - 2} - \frac{1}{2}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{1+\Delta x-1}{1+2\Delta x+(\Delta x)^2-1} - \frac{1}{2}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{2\Delta x+(\Delta x)^2} - \frac{1}{2}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{\Delta x(2+\Delta x)} - \frac{1}{2}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{2+\Delta x} - \frac{1}{2}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{2-(2+\Delta x)}{2(2+\Delta x)}}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{-\Delta x}{2(2+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{2(2+\Delta x)} \cdot \frac{1}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{-1}{2(2+\Delta x)} \\
&= -\frac{1}{4}
\end{aligned}$$

and

$$\begin{aligned}
f_y(1,1) &= \lim_{\Delta y \rightarrow 0} \frac{f(1,1+\Delta y) - f(1,1)}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{(1)(1+\Delta y)-1}{1^2+(1+\Delta y)^2-2} - \frac{1}{2}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{1+\Delta y-1}{1+1+2\Delta y+(\Delta y)^2-2} - \frac{1}{2}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{\Delta y}{2\Delta y+(\Delta y)^2} - \frac{1}{2}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{\Delta y}{\Delta y(2+\Delta y)} - \frac{1}{2}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{1}{2+\Delta y} - \frac{1}{2}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{2-(2+\Delta y)}{2(2+\Delta y)}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{-\Delta y}{2(2+\Delta y)}}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{-\Delta y}{2(2+\Delta y)} \cdot \frac{1}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{-1}{2(2+\Delta y)} \\
&= -\frac{1}{4}
\end{aligned}$$

So $f_x(1,1)$ and $f_y(1,1)$ both exist (and equal $-\frac{1}{4}$). But, we can see that f is in fact not continuous and so not differentiable, since if a function is differentiable it is continuous. We see this by showing that $\lim_{(x,y) \rightarrow (1,1)} f(x,y)$ does not exist. We do so by computing the limit along two different curves containing $(1,1)$ and showing that we obtain different values:

Along $y = 1$:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (1,1)} \frac{xy-1}{x^2+y^2-2} &= \lim_{x \rightarrow 1} \frac{x(1)-1}{x^2+1^2-2} = \lim_{x \rightarrow 1} \frac{x-1}{x^2-1} \\
&= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+1)} \\
&= \lim_{x \rightarrow 1} \frac{1}{x+1} \\
&= \frac{1}{2}
\end{aligned}$$

Along $y = 2 - x$:

$$\begin{aligned}
\lim_{(x,y) \rightarrow (1,1)} \frac{xy-1}{x^2+y^2-2} &= \lim_{x \rightarrow 1} \frac{x(2-x)-1}{x^2+(2-x)^2-2} = \lim_{x \rightarrow 1} \frac{2x-x^2-1}{x^2+4-4x+x^2-2} \\
&= \lim_{x \rightarrow 1} \frac{-(x^2-2x+1)}{2x^2-4x+2} \\
&= \lim_{x \rightarrow 1} \frac{-(x^2-2x+1)}{2(x^2-2x+1)} \\
&= \lim_{x \rightarrow 1} \frac{-1}{2} \\
&= -\frac{1}{2}
\end{aligned}$$

Since these two limits are not equal, the limit does not exist, which means that f cannot be continuous at $(1,1)$ and so is not differentiable at $(1,1)$.

52. We compute f_x and f_y at $(0,0)$ by using the definition of a partial derivative:

$$\begin{aligned}
f_x(0,0) &= \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^2(0)^2}{(\Delta x)^4+0^4} - 0}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} 0 = 0
\end{aligned}$$

and

$$\begin{aligned}
f_y(0,0) &= \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{\frac{(0)^2(\Delta y)^2}{0^4+(\Delta y)^4} - 0}{\Delta y} \\
&= \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y}
\end{aligned}$$

$$= \lim_{\Delta y \rightarrow 0} 0 = 0$$

So, both $f_x(0,0)$ and $f_y(0,0)$ exist (and equal 0). But, we can see that f is in fact not continuous and so not differentiable, since if a function is differentiable it is continuous. We see this by showing that

$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist. We do so by computing the limit along two different curves containing $(0,0)$ and showing that we obtain different values:

Along $y = 0$:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + y^4} &= \lim_{x \rightarrow 0} \frac{x^2 (0^2)}{x^4 + 0^4} \\ &= \lim_{x \rightarrow 0} 0 = 0 \end{aligned}$$

Along $y = x$:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + y^4} &= \lim_{x \rightarrow 0} \frac{x^2 (x^2)}{x^4 + x^4} \\ &= \lim_{x \rightarrow 0} \frac{x^4}{2x^4} \\ &= \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Since these two limits are not equal, the limit does not exist, which means that f cannot be continuous at $(0,0)$ and so is not differentiable at $(0,0)$.

Challenge Problems

53. (a) We have that $R = \frac{\rho L}{\pi r^2} = \frac{1}{\pi} \rho L r^{-2}$. We compute the partial derivatives of R with respect to ρ , L and r :

$$\begin{aligned} \frac{\partial R}{\partial \rho} &= \frac{1}{\pi} L r^{-2} \\ \frac{\partial R}{\partial L} &= \frac{1}{\pi} \rho r^{-2} \\ \frac{\partial R}{\partial r} &= \frac{-2}{\pi} \rho L r^{-3} \end{aligned}$$

So,

$$\begin{aligned} dR &= \frac{\partial R}{\partial \rho} d\rho + \frac{\partial R}{\partial L} dL + \frac{\partial R}{\partial r} dr = \frac{1}{\pi} (L r^{-2} d\rho + \rho r^{-2} dL - 2\rho L r^{-3} dr) \\ &= \frac{1}{\pi} (L r^{-2} \Delta\rho + \rho r^{-2} \Delta L - 2\rho L r^{-3} \Delta r) \\ &= \frac{1}{\pi} (L r^{-2} \rho_0 K \Delta T + \rho r^{-2} L_0 \alpha \Delta T - 2\rho L r^{-3} r_0 \alpha \Delta T) \end{aligned}$$

When $\rho = \rho_0$, $L = L_0$ and $r = r_0$, this gives

$$dR = \frac{1}{\pi} \rho_0 L_0 r_0^{-2} \Delta T (K + \alpha - 2\alpha) = R_0 \Delta T (K - \alpha),$$

where R_0 is the initial resistance. But we are given that $R_0 = 0.50$ and $\Delta T = 40$, so

$$\Delta R \approx dR = 0.50 \cdot 40 \cdot (K - \alpha) = 0.50 \cdot 40 \cdot (3.93 \times 10^{-3} - 5.1 \times 10^{-5}) = 0.07758.$$

So, the resistance of the wire at 40°C is $R_0 + \Delta R \approx 0.578 \Omega$.

(b)

$$\frac{|\Delta R|}{R_0} \approx \frac{0.07758}{0.50} = 0.15516,$$

so the percent change in the resistance of the wire is approximately 15.516% .

(c) The change in resistance caused by the increase in the resistivity is:

$$\frac{\partial R}{\partial \rho} d\rho = \frac{1}{\pi} L r^{-2} \Delta \rho = \frac{1}{\pi} L r^{-2} \rho_0 K \Delta T$$

When $L = L_0$ and $r = r_0$, this gives

$$\begin{aligned} \frac{\partial R}{\partial \rho} d\rho &= \frac{1}{\pi} \rho_0 L_0 r_0^{-2} \Delta T \cdot K = R_0 K \Delta T \\ &= (0.50) \cdot (3.93 \times 10^{-3}) \cdot (40) = 0.0786 \Omega \end{aligned}$$

The change in resistance caused by the expansion of the wire is:

$$\begin{aligned} \frac{\partial R}{\partial L} dL + \frac{\partial R}{\partial r} dr &= \frac{1}{\pi} (\rho r^{-2} dL - 2\rho L r^{-3} dr) = \frac{1}{\pi} (\rho r^{-2} \Delta L - 2\rho L r^{-2} \Delta r) \\ &= \frac{1}{\pi} (\rho r^{-2} L_0 \alpha \Delta T - 2\rho L r^{-3} r_0 \alpha \Delta T) \end{aligned}$$

When $\rho = \rho_0$, $L = L_0$ and $r = r_0$,

$$\begin{aligned} &= \frac{1}{\pi} \rho_0 L_0 r_0^{-2} \Delta T (\alpha - 2\alpha) \\ &= -R_0 \alpha \Delta T = -(0.50) \cdot (5.1 \times 10^{-5}) \cdot 40 = -0.00102 \Omega \end{aligned}$$

Since the absolute value of the first is greater than the absolute value of the second, we conclude that the increase in the resistivity contributes more to the change in resistance.

54. (a) The differential of the function $w = f(x, y, z) = x^{y^z}$ is $dw = f_x dx + f_y dy + f_z dz$. We compute the required partial derivatives:

$$f_x = y^z x^{y^z-1}$$

$$f_y = (\ln x) x^{y^z} (z y^{z-1})$$

$$f_z = (\ln x) x^{y^z} (\ln y) y^z$$

So, the differential is $dw = y^z x^{y^z-1} dx + (\ln x) x^{y^z} (z y^{z-1}) dy + (\ln x) x^{y^z} (\ln y) y^z dz$.

- (b) The differential of the function $w = g(x, y, z) = (x^y)^z$ is $dw = g_x dx + g_y dy + g_z dz$. We compute the required partial derivatives, by first noting that $w = g(x, y, z) = (x^y)^z = x^{yz}$ by rules of exponents:

$$g_x = yz x^{yz-1}$$

$$g_y = (\ln x) x^{yz} (z) = z(\ln x) x^{yz}$$

$$g_z = (\ln x) x^{yz} (y) = y(\ln x) x^{yz}$$

So, the differential is $\boxed{dw = yz x^{yz-1} dx + z(\ln x) x^{yz} dy + y(\ln x) x^{yz} dz}$.

Chapter 12: Functions of Several Variables

12.5 Chain Rules

Concepts and Vocabulary

1. False. The formula is missing a minus sign; it should be $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$. See the Implicit Differentiation Formulas II, on page 855.
2. False. Each term of the formula is missing a factor; it should be $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$. See Chain Rule I for One Independent Variable, on page 849.

Skill Building

3.

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dt} = -\sin(2t) \cdot \frac{d}{dt}(2t) = -\sin(2t) \cdot 2 = -2\sin(2t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2x \cos t - 4y \sin(2t) = 2 \sin t \cos t - 4 \cos(2t) \sin(2t)$$

$$\boxed{\frac{dz}{dt} = 2x \cos t - 4y \sin(2t) = 2 \sin t \cos t - 4 \cos(2t) \sin(2t)}$$

4.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= -2y \\
\frac{dx}{dt} &= \cos(2t) \cdot \frac{d}{dt}(2t) = \cos(2t) \cdot 2 = 2\cos(2t) \\
\frac{dy}{dt} &= -\sin t \\
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 4x \cos(2t) + 2y \sin t = 4 \sin(2t) \cos(2t) + 2 \cos t \sin t
\end{aligned}$$

$$\boxed{\frac{dz}{dt} = 4x \cos(2t) + 2y \sin t = 4 \sin(2t) \cos(2t) + 2 \sin t \cos t}$$

5.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= 2y \\
\frac{dx}{dt} &= \frac{d}{dt}(te^t) = \frac{d}{dt}(t) \cdot e^t + t \cdot \frac{d}{dt}(e^t) = e^t + te^t \\
\frac{dy}{dt} &= \frac{d}{dt}(te^{-t}) = \frac{d}{dt}(t) \cdot e^{-t} + t \cdot \frac{d}{dt}(e^{-t}) = e^{-t} + t \cdot e^{-t} \cdot \frac{d}{dt}(-t) = e^{-t} - t \cdot e^{-t} \\
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2x(e^t + te^t) + 2y(e^{-t} - t \cdot e^{-t}) \\
&= 2te^t(e^t + te^t) + 2te^{-t}(e^{-t} - t \cdot e^{-t}) = 2te^{2t} + 2t^2e^{2t} + 2te^{-2t} - 2t^2e^{-2t}
\end{aligned}$$

$$\boxed{\frac{dz}{dt} = 2x(e^t + te^t) + 2y(e^{-t} - t \cdot e^{-t}) = 2te^{2t} + 2t^2e^{2t} + 2te^{-2t} - 2t^2e^{-2t}}$$

6.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= -2y \\
\frac{dx}{dt} &= \frac{d}{dt}(te^{-t}) = \frac{d}{dt}(t) \cdot e^{-t} + t \cdot \frac{d}{dt}(e^{-t}) = e^{-t} + t \cdot e^{-t} \cdot \frac{d}{dt}(-t) = e^{-t} - te^{-t} \\
\frac{dy}{dt} &= \frac{d}{dt}(t^2e^{-t}) = \frac{d}{dt}(t^2) \cdot e^{-t} + t^2 \cdot \frac{d}{dt}(e^{-t}) = 2te^{-t} + t^2 \cdot e^{-t} \cdot \frac{d}{dt}(-t) = 2te^{-t} - t^2e^{-t} \\
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2x(e^{-t} - te^{-t}) - 2y(2te^{-t} - t^2e^{-t}) \\
&= 2te^{-t}(e^{-t} - te^{-t}) - 2t^2e^{-t}(2te^{-t} - t^2e^{-t}) = 2te^{-2t} - 2t^2e^{-2t} - 4t^3e^{-2t} + 2t^4e^{-2t}
\end{aligned}$$

$$\boxed{\frac{dz}{dt} = 2x(e^{-t} - te^{-t}) - 2y(2te^{-t} - t^2e^{-t}) = 2te^{-2t} - 2t^2e^{-2t} - 4t^3e^{-2t} + 2t^4e^{-2t}}$$

7.

$$\begin{aligned}
\frac{\partial z}{\partial u} &= e^u \sin v \\
\frac{\partial z}{\partial v} &= e^u \cos v \\
\frac{du}{dt} &= \frac{d}{dt} \left(t^{1/2} \right) = \frac{1}{2} t^{-1/2} \\
\frac{dv}{dt} &= \pi \\
\frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} = e^u \sin v \cdot \frac{1}{2} t^{-1/2} + e^u \cos v \cdot \pi \\
&= \frac{1}{2\sqrt{t}} e^{\sqrt{t}} \sin(\pi t) + \pi e^{\sqrt{t}} \cos(\pi t)
\end{aligned}$$

$$\boxed{\frac{dz}{dt} = \frac{e^u \sin v}{2\sqrt{t}} + \pi e^u \cos v = \frac{e^{\sqrt{t}} \sin(\pi t)}{2\sqrt{t}} + \pi e^{\sqrt{t}} \cos(\pi t)}$$

8.

$$\begin{aligned}
\frac{\partial z}{\partial u} &= e^{u/v} \frac{\partial}{\partial u} \left(\frac{u}{v} \right) = e^{u/v} \frac{1}{v} \\
\frac{\partial z}{\partial v} &= e^{u/v} \frac{\partial}{\partial v} \left(\frac{u}{v} \right) = e^{u/v} \frac{\partial}{\partial v} (uv^{-1}) = e^{u/v} (-uv^{-2}) \\
\frac{du}{dt} &= \frac{d}{dt} \left(t^{1/2} \right) = \frac{1}{2} t^{-1/2} \\
\frac{dv}{dt} &= 3t^2 \\
\frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} = e^{u/v} \frac{1}{v} \cdot \frac{1}{2} t^{-1/2} + e^{u/v} (-uv^{-2}) \cdot 3t^2 \\
&= \frac{1}{2\sqrt{t}(t^3+1)} e^{\sqrt{t}/(t^3+1)} - \frac{3t^2 \sqrt{t}}{(t^3+1)^2} e^{\sqrt{t}/(t^3+1)}
\end{aligned}$$

$$\boxed{\frac{dz}{dt} = \frac{e^{\sqrt{t}/(t^3+1)}}{2v\sqrt{t}} - \frac{3t^2 u e^{\sqrt{t}/(t^3+1)}}{v^2} = \frac{e^{\sqrt{t}/(t^3+1)}}{2\sqrt{t}(t^3+1)} - \frac{3t^2 \sqrt{t} e^{\sqrt{t}/(t^3+1)}}{(t^3+1)^2}}$$

9.

$$\begin{aligned}
\frac{\partial z}{\partial u} &= e^{u/v} \frac{\partial}{\partial u} \left(\frac{u}{v} \right) = e^{u/v} \frac{1}{v} \\
\frac{\partial z}{\partial v} &= e^{u/v} \frac{\partial}{\partial v} \left(\frac{u}{v} \right) = e^{u/v} \frac{\partial}{\partial v} (uv^{-1}) = e^{u/v} (-uv^{-2}) \\
\frac{du}{dt} &= \frac{d}{dt} (te^t) = \frac{d}{dt} (t) \cdot e^t + t \cdot \frac{d}{dt} (e^t) = e^t + te^t \\
\frac{dv}{dt} &= e^{t^2} \cdot \frac{d}{dt} (t^2) = 2te^{t^2} \\
\frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} = e^{u/v} \frac{1}{v} \cdot (e^t + te^t) + e^{u/v} (-uv^{-2}) (2te^{t^2}) \\
&= \frac{e^{te^{t-t^2}}}{e^{t^2}} (e^t + te^t) - \frac{te^t e^{te^{t-t^2}}}{(e^{t^2})^2} \cdot 2te^{t^2} = \frac{e^{t+te^{t-t^2}} + te^{t+te^{t-t^2}}}{e^{t^2}} - \frac{2t^2 e^{t+t^2+te^{t-t^2}}}{e^{2t^2}}
\end{aligned}$$

$$\frac{dz}{dt} = \frac{e^{u/v}(e^t + te^t)}{v} - \frac{ue^{u/v}(2te^{t^2})}{v^2} = \frac{e^{t+te^{t-t^2}} + te^{t+te^{t-t^2}}}{e^{t^2}} - \frac{2t^2e^{t+t^2+te^{t-t^2}}}{e^{2t^2}}$$

10.

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{1}{uv} \frac{\partial}{\partial u} (uv) = \frac{v}{uv} = \frac{1}{u} \\ \frac{\partial z}{\partial v} &= \frac{1}{uv} \frac{\partial}{\partial v} (uv) = \frac{u}{uv} = \frac{1}{v} \\ \frac{du}{dt} &= 5t^4 \\ \frac{dv}{dt} &= \frac{d}{dt} \left((t+1)^{1/2} \right) = \frac{1}{2}(t+1)^{-1/2} \frac{d}{dt} (t+1) = \frac{1}{2\sqrt{t+1}} \\ \frac{dz}{dt} &= \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} = \frac{5t^4}{u} + \frac{1}{2v\sqrt{t+1}} = \frac{5t^4}{t^5} + \frac{1}{2\sqrt{t+1}\sqrt{t+1}} = \frac{5}{t} + \frac{1}{2(t+1)}\end{aligned}$$

$$\frac{dz}{dt} = \frac{5t^4}{u} + \frac{1}{2v\sqrt{t+1}} = \frac{5}{t} + \frac{1}{2(t+1)}$$

11.

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^{x^2+y^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) = 2xe^{x^2+y^2} \\ \frac{\partial z}{\partial y} &= e^{x^2+y^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) = 2ye^{x^2+y^2} \\ \frac{dx}{dt} &= \cos(2t) \frac{d}{dt} (2t) = 2\cos(2t) \\ \frac{dy}{dt} &= -\sin t \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 2xe^{x^2+y^2} \cdot 2\cos(2t) - 2ye^{x^2+y^2} \sin t \\ &= 4\sin(2t) \cos(2t) e^{\sin^2(2t) + \cos^2 t} - 2\sin t \cos t e^{\sin^2(2t) + \cos^2 t}\end{aligned}$$

$$\frac{dz}{dt} = 2xe^{x^2+y^2} \cdot 2\cos(2t) - 2ye^{x^2+y^2} \sin t = 4\sin(2t) \cos(2t) e^{\sin^2(2t) + \cos^2 t} - 2\sin t \cos t e^{\sin^2(2t) + \cos^2 t}$$

12.

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^{x^2-y^2} \cdot \frac{\partial}{\partial x} (x^2 - y^2) = 2xe^{x^2-y^2} \\ \frac{\partial z}{\partial y} &= e^{x^2-y^2} \cdot \frac{\partial}{\partial y} (x^2 - y^2) = -2ye^{x^2-y^2} \\ \frac{dx}{dt} &= \cos(2t) \frac{d}{dt} (2t) = 2\cos(2t) \\ \frac{dy}{dt} &= -\sin(2t) \frac{d}{dt} (2t) = -2\sin(2t) \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 4xe^{x^2-y^2} \cos(2t) + 4ye^{x^2-y^2} \sin(2t) \\ &= 4\sin(2t) \cos(2t) e^{\sin^2(2t) - \cos^2(2t)} + 4\sin(2t) \cos(2t) e^{\sin^2(2t) - \cos^2(2t)} \\ &= 8\sin(2t) \cos(2t) e^{\sin^2(2t) - \cos^2(2t)}\end{aligned}$$

$$\frac{dz}{dt} = 4xe^{x^2-y^2} \cos(2t) + 4ye^{x^2-y^2} \sin(2t) = 8 \sin(2t) \cos(2t) e^{\sin^2(2t) - \cos^2(2t)}$$

13.

$$\begin{aligned} \frac{\partial z}{\partial x} &= y \cdot \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = y \cdot \frac{\frac{\partial}{\partial x}(x) \cdot (x^2 + y^2) - x \cdot \frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= y \cdot \frac{x^2 + y^2 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y(-x^2 + y^2)}{(x^2 + y^2)^2} \\ \frac{\partial z}{\partial y} &= x \cdot \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = x \cdot \frac{\frac{\partial}{\partial y}(y) \cdot (x^2 + y^2) - y \cdot \frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= x \cdot \frac{x^2 + y^2 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \\ \frac{dx}{dt} &= \cos t \\ \frac{dy}{dt} &= -\sin t \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{y(-x^2 + y^2) \cos t}{(x^2 + y^2)^2} - \frac{x(x^2 - y^2) \sin t}{(x^2 + y^2)^2} \\ &= \frac{\cos t(-\sin^2 t + \cos^2 t) \cos t}{(\sin^2 t + \cos^2 t)^2} - \frac{\sin t(\sin^2 t - \cos^2 t) \sin t}{(\sin^2 t + \cos^2 t)^2} \\ &= \frac{-\cos^2 t \sin^2 t + \cos^4 t}{1^2} + \frac{-\sin^4 t + \sin^2 t \cos^2 t}{1^2} \\ &= \cos^4 t - \sin^4 t = (\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t) = 1 \cdot (\cos^2 t - \sin^2 t) \end{aligned}$$

$$\frac{dz}{dt} = \frac{y(-x^2 + y^2) \cos t}{(x^2 + y^2)^2} - \frac{x(x^2 - y^2) \sin t}{(x^2 + y^2)^2} = \cos^2 t - \sin^2 t$$

14.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{y}{x} + y \\ \frac{\partial z}{\partial y} &= \ln x + x + \sec^2 y \\ \frac{dx}{dt} &= \frac{\frac{\partial}{\partial t}(t) \cdot (t+1) - t \cdot \frac{d}{dt}(t+1)}{(t+1)^2} = \frac{(t+1) - t}{(t+1)^2} = \frac{1}{(t+1)^2} \\ \frac{dy}{dt} &= 3t^2 - 1 \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \frac{\frac{y}{x} + y}{(t+1)^2} + (\ln x + x + \sec^2 y)(3t^2 - 1) \\ &= \frac{\frac{(t^3-t)(t+1)}{t} + t^3 - t}{(t+1)^2} + \left(\ln \left(\frac{t}{t+1} \right) + \frac{t}{t+1} + \sec^2(t^3 - t) \right) (3t^2 - 1) \\ &= \frac{(t^2 - 1)(t+1) + t^3 - t}{(t+1)^2} + \left(\ln \left(\frac{t}{t+1} \right) + \frac{t}{t+1} + \sec^2(t^3 - t) \right) (3t^2 - 1) \\ &= \frac{2t^3 + t^2 - 2t - 1}{(t+1)^2} + \left(\ln \left(\frac{t}{t+1} \right) + \frac{t}{t+1} + \sec^2(t^3 - t) \right) (3t^2 - 1) \end{aligned}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\frac{y}{x} + y}{(t+1)^2} + (\ln x + x + \sec^2 y)(3t^2 - 1) \\ &= \frac{2t^3 + t^2 - 2t - 1}{(t+1)^2} + \left(\ln\left(\frac{t}{t+1}\right) + \frac{t}{t+1} + \sec^2(t^3 - t) \right)(3t^2 - 1)\end{aligned}$$

15.

$$\begin{aligned}\frac{\partial p}{\partial x} &= 2x \\ \frac{\partial p}{\partial y} &= 2y \\ \frac{\partial p}{\partial z} &= -2z \\ \frac{dx}{dt} &= \frac{d}{dt}(te^t) = \frac{d}{dt}(t) \cdot e^t + t \cdot \frac{d}{dt}(e^t) = e^t + te^t \\ \frac{dy}{dt} &= \frac{d}{dt}(te^{-t}) = \frac{d}{dt}(t) \cdot e^{-t} + t \cdot \frac{d}{dt}(e^{-t}) = e^{-t} + t \cdot e^{-t} \cdot \frac{d}{dt}(-t) = e^{-t} - t \cdot e^{-t} \\ \frac{dz}{dt} &= e^{2t} \cdot \frac{d}{dt}(2t) = 2e^{2t} \\ \frac{dp}{dt} &= \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial y} \frac{dy}{dt} + \frac{\partial p}{\partial z} \frac{dz}{dt} = 2x(e^t + te^t) + 2y(e^{-t} - te^{-t}) - 4ze^{2t} \\ &= 2te^t(e^t + te^t) + 2te^{-t}(e^{-t} - te^{-t}) - 4e^{2t} \cdot e^{2t} = 2te^{2t} + 2t^2e^{2t} + 2te^{-2t} - 2t^2e^{-2t} - 4e^{4t}\end{aligned}$$

$$\frac{dp}{dt} = 2x(e^t + te^t) + 2y(e^{-t} - te^{-t}) - 4ze^{2t} = 2te^{2t} + 2t^2e^{2t} + 2te^{-2t} - 2t^2e^{-2t} - 4e^{4t}$$

16.

$$\begin{aligned}\frac{\partial p}{\partial x} &= 2x \\ \frac{\partial p}{\partial y} &= -2y \\ \frac{\partial p}{\partial z} &= -2z \\ \frac{dx}{dt} &= \frac{d}{dt}(te^{-t}) = \frac{d}{dt}(t) \cdot e^{-t} + t \cdot \frac{d}{dt}(e^{-t}) = e^{-t} + t \cdot e^{-t} \cdot \frac{d}{dt}(-t) = e^{-t} - t \cdot e^{-t} \\ \frac{dy}{dt} &= \frac{d}{dt}(t^2e^{-t}) = \frac{d}{dt}(t^2) \cdot e^{-t} + t^2 \cdot \frac{d}{dt}(e^{-t}) = 2te^{-t} + t^2 \cdot e^{-t} \cdot \frac{d}{dt}(-t) = 2te^{-t} - t^2e^{-t} \\ \frac{dz}{dt} &= e^{-t} \cdot \frac{d}{dt}(-t) = -e^{-t} \\ \frac{dp}{dt} &= \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial y} \frac{dy}{dt} + \frac{\partial p}{\partial z} \frac{dz}{dt} = 2x(e^{-t} - t \cdot e^{-t}) - 2y(2te^{-t} - t^2e^{-t}) + 2ze^{-t} \\ &= 2te^{-t}(e^{-t} - t \cdot e^{-t}) - 2t^2e^{-t}(2te^{-t} - t^2e^{-t}) + 2e^{-t}e^{-t} \\ &= 2te^{-2t} - 2t^2e^{-2t} - 4t^3e^{-2t} + 2t^4e^{-2t} + 2e^{-2t}\end{aligned}$$

$$\frac{dp}{dt} = 2x(e^{-t} - t \cdot e^{-t}) - 2y(2te^{-t} - t^2e^{-t}) + 2ze^{-t} = 2te^{-2t} - 2t^2e^{-2t} - 4t^3e^{-2t} + 2t^4e^{-2t} + 2e^{-2t}$$

17.

$$\begin{aligned}
\frac{\partial p}{\partial x} &= e^x \sin y \cos z \\
\frac{\partial p}{\partial y} &= e^x \cos y \cos z \\
\frac{\partial p}{\partial z} &= -e^x \sin y \sin z \\
\frac{dx}{dt} &= \frac{d}{dt} \left(t^{1/2} \right) = \frac{1}{2} \cdot t^{-1/2} = \frac{1}{2\sqrt{t}} \\
\frac{dy}{dt} &= \pi \\
\frac{dz}{dt} &= \frac{1}{2} \\
\frac{dp}{dt} &= \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial y} \frac{dy}{dt} + \frac{\partial p}{\partial z} \frac{dz}{dt} = \frac{e^x \sin y \cos z}{2\sqrt{t}} + \pi e^x \cos y \cos z - \frac{e^x \sin y \sin z}{2} \\
&= \frac{e^{\sqrt{t}} \sin(\pi t) \cos\left(\frac{t}{2}\right)}{2\sqrt{t}} + \pi e^{\sqrt{t}} \cos(\pi t) \cos\left(\frac{t}{2}\right) - \frac{e^{\sqrt{t}} \sin(\pi t) \sin\left(\frac{t}{2}\right)}{2}
\end{aligned}$$

$ \begin{aligned} \frac{dp}{dt} &= \frac{e^x \sin y \cos z}{2\sqrt{t}} + \pi e^x \cos y \cos z - \frac{e^x \sin y \sin z}{2} \\ &= \frac{e^{\sqrt{t}} \sin(\pi t) \cos\left(\frac{t}{2}\right)}{2\sqrt{t}} + \pi e^{\sqrt{t}} \cos(\pi t) \cos\left(\frac{t}{2}\right) - \frac{e^{\sqrt{t}} \sin(\pi t) \sin\left(\frac{t}{2}\right)}{2} \end{aligned} $
--

18.

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{1}{xyz} \frac{\partial}{\partial x} (xyz) = \frac{yz}{xyz} = \frac{1}{x} \\
\frac{\partial p}{\partial y} &= \frac{1}{xyz} \frac{\partial}{\partial y} (xyz) = \frac{xz}{xyz} = \frac{1}{y} \\
\frac{\partial p}{\partial z} &= \frac{1}{xyz} \frac{\partial}{\partial z} (xyz) = \frac{xy}{xyz} = \frac{1}{z} \\
\frac{dx}{dt} &= 5t^4 \\
\frac{dy}{dt} &= \frac{d}{dt} \left((t+1)^{1/2} \right) = \frac{1}{2} (t+1)^{-1/2} \frac{d}{dt} (t+1) = \frac{1}{2\sqrt{t+1}} \\
\frac{dz}{dt} &= 2t \\
\frac{dp}{dt} &= \frac{\partial p}{\partial x} \frac{dx}{dt} + \frac{\partial p}{\partial y} \frac{dy}{dt} + \frac{\partial p}{\partial z} \frac{dz}{dt} = \frac{5t^4}{x} + \frac{1}{2y\sqrt{t+1}} + \frac{2t}{z} \\
&= \frac{5t^4}{t^5} + \frac{1}{2\sqrt{t+1}\sqrt{t+1}} + \frac{2t}{t^2} = \frac{5}{t} + \frac{1}{2(t+1)} + \frac{2}{t} = \frac{7}{t} + \frac{1}{2(t+1)}
\end{aligned}$$

$ \frac{dp}{dt} = \frac{5t^4}{x} + \frac{1}{2y\sqrt{t+1}} + \frac{2t}{z} = \frac{7}{t} + \frac{1}{2(t+1)} $

19.

$$\begin{aligned}
\frac{\partial p}{\partial u} &= w \cdot \frac{1}{\frac{u}{v}} \cdot \frac{\partial}{\partial u} \left(\frac{u}{v} \right) = w \cdot \frac{1}{\frac{u}{v}} \cdot \frac{1}{v} = \frac{w}{u} \\
\frac{\partial p}{\partial v} &= -w \cdot \frac{1}{\frac{u}{v}} \cdot \frac{\partial}{\partial v} \left(\frac{u}{v} \right) = w \cdot \frac{1}{\frac{u}{v}} \cdot \frac{-u}{v^2} = -\frac{w}{v} \\
\frac{\partial p}{\partial w} &= \ln \left(\frac{u}{v} \right) \\
\frac{du}{dt} &= \frac{d}{dt} (te^t) = \frac{d}{dt} (t) \cdot e^t + t \cdot \frac{d}{dt} (e^t) = e^t + te^t \\
\frac{dv}{dt} &= e^{t^2} \cdot \frac{d}{dt} (t^2) = 2te^{t^2} \\
\frac{dw}{dt} &= e^{2t} \cdot \frac{\partial}{\partial t} (2t) = 2e^{2t} \\
\frac{dp}{dt} &= \frac{\partial p}{\partial u} \frac{du}{dt} + \frac{\partial p}{\partial v} \frac{dv}{dt} + \frac{\partial p}{\partial w} \frac{dw}{dt} = \frac{w(e^t + te^t)}{u} - \frac{2wte^{t^2}}{v} + \ln \left(\frac{u}{v} \right) \cdot 2e^{2t} \\
&= \frac{e^{2t}(e^t + te^t)}{te^t} - \frac{2te^{2t}e^{t^2}}{e^{t^2}} + \ln \left(\frac{te^t}{e^{t^2}} \right) \cdot 2e^{2t} \\
&= e^{2t} + \frac{e^{2t}}{t} - 2te^{2t} + \ln \left(te^{t-t^2} \right) \cdot 2e^{2t} = e^{2t} + \frac{e^{2t}}{t} - 2te^{2t} + 2e^{2t} \left(\ln t + \ln \left(e^{t-t^2} \right) \right) \\
&= e^{2t} + \frac{e^{2t}}{t} - 2te^{2t} + 2e^{2t} \ln t + 2e^{2t}(t - t^2) = e^{2t} + \frac{e^{2t}}{t} + 2e^{2t} \ln t - 2t^2e^{2t}
\end{aligned}$$

$$\frac{dp}{dt} = \frac{w(e^t + te^t)}{u} - \frac{2wte^{t^2}}{v} + \ln \left(\frac{u}{v} \right) \cdot 2e^{2t} = e^{2t} + \frac{e^{2t}}{t} + 2e^{2t} \ln t - 2t^2e^{2t}$$

20.

$$\begin{aligned}
\frac{\partial p}{\partial u} &= we^{u/v} \frac{\partial}{\partial u} \left(\frac{u}{v} \right) = e^{u/v} \frac{w}{v} \\
\frac{\partial p}{\partial v} &= we^{u/v} \frac{\partial}{\partial v} \left(\frac{u}{v} \right) = we^{u/v} \frac{\partial}{\partial v} (uv^{-1}) = e^{u/v} (-u w v^{-2}) \\
\frac{\partial p}{\partial w} &= e^{u/v} \\
\frac{du}{dt} &= \frac{d}{dt} (t^{1/2}) = \frac{1}{2} t^{-1/2} \\
\frac{dv}{dt} &= 3t^2 \\
\frac{dw}{dt} &= e^t \\
\frac{dp}{dt} &= \frac{\partial p}{\partial u} \frac{du}{dt} + \frac{\partial p}{\partial v} \frac{dv}{dt} + \frac{\partial p}{\partial w} \frac{dw}{dt} = e^{u/v} \frac{w}{v} \cdot \frac{1}{2} t^{-1/2} + e^{u/v} (-u w v^{-2}) \cdot 3t^2 + e^{u/v} e^t \\
&= \frac{e^t}{2\sqrt{t}(t^3+1)} e^{\sqrt{t}/(t^3+1)} - \frac{3e^t t^2 \sqrt{t}}{(t^3+1)^2} e^{\sqrt{t}/(t^3+1)} + e^t e^{\sqrt{t}/(t^3+1)} \\
&= \frac{e^{t+\sqrt{t}/(t^3+1)}}{2\sqrt{t}(t^3+1)} - \frac{3t^2 \sqrt{t} e^{t+\sqrt{t}/(t^3+1)}}{(t^3+1)^2} + e^{t+\sqrt{t}/(t^3+1)}
\end{aligned}$$

$$\begin{aligned}\frac{dp}{dt} &= e^{u/v} \frac{w}{v} \cdot \frac{1}{2} t^{-1/2} + e^{u/v} (-uvw^{-2}) \cdot 3t^2 + e^{u/v} e^t \\ &= \frac{e^{t+\sqrt{t}/(t^3+1)}}{2\sqrt{t}(t^3+1)} - \frac{3t^2\sqrt{t}e^{t+\sqrt{t}/(t^3+1)}}{(t^3+1)^2} + e^{t+\sqrt{t}/(t^3+1)}\end{aligned}$$

21.

$$\begin{aligned}\frac{\partial p}{\partial u} &= 2uvw \\ \frac{\partial p}{\partial v} &= u^2w \\ \frac{\partial p}{\partial w} &= u^2v \\ \frac{du}{dt} &= \cos t \\ \frac{dv}{dt} &= -\sin t \\ \frac{dw}{dt} &= e^t \\ \frac{dz}{dt} &= \frac{\partial p}{\partial u} \frac{du}{dt} + \frac{\partial p}{\partial v} \frac{dv}{dt} + \frac{\partial p}{\partial w} \frac{dw}{dt} = 2uvw \cos t - u^2w \sin t + u^2ve^t \\ &= 2e^t \sin t \cos^2 t - e^t \sin^3 t + e^t \sin^2 t \cos t\end{aligned}$$

$$\frac{dp}{dt} = 2uvw \cos t - u^2w \sin t + u^2ve^t = 2e^t \sin t \cos^2 t - e^t \sin^3 t + e^t \sin^2 t \cos t$$

22.

$$\begin{aligned}\frac{\partial p}{\partial u} &= \sqrt{vw} \frac{\partial}{\partial u} (u^{1/2}) = \sqrt{vw} \left(\frac{1}{2} u^{-1/2} \right) = \frac{\sqrt{vw}}{2\sqrt{u}} \\ \frac{\partial p}{\partial v} &= \sqrt{uw} \frac{\partial}{\partial v} (v^{1/2}) = \sqrt{uw} \left(\frac{1}{2} v^{-1/2} \right) = \frac{\sqrt{uw}}{2\sqrt{v}} \\ \frac{\partial p}{\partial w} &= \sqrt{uv} \frac{\partial}{\partial w} (w^{1/2}) = \sqrt{uv} \left(\frac{1}{2} w^{-1/2} \right) = \frac{\sqrt{uv}}{2\sqrt{w}} \\ \frac{du}{dt} &= e^t \\ \frac{dv}{dt} &= \frac{d}{dt} (te^t) = \frac{d}{dt} (t) \cdot e^t + t \cdot \frac{d}{dt} (e^t) = e^t + te^t \\ \frac{dw}{dt} &= \frac{d}{dt} (t^2e^t) = \frac{d}{dt} (t^2) \cdot e^t + t^2 \cdot \frac{d}{dt} (e^t) = 2te^t + t^2e^t \\ \frac{dz}{dt} &= \frac{\partial p}{\partial u} \frac{du}{dt} + \frac{\partial p}{\partial v} \frac{dv}{dt} + \frac{\partial p}{\partial w} \frac{dw}{dt} = \frac{e^t\sqrt{vw}}{2\sqrt{u}} + \frac{(e^t + te^t)\sqrt{uw}}{2\sqrt{v}} + \frac{(2te^t + t^2e^t)\sqrt{uv}}{2\sqrt{w}} \\ &= \frac{e^t\sqrt{t^3e^{3t}}}{2\sqrt{e^t}} + \frac{(e^t + te^t)\sqrt{t^2e^{3t}}}{2\sqrt{te^t}} + \frac{(2te^t + t^2e^t)\sqrt{te^{2t}}}{2\sqrt{t^2e^{2t}}} \\ &= \frac{e^{2t}\sqrt{t^3}}{2} + \frac{(e^{2t} + te^{2t})\sqrt{t}}{2} + \frac{2te^t + t^2e^t}{2\sqrt{t}}\end{aligned}$$

$$\frac{dp}{dt} = \frac{e^t\sqrt{vw}}{2\sqrt{u}} + \frac{(e^t + te^t)\sqrt{uw}}{2\sqrt{v}} + \frac{(2te^t + t^2e^t)\sqrt{uv}}{2\sqrt{w}} = \frac{e^{2t}\sqrt{t^3}}{2} + \frac{(e^{2t} + te^{2t})\sqrt{t}}{2} + \frac{2te^t + t^2e^t}{2\sqrt{t}}$$

23.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= 2y \\
\frac{\partial x}{\partial u} &= e^v \\
\frac{\partial x}{\partial v} &= ue^v \\
\frac{\partial y}{\partial u} &= ve^u \\
\frac{\partial y}{\partial v} &= e^u \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2xe^v + 2yve^u = 2ue^{2v} + 2v^2e^{2u} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = 2xue^v + 2ye^u = 2u^2e^{2v} + 2ve^{2u}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = 2xe^v + 2yve^u = 2ue^{2v} + 2v^2e^{2u}}$$

$$\boxed{\frac{\partial z}{\partial v} = 2xue^v + 2ye^u = 2u^2e^{2v} + 2ve^{2u}}$$

24.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= -2y \\
\frac{\partial x}{\partial u} &= \ln v \\
\frac{\partial x}{\partial v} &= \frac{u}{v} \\
\frac{\partial y}{\partial u} &= \frac{v}{u} \\
\frac{\partial y}{\partial v} &= \ln u \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2x \ln v - \frac{2yv}{u} = 2u (\ln v)^2 - \frac{2v^2 \ln u}{u} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{2xu}{v} - 2y \ln u = \frac{2u^2 \ln v}{v} - 2v (\ln u)^2
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = 2x \ln v - \frac{2yv}{u} = 2u (\ln v)^2 - \frac{2v^2 \ln u}{u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{2xu}{v} - 2y \ln u = \frac{2u^2 \ln v}{v} - 2v (\ln u)^2}$$

25.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= e^x \sin y \\
\frac{\partial z}{\partial y} &= e^x \cos y \\
\frac{\partial x}{\partial u} &= 2uv \\
\frac{\partial x}{\partial v} &= u^2 \\
\frac{\partial y}{\partial u} &= \frac{1}{uv} \frac{\partial}{\partial u} (uv) = \frac{v}{uv} = \frac{1}{u} \\
\frac{\partial y}{\partial v} &= \frac{1}{uv} \frac{\partial}{\partial v} (uv) = \frac{u}{uv} = \frac{1}{v} \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (e^x \sin y) (2uv) + (e^x \cos y) \left(\frac{1}{u} \right) = 2uve^{u^2v} \sin(\ln(uv)) + \frac{e^{u^2v} \cos(\ln(uv))}{u} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (e^x \sin y) (u^2) + (e^x \cos y) \left(\frac{1}{v} \right) = u^2 e^{u^2v} \sin(\ln(uv)) + \frac{e^{u^2v} \cos(\ln(uv))}{v}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = (e^x \sin y) (2uv) + (e^x \cos y) \left(\frac{1}{u} \right) = 2uve^{u^2v} \sin(\ln(uv)) + \frac{e^{u^2v} \cos(\ln(uv))}{u}}$$

$$\boxed{\frac{\partial z}{\partial v} = (e^x \sin y) (u^2) + (e^x \cos y) \left(\frac{1}{v} \right) = u^2 e^{u^2v} \sin(\ln(uv)) + \frac{e^{u^2v} \cos(\ln(uv))}{v}}$$

26.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{xy} \\
\frac{\partial z}{\partial y} &= \ln x \frac{\partial}{\partial y} (y^{-1}) = \ln x (-y^{-2}) = -\frac{\ln x}{y^2} \\
\frac{\partial x}{\partial u} &= \sqrt{v} \frac{\partial}{\partial u} (u^{1/2}) = \sqrt{v} \left(\frac{1}{2} u^{-1/2} \right) = \frac{\sqrt{v}}{2\sqrt{u}} \\
\frac{\partial x}{\partial v} &= \sqrt{u} \frac{\partial}{\partial v} (v^{1/2}) = \sqrt{u} \left(\frac{1}{2} v^{-1/2} \right) = \frac{\sqrt{u}}{2\sqrt{v}} \\
\frac{\partial y}{\partial u} &= v \frac{\partial}{\partial u} (u^{-1}) = v(-u^{-2}) = -\frac{v}{u^2} \\
\frac{\partial y}{\partial v} &= \frac{1}{u} \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\sqrt{v}}{2xy\sqrt{u}} + \frac{v \ln x}{u^2 y^2} = \frac{u\sqrt{v}}{2v\sqrt{uv}\sqrt{u}} + \frac{\ln \sqrt{uv}}{v} = \frac{1}{2v} + \frac{\ln \sqrt{uv}}{v} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\sqrt{u}}{2xy\sqrt{v}} - \frac{\ln x}{uy^2} = \frac{u\sqrt{u}}{2v\sqrt{v}\sqrt{uv}} - \frac{u \ln \sqrt{uv}}{v^2} = \frac{u}{2v^2} - \frac{u \ln \sqrt{uv}}{v^2}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\sqrt{v}}{2xy\sqrt{u}} + \frac{v \ln x}{u^2 y^2} = \frac{1}{2v} + \frac{\ln \sqrt{uv}}{v}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\sqrt{u}}{2xy\sqrt{v}} - \frac{\ln x}{uy^2} = \frac{u}{2v^2} - \frac{u \ln \sqrt{uv}}{v^2}}$$

27.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial x} (x^2 + y^2) = \frac{2x}{x^2 + y^2} \\ \frac{\partial z}{\partial y} &= \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y} (x^2 + y^2) = \frac{2y}{x^2 + y^2} \\ \frac{\partial x}{\partial u} &= \frac{\partial}{\partial u} (u^{-1}v^2) = -u^{-2}v^2 = -\frac{v^2}{u^2} \\ \frac{\partial x}{\partial v} &= \frac{2v}{u} \\ \frac{\partial y}{\partial u} &= \frac{1}{v^2} \\ \frac{\partial y}{\partial v} &= \frac{\partial}{\partial v} (uv^{-2}) = u(-2v^{-3}) = -\frac{2u}{v^3} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = -\frac{2v^2x}{u^2(x^2 + y^2)} + \frac{2y}{v^2(x^2 + y^2)} \\ &= -\frac{2v^2v^2}{uv^4 + \frac{u^5}{v^4}} + \frac{2u}{v^4 \left(\frac{v^4}{u^2} + \frac{u^2}{v^4} \right)} = -\frac{2v^8}{uv^8 + u^5} + \frac{2u^3}{v^8 + u^4} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{4vx}{u(x^2 + y^2)} - \frac{4uy}{v^3(x^2 + y^2)} \\ &= \frac{4vv^2}{v^4 + \frac{u^4}{v^4}} - \frac{4u^2}{v^5 \left(\frac{v^4}{u^2} + \frac{u^2}{v^4} \right)} = \frac{4v^7}{v^8 + u^4} - \frac{4u^4}{v^9 + u^4v} \end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = -\frac{2v^2x}{u^2(x^2 + y^2)} + \frac{2y}{v^2(x^2 + y^2)} = -\frac{2v^8}{u(v^8 + u^4)} + \frac{2u^3}{v^8 + u^4}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{4vx}{u(x^2 + y^2)} - \frac{4uy}{v^3(x^2 + y^2)} = \frac{4v^7}{v^8 + u^4} - \frac{4u^4}{v(v^8 + u^4)}}$$

28.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \sin y - y \cos x \\
\frac{\partial z}{\partial y} &= x \cos y - \sin x \\
\frac{\partial x}{\partial u} &= 2uv \\
\frac{\partial x}{\partial v} &= u^2 \\
\frac{\partial y}{\partial u} &= v^2 \\
\frac{\partial y}{\partial v} &= 2uv \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2uv(\sin y - y \cos x) + v^2(x \cos y - \sin x) \\
&= 2uv \sin(uv^2) - 2u^2v^3 \cos(u^2v) + u^2v^3 \cos(uv^2) - v^2 \sin(u^2v) \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = u^2(\sin y - y \cos x) + 2uv(x \cos y - \sin x) \\
&= u^2 \sin(uv^2) - u^3v^2 \cos(u^2v) + 2u^3v^2 \cos(uv^2) - 2uv \sin(u^2v)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial u} &= 2uv(\sin y - y \cos x) + v^2(x \cos y - \sin x) \\
&= 2uv \sin(uv^2) - 2u^2v^3 \cos(u^2v) + u^2v^3 \cos(uv^2) - v^2 \sin(u^2v)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= u^2(\sin y - y \cos x) + 2uv(x \cos y - \sin x) \\
&= u^2 \sin(uv^2) - u^3v^2 \cos(u^2v) + 2u^3v^2 \cos(uv^2) - 2uv \sin(u^2v)
\end{aligned}$$

29.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= 2y \\
\frac{\partial x}{\partial u} &= \cos(u-v) \frac{\partial}{\partial u} (u-v) = \cos(u-v) \\
\frac{\partial x}{\partial v} &= \cos(u-v) \frac{\partial}{\partial v} (u-v) = -\cos(u-v) \\
\frac{\partial y}{\partial u} &= -\sin(u+v) \frac{\partial}{\partial u} (u+v) = -\sin(u+v) \\
\frac{\partial y}{\partial v} &= -\sin(u+v) \frac{\partial}{\partial v} (u+v) = -\sin(u+v) \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2x \cos(u-v) - 2y \sin(u+v) \\
&= 2 \sin(u-v) \cos(u-v) - 2 \cos(u+v) \sin(u+v) \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = -2x \cos(u-v) - 2y \sin(u+v) \\
&= -2 \sin(u-v) \cos(u-v) - 2 \cos(u+v) \sin(u+v)
\end{aligned}$$

$$\frac{\partial z}{\partial u} = 2x \cos(u - v) - 2y \sin(u + v) = 2 \sin(u - v) \cos(u - v) - 2 \sin(u + v) \cos(u + v)$$

$$\frac{\partial z}{\partial v} = -2x \cos(u - v) - 2y \sin(u + v) = -2 \sin(u - v) \cos(u - v) - 2 \sin(u + v) \cos(u + v)$$

30.

$$\begin{aligned} \frac{\partial z}{\partial x} &= e^x \\ \frac{\partial z}{\partial y} &= 1 \\ \frac{\partial x}{\partial u} &= \frac{\partial}{\partial u} \left(\tan^{-1} \left(\frac{u}{v} \right) \right) = \frac{1}{1 + \left(\frac{u}{v} \right)^2} \cdot \frac{\partial}{\partial u} \left(\frac{u}{v} \right) = \frac{1}{1 + \left(\frac{u}{v} \right)^2} \cdot \frac{1}{v} = \frac{v}{u^2 + v^2} \\ \frac{\partial x}{\partial v} &= \frac{\partial}{\partial v} \left(\tan^{-1} \left(\frac{u}{v} \right) \right) = \frac{1}{1 + \left(\frac{u}{v} \right)^2} \cdot \frac{\partial}{\partial v} (uv^{-1}) = \frac{1}{1 + \left(\frac{u}{v} \right)^2} \cdot (-uv^{-2}) = \frac{-u}{u^2 + v^2} \\ \frac{\partial y}{\partial u} &= \frac{1}{u + v} \frac{\partial}{\partial u} (u + v) = \frac{1}{u + v} \\ \frac{\partial y}{\partial v} &= \frac{1}{u + v} \frac{\partial}{\partial v} (u + v) = \frac{1}{u + v} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{ve^x}{u^2 + v^2} + \frac{1}{u + v} = \frac{ve^{\tan^{-1}(u/v)}}{u^2 + v^2} + \frac{1}{u + v} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = -\frac{ue^x}{u^2 + v^2} + \frac{1}{u + v} = -\frac{ue^{\tan^{-1}(u/v)}}{u^2 + v^2} + \frac{1}{u + v} \end{aligned}$$

$$\frac{\partial z}{\partial u} = \frac{ve^x}{u^2 + v^2} + \frac{1}{u + v} = \frac{ve^{\tan^{-1}(u/v)}}{u^2 + v^2} + \frac{1}{u + v}$$

$$\frac{\partial z}{\partial v} = -\frac{ue^x}{u^2 + v^2} + \frac{1}{u + v} = -\frac{ue^{\tan^{-1}(u/v)}}{u^2 + v^2} + \frac{1}{u + v}$$

31.

$$\begin{aligned} \frac{\partial z}{\partial r} &= se^r \\ \frac{\partial z}{\partial s} &= e^r \\ \frac{\partial r}{\partial u} &= 2u \\ \frac{\partial r}{\partial v} &= 2v \\ \frac{\partial s}{\partial u} &= \frac{\partial}{\partial u} (vu^{-1}) = -vu^{-2} = \frac{-v}{u^2} \\ \frac{\partial s}{\partial v} &= \frac{1}{u} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial u} = (se^r)(2u) + (e^r) \left(\frac{-v}{u^2} \right) = 2ve^{u^2+v^2} - \frac{ve^{u^2+v^2}}{u^2} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial v} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial v} = (se^r)(2v) + (e^r) \left(\frac{1}{u} \right) = \frac{2v^2e^{u^2+v^2}}{u} + \frac{e^{u^2+v^2}}{u} \end{aligned}$$

$$\frac{\partial z}{\partial u} = (se^r)(2u) + (e^r) \left(\frac{-v}{u^2} \right) = 2ve^{u^2+v^2} - \frac{ve^{u^2+v^2}}{u^2}$$

$$\frac{\partial z}{\partial v} = (se^r)(2v) + (e^r) \left(\frac{1}{u} \right) = \frac{2v^2e^{u^2+v^2}}{u} + \frac{e^{u^2+v^2}}{u}$$

32.

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial}{\partial r} \left((s^2 + r^2)^{1/2} \right) = \frac{1}{2} (s^2 + r^2)^{-1/2} \frac{\partial}{\partial r} (s^2 + r^2) = \frac{r}{\sqrt{s^2 + r^2}} \\ \frac{\partial z}{\partial s} &= \frac{\partial}{\partial s} \left((s^2 + r^2)^{1/2} \right) = \frac{1}{2} (s^2 + r^2)^{-1/2} \frac{\partial}{\partial s} (s^2 + r^2) = \frac{s}{\sqrt{s^2 + r^2}} \\ \frac{\partial r}{\partial u} &= \sqrt{v} \frac{\partial}{\partial u} \left(u^{1/2} \right) = \sqrt{v} \left(\frac{1}{2} u^{-1/2} \right) = \frac{\sqrt{v}}{2\sqrt{u}} \\ \frac{\partial r}{\partial v} &= \sqrt{u} \frac{\partial}{\partial v} \left(v^{1/2} \right) = \sqrt{u} \left(\frac{1}{2} v^{-1/2} \right) = \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{\partial s}{\partial u} &= \frac{1}{uv} \frac{\partial}{\partial u} (uv) = \frac{v}{uv} = \frac{1}{u} \\ \frac{\partial s}{\partial v} &= \frac{1}{uv} \frac{\partial}{\partial v} (uv) = \frac{u}{uv} = \frac{1}{v} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial u} = \frac{r\sqrt{v}}{2\sqrt{u}\sqrt{s^2 + r^2}} + \frac{s}{u\sqrt{s^2 + r^2}} \\ &= \frac{\sqrt{uv}\sqrt{v}}{2\sqrt{u}\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{u\sqrt{\ln^2(uv) + uv}} = \frac{v}{2\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{u\sqrt{\ln^2(uv) + uv}} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial v} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial v} = \frac{r\sqrt{u}}{2\sqrt{v}\sqrt{s^2 + r^2}} + \frac{s}{v\sqrt{s^2 + r^2}} \\ &= \frac{\sqrt{u}\sqrt{uv}}{2\sqrt{v}\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{v\sqrt{\ln^2(uv) + uv}} = \frac{u}{2\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{v\sqrt{\ln^2(uv) + uv}} \end{aligned}$$

$$\frac{\partial z}{\partial u} = \frac{r\sqrt{v}}{2\sqrt{u}\sqrt{s^2 + r^2}} + \frac{s}{u\sqrt{s^2 + r^2}} = \frac{v}{2\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{u\sqrt{\ln^2(uv) + uv}}$$

$$\frac{\partial z}{\partial v} = \frac{r\sqrt{u}}{2\sqrt{v}\sqrt{s^2 + r^2}} + \frac{s}{v\sqrt{s^2 + r^2}} = \frac{u}{2\sqrt{\ln^2(uv) + uv}} + \frac{\ln(uv)}{v\sqrt{\ln^2(uv) + uv}}$$

33.

$$\frac{\partial z}{\partial x} = y^2 w^3$$

$$\frac{\partial z}{\partial y} = 2xyw^3$$

$$\frac{\partial z}{\partial w} = 3xy^2w^2$$

$$\frac{\partial x}{\partial u} = 2$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = 5$$

$$\frac{\partial y}{\partial v} = -3$$

$$\frac{\partial w}{\partial u} = 2$$

$$\frac{\partial w}{\partial v} = 3$$

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial u} = 2y^2w^3 + 10xyw^3 + 6xy^2w^2 \\ &= 2(5u-3v)^2(2u+3v)^3 + 10(2u+v)(5u-3v)(2u+3v)^3 + 6(2u+v)(5u-3v)^2(2u+3v)^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial v} = y^2w^3 - 6xyw^3 + 9xy^2w^2 \\ &= (5u-3v)^2(2u+3v)^3 - 6(2u+v)(5u-3v)(2u+3v)^3 + 9(2u+v)(5u-3v)^2(2u+3v)^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial u} &= 2y^2w^3 + 10xyw^3 + 6xy^2w^2 \\ &= 2(5u-3v)^2(2u+3v)^3 + 10(2u+v)(5u-3v)(2u+3v)^3 + 6(2u+v)(5u-3v)^2(2u+3v)^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial v} &= y^2w^3 - 6xyw^3 + 9xy^2w^2 \\ &= (5u-3v)^2(2u+3v)^3 - 6(2u+v)(5u-3v)(2u+3v)^3 + 9(2u+v)(5u-3v)^2(2u+3v)^2 \end{aligned}$$

34.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x \\
\frac{\partial z}{\partial y} &= -2y \\
\frac{\partial z}{\partial w} &= 1 \\
\frac{\partial x}{\partial u} &= e^{u+v} \cdot \frac{\partial}{\partial u} (u+v) = e^{u+v} \\
\frac{\partial x}{\partial v} &= e^{u+v} \cdot \frac{\partial}{\partial v} (u+v) = e^{u+v} \\
\frac{\partial y}{\partial u} &= v \\
\frac{\partial y}{\partial v} &= u \\
\frac{\partial w}{\partial u} &= v \frac{\partial}{\partial u} (u^{-1}) = v(-u^{-2}) = -\frac{v}{u^2} \\
\frac{\partial w}{\partial v} &= \frac{1}{u} \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial u} = 2xe^{u+v} - 2vy - \frac{v}{u^2} = 2e^{2u+2v} - 2uv^2 - \frac{v}{u^2} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial v} = 2xe^{u+v} - 2uy + \frac{1}{u} = 2e^{2u+2v} - 2u^2v + \frac{1}{u}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial u} = 2xe^{u+v} - 2vy - \frac{v}{u^2} = 2e^{2u+2v} - 2uv^2 - \frac{v}{u^2}}$$

$$\boxed{\frac{\partial z}{\partial v} = 2xe^{u+v} - 2uy + \frac{1}{u} = 2e^{2u+2v} - 2u^2v + \frac{1}{u}}$$

35.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

$$\frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = e^{u+2v+3w} \frac{\partial}{\partial u} (u + 2v + 3w) = e^{u+2v+3w}$$

$$\frac{\partial y}{\partial v} = e^{u+2v+3w} \frac{\partial}{\partial v} (u + 2v + 3w) = 2e^{u+2v+3w}$$

$$\frac{\partial y}{\partial w} = e^{u+2v+3w} \frac{\partial}{\partial w} (u + 2v + 3w) = 3e^{u+2v+3w}$$

$$\frac{\partial z}{\partial u} = 0$$

$$\frac{\partial z}{\partial v} = 2$$

$$\frac{\partial z}{\partial w} = 3$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = 2xv + 2ye^{u+2v+3w} = 2uv^2 + 2e^{2u+4v+6w}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = 2xu + 4ye^{u+2v+3w} + 4z = 2u^2v + 4e^{2u+4v+6w} + 4(2v + 3w)$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = 6ye^{u+2v+3w} + 6z = 6e^{2u+4v+6w} + 6(2v + 3w)$$

$$\boxed{\frac{\partial f}{\partial u} = 2xv + 2ye^{u+2v+3w} = 2uv^2 + 2e^{2u+4v+6w}}$$

$$\boxed{\frac{\partial f}{\partial v} = 2xu + 4ye^{u+2v+3w} + 4z = 2u^2v + 4e^{2u+4v+6w} + 4(2v + 3w)}$$

$$\boxed{\frac{\partial f}{\partial w} = 6ye^{u+2v+3w} + 6z = 6e^{2u+4v+6w} + 6(2v + 3w)}$$

36.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 1 \\
\frac{\partial f}{\partial y} &= -2y \\
\frac{\partial f}{\partial z} &= 2z \\
\frac{\partial x}{\partial u} &= \frac{\partial}{\partial u} \left((u+v)^{1/2} \right) = \frac{1}{2}(u+v)^{-1/2} \cdot \frac{\partial}{\partial u} (u+v) = \frac{1}{2\sqrt{u+v}} \\
\frac{\partial x}{\partial v} &= \frac{\partial}{\partial v} \left((u+v)^{1/2} \right) = \frac{1}{2}(u+v)^{-1/2} \cdot \frac{\partial}{\partial v} (u+v) = \frac{1}{2\sqrt{u+v}} \\
\frac{\partial x}{\partial w} &= 0 \\
\frac{\partial y}{\partial u} &= \ln v \\
\frac{\partial y}{\partial v} &= \frac{u+w}{v} \\
\frac{\partial y}{\partial w} &= \ln v \\
\frac{\partial z}{\partial u} &= 0 \\
\frac{\partial z}{\partial v} &= -1 \\
\frac{\partial z}{\partial w} &= 3 \\
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = \frac{1}{2\sqrt{u+v}} - 2y \ln v = \frac{1}{2\sqrt{u+v}} - 2(u+w) (\ln v)^2 \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = \frac{1}{2\sqrt{u+v}} - \frac{2y(u+w)}{v} - 2z \\
&= \frac{1}{2\sqrt{u+v}} - \frac{2(u+w)^2 \ln v}{v} - 4 + 2v - 6w \\
\frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = -2y \ln v + 6z = -2(u+w) (\ln v)^2 + 12 - 6v + 18w
\end{aligned}$$

$$\boxed{\frac{\partial f}{\partial u} = \frac{1}{2\sqrt{u+v}} - 2y \ln v = \frac{1}{2\sqrt{u+v}} - 2(u+w) (\ln v)^2}$$

$$\boxed{\frac{\partial f}{\partial v} = \frac{1}{2\sqrt{u+v}} - \frac{2y(u+w)}{v} - 2z = \frac{1}{2\sqrt{u+v}} - \frac{2(u+w)^2 \ln v}{v} - 4 + 2v - 6w}$$

$$\boxed{\frac{\partial f}{\partial w} = -2y \ln v + 6z = -2(u+w) (\ln v)^2 + 12 - 6v + 18w}$$

37.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos y + z \sin x + 2xyz \\
\frac{\partial f}{\partial y} &= -x \sin y + x^2 z \\
\frac{\partial f}{\partial z} &= -\cos x + x^2 y \\
\frac{\partial x}{\partial u} &= vw \\
\frac{\partial x}{\partial v} &= uw \\
\frac{\partial x}{\partial w} &= uv \\
\frac{\partial y}{\partial u} &= 2u \\
\frac{\partial y}{\partial v} &= 2v \\
\frac{\partial y}{\partial w} &= 2w \\
\frac{\partial z}{\partial u} &= 0 \\
\frac{\partial z}{\partial v} &= 0 \\
\frac{\partial z}{\partial w} &= 1
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = (\cos y + z \sin x + 2xyz)vw + (-x \sin y + x^2 z)2u \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))vw + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2u
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = (\cos y + z \sin x + 2xyz)uw + (-x \sin y + x^2 z)2v \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))uw + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2v
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = (\cos y + z \sin x + 2xyz)uv + (-x \sin y + x^2 z)2w - \cos x + x^2 y \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))uv + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2w - \cos(uvw) + u^2 v^2 w^2(u^2 + v^2 + w^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial u} &= (\cos y + z \sin x + 2xyz)vw + (-x \sin y + x^2 z)2u \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))vw + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2u
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial v} &= (\cos y + z \sin x + 2xyz)uw + (-x \sin y + x^2 z)2v \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))uw + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2v
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial w} &= (\cos y + z \sin x + 2xyz)uv + (-x \sin y + x^2 z)2w - \cos x + x^2 y \\
&= (\cos(u^2 + v^2 + w^2) + w \sin(uvw) + 2uvw^2(u^2 + v^2 + w^2))uv + (-uvw \sin(u^2 + v^2 + w^2) + u^2 v^2 w^3)2w - \cos(uvw) + u^2 v^2 w^2(u^2 + v^2 + w^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x \\
\frac{\partial f}{\partial y} &= 2y \\
\frac{\partial f}{\partial z} &= 0 \\
\frac{\partial x}{\partial u} &= \cos(u-v) \frac{\partial}{\partial u} (u-v) = \cos(u-v) \\
\frac{\partial x}{\partial v} &= \cos(u-v) \frac{\partial}{\partial v} (u-v) = -\cos(u-v) \\
\frac{\partial x}{\partial w} &= 0 \\
\frac{\partial y}{\partial u} &= -\sin(u+v) \frac{\partial}{\partial u} (u+v) = -\sin(u+v) \\
\frac{\partial y}{\partial v} &= -\sin(u+v) \frac{\partial}{\partial v} (u+v) = -\sin(u+v) \\
\frac{\partial y}{\partial w} &= 0 \\
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = 2x \cos(u-v) - 2y \sin(u+v) \\
&= 2 \sin(u-v) \cos(u-v) - 2 \sin(u+v) \cos(u+v) \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = -2x \cos(u-v) - 2y \sin(u+v) \\
&= -2 \sin(u-v) \cos(u-v) - 2 \sin(u+v) \cos(u+v) \\
\frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = 0
\end{aligned}$$

$$\boxed{\frac{\partial f}{\partial u} = 2x \cos(u-v) - 2y \sin(u+v) = 2 \sin(u-v) \cos(u-v) - 2 \sin(u+v) \cos(u+v)}$$

$$\boxed{\frac{\partial f}{\partial v} = -2x \cos(u-v) - 2y \sin(u+v) = -2 \sin(u-v) \cos(u-v) - 2 \sin(u+v) \cos(u+v)}$$

$$\boxed{\frac{\partial f}{\partial w} = 0}$$

39.

$$\frac{\partial f}{\partial x} = 1$$

$$\frac{\partial f}{\partial y} = 4y$$

$$\frac{\partial f}{\partial z} = -2z$$

$$\frac{\partial x}{\partial u} = t$$

$$\frac{\partial x}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 0$$

$$\frac{\partial x}{\partial t} = u$$

$$\frac{\partial y}{\partial u} = e^{u+2v+3w+4t} \frac{\partial}{\partial u} (u+2v+3w+4t) = e^{u+2v+3w+4t}$$

$$\frac{\partial y}{\partial v} = e^{u+2v+3w+4t} \frac{\partial}{\partial v} (u+2v+3w+4t) = 2e^{u+2v+3w+4t}$$

$$\frac{\partial y}{\partial w} = e^{u+2v+3w+4t} \frac{\partial}{\partial w} (u+2v+3w+4t) = 3e^{u+2v+3w+4t}$$

$$\frac{\partial y}{\partial t} = e^{u+2v+3w+4t} \frac{\partial}{\partial t} (u+2v+3w+4t) = 4e^{u+2v+3w+4t}$$

$$\frac{\partial z}{\partial u} = 1$$

$$\frac{\partial z}{\partial v} = \frac{1}{2}$$

$$\frac{\partial z}{\partial w} = 0$$

$$\frac{\partial z}{\partial t} = 4$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = t + 4ye^{u+2v+3w+4t} - 2z = t + 4e^{2u+4v+6w+8t} - 2u - v - 8t$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = 8ye^{u+2v+3w+4t} - z = 8e^{2u+4v+6w+8t} - u - \frac{1}{2}v - 4t$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = 12ye^{u+2v+3w+4t} = 12e^{2u+4v+6w+8t}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = u + 16ye^{u+2v+3w+4t} - 8z = u + 16e^{2u+4v+6w+8t} - 8u - 4v - 32t$$

$$\boxed{\frac{\partial f}{\partial u} = t + 4ye^{u+2v+3w+4t} - 2z = t + 4e^{2u+4v+6w+8t} - 2u - v - 8t}$$

$$\boxed{\frac{\partial f}{\partial v} = 8ye^{u+2v+3w+4t} - z = 8e^{2u+4v+6w+8t} - u - \frac{1}{2}v - 4t}$$

$$\boxed{\frac{\partial f}{\partial w} = 12ye^{u+2v+3w+4t} = 12e^{2u+4v+6w+8t}}$$

$$\frac{\partial f}{\partial t} = u + 16ye^{u+2v+3w+4t} - 8z = u + 16e^{2u+4v+6w+8t} - 8u - 4v - 32t$$

40.

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 2y \\ \frac{\partial f}{\partial z} &= 1 \\ \frac{\partial x}{\partial u} &= \cos(u+t) \frac{\partial}{\partial u} (u+t) = \cos(u+t) \\ \frac{\partial x}{\partial v} &= 0 \\ \frac{\partial x}{\partial w} &= 0 \\ \frac{\partial x}{\partial t} &= \cos(u+t) \frac{\partial}{\partial t} (u+t) = \cos(u+t) \\ \frac{\partial y}{\partial u} &= 0 \\ \frac{\partial y}{\partial v} &= -\sin(v-t) \frac{\partial}{\partial v} (v-t) = -\sin(v-t) \\ \frac{\partial y}{\partial w} &= 0 \\ \frac{\partial y}{\partial t} &= -\sin(v-t) \frac{\partial}{\partial t} (v-t) = \sin(v-t) \\ \frac{\partial z}{\partial u} &= w^2 \\ \frac{\partial z}{\partial v} &= 0 \\ \frac{\partial z}{\partial w} &= 2uw \\ \frac{\partial z}{\partial t} &= 0 \\ \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = 2x \cos(u+t) + w^2 = 2 \sin(u+t) \cos(u+t) + w^2 \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = -2y \sin(v-t) = -2 \sin(v-t) \cos(v-t) \\ \frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} = 2uw \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = 2x \cos(u+t) + 2y \sin(v-t) \\ &= 2 \sin(u+t) \cos(u+t) + 2 \sin(v-t) \cos(v-t)\end{aligned}$$

$$\frac{\partial f}{\partial u} = 2x \cos(u+t) + w^2 = 2 \sin(u+t) \cos(u+t) + w^2$$

$$\frac{\partial f}{\partial v} = -2y \sin(v-t) = -2 \sin(v-t) \cos(v-t)$$

$$\frac{\partial f}{\partial w} = 2uw$$

$$\frac{\partial f}{\partial t} = 2x \cos(u+t) + 2y \sin(v-t) = 2 \sin(u+t) \cos(u+t) + 2 \sin(v-t) \cos(v-t)$$

41. First, we find the partial derivatives of F :

$$F_x = 2xy - y^2 + y \text{ and } F_y = x^2 - 2xy + x.$$

Then we use Implicit Differentiation Formula I. If $x^2 - 2xy + x \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy - y^2 + y}{x^2 - 2xy + x}.$$

$$\frac{dy}{dx} = \frac{-(2xy - y^2 + y)}{x^2 - 2xy + x}$$

42. First, we find the partial derivatives of F :

$$F_x = 3x^2y^2 - y + 2xy \text{ and } F_y = 2x^3y - x + x^2.$$

Then we use Implicit Differentiation Formula I. If $2x^3y - x + x^2 \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2y^2 - y + 2xy}{2x^3y - x + x^2}.$$

$$\frac{dy}{dx} = \frac{-(3x^2y^2 - y + 2xy)}{2x^3y - x + x^2}$$

43. First, we find the partial derivatives of F :

$$F_x = \sin y + y \cos x \text{ and } F_y = x \cos y + \sin x.$$

Then we use Implicit Differentiation Formula I. If $x \cos y + \sin x \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\sin y + y \cos x}{x \cos y + \sin x}.$$

$$\frac{dy}{dx} = \frac{-(\sin y + y \cos x)}{x \cos y + \sin x}$$

44. First, we find the partial derivatives of F :

$$F_x = e^y + ye^x - y \text{ and } F_y = xe^y + e^x - x.$$

Then we use Implicit Differentiation Formula I. If $xe^y + e^x - x \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + ye^x - y}{xe^y + e^x - x}.$$

$$\frac{dy}{dx} = \frac{-(e^y + ye^x - y)}{xe^y + e^x - x}$$

45. First, we find the partial derivatives of F :

$$F_x = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \text{ and } F_y = \frac{1}{3}y^{-2/3} = \frac{1}{3y^{2/3}},$$

assuming $x \neq 0$ and $y \neq 0$. Then we use Implicit Differentiation Formula I.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{3x^{2/3}}}{\frac{1}{3y^{2/3}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}}$$

46. First, we find the partial derivatives of F :

$$F_x = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} \text{ and } F_y = \frac{2}{3}y^{-1/3} = \frac{2}{3y^{1/3}},$$

assuming $x \neq 0$ and $y \neq 0$. Then we use Implicit Differentiation Formula I.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{2}{3x^{1/3}}}{\frac{2}{3y^{1/3}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}}$$

47. First, we find the partial derivatives of F :

$$F_x = z + 2xy^3, F_y = 3z^2 + 3x^2y^2 \text{ and } F_z = x + 6yz - 5.$$

Then we use Implicit Differentiation Formulas II. If $x + 6yz - 5 \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z + 2xy^3}{x + 6yz - 5}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3z^2 + 3x^2y^2}{x + 6yz - 5}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-(z + 2xy^3)}{x + 6yz - 5}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-(3z^2 + 3x^2y^2)}{x + 6yz - 5}}$$

48. First, we find the partial derivatives of F :

$$F_x = 2xz + 3x^2y, F_y = 2yz + x^3 \text{ and } F_z = x^2 + y^2 - 10.$$

Then we use Implicit Differentiation Formulas II. If $x^2 + y^2 - 10 \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xz + 3x^2y}{x^2 + y^2 - 10}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2yz + x^3}{x^2 + y^2 - 10}$$

$$\frac{\partial z}{\partial x} = \frac{-(2xz + 3x^2y)}{x^2 + y^2 - 10}$$

$$\frac{\partial z}{\partial y} = \frac{-(2yz + x^3)}{x^2 + y^2 - 10}$$

49. First, we find the partial derivatives of F :

$$F_x = yz, F_y = \cos z + xz \text{ and } F_z = \cos z - y \sin z + xy.$$

Then we use Implicit Differentiation Formulas II. If $\cos z - y \sin z + xy \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz}{\cos z - y \sin z + xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos z + xz}{\cos z - y \sin z + xy}$$

$$\frac{\partial z}{\partial x} = \frac{-yz}{\cos z - y \sin z + xy}$$

$$\frac{\partial z}{\partial y} = \frac{-(\cos z + xz)}{\cos z - y \sin z + xy}$$

50. First, we find the partial derivatives of F :

$$F_x = \sin y + 2xz, F_y = x \cos y \text{ and } F_z = \sin z + x^2.$$

Then we use Implicit Differentiation Formulas II. If $\sin z + x^2 \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sin y + 2xz}{\sin z + x^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x \cos y}{\sin z + x^2}$$

$$\frac{\partial z}{\partial x} = \frac{-(\sin y + 2xz)}{\sin z + x^2}$$

$$\frac{\partial z}{\partial y} = \frac{-x \cos y}{\sin z + x^2}$$

51. First, we find the partial derivatives of F :

$$F_x = e^{yz} + yze^{xz} + yz, F_y = xze^{yz} + e^{xz} + xz \text{ and } F_z = xye^{yz} + xye^{xz} + xy.$$

Then we use Implicit Differentiation Formulas II. If $xye^{yz} + xye^{xz} + xy \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{yz} + yze^{xz} + yz}{xye^{yz} + xye^{xz} + xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{yz} + e^{xz} + xz}{xye^{yz} + xye^{xz} + xy}$$

$$\frac{\partial z}{\partial x} = \frac{-(e^{yz} + yze^{xz} + yz)}{xye^{yz} + xye^{xz} + xy}$$

$$\frac{\partial z}{\partial y} = \frac{-(xze^{yz} + e^{xz} + xz)}{xye^{yz} + xye^{xz} + xy}$$

52. First, we find the partial derivatives of F :

$$F_x = \frac{e^{yz}}{x} + yze^{xz}, F_y = ze^{yz} \ln x + e^{xz} - z \text{ and } F_z = ye^{yz} \ln x + xye^{xz} - y.$$

Then we use Implicit Differentiation Formulas II. If $ye^{yz} \ln x + xye^{xz} - y \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{e^{yz}}{x} + yze^{xz}}{ye^{yz} \ln x + xye^{xz} - y} = -\frac{e^{yz} + xye^{xz}}{xye^{yz} \ln x + x^2ye^{xz} - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{ze^{yz} \ln x + e^{xz} - z}{ye^{yz} \ln x + xye^{xz} - y}$$

$$\frac{\partial z}{\partial x} = \frac{-(e^{yz} + xye^{xz})}{xye^{yz} \ln x + x^2ye^{xz} - xy}$$

$$\frac{\partial z}{\partial y} = \frac{-(ze^{yz} \ln x + e^{xz} - z)}{ye^{yz} \ln x + xye^{xz} - y}$$

53. Recall that to find $\frac{\partial w}{\partial x}$, we treat y and z as constants. This allows to use the power rule (along with the chain rule).

$$\frac{\partial w}{\partial x} = (4z)(2x + 3y)^{4z-1}(2) = 8z(2x + 3y)^{4z-1}$$

To find $\frac{\partial w}{\partial y}$, we treat x and z as constants. This allows to again use the power rule (along with the chain rule).

$$\frac{\partial w}{\partial y} = (4z)(2x + 3y)^{4z-1}(3) = 12z(2x + 3y)^{4z-1}$$

Lastly, to find $\frac{\partial w}{\partial z}$, we treat x and y as constants. We then use the rule for exponential functions,

$$\frac{d}{dx}[a^x] = a^x \ln a \text{ (along with the chain rule).}$$

$$\frac{\partial w}{\partial z} = (2x + 3y)^{4z} \ln(2x + 3y) \cdot 4 = 4(2x + 3y)^{4z} \ln(2x + 3y).$$

$$\frac{\partial w}{\partial x} = 8z(2x + 3y)^{4z-1}$$

$$\frac{\partial w}{\partial y} = 12z(2x + 3y)^{4z-1}$$

$$\frac{\partial w}{\partial z} = 4(2x + 3y)^{4z} \ln(2x + 3y)$$

54. Recall that to find $\frac{\partial w}{\partial x}$, we treat y and z as constants. This allows to use the power rule (along with the chain rule).

$$\frac{\partial w}{\partial x} = (3y + 4z)(2x)^{3y+4z-1}(2) = 2(3y + 4z)(2x)^{3y+4z-1}$$

To find $\frac{\partial w}{\partial y}$, we treat x and z as constants. We then use the rule for exponential functions, $\frac{d}{dx}[a^x] = a^x \ln a$ (along with the chain rule).

$$\frac{\partial w}{\partial y} = (2x)^{3y+4z} \ln(2x) \cdot 3 = 3(2x)^{3y+4z} \ln(2x)$$

Lastly, to find $\frac{\partial w}{\partial z}$, we treat x and y as constants. We again use the rule for exponential functions, $\frac{d}{dx}[a^x] = a^x \ln a$ (along with the chain rule).

$$\frac{\partial w}{\partial z} = (2x)^{3y+4z} \ln(2x) \cdot 4 = 4(2x)^{3y+4z} \ln(2x).$$

$$\frac{\partial w}{\partial x} = 2(3y + 4z)(2x)^{3y+4z-1}$$

$$\frac{\partial w}{\partial y} = 3(2x)^{3y+4z} \ln(2x)$$

$$\frac{\partial w}{\partial z} = 4(2x)^{3y+4z} \ln(2x)$$

Applications and Extensions

55. Let t represent time, in seconds. We solve the equation $PV = 20T$ for V : $V = \frac{20T}{P} = 20TP^{-1}$. Now we will find a formula for $\frac{\partial V}{\partial t}$, using the Chain Rule I:

$$\begin{aligned} \frac{\partial V}{\partial T} &= 20P^{-1} = \frac{20}{P} \\ \frac{\partial V}{\partial P} &= -20TP^{-2} = \frac{-20T}{P^2} \\ \frac{dV}{dt} &= \frac{\partial V}{\partial T} \frac{dT}{dt} + \frac{\partial V}{\partial P} \frac{dP}{dt} = \frac{20}{P} \frac{dT}{dt} - \frac{20T}{P^2} \frac{dP}{dt} \end{aligned}$$

To use the Ideal Gas Law, T must be in kelvins. So, $T = 80 + 273 = 353$. We are looking for the rate of change of V with respect to time (that is, $\frac{dV}{dt}$) when

$$T = 353, \frac{dT}{dt} = 5, P = 10, \frac{dP}{dt} = -2.$$

Using the above formula for $\frac{dV}{dt}$, we get

$$\frac{dV}{dt} = \frac{20}{10} \cdot 5 - \frac{20 \cdot 353}{10^2} \cdot (-2) = 10 + 141.2 = 151.2.$$

Therefore, the rate of change of the volume is $\boxed{151.2 \text{ m}^3/\text{s}}$.

56. (a) Let t represent time, in hours. We note that the surface of the block of ice consists of six rectangles. Two of the rectangles are l by w , two are l by h , and two are w by h . So, the total surface area of the block of ice is $S = 2lw + 2lh + 2wh$. Now we will find a formula for $\frac{\partial S}{\partial t}$, using the Chain Rule I:

$$\begin{aligned}\frac{\partial S}{\partial l} &= 2w + 2h \\ \frac{\partial S}{\partial w} &= 2l + 2h \\ \frac{\partial S}{\partial h} &= 2l + 2w \\ \frac{dS}{dt} &= \frac{\partial S}{\partial l} \frac{dl}{dt} + \frac{\partial S}{\partial w} \frac{dw}{dt} + \frac{\partial S}{\partial h} \frac{dh}{dt} = (2w + 2h) \frac{dl}{dt} + (2l + 2h) \frac{dw}{dt} + (2l + 2w) \frac{dh}{dt}\end{aligned}$$

We are looking for the rate of change of S with respect to time (that is, $\frac{dS}{dt}$) when

$$l = 3, w = 2, h = 1, \frac{dl}{dt} = -1, \frac{dw}{dt} = -1, \frac{dh}{dt} = -0.5.$$

Using the above formula for $\frac{dS}{dt}$, we get

$$\frac{dS}{dt} = (2 \cdot 2 + 2 \cdot 1) \cdot (-1) + (2 \cdot 3 + 2 \cdot 1) \cdot (-1) + (2 \cdot 3 + 2 \cdot 2) \cdot (-0.5) = -19.$$

Therefore, the rate of change of the surface area is $\boxed{-19 \text{ m}^2/\text{h}}$.

- (b) Again, let t represent time, in hours. The volume of the block of ice is given by the formula $V = lwh$. Now we will find a formula for $\frac{\partial V}{\partial t}$, using the Chain Rule I:

$$\begin{aligned}\frac{\partial V}{\partial l} &= wh \\ \frac{\partial V}{\partial w} &= lh \\ \frac{\partial V}{\partial h} &= lw \\ \frac{dV}{dt} &= \frac{\partial V}{\partial l} \frac{dl}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = wh \frac{dl}{dt} + lh \frac{dw}{dt} + lw \frac{dh}{dt}\end{aligned}$$

We are looking for the rate of change of V with respect to time (that is, $\frac{dV}{dt}$) when

$$l = 3, w = 2, h = 1, \frac{dl}{dt} = -1, \frac{dw}{dt} = -1, \frac{dh}{dt} = -0.5.$$

Using the above formula for $\frac{dV}{dt}$, we get

$$\frac{dV}{dt} = 2 \cdot 1 \cdot (-1) + 3 \cdot 1 \cdot (-1) + 3 \cdot 2 \cdot (-0.5) = -8.$$

Therefore, the rate of change of the volume is $\boxed{-8 \text{ m}^3/\text{h}}$.

57. (a) We compute the second partial derivatives of f with respect to x and t :

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (\sin(x+vt)) = \cos(x+vt) \frac{\partial}{\partial x} (x+vt) = \cos(x+vt) \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\cos(x+vt)) = -\sin(x+vt) \frac{\partial}{\partial x} (x+vt) = -\sin(x+vt) \\ \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t} (\sin(x+vt)) = \cos(x+vt) \frac{\partial}{\partial t} (x+vt) = v \cos(x+vt) \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial t} \right) = \frac{\partial}{\partial t} (v \cos(x+vt)) = -v \sin(x+vt) \frac{\partial}{\partial t} (x+vt) = -v^2 \sin(x+vt)\end{aligned}$$

Therefore,

$$\frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} (-v^2 \sin(x+vt)) = -\sin(x+vt) = \frac{\partial^2 f}{\partial x^2}.$$

So f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(b) We compute the second partial derivatives of f with respect to x and t :

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (e^{x-vt}) = e^{x-vt} \frac{\partial}{\partial x} (x-vt) = e^{x-vt} \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (e^{x-vt}) = e^{x-vt} \frac{\partial}{\partial x} (x-vt) = e^{x-vt} \\ \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t} (e^{x-vt}) = e^{x-vt} \frac{\partial}{\partial t} (x-vt) = -ve^{x-vt} \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial t} \right) = \frac{\partial}{\partial t} (-ve^{x-vt}) = -ve^{x-vt} \frac{\partial}{\partial t} (x-vt) = v^2 e^{x-vt}\end{aligned}$$

Therefore,

$$\frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} (v^2 e^{x-vt}) = e^{x-vt} = \frac{\partial^2 f}{\partial x^2}.$$

So f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(c) We compute the second partial derivatives of f with respect to x and t :

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (\sin x + \sin(vt)) = \cos x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\cos x) = -\sin x \\ \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t} (\sin x + \sin(vt)) = \cos(vt) \frac{\partial}{\partial t} (vt) = v \cos(vt) \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial t} \right) = \frac{\partial}{\partial t} (v \cos(vt)) = -v \sin(vt) \frac{\partial}{\partial t} (vt) = -v^2 \sin(vt)\end{aligned}$$

Therefore,

$$\frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} (-v^2 \sin(vt)) = -\sin(vt) \neq -\sin x = \frac{\partial^2 f}{\partial x^2}.$$

To verify that $-\sin(vt)$ and $-\sin x$ are not equal, we can, for example, substitute $t = 0$ and $x = \frac{\pi}{2}$; in that case, $-\sin(vt) = 0$ and $-\sin x = -1$. So they are indeed not equal. So f does not satisfy the equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

- (d) Let $u = x + vt$. Then $f(u) = f(x + vt)$. We compute the second partial derivatives of f with respect to x and t , using Chain Rule II.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du} \cdot 1 = \frac{df}{du} \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{df}{du} \right) = \frac{d}{du} \left(\frac{df}{du} \right) \cdot \frac{\partial u}{\partial x} = \frac{d}{du} \left(\frac{df}{du} \right) \cdot 1 = \frac{d^2 f}{du^2} \\ \frac{\partial f}{\partial t} &= \frac{df}{du} \frac{\partial u}{\partial t} = \frac{df}{du} \cdot v = v \frac{df}{du} \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t} \left(v \frac{df}{du} \right) = v \frac{d}{du} \left(\frac{df}{du} \right) \cdot \frac{\partial u}{\partial t} = v \frac{d}{du} \left(\frac{df}{du} \right) \cdot v = v^2 \frac{d^2 f}{du^2}\end{aligned}$$

Therefore,

$$\frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} \left(v^2 \frac{d^2 f}{du^2} \right) = \frac{d^2 f}{du^2} = \frac{\partial^2 f}{\partial x^2}.$$

So $f(x + vt)$ satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

58. (a) Let t represent time, in years. We will find a formula for $\frac{\partial Q}{\partial t}$, using the Chain Rule I:

$$\begin{aligned}\frac{\partial Q}{\partial L} &= 400M^{0.7} \cdot (0.3)L^{-0.7} = 120M^{0.7}L^{-0.7} \\ \frac{\partial Q}{\partial M} &= 400L^{0.3} \cdot (0.7)M^{-0.3} = 280M^{-0.3}L^{0.3} \\ \frac{dQ}{dt} &= \frac{\partial Q}{\partial L} \frac{dL}{dt} + \frac{\partial Q}{\partial M} \frac{dM}{dt} = 120M^{0.7}L^{-0.7} \frac{dL}{dt} + 280M^{-0.3}L^{0.3} \frac{dM}{dt}\end{aligned}$$

Now, first, we are looking for the rate of change of Q with respect to time (that is, $\frac{dQ}{dt}$) when

$$L = 19, M = 21, \frac{dL}{dt} = -4, \frac{dM}{dt} = 2.$$

Using the above formula for $\frac{dQ}{dt}$, we get

$$\frac{dQ}{dt} = 120 \cdot 21^{0.7} \cdot 19^{-0.7} \cdot (-4) + 280 \cdot 21^{-0.3} \cdot 19^{0.3} \cdot 2 \approx 28.601889.$$

Therefore, since Q represents output in thousands of units, the rate of change of the production is $\boxed{\approx 28601.889 \text{ units/yr}}$.

- (b) Second, we are looking for the rate of change of Q with respect to time (that is, $\frac{dQ}{dt}$) when

$$L = 21, M = 20, \frac{dL}{dt} = -4, \frac{dM}{dt} = 2.$$

Using the above formula for $\frac{dQ}{dt}$, we get

$$\frac{dQ}{dt} = 120 \cdot 20^{0.7} \cdot 21^{-0.7} \cdot (-4) + 280 \cdot 20^{-0.3} \cdot 21^{0.3} \cdot 2 \approx 104.373740.$$

Therefore, since Q represents output in thousands of units, the rate of change of the production is $\boxed{\approx 104373.740 \text{ units/yr}}$.

59. (a) Using Chain Rule II, we have:

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial a} \frac{\partial a}{\partial s} + \frac{\partial y}{\partial b} \frac{\partial b}{\partial s} = f_a(a, b)h_s(s, t) + f_b(a, b)k_s(s, t) = f_a(a, b)\frac{\partial h}{\partial s}(s, t) + f_b(a, b)\frac{\partial k}{\partial s}(s, t).$$

When $s = 1$ and $t = 3$, we have $a = h(1, 3) = 6$ and $b = k(1, 3) = 2$, so

$$\frac{\partial y}{\partial s}(1, 3) = f_a(6, 2)\frac{\partial h}{\partial s}(1, 3) + f_b(6, 2)\frac{\partial k}{\partial s}(1, 3) = 7 \cdot 4 + 2 \cdot (-3) = 22.$$

$$\text{So } \boxed{\frac{\partial y}{\partial s}(1, 3) = 22}.$$

- (b) Using Chain Rule II, we have:

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial a} \frac{\partial a}{\partial t} + \frac{\partial y}{\partial b} \frac{\partial b}{\partial t} = f_a(a, b)h_t(s, t) + f_b(a, b)k_t(s, t) = f_a(a, b)\frac{\partial h}{\partial t}(s, t) + f_b(a, b)\frac{\partial k}{\partial t}(s, t).$$

When $s = 1$ and $t = 3$, we have $a = h(1, 3) = 6$ and $b = k(1, 3) = 2$, so

$$\frac{\partial y}{\partial t}(1, 3) = f_a(6, 2)\frac{\partial h}{\partial t}(1, 3) + f_b(6, 2)\frac{\partial k}{\partial t}(1, 3) = 7 \cdot 1 + 2 \cdot (-5) = -3.$$

$$\text{So } \boxed{\frac{\partial y}{\partial t}(1, 3) = -3}.$$

60. (a) Using Chain Rule II, we have:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = f_x(x, y)g_s(s, t) + f_y(x, y)h_s(s, t) = f_x(x, y)\frac{\partial x}{\partial s}(s, t) + f_y(x, y)\frac{\partial y}{\partial s}(s, t).$$

When $s = 2$ and $t = -1$, we have $x = g(2, -1) = 3$ and $y = h(2, -1) = 4$, so

$$\frac{\partial z}{\partial s}(2, -1) = f_x(3, 4)\frac{\partial x}{\partial s}(2, -1) + f_y(3, 4)\frac{\partial y}{\partial s}(2, -1) = 12 \cdot 5 + 7 \cdot 2 = 74.$$

$$\text{So } \boxed{\frac{\partial z}{\partial s}(2, -1) = 74}.$$

- (b) Using Chain Rule II, we have:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = f_x(x, y)g_t(s, t) + f_y(x, y)h_t(s, t) = f_x(x, y)\frac{\partial x}{\partial t}(s, t) + f_y(x, y)\frac{\partial y}{\partial t}(s, t).$$

When $s = 2$ and $t = -1$, we have $x = g(2, -1) = 3$ and $y = h(2, -1) = 4$, so

$$\frac{\partial z}{\partial t}(2, -1) = f_x(3, 4)\frac{\partial x}{\partial t}(2, -1) + f_y(3, 4)\frac{\partial y}{\partial t}(2, -1) = 12 \cdot (-3) + 7 \cdot (-2) = -50.$$

$$\text{So } \boxed{\frac{\partial z}{\partial t}(2, -1) = -50}.$$

61. We will use Chain Rule II many times to solve this problem:

[illegible]

Therefore, we have the following formulas:

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial u^2} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial u} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial u} \right)^2$$

$$\boxed{\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial v^2} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 z}{\partial x^2} \left(\frac{\partial x}{\partial v} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} + \frac{\partial^2 z}{\partial y^2} \left(\frac{\partial y}{\partial v} \right)^2}$$

62. We will use Chain Rule II to compute the partial derivatives of z with respect to r and θ .

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y} \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 &= \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right)^2 + \frac{1}{r^2} \left(-r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y} \right)^2 \\ &= \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right)^2 + \left(-\sin \theta \frac{\partial z}{\partial x} + \cos \theta \frac{\partial z}{\partial y} \right)^2 \\ &= \cos^2 \theta \left(\frac{\partial z}{\partial x} \right)^2 + 2 \sin \theta \cos \theta \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \sin^2 \theta \left(\frac{\partial z}{\partial y} \right)^2 + \sin^2 \theta \left(\frac{\partial z}{\partial x} \right)^2 - 2 \sin \theta \cos \theta \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \cos^2 \theta \left(\frac{\partial z}{\partial y} \right)^2 \\ &= (\cos^2 \theta + \sin^2 \theta) \left(\frac{\partial z}{\partial x} \right)^2 + (\cos^2 \theta + \sin^2 \theta) \left(\frac{\partial z}{\partial y} \right)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \end{aligned}$$

We conclude that

$$\left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2.$$

63. We will use Chain Rule II to compute the partial derivatives of z with respect to u and v .

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} = -\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 &= \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right)^2 + \left(-\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \right)^2 \\ &= \cos^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 + 2 \sin \theta \cos \theta \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \sin^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 + \sin^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 - 2 \sin \theta \cos \theta \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \cos^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 \\ &= (\cos^2 \theta + \sin^2 \theta) \left(\frac{\partial f}{\partial x} \right)^2 + (\cos^2 \theta + \sin^2 \theta) \left(\frac{\partial f}{\partial y} \right)^2 = \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \end{aligned}$$

We conclude that

$$\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 = \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2.$$

64. If we let $x = u - v$ and $y = v - u$, then we have $z = f(x, y)$, and we can use Chain Rule II to compute the partial derivatives of z with respect to u and v :

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot (-1) = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot (-1) + \frac{\partial z}{\partial y} \cdot 1 = -\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \end{aligned}$$

Therefore,

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

We conclude that

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0.$$

65. If we let $x = u^2 - v^2$ and $y = v$, then we can write $z = yf(x)$, so $\frac{\partial z}{\partial y} = f(x)$. This implies that $y \frac{\partial z}{\partial y} = yf(x) = z$, so $\frac{\partial z}{\partial y} = \frac{z}{y} = \frac{z}{v}$. Then, we can use Chain Rule II to compute the partial derivatives of z with respect to u and v :

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot 2u + \frac{\partial z}{\partial y} \cdot 0 = 2u \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot (-2v) + \frac{\partial z}{\partial y} \cdot 1 = -2v \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \end{aligned}$$

Therefore,

$$v \frac{\partial z}{\partial u} + u \frac{\partial z}{\partial v} = 2uv \frac{\partial z}{\partial x} - 2uv \frac{\partial z}{\partial x} + u \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial y} = \frac{uz}{v}.$$

We conclude that

$$v \frac{\partial z}{\partial u} + u \frac{\partial z}{\partial v} = \frac{uz}{v}.$$

66. We use an extension of Chain Rule II to compute the partial derivatives of w with respect to x , y and z .

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{dw}{du} \frac{\partial u}{\partial x} = \frac{dw}{du} \frac{\partial}{\partial x} \left((x^2 + y^2 + z^2)^{1/2} \right) \\ &= \frac{dw}{du} \cdot \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \frac{dw}{du} \\ \frac{\partial w}{\partial y} &= \frac{dw}{du} \frac{\partial u}{\partial y} = \frac{dw}{du} \frac{\partial}{\partial y} \left((x^2 + y^2 + z^2)^{1/2} \right) \\ &= \frac{dw}{du} \cdot \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \frac{dw}{du} \\ \frac{\partial w}{\partial z} &= \frac{dw}{du} \frac{\partial u}{\partial z} = \frac{dw}{du} \frac{\partial}{\partial z} \left((x^2 + y^2 + z^2)^{1/2} \right) \\ &= \frac{dw}{du} \cdot \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-1/2} \frac{\partial}{\partial z} (x^2 + y^2 + z^2) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{dw}{du} \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 &= \left(\frac{x^2}{x^2 + y^2 + z^2} + \frac{y^2}{x^2 + y^2 + z^2} + \frac{z^2}{x^2 + y^2 + z^2} \right) \left(\frac{dw}{du} \right)^2 \\ &= \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} \left(\frac{dw}{du} \right)^2 = \left(\frac{dw}{du} \right)^2 \end{aligned}$$

We conclude that

$$\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 = \left(\frac{dw}{du} \right)^2.$$

67. Let $u = \frac{x}{y}$. Then $z = f(u)$. We use Chain Rule II to compute the partial derivatives of z with respect to x and y .

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{dz}{du} \frac{\partial u}{\partial x} = \frac{dz}{du} \frac{\partial}{\partial x} \left(\frac{x}{y} \right) = \frac{1}{y} \frac{dz}{du} \\ \frac{\partial z}{\partial y} &= \frac{dz}{du} \frac{\partial u}{\partial y} = \frac{dz}{du} \frac{\partial}{\partial y} (xy^{-1}) = \frac{dz}{du} (-xy^{-2}) = \frac{-x}{y^2} \frac{dz}{du}\end{aligned}$$

Therefore,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{x}{y} \frac{dz}{du} - \frac{x}{y} \frac{dz}{du} = 0.$$

We conclude that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

68. Let $x = \frac{u}{v}$ and $y = \frac{v}{w}$. Then $z = f(x, y)$. We use an extension of Chain Rule II to compute the partial derivatives of z with respect to u , v and w .

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial}{\partial u} \left(\frac{u}{v} \right) + \frac{\partial z}{\partial y} \cdot 0 = \frac{1}{v} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial}{\partial v} \left(\frac{u}{v} \right) + \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial v} \left(\frac{v}{w} \right) = \frac{\partial z}{\partial x} (-uv^{-2}) + \frac{\partial z}{\partial y} \frac{1}{w} = -\frac{u}{v^2} \frac{\partial z}{\partial x} + \frac{1}{w} \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial w} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial w} = \frac{\partial z}{\partial x} \cdot 0 + \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial w} \left(\frac{v}{w} \right) = \frac{\partial z}{\partial y} (-vw^{-2}) = -\frac{v}{w^2} \frac{\partial z}{\partial y}\end{aligned}$$

Therefore,

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} + w \frac{\partial z}{\partial w} = \frac{u}{v} \frac{\partial z}{\partial x} - \frac{u}{v} \frac{\partial z}{\partial x} + \frac{v}{w} \frac{\partial z}{\partial y} - \frac{v}{w} \frac{\partial z}{\partial y} = 0.$$

We conclude that

$$u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} + w \frac{\partial z}{\partial w} = 0.$$

69. We will use Chain Rule II to solve this problem:

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial r^2} &= \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) = \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) \\ &= \cos \theta \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{\partial y}{\partial r} \right) + \sin \theta \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial r} \right) \\ &= \cos \theta \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial r} + \cos \theta \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial r} + \sin \theta \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial r} + \sin \theta \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial r} \\ &= \cos^2 \theta \frac{\partial^2 f}{\partial x^2} + \sin \theta \cos \theta \frac{\partial^2 f}{\partial y \partial x} + \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta} \left(-r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y} \right)\end{aligned}$$

$$\begin{aligned}
&= -r \cos \theta \frac{\partial f}{\partial x} - r \sin \theta \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) - r \sin \theta \frac{\partial f}{\partial y} + r \cos \theta \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial y} \right) \\
&= -r \cos \theta \frac{\partial f}{\partial x} - r \sin \theta \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{\partial y}{\partial \theta} \right) - r \sin \theta \frac{\partial f}{\partial y} + r \cos \theta \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial \theta} \right) \\
&= -r \cos \theta \frac{\partial f}{\partial x} + r^2 \sin^2 \theta \frac{\partial^2 f}{\partial x^2} - r^2 \sin \theta \cos \theta \frac{\partial^2 f}{\partial y \partial x} - r \sin \theta \frac{\partial f}{\partial y} - r^2 \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + r^2 \cos^2 \theta \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \left(\frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 f}{\partial \theta^2} \right) &= \cos^2 \theta \frac{\partial^2 f}{\partial x^2} + \sin \theta \cos \theta \frac{\partial^2 f}{\partial y \partial x} + \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} \\
&\quad + \sin^2 \theta \frac{\partial^2 f}{\partial y^2} + \frac{1}{r} \cos \theta \frac{\partial f}{\partial x} + \frac{1}{r} \sin \theta \frac{\partial f}{\partial y} \\
&\quad - \frac{1}{r} \cos \theta \frac{\partial f}{\partial x} + \sin^2 \theta \frac{\partial^2 f}{\partial x^2} - \sin \theta \cos \theta \frac{\partial^2 f}{\partial y \partial x} \\
&\quad - \frac{1}{r} \sin \theta \frac{\partial f}{\partial y} - \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + \cos \theta \frac{\partial^2 f}{\partial y^2} \\
&= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 f}{\partial x^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2 f}{\partial y^2} \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f.
\end{aligned}$$

We conclude that

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \left(\frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 f}{\partial \theta^2} \right).$$

70. Using the Implicit Differentiation Formulas II, we have $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$ and $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$. We conclude that

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z} \right) \left(-\frac{F_y}{F_x} \right) \left(-\frac{F_z}{F_y} \right) = -\frac{F_x F_y F_z}{F_z F_x F_y} = -1.$$

Challenge Problems

71. (a) We know, from the Implicit Differentiation Formula I, that

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \quad \text{if } F_y \neq 0.$$

We let $u = x$ and $v = y$ (which is a function of x). Then F_x and F_y become functions of u and v , which are both functions of x . So we can use Chain Rule I to compute the derivatives of F_x and F_y with respect to x .

$$\begin{aligned}
\frac{dF_x}{dx} &= \frac{\partial F_x}{\partial u} \frac{du}{dx} + \frac{\partial F_x}{\partial v} \frac{dv}{dx} = \frac{\partial F_x}{\partial x} \cdot 1 + \frac{\partial F_x}{\partial y} \frac{dy}{dx} = F_{xx} + F_{xy} \frac{dy}{dx} \\
\frac{dF_y}{dx} &= \frac{\partial F_y}{\partial u} \frac{du}{dx} + \frac{\partial F_y}{\partial v} \frac{dv}{dx} = \frac{\partial F_y}{\partial x} \cdot 1 + \frac{\partial F_y}{\partial y} \frac{dy}{dx} = F_{yx} + F_{yy} \frac{dy}{dx}
\end{aligned}$$

Now, using the Quotient Rule, we find that

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{F_x}{F_y} \right) = -\frac{\frac{d}{dx}(F_x) \cdot F_y - F_x \cdot \frac{d}{dx}(F_y)}{F_y^2} \\
&= -\frac{(F_{xx} + F_{xy} \frac{dy}{dx})F_y - F_x(F_{yx} + F_{yy} \frac{dy}{dx})}{F_y^2} \\
&= -\frac{\left(F_{xx} + F_{xy} \left(\frac{-F_x}{F_y}\right)\right)F_y - F_x\left(F_{yx} + F_{yy} \left(\frac{-F_x}{F_y}\right)\right)}{F_y^2} \\
&= -\frac{F_{xx}F_y^2 - F_{xy}F_xF_y - F_xF_{yx}F_y + F_{yy}F_x^2}{F_y^3} \\
&= -\frac{F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}}{F_y^3}
\end{aligned}$$

The last equality is correct because F has continuous second-order partial derivatives, so that Clairaut's Theorem applies. Therefore,

$$\frac{d^2y}{dx^2} = -\frac{F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}}{F_y^3}, \text{ if } F_y \neq 0.$$

(b) We take $F(x, y) = x^3 + 3xy - y^3 - 6$. Then,

$$\begin{aligned}
F_x &= 3x^2 + 3y \\
F_y &= 3x - 3y^2 \\
F_{xx} &= 6x \\
F_{xy} &= 3 \\
F_{yy} &= -6y
\end{aligned}$$

So, using the formula from part (a), we get

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}}{F_y^3} \\
&= -\frac{6x(3x - 3y^2)^2 - 6(3x^2 + 3y)(3x - 3y^2) - 6y(3x^2 + 3y)^2}{(3x - 3y^2)^3} \\
&= \frac{-6x(3x - 3y^2)^2 + 6(3x^2 + 3y)(3x - 3y^2) + 6y(3x^2 + 3y)^2}{(3x - 3y^2)^3}
\end{aligned}$$

Therefore, $\boxed{\frac{d^2y}{dx^2} = \frac{-6x(3x - 3y^2)^2 + 6(3x^2 + 3y)(3x - 3y^2) + 6y(3x^2 + 3y)^2}{(3x - 3y^2)^3}}.$

72. Using the Fundamental Theorem of Calculus, Part I, and Chain Rule II, we have:

$$\begin{aligned}
 f_x &= e^{-x^2/(4\lambda t)} \cdot \frac{\partial}{\partial x} \left(\frac{x}{2\sqrt{\lambda t}} \right) = \frac{e^{-x^2/(4\lambda t)}}{2\sqrt{\lambda t}} \\
 f_{xx} &= \frac{1}{2\sqrt{\lambda t}} \cdot \frac{\partial}{\partial x} \left(e^{-x^2/(4\lambda t)} \right) = \frac{1}{2\sqrt{\lambda t}} e^{-x^2/(4\lambda t)} \left(\frac{-2x}{4\lambda t} \right) \\
 &= \frac{-x}{4t^{3/2}\lambda^{3/2}} e^{-x^2/(4\lambda t)} \\
 f_t &= e^{-x^2/(4\lambda t)} \frac{\partial}{\partial t} \left(\frac{xt^{-1/2}}{2\sqrt{\lambda}} \right) \\
 &= e^{-x^2/(4\lambda t)} \cdot \left(\frac{-1}{2} \right) \frac{xt^{-3/2}}{2\sqrt{\lambda}} = \frac{-x}{4t^{3/2}\lambda^{1/2}} e^{-x^2/(4\lambda t)}
 \end{aligned}$$

Therefore,

$$\lambda f_{xx} = \frac{-x}{4t^{3/2}\lambda^{1/2}} e^{-x^2/(4\lambda t)} = f_t.$$

We conclude that $f_t = \lambda f_{xx}$.

Chapter 12: Functions of Several Variables

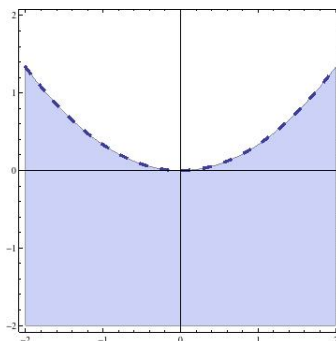
Chapter 12 Review

- $f(1, 1) = e^1 \ln 1 = e(0) = 0$. So, $\boxed{f(1, 1) = 0}$.
 - $\boxed{f(x + \Delta x, y) = e^{x+\Delta x} \ln y}$.
 - $\boxed{f(x, y + \Delta y) = e^x \ln(y + \Delta y)}$.
- $f(1, 1) = 2(1)^2 + 6(1)(1) - (1)^3 = 7$. So, $\boxed{f(1, 1) = 7}$.
 - $\boxed{f(x + \Delta x, y) = 2(x + \Delta x)^2 + 6(x + \Delta x)y - y^3}$.
 - $\boxed{f(x, y + \Delta y) = 2x^2 + 6x(y + \Delta y) - (y + \Delta y)^3}$.
- $f\left(\ln 3, \frac{1}{2}, \frac{1}{4}\right) = e^{\ln 3} \sin^{-1} \left[\frac{1}{2} + 2 \left(\frac{1}{4} \right) \right] = 3 \sin^{-1} \left(\frac{1}{2} + \frac{1}{2} \right) = 3 \sin^{-1}(1) = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}$. So, $\boxed{f\left(\ln 3, \frac{1}{2}, \frac{1}{4}\right) = \frac{3\pi}{2}}$.
 - $f\left(1, 0, \frac{1}{4}\right) = e^1 \sin^{-1} \left[0 + 2 \left(\frac{1}{4} \right) \right] = e \sin^{-1} \left(\frac{1}{2} \right) = e \cdot \frac{\pi}{6} = \frac{e\pi}{6}$. So, $\boxed{f\left(1, 0, \frac{1}{4}\right) = \frac{e\pi}{6}}$.
 - $f(0, 0, 0) = e^0 \sin^{-1}(0 + 2(0)) = (1) \sin^{-1}(0) = (1)(0) = 0$. So, $\boxed{f(0, 0, 0) = 0}$.
- A logarithm may not take a negative number as its input; so, we require that $x^2 - 3y > 0$. Equivalently:

$$\begin{aligned}
 x^2 &> 3y \\
 \frac{x^2}{3} &> y
 \end{aligned}$$

$$y < \frac{x^2}{3}$$

So, the domain is the set of points (x, y) satisfying $y < \frac{x^2}{3}$, or the set $\boxed{\{(x, y) | y < \frac{x^2}{3}\}}$. This is the portion of the xy -plane that lies strictly below the parabola $y = \frac{x^2}{3}$. See the graph below:



5. In order for f to be defined, the quantity underneath the square root must be nonnegative. So, we require that $9 - x^2 - 4y^2 \geq 0$. Equivalently, $9 \geq x^2 + 4y^2$. So the domain is the set $\boxed{\{(x, y) | x^2 + 4y^2 \leq 9\}}$.

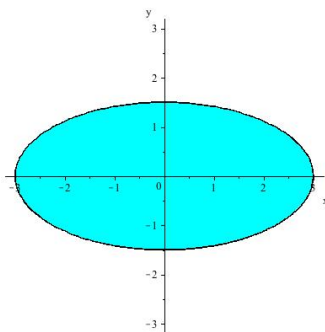
To find the graph, we transform the above inequality:

$$1 \geq \frac{x^2}{9} + \frac{4y^2}{9}$$

$$1 \geq \frac{x^2}{9} + \frac{y^2}{9/4}$$

$$\frac{x^2}{9} + \frac{y^2}{9/4} \leq 1$$

So, the domain is the set of points (x, y) satisfying $\frac{x^2}{9} + \frac{y^2}{9/4} \leq 1$. This is an ellipse centered at the origin with semi-major axis 3 and semi-minor axis $\frac{3}{2}$, and the region inside it. See the graph below:



6. In order for f to be defined, the quantity underneath the square root must be nonnegative. Because it is in the denominator, it must also be nonzero. So, we require that $5 - y^2 > 0$. Equivalently:

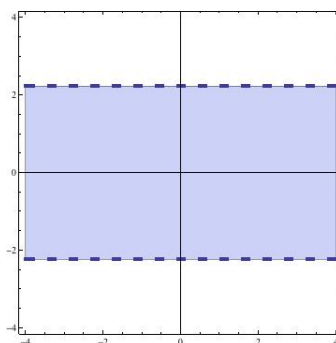
$$5 > y^2$$

$$y^2 < 5$$

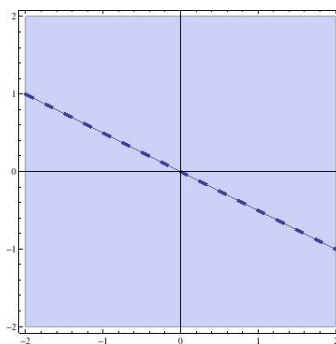
$$-\sqrt{5} < y < \sqrt{5}$$

So, the domain is the set of points (x, y) satisfying $-\sqrt{5} < y < \sqrt{5}$, or the set $\{(x, y) \mid -\sqrt{5} < y < \sqrt{5}\}$.

This is the horizontal “strip” of the plane that lies strictly between the horizontal lines $y = \sqrt{5}$ and $y = -\sqrt{5}$. See the graph below:

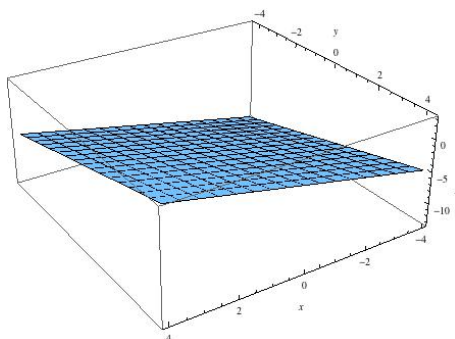


7. Because this function is rational, with a polynomial as both its numerator and denominator, it will only be undefined where the denominator is 0, or when $x + 2y = 0$. This occurs along the line $x = -2y$, or $y = -\frac{x}{2}$. So, the domain is the set of points (x, y) satisfying $y \neq -\frac{x}{2}$, or the set $\{(x, y) \mid y \neq -\frac{x}{2}\}$. This is all of $\mathbb{R}^2$ except for the line $y = -\frac{x}{2}$. See the graph below:

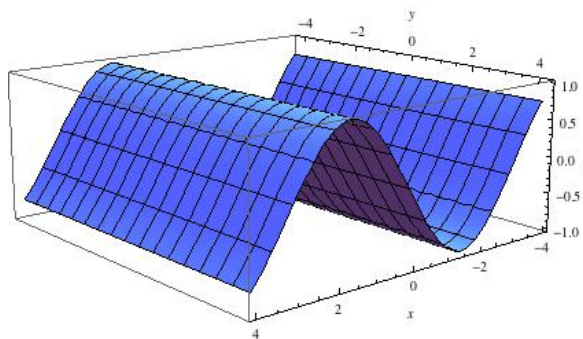


8. Because this function is rational, with a polynomial as both its numerator and denominator, it will only be undefined where the denominator is 0, or when $x^2 = 0$. This occurs only at $x = 0$. So, the domain is the set of points (x, y, z) satisfying $x \neq 0$, or the set $\{(x, y, z) \mid x \neq 0\}$. This is all of $\mathbb{R}^3$ except for the plane $x = 0$.
9. A logarithm may not take a negative number as its input; so, we require that $z > 0$. So, the domain is the set of points (x, y, z) satisfying $z > 0$, or the set $\{(x, y, z) \mid z > 0\}$. This is the part of $\mathbb{R}^3$ that lies strictly above the plane $z = 0$.

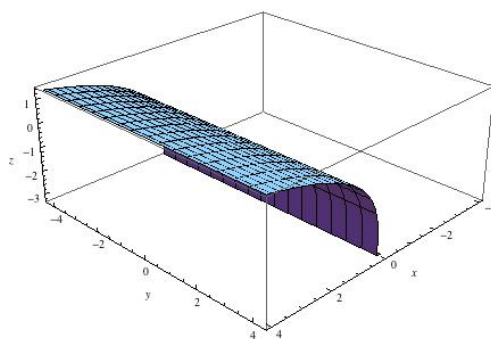
10. This is the plane $z = x - y + 5$, or $x - y - z + 5 = 0$, so it has normal vector $\langle 1, -1, -1 \rangle$. Also, it has x -intercept -5 (which we find by setting y, z to 0), y -intercept 5 (which we find by setting x, z to 0), and z -intercept 5 (which we find by setting x, y to 0).



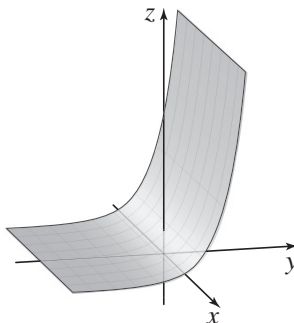
11. In this function, y is an unrestricted variable, so we may sketch the graph by sketching $z = \sin x$ in the xz -plane, and extending freely in y .



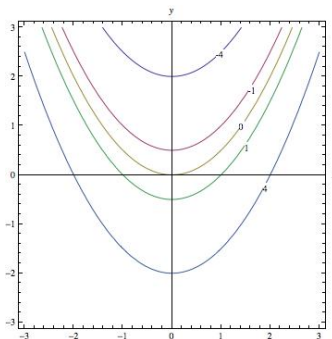
12. In this function, y is an unrestricted variable, so we may sketch the graph by sketching $z = \ln x$ in the xz -plane, and extending freely in y .



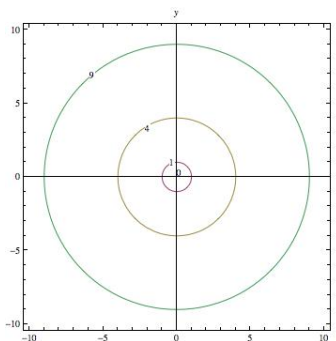
13. In this function, x is an unrestricted variable, so we may sketch the graph by sketching $z = e^y$ in the yz -plane, and extending freely in x .



14. We set $z = c$, and we get $c = x^2 - 2y$, which is equivalent to $y = \frac{x^2 - c}{2} = \frac{x^2}{2} - \frac{c}{2}$. We recognize these equations; their graphs are parabolas shrunk vertically by a factor of 2, and shifted vertically according to the value of $\frac{c}{2}$. For $c > 0$, they are shifted down $\frac{c}{2}$; for $c < 0$, they are shifted up $\frac{|c|}{2}$.



15. We set $z = c$, and we get $c = \sqrt{x^2 + y^2}$, which is equivalent to $c^2 = x^2 + y^2$. We recognize these equations; their graphs are circles centered at the origin, of radius c (note that c must be nonnegative), for $c \neq 0$. For $c = 0$, the equation becomes $x^2 + y^2 = 0$, which is satisfied only when $x = y = 0$, or at the point $(0, 0)$. So, the level curves for $c = 1, 4$, and 9 are circles of radius c centered at the origin, and the level curve for $c = 0$ is only the origin itself.

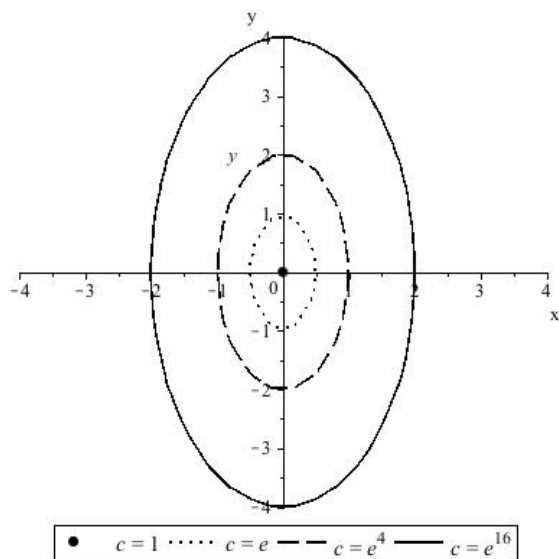


16. We set $z = c$, and we get $c = e^{4x^2 + y^2}$, which is equivalent to $\ln c = 4x^2 + y^2$. For $c = 1$, we have $\ln 1 = 4x^2 + y^2$, or $0 = 4x^2 + y^2$, which is satisfied only when $x = y = 0$, or at the point $(0, 0)$. When $c = e^k$ for $k > 0$, we have $\ln(e^k) = 4x^2 + y^2$, or $k = 4x^2 + y^2$. Equivalently:

$$1 = \frac{4x^2}{k} + \frac{y^2}{k}$$

$$1 = \frac{x^2}{k/4} + \frac{y^2}{k}$$

We recognize these equations; their graphs are ellipses centered at the origin, while the level curve for $c = 1$ is only the origin itself.



17. We consider surfaces of the form $f(x, y, z) = c = 4x^2 + y^2 + z^2$, where c is a constant (which in this case, cannot be negative since $4x^2 + y^2 + z^2 \geq 0$ for all x, y, z). Equivalently:

$$1 = \frac{4x^2}{c} + \frac{y^2}{c} + \frac{z^2}{c} \quad \text{if } c \neq 0$$

$$1 = \frac{x^2}{c/4} + \frac{y^2}{c} + \frac{z^2}{c} \quad \text{if } c \neq 0$$

We recognize these surfaces as ellipsoids centered at the origin (if $c \neq 0$), of increasing radii as c increases. If $c = 0$, our level curve has equation $0 = 4x^2 + y^2 + z^2$, which is only satisfied when $(x, y, z) = (0, 0, 0)$. Thus, our level curve for $c = 0$ is the point $(0, 0, 0)$.

One of the level surfaces associated with the function $w = 4x^2 + y^2 + z^2$ is the point $(0, 0, 0)$, and the others are all ellipsoids in space.

18. We consider surfaces of the form $f(x, y, z) = c = x + y + 2z$, where c is a constant. We observe that this surface forms a plane perpendicular to the vector $\langle 1, 1, 2 \rangle$. We can rewrite this surface as $z = -\frac{x}{2} - \frac{y}{2} + \frac{c}{2}$, from which we can see that increasing values of c increase the z -intercept of the plane (i.e. move it up).

19. We use algebraic properties of limits:

$$\lim_{(x,y) \rightarrow (\pi/2, 0)} \frac{\sin x \cos y}{x} = \left(\lim_{x \rightarrow \pi/2} \frac{\sin x}{x} \right) \left(\lim_{y \rightarrow 0} \cos y \right) = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} \cdot 1 = \frac{1}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}.$$

20. Look at the denominator. Since

$$\lim_{(x,y) \rightarrow (1,2)} (2x + y) = \left(\lim_{x \rightarrow 1} 2x \right) + \left(\lim_{y \rightarrow 2} y \right) = 2 + 2 = 4 \neq 0,$$

we can use the limit of a quotient property. Now look at the numerator:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} (4x^2 + y^2) &= \left(\lim_{x \rightarrow 1} 4x^2 \right) + \left(\lim_{y \rightarrow 2} y^2 \right) \\ &= 4 + 4 = 8. \end{aligned}$$

So, by the limit of a quotient property,

$$\lim_{(x,y) \rightarrow (1,2)} \frac{4x^2 + y^2}{2x + y} = \frac{\lim_{(x,y) \rightarrow (1,2)} (4x^2 + y^2)}{\lim_{(x,y) \rightarrow (1,2)} (2x + y)} = \frac{8}{4} = \boxed{2}.$$

21. (a) We have that, along the lines $y = mx$,

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{3xy^2}{x^2 + y^4} &= \lim_{x \rightarrow 0} \frac{3x(mx)^2}{x^2 + (mx)^4} \\ &= \lim_{x \rightarrow 0} \frac{3m^2x^3}{x^2 + m^4x^4} \\ &= \lim_{x \rightarrow 0} \frac{3m^2x^3}{x^2(1 + m^4x^2)} \\ &= \lim_{x \rightarrow 0} \frac{3m^2x}{1 + m^4x^2} = \frac{0}{1} = 0 \end{aligned}$$

(b) We have that, along the parabola $x = y^2$,

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{3xy^2}{x^2 + y^4} &= \lim_{y \rightarrow 0} \frac{3(y^2)y^2}{(y^2)^2 + y^4} \\ &= \lim_{y \rightarrow 0} \frac{3y^4}{y^4 + y^4} \\ &= \lim_{y \rightarrow 0} \frac{3y^4}{2y^4} \\ &= \lim_{y \rightarrow 0} \frac{3}{2} = \boxed{\frac{3}{2}} \end{aligned}$$

(c) Since the limits differ along different curves containing $(0,0)$, we are forced to conclude that the limit does not exist.

22. We consider the limit along different curves in $\mathbb{R}^3$ that contain the point $(0,0,0)$. We first consider the limit along the x -axis, described by $y = z = 0$:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{4xy}{x^2 + y^2 + z^2} &= \lim_{x \rightarrow 0} \frac{4x(0)}{x^2 + 0^2 + 0^2} \\ &= \lim_{x \rightarrow 0} \frac{0}{x^2} \end{aligned}$$

$$= \lim_{x \rightarrow 0} 0 = 0$$

We now consider the limit along the curve described by $y = x, z = 0$:

$$\begin{aligned} \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{4xy}{x^2 + y^2 + z^2} &= \lim_{x \rightarrow 0} \frac{4x(x)}{x^2 + x^2 + 0^2} \\ &= \lim_{x \rightarrow 0} \frac{4x^2}{2x^2} \\ &= \lim_{x \rightarrow 0} \frac{4}{2} \\ &= \lim_{x \rightarrow 0} 2 = 2 \neq 0 \end{aligned}$$

Since the limits differ along two different curves containing $(0,0,0)$, we are forced to conclude that the limit does not exist.

23. Because this function is a polynomial function, it is continuous at all points (x, y) in the plane.
24. Because this function is a rational function; that is, it is a ratio of polynomial functions, it will be continuous everywhere except where its denominator is 0. So, this function is continuous for all (x, y) such that $x^2 - y^2 \neq 0$. Equivalently, where $x^2 \neq y^2$, or where $x \neq \pm y$. So, this function is continuous on the set $\{(x, y) | x \neq \pm y\}$.
25. (a) This function is a composition of the function $g(t) = \tan^{-1} t$ and $h(x, y) = \frac{1}{x^2 + y^2}$. g is continuous everywhere since $\tan^{-1} t$ is defined for all real numbers t , and so we only need to determine where h is continuous. h is a rational function, so we only need to determine where its denominator is 0. $x^2 + y^2 = 0$ only at the point $(x, y) = (0, 0)$, so h is defined and therefore continuous on the set $\{(x, y) | (x, y) \neq (0, 0)\}$. So, the entire function f is continuous on $\{(x, y) | (x, y) \neq (0, 0)\}$.
- (b) Because $f(x, y) = \tan^{-1} \left(\frac{1}{x^2 + y^2} \right)$ is continuous at $(0, 1)$, we may evaluate the limit by evaluating the function at $(0, 1)$:

$$\lim_{(x,y) \rightarrow (0,1)} \tan^{-1} \left(\frac{1}{x^2 + y^2} \right) = \tan^{-1} \left(\frac{1}{0^2 + 1^2} \right) = \tan^{-1}(1) = \boxed{\frac{\pi}{4}}.$$

$$\begin{aligned} 26. \quad f_x &= e^{x^2+y^2} (2x) \sin(xy) + e^{x^2+y^2} \cos(xy)(y) \\ &= 2xe^{x^2+y^2} \sin(xy) + ye^{x^2+y^2} \cos(xy) \\ &= e^{x^2+y^2} [2x \sin(xy) + y \cos(xy)] \end{aligned}$$

and

$$\begin{aligned} f_y &= e^{x^2+y^2} (2y) \sin(xy) + e^{x^2+y^2} \cos(xy)(x) \\ &= 2ye^{x^2+y^2} \sin(xy) + xe^{x^2+y^2} \cos(xy) \\ &= e^{x^2+y^2} [2y \sin(xy) + x \cos(xy)] \end{aligned}$$

$$\text{so } \boxed{f_x = e^{x^2+y^2} [2x \sin(xy) + y \cos(xy)]} \text{ and } \boxed{f_y = e^{x^2+y^2} [2y \sin(xy) + x \cos(xy)]}.$$

27. We first rewrite the function as $f(x, y) = \frac{x+y}{y} = \frac{x}{y} + 1$ to simplify the computations. Then,

$$f_x = \frac{1}{y}$$

and

$$f_y = x(-y^{-2}) = -\frac{x}{y^2}$$

$$\text{so, } \boxed{f_x = \frac{1}{y}} \text{ and } \boxed{f_y = -\frac{x}{y^2}}.$$

28. We have that

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x - 2y^2)^{-1/2}(1) = \frac{1}{2\sqrt{x - 2y^2}}$$

and

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x - 2y^2)^{-1/2}(-4y) = \frac{-4y}{2\sqrt{x - 2y^2}} = \frac{-2y}{\sqrt{x - 2y^2}}$$

$$\text{So, } \boxed{\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x - 2y^2}}} \text{ and } \boxed{\frac{\partial z}{\partial y} = \frac{-2y}{\sqrt{x - 2y^2}}}.$$

29. We have that

$$\frac{\partial z}{\partial x} = e^x \ln(5x + 2y) + e^x \frac{1}{5x + 2y}(5) = e^x \left[\ln(5x + 2y) + \frac{5}{5x + 2y} \right]$$

and

$$\frac{\partial z}{\partial y} = e^x \frac{1}{5x + 2y}(2) = \frac{2e^x}{5x + 2y}$$

$$\text{So, } \boxed{\frac{\partial z}{\partial x} = e^x \left[\ln(5x + 2y) + \frac{5}{5x + 2y} \right]} \text{ and } \boxed{\frac{\partial z}{\partial y} = \frac{2e^x}{5x + 2y}}.$$

30. We first compute $f_x(x, y)$ and $f_y(x, y)$ and then evaluate these at $(2, 1)$ and $(2, -1)$ respectively.

$$f_x = \frac{1}{2}(x^2 - y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - y^2}}$$

$$f_x(2, 1) = \frac{2}{\sqrt{2^2 - 1^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

And

$$f_y = \frac{1}{2}(x^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{x^2 - y^2}}$$

$$f_y(2, -1) = \frac{-(-1)}{\sqrt{2^2 - (-1)^2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

$$\text{So, } \boxed{f_x(2, 1) = f_y(2, -1) = \frac{2\sqrt{3}}{3}}.$$

31. We first compute $F_x(x, y)$ and $F_y(x, y)$ and then evaluate each of these at $\left(0, \frac{\pi}{6}\right)$.

$$F_x = e^x \sin y$$

$$F_x\left(0, \frac{\pi}{6}\right) = e^0 \sin \frac{\pi}{6} = (1) \left(\frac{1}{2}\right) = \frac{1}{2}.$$

Also,

$$F_y = e^x \cos y$$

$$F_y\left(0, \frac{\pi}{6}\right) = e^0 \cos \frac{\pi}{6} = (1) \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

$$\text{So, } \boxed{F_x\left(0, \frac{\pi}{6}\right) = \frac{1}{2}} \text{ and } \boxed{F_y\left(0, \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}}.$$

32. The slope of the tangent line to the curve of intersection of $\frac{x^2}{24} + \frac{y^2}{12} + \frac{z^2}{6} = 1$ and $y = 1$, where $x = 4$ and $z > 0$, can be found by first solving our equation for z and then finding its partial derivative with respect to x :

$$\frac{x^2}{24} + \frac{y^2}{12} + \frac{z^2}{6} = 1$$

$$\frac{z^2}{6} = 1 - \frac{x^2}{24} - \frac{y^2}{12}$$

$$z^2 = 6 - \frac{x^2}{4} - \frac{y^2}{2}$$

$$z = \pm \sqrt{6 - \frac{x^2}{4} - \frac{y^2}{2}}$$

Because we were told to consider $z > 0$, we take the positive square root, or

$$z = \sqrt{6 - \frac{x^2}{4} - \frac{y^2}{2}} = f(x, y). \text{ Then,}$$

$$f_x(x, 1) = \frac{1}{2} \left(6 - \frac{x^2}{4} - \frac{1^2}{2}\right)^{-1/2} \left(-\frac{2x}{4}\right) = \frac{-x}{4\sqrt{\frac{11}{2} - \frac{x^2}{4}}} = \frac{-x}{4\sqrt{\frac{22 - x^2}{4}}} = \frac{-x}{4\frac{\sqrt{22 - x^2}}{2}} = \frac{-x}{2\sqrt{22 - x^2}}$$

so

at $x = 4$, the slope is $\frac{-4}{2\sqrt{22 - 4^2}} = \frac{-2}{\sqrt{6}}$. This tangent line lies on the plane $y = 1$, and when $x = 4, y = 1$, we have $z = \sqrt{6 - \frac{4^2}{4} - \frac{1^2}{2}} = \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}}$. Symmetric equations of the tangent line are then

$$\frac{z - \sqrt{\frac{3}{2}}}{1} = \frac{x - 4}{-\frac{\sqrt{6}}{2}} \quad y = 1$$

33. (a) The slope of the tangent line to the curve of intersection of $z = 4x^2 - y^2 + 7$ and $y = -2$ at any point is $f_x(x, -2) = 8x$. At the point $(1, -2, 7)$, the slope is 8. This tangent line lies on the plane $y = -2$. Symmetric equations of the tangent line are

$$\frac{z - 7}{1} = \frac{x - 1}{\frac{1}{8}} \quad y = -2$$

- (b) The slope of the tangent line to the curve of intersection of $z = 4x^2 - y^2 + 7$ and $x = 1$ at any point is $f_y(1, y) = -2y$. At the point $(1, -2, 7)$, the slope is 4. This tangent line lies on the plane $x = 1$. Symmetric equations of the tangent line are

$$\frac{z - 7}{1} = \frac{y + 2}{\frac{1}{4}} \quad x = 1$$

34. Since it is stated that V varies directly with T and inversely with P , we can write that $V = k \frac{T}{P}$ for some constant k . We use this to answer the following questions.

- (a) The rate of change of V with respect to T is given by the partial derivative of V with respect to

$$T, V_T. \text{ We have that } V_T = k \left(\frac{1}{P} \right).$$

- (b) The rate of change of V with respect to P is given by the partial derivative of V with respect to

$$P, V_P. \text{ We have that } V_P = kT(-P^{-2}) = -k \left(\frac{T}{P^2} \right), \text{ so } V_P = -k \left(\frac{T}{P^2} \right).$$

- (c) We use Chain Rule I to find $\frac{dV}{dt}$. Since V depends on P and T , which in turn each depend on t , we have that

$$\frac{dV}{dt} = \frac{\partial V}{\partial P} \frac{dP}{dt} + \frac{\partial V}{\partial T} \frac{dT}{dt}.$$

35. $f_x = (1)e^{3x} + (x + y^2)e^{3x} \cdot 3 = e^{3x} + 3(x + y^2)e^{3x}$

$$f_{xx} = 3e^{3x} + 3[(1)e^{3x} + (x + y^2)e^{3x} \cdot 3] = 3e^{3x} + 3e^{3x} + 9(x + y^2)e^{3x} = 6e^{3x} + 9(x + y^2)e^{3x}$$

$$f_{xx} = 6e^{3x} + 9(x + y^2)e^{3x}$$

$$f_{xy} = 3(2y)e^{3x} = 6ye^{3x}$$

$$f_{xy} = 6ye^{3x}$$

$$f_y = 2ye^{3x}$$

$$f_{yx} = 2ye^{3x} \cdot 3 = 6ye^{3x}$$

$$\boxed{f_{yx} = 6ye^{3x}} \text{ (note that this is equal to } f_{xy} \text{)}$$

$$\boxed{f_{yy} = 2e^{3x}}$$

36. $f_x = \sec(xy) \tan(xy) \cdot y = y \sec(xy) \tan(xy)$

$$f_{xx} = [y \sec(xy) \tan(xy) \cdot y] \tan(xy) + [y \sec(xy)] \sec^2(xy) \cdot y = y^2 \sec(xy) \tan^2(xy) + y^2 \sec^3(xy)$$

$$\boxed{f_{xx} = y^2 \sec(xy) \tan^2(xy) + y^2 \sec^3(xy)}$$

$$f_{xy} = [\sec(xy) + y \sec(xy) \tan(xy) \cdot x] \tan(xy) + y \sec(xy) \sec^2(xy) \cdot x \\ = \sec(xy) \tan(xy) + xy \sec(xy) \tan^2(xy) + xy \sec^3(xy)$$

$$\boxed{f_{xy} = \sec(xy) \tan(xy) + xy \sec(xy) \tan^2(xy) + xy \sec^3(xy)}$$

To find f_{yx} , it is easiest to note that by Clairaut's Theorem,

$$f_{yx} = f_{xy} = \sec(xy) \tan(xy) + xy \sec(xy) \tan^2(xy) + xy \sec^3(xy), \text{ and so}$$

$$\boxed{f_{yx} = \sec(xy) \tan(xy) + xy \sec(xy) \tan^2(xy) + xy \sec^3(xy)}$$

$$f_y = \sec(xy) \tan(xy) \cdot x = x \sec(xy) \tan(xy)$$

$$f_{yy} = [x \sec(xy) \tan(xy) \cdot x] \tan(xy) + [x \sec(xy)] \sec^2(xy) \cdot x = x^2 \sec(xy) \tan^2(xy) + x^2 \sec^3(xy)$$

$$\boxed{f_{yy} = x^2 \sec(xy) \tan^2(xy) + x^2 \sec^3(xy)}$$

37. $f_x = e^{xyz}(yz)$

$$\boxed{f_x = yze^{xyz}}$$

$$f_y = e^{xyz}(xz)$$

$$\boxed{f_y = xze^{xyz}}$$

$$f_z = e^{xyz}(xy)$$

$$\boxed{f_z = xye^{xyz}}$$

38. $f_x = ze^{xy} \cdot y$

$$\boxed{f_x = yze^{xy}}$$

$$f_y = ze^{xy} \cdot x$$

$$\boxed{f_y = xze^{xy}}$$

$$\boxed{f_z = e^{xy}}$$

39. $f_x = e^x \sin y$

$$f_y = e^x \cos y + e^y \sin z$$

$$f_z = e^y \cos z$$

40. $f_x = z \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right) \cdot y(-x^{-2})$

$$= \frac{-yz}{x^2 \left(1 + \frac{y^2}{x^2}\right)} = \frac{-yz}{x^2 + y^2}$$

$$f_x = \frac{-yz}{x^2 + y^2}$$

$$f_y = z \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2} \right) \cdot \frac{1}{x}$$

$$= \frac{z}{x \left(1 + \frac{y^2}{x^2}\right)} = \frac{z}{x + \frac{y^2}{x}} = \frac{xz}{x^2 + y^2}$$

$$f_y = \frac{xz}{x^2 + y^2}$$

$$f_z = \tan^{-1} \left(\frac{y}{x} \right)$$

41. We first compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x} = 3x^2y^2 - 2y^4 + 6xy^3$$

$$\frac{\partial z}{\partial y} = 2x^3y - 8xy^3 + 9x^2y^2$$

We now evaluate $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$ using these:

$$\begin{aligned} x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= x(3x^2y^2 - 2y^4 + 6xy^3) + y(2x^3y - 8xy^3 + 9x^2y^2) \\ &= 3x^3y^2 - 2xy^4 + 6x^2y^3 + 2x^3y^2 - 8xy^4 + 9x^2y^3 \\ &= 5x^3y^2 - 10xy^4 + 15x^2y^3 \\ &= 5(x^3y^2 - 2xy^4 + 3x^2y^3) = 5z \quad \checkmark \end{aligned}$$

42. We have that the change in z , Δz , is given by $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$, which we compute as follows:

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = (x_0 + \Delta x)(y_0 + \Delta y)^2 + 2 - (x_0y_0^2 + 2)$$

$$\begin{aligned}
&= (x_0 + \Delta x)(y_0^2 + 2y_0\Delta y + (\Delta y)^2) + 2 - x_0y_0^2 - 2 \\
&= x_0y_0^2 + 2x_0y_0\Delta y + x_0(\Delta y)^2 + y_0^2\Delta x + 2y_0\Delta x\Delta y + \Delta x(\Delta y)^2 - x_0y_0^2 \\
&= 2x_0y_0\Delta y + x_0(\Delta y)^2 + y_0^2\Delta x + 2y_0\Delta x\Delta y + \Delta x(\Delta y)^2
\end{aligned}$$

So $\boxed{\Delta z = 2x_0y_0\Delta y + x_0(\Delta y)^2 + y_0^2\Delta x + 2y_0\Delta x\Delta y + \Delta x(\Delta y)^2}$.

We use this answer to calculate the change in z from $(1,0)$ to $(0.9,0.2)$, which yield $x_0 = 1, y_0 = 0, \Delta x = 0.9 - 1 = -0.1, \Delta y = 0.2 - 0 = 0.2$:

$$\begin{aligned}
\Delta z &= 2(1)(0)(0.2) + (1)(0.2)^2 + (0)^2(-0.1) + 2(0)(-0.1)(0.2) + (-0.1)(0.2)^2 \\
&= 0.04 - 0.004 = 0.036
\end{aligned}$$

So, $\boxed{\Delta z = 0.036}$.

43. (a) We have that the change in z , Δz , is given by $f(x + \Delta x, y + \Delta y) - f(x, y)$, which we compute as follows:

$$\begin{aligned}
f(x + \Delta x, y + \Delta y) - f(x, y) &= (x + \Delta x)(y + \Delta y) - 5(y + \Delta y)^2 - (xy - 5y^2) \\
&= xy + x\Delta y + y\Delta x + \Delta x\Delta y - 5(y^2 + 2y\Delta y + (\Delta y)^2) - xy + 5y^2 \\
&= xy + x\Delta y + y\Delta x + \Delta x\Delta y - 5y^2 - 10y\Delta y - 5(\Delta y)^2 - xy + 5y^2 \\
&= x\Delta y + y\Delta x + \Delta x\Delta y - 10y\Delta y - 5(\Delta y)^2 \\
&= y\Delta x + (x - 10y)\Delta y + (\Delta y)\Delta x + (-5\Delta y)\Delta y
\end{aligned}$$

So, $\boxed{\Delta z = y\Delta x + (x - 10y)\Delta y + (\Delta y)\Delta x + (-5\Delta y)\Delta y}$.

- (b) Note that $f_x(x, y) = y$ and $f_y(x, y) = x - 10y$. So, we see from our manipulations from part (a) that we may choose η_1 and η_2 as follows:

$$\boxed{\eta_1 = \Delta y}$$

$$\boxed{\eta_2 = -5\Delta y},$$

which will yield $\Delta z = y\Delta x + (x - 10y)\Delta y + \eta_1\Delta x + \eta_2\Delta y$. (Note that had we rearranged Δz differently in part (a), we may have had a different choice of η_1 and η_2 .)

- (c) We now show that $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = 0$ and $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = 0$. We have that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_1 = \lim_{\Delta y \rightarrow 0} \Delta y = 0$$

and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \eta_2 = \lim_{\Delta y \rightarrow 0} -5\Delta y = 0.$$

44. The differential dz is given by $dz = f_x(x, y)dx + f_y(x, y)dy$. So, we compute $f_x(x, y)$ and $f_y(x, y)$:
 $f_x(x, y) = \sqrt{1 + y^2}$

and

$$f_y(x, y) = x \left[\frac{1}{2}(1 + y^2)^{-1/2}(2y) \right] = \frac{xy}{\sqrt{1 + y^2}}$$

so, we have that $\boxed{dz = \sqrt{1 + y^2} dx + \frac{xy}{\sqrt{1 + y^2}} dy}.$

45. The differential dz is given by $dz = f_x(x, y)dx + f_y(x, y)dy$. So, we compute $f_x(x, y)$ and $f_y(x, y)$:

$$f_x(x, y) = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}$$

$$= \frac{1}{y\sqrt{1 - \frac{x^2}{y^2}}} = \frac{1}{\sqrt{y^2 \left(1 - \frac{x^2}{y^2}\right)}} \quad \text{since } y > 0$$

$$= \frac{1}{\sqrt{y^2 - x^2}}$$

and

$$f_y(x, y) = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot x(-y^{-2})$$

$$= \frac{-x}{y^2\sqrt{1 - \frac{x^2}{y^2}}} = \frac{-x}{y\sqrt{y^2 \left(1 - \frac{x^2}{y^2}\right)}} \quad \text{since } y > 0$$

$$= \frac{-x}{y\sqrt{y^2 - x^2}}$$

so, we have that $\boxed{dz = \frac{1}{\sqrt{y^2 - x^2}} dx + \frac{-x}{y\sqrt{y^2 - x^2}} dy}.$

46. The differential dw is given by $dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$. Therefore, we compute $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$:

$$f_x(x, y, z) = ze^{xy} \cdot y = yze^{xy}$$

and

$$f_y(x, y, z) = ze^{xy} \cdot x = xze^{xy}$$

and

$$f_z(x, y, z) = e^{xy}$$

so, we have that $\boxed{dw = yze^{xy} dx + xze^{xy} dy + e^{xy} dz}$.

47. The function f is defined everywhere in space except the surface $xyz = 0$. The differential dw is given by $dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$. Therefore, we compute $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$.

$$f_x(x, y, z) = \frac{1}{xyz}(yz) = \frac{1}{x}$$

and

$$f_y(x, y, z) = \frac{1}{xyz}(xz) = \frac{1}{y}$$

and

$$f_z(x, y, z) = \frac{1}{xyz}(xy) = \frac{1}{z}$$

so, we have that $\boxed{dw = \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}$.

48. We compute the differential dz for the function $z = f(x, y) = x\sqrt{1+y^2}$ by first finding its partial derivatives at the point $(x_0, y_0) = (4, 0)$:

$$\begin{aligned} f_x(x, y) &= \sqrt{1+y^2} \\ f_x(4, 0) &= \sqrt{1+0^2} = 1 \end{aligned}$$

$$f_y(x, y) = x \left[\frac{1}{2}(1+y^2)^{-1/2}(2y) \right] = \frac{xy}{\sqrt{1+y^2}}$$

$$f_y(4, 0) = \frac{(4)(0)}{\sqrt{1+0^2}} = 0$$

Also, $dx = \Delta x = 4.1 - 4 = 0.1$ and $dy = \Delta y = 0.1 - 0 = 0.1$. The differential is given by

$$\begin{aligned} dz &= f_x(4, 0)dx + f_y(4, 0)dy \\ &= 1(0.1) + 0(0.1) \\ &= 0.1. \end{aligned}$$

So, the change in z is approximately $\boxed{0.1}$.

49. We compute the differential dz for the function $z = f(x, y) = \sin^{-1}\left(\frac{x}{y}\right)$ by first finding its partial derivatives at the point $(x_0, y_0) = (0, 1)$:

$$f_x(x, y) = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y}$$

$$\begin{aligned}
&= \frac{1}{y\sqrt{1-\frac{x^2}{y^2}}} = \frac{1}{\sqrt{y^2\left(1-\frac{x^2}{y^2}\right)}} \quad \text{since } y > 0 \\
&= \frac{1}{\sqrt{y^2-x^2}} \\
f_x(0,1) &= \frac{1}{\sqrt{1^2-0^2}} = 1 \\
f_y(x,y) &= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot x(-y^{-2}) \\
&= \frac{-x}{y^2\sqrt{1-\frac{x^2}{y^2}}} = \frac{-x}{y\sqrt{y^2\left(1-\frac{x^2}{y^2}\right)}} \quad \text{since } y > 0 \\
&= \frac{-x}{y\sqrt{y^2-x^2}} \\
f_y(0,1) &= \frac{-0}{1\sqrt{1^2-0^2}} = 0
\end{aligned}$$

Also, $dx = \Delta x = 0.1 - 0 = 0.1$ and $dy = \Delta y = 1.1 - 1 = 0.1$. The differential is given by

$$\begin{aligned}
dz &= f_x(0,1)dx + f_y(0,1)dy \\
&= 1(0.1) + 0(0.1) \\
&= 0.1.
\end{aligned}$$

So, the change in z is approximately $\boxed{0.1}$.

50. We observe that we can approximate $f(0.05, 1.98)$ by approximating the change in z from $(0,2)$ to $(0.05, 1.98)$ using the differential. So we compute, for the function $z = f(x, y) = y^2 \cos x$, the differential $dz = f_x(0, 2)dx + f_y(0, 2)dy$. We start by finding the partial derivatives f_x and f_y at the point $(0, 2)$:

$$\begin{aligned}
f_x(x, y) &= y^2(-\sin x) = -y^2 \sin x \\
f_x(0, 2) &= -2^2 \sin 0 = (-4)(0) = 0
\end{aligned}$$

and

$$\begin{aligned}
f_y(x, y) &= 2y \cos x \\
f_y(0, 2) &= 2(2) \cos 0 = (4)(1) = 4.
\end{aligned}$$

Also, $dx = \Delta x = 0.05 - 0 = 0.05$ and $dy = \Delta y = 1.98 - 2 = -0.02$. So, the change in z from $(0, 2)$ to $(0.05, 1.98)$ is approximately equal to the differential:

$$\begin{aligned}
dz &= f_x(0, 2)dx + f_y(0, 2)dy \\
&= 0(0.05) + 4(-0.02) \\
&= -0.08.
\end{aligned}$$

We can then approximate $f(0.05, 1.98)$ by adding the approximate change in z (-0.08) to the value of $f(0, 2) = 2^2 \cos 0 = 4$. This yields that $f(0.05, 1.98) \approx 4 + (-0.08) = 3.92$. So $\boxed{f(0.05, 1.98) \approx 3.92}$.

51. We approximate the maximum variation of the total resistance R by using the differential dR . Note that $\frac{|\Delta L|}{L} = \frac{|dL|}{L} = 0.01$ and $\frac{|\Delta D|}{D} = \frac{|dD|}{D} = 0.02$.

$$\begin{aligned} \text{We have that } \frac{|\Delta R|}{R} &\approx \frac{|dR|}{R} = \frac{\left| \frac{k}{D^2} dL + \frac{-2kL}{D^3} dD \right|}{\frac{kL}{D^2}} \leq \frac{\left| \frac{k}{D^2} dL \right|}{\frac{kL}{D^2}} + \frac{\left| \frac{2kL}{D^3} dD \right|}{\frac{kL}{D^2}} \\ &= \frac{|dL|}{L} + 2 \frac{|dD|}{D} \\ &= 0.01 + 2(0.02) = 0.05. \end{aligned}$$

So, the maximum variation of the total resistance R is $\boxed{\approx 5\%}$

52. We compute $\frac{dz}{dt}$ by applying Chain Rule I. Since z is a function of x and y , which are in turn each functions of t , we have the following:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

We compute the four required derivatives:

$$\frac{\partial z}{\partial x} = \cos(xy) \cdot y - \sin y = y \cos(xy) - \sin y$$

$$\frac{\partial z}{\partial y} = \cos(xy) \cdot x - x \cos y = x \cos(xy) - x \cos y$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = (1)e^t + te^t = e^t + te^t$$

So then

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (y \cos(xy) - \sin y)e^t + (x \cos(xy) - x \cos y)(e^t + te^t) \\ &= [(te^t) \cos(e^t \cdot te^t) - \sin(te^t)]e^t + [e^t \cos(e^t \cdot te^t) - (e^t) \cos(te^t)](e^t + te^t) \\ &= te^{2t} \cos(te^{2t}) - e^t \sin(te^t) + e^{2t} \cos(te^{2t}) + te^{2t} \cos(te^{2t}) - e^{2t} \cos(te^t) - te^{2t} \cos(te^t) \\ &= 2te^{2t} \cos(te^{2t}) - e^t \sin(te^t) + e^{2t} \cos(te^{2t}) - e^{2t} \cos(te^t) - te^{2t} \cos(te^t) \end{aligned}$$

$$\text{So, } \boxed{\frac{dz}{dt} = 2te^{2t} \cos(te^{2t}) - e^t \sin(te^t) + e^{2t} \cos(te^{2t}) - e^{2t} \cos(te^t) - te^{2t} \cos(te^t)}.$$

53. We compute $\frac{dw}{dt}$ by applying Chain Rule I. Since W is a function of x, y , and z , which are in turn each functions of t , we have the following:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

We compute the six required derivatives:

$$\frac{\partial w}{\partial x} = \frac{1}{y} + z(-x^{-2}) = \frac{1}{y} - \frac{z}{x^2}$$

$$\frac{\partial w}{\partial y} = x(-y^{-2}) + \frac{1}{z} = -\frac{x}{y^2} + \frac{1}{z}$$

$$\frac{\partial w}{\partial z} = y(-z^{-2}) + \frac{1}{x} = -\frac{y}{z^2} + \frac{1}{x}$$

$$\frac{dx}{dt} = -t^{-2} = -\frac{1}{t^2}$$

$$\frac{dy}{dt} = -2t^{-3} = -\frac{2}{t^3}$$

$$\frac{dz}{dt} = -3t^{-4} = -\frac{3}{t^4}$$

So then

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= \left(\frac{1}{y} - \frac{z}{x^2} \right) \left(-\frac{1}{t^2} \right) + \left(-\frac{x}{y^2} + \frac{1}{z} \right) \left(-\frac{2}{t^3} \right) + \left(-\frac{y}{z^2} + \frac{1}{x} \right) \left(-\frac{3}{t^4} \right) \\ &= \left(\frac{1}{\frac{1}{t^2}} - \frac{\frac{1}{t^3}}{\left(\frac{1}{t} \right)^2} \right) \left(-\frac{1}{t^2} \right) + \left(-\frac{\frac{1}{t}}{\left(\frac{1}{t^2} \right)^2} + \frac{1}{\frac{1}{t^3}} \right) \left(-\frac{2}{t^3} \right) + \left(-\frac{\frac{1}{t^2}}{\left(\frac{1}{t^3} \right)^2} + \frac{1}{\frac{1}{t}} \right) \left(-\frac{3}{t^4} \right) \\ &= \left(t^2 - \frac{1}{t} \right) \left(-\frac{1}{t^2} \right) + (-t^3 + t^3) \left(-\frac{2}{t^3} \right) + (-t^4 + t) \left(-\frac{3}{t^4} \right) \\ &= -1 + \frac{1}{t^3} + 0 + 3 - \frac{3}{t^3} = 2 - \frac{2}{t^3} \end{aligned}$$

$$\text{So, } \boxed{\frac{dw}{dt} = \left(\frac{1}{y} - \frac{z}{x^2} \right) \left(-\frac{1}{t^2} \right) + \left(-\frac{x}{y^2} + \frac{1}{z} \right) \left(-\frac{2}{t^3} \right) + \left(-\frac{y}{z^2} + \frac{1}{x} \right) \left(-\frac{3}{t^4} \right) = 2 - \frac{2}{t^3}}.$$

54. We compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ by applying Chain Rule II. Since w is a function of x, y , and z , which are in turn each functions of u and v , we have the following:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$

We compute the nine required derivatives:

$$\frac{\partial w}{\partial x} = y - z$$

$$\frac{\partial w}{\partial y} = x + z$$

$$\frac{\partial w}{\partial z} = y - x$$

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial y}{\partial u} = v$$

$$\frac{\partial z}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial v} = u$$

$$\frac{\partial z}{\partial v} = 1$$

So then

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\ &= (y - z)(1) + (x + z)(v) + (y - x)(0) = y - z + xv + zv \\ &= uv - v + (u + v)v + (v)v = uv - v + uv + v^2 + v^2 \\ &= 2uv + 2v^2 - v\end{aligned}$$

$$\text{So, } \boxed{\frac{\partial w}{\partial u} = (y - z) + (x + z)(v) = 2uv + 2v^2 - v}. \text{ Also,}$$

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\ &= (y - z)(1) + (x + z)(u) + (y - x)(1) = y - z + xu + zu + y - x \\ &= 2y - x - z + xu + zu = 2uv - (u + v) - v + (u + v)u + vu \\ &= 2uv - u - v - v + u^2 + uv + uv = 4uv - u - 2v + u^2\end{aligned}$$

$$\text{So, } \boxed{\frac{\partial w}{\partial v} = (y - z) + (x + z)(u) + (y - x) = 4uv - u - 2v + u^2}.$$

55. We compute $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$ by applying Chain Rule II. Since u is a function of x, y , and z , which are in turn each functions of r and s , we have the following:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}$$

and

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

We compute the nine required derivatives:

$$\frac{\partial u}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial x}{\partial r} = \cos s$$

$$\frac{\partial y}{\partial r} = \sin s$$

$$\frac{\partial z}{\partial r} = \frac{1}{2}(r^2 + s^2)^{-1/2}(2r) = \frac{r}{\sqrt{r^2 + s^2}}$$

$$\frac{\partial x}{\partial s} = -r \sin s$$

$$\frac{\partial y}{\partial s} = r \cos s$$

$$\frac{\partial z}{\partial s} = \frac{1}{2}(r^2 + s^2)^{-1/2}(2s) = \frac{s}{\sqrt{r^2 + s^2}}$$

So then

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} \\ &= \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cos s + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \sin s + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{r}{\sqrt{r^2 + s^2}} \\ &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[x \cos s + y \sin s + z \frac{r}{\sqrt{r^2 + s^2}} \right] \\ &= \frac{1}{\sqrt{(r \cos s)^2 + (r \sin s)^2 + (\sqrt{r^2 + s^2})^2}} \left[r \cos s \cos s + r \sin s \sin s + \sqrt{r^2 + s^2} \frac{r}{\sqrt{r^2 + s^2}} \right] \\ &= \frac{1}{\sqrt{r^2 \cos^2 s + r^2 \sin^2 s + r^2 + s^2}} [r \cos^2 s + r \sin^2 s + r] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{r^2(\cos^2 s + \sin^2 s) + r^2 + s^2}} [r(\cos^2 s + \sin^2 s) + r] \\
&= \frac{1}{\sqrt{r^2 + r^2 + s^2}} (2r) \\
&= \frac{1}{\sqrt{2r^2 + s^2}} (2r) \\
&= \frac{2r}{\sqrt{2r^2 + s^2}} \\
\text{So, } \boxed{\frac{\partial u}{\partial r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cos s + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \sin s + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{r}{\sqrt{r^2 + s^2}} = \frac{2r}{\sqrt{2r^2 + s^2}}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s} \\
&= \frac{x}{\sqrt{x^2 + y^2 + z^2}} (-r \sin s) + \frac{y}{\sqrt{x^2 + y^2 + z^2}} (r \cos s) + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{s}{\sqrt{r^2 + s^2}} \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[-xr \sin s + yr \cos s + z \frac{s}{\sqrt{r^2 + s^2}} \right] \\
&= \frac{1}{\sqrt{(r \cos s)^2 + (r \sin s)^2 + (\sqrt{r^2 + s^2})^2}} \left[-r \cos s \cdot r \sin s + r \sin s \cdot r \cos s + \sqrt{r^2 + s^2} \frac{s}{\sqrt{r^2 + s^2}} \right] \\
&= \frac{1}{\sqrt{r^2 \cos^2 s + r^2 \sin^2 s + r^2 + s^2}} (s) \\
&= \frac{s}{\sqrt{r^2(\cos^2 s + \sin^2 s) + r^2 + s^2}} \\
&= \frac{s}{\sqrt{r^2 + r^2 + s^2}} \\
&= \frac{s}{\sqrt{2r^2 + s^2}} \\
\text{So, } \boxed{\frac{\partial u}{\partial s} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} (-r \sin s) + \frac{y}{\sqrt{x^2 + y^2 + z^2}} (r \cos s) + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{s}{\sqrt{r^2 + s^2}} = \frac{s}{\sqrt{2r^2 + s^2}}}.
\end{aligned}$$

56. First, we find the partial derivatives of F :

$$F_x = 2x - 2yz, F_y = 2y - 2xz \text{ and } F_z = -2xy.$$

Then we use Implicit Differentiation Formulas II. If $-2xy \neq 0$,

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{2x - 2yz}{-2xy} \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{2y - 2xz}{-2xy}
\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{2x - 2yz}{2xy}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{2y - 2xz}{2xy}}$$

57. First, we find the partial derivatives of F :

$$F_x = 2 \sin y + 2y \cos x + 2yz, F_y = 2x \cos y + 2 \sin x + 2xz \text{ and } F_z = 2xy.$$

Then we use Implicit Differentiation Formulas II. If $2xy \neq 0$,

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{2 \sin y + 2y \cos x + 2yz}{2xy} = -\frac{2(\sin y + y \cos x + yz)}{2xy} \\ &= -\frac{\sin y + y \cos x + yz}{xy} = \frac{-\sin y - y \cos x - yz}{xy} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{2x \cos y + 2 \sin x + 2xz}{2xy} = -\frac{2(x \cos y + \sin x + xz)}{2xy} \\ &= -\frac{x \cos y + \sin x + xz}{xy} = \frac{-x \cos y - \sin x - xz}{xy} \end{aligned}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\sin y - y \cos x - yz}{xy}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-x \cos y - \sin x - xz}{xy}}$$

58. We write the given function $z = uf(u^2 + v^2)$ as a function of two variables, u and w , by making the substitution $w = u^2 + v^2$. This yields $z = uf(w)$. (Note that f is now a function of one variable.) We compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using Chain Rule II:

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial u} \\ &= f(w) + \left(u \frac{df}{dw}\right)(2u) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial w} \frac{\partial w}{\partial v} \\ &= \left(u \frac{df}{dw}\right)(2v) \end{aligned}$$

then

$$2v \frac{\partial z}{\partial u} - 2u \frac{\partial z}{\partial v} = 2v \left[f(w) + \left(u \frac{df}{dw}\right)(2u) \right] - 2u \left[\left(u \frac{df}{dw}\right)(2v) \right]$$

$$\begin{aligned}
&= 2vf(w) + 4u^2v \frac{df}{dw} - 4u^2v \frac{df}{dw} \\
&= 2vf(w)
\end{aligned}$$

Multiplying top and bottom by u yields $\frac{2vuf(w)}{u} = \frac{2vz}{u}$. ✓

Note that this holds for $u \neq 0$. But, if $u = 0$, then z and all its partials are also 0, so the equation holds as well.

Chapter 13

Directional Derivatives, Gradients, and Extrema

13.1 Directional Derivatives; Gradients

Concepts and Vocabulary

1. **True**.
2. **True**.
3. **False**. The directional derivative is the dot product of two vectors so it is a **scalar**.
4. **True**.
5. **False**. The gradient of f is $f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \sin y\mathbf{i} + x \cos y\mathbf{j}$.
6. The directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ of f at (x_0, y_0) is a maximum when $\mathbf{u} = \frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$.
7. The value of $z = f(x, y)$ at (x_0, y_0) decreases most rapidly in the direction of $-\nabla f(x_0, y_0)$.
8. For a differentiable function $z = f(x, y)$, the maximum value of $D_{\mathbf{u}}f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$.

Skill Building

9. (a) The partial derivatives are

$$f_x(x, y) = y^2 + 2x \quad \text{and} \quad f_y(x, y) = 2xy$$

and are continuous, so f is differentiable. Using the unit vector $\mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= (y^2 + 2x) \cdot \frac{1}{2} + (2xy) \cdot \frac{\sqrt{3}}{2} \\ &= \frac{1}{2}y^2 + x + xy\sqrt{3}. \end{aligned}$$

$$\text{Then } D_{\mathbf{u}}f(-1, 2) = \frac{1}{2}(2)^2 + (-1) + (-1)(2)\sqrt{3} = 1 - 2\sqrt{3}.$$

(b) At the point $(-1, 2)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(-1, 2) = 1 - 2\sqrt{3}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x, y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(-1, 2, 0)$ in the direction $\mathbf{u}$ is $1 - 2\sqrt{3}$.

10. (a) The partial derivatives are

$$f_x(x, y) = 3y \quad \text{and} \quad f_y(x, y) = 3x + 2y$$

and are continuous, so f is differentiable. Using the unit vector $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= (3y) \cdot \frac{\sqrt{2}}{2} + (3x + 2y) \cdot \frac{\sqrt{2}}{2} \\ &= \frac{5\sqrt{2}}{2}y + \frac{3\sqrt{2}}{2}x. \end{aligned}$$

Then $D_{\mathbf{u}}f(2, 1) = \frac{5\sqrt{2}}{2}(1) + \frac{3\sqrt{2}}{2}(2) = \boxed{\frac{11\sqrt{2}}{2}}$.

(b) At the point $(2, 1)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(2, 1) = \frac{11\sqrt{2}}{2}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x, y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(2, 1, 0)$ in the direction $\mathbf{u}$ is $\frac{11\sqrt{2}}{2}$.

11. (a) The partial derivatives are

$$f_x(x, y) = 2y \quad \text{and} \quad f_y(x, y) = 2x - 2y$$

and are continuous, so f is differentiable. Using formula (1) with $\theta = \frac{2\pi}{3}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= (2y) \cdot \cos \frac{2\pi}{3} + (2x - 2y) \cdot \sin \frac{2\pi}{3} = (2y) \cdot \left(-\frac{1}{2}\right) + (2x - 2y) \cdot \frac{\sqrt{3}}{2} \\ &= -y + x\sqrt{3} - y\sqrt{3}. \end{aligned}$$

where $\mathbf{u} = \cos \frac{2\pi}{3}\mathbf{i} + \sin \frac{2\pi}{3}\mathbf{j}$. Then $D_{\mathbf{u}}f(-1, 3) = -(3) + (-1)\sqrt{3} - (3)\sqrt{3} = \boxed{-3 - 4\sqrt{3}}$.

(b) At the point $(-1, 3)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(-1, 3) = -3 - 4\sqrt{3}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x, y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(-1, 3, 0)$ in the direction $\mathbf{u}$ is $-3 - 4\sqrt{3}$.

12. (a) The partial derivatives are

$$f_x(x, y) = 2y + 2x \quad \text{and} \quad f_y(x, y) = 2x$$

and are continuous, so f is differentiable. Using formula (1) with $\theta = \frac{4\pi}{3}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= (2y + 2x) \cdot \cos \frac{4\pi}{3} + (2x) \cdot \sin \frac{4\pi}{3} = (2y + 2x) \cdot \left(-\frac{1}{2}\right) + (2x) \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= -y - x - x\sqrt{3}. \end{aligned}$$

where $\mathbf{u} = \cos \frac{4\pi}{3}\mathbf{i} + \sin \frac{4\pi}{3}\mathbf{j}$. Then $D_{\mathbf{u}}f(0, 3) = -(3) - 0 - 0\sqrt{3} = \boxed{-3}$.

(b) At the point $(0, 3)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(0, 3) = -3$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x, y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(0, 3, 0)$ in the direction $\mathbf{u}$ is -3 .

13. (a) The partial derivatives are

$$f_x(x, y) = e^y + ye^x \quad \text{and} \quad f_y(x, y) = xe^y + e^x$$

and are continuous, so f is differentiable. Using formula (1) with $\theta = \frac{\pi}{6}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= (e^y + ye^x) \cdot \cos \frac{\pi}{6} + (xe^y + e^x) \cdot \sin \frac{\pi}{6} = (e^y + ye^x) \cdot \left(\frac{\sqrt{3}}{2}\right) + (xe^y + e^x) \cdot \left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{2}e^y + \frac{\sqrt{3}}{2}ye^x + \frac{1}{2}xe^y + \frac{1}{2}e^x. \end{aligned}$$

where $\mathbf{u} = \cos \frac{\pi}{6} \mathbf{i} + \sin \frac{\pi}{6} \mathbf{j}$. Then $D_{\mathbf{u}}f(0,0) = \boxed{\frac{1+\sqrt{3}}{2}}$.

(b) At the point $(0,0)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(0,0) = \frac{\sqrt{3}+1}{2}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x,y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(0,0,0)$ in the direction $\mathbf{u}$ is $\frac{\sqrt{3}+1}{2}$.

14. (a) The partial derivatives are

$$f_x(x,y) = \ln y \quad \text{and} \quad f_y(x,y) = \frac{x}{y}$$

and are continuous when $y \neq 0$, so f is differentiable when $y \neq 0$. Using formula (1) with $\theta = \frac{\pi}{4}$, we have

$$D_{\mathbf{u}}f(x,y) = \ln y \cdot \cos \frac{\pi}{4} + \frac{x}{y} \cdot \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \left(\ln y + \frac{x}{y} \right)$$

where $\mathbf{u} = \cos \frac{\pi}{4} \mathbf{i} + \sin \frac{\pi}{4} \mathbf{j}$. Then $D_{\mathbf{u}}f(5,1) = \frac{\sqrt{2}}{2} \left(\ln(1) + \frac{5}{1} \right) = \boxed{\frac{5\sqrt{2}}{2}}$.

(b) At the point $(5,1)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(5,1) = \frac{5\sqrt{2}}{2}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x,y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(5,1,0)$ in the direction $\mathbf{u}$ is $\frac{5\sqrt{2}}{2}$.

15. (a) First we note that $\mathbf{a}$ is not a unit vector. The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{\|\mathbf{a}\|}$. Then

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{(3)^2 + (-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of $f(x,y) = \tan^{-1}(yx^{-1})$ are

$$f_x(x,y) = \frac{1}{1+(yx^{-1})^2} \cdot -yx^{-2} = -\frac{y}{x^2+y^2} \quad \text{and} \quad f_y(x,y) = \frac{1}{1+(yx^{-1})^2} \cdot x^{-1} = \frac{x}{x^2+y^2}$$

and are continuous when $(x,y) \neq (0,0)$, so f is differentiable when $(x,y) \neq (0,0)$. Using the unit vector $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= -\frac{y}{x^2+y^2} \cdot \frac{3}{5} + \frac{x}{x^2+y^2} \cdot -\frac{4}{5} \\ &= \frac{-3y-4x}{5(x^2+y^2)}. \end{aligned}$$

Then $D_{\mathbf{u}}f(1,1) = \frac{-3(1)-4(1)}{5((1)^2+(1)^2)} = \boxed{-\frac{7}{10}}$.

(b) At the point $(1,1)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(1,1) = -\frac{7}{10}$.

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x,y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(1,1,0)$ in the direction $\mathbf{u}$ is $-\frac{7}{10}$.

16. (a) First we note that $\mathbf{a}$ is not a unit vector. The unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ is $\frac{\mathbf{a}}{\|\mathbf{a}\|}$. Then

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{5\mathbf{i} + 12\mathbf{j}}{\sqrt{(5)^2 + (12)^2}} = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$

The partial derivatives of $f(x, y) = \ln \left((x^2 + y^2)^{1/2} \right) = \frac{1}{2} \ln (x^2 + y^2)$ are

$$f_x(x, y) = \frac{1}{2} \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = \frac{1}{2} \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2}$$

and are continuous when $(x, y) \neq (0, 0)$, so f is differentiable when $(x, y) \neq (0, 0)$. Using the unit vector $\mathbf{u} = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$, we have

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= \frac{x}{x^2 + y^2} \cdot \frac{5}{13} + \frac{y}{x^2 + y^2} \cdot \frac{12}{13} \\ &= \frac{5x + 12y}{13(x^2 + y^2)}. \end{aligned}$$

Then $D_{\mathbf{u}}f(3, 4) = \frac{5(3) + 12(4)}{13((3)^2 + (4)^2)} = \boxed{\frac{63}{325}}.$

(b) At the point $(3, 4)$, in the direction of $\mathbf{u}$, the rate of change of the function is $D_{\mathbf{u}}f(3, 4) = \frac{63}{325}.$

(c) The slope of the tangent line to the curve C that is the intersection of the surface $f(x, y)$ and the plane perpendicular to the xy -plane that contains the line through the point $(3, 4, 0)$ in the direction $\mathbf{u}$ is $\frac{63}{325}.$

17. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (y^2 + 2x)\mathbf{i} + (2xy)\mathbf{j}.$$

The gradient of f at the point $(1, 2)$ is

$$\boxed{\nabla f(1, 2) = 6\mathbf{i} + 4\mathbf{j}}.$$

(b) The unit vector from $(1, 2)$ to $(2, 4)$ is

$$\mathbf{u} = \frac{(2-1)\mathbf{i} + (4-2)\mathbf{j}}{\sqrt{(2-1)^2 + (4-2)^2}} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}f(1, 2)$, we have

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = (6\mathbf{i} + 4\mathbf{j}) \cdot \left(\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j} \right) = \boxed{\frac{14\sqrt{5}}{5}}.$$

18. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (2y + 2x)\mathbf{i} + (2x)\mathbf{j}.$$

The gradient of f at the point $(-1, 1)$ is

$$\boxed{\nabla f(-1, 1) = -2\mathbf{j}}.$$

(b) The unit vector from $(-1, 1)$ to $(1, 2)$ is

$$\mathbf{u} = \frac{(1 - (-1))\mathbf{i} + (2 - 1)\mathbf{j}}{\sqrt{(1 - (-1))^2 + (2 - 1)^2}} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}f(-1, 1)$, we have

$$D_{\mathbf{u}}f(-1, 1) = \nabla f(-1, 1) \cdot \mathbf{u} = (-2\mathbf{j}) \cdot \left(\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j} \right) = \boxed{-\frac{2\sqrt{5}}{5}}.$$

19. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (2y + 2x)\mathbf{i} + (2x)\mathbf{j}.$$

The gradient of f at the point $(0, 3)$ is

$$\boxed{\nabla f(0, 3) = 6\mathbf{i}}.$$

(b) The unit vector from $(0, 3)$ to $(4, 1)$ is

$$\mathbf{u} = \frac{(4-0)\mathbf{i} + (1-3)\mathbf{j}}{\sqrt{(4-0)^2 + (1-3)^2}} = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}(0, 3)$, we have

$$D_{\mathbf{u}}(0, 3) = \nabla f(0, 3) \cdot \mathbf{u} = (6\mathbf{i}) \cdot \left(\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j} \right) = \boxed{\frac{12\sqrt{5}}{5}}.$$

20. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (3y)\mathbf{i} + (3x + 2y)\mathbf{j}.$$

The gradient of f at the point $(2, 1)$ is

$$\boxed{\nabla f(2, 1) = 3\mathbf{i} + 8\mathbf{j}}.$$

(b) The unit vector from $(2, 1)$ to $(4, 1)$ is

$$\mathbf{u} = \frac{(4-2)\mathbf{i} + (1-1)\mathbf{j}}{\sqrt{(4-2)^2 + (1-1)^2}} = \mathbf{i}.$$

Using formula (3) to find $D_{\mathbf{u}}(2, 1)$, we have

$$D_{\mathbf{u}}(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = (3\mathbf{i} + 8\mathbf{j}) \cdot (\mathbf{i}) = \boxed{3}.$$

21. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (y + \cos x)\mathbf{i} + (x)\mathbf{j}.$$

The gradient of f at the point $(0, 1)$ is

$$\boxed{\nabla f(0, 1) = 2\mathbf{i}}.$$

(b) The unit vector from $(0, 1)$ to $(\pi, 2)$ is

$$\mathbf{u} = \frac{(\pi-0)\mathbf{i} + (2-1)\mathbf{j}}{\sqrt{(\pi-0)^2 + (2-1)^2}} = \frac{\pi}{\sqrt{\pi^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{\pi^2 + 1}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}(0, 1)$, we have

$$D_{\mathbf{u}}(0, 1) = \nabla f(0, 1) \cdot \mathbf{u} = (2\mathbf{i}) \cdot \left(\frac{\pi}{\sqrt{\pi^2 + 1}}\mathbf{i} + \frac{1}{\sqrt{\pi^2 + 1}}\mathbf{j} \right) = \boxed{\frac{2\pi}{\sqrt{\pi^2 + 1}}}.$$

22. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = ye^{xy}\mathbf{i} + (xe^{xy} + \cos y)\mathbf{j}.$$

The gradient of f at the point $\left(0, \frac{\pi}{2}\right)$ is

$$\nabla f\left(0, \frac{\pi}{2}\right) = \frac{\pi}{2} \mathbf{i}.$$

(b) The unit vector from $\left(0, \frac{\pi}{2}\right)$ to $(1, 0)$ is

$$\mathbf{u} = \frac{(1-0)\mathbf{i} + \left(0 - \frac{\pi}{2}\right)\mathbf{j}}{\sqrt{(1-0)^2 + \left(0 - \frac{\pi}{2}\right)^2}} = \frac{2}{\sqrt{\pi^2 + 4}}\mathbf{i} - \frac{\pi}{\sqrt{\pi^2 + 4}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}\left(0, \frac{\pi}{2}\right)$, we have

$$D_{\mathbf{u}}\left(0, \frac{\pi}{2}\right) = \nabla f\left(0, \frac{\pi}{2}\right) \cdot \mathbf{u} = \left(\frac{\pi}{2}\mathbf{i}\right) \cdot \left(\frac{2}{\sqrt{\pi^2 + 4}}\mathbf{i} - \frac{\pi}{\sqrt{\pi^2 + 4}}\mathbf{j}\right) = \boxed{\frac{\pi}{\sqrt{\pi^2 + 4}}}.$$

23. (a) The partial derivatives were computed in Problem 15, so the gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \left(-\frac{y}{x^2 + y^2}\right)\mathbf{i} + \left(\frac{x}{x^2 + y^2}\right)\mathbf{j}.$$

The gradient of f at the point $(1, 0)$ is

$$\nabla f(1, 0) = \mathbf{j}.$$

(b) The unit vector from $(1, 0)$ to $(4, \pi)$ is

$$\mathbf{u} = \frac{(4-1)\mathbf{i} + (\pi-0)\mathbf{j}}{\sqrt{(4-1)^2 + (\pi-0)^2}} = \frac{3}{\sqrt{\pi^2 + 9}}\mathbf{i} + \frac{\pi}{\sqrt{\pi^2 + 9}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}(1, 0)$, we have

$$D_{\mathbf{u}}(1, 0) = \nabla f(1, 0) \cdot \mathbf{u} = (\mathbf{j}) \cdot \left(\frac{3}{\sqrt{\pi^2 + 9}}\mathbf{i} + \frac{\pi}{\sqrt{\pi^2 + 9}}\mathbf{j}\right) = \boxed{\frac{\pi}{\sqrt{\pi^2 + 9}}}.$$

24. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \left(\frac{x}{x^2 + y^2}\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2}\right)\mathbf{j}.$$

The gradient of f at the point $(3, 4)$ is

$$\nabla f(3, 4) = \frac{3}{25}\mathbf{i} + \frac{4}{25}\mathbf{j}.$$

(b) The unit vector from $(3, 4)$ to $(0, 5)$ is

$$\mathbf{u} = \frac{(0-3)\mathbf{i} + (5-4)\mathbf{j}}{\sqrt{(0-3)^2 + (5-4)^2}} = -\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}(3, 4)$, we have

$$D_{\mathbf{u}}(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \left(\frac{3}{25}\mathbf{i} + \frac{4}{25}\mathbf{j}\right) \cdot \left(-\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}\right) = \boxed{-\frac{1}{5\sqrt{10}}}.$$

25. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (2xe^y)\mathbf{i} + (x^2e^y)\mathbf{j}.$$

The gradient of f at the point $(2, 0)$ is

$$\boxed{\nabla f(2, 0) = 4\mathbf{i} + 4\mathbf{j}}.$$

(b) The unit vector from $(2, 0)$ to $(3, 0)$ is

$$\mathbf{u} = \frac{(3-2)\mathbf{i} + (0-0)\mathbf{j}}{\sqrt{(3-2)^2 + (0-0)^2}} = \mathbf{i}.$$

Using formula (3) to find $D_{\mathbf{u}}(2, 0)$, we have

$$D_{\mathbf{u}}(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = (4\mathbf{i} + 4\mathbf{j}) \cdot (\mathbf{i}) = \boxed{4}.$$

26. (a) The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (2xe^{x^2+y^2})\mathbf{i} + (2ye^{x^2+y^2})\mathbf{j}.$$

The gradient of f at the point $(1, 2)$ is

$$\boxed{\nabla f(1, 2) = 2e^5\mathbf{i} + 4e^5\mathbf{j}}.$$

(b) The unit vector from $(1, 2)$ to $(2, 3)$ is

$$\mathbf{u} = \frac{(2-1)\mathbf{i} + (3-2)\mathbf{j}}{\sqrt{(2-1)^2 + (3-2)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

Using formula (3) to find $D_{\mathbf{u}}(1, 2)$, we have

$$D_{\mathbf{u}}(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = (2e^5\mathbf{i} + 4e^5\mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = \boxed{2\sqrt{2}e^5}.$$

27. (a) The gradient of f at (x, y, z) is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = (2xy - yz^2)\mathbf{i} + (x^2 - xz^2)\mathbf{j} + (-2xyz)\mathbf{k}.$$

The gradient of f at the point $(0, 1, 2)$ is

$$\boxed{\nabla f(0, 1, 2) = -4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = -4\mathbf{i}}.$$

(b) The unit vector from $(0, 1, 2)$ to $(1, 4, 3)$ is

$$\mathbf{u} = \frac{(1-0)\mathbf{i} + (4-1)\mathbf{j} + (3-2)\mathbf{k}}{\sqrt{(1-0)^2 + (4-1)^2 + (3-2)^2}} = \frac{1}{\sqrt{11}}\mathbf{i} + \frac{3}{\sqrt{11}}\mathbf{j} + \frac{1}{\sqrt{11}}\mathbf{k}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(0, 1, 2)$, we have

$$D_{\mathbf{u}}(0, 1, 2) = \nabla f(0, 1, 2) \cdot \mathbf{u} = (-4\mathbf{i}) \cdot \left(\frac{1}{\sqrt{11}}\mathbf{i} + \frac{3}{\sqrt{11}}\mathbf{j} + \frac{1}{\sqrt{11}}\mathbf{k}\right) = \boxed{-\frac{4\sqrt{11}}{11}}.$$

28. (a) The gradient of f at (x, y, z) is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}.$$

The gradient of f at the point $(1, 2, -1)$ is

$$\nabla f(1, 2, -1) = 5\mathbf{i} + -3\mathbf{j} + 2\mathbf{k}.$$

(b) The unit vector from $(1, 2, -1)$ to $(2, 0, 1)$ is

$$\mathbf{u} = \frac{(2-1)\mathbf{i} + (0-2)\mathbf{j} + (1-(-1))\mathbf{k}}{\sqrt{(2-1)^2 + (0-2)^2 + (1-(-1))^2}} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(1, 2, -1)$, we have

$$D_{\mathbf{u}}(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = (5\mathbf{i} + -3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) = \boxed{5}.$$

29. (a) The partial derivatives of $f(x, y, z) = z \tan^{-1}(yx^{-1})$ are

$$\begin{aligned} f_x(x, y, z) &= z \cdot \frac{1}{1 + (yx^{-1})^2} \cdot (-x^{-2}) = -\frac{zy}{x^2 + y^2} \\ f_y(x, y, z) &= z \cdot \frac{1}{1 + (yx^{-1})^2} \cdot x^{-1} = \frac{zx}{x^2 + y^2} \\ f_z(x, y, z) &= \tan^{-1} \frac{y}{x}. \end{aligned}$$

The gradient of f at (x, y, z) is

$$\nabla f(x, y) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \left(-\frac{zy}{x^2 + y^2}\right)\mathbf{i} + \left(\frac{zx}{x^2 + y^2}\right)\mathbf{j} + \left(\tan^{-1} \frac{y}{x}\right)\mathbf{k}.$$

The gradient of f at the point $(1, 1, 3)$ is

$$\nabla f(1, 1, 3) = -\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.$$

(b) The unit vector from $(1, 1, 3)$ to $(2, 0, -2)$ is

$$\mathbf{u} = \frac{(2-1)\mathbf{i} + (0-1)\mathbf{j} + (-2-3)\mathbf{k}}{\sqrt{(2-1)^2 + (0-1)^2 + (-2-3)^2}} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{1}{3\sqrt{3}}\mathbf{j} - \frac{5}{3\sqrt{3}}\mathbf{k}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(1, 1, 3)$, we have

$$D_{\mathbf{u}}(1, 1, 3) = \nabla f(1, 1, 3) \cdot \mathbf{u} = \left(-\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}\right) \cdot \left(\frac{1}{3\sqrt{3}}\mathbf{i} - \frac{1}{3\sqrt{3}}\mathbf{j} - \frac{5}{3\sqrt{3}}\mathbf{k}\right) = -\frac{1}{\sqrt{3}} - \frac{5\pi}{12\sqrt{3}} = \boxed{-\frac{(12 + 5\pi)\sqrt{3}}{36}}.$$

30. (a) The gradient of f at (x, y, z) is

$$\nabla f(x, y) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = (\cos x \cos(y + z))\mathbf{i} + (-\sin x \sin(y + z))\mathbf{j} + (-\sin x \sin(y + z))\mathbf{k}.$$

The gradient of f at the point $(1, 1, 1)$ is

$$\nabla f(1, 1, 1) = \cos(1) \cos(2)\mathbf{i} - \sin(1) \sin(2)\mathbf{j} - \sin(1) \sin(2)\mathbf{j}.$$

(b) The unit vector from $(1, 1, 1)$ to $(2, -1, 0)$ is

$$\mathbf{u} = \frac{(2-1)\mathbf{i} + (-1-1)\mathbf{j} + (0-1)\mathbf{k}}{\sqrt{(2-1)^2 + (-1-1)^2 + (0-1)^2}} = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(1, 1, 1)$, we have

$$\begin{aligned} D_{\mathbf{u}}(1, 1, 1) &= \nabla f(1, 1, 1) \cdot \mathbf{u} = (\cos(1) \cos(2) \mathbf{i} - \sin(1) \sin(2) \mathbf{j} - \sin(1) \sin(2) \mathbf{j}) \cdot \left(\frac{1}{\sqrt{6}} \mathbf{i} - \frac{2}{\sqrt{6}} \mathbf{j} - \frac{1}{\sqrt{6}} \mathbf{k} \right) \\ &= \boxed{\frac{\cos(1) \cos(2)}{\sqrt{6}} + \frac{3 \sin(1) \sin(2)}{\sqrt{6}}}. \end{aligned}$$

31. (a) The gradient of f at (x, y, z) is

$$\nabla f(x, y) = f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k} = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{i} + \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{j} + \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{k}.$$

The gradient of f at the point $(3, 4, 0)$ is

$$\boxed{\nabla f(3, 4, 0) = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}}.$$

(b) The unit vector from $(3, 4, 0)$ to $(1, -1, 1)$ is

$$\mathbf{u} = \frac{(1-3) \mathbf{i} + (-1-4) \mathbf{j} + (1-0) \mathbf{k}}{\sqrt{(1-3)^2 + (-1-4)^2 + (1-0)^2}} = -\frac{2}{\sqrt{30}} \mathbf{i} - \frac{5}{\sqrt{30}} \mathbf{j} + \frac{1}{\sqrt{30}} \mathbf{k}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(3, 4, 0)$, we have

$$D_{\mathbf{u}}(3, 4, 0) = \nabla f(3, 4, 0) \cdot \mathbf{u} = \left(\frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right) \cdot \left(-\frac{2}{\sqrt{30}} \mathbf{i} - \frac{5}{\sqrt{30}} \mathbf{j} + \frac{1}{\sqrt{30}} \mathbf{k} \right) = -\frac{26}{5\sqrt{30}} = \boxed{-\frac{13\sqrt{30}}{75}}.$$

32. (a) The partial derivatives of $f(x, y, z) = x(x^2 + 2y^2 + 3z^2)^{-1/2}$ are

$$\begin{aligned} f_x(x, y, z) &= (x^2 + 2y^2 + 3z^2)^{-1/2} - \frac{1}{2} x (x^2 + 2y^2 + 3z^2)^{-3/2} \cdot 2x = \frac{2y^2 + 3z^2}{(x^2 + 2y^2 + 3z^2)^{3/2}} \\ f_y(x, y, z) &= -\frac{1}{2} x (x^2 + 2y^2 + 3z^2)^{-3/2} \cdot 4y = -\frac{2xy}{(x^2 + 2y^2 + 3z^2)^{3/2}} \\ f_z(x, y, z) &= -\frac{1}{2} x (x^2 + 2y^2 + 3z^2)^{-3/2} \cdot 6z = -\frac{3xz}{(x^2 + 2y^2 + 3z^2)^{3/2}}. \end{aligned}$$

The gradient of f at (x, y, z) is

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k} \\ &= \left(\frac{2y^2 + 3z^2}{(x^2 + 2y^2 + 3z^2)^{3/2}} \right) \mathbf{i} + \left(-\frac{2xy}{(x^2 + 2y^2 + 3z^2)^{3/2}} \right) \mathbf{j} + \left(-\frac{3xz}{(x^2 + 2y^2 + 3z^2)^{3/2}} \right) \mathbf{k}. \end{aligned}$$

The gradient of f at the point $(1, 2, 1)$ is

$$\boxed{\nabla f(1, 2, 1) = \frac{11\sqrt{3}}{72} \mathbf{i} - \frac{\sqrt{3}}{18} \mathbf{j} - \frac{\sqrt{3}}{24} \mathbf{k}}.$$

(b) The unit vector from $(1, 2, 1)$ to $(-1, 1, 1)$ is

$$\mathbf{u} = \frac{(-1-1) \mathbf{i} + (1-2) \mathbf{j} + (1-1) \mathbf{k}}{\sqrt{(-1-1)^2 + (1-2)^2 + (1-1)^2}} = -\frac{2}{\sqrt{5}} \mathbf{i} - \frac{1}{\sqrt{5}} \mathbf{j}.$$

Using the formula for the directional derivative at the end of the section to find $D_{\mathbf{u}}(1, 2, 1)$, we have

$$D_{\mathbf{u}}(1, 2, 1) = \nabla f(1, 2, 1) \cdot \mathbf{u} = \left(\frac{11\sqrt{3}}{72}\mathbf{i} - \frac{\sqrt{3}}{18}\mathbf{j} - \frac{\sqrt{3}}{24}\mathbf{k} \right) \cdot \left(-\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j} \right) = \boxed{-\frac{\sqrt{15}}{20}}.$$

33. (a) We begin by finding the gradient of f at $(-1, 2)$.

$$\begin{aligned}\nabla f(x, y) &= (y^2 + 2x)\mathbf{i} + 2xy\mathbf{j} \\ \nabla f(-1, 2) &= 2\mathbf{i} - 4\mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(-1, 2)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\mathbf{u} = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(-1, 2)\| = \sqrt{4 + 16} = \boxed{2\sqrt{5}}.$$

34. (a) We begin by finding the gradient of f at $(2, 1)$.

$$\begin{aligned}\nabla f(x, y) &= 3y\mathbf{i} + (3x + 2y)\mathbf{j} \\ \nabla f(2, 1) &= 3\mathbf{i} + 8\mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(2, 1)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\mathbf{u} = \frac{3\sqrt{73}}{73}\mathbf{i} + \frac{8\sqrt{73}}{73}\mathbf{j}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(2, 1)\| = \sqrt{9 + 64} = \boxed{\sqrt{73}}.$$

35. (a) We begin by finding the gradient of f at $(0, 0)$.

$$\begin{aligned}\nabla f(x, y) &= (e^y + ye^x)\mathbf{i} + (xe^y + e^x)\mathbf{j} \\ \nabla f(0, 0) &= \mathbf{i} + \mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(0, 0)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(0, 0)\| = \sqrt{1 + 1} = \boxed{\sqrt{2}}.$$

36. (a) We begin by finding the gradient of f at $(5, 1)$.

$$\begin{aligned}\nabla f(x, y) &= \ln y\mathbf{i} + \frac{x}{y}\mathbf{j} \\ \nabla f(5, 1) &= 5\mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(5, 1)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\boxed{\mathbf{u} = \mathbf{j}}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(5, 1)\| = \sqrt{0 + 25} = \boxed{5}.$$

37. (a) We begin by finding the gradient of f at $(1, 2)$.

$$\begin{aligned}\nabla f(x, y) &= \frac{-x^2 + y^2}{(x^2 + y^2)^2} \mathbf{i} + \frac{-2xy}{(x^2 + y^2)^2} \mathbf{j} \\ \nabla f(1, 2) &= \frac{3}{25} \mathbf{i} - \frac{4}{25} \mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(1, 2)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\boxed{\mathbf{u} = \frac{3}{5} \mathbf{i} - \frac{4}{5} \mathbf{j}}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(1, 2)\| = \sqrt{\frac{9}{625} + \frac{16}{625}} = \boxed{\frac{1}{5}}.$$

38. (a) We begin by finding the gradient of f at $(3, 4)$.

$$\begin{aligned}\nabla f(x, y) &= \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} \\ \nabla f(3, 4) &= \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(3, 4)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\boxed{\mathbf{u} = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(3, 4)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \boxed{1}.$$

39. (a) The partial derivatives of f were computed in Problem 29 above. We begin by finding the gradient of f at $(1, 1, 3)$.

$$\begin{aligned}\nabla f(x, y, z) &= \left(-\frac{zy}{x^2 + y^2} \right) \mathbf{i} + \left(\frac{zx}{x^2 + y^2} \right) \mathbf{j} + \left(\tan^{-1} \frac{y}{x} \right) \mathbf{k} \\ \nabla f(1, 1, 3) &= -\frac{3}{2} \mathbf{i} + \frac{3}{2} \mathbf{j} + \frac{\pi}{4} \mathbf{k}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(1, 1, 3)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\boxed{\mathbf{u} = -\frac{6}{\sqrt{72 + \pi^2}} \mathbf{i} + \frac{6}{\sqrt{72 + \pi^2}} \mathbf{j} + \frac{\pi}{\sqrt{72 + \pi^2}} \mathbf{k}}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

$$\|\nabla f(1, 1, 3)\| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{\pi^2}{16}} = \sqrt{\frac{9}{2} + \frac{\pi^2}{16}}.$$

40. (a) We begin by finding the gradient of f at $(3, 4, 0)$.

$$\begin{aligned}\nabla f(x, y, z) &= \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{i} + \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{j} + \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{k} \\ \nabla f(3, 4, 0) &= \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}\end{aligned}$$

The direction $\mathbf{u}$ for which $D_{\mathbf{u}}f(3, 4, 0)$ is maximum occurs when ∇f and $\mathbf{u}$ have the same direction. That is, the directional derivative is maximized in the direction

$$\mathbf{u} = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}.$$

(b) The maximum value of the directional derivative equals the magnitude of the gradient, namely

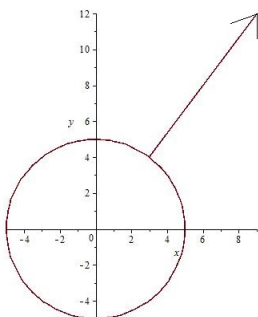
$$\|\nabla f(3, 4, 0)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \boxed{1}.$$

41. Because $f(3, 4) = 3^2 + 4^2 = 25$, the point P is on the level curve $x^2 + y^2 = 25$. Since

$$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

the gradient at P is

$$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}.$$

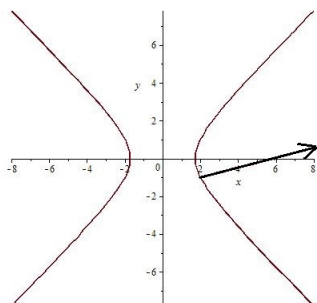


42. Because $f(2, -1) = 2^2 - (-1)^2 = 3$, the point P is on the level curve $x^2 - y^2 = 3$. Since

$$\nabla f(x, y) = 2x\mathbf{i} - 2y\mathbf{j}$$

the gradient at P is

$$\nabla f(2, -1) = 4\mathbf{i} + 2\mathbf{j}.$$

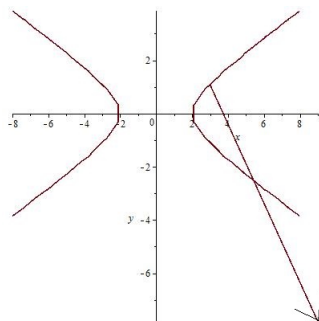


43. Because $f\left(3, \frac{\sqrt{5}}{2}\right) = 3^2 - 4\left(\frac{\sqrt{5}}{2}\right)^2 = 4$, the point P is on the level curve $x^2 - 4y^2 = 4$. Since

$$\nabla f(x, y) = 2x\mathbf{i} - 8y\mathbf{j}$$

the gradient at P is

$$\nabla f\left(3, \frac{\sqrt{5}}{2}\right) = 6\mathbf{i} - 4\sqrt{5}\mathbf{j}.$$

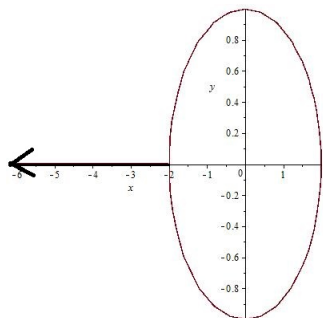


44. Because $f(-2, 0) = (-2)^2 + 4(0)^2 = 4$, the point P is on the level curve $x^2 + 4y^2 = 4$. Since

$$\nabla f(x, y) = 2x\mathbf{i} + 8y\mathbf{j}$$

the gradient at P is

$$\nabla f(-2, 0) = -4\mathbf{i}.$$

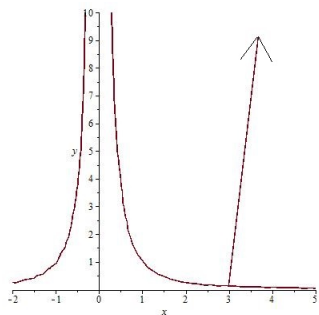


45. Because $f\left(3, \frac{1}{9}\right) = 3^2 \cdot \frac{1}{9} = 1$, the point P is on the level curve $x^2y = 1$. Since

$$\nabla f(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$$

the gradient at P is

$$\nabla f\left(3, \frac{1}{9}\right) = \frac{2}{3}\mathbf{i} + 9\mathbf{j}.$$

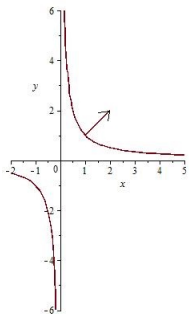


46. Because $f(1, 1) = 1 \cdot 1 = 1$, the point P is on the level curve $xy = 1$. Since

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

the gradient at P is

$$\nabla f(1, 1) = \mathbf{i} + \mathbf{j}.$$



Applications and Extensions

47. (a) The rate of change in temperature at $(0, 0)$ is $D_{\mathbf{u}}T(0, 0)$ where $\mathbf{u}$ is the unit vector in the direction of $3\mathbf{i} - 4\mathbf{j}$. That is, $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$. To compute the directional derivative, we first compute the gradient of T .

$$\begin{aligned} T_x(x, y) &= e^x(\sin x + \sin y) + e^x(\cos x) \\ T_y(x, y) &= e^x \cos y, \end{aligned}$$

so

$$\nabla T(x, y) = (e^x(\sin x + \sin y) + e^x(\cos x))\mathbf{i} + e^x \cos y\mathbf{j}.$$

Then

$$D_{\mathbf{u}}T(0, 0) = \nabla T(0, 0) \cdot \mathbf{u} = (\mathbf{i} + \mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = -\frac{1}{5}.$$

The rate of change in temperature at $(0, 0)$ in the direction of $3\mathbf{i} - 4\mathbf{j}$ is $\boxed{-\frac{1}{5}^\circ\text{C}}$.

(b) The greatest temperature increase at $(0, 0)$ occurs in the direction of the gradient at $(0, 0)$, namely

$$\nabla T(0, 0) = \mathbf{i} + \mathbf{j}.$$

The unit vector in this direction is

$$\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.$$

(c) The greatest temperature decrease at $(0, 0)$ occurs in the opposite direction of the gradient at $(0, 0)$, namely

$$-\nabla T(0, 0) = -\mathbf{i} - \mathbf{j}.$$

The unit vector in this direction is

$$\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}.$$

(d) The rate of change in T at $(0, 0)$ remains the same for directions orthogonal to the gradient $\nabla T(0, 0) = \mathbf{i} + \mathbf{j}$. Those two directions are $\boxed{\pm(\mathbf{i} - \mathbf{j})}$.

48. (a) The vector from (x, y) to $(0, 0)$ is $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$ and the unit vector in this same direction is $\mathbf{u} = -\frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$. The rate of change of V in the direction of $\mathbf{u}$, namely $D_{\mathbf{u}}V(x, y)$ is

$$\begin{aligned} D_{\mathbf{u}}V(x, y) &= \nabla V(x, y) \cdot \mathbf{u} = \left(\frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} \right) \cdot \left(-\frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} \right) \\ &= -\frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \boxed{-\frac{1}{\sqrt{x^2 + y^2}}}, \end{aligned}$$

where the partial derivatives were computed in Problem 16 above.

(b) The two unit vectors orthogonal to the direction in part (a) are $\pm \left(\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j} \right)$. We will label them $\mathbf{u}_1 = \frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$ and $\mathbf{u}_2 = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$. Then the rate of change of V in the direction of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, are

$$\begin{aligned} D_{\mathbf{u}_1}V(x, y) &= \nabla V(x, y) \cdot \mathbf{u}_1 = \left(\frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} \right) \cdot \left(\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j} \right) = \boxed{0} \\ D_{\mathbf{u}_2}V(x, y) &= \nabla V(x, y) \cdot \mathbf{u}_2 = \left(\frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} \right) \cdot \left(-\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j} \right) = \boxed{0} \end{aligned}$$

(c) The greatest increase in potential V occurs in the direction of the gradient, namely

$$\boxed{\nabla V(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}}.$$

(d) The greatest decrease in potential V occurs in the opposite direction of the gradient, namely

$$\boxed{-\nabla V(x, y) = -\frac{x}{x^2 + y^2}\mathbf{i} - \frac{y}{x^2 + y^2}\mathbf{j}}.$$

49. The water will flow in the direction of maximum decrease from the point $(1, 2, 2)$. The direction of maximum decrease occurs in the opposite direction of the gradient of z at the point $(1, 2, 2)$. The gradient of z is

$$\nabla z(x, y) = -4x\mathbf{i} - 2y\mathbf{j},$$

so the gradient at $(1, 2, 2)$ is $\nabla z(1, 2) = -4\mathbf{i} - 4\mathbf{j}$. This means the water will flow in the direction of $4\mathbf{i} + 4\mathbf{j}$. The unit vector in this direction is $\boxed{\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}$.

50. (a) The unit vector from $(3, 4)$ to $(4, 3)$ is $\mathbf{u} = \frac{(4-3)\mathbf{i} + (3-4)\mathbf{j}}{\sqrt{(4-3)^2 + (3-4)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$. The gradient of T is $\nabla T(x, y) = 6x\mathbf{i} + 8y\mathbf{j}$. The rate of change of the temperature is

$$\begin{aligned} D_{\mathbf{u}}T(3, 4) &= \nabla T(3, 4) \cdot \mathbf{u} = (18\mathbf{i} + 32\mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} \right) \\ &= -\frac{14}{\sqrt{2}} = -7\sqrt{2}. \end{aligned}$$

We are asked for how fast the temperature is changing, so we find the absolute value of the rate of change, which means the temperature is changing by $\boxed{7\sqrt{2}^\circ\text{C per unit of distance}}$.

(b) If we want to cool off as fast as possible, we want the temperature to decrease as quickly as possible, which is in the opposite direction of the gradient at $(4, 3)$. So we should move in the direction of

$$\boxed{-\nabla T(4, 3) = -24\mathbf{i} - 24\mathbf{j}}.$$

(c) If we want to stay at a constant temperature, we want to move along points (x, y) such that $T(x, y) = T(4, 3) = 89$. This means, we must be on the path

$$\begin{aligned} T(x, y) &= 89 \\ 3x^2 + 4y^2 + 5 &= 89 \end{aligned}$$

$$\boxed{\frac{x^2}{28} + \frac{y^2}{21} = 1}$$

to remain at a constant temperature.

(d) At the origin, the gradient is $\nabla T(0, 0) = 0\mathbf{i} + 0\mathbf{j}$ so the directional derivative, in any direction, will be 0.

51. The gradient of T is $\nabla T(x, y) = \sin(2y)\mathbf{i} + 2x \cos(2y)\mathbf{j}$. The unit vector in the direction making an angle of $\frac{\pi}{6}$ with the positive x -axis is $\mathbf{u} = \cos \frac{\pi}{6}\mathbf{i} + \sin \frac{\pi}{6}\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$. The rate of change of the temperature at the point $\left(1, \frac{\pi}{4}\right)$ in the direction $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}T\left(1, \frac{\pi}{4}\right) &= \nabla T\left(1, \frac{\pi}{4}\right) \cdot \mathbf{u} = (\mathbf{i}) \cdot \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) \\ &= \boxed{\frac{\sqrt{3}}{2}^\circ\text{C per unit of distance}}. \end{aligned}$$

52. This question can be rephrased to find the unit vector(s) $\mathbf{u}$ such that

$$D_{\mathbf{u}}z(-1, 1) = \nabla z(-1, 1) \cdot \mathbf{u} = 2.$$

The gradient of z is $\nabla z(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ so $\nabla z(-1, 1) = \mathbf{i} - 2\mathbf{j}$. If we let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, then

$$\begin{aligned} \nabla z(-1, 1) \cdot \mathbf{u} &= 2 \\ (\mathbf{i} - 2\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j}) &= 2 \\ a - 2b &= 2. \end{aligned}$$

Recalling that $\mathbf{u}$ is a unit vector $\sqrt{a^2 + b^2} = 1$ or $a = \pm\sqrt{1 - b^2}$, we can plug each of these in to solve for a and b . First we plug in the positive square root:

$$\begin{aligned} \sqrt{1 - b^2} - 2b &= 2 \\ 1 - b^2 &= (2 + 2b)^2 \\ 1 - b^2 &= 4 + 8b + 4b^2 \\ b &= -1, -\frac{3}{5}. \end{aligned}$$

Using the equation $a - 2b = 2$, or $a = 2 + 2b$, we find that the a value corresponding to $b = -1$ is $a = 0$, and the a value corresponding to $b = -\frac{3}{5}$ is $a = \frac{4}{5}$. This leads to the two unit vectors $-\mathbf{j}$ and $\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$. If we plug in the negative square root, we will get the same equation to solve so we will get the same unit vectors. So

if we want a rate of change of z to be 2, we move in either the direction of $\boxed{-\mathbf{j}}$ or $\boxed{\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}}$.

53. (a) The gradient of z is $\nabla z = 8x\mathbf{i} + 18y\mathbf{j}$. The maximum rate of change at $(2, 1)$ occurs in the direction of the gradient at that point, so $\nabla z(2, 1) = 16\mathbf{i} + 18\mathbf{j}$. The unit vector in this direction is $\boxed{\frac{8\sqrt{145}}{145}\mathbf{i} + \frac{9\sqrt{145}}{145}\mathbf{j}}$.

(b) To verify that this vector is normal, we will compute the direction of the tangent line and verify that the directions of the tangent line and of the gradient are normal. First we find $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned} 8x + 18y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{8x}{18y}. \end{aligned}$$

This means the slope of the tangent line to the curve at $(2, 1)$ is $-\frac{16}{18}$. The direction of the tangent line is then $-16\mathbf{i} + 16\mathbf{j}$. By taking the dot product of this vector and the gradient found in part (a), we will obtain 0 which means the gradient is normal to the curve at $(2, 1)$.

(c) The maximum rate of change at $(2, 1)$ is the magnitude of the gradient, so

$$\|16\mathbf{i} + 18\mathbf{j}\| = \sqrt{16^2 + 18^2} = \boxed{2\sqrt{145}}.$$

54. To verify that the level curves are orthogonal, it is enough to verify that their tangent lines are orthogonal at each point. The slope of the tangent line to the level curve $f(x, y) = x^2 - y^2 = k$ is found using implicit differentiation.

$$\begin{aligned} 2x - 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \end{aligned}$$

The tangent line at a general point (x_0, y_0) to this level curve is then parallel to the vector $y_0\mathbf{i} + x_0\mathbf{j}$. The slope of the tangent line to the level curve $f(x, y) = xy = \ell$ is found using implicit differentiation.

$$\begin{aligned} y + x \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

The tangent line at a general point (x_0, y_0) to this level curve is then parallel to the vector $-x_0\mathbf{i} + y_0\mathbf{j}$. To verify that these two vectors are orthogonal, we take their dot product

$$(y_0\mathbf{i} + x_0\mathbf{j}) \cdot (-x_0\mathbf{i} + y_0\mathbf{j}) = 0.$$

55. A general vector that is normal to the level curve of f through the point P is the gradient $\nabla f(1, -2)$. The gradient of f is

$$\nabla f(x, y) = 8xy\mathbf{i} + 4x^2\mathbf{j}$$

so $\nabla f(1, -2) = -16\mathbf{i} + 4\mathbf{j}$. If we want a unit vector $\mathbf{u}$ that is normal to the curve at this point, we will divide by the magnitude.

$$\mathbf{u} = \frac{1}{4\sqrt{17}}(-16\mathbf{i} + 4\mathbf{j}) = \boxed{-\frac{4\sqrt{17}}{17}\mathbf{i} + \frac{\sqrt{17}}{17}\mathbf{j}}$$

Another unit vector that is normal to the curve at the point P is

$$-\mathbf{u} = \boxed{\frac{4\sqrt{17}}{17}\mathbf{i} - \frac{\sqrt{17}}{17}\mathbf{j}}.$$

56. A general vector that is normal to the level curve of f through the point P is the gradient $\nabla f(1, 1)$. The gradient of f is

$$\nabla f(x, y) = 4x\mathbf{i} + 2y\mathbf{j}$$

so $\nabla f(1, 1) = 4\mathbf{i} + 2\mathbf{j}$. If we want a unit vector $\mathbf{u}$ that is normal to the curve at this point, we will divide by the magnitude.

$$\mathbf{u} = \frac{1}{2\sqrt{5}} (4\mathbf{i} + 2\mathbf{j}) = \boxed{\frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}}$$

Another unit vector that is normal to the curve at the point P is

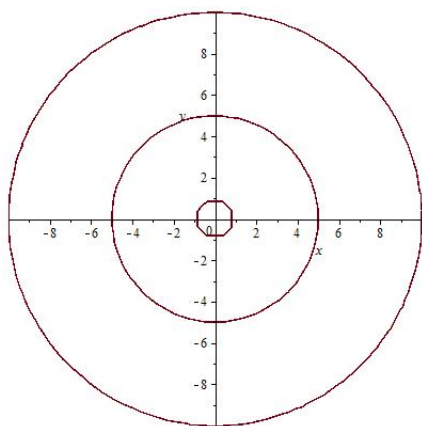
$$-\mathbf{u} = \boxed{-\frac{2\sqrt{5}}{5}\mathbf{i} - \frac{\sqrt{5}}{5}\mathbf{j}}.$$

57. (a) The level curve for a general $T > 0$ is

$$\begin{aligned} \frac{250}{\sqrt{x^2 + y^2}} &= T \\ \sqrt{x^2 + y^2} &= \frac{250}{T} \\ x^2 + y^2 &= \left(\frac{250}{T}\right)^2 \end{aligned}$$

which is a circle centered at the origin of radius $\frac{250}{T}$.

(b) The level curves are drawn for $T = 25$, 50, and 250 with the smallest circle representing $T = 250$, the middle circle representing $T = 50$, and the largest circle representing $T = 25$.



(c) We have $T(-3, 4) = \frac{250}{\sqrt{(-3)^2 + (4)^2}} = 50$ so the level curve through $(-3, 4)$ is $\frac{250}{\sqrt{x^2 + y^2}} = 50$ or

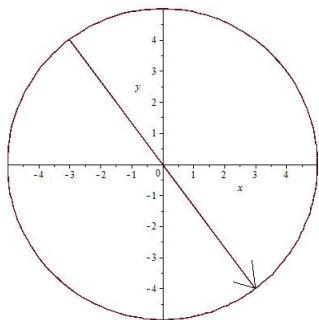
$$\boxed{x^2 + y^2 = 25}.$$

(d) The gradient of $T(x, y) = 250(x^2 + y^2)^{-1/2}$ is

$$\nabla T(x, y) = -\frac{250x}{(x^2 + y^2)^{3/2}}\mathbf{i} - \frac{250y}{(x^2 + y^2)^{3/2}}\mathbf{j},$$

so

$$\boxed{\nabla T(-3, 4) = 6\mathbf{i} - 8\mathbf{j}}.$$



(e) An isothermal curve is a curve at which the temperature is the same for all points on the curve.

58. (a) The gradient of $z = -x^2 - y^2 + 5x$ is $\nabla z(x, y) = (-2x + 5)\mathbf{i} + (-2y)\mathbf{j}$. The unit vector in the northeast direction is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. The rate of change of altitude $D_{\mathbf{u}}z$ at the point $(1, 1, 3)$ is

$$\begin{aligned} D_{\mathbf{u}}z(1, 1) &= \nabla z(1, 1) \cdot \mathbf{u} = (3\mathbf{i} - 2\mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \right) \\ &= \boxed{\frac{1}{\sqrt{2}}}. \end{aligned}$$

(b) The climber wants to start moving in the direction of maximum ascent from the point $(1, 2, 0)$ which is in the direction of the gradient at that point. So the climber should walk in the direction of

$$\nabla f(1, 2) = \boxed{3\mathbf{i} - 4\mathbf{j}}.$$

(c) To remain at the same altitude, there should be no change in z which means the climber should move in a direction perpendicular to the gradient at $(2, 1)$. The gradient at $(2, 1)$ is

$$\nabla f(2, 1) = \mathbf{i} - 2\mathbf{j}.$$

The directions perpendicular to this direction are $\boxed{\pm(2\mathbf{i} + \mathbf{j})}$.

59. The direction of greatest potential drop (i.e. the direction that a charge moves) from the point $(1, -1, 2)$ is in the opposite direction of the gradient at that point. The gradient is

$\nabla V(x, y, z) = yze^{xyz}\mathbf{i} + xze^{xyz}\mathbf{j} + xye^{xyz}\mathbf{k}$, so the charge moves in the following direction:

$$-\nabla V(1, -1, 2) = -(-2e^{-2}\mathbf{i} + 2e^{-2}\mathbf{j} - e^{-2}\mathbf{k}) = e^{-2}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

or, equivalently, in the direction of the unit vector

$$\mathbf{u} = \boxed{\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}}.$$

The charge is moving at a speed equal to directional derivative of V at $(1, -1, 2)$ in the direction $\mathbf{u}$, so the electrical potential changes at a rate of

$$\nabla V(1, -1, 2) \cdot \mathbf{u} = (-e^{-2}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})) \cdot \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) = \boxed{-3e^{-2} \text{ volts per unit change}}.$$

60. (a)

$$\begin{aligned} \mathbf{E} &= -\nabla V(x, y, z) = -\left(-\frac{kQx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} - \frac{kQy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} - \frac{kQz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \right) \\ &= \boxed{\frac{kQx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{kQy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{kQz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}}. \end{aligned}$$

(b) If r is the distance from the origin to (x, y, z) , then $r^2 = x^2 + y^2 + z^2$. Then

$$\begin{aligned}\|\mathbf{E}\| &= \sqrt{\left(\frac{kQx}{(x^2 + y^2 + z^2)^{3/2}}\right)^2 + \left(\frac{kQy}{(x^2 + y^2 + z^2)^{3/2}}\right)^2 + \left(\frac{kQz}{(x^2 + y^2 + z^2)^{3/2}}\right)^2} \\ &= \sqrt{\frac{(kQ)^2 r^2}{r^6}} = \frac{kQ}{r^2}.\end{aligned}$$

(c) An equipotential surface looks like $\frac{kQ}{\sqrt{x^2 + y^2 + z^2}} = C$ for some constant C , which can be rewritten as

$$x^2 + y^2 + z^2 = \left(\frac{kQ}{C}\right)^2.$$

Let $f(x, y, z) = x^2 + y^2 + z^2 - \left(\frac{kQ}{C}\right)^2$. The tangent plane to f at the point (x_0, y_0, z_0) is given by $\nabla f(x_0, y_0, z_0) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$, so any vector normal to the surface must be parallel to $\nabla f(x_0, y_0, z_0) = 2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$, or $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Note that $(x_0^2 + y_0^2 + z_0^2)^{3/2} = \left(\frac{kQ}{C}\right)^3 = C'$ for some constant C' . From part (a),

$$\mathbf{E}(x_0, y_0, z_0) = \frac{kQ}{C'}(x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}),$$

which means $\mathbf{E}(x_0, y_0, z_0)$ is parallel to $\nabla f(x_0, y_0, z_0)$, so $\mathbf{E}$ is normal to the surface at any point.

61. The vector from $(0, 1, 2)$ to the point $(2, 1, 2)$ is $(2, 0, 0)$ and the vector $\mathbf{i}$ is the unit vector in this same direction. So the rate of change of T in this direction is $\nabla T(x, y, z) \cdot \mathbf{i} = T_x(x, y, z) = -\frac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$.

If we go straight from $(0, 1, 2)$ to $(2, 1, 2)$, then every point connecting these two points has $y = 1$ and $z = 2$. So a general point $(x, 1, 2)$ for $0 \leq x \leq 2$ has rate of change

$$\boxed{-\frac{x}{\sqrt{4 - x^2}}}.$$

62. First we find the gradient.

$$\nabla f(x, y, z) = -\frac{2x}{(x^2 + y^2 + z^2)^2}\mathbf{i} - \frac{2y}{(x^2 + y^2 + z^2)^2}\mathbf{j} - \frac{2z}{(x^2 + y^2 + z^2)^2}\mathbf{k}.$$

Since the cell will move in the direction of the gradient, the cell will move in the following direction

$$\nabla f(1, 2, 3) = \boxed{-\frac{1}{98}\mathbf{i} - \frac{1}{49}\mathbf{j} - \frac{3}{98}\mathbf{k}}.$$

63. The function $U_g(x, y, z) = mgz$ has gradient $\nabla U_g = 0\mathbf{i} + 0\mathbf{j} + mg\mathbf{k}$, so

$$\mathbf{F}_g = -\nabla U_g = -mg\mathbf{k}.$$

The gravitational force vector points in the negative z direction (i.e. downward) and has magnitude mg .

64. Since one object is at the origin and the other is at (x, y, z) , we have $r = \sqrt{x^2 + y^2 + z^2}$ so

$U_g = -G\frac{mM}{\sqrt{x^2 + y^2 + z^2}} = -GmM(x^2 + y^2 + z^2)^{-1/2}$. Computing the opposite of the gradient of U_g , we have

$$\begin{aligned}\mathbf{F}_g = -\nabla U_g &= -\frac{GmMx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} - \frac{GmMy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} - \frac{GmMz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\ &= -GmM\left(\frac{x}{r^3}\mathbf{i} + \frac{y}{r^3}\mathbf{j} + \frac{z}{r^3}\mathbf{k}\right).\end{aligned}$$

Now we compute the magnitude

$$\begin{aligned}\|\mathbf{F}_g\| &= |-GmM| \sqrt{\frac{x^2}{r^6} + \frac{y^2}{r^6} + \frac{z^2}{r^6}} \\ &= GmM \sqrt{\frac{r^2}{r^6}} = G \frac{mM}{r^2}.\end{aligned}$$

65. Since $\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} \neq \mathbf{0}$, the two directions orthogonal to $\nabla f(x_0, y_0)$ are $\pm (f_y(x_0, y_0)\mathbf{i} - f_x(x_0, y_0)\mathbf{j})$. The two unit vectors in these directions are

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{\|\nabla f(x_0, y_0)\|} (f_y(x_0, y_0)\mathbf{i} - f_x(x_0, y_0)\mathbf{j}) \\ \mathbf{u}_2 &= \frac{1}{\|\nabla f(x_0, y_0)\|} (-f_y(x_0, y_0)\mathbf{i} + f_x(x_0, y_0)\mathbf{j}).\end{aligned}$$

Now we compute the directional derivatives in both unit vectors' directions.

$$\begin{aligned}D_{\mathbf{u}_1}f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \mathbf{u}_1 = (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \frac{1}{\|\nabla f(x_0, y_0)\|} (f_y(x_0, y_0)\mathbf{i} - f_x(x_0, y_0)\mathbf{j}) \\ &= \frac{1}{\|\nabla f(x_0, y_0)\|} (f_x(x_0, y_0)f_y(x_0, y_0) - f_y(x_0, y_0)f_x(x_0, y_0)) = 0 \\ D_{\mathbf{u}_2}f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \mathbf{u}_2 = (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \frac{1}{\|\nabla f(x_0, y_0)\|} (-f_y(x_0, y_0)\mathbf{i} + f_x(x_0, y_0)\mathbf{j}) \\ &= \frac{1}{\|\nabla f(x_0, y_0)\|} (-f_x(x_0, y_0)f_y(x_0, y_0) + f_y(x_0, y_0)f_x(x_0, y_0)) = 0\end{aligned}$$

66. (a)

$$\begin{aligned}\nabla(ku) &= \nabla(kf(x, y)) = kf_x(x, y)\mathbf{i} + kf_y(x, y)\mathbf{j} \\ &= k[f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] = k\nabla u\end{aligned}$$

(b)

$$\begin{aligned}\nabla(u + v) &= \nabla(f(x, y) + g(x, y)) = [f_x(x, y) + g_x(x, y)]\mathbf{i} + [f_y(x, y) + g_y(x, y)]\mathbf{j} \\ &= [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] + [g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j}] = \nabla u + \nabla v\end{aligned}$$

(c)

$$\begin{aligned}\nabla(uv) &= \nabla(f(x, y) \cdot g(x, y)) = [f_x(x, y) \cdot g(x, y) + f(x, y) \cdot g_x(x, y)]\mathbf{i} + [f_y(x, y) \cdot g(x, y) + f(x, y) \cdot g_y(x, y)]\mathbf{j} \\ &= [f(x, y) \cdot g_x(x, y)\mathbf{i} + f(x, y) \cdot g_y(x, y)\mathbf{j}] + [g(x, y) \cdot f_x(x, y)\mathbf{i} + g(x, y) \cdot f_y(x, y)\mathbf{j}] \\ &= f(x, y) [g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j}] + g(x, y) [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \\ &= u\nabla v + v\nabla u\end{aligned}$$

(d)

$$\begin{aligned}\nabla\left(\frac{u}{v}\right) &= \nabla\left(\frac{f(x, y)}{g(x, y)}\right) = \left(\frac{g(x, y)f_x(x, y) - f(x, y)g_x(x, y)}{(g(x, y))^2}\right)\mathbf{i} + \left(\frac{g(x, y)f_y(x, y) - f(x, y)g_y(x, y)}{(g(x, y))^2}\right)\mathbf{j} \\ &= \frac{g(x, y)f_x(x, y)\mathbf{i} - f(x, y)g_x(x, y)\mathbf{i} + g(x, y)f_y(x, y)\mathbf{j} - f(x, y)g_y(x, y)\mathbf{j}}{(g(x, y))^2} \\ &= \frac{g(x, y) [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] - f(x, y) [g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j}]}{v^2} = \frac{v\nabla u - u\nabla v}{v^2}\end{aligned}$$

(e)

$$\begin{aligned}\nabla u^a &= \nabla (f(x, y))^a = (af(x, y)^{a-1}f_x(x, y))\mathbf{i} + (af(x, y)^{a-1}f_y(x, y))\mathbf{j} \\ &= af(x, y)^{a-1} [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] = au^{a-1}\nabla u\end{aligned}$$

67. Using formula (3)

$$\begin{aligned} D_{\mathbf{u}}f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} = \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right) \cdot \frac{1}{\sqrt{a_1^2 + a_2^2}} (a_1 \mathbf{i} + a_2 \mathbf{j}) \\ &= \frac{a_1 \frac{\partial f}{\partial x} + a_2 \frac{\partial f}{\partial y}}{\sqrt{a_1^2 + a_2^2}}. \end{aligned}$$

68. If $F(x, y) = 0$, then taking the derivative with respect to x , we have $F_x(x, y) + F_y(x, y) \frac{dy}{dx} = 0$ so

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

The slope of the tangent line at (x_0, y_0) is $-\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}$. One vector that is parallel to the tangent line of $F(x, y) = 0$ at (x_0, y_0) is $F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j}$. Now we show that the dot product between this vector and $\nabla F(x_0, y_0)$ is 0, which means that $\nabla F(x_0, y_0)$ is normal to $F(x, y) = 0$ at (x_0, y_0) .

$$(F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j}) \cdot \nabla F(x_0, y_0) = (F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j}) \cdot (F_x(x_0, y_0)\mathbf{i} + F_y(x_0, y_0)\mathbf{j}) = 0$$

69. In Problem 68, we showed that $F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j}$ is a vector parallel to the tangent line of $F(x, y) = 0$ at (x_0, y_0) . If $b = F_y(x_0, y_0) \neq 0$, then the slope of this tangent line is $-\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}$. Using point-slope form, the equation of the tangent line is

$$\begin{aligned} y - y_0 &= -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}(x - x_0) \\ F_y(x_0, y_0)(y - y_0) &= -F_x(x_0, y_0)(x - x_0) \\ a(x - x_0) + b(y - y_0) &= 0. \end{aligned}$$

If $b = F_y(x_0, y_0) = 0$, then we know that $a = F_x(x_0, y_0) \neq 0$ so the tangent line is vertical since it must be parallel to the vector $F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j} = -F_x(x_0, y_0)\mathbf{j}$. This means the tangent line is

$$\begin{aligned} x - x_0 &= 0 \\ a(x - x_0) + 0(y - y_0) &= 0 \\ a(x - x_0) + b(y - y_0) &= 0. \end{aligned}$$

70. (a) Using Problem 69, $F(x, y) = x^2 - y^2 - 16$, so $F_x(x, y) = 2x$ and $F_y(x, y) = -2y$. Then the tangent line is

$$\begin{aligned} F_x(5, 3)(x - 5) + F_y(5, 3)(y - 3) &= 0 \\ 10(x - 5) - 6(y - 3) &= 0 \\ y &= \frac{5}{3}x - \frac{16}{3}. \end{aligned}$$

If we use implicit differentiation, then

$$\begin{aligned} 2x - 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \\ \frac{dy}{dx} \Big|_{(5, 3)} &= \frac{5}{3}, \end{aligned}$$

which agrees with the slope we found using Problem 69.

(b) The tangent line is

$$\begin{aligned} F_x(4,0)(x-4) + F_y(4,0)(y-0) &= 0 \\ 8(x-4) &= 0 \\ x &= 4. \end{aligned}$$

71. No, f is not necessarily differentiable. One example of such a function is $f(x, y) = (x^2y)^{1/3}$. Note that $D_{\mathbf{u}}f(0,0)$ in the direction of $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is $(\cos^2\theta\sin\theta)^{1/3}$. Also, $f_x(0,0) = f_y(0,0) = 0$. If f were differentiable at $(0,0)$, then $D_{\mathbf{u}}f(0,0)$ would equal $\nabla f(0,0) \cdot \mathbf{u} = 0$ for every unit vector $\mathbf{u}$. However, $D_{\mathbf{u}}f(0,0) = (\cos^2\theta\sin\theta)^{1/3}$ is not identically zero, so f is not differentiable at the origin.

72. The points on the line $x-1 = y-2 = z-1$ can be parametrized by $(x(t), y(t), z(t)) = (t, t+1, t)$ which points in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$. The unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Now the gradient of f is $\nabla f(x, y, z) = 3z\mathbf{i} - 2y\mathbf{j} + (3z^2 + 3x)\mathbf{k}$, so the directional derivative at $(1, 2, 1)$ in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}f(1, 2, 1) &= \nabla f(1, 2, 1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) \cdot \left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \right) \\ &= \boxed{\frac{5}{\sqrt{3}}}. \end{aligned}$$

Similarly, we obtain $\boxed{-\frac{5}{\sqrt{3}}}$ if we use $-\mathbf{u}$ for the direction of the line.

Challenge Problems

73. (a) Since $r = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$, we have $r^n = (x^2 + y^2)^{n/2}$. Then

$$\begin{aligned} \nabla r^n &= \frac{n}{2} (x^2 + y^2)^{n/2-1} \cdot 2x\mathbf{i} + \frac{n}{2} (x^2 + y^2)^{n/2-1} \cdot 2y\mathbf{j} \\ &= nx \left(\sqrt{x^2 + y^2} \right)^{n-2} \mathbf{i} + ny \left(\sqrt{x^2 + y^2} \right)^{n-2} \mathbf{j} = n \left(\sqrt{x^2 + y^2} \right)^{n-2} (x\mathbf{i} + y\mathbf{j}) \\ &= nr^{n-2} \mathbf{r}. \end{aligned}$$

(b) Using the Chain Rule

$$\begin{aligned} g_x(r) &= g'(r) \cdot r_x = g'(r) \cdot (x^2 + y^2)^{-1/2} \cdot x = \frac{g'(r)}{r} x \\ g_y(r) &= g'(r) \cdot r_y = g'(r) \cdot (x^2 + y^2)^{-1/2} \cdot y = \frac{g'(r)}{r} y. \end{aligned}$$

Then

$$\nabla g(r) = g_x(r)\mathbf{i} + g_y(r)\mathbf{j} = \frac{g'(r)}{r} (x\mathbf{i} + y\mathbf{j}) = \frac{g'(r)}{r} \mathbf{r}.$$

(c) Using formula (3) and part (b)

$$D_{\mathbf{u}}[g(r)] = \nabla g(r) \cdot \mathbf{u} = \frac{g'(r)}{r} \mathbf{r} \cdot \mathbf{u} = \frac{g'(r)}{r} (\mathbf{r} \cdot \mathbf{u}).$$

(d) Let $f(x, y) = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} = x(x^2 + y^2)^{-1/2}$. Then

$$\begin{aligned} f_x(x, y) &= (x^2 + y^2)^{-1/2} - x^2 (x^2 + y^2)^{-3/2} = (x^2 + y^2)^{-3/2} ((x^2 + y^2) - x^2) \\ &= \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{y^2}{r^3} \\ f_y(x, y) &= -xy (x^2 + y^2)^{-3/2} = -\frac{xy}{(x^2 + y^2)^{3/2}} = -\frac{xy}{r^3}. \end{aligned}$$

Now,

$$\begin{aligned}
 \frac{\mathbf{i}}{r} - \left(\frac{x}{r}\right) \left(\frac{\mathbf{r}}{r^2}\right) &= \frac{1}{r} \mathbf{i} - \left(\frac{x}{r^3}\right) (x\mathbf{i} + y\mathbf{j}) = \left(\frac{1}{r} - \frac{x^2}{r^3}\right) \mathbf{i} - \frac{xy}{r^3} \mathbf{j} \\
 &= \frac{r^2 - x^2}{r^3} \mathbf{i} - \frac{xy}{r^3} \mathbf{j} = \frac{y^2}{r^3} \mathbf{i} - \frac{xy}{r^3} \mathbf{j} \\
 &= f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} = \nabla f(x, y) \\
 &= \nabla \left(\frac{x}{r}\right).
 \end{aligned}$$

74. To compute the directional derivative at $(0, 0)$, we need to first compute $f_x(0, 0)$ and $f_y(0, 0)$ which we will compute using the definition of the partial derivative. In each step where the equal symbol is labelled with an asterisk (*), we used L'Hopital's Rule.

$$\begin{aligned}
 f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(h^2)}{h^2} - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(h^2) - h^2}{h^3} \stackrel{*}{=} \lim_{h \rightarrow 0} \frac{\cos(h^2) \cdot 2h - 2h}{3h^2} = \lim_{h \rightarrow 0} \frac{2\cos(h^2) - 2}{3h} \\
 &\stackrel{*}{=} \lim_{h \rightarrow 0} \frac{-2\sin(h^2) \cdot 2h}{3} = 0
 \end{aligned}$$

Similarly

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(h^2)}{h^2} - 1}{h} = 0$$

Since $f_x(0, 0) = 0 = f_y(0, 0)$, the directional derivative in any direction at $(0, 0)$ is 0.

75. Since $\nabla f(x, y) = c(x\mathbf{i} + y\mathbf{j})$ we have that $f_x(x, y) = cx$ and $f_y(x, y) = cy$. By integrating f_x with respect to x , we find that $f(x, y) = \frac{1}{2}cx^2 + g(y)$ for some function g depending only on y . Differentiating $f(x, y) = \frac{1}{2}cx^2 + g(y)$ with respect to y gives

$$cy = f_y(x, y) = g'(y)$$

which means $g(y) = \frac{1}{2}cy^2 + C$ for some constant C . Putting this all together, we find that

$$f(x, y) = \frac{1}{2}cx^2 + \frac{1}{2}cy^2 + C = \frac{1}{2}c(x^2 + y^2) + C.$$

If (x_0, y_0) lies on a circle centered at $(0, 0)$ of radius k , then $x_0^2 + y_0^2 = k^2$ so plugging this point into f , we find that

$$f(x_0, y_0) = \frac{1}{2}c(x_0^2 + y_0^2) + C = \frac{1}{2}ck^2 + C,$$

which is a constant.

76. For the gradient to be normal to the level surface f through P_0 means that it is perpendicular to any curve that lies on the surface and goes through P_0 . Suppose that some curve on the surface $f(x, y, z) = k$ goes through P_0 and is represented parametrically by $x = x(t)$, $y = y(t)$, and $z = z(t)$, with $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$. Then

$$f(x(t), y(t), z(t)) = k.$$

Differentiating with respect to t , we get

$$f_x(x(t), y(t), z(t)) \cdot x'(t) + f_y(x(t), y(t), z(t)) \cdot y'(t) + f_z(x(t), y(t), z(t)) \cdot z'(t) = 0,$$

or, equivalently,

$$\nabla f(x, y, z) \cdot [x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = 0.$$

We know that $x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$ is tangent to the curve at the point P_0 , so $\nabla f(x_0, y_0, z_0)$ is normal to any curve on the surface, which then means it is normal to the surface at point P_0 .

77. Since $g(x, y) = D_{\mathbf{u}}f(x, y)$ we know that

$$g(x, y) = u_1 f_x(x, y) + u_2 f_y(x, y).$$

Then using formula(3) again to compute $D_{\mathbf{u}}g(x, y)$, we have

$$\begin{aligned} D_{\mathbf{u}}g(x, y) &= v_1 g_x(x, y) + v_2 g_y(x, y) \\ &= v_1 (u_1 f_{xx}(x, y) + u_2 f_{yx}(x, y)) + v_2 (u_1 f_{xy}(x, y) + u_2 f_{yy}(x, y)) \\ &= u_1 v_1 f_{xx}(x, y) + (u_1 v_2 + u_2 v_1) f_{xy}(x, y) + u_2 v_2 f_{yy}(x, y) \end{aligned}$$

where in the last step we used that $f_{xy} = f_{yx}$ since f has continuous second-order partial derivatives.

78. In Problem 68, we showed that $F_y(x_0, y_0)\mathbf{i} - F_x(x_0, y_0)\mathbf{j}$ is a vector parallel to the tangent line of $F(x, y) = 0$ at (x_0, y_0) . If $b = F_y(x_0, y_0) \neq 0$, then the slope of the tangent line is $m = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}$.

13.2 Tangent Planes

Concepts and Vocabulary

1. **True**. The tangent plane has equation $\nabla F(x_0, y_0, z_0) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$.

2. **True**.

Skill Building

3. Using the formula at the end of the section, an equation of the tangent plane is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The partial derivatives and their values at the given point are as follows

$$\begin{aligned} f_x(x, y) &= -4x & \text{and} & & f_y(x, y) &= -2y \\ f_x(0, 2) &= 0 & \text{and} & & f_y(0, 2) &= -4 \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} z - 6 &= 0(x - 0) - 4(y - 2) \\ 4y + z &= 14. \end{aligned}$$

4. Using the formula at the end of the section, an equation of the tangent plane is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The partial derivatives and their values at the given point are as follows

$$\begin{aligned} f_x(x, y) &= 2x & \text{and} & & f_y(x, y) &= 2y \\ f_x(1, -2) &= 2 & \text{and} & & f_y(1, -2) &= -4 \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} z - 9 &= 2(x - 1) - 4(y + 2) \\ -2x + 4y + z &= -1. \end{aligned}$$

5. Using the formula at the end of the section, an equation of the tangent plane is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The partial derivatives and their values at the given point are as follows

$$\begin{aligned} f_x(x, y) &= 8x & \text{and} & & f_y(x, y) &= 18y \\ f_x(1, 1) &= 8 & \text{and} & & f_y(1, 1) &= 18 \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} z - 13 &= 8(x - 1) + 18(y - 1) \\ 8x + 18y - z &= 13. \end{aligned}$$

6. Using the formula at the end of the section, an equation of the tangent plane is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The partial derivatives and their values at the given point are as follows

$$\begin{aligned} f_x(x, y) &= \frac{x}{\sqrt{x^2 + 3y^2}} & \text{and} & & f_y(x, y) &= \frac{3y}{\sqrt{x^2 + 3y^2}} \\ f_x(1, 1) &= \frac{1}{2} & \text{and} & & f_y(1, 1) &= \frac{3}{2} \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} z - 2 &= \frac{1}{2}(x - 1) + \frac{3}{2}(y - 1) \\ -\frac{1}{2}x - \frac{3}{2}y + z &= 0. \end{aligned}$$

7. (a) To use the theorem from this section with $F(x, y, z) = x^2 + y^2 + z^2 - 14 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(1, -2, 3) = 2 \\ F_y(x, y, z) = 2y & F_y(1, -2, 3) = -4 \\ F_z(x, y, z) = 2z & F_z(1, -2, 3) = 6 \end{array}$$

Then an equation of the tangent plane is

$$2(x - 1) - 4(y + 2) + 6(z - 3) = 0$$

$$\boxed{x - 2y + 3z = 14}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x - 1}{1} = \frac{y + 2}{-2} = \frac{z - 3}{3}}.$$

8. (a) To use the theorem from this section with $F(x, y, z) = 2x^2 + 3y^2 + z^2 - 12 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 4x & F_x(2, 1, -1) = 8 \\ F_y(x, y, z) = 6y & F_y(2, 1, -1) = 6 \\ F_z(x, y, z) = 2z & F_z(2, 1, -1) = -2 \end{array}$$

Then an equation of the tangent plane is

$$8(x - 2) + 6(y - 1) - 2(z + 1) = 0$$

$$\boxed{4x + 3y - z = 12}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 1}{-1}}.$$

9. (a) To use the theorem from this section with $F(x, y, z) = 4x^2 + y^2 - 2z^2 - 3 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 8x & F_x(1, 1, -1) = 8 \\ F_y(x, y, z) = 2y & F_y(1, 1, -1) = 2 \\ F_z(x, y, z) = -4z & F_z(1, 1, -1) = 4 \end{array}$$

Then an equation of the tangent plane is

$$8(x - 1) + 2(y - 1) + 4(z + 1) = 0$$

$$\boxed{4x + y + 2z = 3}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x - 1}{4} = \frac{y - 1}{1} = \frac{z + 1}{2}}.$$

10. (a) To use the theorem from this section with $F(x, y, z) = x^2 + 3y^2 - z^2 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(1, -1, -2) = 2 \\ F_y(x, y, z) = 6y & F_y(1, -1, -2) = -6 \\ F_z(x, y, z) = -2z & F_z(1, -1, -2) = 4 \end{array}$$

Then an equation of the tangent plane is

$$2(x-1) - 6(y+1) + 4(z+2) = 0$$

$$\boxed{x - 3y + 2z = 0}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-1}{1} = \frac{y+1}{-3} = \frac{z+2}{2}}.$$

11. (a) To use the theorem from this section with $F(x, y, z) = x^2 + y^2 - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(-2, 1, 5) = -4 \\ F_y(x, y, z) = 2y & F_y(-2, 1, 5) = 2 \\ F_z(x, y, z) = -1 & F_z(-2, 1, 5) = -1 \end{array}$$

Then an equation of the tangent plane is

$$-4(x+2) + 2(y-1) - 1(z-5) = 0$$

$$\boxed{4x - 2y + z = -5}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x+2}{-4} = \frac{y-1}{2} = \frac{z-5}{-1}}.$$

12. (a) To use the theorem from this section with $F(x, y, z) = 2x^2 - 3y^2 - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 4x & F_x(2, 1, 5) = 8 \\ F_y(x, y, z) = -6y & F_y(2, 1, 5) = -6 \\ F_z(x, y, z) = -1 & F_z(2, 1, 5) = -1 \end{array}$$

Then an equation of the tangent plane is

$$8(x-2) - 6(y-1) - 1(z-5) = 0$$

$$\boxed{8x - 6y - z = 5}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-2}{8} = \frac{y-1}{-6} = \frac{z-5}{-1}}.$$

13. (a) To use the theorem from this section with $F(x, y, z) = 2x^2 - y^2 - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 4x & F_x(1, 0, 2) = 4 \\ F_y(x, y, z) = -2y & F_y(1, 0, 2) = 0 \\ F_z(x, y, z) = -1 & F_z(1, 0, 2) = -1 \end{array}$$

Then an equation of the tangent plane is

$$4(x-1) + 0(y-0) - 1(z-2) = 0$$

$$\boxed{4x - z = 2}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-1}{4} = \frac{z-2}{-1}; y=0}.$$

14. (a) To use the theorem from this section with $F(x, y, z) = x^2 + y^2 - 2z^2 + 13 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(2, 1, 3) = 4 \\ F_y(x, y, z) = 2y & F_y(2, 1, 3) = 2 \\ F_z(x, y, z) = -4z & F_z(2, 1, 3) = -12 \end{array}$$

Then an equation of the tangent plane is

$$4(x-2) + 2(y-1) - 12(z-3) = 0$$

$$\boxed{2x + y - 6z = -13}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-2}{2} = \frac{y-1}{1} = \frac{z-3}{-6}}.$$

15. (a) To use the theorem from this section with $F(x, y, z) = x^2 - 3y^2 - 4z^2 - 2 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(3, 1, 1) = 6 \\ F_y(x, y, z) = -6y & F_y(3, 1, 1) = -6 \\ F_z(x, y, z) = -8z & F_z(3, 1, 1) = -8 \end{array}$$

Then an equation of the tangent plane is

$$6(x-3) - 6(y-1) - 8(z-1) = 0$$

$$\boxed{3x - 3y - 4z = 2}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-3}{3} = \frac{y-1}{-3} = \frac{z-1}{-4}}.$$

16. (a) To use the theorem from this section with $F(x, y, z) = x^2 + 2y^2 - 5z^2 - 4 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(1, 2, 1) = 2 \\ F_y(x, y, z) = 4y & F_y(1, 2, 1) = 8 \\ F_z(x, y, z) = -10z & F_z(1, 2, 1) = -10 \end{array}$$

Then an equation of the tangent plane is

$$2(x-1) + 8(y-2) - 10(z-1) = 0$$

$$\boxed{x + 4y - 5z = 4}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-1}{1} = \frac{y-2}{4} = \frac{z-1}{-5}}.$$

17. (a) To use the theorem from this section with $F(x, y, z) = \ln(\sec(x^2 + y^2)) - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= \frac{1}{\sec(x^2 + y^2)} \cdot \sec(x^2 + y^2) \tan(x^2 + y^2) \cdot 2x = 2x \tan(x^2 + y^2) \\ F_x\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}, \ln \sqrt{2}\right) &= 2\sqrt{\frac{\pi}{8}} \tan \frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{2}} \\ F_y(x, y, z) &= \frac{1}{\sec(x^2 + y^2)} \cdot \sec(x^2 + y^2) \tan(x^2 + y^2) \cdot 2y = 2y \tan(x^2 + y^2) \\ F_y\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}, \ln \sqrt{2}\right) &= 2\sqrt{\frac{\pi}{8}} \tan \frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{2}} \\ F_z(x, y, z) &= -1 \\ F_z\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}, \ln \sqrt{2}\right) &= -1 \end{aligned}$$

Then an equation of the tangent plane is

$$\frac{\sqrt{\pi}}{\sqrt{2}} \left(x - \sqrt{\frac{\pi}{8}}\right) + \frac{\sqrt{\pi}}{\sqrt{2}} \left(y - \sqrt{\frac{\pi}{8}}\right) - 1(z - \ln \sqrt{2}) = 0$$

$$\frac{\sqrt{\pi}}{\sqrt{2}}x + \frac{\sqrt{\pi}}{\sqrt{2}}y - z = \frac{\pi}{2} - \ln \sqrt{2}.$$

(b) Symmetric equations of the normal line are

$$\frac{x - \sqrt{\frac{\pi}{8}}}{\frac{\sqrt{\pi}}{\sqrt{2}}} = \frac{y - \sqrt{\frac{\pi}{8}}}{\frac{\sqrt{\pi}}{\sqrt{2}}} = \frac{z - \ln \sqrt{2}}{-1}.$$

18. (a) To use the theorem from this section with $F(x, y, z) = \frac{x^2 - y^2}{x^2 + y^2} - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= \frac{(x^2 + y^2) \cdot 2x - (x^2 - y^2) \cdot 2x}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2} \\ F_x\left(1, 2, -\frac{3}{5}\right) &= \frac{16}{25} \\ F_y(x, y, z) &= \frac{(x^2 + y^2) \cdot (-2y) - (x^2 - y^2) \cdot 2y}{(x^2 + y^2)^2} = -\frac{4yx^2}{(x^2 + y^2)^2} \\ F_y\left(1, 2, -\frac{3}{5}\right) &= -\frac{8}{25} \\ F_z(x, y, z) &= -1 \\ F_z\left(1, 2, -\frac{3}{5}\right) &= -1 \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} \frac{16}{25}(x - 1) - \frac{8}{25}(y - 2) - \left(z + \frac{3}{5}\right) &= 0 \\ \frac{16}{25}x - \frac{8}{25}y - z &= \frac{3}{5} \\ \boxed{16x - 8y - 25z = 15}. \end{aligned}$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-1}{16} = \frac{y-2}{-8} = \frac{z+\frac{3}{5}}{-25}}.$$

19. (a) To use the theorem from this section with $F(x, y, z) = e^x \cos y - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= e^x \cos y & F_x\left(0, \frac{\pi}{2}, 0\right) &= 0 \\ F_y(x, y, z) &= -e^x \sin y & F_y\left(0, \frac{\pi}{2}, 0\right) &= -1 \\ F_z(x, y, z) &= -1 & F_z\left(0, \frac{\pi}{2}, 0\right) &= -1 \end{aligned}$$

Then an equation of the tangent plane is

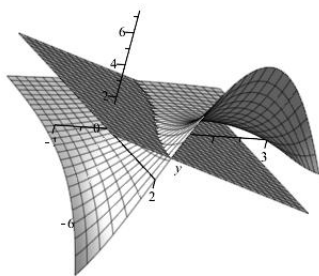
$$0(x-0) - 1\left(y - \frac{\pi}{2}\right) - 1(z-0) = 0$$

$$\boxed{y + z = \frac{\pi}{2}}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{y - \frac{\pi}{2}}{-1} = \frac{z}{-1}; x = 0}.$$

(c)



20. (a) To use the theorem from this section with $F(x, y, z) = \ln(x^2 + y^2) - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= \frac{2x}{x^2 + y^2} & F_x(1, -1, \ln 2) &= 1 \\ F_y(x, y, z) &= \frac{2y}{x^2 + y^2} & F_y(1, -1, \ln 2) &= -1 \\ F_z(x, y, z) &= -1 & F_z(1, -1, \ln 2) &= -1 \end{aligned}$$

Then an equation of the tangent plane is

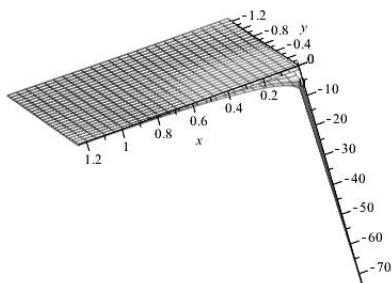
$$1(x-1) - 1(y+1) - 1(z - \ln 2) = 0$$

$$\boxed{x - y - z = 2 - \ln 2}.$$

(b) Symmetric equations of the normal line are

$$\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-\ln 2}{-1}.$$

(c)



21. (a) To use the theorem from this section with $F(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 9 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= \frac{2}{3}x^{-1/3} & F_x(1, 8, -8) &= \frac{2}{3} \\ F_y(x, y, z) &= \frac{2}{3}y^{-1/3} & F_y(1, 8, -8) &= \frac{1}{3} \\ F_z(x, y, z) &= \frac{2}{3}z^{-1/3} & F_z(1, 8, -8) &= -\frac{1}{3} \end{aligned}$$

Then an equation of the tangent plane is

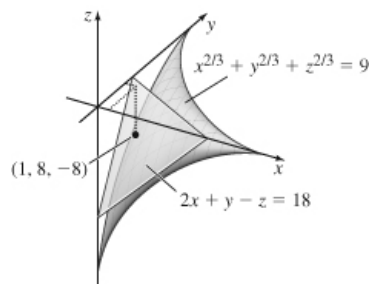
$$\frac{2}{3}(x-1) + \frac{1}{3}(y-8) - \frac{1}{3}(z+8) = 0$$

$$\boxed{2x + y - z = 18}.$$

(b) Symmetric equations of the normal line are

$$\frac{x-1}{2} = \frac{y-8}{1} = \frac{z+8}{-1}.$$

(c)



22. (a) To use the theorem from this section with $F(x, y, z) = x^{1/2} + y^{1/2} + z^{1/2} - 6 = 0$, we begin by

finding the partial derivatives of F and their values at the given point.

$$\begin{aligned} F_x(x, y, z) &= \frac{1}{2}x^{-1/2} & F_x(1, 4, 9) &= \frac{1}{2} \\ F_y(x, y, z) &= \frac{1}{2}y^{-1/2} & F_y(1, 4, 9) &= \frac{1}{4} \\ F_z(x, y, z) &= \frac{1}{2}z^{-1/2} & F_z(1, 4, 9) &= \frac{1}{6} \end{aligned}$$

Then an equation of the tangent plane is

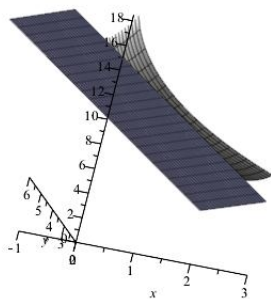
$$\frac{1}{2}(x-1) + \frac{1}{4}(y-4) + \frac{1}{6}(z-9) = 0$$

$$\boxed{6x + 3y + 2z = 36}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x-1}{6} = \frac{y-4}{3} = \frac{z-9}{2}}.$$

(c)



Applications and Extensions

23. Using $z = f(x, y) = \sqrt{x^2 + y^2 - 24}$, the partial derivatives and their values at the point $(3, -4, 1)$ are as follows

$$\begin{aligned} f_x(x, y) &= \frac{x}{\sqrt{x^2 + y^2 - 24}} & \text{and} & & f_y(x, y) &= \frac{y}{\sqrt{x^2 + y^2 - 24}} \\ f_x(3, -4) &= 3 & \text{and} & & f_y(3, -4) &= -4 \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} z - 1 &= 3(x - 3) - 4(y + 4) \\ 3x - 4y - z &= 24, \end{aligned}$$

which agrees with Example 1.

24. In Problem 23, we found the coefficients of x , y , and z of the tangent plane to be 3, -4 , and -1 , respectively. This means that symmetric equations of the normal line at the point $(3, -4, 1)$ are

$$\frac{x-3}{3} = \frac{y+4}{-4} = \frac{z-1}{-1},$$

which agrees with Example 2.

25. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 6 - 2x = 0 \\ f_y(x, y) &= -4 - 4y = 0. \end{aligned}$$

The only point that simultaneously makes both partial derivatives vanish is $(3, -1)$. This means that $(3, -1, 11)$ is the only point where the tangent plane is parallel to the xy -plane.

26. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 4 + 2x = 0 \\ f_y(x, y) &= -2 + 2y = 0. \end{aligned}$$

The only point that simultaneously makes both partial derivatives vanish is $(-2, 1)$. This means that $(-2, 1, -5)$ is the only point where the tangent plane is parallel to the xy -plane.

27. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 2x + 2y = 0 \\ f_y(x, y) &= 2x + 2y = 0. \end{aligned}$$

Every point $(x, -x)$ makes both partial derivatives vanish. This means that there are an infinite number of points where the tangent plane is parallel to the xy -plane, and they are $(x, -x, 0)$ for any real number x .

28. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 2x - 3y = 0 \\ f_y(x, y) &= -3x + 2y = 0. \end{aligned}$$

Solving the f_x equation for x and substituting it into the f_y equation, we have

$$\begin{aligned} -3\left(\frac{3y}{2}\right) + 2y &= 0 \\ y &= 0, \end{aligned}$$

so $x = 0$ too. This means that $(0, 0, 0)$ is the only point where the tangent plane is parallel to the xy -plane.

29. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 8x^3 - 2x = 0 \\ f_y(x, y) &= -2y - 2 = 0. \end{aligned}$$

Factoring the first equation, we have

$$\begin{aligned} 2x(4x^2 - 1) &= 0 \\ x &= 0, \pm \frac{1}{2}, \end{aligned}$$

and solving the second equation $y = -1$. So the points where the tangent plane is parallel to the x -plane are

$$\left(0, -1, 1\right), \left(\frac{1}{2}, -1, \frac{7}{8}\right), \left(-\frac{1}{2}, -1, \frac{7}{8}\right).$$

30. We will use the fact that the tangent plane is parallel to the xy -plane at $P = (x_0, y_0)$ when the normal line at P is perpendicular to the xy -plane, or, equivalently, when $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Finding the partial derivatives of f and setting them equal to 0, we have

$$\begin{aligned} f_x(x, y) &= 2x - 2 = 0 \\ f_y(x, y) &= 4y^3 - 8y = 0. \end{aligned}$$

From the f_x equation, we see that $x = 1$. Factoring the f_y equation, we have

$$\begin{aligned} 4y(y^2 - 2) &= 0 \\ y &= 0, \pm\sqrt{2}. \end{aligned}$$

So the points where the tangent plane is parallel to the x -plane are $\left(1, 0, -1\right), \left(1, \sqrt{2}, -5\right), \left(1, -\sqrt{2}, -5\right)$.

31. She should bolt the solar panel in the direction of the normal line, which is in the direction of the gradient of $F(x, y, z) = 4 - x^2 - y^2 - z$ at the point $(1, 1, 2)$. Since $\nabla F(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$, she should bolt in the direction of $\nabla F(1, 1, 2) = -2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

32. (a) To find the tangent plane of the first equation, we will call the left side of the equation $F(x, y, z)$. Now we find the partial derivatives of F and their values at the point $(0, -1, 2)$.

$$\begin{aligned} F_x(x, y, z) &= 2x & F_x(0, -1, 2) &= 0 \\ F_y(x, y, z) &= 4 & F_y(0, -1, 2) &= 4 \\ F_z(x, y, z) &= 2z & F_z(0, -1, 2) &= 4 \end{aligned}$$

The equation of the tangent plane is then

$$\begin{aligned} 0(x - 0) - 4(y + 1) - 4(z - 2) &= 0 \\ 4y + 4z &= 4 \end{aligned}$$

To find the tangent plane of the second equation, we will call the left side of the equation $G(x, y, z)$. Now we find the partial derivatives of F and their values at the point $(0, -1, 2)$.

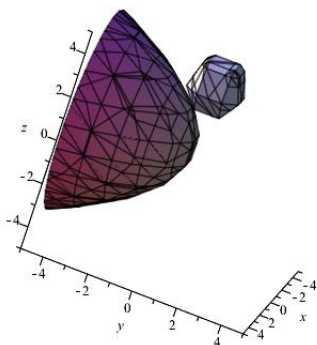
$$\begin{aligned} G_x(x, y, z) &= 2x & G_x(0, -1, 2) &= 0 \\ G_y(x, y, z) &= 2y & G_y(0, -1, 2) &= -2 \\ G_z(x, y, z) &= 2z - 6 & G_z(0, -1, 2) &= -2 \end{aligned}$$

The equation of the tangent plane is then

$$\begin{aligned} 0(x - 0) - 2(y + 1) - 2(z - 2) &= 0 \\ 2y + 2z &= 2 \end{aligned}$$

Both tangent planes are the same, so the two surfaces are tangent at the common point.

(b)



33. (a) To find the points of intersection, we solve for $x^2 + y^2$ in the first equation and plug it into the second. Then

$$\begin{aligned}(1 - z^2) - z^2 &= 1 \\ z &= 0.\end{aligned}$$

This means the two surfaces intersect in the xy -plane when $x^2 + y^2 = 1$. Let $P = (x_0, y_0, 0)$ be one such point. To find the tangent plane of the first equation at P , we will write it as $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Now we find the partial derivatives of F and their values at the point P .

$$\begin{aligned}F_x(x, y, z) &= 2x & F_x(x_0, y_0, 0) &= 2x_0 \\ F_y(x, y, z) &= 2y & F_y(x_0, y_0, 0) &= 2y_0 \\ F_z(x, y, z) &= 2z & F_z(x_0, y_0, 0) &= 0\end{aligned}$$

The equation of the tangent plane at (x_0, y_0) is then

$$\begin{aligned}2x_0(x - x_0) + 2y_0(y - y_0) &= 0 \\ x_0x + y_0y &= x_0^2 + y_0^2 = 1.\end{aligned}$$

To find the tangent plane of the second equation at P , we will write it as $G(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$. Now we find the partial derivatives of F and their values at the point P .

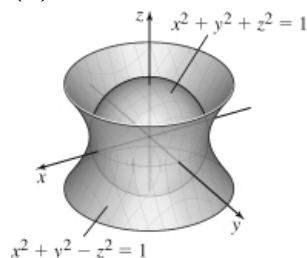
$$\begin{aligned}G_x(x, y, z) &= 2x & G_x(x_0, y_0, 0) &= 2x_0 \\ G_y(x, y, z) &= 2y & G_y(x_0, y_0, 0) &= 2y_0 \\ G_z(x, y, z) &= -2z & G_z(x_0, y_0, 0) &= 0\end{aligned}$$

The equation of the tangent plane is then

$$\begin{aligned}2x_0(x - x_0) + 2y_0(y - y_0) &= 0 \\ x_0x + y_0y &= x_0^2 + y_0^2 = 1\end{aligned}$$

Both tangent planes are the same at P , so the two surfaces are tangent at any point of intersection.

(b)



34. (a) If the two surfaces are orthogonal, then their normal lines are orthogonal, which means their gradient vectors are orthogonal. We will show that the gradient vectors to both surfaces at any point of intersection are orthogonal.

To find the points of intersection, we solve for $x^2 + y^2$ in the first equation and plug it into the second. Then

$$\begin{aligned}(4 - z^2) - z^2 &= 0 \\ z^2 &= 2 \\ z &= \pm\sqrt{2}.\end{aligned}$$

This means the two surfaces intersect in the planes $z = \pm\sqrt{2}$ when $x^2 + y^2 = 2$. Let $P = (x_0, y_0, z_0)$ be such a point; that is $x_0^2 + y_0^2 = 2$ and $z_0^2 = 2$. We start by writing the first equation as $F(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$ and find its gradient.

$$\begin{aligned}\nabla F(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ \nabla F(x_0, y_0, z_0) &= 2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}\end{aligned}$$

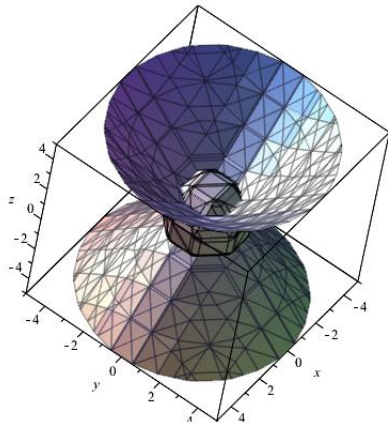
If we denote the second equation as $G(x, y, z) = x^2 + y^2 - z^2 = 0$, then

$$\begin{aligned}\nabla G(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \\ \nabla G(x_0, y_0, z_0) &= 2x_0\mathbf{i} + 2y_0\mathbf{j} - 2z_0\mathbf{k}.\end{aligned}$$

Now computing the dot product to show the two gradient vectors are orthogonal, we have

$$\begin{aligned}\nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) &= (2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}) \cdot (2x_0\mathbf{i} + 2y_0\mathbf{j} - 2z_0\mathbf{k}) \\ &= 4(x_0^2 + y_0^2) - 4z_0^2 = 4(2) - 4(2) \\ &= 0.\end{aligned}$$

(b)



35. Denoting the sphere by $F(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$, the partial derivatives at a general point $P = (x_0, y_0, z_0)$ on the sphere are

$$\begin{aligned}F_x(x_0, y_0, z_0) &= 2x_0 \\ F_y(x_0, y_0, z_0) &= 2y_0 \\ F_z(x_0, y_0, z_0) &= 2z_0.\end{aligned}$$

Then symmetric equations of the normal line are

$$\frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0}.$$

Notice if we plug in the origin $(0, 0, 0)$ to these equations, each fraction simplifies to $-\frac{1}{2}$ (since x_0, y_0, z_0 are all not equal to 0 so they can be cancelled), which means the normal line passes through the origin.

36. From Problem 35, we found the symmetric equations of the normal line to be

$$\frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0}.$$

Plugging in the point $(-x_0, -y_0, -z_0)$ results in each fraction simplifying to -1 , which means the normal line passes through this point.

37. Denote the equation by $F(x, y, z) = \sin(x + y) + \sin(y + z) - 1 = 0$. Now we find the partial derivatives of F and their values at the point $(0, \frac{\pi}{2}, \frac{\pi}{2})$.

$$\begin{aligned} F_x(x, y, z) &= \cos(x + y) & F_x\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) &= 0 \\ F_y(x, y, z) &= \cos(x + y) + \cos(y + z) & F_y\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) &= -1 \\ F_z(x, y, z) &= \cos(y + z) & F_z\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) &= -1 \end{aligned}$$

Then an equation of the tangent plane is

$$0(x - 0) - 1\left(y - \frac{\pi}{2}\right) - \left(z - \frac{\pi}{2}\right) = 0$$

$y + z = \pi.$

Symmetric equations of the normal line are

$\frac{y - \frac{\pi}{2}}{-1} = \frac{z - \frac{\pi}{2}}{-1}; x = 0.$

38. To be parallel to the plane $2x - 3y + z = 4$ at the point $P = (x_0, y_0, z_0)$ means the normal line at P is parallel to the vector $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. The normal line at P has direction $\nabla F(x_0, y_0, z_0)$ where $F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} - 1 = 0$. Then

$$\nabla F(x, y, z) = \frac{2}{9}x\mathbf{i} + \frac{1}{2}y\mathbf{j} + \frac{1}{18}z\mathbf{k},$$

So if the normal line at P is parallel to $\mathbf{u}$, then $\nabla F(x_0, y_0, z_0) = c\mathbf{u}$ for some constant c , or

$$\frac{2}{9}x_0 = 2c \quad \frac{1}{2}y_0 = -3c \quad \frac{1}{18}z_0 = c.$$

Solving the first equation for c and plugging it into the second, we have $y_0 = -\frac{2}{3}x_0$. Similarly plugging it into the third equation, we have $z_0 = 2x_0$. If this point must be on the given surface, then

$$\begin{aligned} \frac{x_0^2}{9} + \frac{\left(-\frac{2}{3}x_0\right)^2}{4} + \frac{(2x_0)^2}{36} &= 1 \\ \frac{x_0^2}{3} &= 1 \\ x_0 &= \pm\sqrt{3}. \end{aligned}$$

This means there are two points where the tangent plane is parallel to the plane $2x - 3y + z = 4$, and they

are $\left(\sqrt{3}, -\frac{2}{3}\sqrt{3}, 2\sqrt{3}\right)$ and $\left(-\sqrt{3}, \frac{2}{3}\sqrt{3}, -2\sqrt{3}\right)$.

39. The cylinder can be written as $F(x, y, z) = x^2 + y^2 - a^2 = 0$. Then

$$\begin{aligned} F_x(x, y, z) &= 2x & F_x(x_0, y_0, z_0) &= 2x_0 \\ F_y(x, y, z) &= 2y & F_y(x_0, y_0, z_0) &= 2y_0 \\ F_z(x, y, z) &= 0 & F_z(x_0, y_0, z_0) &= 0 \end{aligned}$$

Then an equation of the tangent plane is

$$2x_0(x - x_0) + 2y_0(y - y_0) + 0(z - z_0) = 0$$

$$\boxed{x_0x + y_0y = x_0^2 + y_0^2 = a^2}.$$

40. The cone can be written as $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$. Then

$$\begin{aligned} F_x(x, y, z) &= \frac{2x}{a^2} & F_x(x_0, y_0, z_0) &= \frac{2x_0}{a^2} \\ F_y(x, y, z) &= \frac{2y}{b^2} & F_y(x_0, y_0, z_0) &= \frac{2y_0}{b^2} \\ F_z(x, y, z) &= -\frac{2z}{c^2} & F_z(x_0, y_0, z_0) &= -\frac{2z_0}{c^2} \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) &= 0 \\ \frac{x_0}{a^2}x + \frac{y_0}{b^2}y - \frac{z_0}{c^2}z &= \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 0 \\ \boxed{\frac{x_0}{a^2}x + \frac{y_0}{b^2}y &= \frac{z_0}{c^2}z}. \end{aligned}$$

The point $(0, 0, 0)$ needs to be treated separately because at this point all the partial derivatives vanish.

41. We begin by finding the partial derivatives of $F(x, y, z) = x^{1/2} + y^{1/2} + z^{1/2} - a^{1/2} = 0$ and their values at a general point (x_0, y_0, z_0) .

$$\begin{aligned} F_x(x, y, z) &= \frac{1}{2}x^{-1/2} & F_x(x_0, y_0, z_0) &= \frac{1}{2\sqrt{x_0}} \\ F_y(x, y, z) &= \frac{1}{2}y^{-1/2} & F_y(x_0, y_0, z_0) &= \frac{1}{2\sqrt{y_0}} \\ F_z(x, y, z) &= \frac{1}{2}z^{-1/2} & F_z(x_0, y_0, z_0) &= \frac{1}{2\sqrt{z_0}} \end{aligned}$$

Then an equation of the tangent plane is

$$\begin{aligned} \frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) &= 0 \\ \frac{1}{\sqrt{x_0}}(x - x_0) + \frac{1}{\sqrt{y_0}}(y - y_0) + \frac{1}{\sqrt{z_0}}(z - z_0) &= 0. \end{aligned}$$

The intercept on the x -axis is the point where $y = z = 0$, so plugging $y = z = 0$ into the tangent plane, we find that

$$\begin{aligned} \frac{1}{\sqrt{x_0}}(x - x_0) - \frac{y_0}{\sqrt{y_0}} - \frac{z_0}{\sqrt{z_0}} &= 0 \\ \frac{x}{\sqrt{x_0}} &= \frac{x_0}{\sqrt{x_0}} + \frac{y_0}{\sqrt{y_0}} + \frac{z_0}{\sqrt{z_0}} \\ \frac{x}{\sqrt{x_0}} &= x_0^{1/2} + y_0^{1/2} + z_0^{1/2} = a^{1/2}. \end{aligned}$$

So the x -intercept is $a^{1/2}\sqrt{x_0}$. Similarly, the y - and z -intercepts are $a^{1/2}\sqrt{y_0}$ and $a^{1/2}\sqrt{z_0}$, respectively. If we add the three intercepts together, we get

$$a^{1/2}\sqrt{x_0} + a^{1/2}\sqrt{y_0} + a^{1/2}\sqrt{z_0} = a^{1/2}(x_0^{1/2} + y_0^{1/2} + z_0^{1/2}) = a^{1/2}a^{1/2} = a,$$

which is a constant.

Challenge Problems

42. Let $F(x, y, z) = x \sin(yz) - 1 = 0$ and $G(x, y, z) = ze^{y^2-x^2} - \frac{\pi}{2} = 0$. The tangent line at $\left(1, 1, \frac{\pi}{2}\right)$ is perpendicular to both $\nabla F\left(1, 1, \frac{\pi}{2}\right)$ and $\nabla G\left(1, 1, \frac{\pi}{2}\right)$. One vector $\mathbf{v}$ that is perpendicular to both gradient vectors is the cross product

$$\mathbf{v} = \nabla F\left(1, 1, \frac{\pi}{2}\right) \times \nabla G\left(1, 1, \frac{\pi}{2}\right).$$

We now compute the two gradient vectors.

$$\begin{aligned} F_x(x, y, z) &= \sin(yz) & F_x\left(1, 1, \frac{\pi}{2}\right) &= 1 \\ F_y(x, y, z) &= xz \cos(yz) & F_y\left(1, 1, \frac{\pi}{2}\right) &= 0 \\ F_z(x, y, z) &= xy \cos(yz) & F_z\left(1, 1, \frac{\pi}{2}\right) &= 0, \end{aligned}$$

so $\nabla F\left(1, 1, \frac{\pi}{2}\right) = \mathbf{i}$. Also,

$$\begin{aligned} G_x(x, y, z) &= -2xze^{y^2-x^2} & G_x\left(1, 1, \frac{\pi}{2}\right) &= -\pi \\ G_y(x, y, z) &= 2yze^{y^2-x^2} & G_y\left(1, 1, \frac{\pi}{2}\right) &= \pi \\ G_z(x, y, z) &= e^{y^2-x^2} & G_z\left(1, 1, \frac{\pi}{2}\right) &= 1, \end{aligned}$$

so $\nabla G\left(1, 1, \frac{\pi}{2}\right) = -\pi\mathbf{i} + \pi\mathbf{j} + \mathbf{k}$. Then

$$\mathbf{v} = \nabla F\left(1, 1, \frac{\pi}{2}\right) \times \nabla G\left(1, 1, \frac{\pi}{2}\right) = -\mathbf{j} + \pi\mathbf{k}.$$

This means an equation of the tangent line is

$$\boxed{\mathbf{r}(t) = \left(\mathbf{i} + \mathbf{j} + \frac{\pi}{2}\mathbf{k}\right) + t\mathbf{v} = \left(\mathbf{i} + \mathbf{j} + \frac{\pi}{2}\mathbf{k}\right) + t(-\mathbf{j} + \pi\mathbf{k}).}$$

43. Let $F(x, y, z) = (yz)^{xz}$. To find the partial derivatives of F , we will use logarithmic differentiation. Take the natural log of F , so

$$\ln F(x, y, z) = \ln((yz)^{xz}) = xz \ln(yz). \quad (13.1)$$

Now differentiate both sides of (13.1) with respect to x .

$$\begin{aligned} \frac{1}{F(x, y, z)} \cdot F_x(x, y, z) &= z \ln(yz) \\ F_x(x, y, z) &= (yz)^{xz} \cdot z \ln(yz) \end{aligned}$$

Differentiate both sides of (13.1) with respect to y .

$$\begin{aligned} \frac{1}{F(x, y, z)} \cdot F_y(x, y, z) &= xz \cdot \frac{z}{yz} = \frac{xz}{y} \\ F_y(x, y, z) &= (yz)^{xz} \cdot \frac{xz}{y} \end{aligned}$$

Differentiate both sides of (13.1) with respect to z .

$$\begin{aligned} \frac{1}{F(x, y, z)} \cdot F_z(x, y, z) &= x \ln(yz) + xz \cdot \frac{y}{yz} = x \ln(yz) + x \\ F_z(x, y, z) &= (yz)^{xz} (x \ln(yz) + x) \end{aligned}$$

If we plug in the point $(2, 1, 2)$ to each of the partial derivatives, we have

$$\begin{aligned} F_x(2, 1, 2) &= 32 \ln(2) \\ F_y(2, 1, 2) &= 64 \\ F_z(2, 1, 2) &= 32 \ln(2) + 32. \end{aligned}$$

Then an equation of the tangent plane is

$$32 \ln(2)(x - 2) + 64(y - 1) + (32 \ln(2) + 32)(z - 2) = 0$$

$$\boxed{\ln(2)x + 2y + (\ln(2) + 1)z = 4 + 4 \ln(2)}.$$

Symmetric equations of the normal line are

$$\boxed{\frac{x - 2}{\ln(2)} = \frac{y - 1}{2} = \frac{z - 2}{\ln(2) + 1}}.$$

44. No, the two vectors do not necessarily have to be at right angles. For example, take $f(x, y) = x^2 + y^2$ and write the equation of the tangent plane at $\left(\frac{1}{2}, \frac{1}{2}\right)$. The partial derivatives are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, so an equation of the tangent plane is

$$\begin{aligned} \left(x - \frac{1}{2}\right) + \left(y - \frac{1}{2}\right) &= z - \frac{1}{2} \\ x + y - z &= \frac{1}{2}. \end{aligned}$$

The two vectors $\mathbf{v}_1 = \mathbf{i} + \mathbf{k}$ and $\mathbf{v}_2 = \mathbf{j} + \mathbf{k}$ both lie in the tangent plane since they are orthogonal to the normal vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$. However $\mathbf{v}_1$ and $\mathbf{v}_2$ are not at a right angle since $\mathbf{v}_1 \cdot \mathbf{v}_2 \neq 0$.

13.3 Extrema of Functions of Two Variables

Concepts and Vocabulary

1. Suppose $z = f(x, y)$ is a function with domain D , and (x_0, y_0) is an interior point of D . If $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) within some disk centered at (x_0, y_0) , then f has **(a) a local maximum** at (x_0, y_0) .
2. If $z = f(x, y)$ is a function of two variables whose domain is an open set D containing the point (x_0, y_0) , then the point (x_0, y_0) is a critical point of f if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or if $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.
3. **False**. See the theorem preceding Example 1.
4. If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0) , then the point $(x_0, y_0, f(x_0, y_0))$ is called **(c) a saddle point**.
5. **True**. See the theorem in subsection 3.
6. If $z = f(x, y)$ is a function of two variables that is continuous on a closed, bounded set D , then the absolute extrema are either at the critical points of f in D or on the **boundary** of D .

Skill Building

7. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 4x^3 - 4x = 0 \\ f_y(x, y) = 2y = 0 \end{cases}$$

From f_y , we see that $y = 0$ and f_x factors as $4x(x^2 - 1) = 0$ so the critical points are $(0, 0), (1, 0), (-1, 0)$.

8. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x + 6 = 0 \\ f_y(x, y) = -2y - 2 = 0 \end{cases}$$

The only critical point is $(-3, -1)$.

9. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 4y - 4x^3 = 0 \\ f_y(x, y) = 4x - 4y^3 = 0 \end{cases}$$

From the first equation, $y = x^3$ and plugging it into the f_y equation, we have

$$\begin{aligned} 4x - 4(x^3)^3 &= 0 \\ x(1 - x^8) &= 0 \\ x &= 0, \pm 1. \end{aligned}$$

This means the critical points are $(0, 0), (1, 1), (-1, -1)$.

10. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 + 6y = 0 \\ f_y(x, y) = 6x + 6y = 0 \end{cases}$$

From the second equation, $x = -y$ and plugging it into the f_x equation, we have

$$\begin{aligned} 3(-y)^2 + 6y &= 0 \\ 3y(y + 2) &= 0 \\ y &= 0, -2. \end{aligned}$$

This means the critical points are $(0, 0), (2, -2)$.

11. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = -e^{-x}(x^2 + y) + 2xe^{-x} = 0 \\ f_y(x, y) = e^{-x} = 0 \end{cases}$$

Since e^{-x} is never equal to 0, there are no solutions to the bottom equation, which means there are no critical points.

12. First we find the partial derivatives.

$$\begin{cases} f_x(x, y) = y - \frac{2}{x^2} = 0 \\ f_y(x, y) = x - \frac{4}{y^2} = 0 \end{cases}$$

Notice that neither f_x nor f_y are defined when $x = 0$ or $y = 0$, however these values are not in the domain of f , so the only critical points come from solving the system above. Solving for y in the f_x equation and plugging it into the f_y equation, we have

$$\begin{aligned} x - \frac{4}{(2/x^2)^2} &= 0 \\ x - x^4 = x(1 - x^3) &= 0 \\ x &= 1, \end{aligned}$$

where we excluded the solution $x = 0$ since any point with x -coordinate 0 is not in the domain of f . This means the only critical point is (1, 2).

13. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 2 = 0 \\ f_y(x, y) = 2y + 4 = 0 \end{cases}$$

The only solution is $x = 1, y = -2$, so the only critical point is $(1, -2)$. The second-order partial derivatives and their values at $(1, -2)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A &= f_{xx}(1, -2) = 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B &= f_{xy}(1, -2) = 0 \\ f_{yy}(x, y) &= 2 & C &= f_{yy}(1, -2) = 2 \\ AC - B^2 &= 4 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, $(1, -2)$ is a local minimum. The local minimum is at (1, -2) and the local minimum is $z = -3$.

14. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 4 = 0 \\ f_y(x, y) = 2y + 2 = 0 \end{cases}$$

The only solution is $x = 2, y = -1$, so the only critical point is $(2, -1)$. The second-order partial derivatives and their values at $(2, -1)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A &= f_{xx}(2, -1) = 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B &= f_{xy}(2, -1) = 0 \\ f_{yy}(x, y) &= 2 & C &= f_{yy}(2, -1) = 2 \\ AC - B^2 &= 4 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, $(2, -1)$ is a local minimum. The local minimum is at (2, -1) and the local minimum is $z = -9$.

15. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 4 = 0 \\ f_y(x, y) = 8y + 8 = 0 \end{cases}$$

The only solution is $x = 2, y = -1$, so the only critical point is $(2, -1)$. The second-order partial derivatives and their values at $(2, -1)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A &= f_{xx}(2, -1) = 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B &= f_{xy}(2, -1) = 0 \\ f_{yy}(x, y) &= 8 & C &= f_{yy}(2, -1) = 8 \\ AC - B^2 &= 16 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, $(2, -1)$ is a local minimum. The local minimum is at $(2, -1)$ and the local minimum is $z = -9$.

16. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 4x + 4 = 0 \\ f_y(x, y) = -2y - 4 = 0 \end{cases}$$

The only solution is $x = -1, y = -2$, so the only critical point is $(-1, -2)$. The second-order partial derivatives and their values at $(-1, -2)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 4 & A &= f_{xx}(-1, -2) = 4 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B &= f_{xy}(-1, -2) = 0 \\ f_{yy}(x, y) &= -2 & C &= f_{yy}(-1, -2) = -2 \\ AC - B^2 &= -8 \end{aligned}$$

Since $AC - B^2 < 0$, $(-1, -2, 10)$ is a saddle point.

17. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 - 6y = 0 \\ f_y(x, y) = -6x + 3y^2 = 0 \end{cases}$$

Solving for y in the f_x equation and substituting it into the f_y equation, we have

$$\begin{aligned} -6x + 3\left(\frac{1}{2}x^2\right)^2 &= 0 \\ -24x + 3x^4 &= 3x(-8 + x^3) = 0 \\ x &= 0, 2. \end{aligned}$$

The two critical points are then $(0, 0), (2, 2)$. The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 6x & f_{xx}(0, 0) &= 0 & f_{xx}(2, 2) &= 12 \\ f_{xy}(x, y) &= f_{yx}(x, y) = -6 & f_{xy}(0, 0) &= -6 & f_{xy}(2, 2) &= -6 \\ f_{yy}(x, y) &= 6y & f_{yy}(0, 0) &= 0 & f_{yy}(2, 2) &= 12 \end{aligned}$$

For the critical point $(0, 0)$, $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = -36 < 0$, so $(0, 0, 0)$ is a saddle point.

For the critical point $(2, 2)$, $f_{xx}(2, 2) \cdot f_{yy}(2, 2) - [f_{xy}(2, 2)]^2 = 108 > 0$ and $f_{xx}(2, 2) > 0$, so $(2, 2)$ is a local minimum. The local minimum is at $(2, 2)$ and the local minimum is $z = -8$.

18. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 3y = 0 \\ f_y(x, y) = -3x - 2y = 0 \end{cases}$$

The only solution to this system is $(0, 0)$. The second-order partial derivatives and their values at the critical point are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A &= f_{xx}(0, 0) = 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = -3 & B &= f_{xy}(0, 0) = -3 \\ f_{yy}(x, y) &= -2 & C &= f_{yy}(0, 0) = -2 \\ AC - B^2 &= -13 \end{aligned}$$

Since $AC - B^2 < 0$, $\boxed{(0, 0, 0) \text{ is a saddle point}}$.

19. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x + 3y - 6 = 0 \\ f_y(x, y) = 3x - 2y + 4 = 0 \end{cases}$$

Solving for x in f_x and plugging it into f_y , we have

$$\begin{aligned} 3\left(\frac{6-3y}{2}\right) - 2y + 4 &= 0 \\ 18 - 9y - 4y &= -8 \\ y &= 2. \end{aligned}$$

The only critical point is then $(0, 2)$. The second-order partial derivatives and their values at the critical point are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A &= f_{xx}(0, 2) = 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 3 & B &= f_{xy}(0, 2) = 3 \\ f_{yy}(x, y) &= -2 & C &= f_{yy}(0, 2) = -2 \\ AC - B^2 &= -13 \end{aligned}$$

Since $AC - B^2 < 0$, $\boxed{(0, 2, 4) \text{ is a saddle point}}$.

20. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 4x + y - 2 = 0 \\ f_y(x, y) = x + 2y + 3 = 0 \end{cases}$$

Solving for y in f_x and plugging it into f_y , we have

$$\begin{aligned} x + 2(2 - 4x) + 3 &= 0 \\ -7x + 7 &= 0 \\ x &= 1. \end{aligned}$$

The only critical point is then $(1, -2)$. The second-order partial derivatives and their values at the critical point are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 4 & A &= f_{xx}(1, -2) = 4 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 1 & B &= f_{xy}(1, -2) = 1 \\ f_{yy}(x, y) &= 2 & C &= f_{yy}(1, -2) = 2 \\ AC - B^2 &= 7 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, the critical point $(1, -2)$ is a local minimum. The local minimum is at $\boxed{(1, -2)}$ and the local minimum is $\boxed{z = 2}$.

21. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 + 3y = 0 \\ f_y(x, y) = 3x + 3y^2 = 0 \end{cases}$$

Solving for y in f_x and plugging it into f_y , we have

$$\begin{aligned} 3x + 3(-x^2)^2 &= 0 \\ x(1 + x^3) &= 0 \\ x &= 0, -1. \end{aligned}$$

The two critical points are then $(0, 0)$ and $(-1, -1)$. The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 6x & f_{xx}(0, 0) &= 0 & f_{xx}(-1, -1) &= -6 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 3 & f_{xy}(0, 0) &= 3 & f_{xy}(-1, -1) &= 3 \\ f_{yy}(x, y) &= 6y & f_{yy}(0, 0) &= 0 & f_{yy}(-1, -1) &= -6 \end{aligned}$$

For the critical point $(0, 0)$, $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = -9 < 0$, so $(0, 0, 0)$ is a saddle point.

For the critical point $(-1, -1)$, $f_{xx}(-1, -1) \cdot f_{yy}(-1, -1) - [f_{xy}(-1, -1)]^2 = 27 > 0$ and $f_{xx}(-1, -1) < 0$, so $(-1, -1)$ is a local maximum. The local maximum is at $(-1, -1)$ and the local maximum is $z = 1$.

22. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 - 3y = 0 \\ f_y(x, y) = -3x - 3y^2 = 0 \end{cases}$$

Solving for y in f_x and plugging it into f_y , we have

$$\begin{aligned} -3x - 3(x^2)^2 &= 0 \\ x(1 + x^3) &= 0 \\ x &= 0, -1. \end{aligned}$$

The two critical points are then $(0, 0)$ and $(-1, 1)$. The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 6x & f_{xx}(0, 0) &= 0 & f_{xx}(-1, 1) &= -6 \\ f_{xy}(x, y) &= f_{yx}(x, y) = -3 & f_{xy}(0, 0) &= -3 & f_{xy}(-1, 1) &= -3 \\ f_{yy}(x, y) &= -6y & f_{yy}(0, 0) &= 0 & f_{yy}(-1, 1) &= -6 \end{aligned}$$

For the critical point $(0, 0)$, $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 9 < 0$, so $(0, 0, 0)$ is a saddle point.

For the critical point $(-1, 1)$, $f_{xx}(-1, 1) \cdot f_{yy}(-1, 1) - [f_{xy}(-1, 1)]^2 = 27 > 0$ and $f_{xx}(-1, 1) < 0$, so $(-1, 1)$ is a local maximum. The local maximum is at $(-1, 1)$ and the local maximum is $z = 1$.

23. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 + 2xy = 0 \\ f_y(x, y) = x^2 + 2y = 0 \end{cases}$$

Solving for y in f_y and plugging it into f_x , we have

$$\begin{aligned} 3x^2 + 2x\left(-\frac{x^2}{2}\right) &= 0 \\ x^2(3 - x) &= 0 \\ x &= 0, 3. \end{aligned}$$

The two critical points are then $(0, 0)$ and $\left(3, -\frac{9}{2}\right)$. The second-order partial derivatives and their values

at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 6x + 2y & f_{xx}(0, 0) &= 0 & f_{xx}\left(3, -\frac{9}{2}\right) &= 9 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 2x & f_{xy}(0, 0) &= 0 & f_{xy}\left(3, -\frac{9}{2}\right) &= 6 \\ f_{yy}(x, y) &= 2 & f_{yy}(0, 0) &= 2 & f_{yy}\left(3, -\frac{9}{2}\right) &= 2 \end{aligned}$$

For the critical point $(0, 0)$, $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0$, so the

second partial derivative test is inconclusive for the point $(0, 0)$

For the critical point $\left(3, -\frac{9}{2}\right)$, $f_{xx}\left(3, -\frac{9}{2}\right) \cdot f_{yy}\left(3, -\frac{9}{2}\right) - [f_{xy}\left(3, -\frac{9}{2}\right)]^2 = -18 < 0$, so

$\left(3, -\frac{9}{2}, \frac{27}{4}\right)$ is a saddle point.

24. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = -2xy + 1 = 0 \\ f_y(x, y) = 9y^2 - x^2 = 0 \end{cases}$$

Note that neither x nor y can be equal to 0 (because of the first equation) so solving for y in f_x and plugging it into f_y , we have

$$\begin{aligned} 9\left(\frac{1}{2x}\right)^2 - x^2 &= 0 \\ \frac{9}{4x^2} - x^2 &= 0 \\ \frac{9}{4} - x^4 &= 0 \\ x &= \pm \sqrt[4]{\frac{9}{4}} = \pm \sqrt{\frac{3}{2}} = \pm \frac{\sqrt{6}}{2}. \end{aligned}$$

Since $y = \frac{1}{2x}$ the two critical points are then $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right)$ and $\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right)$. The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= -2y & f_{xx}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) &= -\frac{\sqrt{6}}{3} & f_{xx}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) &= \frac{\sqrt{6}}{3} \\ f_{xy}(x, y) &= f_{yx}(x, y) = -2x & f_{xy}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) &= -\sqrt{6} & f_{xy}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) &= \sqrt{6} \\ f_{yy}(x, y) &= 18y & f_{yy}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) &= 3\sqrt{6} & f_{yy}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) &= -3\sqrt{6} \end{aligned}$$

For the critical point $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right)$,

$$f_{xx}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) \cdot f_{yy}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) - [f_{xy}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right)]^2 = -12 < 0,$$

so $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)$ is a saddle point.

For the critical point $\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right)$, $f_{xx}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) \cdot f_{yy}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) - \left[f_{xy}\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right)\right]^2 = -12 < 0$, so $\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}\right)$ is a saddle point.

25. The partial derivatives are

$$\begin{aligned} f_x(x, y) &= \frac{(x+y)(0) - (y)(1)}{(x+y)^2} = -\frac{y}{(x+y)^2} \\ f_y(x, y) &= \frac{(x+y)(1) - (y)(1)}{(x+y)^2} = \frac{x}{(x+y)^2}. \end{aligned}$$

The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = -\frac{y}{(x+y)^2} = 0 \\ f_y(x, y) = \frac{x}{(x+y)^2} = 0 \end{cases}$$

or where f_x or f_y are undefined. However, all points where f_x or f_y are undefined are not in the domain of f , so we need only find solutions of the system to find critical points. The only solution is $x = 0, y = 0$, but this point is not in the domain, so there are no critical points.

26. The partial derivatives are

$$\begin{aligned} f_x(x, y) &= \frac{(x+y)(1) - (x)(1)}{(x+y)^2} = \frac{y}{(x+y)^2} \\ f_y(x, y) &= \frac{(x+y)(0) - (x)(1)}{(x+y)^2} = -\frac{x}{(x+y)^2}. \end{aligned}$$

The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = \frac{y}{(x+y)^2} = 0 \\ f_y(x, y) = -\frac{x}{(x+y)^2} = 0 \end{cases}$$

or where f_x or f_y are undefined. However, all points where f_x or f_y are undefined are not in the domain of f , so we need only find solutions of the system to find critical points. The only solution is $x = 0, y = 0$, but this point is not in the domain, so there are no critical points.

27. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = -\sin x = 0 \\ f_y(x, y) = -\sin y = 0 \end{cases}$$

The critical points are then $(m\pi, n\pi)$ for any integers m, n . The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{array}{lll} f_{xx}(x, y) = -\cos x & f_{xx}(m\pi, n\pi) = 1 \text{ if } m \text{ is odd} & f_{xx}(m\pi, n\pi) = -1 \text{ if } m \text{ is even} \\ f_{xy}(x, y) = f_{yx}(x, y) = 0 & & f_{xy}(m\pi, n\pi) = 0 \\ f_{yy}(x, y) = -\cos y & f_{yy}(m\pi, n\pi) = 1 \text{ if } n \text{ is odd} & f_{yy}(m\pi, n\pi) = -1 \text{ if } n \text{ is even} \end{array}$$

If m and n are odd, then $f_{xx}(m\pi, n\pi) \cdot f_{yy}(m\pi, n\pi) - (f_{xy}(m\pi, n\pi))^2 = 1 > 0$ and $f_{xx}(m\pi, n\pi) > 0$ so $(m\pi, n\pi)$ is a local minimum. The local minimum is at $(m\pi, n\pi)$ for m and n odd and the local minimum is $z = -2$

If m and n are even, then $f_{xx}(m\pi, n\pi) \cdot f_{yy}(m\pi, n\pi) - (f_{xy}(m\pi, n\pi))^2 = 1 > 0$ and $f_{xx}(m\pi, n\pi) < 0$ so $(m\pi, n\pi)$ is a local maximum. The local maximum is at $(m\pi, n\pi)$ for m and n even and the local maximum is $z = 2$.

If exactly one of m or n is odd and the other is even, then

$f_{xx}(m\pi, n\pi) \cdot f_{yy}(m\pi, n\pi) - (f_{xy}(m\pi, n\pi))^2 = -1 < 0$, so $(m\pi, n\pi, 0)$ is a saddle point when exactly one of m or n is odd.

28. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 4 \cos y = 0 \\ f_y(x, y) = 4x \sin y = 0 \end{cases}$$

From the second equation, we see that $x = 0$ or $y = n\pi$ for some integer n . If $x = 0$, then the first equation tells us that $y = m\pi + \frac{\pi}{2}$ for some integer m .

If $y = n\pi$, then the first equation tells us that $x = 2$ when n is even and $x = -2$ when n is odd. So the critical points are

$$\begin{aligned} \text{Case 1 : } & \left(0, m\pi + \frac{\pi}{2}\right) \quad \text{for } m \text{ an integer} \\ \text{Case 2 : } & (2, n\pi) \quad \text{for } n \text{ an even integer} \\ \text{Case 3 : } & (-2, n\pi) \quad \text{for } n \text{ an odd integer.} \end{aligned}$$

The second-order partial derivatives are

$$f_{xx}(x, y) = 2 \quad f_{xy}(x, y) = f_{yx}(x, y) = 4 \sin y \quad f_{yy}(x, y) = 4x \cos y.$$

In Case 1, $f_{xx}\left(0, m\pi + \frac{\pi}{2}\right) \cdot f_{yy}\left(0, m\pi + \frac{\pi}{2}\right) - [f_{xy}\left(0, m\pi + \frac{\pi}{2}\right)]^2 = -16 < 0$ so the points

$\left(0, m\pi + \frac{\pi}{2}, 4\right)$ are saddle points.

In Case 2, $f_{xx}(2, n\pi) \cdot f_{yy}(2, n\pi) - [f_{xy}(2, n\pi)]^2 = 16 > 0$ and $f_{xx}(2, n\pi) > 0$ so these points are local minimums. The local minima are at $(2, n\pi)$ for n an even integer and the local minima is $z = 0$.

In Case 3, $f_{xx}(-2, n\pi) \cdot f_{yy}(-2, n\pi) - [f_{xy}(-2, n\pi)]^2 = 16 > 0$ and $f_{xx}(-2, n\pi) > 0$ so these points are local minimums. The local minima are at $(-2, n\pi)$ for n an odd integer and the local minima is $z = 0$.

29. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 6y \sin x = 0 \\ f_y(x, y) = 2y - 6 \cos x = 0 \end{cases}$$

From the first equation, we see that $y = 0$ or $x = n\pi$ for some integer n . If $y = 0$, then the second equation tells us that $x = m\pi + \frac{\pi}{2}$ for some integer m .

If $x = n\pi$, then the second equation tells us that $y = 3$ when n is even and $y = -3$ when n is odd. So the critical points are

$$\begin{aligned} \text{Case 1 : } & \left(m\pi + \frac{\pi}{2}, 0\right) \quad \text{for } m \text{ an integer} \\ \text{Case 2 : } & (n\pi, 3) \quad \text{for } n \text{ an even integer} \\ \text{Case 3 : } & (n\pi, -3) \quad \text{for } n \text{ an odd integer.} \end{aligned}$$

The second-order partial derivatives are

$$f_{xx}(x, y) = 6y \cos x \quad f_{xy}(x, y) = f_{yx}(x, y) = 6 \sin x \quad f_{yy}(x, y) = 2.$$

In Case 1, $f_{xx}\left(m\pi + \frac{\pi}{2}, 0\right) \cdot f_{yy}\left(m\pi + \frac{\pi}{2}, 0\right) - [f_{xy}\left(m\pi + \frac{\pi}{2}, 0\right)]^2 = -36 < 0$ so the points

$\left(m\pi + \frac{\pi}{2}, 0, 6\right)$ are saddle points.

In Case 2, $f_{xx}(n\pi, 3) \cdot f_{yy}(n\pi, 3) - [f_{xy}(n\pi, 3)]^2 = 36 > 0$ and $f_{xx}(n\pi, 3) = 18 > 0$ so these points are local minimums. The local minima are at $(n\pi, 3)$ for n an even integer and the local minima is $z = -3$.

In Case 3, $f_{xx}(n\pi, -3) \cdot f_{yy}(n\pi, -3) - [f_{xy}(n\pi, -3)]^2 = 16 > 0$ and $f_{xx}(n\pi, -3) = 18 > 0$ so these points are local minimums. The local minima are at $(n\pi, -3)$ for n an odd integer and the local minima is $z = -3$.

30. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 6\sin y = 0 \\ f_y(x, y) = -6x\cos y = 0 \end{cases}$$

From the second equation, we see that $x = 0$ or $y = n\pi + \frac{\pi}{2}$ for some integer n . If $x = 0$, then the first equation tells us that $y = m\pi$ for some integer m .

If $y = n\pi + \frac{\pi}{2}$, then the first equation tells us that $x = 3$ when n is even and $x = -3$ when n is odd. So the critical points are

$$\begin{aligned} \text{Case 1 :} & \quad (0, m\pi) \quad \text{for } m \text{ an integer} \\ \text{Case 2 :} & \quad \left(3, n\pi + \frac{\pi}{2}\right) \quad \text{for } n \text{ an even integer} \\ \text{Case 3 :} & \quad \left(-3, n\pi + \frac{\pi}{2}\right) \quad \text{for } n \text{ an odd integer.} \end{aligned}$$

The second-order partial derivatives are

$$f_{xx}(x, y) = 2 \quad f_{xy}(x, y) = f_{yx}(x, y) = -6\cos y \quad f_{yy}(x, y) = 6x\sin y.$$

In Case 1, $f_{xx}(0, m\pi) \cdot f_{yy}(0, m\pi) - [f_{xy}(0, m\pi)]^2 = -36 < 0$ so the points $(0, m\pi, 2)$ are saddle points.

In Case 2, $f_{xx}\left(3, n\pi + \frac{\pi}{2}\right) \cdot f_{yy}\left(3, n\pi + \frac{\pi}{2}\right) - \left[f_{xy}\left(3, n\pi + \frac{\pi}{2}\right)\right]^2 = 36 > 0$ and $f_{xx}\left(3, n\pi + \frac{\pi}{2}\right) > 0$ so these points are local minimums. The local minima are at $\left(3, n\pi + \frac{\pi}{2}\right)$ for n an even integer and the local minima is $z = -7$.

In Case 3, $f_{xx}\left(-3, n\pi + \frac{\pi}{2}\right) \cdot f_{yy}\left(-3, n\pi + \frac{\pi}{2}\right) - \left[f_{xy}\left(-3, n\pi + \frac{\pi}{2}\right)\right]^2 = 36 > 0$ and $f_{xx}\left(-3, n\pi + \frac{\pi}{2}\right) > 0$ so these points are local minimums. The local minima are at $\left(-3, n\pi + \frac{\pi}{2}\right)$ for n an odd integer and the local minima is $z = -7$.

31. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = ye^{xy} = 0 \\ f_y(x, y) = xe^{xy} = 0 \end{cases}$$

Since e^{xy} is never equal to 0, the only critical point is $(0, 0)$. The second-order partial derivatives and their values at the critical point are as follows.

$$\begin{aligned} f_{xx}(x, y) &= y^2e^{xy} & A &= f_{xx}(0, 0) = 0 \\ f_{xy}(x, y) &= f_{yx}(x, y) = e^{xy} + xye^{xy} & B &= f_{xy}(0, 0) = 1 \\ f_{yy}(x, y) &= x^2e^{xy} & C &= f_{yy}(0, 0) = 0 \\ AC - B^2 &= -1 \end{aligned}$$

Since $AC - B^2 < 0$, $(0, 0, 1)$ is a saddle point.

32. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = ye^{-(x+y)} - xye^{-(x+y)} = 0 \\ f_y(x, y) = xe^{-(x+y)} - xye^{-(x+y)} = 0 \end{cases}$$

Since $e^{-(x+y)}$ is never equal to 0, we can cancel it and obtain the new system to solve.

$$\begin{cases} y - xy = y(1 - x) = 0 \\ x - xy = x(1 - y) = 0 \end{cases}$$

From here, we see the only critical points are $(0,0)$ and $(1,1)$. The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{array}{lll} f_{xx}(x,y) = -2ye^{-(x+y)} + xye^{-(x+y)} & f_{xx}(1,1) = -e^{-2} & f_{xx}(0,0) = 0 \\ f_{xy}(x,y) = f_{yx}(x,y) = e^{-(x+y)} - xe^{-(x+y)} - ye^{-(x+y)} + xye^{-(x+y)} & f_{xy}(1,1) = 0 & f_{xy}(0,0) = 1 \\ f_{yy}(x,y) = -2xe^{-(x+y)} + xye^{-(x+y)} & f_{yy}(1,1) = -e^{-2} & f_{yy}(0,0) = 0 \end{array}$$

For the critical point $(0,0)$, $f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -1 < 0$, so the point $(0,0,0)$ is a saddle point.

For the critical point $(1,1)$, $f_{xx}(1,1) \cdot f_{yy}(1,1) - [f_{xy}(1,1)]^2 = e^{-4} > 0$ and $f_{xx}(1,1) < 0$, so the point $(1,1)$ is a local maximum. The local maximum is at $(1,1)$ and the local maximum is $z = e^{-2}$.

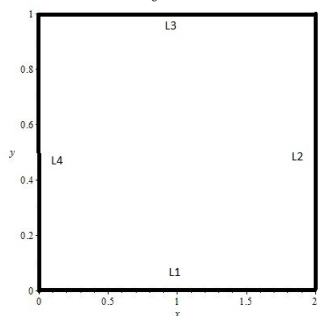
33. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x,y) = 2x - 4y = 0 \\ f_y(x,y) = -4x + 4 = 0 \end{cases}$$

The only critical point is then $\left(1, \frac{1}{2}\right)$. The value of f at this point is

$$f\left(1, \frac{1}{2}\right) = 1.$$

The boundary of D is made up of the four line segments L_1 , L_2 , L_3 , and L_4 as drawn in the following picture.



On L_1 : $y = 0$ and $0 \leq x \leq 2$. The function $f(x,y) = f(x,0) = x^2$ is increasing on $0 \leq x \leq 2$ and its extreme values are $f(0,0) = 0$ and $f(2,0) = 4$.

On L_2 : $x = 2$ and $0 \leq y \leq 1$. The function $f(x,y) = f(2,y) = 4 - 4y$ is decreasing on $0 \leq y \leq 1$ and its extreme values are $f(2,1) = 0$ and $f(2,0) = 4$.

On L_3 : $y = 1$ and $0 \leq x \leq 2$. The function $f(x,y) = f(x,1) = x^2 - 4x + 4$ can be thought of as a function of a single variables, namely $g(x) = x^2 - 4x + 4$. The extreme of this function can be found by finding the critical number(s) of g , namely where $g'(x) = 2x - 4 = 0$, or $x = 2$. This means the extrema occur at the endpoints, and those extreme values are $g(2) = f(2,1) = 0$ and $g(0) = f(0,1) = 4$.

On L_4 : $x = 0$ and $0 \leq y \leq 1$. Then function $f(x,y) = f(0,y) = 4y$ is increasing on $0 \leq y \leq 1$ and its extreme values are $f(0,0) = 0$ and $f(0,1) = 4$.

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is 0 and the absolute maximum is 4.

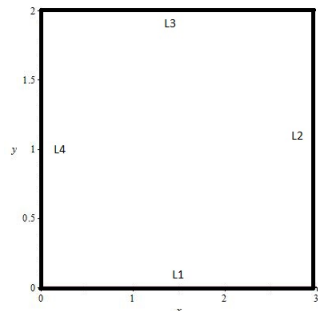
34. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x,y) = 2x - 4y = 0 \\ f_y(x,y) = -4x + 4 = 0 \end{cases}$$

The only critical point is then $\left(1, \frac{1}{2}\right)$. The value of f at this point is

$$f\left(1, \frac{1}{2}\right) = 1.$$

The boundary of D is made up of the four line segments L_1 , L_2 , L_3 , and L_4 as drawn in the following picture.



On L_1 : $y = 0$ and $0 \leq x \leq 3$. The function $f(x, y) = f(x, 0) = x^2$ is increasing on $0 \leq x \leq 3$ and its extreme values are $f(0, 0) = 0$ and $f(3, 0) = 9$.

On L_2 : $x = 3$ and $0 \leq y \leq 2$. The function $f(x, y) = f(3, y) = 9 - 8y$ is decreasing on $0 \leq y \leq 2$ and its extreme values are $f(3, 2) = -7$ and $f(3, 0) = 9$.

On L_3 : $y = 2$ and $0 \leq x \leq 3$. The function $f(x, y) = f(x, 2) = x^2 - 8x + 8$ can be thought of as a function of a single variable, namely $g(x) = x^2 - 8x + 8$. The extreme of this function can be found by finding the critical number(s) of g , namely where $g'(x) = 2x - 8 = 0$, or $x = 4$. This value is not in the interval $0 \leq x \leq 3$, so the extrema occur at the endpoints, and those extreme values are $g(3) = f(3, 2) = -7$ and $g(0) = f(0, 2) = 8$.

On L_4 : $x = 0$ and $0 \leq y \leq 2$. Then function $f(x, y) = f(0, y) = 4y$ is increasing on $0 \leq y \leq 2$ and its extreme values are $f(0, 0) = 0$ and $f(0, 2) = 8$.

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -7 and the absolute maximum is 9 .

35. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x + 2 = 0 \\ f_y(x, y) = 2y - 2 = 0 \end{cases}$$

The only critical point is then $(-1, 1)$. The value of f at this critical point is $f(-1, 1) = -2$.

The boundary of D is a circle of radius 2 centered at the origin which can be parameterized by $x(t) = 2 \cos t$ and $y = 2 \sin t$ for $0 \leq t \leq 2\pi$. On the boundary

$$z = f(x, y) = f(2 \cos t, 2 \sin t) = 4 \cos^2 t + 4 \cos t + 4 \sin^2 t - 4 \sin t = 4(\cos t - \sin t + 1).$$

Now we find where $z'(t) = 0$ on $0 \leq t \leq 2\pi$.

$$\begin{aligned} 4(-\sin t - \cos t) &= 0 \\ -\sin t &= \cos t \\ t &= \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

At these two critical values and the endpoints $t = 0, 2\pi$, the values of f are

$$\begin{aligned} f\left(2 \cos \frac{3\pi}{4}, 2 \sin \frac{3\pi}{4}\right) &= 4 - 4\sqrt{2} \\ f\left(2 \cos \frac{7\pi}{4}, 2 \sin \frac{7\pi}{4}\right) &= 4 + 4\sqrt{2} \\ f(2 \cos 0, 2 \sin 0) &= 8 \\ f(2 \cos 2\pi, 2 \sin 2\pi) &= 8 \end{aligned}$$

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -2

and the absolute maximum is $4 + 4\sqrt{2}$.

36. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 4 = 0 \\ f_y(x, y) = 2y - 4 = 0 \end{cases}$$

The only critical point is then $(2, 2)$. The value of f at this critical point is $f(2, 2) = -8$.

The boundary of D is a circle of radius 3 centered at the origin which can be parameterized by $x(t) = 3 \cos t$ and $y = 3 \sin t$ for $0 \leq t \leq 2\pi$. On the boundary

$$z = f(x, y) = f(3 \cos t, 3 \sin t) = 9 \cos^2 t - 12 \cos t + 9 \sin^2 t - 12 \sin t = -12 \cos t - 12 \sin t + 9.$$

Now we find where $z'(t) = 0$ on $0 \leq t \leq 2\pi$.

$$\begin{aligned} 12 \sin t - 12 \cos t &= 0 \\ \sin t &= \cos t \\ t &= \frac{\pi}{4}, \frac{5\pi}{4}. \end{aligned}$$

At these two critical values and the endpoints $t = 0, 2\pi$, the values of f are

$$\begin{aligned} f\left(3 \cos \frac{\pi}{4}, 3 \sin \frac{\pi}{4}\right) &= 9 - 12\sqrt{2} \\ f\left(3 \cos \frac{5\pi}{4}, 3 \sin \frac{5\pi}{4}\right) &= 9 + 12\sqrt{2} \\ f(2 \cos 0, 2 \sin 0) &= -3 \\ f(2 \cos 2\pi, 2 \sin 2\pi) &= -3 \end{aligned}$$

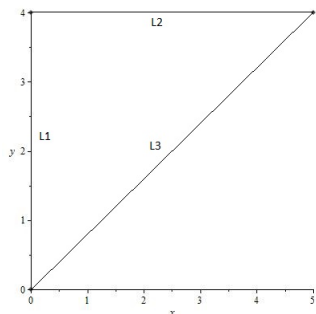
Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -8

and the absolute maximum is $12 + 12\sqrt{2}$.

37. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = y + 4 = 0 \\ f_y(x, y) = x - 3 = 0 \end{cases}$$

The only critical point is then $(3, -4)$. The value of f at this critical point is $f(3, -4) = 17$. The boundary of D is made up of the three line segments L_1 , L_2 , and L_3 as drawn in the following picture.



On L_1 : $x = 0$ and $0 \leq y \leq 4$. The function $f(x, y) = f(0, y) = 5 - 3y$ is decreasing on $0 \leq y \leq 4$ and its extreme values are $f(0, 4) = -7$ and $f(0, 0) = 5$.

On L_2 : $y = 4$ and $0 \leq x \leq 5$. The function $f(x, y) = f(x, 4) = 8x - 7$ is increasing on $0 \leq x \leq 5$ and its extreme values are $f(0, 4) = -7$ and $f(5, 4) = 33$.

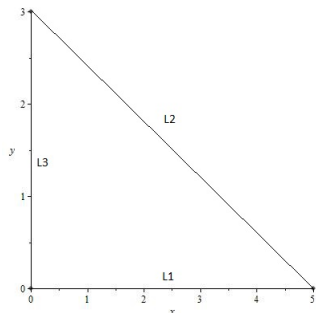
On L_3 : $y = \frac{4}{5}x$ and $0 \leq x \leq 5$. The function $f(x, y) = f\left(x, \frac{4}{5}x\right) = \frac{4}{5}x^2 + \frac{8}{5}x + 5$ can be thought of as a function of a single variable, namely $g(x) = \frac{4}{5}x^2 + \frac{8}{5}x + 5$. The extrema of g occur at the endpoints of the interval or at a critical number, which are found by setting $g'(x) = \frac{8}{5}x + \frac{8}{5} = 0$, namely $x = -1$. Since this critical number is not in the interval $0 \leq x \leq 5$ the extreme values are $g(0) = f(0, 0) = 5$ and $g(5) = f(5, 4) = 33$.

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -7 and the absolute maximum is 33 .

38. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3y - 3 = 0 \\ f_y(x, y) = 3x - 6 = 0 \end{cases}$$

The only critical point is then $(2, 1)$. The value of f at this critical point is $f(2, 1) = -5$. The boundary of D is made up of the three line segments L_1 , L_2 , and L_3 as drawn in the following picture.



On L_1 : $y = 0$ and $0 \leq x \leq 5$. The function $f(x, y) = f(x, 0) = 1 - 3x$ is decreasing on $0 \leq x \leq 5$ and its extreme values are $f(0, 0) = 1$ and $f(5, 0) = -14$.

On L_2 : $y = 3 - \frac{3}{5}x$ and $0 \leq x \leq 5$. The function $f(x, y) = f\left(x, 3 - \frac{3}{5}x\right) = -\frac{9}{5}x^2 + \frac{48}{5}x - 17$ has a maximum when $f'\left(x, 3 - \frac{3}{5}x\right) = -\frac{18}{5}x + \frac{48}{5} = 0$, namely $x = \frac{8}{3}$. The value at this point is $f\left(\frac{8}{3}\right) = -\frac{21}{5}$. Testing the endpoints of the interval $0 \leq x \leq 5$, we find that $f(0, 3) = -17$ and $f(5, 0) = -14$. Then the extreme values on L_2 are $f(0, 3) = -17$ and $f\left(\frac{8}{3}\right) = -\frac{21}{5}$.

On L_3 : $x = 0$ and $0 \leq y \leq 3$. The function $f(x, y) = f(0, y) = 1 - 6y$ is decreasing on $0 \leq y \leq 3$ and its extreme values are $f(0, 3) = -17$ and $f(0, 0) = 1$.

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -17 and the absolute maximum is 1 .

Applications and Extensions

39. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = \cos x \sin y = 0 \\ f_y(x, y) = \sin x \cos y = 0 \end{cases}$$

From the first equation, we see that $x = n\pi + \frac{\pi}{2}$ or $y = n\pi$ for any integer n .

If $x = n\pi + \frac{\pi}{2}$, then the second equation implies $y = m\pi + \frac{\pi}{2}$ for any integer m .

If $y = n\pi$, then the second equation implies $x = m\pi$ for any integer m .

The second-order partial derivatives are as follows.

$$\begin{aligned} f_{xx}(x, y) &= -\sin x \sin y \\ f_{xy}(x, y) &= f_{yx}(x, y) = \cos x \cos y \\ f_{yy}(x, y) &= -\sin x \sin y \end{aligned}$$

For the critical points $\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)$ and $\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)$, we have that

$$f_{xx}\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) \cdot f_{yy}\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) - \left[f_{xy}\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)\right]^2 = 1 > 0$$

$$f_{xx}\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) \cdot f_{yy}\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) - \left[f_{xy}\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)\right]^2 = 1 > 0$$

$$f_{xx}\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) = 1 > 0 \text{ and } f_{xx}\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) = 1 > 0.$$

This means we have local minima at the points $\left(-\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)$ and $\left(\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)$ and

the local minimum is $z = -1$.

For the critical points $\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)$ and $\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)$, we have that

$$f_{xx}\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) \cdot f_{yy}\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) - \left[f_{xy}\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)\right]^2 = 1 > 0$$

$$f_{xx}\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) \cdot f_{yy}\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) - \left[f_{xy}\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)\right]^2 = 1 > 0$$

$$f_{xx}\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right) = -1 < 0 \text{ and } f_{xx}\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right) = -1 < 0.$$

This means we have local maxima at the points $\left(-\frac{\pi}{2} + 2\pi m, -\frac{\pi}{2} + 2\pi n\right)$ and $\left(\frac{\pi}{2} + 2\pi m, \frac{\pi}{2} + 2\pi n\right)$ and

the local maximum is $z = 1$.

For the critical points $(m\pi, n\pi)$,

$$f_{xx}(m\pi, n\pi) \cdot f_{yy}(m\pi, n\pi) - [f_{xy}(m\pi, n\pi)]^2 = -1 < 0,$$

so the points $(m\pi, n\pi, 0)$ are saddle points.

40. (a) By completing the square for both x and y , the equation becomes

$$\begin{aligned} (x-3)^2 + (y+2)^2 + z &= -12 + 9 + 4 \\ z &= -(x-3)^2 - (y+2)^2 + 1. \end{aligned}$$

The value of z is largest when $x = 3$ and $y = -2$, so the vertex is $(3, -2, 1)$.

(b) To find the vertex, we will find the local maximum. We start by finding the critical point, which is the solution of the system of equations

$$\begin{cases} f_x(x, y) = -2x + 6 = 0 \\ f_y(x, y) = -2y - 4 = 0 \end{cases}$$

The only critical point is then $(3, -2)$. The second-order partial derivatives and their values at $(3, -2)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= -2 & A &= f_{xx}(3, -2) = -2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B &= f_{xy}(3, -2) = 0 \\ f_{yy}(x, y) &= -2 & C &= f_{yy}(3, -2) = -2 \\ AC - B^2 &= 4 \end{aligned}$$

Since $AC - B^2 > 0$ and $f_{xx}(3, -2) < 0$, the point $(3, -2, 1)$ is indeed the local maximum, which is the vertex of the paraboloid.

41. The highest point on the mountain is the vertex, which is the critical point, found by solving the system of equations

$$\begin{cases} f_x(x, y) = 2y - 4x - 8 = 0 \\ f_y(x, y) = 2x - 2y + 6 = 0 \end{cases}$$

Solving for y in the f_x equation and plugging it into the f_y equation, we have

$$\begin{aligned} 2x - 2(2x + 4) + 6 &= 0 \\ -2x &= 2 \\ x &= -1 \end{aligned}$$

so that $y = 2$. This means at its highest peak, the mountain is $\boxed{z(-1, 2) = 14 \text{ kilometers}}$ above sea level.

42. (a) Instead of minimizing the distance from the plane to the origin, we will, equivalently, minimize the *square* of the distance from the plane to the origin. Let (x, y, z) be some point on the plane, so $z = 14 - 3x - 2y$. The square of the distance from (x, y, z) to the origin is

$$f(x, y) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + (14 - 3x - 2y)^2.$$

We now need to find the absolute minimum of f on the real plane. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 20x + 12y - 84 = 0 \\ f_y(x, y) = 12x + 10y - 56 = 0 \end{cases}$$

Solving for y in the f_y equation and plugging it into the f_x equation, we have

$$\begin{aligned} 20x + 12\left(\frac{56 - 12x}{10}\right) - 84 &= 0 \\ 28x - 84 &= 0 \\ x &= 3, \end{aligned}$$

which means $y = 2$. The only critical point is $(3, 2)$. The second-order partial derivatives and their values at $(3, 2)$ are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 20 & A &= f_{xx}(3, 2) = 20 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 12 & B &= f_{xy}(3, 2) = 12 \\ f_{yy}(x, y) &= 10 & C &= f_{yy}(3, 2) = 10 \\ AC - B^2 &= 56 \end{aligned}$$

Since $AC - B^2 > 0$ and $f_{xx}(3, 2) = 20 > 0$, the point $(3, 2)$ is a local minimum. Since the physical properties of the problem require that the absolute minimum exists, the local minimum is also the absolute minimum. So the point in the plane closest to the origin is $\boxed{(3, 2, 1)}$.

(b) The distance from the origin to the point $(3, 2, 1)$ is

$$\sqrt{3^2 + 2^2 + 1^2} = \boxed{\sqrt{14}}.$$

43. If we let l , w , and h be the length, width, and height of the box, respectively, then on one hand the surface area of the box is S and on the other hand it is $2lh + 2wh + lw$, so

$$S = 2lh + 2wh + lw.$$

The volume of the box is

$$V(l, w, h) = lwh \quad l > 0, \quad w > 0, \quad h > 0.$$

Solving for h in the surface area equation, we have $h = \frac{S - lw}{2l + 2w}$. Plugging this into the volume function, we have

$$V(l, w) = \frac{Slw - l^2w^2}{2l + 2w}.$$

Since $l > 0$ and $w > 0$, the only critical points of V will come from solving the system of equations

$$\begin{cases} V_l(l, w) = \frac{(2l + 2w)(Sw - 2lw^2) - (Slw - l^2w^2)(2)}{(2l + 2w)^2} = \frac{w^2(S - 2lw - l^2)}{2(l + w)^2} = 0 \\ V_w(l, w) = \frac{(2l + 2w)(Sl - 2l^2w) - (Slw - l^2w^2)(2)}{(2l + 2w)^2} = \frac{l^2(S - 2lw - w^2)}{2(l + w)^2} = 0 \end{cases}$$

Since both l and w are positive, we need to solve the system

$$\begin{cases} S - 2lw - l^2 = 0 \\ S - 2lw - w^2 = 0 \end{cases}$$

Subtracting the first equation from the second, we find that $l^2 = w^2$. Again since l and w are both positive, it must be that $l = w$. Then using the first equation in the system and the fact that $l = w$,

$$\begin{aligned} S - 2l^2 - l^2 &= 0 \\ 3l^2 &= S \\ l &= \sqrt{\frac{S}{3}} \\ w &= \sqrt{\frac{S}{3}}. \end{aligned}$$

Then

$$h = \frac{S - l^2}{2l + 2l} = \frac{S - l^2}{4l} = \frac{S - \frac{S}{3}}{4\sqrt{\frac{S}{3}}} = \frac{\sqrt{3S}}{6}.$$

The dimensions that maximize volume are when the length and width are $\sqrt{\frac{S}{3}}$ and the height is $\frac{\sqrt{3S}}{6} = \frac{1}{2}\sqrt{\frac{S}{3}}$.

44. If we let l , w , and h be the length, width, and height of the box, respectively, then on one hand the surface area of the box is S and on the other hand it is $2lh + 2wh + 2lw$, so

$$S = 2lh + 2wh + 2lw.$$

The volume of the box is

$$V(l, w, h) = lwh \quad l > 0, \quad w > 0, \quad h > 0.$$

Solving for h in the surface area equation, we have $h = \frac{S - 2lw}{2l + 2w}$. Plugging this into the volume function, we have

$$V(l, w) = \frac{Slw - 2l^2w^2}{2l + 2w}.$$

Since $l > 0$ and $w > 0$, the only critical points of V will come from solving the system of equations

$$\begin{cases} V_l(l, w) = \frac{(2l + 2w)(Sw - 4lw^2) - (Slw - 2l^2w^2)(2)}{(2l + 2w)^2} = \frac{w^2(S - 4lw - 2l^2)}{2(l + w)^2} = 0 \\ V_w(l, w) = \frac{(2l + 2w)(Sl - 4l^2w) - (Slw - 2l^2w^2)(2)}{(2l + 2w)^2} = \frac{l^2(S - 4lw - 2w^2)}{2(l + w)^2} = 0 \end{cases}$$

Since both l and w are positive, we need to solve the system

$$\begin{cases} S - 4lw - 2l^2 = 0 \\ S - 4lw - 2w^2 = 0 \end{cases}$$

Subtracting the first equation from the second, we find that $2l^2 = 2w^2$. Again since l and w are both positive, it must be that $l = w$. Then using the first equation in the system and the fact that $l = w$,

$$\begin{aligned} S - 4l^2 - 2l^2 &= 0 \\ 6l^2 &= S \\ l &= \sqrt{\frac{S}{6}} \\ w &= \sqrt{\frac{S}{6}}. \end{aligned}$$

Then

$$h = \frac{S - 2l^2}{2l + 2l} = \frac{S - 2l^2}{4l} = \frac{S - 2 \cdot \frac{S}{6}}{4\sqrt{\frac{S}{6}}} = \sqrt{\frac{S}{6}}.$$

The dimensions that maximize volume are when the length, width, and height are all $\sqrt{\frac{S}{6}}$.

45. It will be beneficial to read the solution to Example 7 before reading this solution. Since the box is closed, the surface area S function is now

$$S = 2xy + 2xz + 2yz.$$

The volume V equation is still $V = 500 = xyz$ so $z = \frac{500}{xy}$, making the surface area function

$$S(x, y) = 2xy + \frac{1000}{y} + \frac{1000}{x}.$$

The critical points of S satisfy the system of equations

$$\begin{cases} S_x(x, y) = 2y - \frac{1000}{x^2} = 0 \\ S_y(x, y) = 2x - \frac{1000}{y^2} = 0 \end{cases}$$

Solving for y in the first equation, we have $y = \frac{500}{x^2}$. Plugging this into the second equation, we have

$$\begin{aligned} 2x - \frac{1000}{(500/x^2)^2} &= 0 \\ 2x - \frac{1}{250}x^4 &= 0 \\ 2 - \frac{1}{250}x^3 &= 0 \\ x &= \sqrt[3]{500}, \end{aligned}$$

so $y = \sqrt[3]{500}$ too.

The second-order partial derivatives of S are

$$S_{xx} = \frac{2000}{x^3} \quad S_{xy} = S_{yx} = 2 \quad S_{yy} = \frac{2000}{y^3}.$$

At the critical point $(\sqrt[3]{500}, \sqrt[3]{500})$,

$$A = S_{xx}(\sqrt[3]{500}, \sqrt[3]{500}) = 4 \quad B = S_{xy}(\sqrt[3]{500}, \sqrt[3]{500}) = 2 \quad C = S_{yy}(\sqrt[3]{500}, \sqrt[3]{500}) = 4,$$

so $AC - B^2 > 0$ and $A > 0$ which means $(\sqrt[3]{500}, \sqrt[3]{500})$ is a local minimum. Since the physical properties of the problem require that the absolute minimum exists, the local minimum is the absolute minimum. Using the equation $z = \frac{500}{xy}$, we find that $z = \sqrt[3]{500}$. The dimensions of the box that yield the least possible

amount of material are when $x = y = z = \sqrt[3]{500}$ centimeters.

46. If x, y, z are the length, width, and height of the box, then the cost function is given by

$$C(x, y, z) = 4xy + 3xz + 2(xz + 2yz) = 4xy + 5xz + 4yz,$$

where the side costing \$3 per square meter is the side with area xz . The volume V equation is $V = 80 = xyz$ so $z = \frac{80}{xy}$, making the cost function

$$C(x, y) = 4xy + \frac{400}{y} + \frac{320}{x}.$$

The critical points of C satisfy the system of equations

$$\begin{cases} C_x(x, y) = 4y - \frac{320}{x^2} = 0 \\ C_y(x, y) = 4x - \frac{400}{y^2} = 0 \end{cases}$$

Solving for y in the first equation, we have $y = \frac{80}{x^2}$. Plugging this into the second equation, we have

$$\begin{aligned} 4x - \frac{400}{(80/x^2)^2} &= 0 \\ 4x - \frac{1}{16}x^4 &= 0 \\ 4 - \frac{1}{16}x^3 &= 0 \\ x &= 4, \end{aligned}$$

so this makes $y = 5$.

The second-order partial derivatives of C are

$$C_{xx} = \frac{640}{x^3} \quad C_{xy} = C_{yx} = 4 \quad C_{yy} = \frac{800}{y^3}.$$

At the critical point $(4, 5)$,

$$A = C_{xx}(4, 5) = 10 \quad B = C_{xy}(4, 5) = 4 \quad C = C_{yy}(4, 5) = \frac{32}{5},$$

so $AC - B^2 > 0$ and $A > 0$ which means $(4, 5)$ is a local minimum. Since the physical properties of the problem require that the absolute minimum exists, the local minimum is the absolute minimum. The dimensions of the box that yield the least possible amount of material are when

$$\text{length is } x = 4 \text{ meters, width is } y = 5 \text{ meters, and height is } z = \frac{80}{4 \cdot 5} = 4 \text{ meters}.$$

47. (a) If x, y, z are the length, width, and height of the box, then the length plus the girth is $x + 2y + 2z$. In order to maximize the volume, we want the length plus the girth to be as large as possible, which means $x + 2y + 2z = 130$, or $x = 130 - 2y - 2z$. The volume function V is then

$$\begin{aligned} V(x, y, z) &= xyz \\ V(y, z) &= yz(130 - 2y - 2z) = 130yz - 2zy^2 - 2yz^2. \end{aligned}$$

The critical points of V satisfy the system of equations

$$\begin{cases} V_y(y, z) = 130z - 4yz - 2z^2 = 2z(65 - 2y - z) = 0 \\ V_z(y, z) = 130y - 2y^2 - 4yz = 2y(65 - y - 2z) = 0 \end{cases}$$

Since $y > 0$ and $z > 0$ we need to solve the system

$$\begin{cases} 65 - 2y - z = 0 \\ 65 - y - 2z = 0 \end{cases}$$

Using substitution, $y = z = \frac{65}{3}$. This means $x = \frac{130}{3}$. The second-order partial derivatives of V and their values at the critical point $(y, z) = \left(\frac{65}{3}, \frac{65}{3}\right)$ are

$$\begin{aligned} V_{yy}(y, z) &= -4z & A &= V_{yy}\left(\frac{65}{3}, \frac{65}{3}\right) = -\frac{260}{3} \\ V_{yz}(y, z) &= V_{zy}(y, z) = 130 - 4y - 4z & B &= V_{yz}\left(\frac{65}{3}, \frac{65}{3}\right) = -\frac{130}{3} \\ V_{zz}(y, z) &= -4y & C &= V_{zz}\left(\frac{65}{3}, \frac{65}{3}\right) = -\frac{260}{3} \end{aligned}$$

Since $AC - B^2 = \frac{16900}{3} > 0$ and $A < 0$, the critical point is a local maximum. Since the physical properties of the problem require that the absolute maximum exists, the local maximum is the absolute maximum. This means the dimensions that yield maximum possible volume of a box to be shipped are

$$\boxed{\frac{65}{3} \text{ inches for the width and height, and } \frac{130}{3} \text{ inches for the length.}}$$

(b) If the package is cylindrical, then let x be the length and r the radius of a cross sectional circle. Then the length plus the girth is $x + 2\pi r$. In order to maximize the volume, we want the length plus the girth to be as large as possible, which means $x + 2\pi r = 130$, or $x = 130 - 2\pi r$. The volume function V is then

$$\begin{aligned} V(x, r) &= \pi r^2 x \\ V(r) &= \pi r^2 (130 - 2\pi r) = 130\pi r^2 - 2\pi^2 r^3. \end{aligned}$$

The critical values are the solutions to

$$V'(r) = 260\pi r - 6\pi^2 r^2 = \pi r(260 - 6\pi r) = 0.$$

Since $r > 0$, it must be that $r = \frac{130}{3\pi}$. Now we find the second derivative and it's value at $r = \frac{130}{3\pi}$.

$$V''(r) = 260\pi - 12\pi^2 r \quad V''\left(\frac{130}{3\pi}\right) = -260\pi < 0$$

so by the Second-Derivatives Test, this critical value is a local maximum, but must also be the absolute maximum since it is the only critical value. This means when the

radius of the package is $\frac{130}{3\pi}$ inches and the

length is $130 - \frac{2\pi \cdot 130}{3\pi} = \frac{130}{3}$ inches, the package will have maximum possible volume.

48. The critical points are solutions of the system of equations

$$\begin{cases} y_x(t, x) = 2x(a - x)te^{-t} - x^2te^{-t} = te^{-t}x(-3x + 2a) = 0 \\ y_t(t, x) = x^2(a - x)e^{-t} - x^2(a - x)te^{-t} = x^2(a - x)e^{-t}(1 - t) = 0 \end{cases}$$

From the first equation, we see that either $t = 0$, $x = 0$, or $x = \frac{2a}{3}$.

If $t = 0$, then the y_t equation simplifies to $x^2(a - x) = 0$, which has solutions $x = 0$ and $x = a$. So two critical points are $(0, 0)$ and $(0, a)$. The values of y at these point are $y(0, 0) = 0$ and $y(0, a) = 0$.

If $x = 0$, then the y_t equation is true for any value of t , so the points $(t, 0)$ are critical points for $t \geq 0$. However, $y(t, x) \geq 0$, and if $x = 0$, then $y(t, 0) = 0$, so any point where $x = 0$ cannot maximize y .

If $x = \frac{2a}{3}$, then the y_t equation simplifies to $\frac{4a^3}{27}e^{-t}(1 - t)$, which has solution $t = 1$. So the last critical point is $\left(1, \frac{2a}{3}\right)$. The value of y at this critical point is $y\left(1, \frac{2a}{3}\right) = \frac{4a^3}{27}e^{-1}$. The value $\frac{4a^3}{27}e^{-1}$ is the maximum value of y and can be seen as follows. The function y is the product of $x^2(a - x)$ and te^{-t} , which are two functions independent of each other in terms of the variables t and x . As we found above, the first function $x^2(a - x)$ is maximized when $x = \frac{2a}{3}$ and the second is maximized when $t = 1$.

49. (a) If x is fixed, say at x_0 , then

$$f(y) = R(x_0, y) = x_0^2 y^2 (a - x_0)(b - y).$$

To maximize y , we find f' and set it equal to 0.

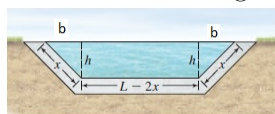
$$\begin{aligned} f'(y) &= 2x_0^2 y(a - x_0)(b - y) - x_0^2 y^2(a - x_0) = 0 \\ x_0^2 y(a - x_0)(2(b - y) - y) &= 0 \\ x_0^2 y(a - x_0)(2b - 3y) &= 0 \\ y &= 0, \frac{2b}{3}. \end{aligned}$$

If we plug in $y = 0$, then $R(x_0, 0) = 0$, so the maximum must occur when $y = \frac{2b}{3}$.

(b) By symmetry of the function, it must also be that the maximum must occur when $x = \frac{2a}{3}$.

(c) Since $R(x, y) = (x^2(a - x))(y^2(b - x))$, maximizing R amounts to maximizing $x^2(a - x)$ and $y^2(b - x)$, which we did in parts (a) and (b), so the amount that maximizes the reaction is when $x = \frac{2a}{3}$ and $y = \frac{2b}{3}$.

50. (a) For maximum possible flow, we want the area of the cross section shown in the picture to be maximized. The length of the top is $2b + L - 2x$ where b is labelled in the following picture.



The area of a trapezoid with base lengths b_1 and b_2 and height h is $\frac{1}{2}(b_1 + b_2)h$, so the area of the cross section is

$$A(h, x) = \frac{1}{2}(2b + L - 2x)h = \frac{1}{2}(2\sqrt{x^2 - h^2} + L - 2x)h, \quad h > 0, \quad x > 0.$$

The critical points are solutions of the system of equations

$$\begin{cases} A_x(h, x) = \frac{1}{2} \left(\frac{2x}{\sqrt{x^2 - h^2}} - 4 \right) h = \frac{x - 2\sqrt{x^2 - h^2}}{\sqrt{x^2 - h^2}} \cdot h = 0 \\ A_h(h, x) = -\frac{h^2}{\sqrt{x^2 - h^2}} + \sqrt{x^2 - h^2} + L - 2x = \frac{-2h^2 + x^2 + L\sqrt{x^2 - h^2} - 2x\sqrt{x^2 - h^2}}{\sqrt{x^2 - h^2}} = 0 \end{cases}$$

Since $b = \sqrt{x^2 - h^2}$ could be 0, we have to consider the case $x = h$ separately from $x \neq h$.

If $x = h$ (then the opening is rectangular, and not trapezoidal), then $A(h, x) = xL - 2x^2$ which is maximized when $\frac{d(xL - 2x^2)}{dx} = L - 4x = 0$, or $x = \frac{L}{4}$. In this case $A\left(\frac{L}{4}, \frac{L}{4}\right) = \frac{L^2}{8}$.

If $x \neq h$, then the system above becomes

$$\begin{cases} A_x(h, x) = (x - 2\sqrt{x^2 - h^2}) \cdot h = 0 \\ A_h(h, x) = -2h^2 + x^2 + L\sqrt{x^2 - h^2} - 2x\sqrt{x^2 - h^2} = 0 \end{cases}$$

From the first equation, either $x - 2\sqrt{x^2 - h^2} = 0$ or $h = 0$. Since $h > 0$, we must have

$$\begin{aligned} x - 2\sqrt{x^2 - h^2} &= 0 \\ x &= 2\sqrt{x^2 - h^2} \\ x^2 &= 4x^2 - 4h^2 \\ 3x^2 &= 4h^2 \\ x &= \frac{2}{\sqrt{3}}h. \end{aligned}$$

Plugging this into the second equation

$$\begin{aligned} -2h^2 + x^2 + L\sqrt{x^2 - h^2} - 2x\sqrt{x^2 - h^2} &= 0 \\ -\frac{2}{3}h^2 + \frac{Lh}{\sqrt{3}} - \frac{4}{3}h^2 &= 0 \\ \frac{1}{3}h(-6h + L\sqrt{3}) &= 0 \\ h &= \frac{L\sqrt{3}}{6}, \end{aligned}$$

since $h > 0$. So $x = \frac{2}{\sqrt{3}}h = \frac{1}{3}L$. In this case

$$\begin{aligned} A\left(\frac{L\sqrt{3}}{6}, \frac{1}{3}L\right) &= \frac{1}{2}\left(2\sqrt{\frac{L^2}{9} - \frac{L^2}{12}} + 2\left(L - \frac{2}{3}L\right)\right) \cdot \frac{L\sqrt{3}}{6} \\ &= \frac{L\sqrt{3}}{12}\left(\frac{1}{3}L + \frac{2}{3}L\right) = \frac{\sqrt{3}}{12}L^2 > \frac{L^2}{8}. \end{aligned}$$

Comparing the values of A for the $x = h$ and $x \neq h$ cases, we see that the maximum possible flow is allowed

when $\boxed{h = \frac{L\sqrt{3}}{6}}$ and $\boxed{x = \frac{L}{3}}$.

(b) If we created a semicircular channel, then the length of semicircle would be $L = \pi r$, where r is the radius of the channel. The area of a cross section would be

$$A = \frac{1}{2}\pi\left(\frac{L}{\pi}\right)^2 = \frac{L^2}{2\pi} > \frac{\sqrt{3}}{12}L^2,$$

which means $\boxed{\text{the semicircular shape would be preferable to the trapezoidal shape}}.$

51. We first find the critical points of F , namely the solutions of the system of equations

$$\begin{cases} F_x(x, y) = 12x - 4 = 0 \\ f_y(x, y) = 12y - 6 = 0 \end{cases}$$

The only critical point is then $\left(\frac{1}{3}, \frac{1}{2}\right)$. The second-order partial derivatives and their values at $\left(\frac{1}{3}, \frac{1}{2}\right)$ are

as follows.

$$\begin{aligned} F_{xx}(x, y) &= 12 & A &= F_{xx}\left(\frac{1}{3}, \frac{1}{2}\right) = 12 \\ F_{xy}(x, y) &= F_{yx}(x, y) = 0 & B &= F_{xy}\left(\frac{1}{3}, \frac{1}{2}\right) = 0 \\ F_{yy}(x, y) &= 12 & C &= F_{yy}\left(\frac{1}{3}, \frac{1}{2}\right) = 12 \\ AC - B^2 &= 144 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, $\left(\frac{1}{3}, \frac{1}{2}\right)$ is a local minimum. Since the physical properties of the problem require that the absolute minimum exists, the local minimum is also the absolute minimum. So the fuel reservoir should be located at the point $\boxed{\left(\frac{1}{3}, \frac{1}{2}\right)}$.

52. If l, w, h represent the length, width, and height of the shipping container, then the top and bottom have surface area lw , two of the sides have surface area lh , and the other two sides have surface area wh . The cost C of the box is then

$$C(l, w, h) = a(2lh + 2wh) + \frac{3}{2}a(2lw).$$

Since the volume is to be $\frac{3}{2}$, we have $lwh = \frac{3}{2}$, or $h = \frac{3}{2lw}$. With this equation, the cost function becomes

$$C(l, w) = \frac{3a}{w} + \frac{3a}{l} + 3alw, \quad w > 0, \quad l > 0.$$

We first find the critical points of C , namely the solutions of the system of equations

$$\begin{cases} C_l(l, w) = -3al^{-2} + 3aw = 0 \\ C_w(l, w) = -3aw^{-2} + 3al = 0 \end{cases}$$

Solving for w in the C_l equation and plugging it into the C_w equation, we have

$$\begin{aligned} -3a\left(\frac{1}{l^2}\right) + 3al &= 0 \\ l &= \frac{1}{l^2} \\ l &= 1, \end{aligned}$$

so $w = 1$. The second-order partial derivatives and their values at $(1, 1)$ are as follows.

$$\begin{aligned} C_{ll}(l, w) &= 6al^{-3} & A &= C_{ll}(1, 1) = 6a \\ C_{lw}(l, w) &= C_{wl} = 3a & B &= C_{lw}(1, 1) = 3a \\ C_{ww}(l, w) &= 6aw^{-3} & C &= C_{ww}(1, 1) = 6a \\ AC - B^2 &= 27a^2 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$ (since $a > 0$), the point $(1, 1)$ is a local minimum. Since the physical properties of the problem require that the absolute minimum exists, the local minimum is also the absolute minimum. The dimensions of the container that yield minimum cost are

$$\boxed{\text{length is 1 foot, width is 1 foot, and height is } \frac{3}{2 \cdot 1 \cdot 1} = \frac{3}{2} \text{ feet}}.$$

53. We first find the critical points of $M(x, y) = yx - yx^2 - y^2$, namely the solutions of the system of equations

$$\begin{cases} M_x(x, y) = y(1 - 2x) = 0 \\ M_y(x, y) = x - x^2 - 2y = 0 \end{cases}$$

Solving for y in the M_y equation and plugging it into the M_x equation, we have

$$\begin{aligned}\left(\frac{x-x^2}{2}\right)(1-2x) &= 0 \\ x(1-x)(1-2x) &= 0 \\ x &= 0, 1, \frac{1}{2}.\end{aligned}$$

This means the critical points are $(0, 0), (1, 0), \left(\frac{1}{2}, \frac{1}{8}\right)$. The values of M at these critical points are

$$M(0, 0) = 0 \quad M(1, 0) = 0 \quad M\left(\frac{1}{2}, \frac{1}{8}\right) = \frac{1}{64}.$$

Along the boundary where $x = 0$ or $y = 0$, $M \leq 0$ so the highest reading has to occur at the point $\boxed{\left(\frac{1}{2}, \frac{1}{8}\right)}$.

54. First,

$$\begin{aligned}P = R - C &= \left(2700x - \frac{3}{20}x^2 + 1000y - y^2 + \frac{1}{2}xy + 10000\right) - \left(\frac{1}{20}x^2 + 700x + y^2 - 150y - \frac{1}{2}xy\right) \\ &= 2000x - \frac{1}{5}x^2 + 1150y - 2y^2 + xy + 10000.\end{aligned}$$

We find the critical points of P , namely the solutions of the system of equations

$$\begin{cases} P_x(x, y) = 2000 - \frac{2}{5}x + y = 0 \\ P_y(x, y) = 1150 - 4y + x = 0 \end{cases}$$

Solving for y in the P_x equation and plugging it into the P_y equation, we have

$$\begin{aligned}1150 - 4\left(\frac{2}{5}x - 2000\right) + x &= 0 \\ x &= 15250,\end{aligned}$$

so $y = 4100$. The second-order partial derivatives and their values at $(15250, 4100)$ are as follows.

$$\begin{aligned}P_{xx}(x, y) &= -\frac{2}{5} & A = P_{xx}(15250, 4100) &= -\frac{2}{5} \\ P_{xy}(x, y) &= P_{yx}(x, y) = 1 & B = P_{xy}(15250, 4100) &= 1 \\ P_{yy}(x, y) &= -4 & C = P_{yy}(15250, 4100) &= -4 \\ AC - B^2 &= \frac{3}{5}\end{aligned}$$

Since $AC - B^2 > 0$ and $A < 0$, the point $(15250, 4100)$ is a local maximum, which is then the absolute maximum based on the nature of the problem. When $\boxed{15250 \text{ kilograms of Grade A steel}}$ and $\boxed{4100 \text{ kilograms of Grade B steel}}$ is used in the manufacturing, the profit will be maximized.

55. The revenue R can be found by multiplying the price of each item by the number of such items sold. That is,

$$R(x, y) = x(12 - x) + y(8 - y) = 12x - x^2 + 8y - y^2.$$

Then the profit function is

$$P(x, y) = R(x, y) - C(x, y) = 12x - 2x^2 + 8y - 4y^2 - 2xy.$$

We find the critical points of P , namely the solutions of the system of equations

$$\begin{cases} P_x(x, y) = 12 - 4x - 2y = 0 \\ P_y(x, y) = 8 - 8y - 2x = 0 \end{cases}$$

Solving for y in the P_x equation and plugging it into the P_y equation, we have

$$\begin{aligned} 8 - 8(6 - 2x) - 2x &= 0 \\ 14x &= 40 \\ x &= \frac{20}{7}, \end{aligned}$$

so $y = \frac{2}{7}$. The second-order partial derivatives and their values at $\left(\frac{20}{7}, \frac{2}{7}\right)$ are as follows.

$$\begin{aligned} P_{xx}(x, y) &= -4 & A = P_{xx}\left(\frac{20}{7}, \frac{2}{7}\right) &= -4 \\ P_{xy}(x, y) = P_{yx}(x, y) &= -2 & B = P_{xy}\left(\frac{20}{7}, \frac{2}{7}\right) &= -2 \\ P_{yy}(x, y) &= -8 & C = P_{yy}\left(\frac{20}{7}, \frac{2}{7}\right) &= -8 \\ AC - B^2 &= 28 \end{aligned}$$

Since $AC - B^2 > 0$ and $A < 0$, the point $\left(\frac{20}{7}, \frac{2}{7}\right)$ is a local maximum, which is then the absolute maximum

based on the nature of the problem. The maximum profit occurs when $x = \frac{20}{7}$ thousand units are sold and

$y = \frac{2}{7}$ thousand units are sold which makes the prices $p = \frac{64}{7}$ thousand dollars and $q = \frac{54}{7}$ thousand dollars, and the maximum profit is

$$P\left(\frac{20}{7}, \frac{2}{7}\right) = \frac{128}{7} \approx \$18,286.$$

56. We find the critical points of Q , namely the solutions of the system of equations

$$\begin{cases} Q_x(x, y) = 2x - y - 3 = 0 \\ Q_y(x, y) = 2y - x - 6 = 0 \end{cases}$$

Solving for y in the Q_x equation and plugging it into the Q_y equation, we have

$$\begin{aligned} 2(2x - 3) - x - 6 &= 0 \\ 3x &= 12 \\ x &= 4, \end{aligned}$$

so $y = 5$. The second-order partial derivatives and their values at $(4, 5)$ are as follows.

$$\begin{aligned} Q_{xx}(x, y) &= 2 & A = Q_{xx}(4, 5) &= 2 \\ Q_{xy}(x, y) = Q_{yx}(x, y) &= -1 & B = Q_{xy}(4, 5) &= -1 \\ Q_{yy}(x, y) &= 2 & C = Q_{yy}(4, 5) &= 2 \\ AC - B^2 &= 3 \end{aligned}$$

Since $AC - B^2 > 0$ and $Q_{xx}(4, 5) > 0$, the point $(4, 5)$ is a local minimum. This means that labor cost is minimized when $x = 4$ and $y = 5$.

57. We find the critical points of g , namely the solutions of the system of equations

$$\begin{cases} g_x(x, y) = 2Ax + 2By = 0 \\ g_y(x, y) = 2Bx + 2Cy = 0 \end{cases}$$

Solving for x in the g_x equation (assuming $A \neq 0$) and plugging it into the g_y equation, we have

$$\begin{aligned} 2B \left(-\frac{By}{A} \right) + 2Cy &= 0 \\ y \left(-\frac{B^2}{A} + C \right) &= 0 \\ y &= 0, \end{aligned}$$

so $x = 0$ too making $(0, 0)$ the only critical point. Now we find the second-order partial derivatives, and their values at $(0, 0)$.

$$\begin{aligned} g_{xx}(x, y) &= 2A & g_{xx}(0, 0) &= 2A \\ g_{xy}(x, y) &= g_{yx}(x, y) = 2B & g_{xy}(0, 0) &= 2B \\ g_{yy}(x, y) &= 2C & g_{yy}(0, 0) &= 2C \end{aligned}$$

Notice that

$$g_{xx}(0, 0) \cdot g_{yy}(0, 0) - (g_{xy}(0, 0))^2 = 4(AC - B^2).$$

Call this quantity D . If $AC - B^2 > 0$ and $A < 0$, then $D > 0$ and $g_{xx}(0, 0) < 0$, so $(0, 0)$ is a local maximum. If $AC - B^2 > 0$ and $A > 0$, then $D > 0$ and $g_{xx}(0, 0) > 0$, so $(0, 0)$ is a local minimum.

(b) If $AC - B^2 < 0$, then $D < 0$ so $(0, 0, 0)$ is a saddle point.

Challenge Problems

58. In this problem, we will, equivalently, minimize the *square* of the distance between the two lines. A general point on L_1 looks like $(t - 6, t, 2t)$ for some value of t and a general point on L_2 looks like $(s, s, -s)$ for some value of s . Then the square of the distance between these general points is

$$f(s, t) = (t - 6 - s)^2 + (t - s)^2 + (2t + s)^2 = 6t^2 - 12t + 36 + 12s + 3s^2.$$

The critical points are solutions of the system of equations

$$\begin{cases} f_s(s, t) = 12 + 6s = 0 \\ f_t(s, t) = 12t - 12 = 0 \end{cases}$$

The only solution is $s = -2, t = 1$, so the only critical point is $(-2, 1)$. The second-order partial derivatives and their values at $(-2, 1)$ are as follows.

$$\begin{aligned} f_{ss}(s, t) &= 6 & A &= f_{ss}(-2, 1) = 6 \\ f_{st}(s, t) &= f_{ts}(s, t) = 0 & B &= f_{st}(-2, 1) = 0 \\ f_{tt}(s, t) &= 12 & C &= f_{tt}(-2, 1) = 12 \\ AC - B^2 &= 72 \end{aligned}$$

Since $AC - B^2 > 0$ and $f_{ss}(-2, 1) > 0$, this critical point is a local minimum, which by the characteristics of the problem guarantee it is also an absolute minimum. This means the distance between the two lines is the distance between the two points: on L_1 when $t = 1$, $(-5, 1, 2)$ and on L_2 when $s = -2$, $(-2, -2, 2)$. The distance between the two lines is then

$$\sqrt{(-5 - (-2))^2 + (1 - (-2))^2 + (2 - 2)^2} = \boxed{3\sqrt{2}}.$$

59. (a) Define

$$f(b, m) = \sum_{k=1}^n (mx_k + b - y_k)^2.$$

In order to minimize f , we will find the critical points by solving the system

$$\begin{cases} f_b(b, m) = \sum_{k=1}^n 2(mx_k + b - y_k) = 0 \\ f_m(b, m) = \sum_{k=1}^n 2x_k(mx_k + b - y_k) = 0 \end{cases}$$

Using properties of summations, we can rewrite these equations

$$\begin{aligned} f_b(b, m) &= 2m \sum_{k=1}^n x_k + 2bn - 2 \sum_{k=1}^n y_k = 0 \\ f_m(b, m) &= 2m \sum_{k=1}^n x_k^2 + 2b \sum_{k=1}^n x_k - 2 \sum_{k=1}^n x_k y_k = 0 \end{aligned}$$

We can cancel the factor of 2 from both equations. Then solving for b in the f_b equation and plugging it into the f_m equation, we have

$$\begin{aligned} m \sum_{k=1}^n x_k^2 + \frac{1}{n} \left(\sum_{k=1}^n y_k - m \sum_{k=1}^n x_k \right) \sum_{k=1}^n x_k - \sum_{k=1}^n x_k y_k &= 0 \\ \frac{m}{n} \left(n \sum_{k=1}^n x_k^2 - \left(\sum_{k=1}^n x_k \right)^2 \right) &= \frac{n \sum_{k=1}^n x_k y_k - \sum_{k=1}^n x_k \sum_{k=1}^n y_k}{n} \\ m &= \frac{n \sum_{k=1}^n x_k y_k - \sum_{k=1}^n x_k \sum_{k=1}^n y_k}{n \sum_{k=1}^n x_k^2 - \left(\sum_{k=1}^n x_k \right)^2}. \end{aligned}$$

Plugging this back into the f_b equation and solving for b , we find that

$$b = \frac{1}{n} \left(\sum_{k=1}^n y_k - m \sum_{k=1}^n x_k \right).$$

To verify that these values of b and m produce a minimum, we find the second order partial derivatives

$$f_{bb}(b, m) = 2n \quad f_{bm}(b, m) = f_{mb}(b, m) = 2 \sum_{k=1}^n x_k \quad f_{mm}(b, m) = 2 \sum_{k=1}^n x_k^2.$$

Then for the values of b and m we found above,

$$f_{bb}(b, m) \cdot f_{mm}(b, m) - (f_{bm}(b, m))^2 = 4 \left(n \sum_{k=1}^n x_k^2 - \left(\sum_{k=1}^n x_k \right)^2 \right) > 0,$$

by the Cauchy-Schwarz inequality (you may want to look at Problem 60 in Section 10.4) and $f_{bb}(b, m) = 2n > 0$, so these values of m and b do yield a local minimum. Based on the physical aspects of the problem, the same values of m and b also produce an absolute minimum.

(b) We start by computing the following quantities.

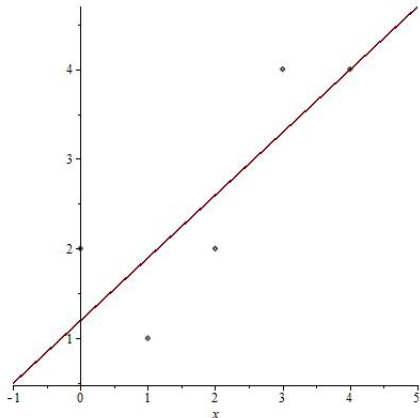
$$\begin{aligned} n &= 5 \\ \sum_{k=1}^5 x_k &= 0 + 1 + 2 + 3 + 4 = 10 \\ \sum_{k=1}^5 y_k &= 2 + 1 + 2 + 4 + 4 = 13 \\ \sum_{k=1}^5 x_k y_k &= 0 + 1 + 4 + 12 + 16 = 33 \\ \sum_{k=1}^5 x_k^2 &= 0 + 1 + 4 + 9 + 16 = 30. \end{aligned}$$

Then

$$\begin{aligned} m &= \frac{5 \cdot 33 - 10 \cdot 13}{5 \cdot 30 - (10)^2} = \frac{7}{10} \\ b &= \frac{1}{5} \left(13 - \frac{7}{10} \cdot 10 \right) = \frac{6}{5}. \end{aligned}$$

This means the line of best fit is

$$y = \frac{7}{10}x + \frac{6}{5}.$$



60. (a) If $\mathbf{u} = \mathbf{v}$, then $u_1 = v_1$ and $u_2 = v_2$ and the second-order directional derivative at (x_0, y_0) is

$$\begin{aligned} D_{\mathbf{u}}g(x_0, y_0) &= u_1^2 f_{xx}(x_0, y_0) + 2u_1 u_2 f_{xy}(x_0, y_0) + u_2^2 f_{yy}(x_0, y_0) \\ &= u_1^2 A + 2u_1 u_2 B + u_2^2 C. \end{aligned}$$

(b) If we take $Au_1^2 + 2Bu_1u_2 + Cu_2^2$ and complete the square, we have

$$\begin{aligned} A \left(u_1^2 + \frac{2Bu_1u_2}{A} \right) + Cu_2^2 &= A \left(u_1 + \frac{Bu_2}{A} \right)^2 + Cu_2^2 - \frac{B^2}{A} u_2^2 \\ &= A \left(u_1 + \frac{Bu_2}{A} \right)^2 + \frac{u_2^2}{A} (AC - B^2). \end{aligned}$$

If $AC - B^2 > 0$ and $A > 0$, then since f_{xx} and $f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, there is a disk B with center (x_0, y_0) with a positive radius $\delta > 0$ where for any point (x, y) in B , both $f_{xx}(x, y) > 0$ and $f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) > 0$. This means that if C is the curve obtained by intersecting the surface of f with the vertical plane through $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$, then C is concave upward for all such points. Note that this is true for any direction vector $\mathbf{u}$, so the point (x_0, y_0) must be a local minimum. If $AC - B^2 > 0$ and $A < 0$, then, using the same reasoning as above, every curve C will be concave downward inside the disk B in any direction $\mathbf{u}$, so the point (x_0, y_0) must be a local maximum.

If $AC - B^2 < 0$, then there are two cases. If $A \neq 0$, then we claim that $f(x, y)$ takes both positive and negative values, so cannot be a local minimum nor local maximum. Indeed, if $A \neq 0$, then the quadratic equation $Ax^2 + 2Bx + C = 0$ has two distinct real roots, namely $\frac{-B \pm \sqrt{B^2 - AC}}{A}$, and the graph takes both positive and negative values. If $A = 0$, then $B \neq 0$, and the expression $Ax^2 + 2Bx + C = 2Bx + C$ represents a line with a non-zero slope, so takes positive and negative values, as well. In either case, we have that $(0, 0, 0)$ is a saddle point.

(c) Saddle Point Case: Consider the function $f(x, y) = x^3 + y^3$. Then $f_x(x, y) = 3x^2$ and $f_y(x, y) = 3y^2$ so $(0, 0)$ is a critical point. Note that $f_{xx}(x, y) = 6x$, $f_{yy}(x, y) = 6y$, and $f_{xy}(x, y) = 0$, so

$$f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0.$$

We can see that $(0, 0)$ is a saddle point since along the curve $y = 0$, then function becomes $f(x, 0) = x^3$ which no matter the disk chosen around the point $(0, 0)$, there will always be points $(a, 0)$ and $(-a, 0)$ for $a > 0$ where $f(-a, 0) \leq f(0, 0) \leq f(a, 0)$ meaning that $(0, 0)$ is a saddle point.

Local Minimum Case: Consider the function $f(x, y) = x^4 + y^4$. Then $f_x(x, y) = 4x^3$ and $f_y(x, y) = 4y^3$ so $(0, 0)$ is a critical point. Note that $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$, and $f_{xy}(x, y) = 0$, so

$$f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0.$$

We can see that $(0, 0)$ is a local minimum because it is actually the absolute minimum. Since $x^4, y^4 \geq 0$ for any point (x, y) , this means $f(0, 0) \leq f(x, y)$ for any point (x, y) , so $f(0, 0) = 0$ is the local minimum.

Local Maximum Case: Consider the function $f(x, y) = -x^4 - y^4$. Then $f_x(x, y) = -4x^3$ and $f_y(x, y) = -4y^3$ so $(0, 0)$ is a critical point. Note that $f_{xx}(x, y) = -12x^2$, $f_{yy}(x, y) = -12y^2$, and $f_{xy}(x, y) = 0$, so

$$f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0.$$

We can see that $(0, 0)$ is a local maximum because it is actually the absolute maximum. Since $-x^4, -y^4 \leq 0$ for any point (x, y) , this means $f(0, 0) \geq f(x, y)$ for any point (x, y) , so $f(0, 0) = 0$ is the local maximum.

13.4 Lagrange Multipliers

Concepts and Vocabulary

1. The number λ in the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ is called a **(c) Lagrange multiplier**.
2. **True**.
3. **False**. Lagrange multipliers can be used for functions with any number of variables. See the discussion preceding Example 5.
4. **False**. The number of Lagrange multipliers depends on the number of constraint functions.

Skill Building

5. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 3 = y\lambda \\ 1 = x\lambda \\ xy - 8 = 0 \end{cases}$$

Eliminating λ from the first two equations results in $y = 3x$, so the third equation becomes

$$\begin{aligned} x(3x) - 8 &= 0 \\ x &= \pm\sqrt{\frac{8}{3}}. \end{aligned}$$

The test points are then $\left(\sqrt{\frac{8}{3}}, 3\sqrt{\frac{8}{3}}\right)$ and $\left(-\sqrt{\frac{8}{3}}, -3\sqrt{\frac{8}{3}}\right)$. The corresponding values are

$$\begin{aligned} f\left(\sqrt{\frac{8}{3}}, 3\sqrt{\frac{8}{3}}\right) &= 4\sqrt{6} \\ f\left(-\sqrt{\frac{8}{3}}, -3\sqrt{\frac{8}{3}}\right) &= -4\sqrt{6} \end{aligned}$$

Then $4\sqrt{6}$ is the maximum value and $-4\sqrt{6}$ is the minimum value.

6. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 3 = \lambda y \\ 1 = \lambda x \\ xy - 1 = 0 \end{cases}$$

Eliminating λ from the first two equations results in $y = 3x$, so the third equation becomes

$$\begin{aligned} x(3x) - 1 &= 0 \\ x &= \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}. \end{aligned}$$

The test points are then $\left(\frac{\sqrt{3}}{3}, \sqrt{3}\right)$ and $\left(-\frac{\sqrt{3}}{3}, -\sqrt{3}\right)$. The corresponding values are

$$\begin{aligned} f\left(\frac{\sqrt{3}}{3}, \sqrt{3}\right) &= 2\sqrt{3} + 4 \\ f\left(-\frac{\sqrt{3}}{3}, -\sqrt{3}\right) &= -2\sqrt{3} + 4 \end{aligned}$$

Then $2\sqrt{3} + 4$ is the maximum value and $-2\sqrt{3} + 4$ is the minimum value.

7. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 3 = 2x\lambda \\ 1 = 8y\lambda \\ x^2 + 4y^2 - 1 = 0 \end{cases}$$

Eliminating λ from the first two equations results in $x = 12y$, so the third equation becomes

$$\begin{aligned} (12y)^2 + 4y^2 - 1 &= 0 \\ y &= \pm \frac{1}{2} \sqrt{\frac{1}{37}} = \pm \frac{\sqrt{37}}{74}. \end{aligned}$$

The test points are then $\left(\frac{6\sqrt{37}}{37}, \frac{\sqrt{37}}{74}\right)$ and $\left(-\frac{6\sqrt{37}}{37}, -\frac{\sqrt{37}}{74}\right)$. The corresponding values are

$$\begin{aligned} f\left(\frac{6\sqrt{37}}{37}, \frac{\sqrt{37}}{74}\right) &= \frac{1}{2}\sqrt{37} + 4 \\ f\left(-\frac{6\sqrt{37}}{37}, -\frac{\sqrt{37}}{74}\right) &= -\frac{1}{2}\sqrt{37} + 4 \end{aligned}$$

Then $\boxed{4 + \frac{1}{2}\sqrt{37} \text{ is the maximum value}}$ and $\boxed{4 - \frac{1}{2}\sqrt{37} \text{ is the minimum value}}$.

8. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 3 = 2x\lambda \\ 1 = 2y\lambda \\ x^2 + y^2 - 4 = 0 \end{cases}$$

Eliminating λ from the first two equations results in $x = 3y$, so the third equation becomes

$$\begin{aligned} (3y)^2 + y^2 - 4 &= 0 \\ y &= \pm \frac{\sqrt{10}}{5}. \end{aligned}$$

The test points are then $\left(\frac{3\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right)$ and $\left(-\frac{3\sqrt{10}}{5}, -\frac{\sqrt{10}}{5}\right)$. The corresponding values are

$$\begin{aligned} f\left(\frac{3\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right) &= 2\sqrt{10} \\ f\left(-\frac{3\sqrt{10}}{5}, -\frac{\sqrt{10}}{5}\right) &= -2\sqrt{10} \end{aligned}$$

Then $\boxed{2\sqrt{10} \text{ is the maximum value}}$ and $\boxed{-2\sqrt{10} \text{ is the minimum value}}$.

9. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 1 = 2x\lambda \\ -4y = 2y\lambda \\ x^2 + y^2 - 1 = 0 \end{cases}$$

From the second equation, either $y = 0$, or $\lambda = -2$. If $y = 0$, then the third equation gives us $x = \pm 1$, yielding two test points $(1, 0)$ and $(-1, 0)$. If $\lambda = -2$, then the first equation implies that $x = -\frac{1}{4}$. In this

case, the third equation becomes

$$\begin{aligned}\frac{1}{16} + y^2 - 1 &= 0 \\ y &= \pm \frac{\sqrt{15}}{4},\end{aligned}$$

which yields two more test points $\left(-\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$ and $\left(-\frac{1}{4}, -\frac{\sqrt{15}}{4}\right)$. The corresponding values are

$$\begin{aligned}f(1, 0) &= 1 \\ f(-1, 0) &= -1 \\ f\left(-\frac{1}{4}, \frac{\sqrt{15}}{4}\right) &= -\frac{17}{8} \\ f\left(-\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) &= -\frac{17}{8}\end{aligned}$$

Then 1 is the maximum value and $-\frac{17}{8}$ is the minimum value.

10. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x = 2x\lambda \\ 12y^2 = 4y\lambda \\ x^2 + 2y^2 - 2 = 0 \end{cases}$$

From the first equation, either $x = 0$, or $\lambda = 1$. If $x = 0$, then the third equation gives us $y = \pm 1$, yielding two test points $(0, 1)$ and $(0, -1)$. If $\lambda = 1$, then the second equation implies that $y = 0$ or $y = \frac{1}{3}$. For $y = 0$, the third equation means $x = \pm\sqrt{2}$ leading to the two test points $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. For $y = \frac{1}{3}$, the third equation means $x = \pm\frac{4}{3}$ leading to the two test points $\left(\frac{4}{3}, \frac{1}{3}\right)$ and $\left(-\frac{4}{3}, \frac{1}{3}\right)$. The corresponding values are

$$\begin{aligned}f(0, 1) &= 4 \\ f(0, -1) &= -4 \\ f(\sqrt{2}, 0) &= 2 \\ f(-\sqrt{2}, 0) &= 2 \\ f\left(\frac{4}{3}, \frac{1}{3}\right) &= \frac{52}{27} \\ f\left(-\frac{4}{3}, \frac{1}{3}\right) &= \frac{52}{27}\end{aligned}$$

Then 4 is the maximum value and -4 is the minimum value.

11. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2y = 2x\lambda \\ 2x = 2y\lambda \\ x^2 + y^2 - 2 = 0 \end{cases}$$

Eliminating λ from the first two equations, we have $x^2 = y^2$ for $x, y \neq 0$. Notice if either $x = 0$ or $y = 0$, then the other must also be 0 by the first two equations, but then the third equation would not be satisfied.

So neither x nor y can be equal to 0. Plugging $x^2 = y^2$ into the third equation, we find that

$$\begin{aligned}x^2 + x^2 - 2 &= 0 \\x &= \pm 1\end{aligned}$$

These values yield four test points: $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The corresponding values are

$$\begin{aligned}f(1, 1) &= 2 \\f(1, -1) &= -2 \\f(-1, 1) &= -2 \\f(-1, -1) &= 2\end{aligned}$$

Then 2 is the maximum value and -2 is the minimum value.

12. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} y = \lambda 18x \\ x = \lambda 8y \\ 9x^2 + 4y^2 - 36 = 0 \end{cases}$$

Eliminating λ from the first two equations, we have $9x^2 = 4y^2$ for $x, y \neq 0$. Notice if either $x = 0$ or $y = 0$, then the other must also be 0 by the first two equations, but then the third equation would not be satisfied. So neither x nor y can be equal to 0. Plugging $9x^2 = 4y^2$ into the third equation, we find that

$$\begin{aligned}9x^2 + 9x^2 - 36 &= 0 \\x &= \pm\sqrt{2}\end{aligned}$$

Since $y^2 = \frac{9}{4}x^2$, the values for x we found yield four test points:

$\left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right), \left(\sqrt{2}, -\frac{3\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{3\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{3\sqrt{2}}{2}\right)$. The corresponding values are

$$\begin{aligned}f\left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right) &= 3 \\f\left(\sqrt{2}, -\frac{3\sqrt{2}}{2}\right) &= -3 \\f\left(-\sqrt{2}, \frac{3\sqrt{2}}{2}\right) &= -3 \\f\left(-\sqrt{2}, -\frac{3\sqrt{2}}{2}\right) &= 3\end{aligned}$$

Then 3 is the maximum value and -3 is the minimum value.

13. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x - 4y = 2x\lambda \\ -4x + 8y = 2y\lambda \\ x^2 + y^2 - 4 = 0 \end{cases}$$

Eliminating λ from the first two equations, we have

$$\begin{aligned}\frac{2x - 4y}{2x} &= \frac{-4x + 8y}{2y} \\4xy - 8y^2 &= -8x^2 + 16xy \\2x^2 - 5xy - 2y^2 &= 0.\end{aligned}$$

Using the Quadratic Formula with $a = 2$, $b = -5y$, and $c = -2y^2$, we have

$$x = \frac{-(-5y) \pm \sqrt{(-5y)^2 - 4(2)(-2y^2)}}{2(2)} = \frac{5y \pm \sqrt{9y^2}}{4}$$

$$x = 2y \quad \text{or} \quad x = \frac{1}{2}y$$

Plugging $x = 2y$ into the third equation, we have

$$4y^2 + y^2 - 4 = 0$$

$$y = \pm \frac{2}{\sqrt{5}},$$

which yield the two test points $\left(\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(-\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$. Plugging $x = -\frac{1}{2}y$ into the third equation, we have

$$\frac{1}{4}y^2 + y^2 - 4 = 0$$

$$y = \pm \frac{4}{\sqrt{5}},$$

which yield the other two test points $\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$. The corresponding values are

$$f\left(\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 0$$

$$f\left(-\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = 0$$

$$f\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = 20$$

$$f\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = 20$$

Then 20 is the maximum value and 0 is the minimum value.

14. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 18x - 6y = 2x\lambda \\ -6x + 2y = 2y\lambda \\ x^2 + y^2 - 25 = 0 \end{cases}$$

Eliminating λ from the first two equations, we have

$$\frac{18x - 6y}{2x} = \frac{-6x + 2y}{2y}$$

$$36xy - 12y^2 = -12x^2 + 4xy$$

$$3x^2 + 8xy - 3y^2 = 0.$$

Using the Quadratic Formula with $a = 3$, $b = 8y$, and $c = -3y^2$, we have

$$x = \frac{-(8y) \pm \sqrt{(8y)^2 - 4(3)(-3y^2)}}{2(3)} = \frac{-8y \pm \sqrt{100y^2}}{6}$$

$$x = \frac{1}{3}y \quad \text{or} \quad x = -3y$$

Plugging $x = \frac{1}{3}y$ into the third equation, we have

$$\begin{aligned}\frac{1}{9}y^2 + y^2 - 25 &= 0 \\ y &= \pm \frac{3\sqrt{10}}{2},\end{aligned}$$

which yield the two test points $\left(\frac{\sqrt{10}}{2}, \frac{3\sqrt{10}}{2}\right)$ and $\left(-\frac{\sqrt{10}}{2}, -\frac{3\sqrt{10}}{2}\right)$. Plugging $x = -3y$ into the third equation, we have

$$\begin{aligned}9y^2 + y^2 - 25 &= 0 \\ y &= \pm \frac{\sqrt{10}}{2},\end{aligned}$$

which yield the other two test points $\left(-\frac{3\sqrt{10}}{2}, \frac{\sqrt{10}}{2}\right)$ and $\left(\frac{3\sqrt{10}}{2}, -\frac{\sqrt{10}}{2}\right)$. The corresponding values are

$$\begin{aligned}f\left(\frac{\sqrt{10}}{2}, \frac{3\sqrt{10}}{2}\right) &= 0 \\ f\left(-\frac{\sqrt{10}}{2}, -\frac{3\sqrt{10}}{2}\right) &= 0 \\ f\left(-\frac{3\sqrt{10}}{2}, \frac{\sqrt{10}}{2}\right) &= 250 \\ f\left(\frac{3\sqrt{10}}{2}, -\frac{\sqrt{10}}{2}\right) &= 250\end{aligned}$$

Then 250 is the maximum value and 0 is the minimum value.

15. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 4 = 2x\lambda \\ -3 = 2y\lambda \\ 2 = -6\lambda \\ x^2 + y^2 - 6z = 0 \end{cases}$$

From the third equation, we find that $\lambda = -\frac{1}{3}$ so from the first two equations, we have $x = -6$ and $y = \frac{9}{2}$.

Plugging these values into the fourth equation, we find that $z = \frac{75}{8}$ so the only test point is $\left(-6, \frac{9}{2}, \frac{75}{8}\right)$.

The value of the function at this point is

$$f\left(-6, \frac{9}{2}, \frac{75}{8}\right) = -\frac{75}{4}.$$

Since there is only one test point, there is either no maximum or no minimum. Since $\left(0, 0, -\frac{1}{6}\right)$ is on the constraint curve and $f\left(0, 0, -\frac{1}{6}\right) = -\frac{1}{3}$, we know that $-\frac{75}{4}$ is not the maximum. This means there is

no maximum and $-\frac{75}{4}$ is the minimum.

16. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 2x = 2\lambda \\ 4y = -3\lambda \\ 2z = \lambda \\ 2x - 3y + z - 6 = 0 \end{cases}$$

From the first equations, we see that $x = \lambda, y = -\frac{3}{4}\lambda, z = \frac{1}{2}\lambda$. Plugging these into the fourth equation, we have

$$\begin{aligned} 2\lambda + \frac{9}{4}\lambda + \frac{1}{2}\lambda - 6 &= 0 \\ \lambda &= \frac{24}{19}, \end{aligned}$$

which means the only test point is $\left(\frac{24}{19}, -\frac{18}{19}, \frac{12}{19}\right)$. The value of the function at this point is

$$f\left(\frac{24}{19}, -\frac{18}{19}, \frac{12}{19}\right) = \frac{72}{19}.$$

Since there is only one test point, there is either no maximum or no minimum. Since $(0, 0, 6)$ is on the constraint curve and $f(0, 0, 6) = 36$, we know that $\frac{72}{19}$ is not the maximum. This means there is no maximum

and $\frac{72}{19}$ is the minimum.

17. From the Extreme Value Theorem, f has both an absolute maximum and an absolute minimum on the closed, bounded set $x^2 + y^2 \leq 2$. The critical points are the solutions to the system

$$\begin{cases} f_x(x, y) = 2x + 4y = 0 \\ f_y(x, y) = 2y + 4x = 0 \end{cases}$$

Using elimination, the only critical point, $(0, 0)$, lies inside the constraint set. The only other place extrema can occur is on the boundary $g(x, y) = x^2 + y^2 - 2 = 0$. We use Lagrange multipliers to find the extrema of f subject to the constraint curve $g(x, y) = 0$. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x + 4y = 2x\lambda \\ 2y + 4x = 2y\lambda \\ x^2 + y^2 - 2 = 0 \end{cases}$$

Eliminating λ from the first two equations, we find that

$$\begin{aligned} \frac{2x + 4y}{2x} &= \frac{2y + 4x}{2y} \\ 1 + \frac{2y}{x} &= 1 + \frac{2x}{y} \\ x^2 &= y^2. \end{aligned}$$

Plugging this into the third equation, we have

$$\begin{aligned} x^2 + x^2 - 2 &= 0 \\ x &= \pm 1 \end{aligned}$$

These yield the four test points $(1, 1), (1, -1), (-1, 1), (-1, -1)$. Now we plug the critical points and test

points into f .

$$\begin{aligned} f(0,0) &= 0 \\ f(1,1) &= 6 \\ f(1,-1) &= -2 \\ f(-1,1) &= -2 \\ f(-1,-1) &= 6 \end{aligned}$$

Then 6 is the absolute maximum and -2 is the absolute minimum.

18. From the Extreme Value Theorem, f has both an absolute maximum and an absolute minimum on the closed, bounded set $x^2 + y^2 \leq 4$. The critical points are the solutions to the system

$$\begin{cases} f_x(x, y) = 4x = 0 \\ f_y(x, y) = 2y = 0 \end{cases}$$

The only critical point, $(0,0)$, lies inside the constraint set. The only other place extrema can occur is on the boundary $g(x, y) = x^2 + y^2 - 4 = 0$. We use Lagrange multipliers to find the extrema of f subject to the constraint curve $g(x, y) = 0$. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 4x = 2x\lambda \\ 2y = 2y\lambda \\ x^2 + y^2 - 4 = 0 \end{cases}$$

If $x = 0$, then $y = \pm 2$, so two test points are $(0, 2)$ and $(0, -2)$. If $x \neq 0$, then the first equation implies that $\lambda = 2$ which means, from the second equation, we find that $y = 0$. This means the only other critical points are $(2, 0)$ and $(-2, 0)$. Now we plug the critical points and test points into f .

$$\begin{aligned} f(0,0) &= 0 \\ f(0,2) &= 4 \\ f(0,-2) &= 4 \\ f(2,0) &= 8 \\ f(-2,0) &= 8 \end{aligned}$$

Then 8 is the absolute maximum and 0 is the absolute minimum.

19. We introduce two Lagrange multipliers λ_1 and λ_2 and solve the system

$$\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z), \quad g(x, y, z) = x^2 + y^2 - z^2 = 0, \quad h(x, y, z) = x + y - z + 1 = 0.$$

$$\begin{cases} 2x = \lambda_1 2x + \lambda_2 \\ 2y = \lambda_1 2y + \lambda_2 \\ 2z = -\lambda_1 2z - \lambda_2 \\ x^2 + y^2 - z^2 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

From the first equation, $\lambda_2 = 2x - 2x\lambda_1$, and plugging this into the second equation, we have

$$\begin{aligned} 2y &= 2y\lambda_1 + 2x - 2x\lambda_1 \\ 2y + 2x &= \lambda_1(2y + 2x). \end{aligned}$$

If $x \neq y$, then $\lambda_1 = 1$ which means $\lambda_2 = 0$ from either the first or second equation. The third equation then means that $z = 0$. If $z = 0$, then the final two equations do not have a real solution. So going back, we see that it must be the case that $x = y$. In this case, the final two equations become

$$\begin{cases} 2x^2 - z^2 = 0 \\ 2x - z + 1 = 0 \end{cases}$$

which, using substitution, has solutions $x = -1 + \frac{\sqrt{2}}{2}, z = \sqrt{2} - 1$ and $x = -1 - \frac{\sqrt{2}}{2}, z = -\sqrt{2} - 1$. Then the test points are $\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, \sqrt{2} - 1\right)$ and $\left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -\sqrt{2} - 1\right)$. The corresponding values at the test points are

$$\begin{aligned} f\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, \sqrt{2} - 1\right) &= 6 - 4\sqrt{2} \\ f\left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -\sqrt{2} - 1\right) &= 6 + 4\sqrt{2}. \end{aligned}$$

Then $\boxed{6 + 4\sqrt{2} \text{ is the absolute maximum value}}$ and $\boxed{6 - 4\sqrt{2} \text{ is the absolute minimum value}}$.

20. We introduce two Lagrange multipliers λ_1 and λ_2 and solve the system

$$\nabla w(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z), \quad g(x, y, z) = 2x + y + 2z - 9 = 0, \quad h(x, y, z) = 5x + 5y + 7z - 29 = 0.$$

$$\begin{cases} 2x = 2\lambda_1 + 5\lambda_2 \\ 2y = \lambda_1 + 5\lambda_2 \\ 2z = 2\lambda_1 + 7\lambda_2 \\ 2x + y + 2z - 9 = 0 \\ 5x + 5y + 7z - 29 = 0 \end{cases}$$

Solving for λ_1 in the second equation and plugging it into the first equation, we have

$$\begin{aligned} 2x &= 2(2y - 5\lambda_2) + 5\lambda_2 \\ \lambda_2 &= \frac{2x - 4y}{-5}. \end{aligned}$$

Plugging this into the third equation (along with λ_1 from the third equation), we have

$$\begin{aligned} 2z &= 2(2y - 5\lambda_2) + 7\lambda_2 \\ 2z &= 4y - 3\lambda_2 \\ 2z &= 4y - 3\left(\frac{2x - 4y}{-5}\right) \\ -3x - 4y + 5z &= 0. \end{aligned}$$

Then using substitution to solve the system including this equation and the last two from the system above, we find the only solution is $(2, 1, 2)$. Then the $\boxed{\text{absolute minimum is } w(2, 1, 2) = 9}$.

Applications and Extensions

21. We will minimize the *square* of the distance between a point (x, y) on the given line and the origin. That is, we will minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x - 3y - 6 = 0$. We will use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x = \lambda \\ 2y = -3\lambda \\ x - 3y - 6 = 0 \end{cases}$$

Eliminating λ in the first two equations, we have $y = -3x$. Plugging this into the third equation, we find that

$$x = \frac{3}{5}, \text{ so } y = -\frac{9}{5}. \text{ This means that point } \boxed{\left(\frac{3}{5}, -\frac{9}{5}\right) \text{ is the point closest to the origin on the given line.}}$$

22. We will minimize the *square* of the distance between a point (x, y, z) on the given plane and the origin. That is, we will minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 2x + y - 3z - 6 = 0$. We will use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

and $g(x, y, z) = 0$, namely

$$\begin{cases} 2x = 2\lambda \\ 2y = \lambda \\ 2z = -3\lambda \\ 2x + y - 3z - 6 = 0 \end{cases}$$

Eliminating λ in the first three equations, we have $x = 2y$ and $z = -3y$. Plugging these into the fourth equation, we find that $y = \frac{3}{7}$, so $x = \frac{6}{7}$ and $z = -\frac{9}{7}$. This means that point

$$\left(\frac{6}{7}, \frac{3}{7}, -\frac{9}{7}\right) \text{ is the point closest to the origin on the given plane.}$$

23. We need to minimize $f(x, y) = xy$ subject to the constraint $g(x, y) = x + 2y - 21 = 0$. We will use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} y = \lambda \\ x = 2\lambda \\ x + 2y - 21 = 0 \end{cases}$$

Eliminating λ from the first two equations, we find that $x = 2y$. Plugging this into the third equation, we find that $y = \frac{21}{4}$, so $x = \frac{21}{2}$. The maximum product is $\frac{441}{8}$.

24. We will minimize the quotient $f(x, y) = \frac{x}{y}$ subject to the constraint curve $g(x, y) = 4x + 2y - 100 = 0$. We will use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} \frac{1}{y} = 4\lambda \\ -xy^{-2} = 2\lambda \\ 4x + 2y - 100 = 0 \end{cases}$$

Solving for λ in the first two equations and equating them, we find that

$$\begin{aligned} \frac{1}{4y} &= -\frac{x}{2y^2} \\ 2y^2 &= -4xy \\ y &= -2x. \end{aligned}$$

Plugging this into the third equation, we have that $4x + 2(-2x) - 100 = 0$, which does not have any solutions. This means Lagrange multipliers is not applicable for this problem.

25. (a) To be on the plane $3 - y = 0$, it must be that $y = 3$. So a point on the intersection of the two surfaces must satisfy $g(x, z) = 4x^2 + (3)^2 + z^2 - 16 = 0$. We will minimize the *square* of the distance between a point on this curve and the origin. For such a point $(x, 3, z)$, we seek to minimize $f(x, z) = x^2 + 9 + z^2$ subject to $g(x, z) = 0$. We will use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, z) = \lambda \nabla g(x, z)$ and $g(x, z) = 0$, namely

$$\begin{cases} 2x = 8x\lambda \\ 2z = 2z\lambda \\ 4x^2 + z^2 - 7 = 0 \end{cases}$$

If $z \neq 0$, then the second equation implies that $\lambda = 1$, so that the first equation then yields $x = 0$. If $x = 0$, then from the third equation, we find that $z = \pm\sqrt{7}$ to give us two test points $(0, \sqrt{7}), (0, -\sqrt{7})$.

If $z = 0$, then the third equation gives us $x = \pm\frac{\sqrt{7}}{2}$ which gives two more test points: $(\frac{\sqrt{7}}{2}, 0), (-\frac{\sqrt{7}}{2}, 0)$.

Plugging these into we find that

$$\begin{aligned} f(0, \sqrt{7}) &= 16 \\ f(0, -\sqrt{7}) &= 16 \\ f\left(\frac{\sqrt{7}}{2}, 0\right) &= \frac{43}{4} \\ f\left(\frac{-\sqrt{7}}{2}, 0\right) &= \frac{43}{4} \end{aligned}$$

The points that are farthest from the origin are $\boxed{(0, 3, \sqrt{7}), (0, 3, -\sqrt{7})}$.

(b) The points that are closest to the origin are $\boxed{\left(\frac{\sqrt{7}}{2}, 3, 0\right), \left(-\frac{\sqrt{7}}{2}, 3, 0\right)}$.

26. We will maximize and minimize the *square* of the distance between the origin and the intersection of the surfaces. We want to find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x^2 + 2x + y^2 + z^2 - 16 = 0$ and $h(x, y, z) = 3x + y - z = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z)$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$, namely

$$\begin{cases} 2x = (2x + 2)\lambda_1 + 3\lambda_2 \\ 2y = 2y\lambda_1 + \lambda_2 \\ 2z = 2z\lambda_1 - \lambda_2 \\ x^2 + 2x + y^2 + z^2 - 16 = 0 \\ 3x + y - z = 0 \end{cases}$$

By adding the second and third equations together, we find that $2y + 2z = \lambda_1(2y + 2z)$, or $y + z = \lambda_1(y + z)$. If $y + z \neq 0$, then $\lambda_1 = 1$. If $\lambda_1 = 1$, then from the first equation, we have $0 = 2 + 3\lambda_2$, or $\lambda_2 = -\frac{2}{3}$. But from the second equation, we find that $\lambda_2 = 0$. So it must be that $\lambda_1 \neq 1$, which means $y + z = 0$, or $y = -z$. Plugging this into the fourth and fifth equations, we have the system

$$\begin{cases} x^2 + 2x + 2y^2 - 16 = 0 \\ 3x + 2y = 0 \end{cases}$$

Solving for x in the second equation and substituting it into the first, we have

$$\begin{aligned} \left(-\frac{2}{3}y\right)^2 + 2\left(-\frac{2}{3}y\right) + 2y^2 - 16 &= 0 \\ \frac{22}{9}y^2 - \frac{4}{3}y - 16 &= 0 \\ 11y^2 - 6y - 72 &= 0 \\ y &= \frac{3 \pm 3\sqrt{89}}{11}. \end{aligned}$$

This means our two test points are

$$\left(\frac{-2 + 2\sqrt{89}}{11}, \frac{3 - 3\sqrt{89}}{11}, \frac{-3 + 3\sqrt{89}}{11}\right) \quad \text{and} \quad \left(\frac{-2 - 2\sqrt{89}}{11}, \frac{3 + 3\sqrt{89}}{11}, \frac{-3 - 3\sqrt{89}}{11}\right).$$

Plugging these point into f , we find the values $\frac{180 - 4\sqrt{89}}{11}$ and $\frac{180 + 4\sqrt{89}}{11}$, respectively. This means that

the point $\boxed{\left(\frac{-2 - 2\sqrt{89}}{11}, \frac{3 + 3\sqrt{89}}{11}, \frac{-3 - 3\sqrt{89}}{11}\right)}$ is farthest from the origin and the point

$$\left(\frac{-2 + 2\sqrt{89}}{11}, \frac{3 - 3\sqrt{89}}{11}, \frac{-3 + 3\sqrt{89}}{11} \right) \text{ is closest to the origin.}$$

27. We use Lagrange multipliers to maximize $f(x, y) = xy$ subject to $g(x, y) = x^2 + 2y^2 - 2 = 0$. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} y = 2x\lambda \\ x = 4y\lambda \\ x^2 + 2y^2 - 2 = 0 \end{cases}$$

Substituting the first equation into the second, we have

$$\begin{aligned} x &= 8x\lambda^2 \\ x = 0 \quad \text{or} \quad \lambda &= \pm \sqrt{\frac{1}{8}} = \pm \frac{\sqrt{2}}{4}. \end{aligned}$$

If $x = 0$, then the first equation says $y = 0$, but the point $(0, 0)$ does not satisfy the third equation so there are no test points in this case. If $\lambda = \pm \frac{\sqrt{2}}{4}$, then the first two equations yield $y^2 = 4x^2\lambda^2 = \frac{1}{2}x^2$. Plugging this into the third equation, we have

$$\begin{aligned} x^2 + x^2 - 2 &= 0 \\ x &= \pm 1. \end{aligned}$$

Using the third equation to find the corresponding y -values, we find that the four test points are then

$$\left(1, \frac{\sqrt{2}}{2}\right), \left(1, -\frac{\sqrt{2}}{2}\right), \left(-1, \frac{\sqrt{2}}{2}\right), \left(-1, -\frac{\sqrt{2}}{2}\right). \text{ The maximum product is then } \frac{\sqrt{2}}{2} \text{ at the points } \left(1, \frac{\sqrt{2}}{2}\right)$$

and $\left(-1, -\frac{\sqrt{2}}{2}\right).$

28. If x, y, z represent the length, width, and height, respectively, for the box, then we look to maximize $V(x, y, z) = xyz$ where $x, y, z > 0$ subject to $SA(x, y, z) = xy + 2xz + 2yz - 48 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla V(x, y, z) = \lambda \nabla SA(x, y, z)$ and $SA(x, y, z) = 0$, namely

$$\begin{cases} yz = \lambda(y + 2z) \\ xz = \lambda(x + 2z) \\ xy = \lambda(2x + 2y) \\ xy + 2xz + 2yz - 48 = 0 \end{cases}$$

Eliminating λ in the first three equations, we have

$$\frac{yz}{y + 2z} = \frac{xz}{x + 2z} = \frac{xy}{2x + 2y}.$$

From the left-most equality, we cross multiply to get that $xyz + 2yz^2 = xyz + 2xz^2$, or $x = y$. Then from the third equation, we have $x^2 = 4x\lambda$, or $x = 4\lambda$. Plugging this into the second equation

$$\begin{aligned} 4\lambda z &= \lambda(4\lambda + 2z) \\ 4z &= 4\lambda + 2z \\ z &= 2\lambda = \frac{1}{2}x = \frac{1}{2}y. \end{aligned}$$

Plugging this all into the fourth equation,

$$\begin{aligned} x^2 + x^2 + x^2 - 48 &= 0 \\ x &= 4, \end{aligned}$$

so $y = 4$ and $z = 2$. The dimensions that maximize the volume are the box with square base and dimensions $\boxed{4 \text{ cm} \times 4 \text{ cm} \times 2 \text{ cm}}$.

29. The surface area of a cylindrical can of radius r and height h is $SA(r, h) = 2\pi r^2 + 2\pi rh$ where $r, h > 0$. We need to maximize this subject to the volume equation $V(r, h) = \pi r^2 h - V = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla SA(r, h) = \lambda \nabla V(r, h)$ and $V(r, h) = 0$, namely

$$\begin{cases} 4\pi r + 2\pi h = 2\pi rh\lambda \\ 2\pi r = \pi r^2\lambda \\ \pi r^2 h - V = 0 \end{cases}$$

From the second equation, $\lambda = \frac{2}{r}$. Plugging this into the first equation, we have

$$\begin{aligned} 4\pi r + 2\pi h &= 4\pi h \\ 2r + h &= 2h \\ 2r &= h. \end{aligned}$$

Then, from the third equation, we have

$$\begin{aligned} 2\pi r^3 &= V \\ r &= \sqrt[3]{\frac{V}{2\pi}}. \end{aligned}$$

If the dimensions of the can are $r = \sqrt[3]{\frac{V}{2\pi}}$ and $h = 2\sqrt[3]{\frac{V}{2\pi}}$, then the surface area will be minimized.

30. If x, y, z represent the length, width, and height, respectively, of the container, then we look to minimize the surface area $SA(x, y, z) = 2(xy + xz + yz)$ subject to the volume being fixed at 216, namely $V(x, y, z) = xyz - 216 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla SA(x, y, z) = \lambda \nabla V(x, y, z)$ and $V(x, y, z) = 0$, namely

$$\begin{cases} 2(y + z) = \lambda yz \\ 2(x + z) = \lambda xz \\ 2(x + y) = \lambda xy \\ xyz - 216 = 0 \end{cases}$$

Eliminating λ in the first three equations, we find that

$$\frac{2(y + z)}{yz} = \frac{2(x + z)}{xz} = \frac{2(x + y)}{xy}.$$

Cross multiplying in the first equality, we have $xyz + xz^2 = xyz + yz^2$, or $x = y$. Similarly, cross multiplying in the second equality results in $y = z$, so $x = y = z$. Plugging these into the fourth equation, we find that $x^3 - 216 = 0$, or $x = 6$, so $x = y = z = 6$. The dimensions that will minimize the amount of material are $\boxed{6 \text{ m} \times 6 \text{ m} \times 6 \text{ m}}$.

31. If x, y, z represent the length, width, and height, respectively, of the box, then the cost function we seek to minimize is $C(x, y, z) = \$1(xy + 2xz + 2yz) = xy + 2xz + 2yz$. Since the volume is fixed at 12, we must have $V(x, y, z) = xyz - 12 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla C(x, y, z) = \lambda \nabla V(x, y, z)$ and $V(x, y, z) = 0$, namely

$$\begin{cases} y + 2z = \lambda yz \\ x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz - 12 = 0 \end{cases}$$

Eliminating λ from the first three equations, we have

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy}.$$

Cross multiplying the first equality, we have $xyz + 2xz^2 = xyz + 2yz^2$, or $x = y$. Similarly, cross multiplying the second equality, we have $2x^2z + 2xyz = x^2y + 2xyz$, or $y = 2z$. Plugging these into the fourth equation, we have $\frac{1}{2}y^3 - 12 = 0$, or $y = 2\sqrt[3]{3}$. Then the box with square base and dimensions $\boxed{2\sqrt[3]{3} \text{ m} \times 2\sqrt[3]{3} \text{ m} \times \sqrt[3]{3} \text{ m}}$ will minimize the cost of the material.

32. If x, y, z represent the length, width, and height, respectively, of the box, then the cost function we seek to minimize is $C(x, y, z) = \$2xy + \$1(xy + 2xz + 2yz) = 3xy + 2xz + 2yz$ subject to the constraint $V(x, y, z) = xyz - 18 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla C(x, y, z) = \lambda \nabla V(x, y, z)$ and $V(x, y, z) = 0$, namely

$$\begin{cases} 3y + 2z = \lambda yz \\ 3x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz - 18 = 0 \end{cases}$$

Eliminating λ from the first three equations, we have

$$\frac{3y + 2z}{yz} = \frac{3x + 2z}{xz} = \frac{2x + 2y}{xy}.$$

Cross multiplying the first equality, we have $3xyz + 2xz^2 = 3xyz + 2yz^2$, or $x = y$. Similarly, cross multiplying the second equality, we have $3x^2y + 2xyz = 2x^2z + 2xyz$, or $z = \frac{3}{2}y$. Plugging these into the fourth equation, we have $\frac{3}{2}y^3 - 18 = 0$, or $y = \sqrt[3]{12}$. Then the box with square base and dimensions

$\boxed{\sqrt[3]{12} \text{ m} \times \sqrt[3]{12} \text{ m} \times \frac{3}{2}\sqrt[3]{12} \text{ m}}$ will minimize the cost of the material.

33. If x, y, z represent the length, width, and height, respectively, of the box, then we look to maximize the volume $V(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z - 45 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla V(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x + y + z - 45 = 0 \end{cases}$$

Eliminating λ from the first three equations, we have $x = y = z$. Plugging this into the fourth equation, we have $3x - 45 = 0$, or $x = 15$. Then the luggage with maximum possible volume is a cube with dimensions $\boxed{15 \text{ in} \times 15 \text{ in} \times 15 \text{ in}}$.

34. We look to maximize $T(x, y, z) = 100x^2yz$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla T(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 200xyz = 2x\lambda \\ 100x^2z = 2y\lambda \\ 100x^2y = 2z\lambda \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases}$$

Subtracting the third equation from the second, we find that $50x^2(z - y) = \lambda(y - z)$. This gives the following possibilities:

(1) $\lambda = x = 0$, so $T = 0$;

(2) $\lambda = -50x^2 \neq 0$, so $y = -z$ from the third equation and $\lambda = 100yz = -100y^2$ from the first, and $x^2 = 2y^2$ from these two λ -values. The last equation then gives $4y^2 = 1$, or $y = \pm \frac{1}{2}$, so we have the test points

$$\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{1}{2}, \mp \frac{1}{2} \right);$$

(3) $\lambda = z = 0$, so $T = 0$;

(4) $y = z \neq 0$, so $\lambda = 50x^2$ from the second equation with $\lambda \neq 0$ (or we again have case (1)), thus $100xy^2 = 50x^3$ from the first equation, or $x^2 = 2y^2$, giving test points $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ just like case (2).

From the test points, we see that T has a maximum of 12.5 degrees Celsius at $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ and a minimum of -12.5 degrees Celsius at $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

35. (a) The area of the rectangle is $x \cdot 2x = 2x^2$. If the area of the rectangle must be at least 800, then $2x^2 \geq 800$, which means $x \geq 20$ meters. Similarly, if y represents the side length of the square field, then the area of the square is $y \cdot y = y^2$, and the area of the square must be at least 100, so $y^2 \geq 100$, which means $y \geq 10$. To find the maximum value of x , it must occur when y is minimized, namely $y = 10$. The total fencing is 340 meters, so if we add the perimeter of both fields, we find that

$$\begin{aligned} 2(x + 2x) + 4(y) &= 340 \\ x &= \frac{170}{3} - \frac{2}{3}y. \end{aligned}$$

This means the maximum value of x is $x = \frac{170}{3} - \frac{2}{3} \cdot 10 = 50$ meters. Putting both answers together, $20 \leq x \leq 50$.

(b) To maximize enclosed area, we look to maximize $A(x, y) = 2x^2 + y^2$ subject to the fencing constraint $g(x, y) = 6x + 4y - 340 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla A(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 4x = 6\lambda \\ 2y = 4\lambda \\ 6x + 4y - 340 = 0 \end{cases}$$

Eliminating λ from the first two equations, we find that $y = \frac{4}{3}x$. Plugging this into the third equation, we find that $x = 30$, so $y = 40$. The greatest number of square meters that can be enclosed by the two fields is $A(30, 40) = \text{3400 m}^2$. [Answer does not match BoB.]

36. If x and y represent the length and width, respectively, of the garden, then we seek to maximize $A(x, y) = xy$ subject to the constraint $g(x, y) = \$21.53(2x + 2y) - 5000 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla A(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} y = 43.06\lambda \\ x = 43.06\lambda \\ 21.53(2x + 2y) - 5000 = 0 \end{cases}$$

From the first two equations, we see that $x = y$, so from the third equation, we have $x = \frac{500}{4 \cdot 21.53} \approx 58.058$ feet. The largest area that can be enclosed is approximately $A(58.058, 58.058) \approx \text{3370.731 square feet}$.

37. Let $C(x, y) = 18x^2 + 9y^2$ and $g(x, y) = x + y - 5400$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla C(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 36x = \lambda \\ 18y = \lambda \\ x + y - 5400 = 0 \end{cases}$$

The first two equations imply that $y = 2x$, so plugging this into the third equation, we find that $x = 1800$ which means $y = 3600$. Production levels of $x = 1800$ thousand units and $y = 3600$ thousand units will minimize production cost.

38. Let $P(x, y) = x^2 + 3xy - 6x$ and $g(x, y) = x + y - 40$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla P(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x + 3y - 6 = \lambda \\ 3x = \lambda \\ x + y - 40 = 0 \end{cases}$$

The first two equations imply that $x = 3y - 6$, so plugging this into the third equation, we find that $y = \frac{23}{2}$

which means $x = \frac{57}{2}$. The production levels that maximize production are $\boxed{x = \frac{57}{2}}$ and $\boxed{y = \frac{23}{2}}$.

39. (a) Adding the total cost of capital and labor must be \$51,000 which means $g(K, L) = 175K + 125L - 175000 = 0$. We now seek to maximize $P(K, L) = 1.01K^{0.25}L^{0.75}$ subject to the constraint curve g . Using Lagrange multipliers, the test points satisfy the system of equations $\nabla P(K, L) = \lambda \nabla g(K, L)$ and $g(K, L) = 0$, namely

$$\begin{cases} 0.2525K^{-0.75}L^{0.75} = 175\lambda \\ 0.7575K^{0.25}L^{-0.25} = 125\lambda \\ 175K + 125L - 175000 = 0 \end{cases}$$

Eliminating λ in the first two equations, we find that

$$\begin{aligned} \frac{0.2525K^{-0.75}L^{0.75}}{175} &= \frac{0.7575K^{0.25}L^{-0.25}}{125} \\ L &= \frac{132.5625}{31.5625}K. \end{aligned}$$

Plugging this into the third equation, we solve for K and get $K = 250$, which means $L = 1050$. Total production is maximized when units of $\boxed{\text{capital is 250}}$ and units of $\boxed{\text{labor is 1050}}$.

(b) Under these conditions, the maximum number of units of production that could be generated is $P(250, 1050) \approx 740.8$.

40. We will use Lagrange multipliers to minimize and maximize the *square* of the distance between a point (x, y, z) on the two surfaces and the origin. That is, we will minimize and maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint curves $g(x, y, z) = xyz + 1 = 0$ and $h(x, y, z) = x + y + z - 1 = 0$. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z)$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$, namely

$$\begin{cases} 2x = yz\lambda_1 + \lambda_2 \\ 2y = xz\lambda_1 + \lambda_2 \\ 2z = xy\lambda_1 + \lambda_2 \\ xyz + 1 = 0 \\ x + y + z - 1 = 0 \end{cases}$$

Eliminating λ_2 from the first three equations, we have

$$2x - yz\lambda_1 = 2y - xz\lambda_1 = 2z - xy\lambda_1. \quad (13.2)$$

The first equality can be written as

$$2(x - y) = -z\lambda_1(x - y). \quad (13.3)$$

If $x = y$, then the last two equations of the original system become $x^2z + 1 = 0$ and $2x + z - 1 = 0$. Using substitution, we find that $x = 1, z = -1$. This means one test point is $(1, 1, -1)$. Going back to equation (13.3), now we assume $x \neq y$. In this case, $2 = -z\lambda_1$. Plugging this in to the right-most equality in (13.2), we have that $2(y - z) = -\frac{2}{z}x(y - z)$. If $y \neq z$, then $z = -x$ and the last two equations in the original system become $-x^2y + 1 = 0$ and $y - 1 = 0$. The only two solutions here are $y = 1$ and $x = \pm 1$. Since we assumed that $x \neq y$ in this case, the only test point in this case is then $(-1, 1, 1)$. Finally, if $y = z$, then just as above, the last two equations of the original system yield the final test point $(1, -1, 1)$. Each of the test

points $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ are the same distance from the origin, so they are the points on the two surfaces closest to the origin. However, there is no point that is farther from the origin than all other points.

41. Refer to the solution of Problem 20.

42. Let the fixed surface area of the box be S and the length, width, and height of the box be x, y, z , respectively. We look to maximize $V(x, y, z) = xyz$ subject to $SA(x, y, z) = 2(xy + xz + yz) - S = 0$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla V(x, y, z) = \lambda \nabla SA(x, y, z)$ and $SA(x, y, z) = 0$, namely

$$\begin{cases} yz = 2(y + z)\lambda \\ xz = 2(x + z)\lambda \\ xy = 2(x + y)\lambda \\ 2(xy + xz + yz) - S = 0 \end{cases}$$

Eliminating λ in the first three equations, we find that

$$\frac{2(y + z)}{yz} = \frac{2(x + z)}{xz} = \frac{2(x + y)}{xy}.$$

Cross multiplying in the first equality, we have $xyz + xz^2 = xyz + yz^2$, or $x = y$. Similarly, cross multiplying in the second equality results in $y = z$, so $x = y = z$. This means the maximum volume of a box with fixed surface area is obtained from a box where the length, width, and height are all the same, namely a cube.

43. If r and h denote the radius and height, respectively, of the cylindrical can and S the fixed surface area, then we look to maximize $V(r, h) = \pi r^2 h$ subject to $SA(r, h) = 2\pi r^2 + 2\pi r h - S = 0$. The test points satisfy the system of equations $\nabla V(r, h) = \lambda \nabla SA(r, h)$ and $SA(r, h) = 0$, namely

$$\begin{cases} 2\pi r h = (4\pi r + 2\pi h)\lambda \\ \pi r^2 = 2\pi r \lambda \\ 2\pi r^2 + 2\pi r h - S = 0 \end{cases}$$

Eliminating λ from the first two equations implies that

$$\begin{aligned} \frac{2\pi r h}{4\pi r + 2\pi h} &= \frac{\pi r^2}{2\pi r} \\ \frac{rh}{2r + h} &= \frac{r}{2} \\ h &= 2r. \end{aligned}$$

So since the only test point occurs when the height is double the radius, we see that the can of maximum volume occurs when the height of the can equals the diameter.

44. We will minimize the *square* of the distance between a point (x, y, z) on the intersection of the surfaces and the origin. That is, we will minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint functions $g(x, y, z) = x + y + z - 1 = 0$ and $h(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$. Using Lagrange multipliers, the test points satisfy the system of equations

$$\begin{cases} 2x = 2x\lambda_1 + \lambda_2 \\ 2y = 2y\lambda_1 + \lambda_2 \\ 2z = -2z\lambda_1 + \lambda_2 \\ x + y + z - 1 = 0 \\ x^2 + y^2 - z^2 - 1 = 0 \end{cases}$$

Eliminating λ_2 from the first three equations, we have

$$x(1 - \lambda_1) = y(1 - \lambda_1) = z(1 + \lambda_1).$$

If $\lambda_1 = 1$, then $z = 0$. In this case, the last two equations become $x + y = 1$ and $x^2 + y^2 = 1$. Solving the first equation for y and plugging it into the second, we have

$$\begin{aligned} x^2 + (1 - x)^2 &= 1 \\ 2x(x - 1) &= 0 \\ x &= 0, 1. \end{aligned}$$

In this case the two test points are $(0, 1, 0)$ and $(1, 0, 0)$.

If $\lambda_1 \neq 1$, then $x = y$ and the last two equations of the original system become $2x + z = 1$ and $2x^2 - z^2 = 1$. Solving the first equation for z and plugging it into the second, we have

$$\begin{aligned} 2x^2 - (1 - 2x)^2 &= 1 \\ -2(x - 1)^2 &= 0 \\ x &= 1. \end{aligned}$$

In this case, the only test point is $(1, 1, -1)$.

The two points closest to the origin are $\boxed{(0, 1, 0)}$ and $\boxed{(1, 0, 0)}$.

45. To be on the union of the two curves means you are one or the other, or both. So we will find the extrema of f on both curves separately and then find the extrema of f by comparing both sets of extrema on both curves.

Let $g(x, y) = x^2 + y^2 - 1 = 0$. To find the extrema of f on the constraint curve $g(x, y) = 0$, we use Lagrange multipliers. The test points satisfy the system of equations

$$\begin{cases} 4x^3 = 2x\lambda \\ 4y^3 = 2y\lambda \\ x^2 + y^2 - 1 = 0 \end{cases}$$

From the first equation, we see that either $x = 0$, or $\lambda = 2x^2$. If $x = 0$, then from the third equation we see that $y = \pm 1$, so two test points are $(0, \pm 1)$. If $x \neq 0$, then $\lambda = 2x^2$ so $\lambda \neq 0$. Plugging this into the second equation, we have $4y^3 = 4yx^2$. From here, either $y = 0$, or $y^2 = x^2$, so $x = \pm y$. If $y = 0$, then from the third equation we see that $x = \pm 1$, so two more test points are $(\pm 1, 0)$. If $x = \pm y$, then from the third equation we see that

$$\begin{aligned} 2y^2 &= 1 \\ y &= \pm \frac{\sqrt{2}}{2}, \end{aligned}$$

which leads to the four test points $\left(\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$.

Now let the constraint curve be $h(x, y) = x^3 + y^3 - 1 = 0$. To find the extrema of f when $h(x, y) = 0$, we use Lagrange multipliers again. The test points satisfy the system of equations

$$\begin{cases} 4x^3 = 3x^2\lambda \\ 4y^3 = 3y^2\lambda \\ x^3 + y^3 - 1 = 0 \end{cases}$$

From the first equation, we see that either $x = 0$, or $\lambda = \frac{4}{3}x$. If $x = 0$, then from the third equation we see that $y = 1$, so one test point is $(0, 1)$. If $x \neq 0$, then $\lambda = \frac{4}{3}x$. Plugging this into the second equation, we find that $4y^3 = 4xy^2$. From here, either $y = 0$, or $x = y$. If $y = 0$, then from the third equation we see that $x = 1$, so another test point is $(1, 0)$. If $x = y$, then the third equation becomes

$$\begin{aligned} 2x^3 &= 1 \\ x &= \frac{1}{\sqrt[3]{2}}, \end{aligned}$$

which leads to the final test point $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$.

Plugging each of the test points into $f(x, y)$ we have

$$\begin{aligned} f(\pm 1, 0) = f(0, \pm 1) &= 5 \\ f\left(\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) &= \frac{9}{2} \\ f\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right) &= \frac{1}{\sqrt[3]{2}} + 4 \approx 4.794. \end{aligned}$$

Then f attains its maximum value at the points $\left(\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and its minimum value at the points $(\pm 1, 0)$ and $(0, \pm 1)$

46. Let $g(x, y, z) = x^2 + y^2 + z^2 - 1$. First note that the extreme values must occur where all of $x, y, z \neq 0$ since the function value at such a point would be 0, however the points $(\pm 1, 0, 0)$ yield values above and below 0, so 0 cannot be an extrema. So for the remainder of the problem, we assume $x, y, z \neq 0$. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} yz = 2x\lambda \\ xz = 2y\lambda \\ xy = 2z\lambda \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases}$$

Eliminating λ from the first three equations, we find that $\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$. Cross multiplying in the first equality, we have $2y^2z = 2x^2z$, or $y = \pm x$. Similarly, the second equality yields $y = \pm z$. Plugging these into the fourth equation, we have

$$\begin{aligned} (\pm y)^2 + y^2 + (\pm y)^2 - 1 &= 0 \\ y &= \pm \frac{\sqrt{3}}{3}. \end{aligned}$$

So $x = \pm \frac{\sqrt{3}}{3}$ and $z = \pm \frac{\sqrt{3}}{3}$.

The maximum value of the product of x, y , and z occurs when their product is positive, and the minimum value occurs when their product is negative. So the

absolute maximum is $\left(\frac{\sqrt{3}}{3}\right)^3 = \frac{\sqrt{3}}{9}$ and the

absolute minimum is $-\left(\frac{\sqrt{3}}{3}\right)^3 = -\frac{\sqrt{3}}{9}$.

Challenge Problems

47. To minimize $f(x, y, z) = x^4 + y^4 + z^4$ subject to the constraint curve $g(x, y, z) = Ax + By + Cz - D = 0$, we use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 4x^3 = \lambda A \\ 4y^3 = \lambda B \\ 4z^3 = \lambda C \\ Ax + By + Cz - D = 0 \end{cases}$$

Solving for λ in the first three equations, we find that

$$\frac{4x^3}{A} = \frac{4y^3}{B} = \frac{4z^3}{C}$$

so that $y = \sqrt[3]{\frac{B}{A}}x$ and $z = \sqrt[3]{\frac{C}{A}}x$. Plugging these into the fourth equation, we have

$$Ax + B \cdot \sqrt[3]{\frac{B}{A}}x + C \cdot \sqrt[3]{\frac{C}{A}}x - D = 0$$

$$x = \frac{A^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}.$$

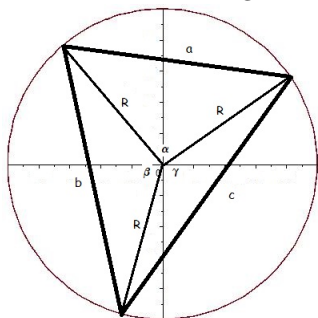
Using this value of x to find y and z , we have the single test point

$$\left(\frac{A^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}, \frac{B^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}, \frac{C^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}} \right).$$

The minimum value is then

$$f\left(\frac{A^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}, \frac{B^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}, \frac{C^{1/3}D}{A^{4/3} + B^{4/3} + C^{4/3}}\right) = \boxed{\frac{D^4}{\left(A^{4/3} + B^{4/3} + C^{4/3}\right)^3}}.$$

48. Consider a triangle inscribed in a circle of radius $R > 0$ with side lengths a, b, c (see the picture below).



Let the central angles corresponding to sides a, b, c be α, β, γ , respectively. Then using the Law of Cosines, we have

$$\begin{aligned} a^2 &= 2R^2 - 2R^2 \cos \alpha = 2R^2 (1 - \cos \alpha) \\ b^2 &= 2R^2 (1 - \cos \beta) \\ c^2 &= 2R^2 (1 - \cos \gamma). \end{aligned}$$

These equations imply that

$$a = \sqrt{2}R(1 - \cos \alpha)^{1/2} \quad a = \sqrt{2}R(1 - \cos \beta)^{1/2} \quad a = \sqrt{2}R(1 - \cos \gamma)^{1/2}.$$

We're looking to minimize

$$P(\alpha, \beta, \gamma) = a + b + c = \sqrt{2}R \left((1 - \cos \alpha)^{1/2} + (1 - \cos \beta)^{1/2} + (1 - \cos \gamma)^{1/2} \right),$$

subject to the constraint $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - 2\pi = 0$. Throughout the remainder of the problem, we also assume that $\alpha, \beta, \gamma > 0$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla P(\alpha, \beta, \gamma) = \lambda \nabla g(\alpha, \beta, \gamma)$ and $g(\alpha, \beta, \gamma) = 0$, namely

$$\begin{cases} \frac{\sqrt{2}}{2}R(1 - \cos \alpha)^{-1/2} \cdot \sin \alpha = \lambda \\ \frac{\sqrt{2}}{2}R(1 - \cos \beta)^{-1/2} \cdot \sin \beta = \lambda \\ \frac{\sqrt{2}}{2}R(1 - \cos \gamma)^{-1/2} \cdot \sin \gamma = \lambda \\ \alpha + \beta + \gamma - 2\pi = 0 \end{cases}$$

Eliminating λ in the first three equations, we have

$$\begin{aligned}\frac{\sin \alpha}{\sqrt{1 - \cos \alpha}} &= \frac{\sin \beta}{\sqrt{1 - \cos \beta}} = \frac{\sin \gamma}{\sqrt{1 - \cos \gamma}} \\ \frac{\sin^2 \alpha}{1 - \cos \alpha} &= \frac{\sin^2 \beta}{1 - \cos \beta} = \frac{\sin^2 \gamma}{1 - \cos \gamma} \\ \frac{1 - \cos^2 \alpha}{1 - \cos \alpha} &= \frac{1 - \cos^2 \beta}{1 - \cos \beta} = \frac{1 - \cos^2 \gamma}{1 - \cos \gamma} \\ 1 + \cos \alpha &= 1 + \cos \beta = 1 + \cos \gamma \\ \cos \alpha &= \cos \beta = \cos \gamma \\ \alpha &= \beta = \gamma.\end{aligned}$$

Since all the inscribed angles are equal, it must be that $a = b = c$, which means the triangle with maximum perimeter is an equilateral triangle.

49. To find the point closest to the paraboloid, we will minimize the *square* of the distance between a point on the paraboloid and $(1, 1, 2)$. For a point (x, y, z) on the paraboloid, the square of the distance between itself at $(1, 1, 2)$ is

$$f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 2)^2.$$

We need to minimize f subject to $g(x, y, z) = x^2 + y^2 + z - 2 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 2(x - 1) = 2x\lambda \\ 2(y - 1) = 2y\lambda \\ 2(z - 2) = \lambda \\ x^2 + y^2 + z - 2 = 0 \end{cases}$$

Solving for λ in the first three equations, we find that

$$\frac{2x - 2}{2x} = \frac{2y - 2}{2y} = 2z - 4$$

so that $y = x$ and $z = \frac{10x - 2}{4x}$. Plugging these into the fourth equation, we find that

$$\begin{aligned}x^2 + x^2 + \frac{10x - 2}{4x} - 2 &= 0 \\ 4x^3 + x - 1 &= 0 \\ (2x - 1)(2x^2 + x + 1) &= 0\end{aligned}$$

for which the only real solution is $x = \frac{1}{2}$. The only test point is then $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$, making it the minimum based on the physical properties of the problem.

50. To find the point closest to the origin, we will minimize the *square* of the distance between a point on the surface and the origin. For a point (x, y, z) on the curve, the square of the distance between itself and the origin is

$$f(x, y, z) = x^2 + y^2 + z^2.$$

We need to minimize f subject to $g(x, y, z) = xy - z^2 - 6y + 36 = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 2x = y\lambda \\ 2y = (x - 6)\lambda \\ 2z = -2z\lambda \\ xy - z^2 - 6y + 36 = 0 \end{cases}$$

Case 1 $z \neq 0$: From the third equation, if $z \neq 0$, then $\lambda = -1$. In this case, the first two equations become $2x = -y$ and $2y = 6 - x$. Using substitution the solution to this system is $x = -2, y = 4$. Plugging these values into the fourth equation, we find that $z = \pm 2$. Both points $\boxed{(-2, 4, 2)}$ and $\boxed{(-2, 4, -2)}$ have the same distance from the origin, namely $2\sqrt{6}$.

Case 2 $z = 0$: From the last equation, it must be that $xy - 6y + 36 = 0$. Writing this equation as $y = \frac{36}{6-x}$. To find the minimum distance between a point on this hyperbola and the origin (since $z = 0$), we need to minimize $x^2 + y^2 = x^2 + \left(\frac{36}{6-x}\right)^2$. Setting the derivative equal to 0 reduces to the quartic equation $x(x-6)^3 = (36)^2$. Using a CAS (or calculator), we find that $x \approx -2.2817$ and $x \approx 10.915$. The corresponding points are then approximately $(-2.2817, 4.347, 0)$ and $(10.915, -7.324, 0)$, which are farther from the origin than those in case 1.

This means the points $\boxed{(-2, 4, \pm 2)}$ are a minimum distance on the surface to the origin.

Chapter Review

1. Since

$$f_x(x, y) = \frac{1}{\sec(x^2 + y^2)} \cdot \sec(x^2 + y^2) \tan(x^2 + y^2) \cdot 2x = 2x \tan(x^2 + y^2)$$

$$f_y(x, y) = \frac{1}{\sec(x^2 + y^2)} \cdot \sec(x^2 + y^2) \tan(x^2 + y^2) \cdot 2y = 2y \tan(x^2 + y^2)$$

the gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2x \tan(x^2 + y^2)\mathbf{i} + 2y \tan(x^2 + y^2)\mathbf{j}.$$

The gradient of f at the point $\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right)$ is

$$\boxed{\nabla f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) = \sqrt{\frac{\pi}{2}}\mathbf{i} + \sqrt{\frac{\pi}{2}}\mathbf{j}}.$$

The unit vector in the direction of $\mathbf{a} = \mathbf{i} + \mathbf{j}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$D_{\mathbf{u}}f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) = \nabla f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) \cdot \mathbf{u} = \left(\sqrt{\frac{\pi}{2}}\mathbf{i} + \sqrt{\frac{\pi}{2}}\mathbf{j}\right) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right)$$

$$= \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}.$$

2. Since

$$f_x(x, y) = \frac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2} = -\frac{4x^2y}{(x^2 + y^2)^2}$$

the gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \frac{4xy^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2}\mathbf{j}.$$

The gradient of f at the point $(1, 1)$ is

$$\boxed{\nabla f(1, 1) = \mathbf{i} - \mathbf{j}}.$$

The unit vector in the direction of $\mathbf{a} = \mathbf{i} - \mathbf{j}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = (\mathbf{i} - \mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \boxed{\sqrt{2}}.$$

3. Since

$$f_x(x, y) = \frac{y}{\sqrt{1 - x^2}}$$

$$f_y(x, y) = \sin^{-1} x$$

the gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \frac{y}{\sqrt{1-x^2}}\mathbf{i} + \sin^{-1}(x)\mathbf{j}.$$

The gradient of f at the point $(0, 1)$ is

$$\boxed{\nabla f(0, 1) = \mathbf{i}}.$$

The unit vector in the direction of $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$\begin{aligned} D_{\mathbf{u}}f(0, 1) &= \nabla f(0, 1) \cdot \mathbf{u} = (\mathbf{i}) \cdot \left(\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j} \right) \\ &= \boxed{\frac{3\sqrt{10}}{10}}. \end{aligned}$$

4. Since $f(x, y, z) = \ln x + \ln y + \ln z$ and

$$\begin{aligned} f_x(x, y, z) &= \frac{1}{x} \\ f_y(x, y, z) &= \frac{1}{y} \\ f_z(x, y, z) &= \frac{1}{z} \end{aligned}$$

the gradient of f at (x, y, z) is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \frac{1}{x}\mathbf{i} + \frac{1}{y}\mathbf{j} + \frac{1}{z}\mathbf{k}.$$

The gradient of f at the point $(1, 1, 3)$ is

$$\boxed{\nabla f(1, 1, 3) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}}.$$

The unit vector in the direction of $\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{11}}\mathbf{i} + \frac{1}{\sqrt{11}}\mathbf{j} + \frac{3}{\sqrt{11}}\mathbf{k}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$\begin{aligned} D_{\mathbf{u}}f(1, 1, 3) &= \nabla f(1, 1, 3) \cdot \mathbf{u} = \left(\mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k} \right) \cdot \left(\frac{1}{\sqrt{11}}\mathbf{i} + \frac{1}{\sqrt{11}}\mathbf{j} + \frac{3}{\sqrt{11}}\mathbf{k} \right) \\ &= \boxed{\frac{3\sqrt{11}}{11}}. \end{aligned}$$

5. Since

$$\begin{aligned} f_x(x, y, z) &= -ye^{-x}(x^2 + y^2 + z^2 + 1) + 2xye^{-x} = ye^{-x}(2x - x^2 - y^2 - z^2 - 1) \\ f_y(x, y, z) &= e^{-x}(x^2 + y^2 + z^2 + 1) + 2y^2e^{-x} = e^{-x}(x^2 + 3y^2 + z^2 + 1) \\ f_z(x, y, z) &= 2zye^{-x} \end{aligned}$$

the gradient of f at (x, y, z) is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = ye^{-x}(2x - x^2 - y^2 - z^2 - 1)\mathbf{i} + e^{-x}(x^2 + 3y^2 + z^2 + 1)\mathbf{j} + 2zye^{-x}\mathbf{k}.$$

The gradient of f at the point $(0, 0, 0)$ is

$$\boxed{\nabla f(0, 0, 0) = \mathbf{j}}.$$

The unit vector in the direction of $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$\begin{aligned} D_{\mathbf{u}}f(0, 0, 0) &= \nabla f(0, 0, 0) \cdot \mathbf{u} = (\mathbf{j}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \right) \\ &= \boxed{\frac{1}{3}}. \end{aligned}$$

6. Since

$$\begin{aligned} f_x(x, y, z) &= 2(2x + y + z) \cdot 2 + yz = 8x + 4y + 4z + yz \\ f_y(x, y, z) &= 2(2x + y + z) + xz = 4x + 2y + 2z + xz \\ f_z(x, y, z) &= 2(2x + y + z) + xy = 4x + 2y + 2z + xy \end{aligned}$$

the gradient of f at (x, y, z) is

$$\nabla f(x, y) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = (8x + 4y + 4z + yz)\mathbf{i} + (4x + 2y + 2z + xz)\mathbf{j} + (4x + 2y + 2z + xy)\mathbf{k}.$$

The gradient of f at the point $(1, 1, 1)$ is

$$\boxed{\nabla f(1, 1, 1) = 17\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}}.$$

The unit vector in the direction of $\mathbf{a} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ is $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$, so the directional derivative of f at the given point in the direction of $\mathbf{a}$ is

$$\begin{aligned} D_{\mathbf{u}}f(1, 1, 1) &= \nabla f(1, 1, 1) \cdot \mathbf{u} = (17\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}) \cdot \left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \right) \\ &= \boxed{-\frac{35\sqrt{3}}{3}}. \end{aligned}$$

7. (a) We recall that the direction in which the directional derivative is maximized is the one in the direction of the gradient at the given point. Also, the maximum value of the directional derivative is the magnitude of the gradient, and the minimum value is the opposite of the magnitude of the gradient. The gradient of f and its value at the given point are

$$\begin{aligned} \nabla f(x, y) &= 2x \sec(x^2 + y^2) \tan(x^2 + y^2)\mathbf{i} + 2y \sec(x^2 + y^2) \tan(x^2 + y^2)\mathbf{j} \\ \nabla f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) &= \sqrt{\pi}\mathbf{i} + \sqrt{\pi}\mathbf{j}. \end{aligned}$$

The direction for which the directional derivative is maximized is then $\boxed{\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}}.$

(b) The maximum value of the directional derivative is

$$\left\| \nabla f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) \right\| = \sqrt{\pi + \pi} = \boxed{\sqrt{2\pi}}$$

and the minimum value of the directional derivative is

$$-\left\| \nabla f\left(\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{8}}\right) \right\| = \boxed{-\sqrt{2\pi}}.$$

8. (a) We recall that the direction in which the directional derivative is maximized is the one in the direction of the gradient at the given point. Also, the maximum value of the directional derivative is the magnitude

of the gradient, and the minimum value is the opposite of the magnitude of the gradient. The gradient of f (see Problem 2 above) and its value at the given point are

$$\begin{aligned}\nabla f(x, y) &= \frac{4xy^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2} \mathbf{j} \\ \nabla f(1, 1) &= \mathbf{i} - \mathbf{j}.\end{aligned}$$

The direction for which the directional derivative is maximized is then $\boxed{\frac{\sqrt{2}}{2} \mathbf{i} - \frac{\sqrt{2}}{2} \mathbf{j}}$.

(b) The maximum value of the directional derivative is

$$\|\nabla f(1, 1)\| = \sqrt{1 + 1} = \boxed{\sqrt{2}}$$

and the minimum value of the directional derivative is

$$-\|\nabla f(1, 1)\| = \boxed{-\sqrt{2}}.$$

9. (a) We recall that the direction in which the directional derivative is maximized is the one in the direction of the gradient at the given point. Also, the maximum value of the directional derivative is the magnitude of the gradient, and the minimum value is the opposite of the magnitude of the gradient. The gradient of f (see Problem 3 above) and its value at the given point are

$$\begin{aligned}\nabla f(x, y) &= \frac{y}{\sqrt{1 - x^2}} \mathbf{i} + \sin^{-1}(x) \mathbf{j} \\ \nabla f(0, 1) &= \mathbf{i}.\end{aligned}$$

The direction for which the directional derivative is maximized is then $\boxed{\mathbf{i}}$.

(b) The maximum value of the directional derivative is

$$\|\nabla f(0, 1)\| = \boxed{1}$$

and the minimum value of the directional derivative is

$$-\|\nabla f(0, 1)\| = \boxed{-1}.$$

10. (a) To use the theorem from this section with $F(x, y, z) = x^2 - y^2 + z^2 - 4 = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 2x & F_x(-1, 1, 2) = -2 \\ F_y(x, y, z) = -2y & F_y(-1, 1, 2) = -2 \\ F_z(x, y, z) = 2z & F_z(-1, 1, 2) = 4 \end{array}$$

Then an equation of the tangent plane is

$$\begin{aligned}-2(x + 1) - 2(y - 1) + 4(z - 2) &= 0 \\ \boxed{-2x - 2y + 4z} &= 8.\end{aligned}$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x + 1}{-2} = \frac{y - 1}{-2} = \frac{z - 2}{4}}.$$

11. (a) To use the theorem from this section with $F(x, y, z) = 2x^2 + y^2 - z = 0$, we begin by finding the partial derivatives of F and their values at the given point.

$$\begin{array}{ll} F_x(x, y, z) = 4x & F_x(1, 0, 2) = 4 \\ F_y(x, y, z) = 2y & F_y(1, 0, 2) = 0 \\ F_z(x, y, z) = -1 & F_z(1, 0, 2) = -1 \end{array}$$

Then an equation of the tangent plane is

$$4(x - 1) + 0(y - 0) - 1(z - 2) = 0$$

$$\boxed{4x - z = 2}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x - 1}{4} = \frac{z - 2}{-1}; y = 0}.$$

12. (a) To use the theorem from this section with $F(x, y, z) = y \sin^{-1} x - z = 0$, we begin by finding the partial derivatives of F and their values at the point $(0, 1, 0)$.

$$\begin{array}{l} F_x(x, y, z) = \frac{y}{\sqrt{1 - x^2}} \\ F_x(0, 1, 0) = 1 \\ F_y(x, y, z) = \sin^{-1} x \\ F_y(0, 1, 0) = 0 \\ F_z(x, y, z) = -1 \\ F_z(0, 1, 0) = -1 \end{array}$$

Then an equation of the tangent plane is

$$(x - 0) + 0(y - 1) - (z - 0) = 0$$

$$\boxed{x - z = 0}.$$

(b) Symmetric equations of the normal line are

$$\boxed{\frac{x}{1} = \frac{z}{-1}; y = 1}.$$

13. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x + y + 6 = 0 \\ f_y(x, y) = x + 2y = 0 \end{cases}$$

From the second equation, $x = -2y$, so substituting this into the first equation, we have

$$\begin{array}{rcl} 2(-2y) + y + 6 & = & 0 \\ y & = & 2. \end{array}$$

The only solution is then $x = -4, y = 2$, so the only critical point is $\boxed{(-4, 2)}$.

14. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3x^2 + 3y = 0 \\ f_y(x, y) = -3y^2 + 3x = 0 \end{cases}$$

From the second equation, $x = y^2$, so substituting this into the first equation, we have

$$\begin{aligned} 3(y^2)^2 + 3y &= 0 \\ y(y^3 + 1) &= 0 \\ y &= 0, -1. \end{aligned}$$

The critical points are then $\boxed{(0, 0)}$ and $\boxed{(1, -1)}$.

15. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = y = 0 \\ f_y(x, y) = x = 0 \end{cases}$$

The only critical point is then $\boxed{(0, 0)}$.

16. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = e^{xy} + xy e^{xy} = 0 \\ f_y(x, y) = x^2 e^{xy} = 0 \end{cases}$$

Since e^{xy} is never equal to 0, the second equation tells us that $x = 0$. From the first equation, we have $1 + xy = 0$, but this cannot hold if $x = 0$. This means there are $\boxed{\text{no critical points}}$.

17. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = \cos x = 0 \\ f_y(x, y) = \cos y = 0 \end{cases}$$

The critical points are then $\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)$ for any integers m, n . The second-order partial derivatives and their values at the critical points are as follows.

$$\begin{aligned} f_{xx}(x, y) &= -\sin x & f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) &= 1 \text{ if } m \text{ is odd} & f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) &= -1 \text{ if } m \text{ is even} \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & f_{xy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) &= 0 \\ f_{yy}(x, y) &= -\sin y & f_{yy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) &= 1 \text{ if } n \text{ is odd} & f_{yy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) &= -1 \text{ if } n \text{ is even} \end{aligned}$$

If m and n are odd, then

$$f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) \cdot f_{yy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) - \left(f_{xy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)\right)^2 = 1 > 0$$

and $f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) > 0$ so $\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)$ is a local minimum. The local minimum is

$$\boxed{f\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) = -2 \text{ for } m \text{ and } n \text{ odd}}.$$

If m and n are even, then

$$f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) \cdot f_{yy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) - \left(f_{xy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)\right)^2 = 1 > 0$$

and $f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) < 0$ so $\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)$ is a local maximum. The local maximum is

$$\boxed{f\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) = 2 \text{ for } m \text{ and } n \text{ even}}.$$

If exactly one of m or n is odd and the other is even, then

$$f_{xx}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) \cdot f_{yy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right) - \left(f_{xy}\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi\right)\right)^2 = -1 < 0, \text{ so}$$

$$\boxed{\left(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi, 0\right) \text{ is a saddle point when exactly one of } m \text{ or } n \text{ is odd}}.$$

18. The critical points are solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x = 0 \\ f_y(x, y) = -9 + 2y = 0 \end{cases}$$

The only critical point is then $\left(0, \frac{9}{2}\right)$. The second-order partial derivatives and their values at the critical point are as follows.

$$\begin{aligned} f_{xx}(x, y) &= 2 & A = f_{xx}\left(0, \frac{9}{2}\right) &= 2 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 0 & B = f_{xy}\left(0, \frac{9}{2}\right) &= 0 \\ f_{yy}(x, y) &= 2 & C = f_{yy}\left(0, \frac{9}{2}\right) &= 2 \\ AC - B^2 &= 4 \end{aligned}$$

Since $AC - B^2 > 0$ and $A > 0$, the critical point $\left(0, \frac{9}{2}\right)$ is a local minimum. The local minimum is

$$\boxed{f\left(0, \frac{9}{2}\right) = -\frac{81}{4}}.$$

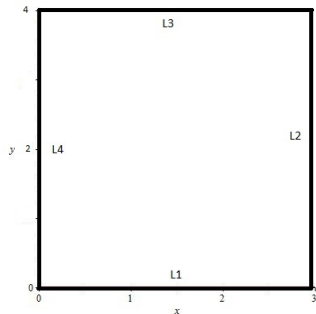
19. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 2x - 2y = 0 \\ f_y(x, y) = -2x + 2 = 0 \end{cases}$$

The only critical point is then $(1, 1)$. The value of f at this point is

$$f(1, 1) = 1.$$

The boundary of D is made up of the four line segments L_1 , L_2 , L_3 , and L_4 as drawn in the following picture.



On L_1 : $y = 0$ and $0 \leq x \leq 3$. The function $f(x, y) = f(x, 0) = x^2$ is increasing on $0 \leq x \leq 3$ and its extreme values are $f(0, 0) = 0$ and $f(3, 0) = 9$.

On L_2 : $x = 3$ and $0 \leq y \leq 4$. The function $f(x, y) = f(3, y) = 9 - 4y$ is decreasing on $0 \leq y \leq 4$ and its extreme values are $f(3, 4) = -7$ and $f(3, 0) = 9$.

On L_3 : $y = 4$ and $0 \leq x \leq 3$. The function $f(x, y) = f(x, 4) = x^2 - 8x + 8$ can be thought of as a function of a single variable, namely $g(x) = x^2 - 8x + 8$. The extrema of this function can be found by finding the critical number(s) of g , namely where $g'(x) = 2x - 8 = 0$, or $x = 4$. Since this value is not in the interval $0 \leq x \leq 3$, the extreme values occur at the endpoints $x = 0$ and $x = 3$. The extreme values of f on L_3 are $g(0) = f(0, 4) = 8$ and $g(3) = f(3, 4) = -7$.

On L_4 : $x = 0$ and $0 \leq y \leq 4$. Then function $f(x, y) = f(0, y) = 2y$ is increasing on $0 \leq y \leq 4$ and its extreme values are $f(0, 0) = 0$ and $f(0, 4) = 8$.

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is -7 and the absolute maximum is 9 .

20. The function f is continuous on its domain, which is a closed, bounded set. To find the absolute extrema, we first find the critical points of f , namely the solutions of the system of equations

$$\begin{cases} f_x(x, y) = 3y^2 = 0 \\ f_y(x, y) = 6xy = 0 \end{cases}$$

From the first equation, we see that $y = 0$, and x can be any value. So $(a, 0)$ is a critical point for any value $-3 \leq a \leq 3$ since the point must lie in D . The value of f at a critical point $(a, 0)$ is $f(a, 0) = 0$.

The boundary of D is a circle of radius 3 centered at the origin which can be parameterized by $x(t) = 3 \cos t$ and $y = 3 \sin t$ for $0 \leq t \leq 2\pi$. On the boundary

$$z = f(x, y) = f(3 \cos t, 3 \sin t) = 81 \cos t \sin^2 t.$$

Now we find where $z'(t) = 0$ on $0 \leq t \leq 2\pi$.

$$\begin{aligned} -81 \sin^3 t + 162 \cos^2 t \sin t &= 0 \\ 81 \sin t (-1 + 3 \cos^2 t) &= 0 \end{aligned}$$

$$t = 0, \pi, 2\pi, \cos^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi - \cos^{-1}\left(\frac{\sqrt{3}}{3}\right).$$

The values of f at these points are

$$\begin{aligned} f(3 \cos 0, 3 \sin 0) &= f(3 \cos \pi, 3 \sin \pi) = f(3 \cos 2\pi, 3 \sin 2\pi) = 0 \\ f\left(3 \cos \cos^{-1}\left(\frac{\sqrt{3}}{3}\right), 3 \sin \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) &= f\left(\sqrt{3}, \frac{\sqrt{6}}{3}\right) = 2\sqrt{3} \\ f\left(3 \cos\left(\pi - \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)\right), 3 \sin\left(\pi - \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)\right)\right) &= f\left(-\sqrt{3}, \frac{\sqrt{6}}{3}\right) = -2\sqrt{3} \end{aligned}$$

Comparing the values of f at the critical points and on the boundary, we see that the absolute minimum is $-2\sqrt{3}$

and the absolute maximum is $2\sqrt{3}$.

21. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 10x + y = 2\lambda \\ 6y + x = -\lambda \\ 2x - y - 20 = 0 \end{cases}$$

Solving for λ in the second equation, and plugging it into the first equation, we have

$$\begin{aligned} 10x + y &= 2(-6y - x) \\ y &= -\frac{12}{13}x. \end{aligned}$$

Plugging this into the third equation, we find that $2x + \frac{12}{13}x - 20 = 0$, or $x = \frac{130}{19}$. The only test point is

then $\left(\frac{130}{19}, -\frac{120}{19}\right)$. The minimum value is $f\left(\frac{130}{19}, -\frac{120}{19}\right) = \frac{5900}{19}$.

22. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} \sqrt{y} = 2\lambda \\ \frac{1}{2}xy^{-1/2} = \lambda \\ 2x + y - 3000 = 0 \end{cases}$$

Solving for λ in the first two equations and equating them yields

$$\begin{aligned} \frac{1}{2}y^{1/2} &= \frac{1}{2}xy^{-1/2} \\ y &= x. \end{aligned}$$

Then the third equation becomes

$$\begin{aligned} 2x + x - 3000 &= 0 \\ x &= 1000. \end{aligned}$$

The only test point is then $(1000, 1000)$. The minimum value is $f(1000, 1000) = 10000\sqrt{10}$.

23. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} 2x = 2\lambda \\ 2y = \lambda \\ 2x + y - 4 = 0 \end{cases}$$

Eliminating λ in the first two equations yields $x = 2y$. Plugging this into the third equation yields $4y + y - 4 = 0$, or $y = \frac{4}{5}$. The only test point is then $\left(\frac{8}{5}, \frac{4}{5}\right)$. The minimum value is $f\left(\frac{8}{5}, \frac{4}{5}\right) = \frac{16}{5}$.

24. The test points satisfy the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, namely

$$\begin{cases} y^2 = 2x\lambda \\ 2xy = 2y\lambda \\ x^2 + y^2 - 1 = 0 \end{cases}$$

From the second equation, we see that $2y(x - \lambda) = 0$, so that either $y = 0$, or $x = \lambda$. If $y = 0$, then the third equation implies that $x = \pm 1$, so two test points are $(1, 0)$ and $(-1, 0)$. If $x = \lambda$, then the first equation says that $y^2 = 2x^2$. Plugging this into the third equation, we find that $x^2 + 2x^2 - 1 = 0$, or $x = \pm \frac{\sqrt{3}}{3}$. Then using $y^2 = 2x^2$, we find the four test points

$$\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right).$$

The corresponding values are

$$\begin{aligned} f(1, 0) &= f(-1, 0) &= 0 \\ f\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right) &= f\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right) &= \frac{2\sqrt{3}}{9} \\ f\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right) &= f\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right) &= -\frac{2\sqrt{3}}{9} \end{aligned}$$

Then $-\frac{2\sqrt{3}}{9}$ is the minimum value and $\frac{2\sqrt{3}}{9}$ is the maximum value.

25. (a) The rate of change in temperature at $(0, 0)$ is $D_{\mathbf{u}}T(0, 0)$ where $\mathbf{u}$ is the unit vector in the direction of $4\mathbf{i} + 3\mathbf{j}$. That is, $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. To compute the directional derivative, we first compute the gradient of T .

$$\begin{aligned} T_x(x, y) &= e^y(\sin x + \sin y) + e^y(\cos x) \\ T_y(x, y) &= e^y \cos y, \end{aligned}$$

so

$$\nabla T(x, y) = (e^y(\sin x + \sin y) + e^y(\cos x))\mathbf{i} + e^y \cos y \mathbf{j}.$$

Then

$$\begin{aligned} D_{\mathbf{u}}T(0, 0) &= \nabla T(0, 0) \cdot \mathbf{u} = (\mathbf{i} + \mathbf{j}) \cdot \left(\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}\right) \\ &= \frac{7}{5}. \end{aligned}$$

The rate of change in temperature at $(0, 0)$ in the direction of $4\mathbf{i} + 3\mathbf{j}$ is $\frac{7}{5}^\circ\text{C}$.

(b) The greatest temperature increase at $(0, 0)$ occurs in the direction of the gradient at $(0, 0)$, namely

$$\nabla T(0, 0) = \mathbf{i} + \mathbf{j},$$

which is the same direction as the unit vector $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$.

(c) The greatest temperature decrease at $(0, 0)$ occurs in the opposite direction of the gradient at $(0, 0)$, namely

$$-\nabla T(0, 0) = -\mathbf{i} - \mathbf{j},$$

which is the same direction as the unit vector $-\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$.

(d) The rate of change in T at $(0, 0)$ remains the same for directions orthogonal to the gradient $\nabla T(0, 0) = \mathbf{i} + \mathbf{j}$. Those two directions are $\pm(\mathbf{i} - \mathbf{j})$, which are the same directions as the unit vectors $\pm\left(\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right)$.

26. (a) The profit function $P(x, y) = R(x, y) - C(x, y)$ is

$$P(x, y) = -3x^2 - 2y^2 + 2xy + 1800x.$$

The critical points are solutions of the system of equations

$$\begin{cases} P_x(x, y) = -6x + 2y + 1800 = 0 \\ f_y(x, y) = -4y + 2x = 0 \end{cases}$$

From the last equation, we see that $x = 2y$, so plugging this into the first equation, we have

$$\begin{aligned} -6(2y) + 2y + 1800 &= 0 \\ y &= 180, \end{aligned}$$

so $x = 360$. The second-order partial derivatives and their values at $(360, 180)$ are as follows.

$$\begin{aligned} P_{xx}(x, y) &= -6 & A &= P_{xx}(360, 180) = -6 \\ P_{xy}(x, y) &= P_{yx}(x, y) = 2 & B &= P_{xy}(360, 180) = 2 \\ P_{yy}(x, y) &= -4 & C &= P_{yy}(360, 180) = -4 \\ AC - B^2 &= 20 \end{aligned}$$

Since $AC - B^2 > 0$ and $A < 0$, the point $(360, 180)$ is a local maximum, which is then the absolute maximum based on the nature of the problem. When the number of units of $x = 360$ and the number of units of $y = 180$ the profit will be maximized.

(b) The maximum profit is $P(360, 180) = 324000$.

27. (a) Adding the total value of capital and labor must be \$51,000 which means $g(K, L) = 50K + 100L - 51000 = 0$. We now seek to maximize $P(K, L) = 10K^{0.3}L^{0.7}$ subject to the constraint curve g . Using Lagrange multipliers, the test points satisfy the system of equations $\nabla P(K, L) = \lambda \nabla g(K, L)$ and $g(K, L) = 0$, namely

$$\begin{cases} 3K^{-0.7}L^{0.7} = 50\lambda \\ 7K^{0.3}L^{-0.3} = 100\lambda \\ 50K + 100L - 51000 = 0 \end{cases}$$

Solving for λ in the first two equations and equating them, we have

$$\begin{aligned} \frac{3L^{0.7}}{50K^{0.7}} &= \frac{7K^{0.3}}{100L^{0.3}} \\ L &= \frac{7}{6}K. \end{aligned}$$

Plugging this into the third equation, we solve for K and get $K = 306$, which means $L = 357$. Total production is maximized when units of capital is 306 and units of labor is 357.

(b) Under these conditions, the maximum number of units of production that could be generated is $P(306, 357)$ ≈ 3408.7 .

28. If (x, y, z) is some point on the given surface that is also a vertex of the rectangular solid where each of $x, y, z > 0$, then the other eight vertices of the solid are $(\pm x, \pm y, \pm z)$. This means the length of the solid is $2x$, the width is $2y$, and the height is $2z$. So we are trying to maximize $f(x, y, z) = 8xyz$ subject to the constraint that $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 8yz = \frac{2x}{a^2}\lambda \\ 8xz = \frac{2y}{b^2}\lambda \\ 8xy = \frac{2z}{c^2}\lambda \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \end{cases}$$

Eliminating λ from the first three equations, we find that

$$\begin{aligned} \frac{8yza^2}{2x} &= \frac{8xzb^2}{2y} = \frac{8xyc^2}{2z} \\ \frac{yza^2}{x} &= \frac{xzb^2}{y} = \frac{xyz^2}{z} \end{aligned}$$

Cross multiplying in the first equality, we have $y^2za^2 = x^2zb^2$, or $x = \frac{a}{b}y$ since $x, y > 0$. Similarly, the second equality yields $z = \frac{c}{b}y$. Plugging these into the fourth equation, we have

$$\begin{aligned} \frac{\frac{a^2}{b^2}y^2}{a^2} + \frac{y^2}{b^2} + \frac{\frac{c^2}{b^2}y^2}{c^2} - 1 &= 0 \\ \frac{3y^2}{b^2} &= 1 \\ y &= \frac{b}{\sqrt{3}}. \end{aligned}$$

So $x = \frac{a}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$.

The maximum volume is then $f\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}abc$.

29. We will use Lagrange multipliers to minimize the *square* of the distance between a point (x, y, z) on the surface and the origin. That is, we will minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint curve $g(x, y, z) = xyz - 1 = 0$. The test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, $g(x, y, z) = 0$, namely

$$\begin{cases} 2x = yz\lambda \\ 2y = xz\lambda \\ 2z = xy\lambda \\ xyz - 1 = 0 \end{cases}$$

Noting that $x, y, z \neq 0$ for otherwise the fourth equation could never hold, we solve for λ in the first three equations and equate them

$$\frac{2x}{yz} = \frac{2y}{xz} = \frac{2z}{xy}.$$

Cross multiplying the first equality results in $2x^2z = 2y^2z$, or $x = \pm y$. Similarly, the second equality yields $z = \pm y$. Plugging these into the fourth equation, we find that $(\pm y) \cdot y \cdot (\pm y) = 1$, or $\pm y^3 = 1$. This equation can only hold if $y = \pm 1$. This means the points on the surface closest to the origin are

$$\boxed{(1, 1, 1), (-1, 1, -1), (-1, -1, 1), (1, -1, -1)}.$$

[Note, for example, that the point $(1, 1, -1)$ is not one of the points since this point is not on the surface $xyz = 1$.]

30. Let (x, y, z) be the point on the plane that is also the vertex of the rectangular solid where $x, y, z > 0$. The volume of the rectangular solid is then $f(x, y, z) = xyz$. So we are trying to maximize $f(x, y, z) = xyz$ subject to the constraint that $g(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$. Using Lagrange multipliers, the test points satisfy the system of equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} yz = \frac{1}{a}\lambda \\ xz = \frac{1}{b}\lambda \\ xy = \frac{1}{c}\lambda \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \end{cases}$$

Eliminating λ from the first three equations, we find that $ayz = bxz = cxy$. In the first equality, we have $x = \frac{a}{b}y$ since $x, y > 0$. Similarly, the second equality yields $z = \frac{c}{b}y$. Plugging these into the fourth equation, we have

$$\begin{aligned} \frac{\frac{a}{b}y}{a} + \frac{y}{b} + \frac{\frac{c}{b}y}{c} - 1 &= 0 \\ \frac{3y}{b} &= 1 \\ y &= \frac{b}{3}. \end{aligned}$$

So $x = \frac{a}{3}$ and $z = \frac{c}{3}$.

The maximum volume is then $f\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = \boxed{\frac{abc}{27}}$.

31. If x, y, z represent the length, width, and height, respectively, of the box, then the cost function we seek to minimize is $C(x, y, z) = 5xy + 1 \cdot (xy + 2xz + 2yz) = 6xy + 2xz + 2yz$ subject to the constraint $g(x, y, z) = xyz - V = 0$. We use Lagrange multipliers. The test points satisfy the system of equations $\nabla C(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, namely

$$\begin{cases} 6y + 2z = \lambda yz \\ 6x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz - V = 0 \end{cases}$$

Eliminating λ from the first three equations, we have

$$\frac{6y + 2z}{yz} = \frac{6x + 2z}{xz} = \frac{2x + 2y}{xy}.$$

Cross multiplying the first equality, we have $6xyz + 2xz^2 = 6xyz + 2yz^2$, or $x = y$. Similarly, cross multiplying the second equality, we have $6x^2y + 2xyz = 2x^2z + 2xyz$, or $z = 3y$. Plugging these into the fourth equation,

we have $3y^3 - V = 0$, or $y = \sqrt[3]{\frac{V}{3}}$. Then the box with square base and dimensions $\sqrt[3]{\frac{V}{3}} \times \sqrt[3]{\frac{V}{3}} \times 3\sqrt[3]{\frac{V}{3}}$ will minimize the cost of the material.

32. We introduce two Lagrange multipliers λ_1 and λ_2 and solve the system

$$\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z), \quad g(x, y, z) = x^2 + y^2 - 1 = 0, \quad h(x, y, z) = y - 3z = 0.$$

$$\begin{cases} yz = 2x\lambda_1 \\ xz = 2y\lambda_1 + \lambda_2 \\ xy = -3\lambda_2 \\ x^2 + y^2 - 1 = 0 \\ y - 3z = 0 \end{cases}$$

If $x = 0$, then from the fourth equation, we see that $y = \pm 1$ and then from the first equation, we see that $z = 0$. But this cannot be since $y = 3z$ from the fifth equation. This means $x \neq 0$. Solving for λ_1 in the first equation and λ_2 in the third equation, and plugging them into the second equation, we have

$$\begin{aligned} xz &= 2y \left(\frac{yz}{2x} \right) - \frac{xy}{3} \\ x^2 z &= y^2 z - \frac{x^2 y}{3} \\ x^2 z &= 9z^3 - x^2 z && \text{using } y = 3z \text{ from the fifth equation} \\ z(2x^2 - 9z^2) &= 0. \end{aligned}$$

If $z = 0$, then $y = 0$ too from equation five, which means $x = \pm 1$ from the fourth equation. So two test points are $(\pm 1, 0, 0)$. Now if $z \neq 0$, then it must be that $2x^2 = 9z^2$, or $z = \pm \frac{\sqrt{2}}{3}x$. From the fifth equation, $y = 3z = \pm\sqrt{2}x$. Plugging $y = \pm\sqrt{2}x$ into the fourth equation, we have

$$\begin{aligned} x^2 + \left(\pm\sqrt{2}x \right)^2 &= 1 \\ 3x^2 &= 1 \\ x &= \pm \frac{1}{\sqrt{3}}. \end{aligned}$$

This gives us four more test points

$$\left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{6}}{9} \right), \left(\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{6}}{9} \right), \left(-\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{6}}{9} \right), \left(-\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{6}}{9} \right).$$

Plugging each of the test points into f , we find that

$$\begin{aligned} f(1, 0, 0) &= f(-1, 0, 0) = 0 \\ f\left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{6}}{9}\right) &= f\left(\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{6}}{9}\right) = \frac{2\sqrt{3}}{27} \\ f\left(-\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{6}}{9}\right) &= f\left(-\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{6}}{9}\right) = -\frac{2\sqrt{3}}{27} \end{aligned}$$

Then $\frac{2\sqrt{3}}{27}$ is the absolute maximum and $-\frac{2\sqrt{3}}{27}$ is the absolute minimum.

Multiple Integrals

14.1 The Double Integral over a Rectangular Region

Concepts and Vocabulary

1. True. First, we integrate partially with respect to y ; since x is treated as a constant with respect to y , by properties of definite integrals we can factor it out:

$$\int_0^2 \left[\int_0^1 xy^2 dy \right] dx = \int_0^2 x \left[\int_0^1 y^2 dy \right] dx.$$

Now in the integral with respect to x , the value of $\left[\int_0^1 y^2 dy \right]$ is a numerical constant, which can be factored out, yielding

$$\int_0^2 x \left[\int_0^1 y^2 dy \right] dx = \left[\int_0^2 x dx \right] \cdot \left[\int_0^1 y^2 dy \right].$$

2. False. If a function f is continuous on a closed rectangular region R defined by $a \leq x \leq b$ and $c \leq y \leq d$, then according to Fubini's Theorem: $\int_c^d \left[\int_a^b f(x, y) dx \right] dy = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$.
3. $\int_1^4 x^2 \sqrt{y} dy$ is a partial integral with respect to y , whose output is (d) a function of x .
4. True. According to the Existence of the Double Integral Theorem, for a function $z = f(x, y)$ that is continuous on a closed rectangular region R , the double integral $\iint_R f(x, y) dA$ exists.

Skill Building

5. Let $f(x, y) = x(3 - y)$, and let R be the closed rectangular region defined by $0 \leq x \leq 3$, $0 \leq y \leq 3$. Partition R into nine congruent subsquares with sides $\Delta x_i = 1$, $i = 1, 2, 3$, and $\Delta y_j = 1$, $j = 1, 2, 3$, each with area $\Delta A_k = 1$, as shown in the figure in Problem 5.
 - (a) Choose (u_k, v_k) , $k = 1, 2, \dots, 9$, as the lower right corner of the k th subsquare: $(u_1, v_1) = (1, 0)$, $(u_2, v_2) = (2, 0)$, $(u_3, v_3) = (3, 0)$, $(u_4, v_4) = (1, 1)$, $(u_5, v_5) = (2, 1)$, $(u_6, v_6) = (3, 1)$, $(u_7, v_7) = (1, 2)$, $(u_8, v_8) = (2, 2)$, and $(u_9, v_9) = (3, 2)$. Then the corresponding Riemann sum is:

$$\begin{aligned} \sum_{k=1}^9 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_9, v_9) \Delta A_9 \\ &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_9, v_9)] \cdot 1 \\ &= [f(1, 0) + f(2, 0) + f(3, 0) + f(1, 1) + f(2, 1) \\ &\quad + f(3, 1) + f(1, 2) + f(2, 2) + f(3, 2)] \cdot 1 \\ &= (3 + 6 + 9 + 2 + 4 + 6 + 1 + 2 + 3) \cdot 1 = \boxed{36}. \end{aligned}$$

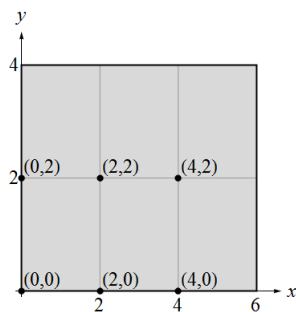
- (b) Choose (u_k, v_k) , $k = 1, 2, \dots, 9$, as the upper left corner of the k th subsquare: $(u_1, v_1) = (0, 1)$, $(u_2, v_2) = (1, 1)$, $(u_3, v_3) = (2, 1)$, $(u_4, v_4) = (0, 2)$, $(u_5, v_5) = (1, 2)$, $(u_6, v_6) = (2, 2)$, $(u_7, v_7) = (0, 3)$, $(u_8, v_8) = (1, 3)$, and $(u_9, v_9) = (2, 3)$. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^9 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_9, v_9) \Delta A_9 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_9, v_9)] \cdot 1 \\
 &= [f(0, 1) + f(1, 1) + f(2, 1) + f(0, 2) + f(1, 2) \\
 &\quad + f(2, 2) + f(0, 3) + f(1, 3) + f(2, 3)] \cdot 1 \\
 &= (0 + 2 + 4 + 0 + 1 + 2 + 0 + 0 + 0) \cdot 1 = \boxed{9}.
 \end{aligned}$$

6. Let $f(x, y) = 3xy^2$, and let R be the closed rectangular region defined by $0 \leq x \leq 6$, $0 \leq y \leq 4$. Partition R into six congruent subsquares with sides $\Delta x_i = 2$, $i = 1, 2, 3$, and $\Delta y_j = 2$, $j = 1, 2$, each with area $\Delta A_k = 4$.

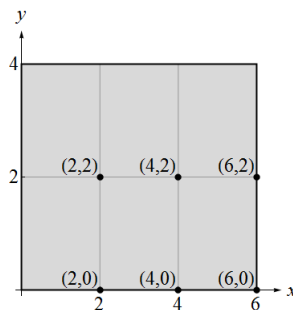
- (a) Choose (u_k, v_k) , $k = 1, 2, \dots, 6$, as the lower left corner of the k th subsquare: $(u_1, v_1) = (0, 0)$, $(u_2, v_2) = (2, 0)$, $(u_3, v_3) = (4, 0)$, $(u_4, v_4) = (0, 2)$, $(u_5, v_5) = (2, 2)$, and $(u_6, v_6) = (4, 2)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^6 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_6, v_6) \Delta A_6 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_6, v_6)] \cdot 4 \\
 &= [f(0, 0) + f(2, 0) + f(4, 0) + f(0, 2) + f(2, 2) + f(4, 2)] \cdot 4 \\
 &= (0 + 0 + 0 + 0 + 24 + 48) \cdot 4 = \boxed{288}.
 \end{aligned}$$



- (b) Choose (u_k, v_k) , $k = 1, 2, \dots, 6$, as the lower right corner of the k th subsquare: $(u_1, v_1) = (2, 0)$, $(u_2, v_2) = (4, 0)$, $(u_3, v_3) = (6, 0)$, $(u_4, v_4) = (2, 2)$, $(u_5, v_5) = (4, 2)$, and $(u_6, v_6) = (6, 2)$. See the figure below. Then the corresponding Riemann sum is:

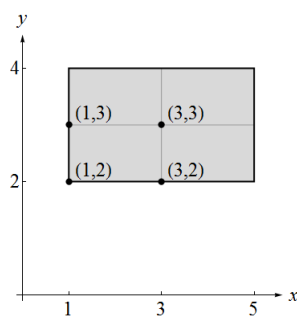
$$\begin{aligned}
 \sum_{k=1}^6 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_6, v_6) \Delta A_6 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_6, v_6)] \cdot 4 \\
 &= [f(2, 0) + f(4, 0) + f(6, 0) + f(2, 2) + f(4, 2) + f(6, 2)] \cdot 4 \\
 &= (0 + 0 + 0 + 24 + 48 + 72) \cdot 4 = \boxed{576}.
 \end{aligned}$$



7. Let $f(x, y) = x^2 + y$, and let R be the closed rectangular region defined by $1 \leq x \leq 5$, $2 \leq y \leq 4$. Partition R into four congruent subrectangles with sides $\Delta x_i = 2$, $i = 1, 2$, and $\Delta y_j = 1$, $j = 1, 2$, each with area $\Delta A_k = 2$.

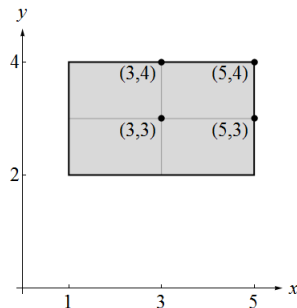
- (a) Choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the lower left corner of the k th subrectangle: $(u_1, v_1) = (1, 2)$, $(u_2, v_2) = (3, 2)$, $(u_3, v_3) = (1, 3)$, and $(u_4, v_4) = (3, 3)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + f(u_3, v_3) \Delta A_3 + f(u_4, v_4) \Delta A_4 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + f(u_3, v_3) + f(u_4, v_4)] \cdot 2 \\
 &= [f(1, 2) + f(3, 2) + f(1, 3) + f(3, 3)] \cdot 2 \\
 &= (3 + 11 + 4 + 12) \cdot 2 = \boxed{60}.
 \end{aligned}$$



- (b) Choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the upper right corner of the k th subrectangle: $(u_1, v_1) = (3, 3)$, $(u_2, v_2) = (5, 3)$, $(u_3, v_3) = (3, 4)$, and $(u_4, v_4) = (5, 4)$. See the figure below. Then the corresponding Riemann sum is:

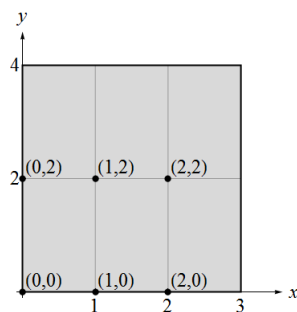
$$\begin{aligned}
 \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + f(u_3, v_3) \Delta A_3 + f(u_4, v_4) \Delta A_4 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + f(u_3, v_3) + f(u_4, v_4)] \cdot 2 \\
 &= [f(3, 3) + f(5, 3) + f(3, 4) + f(5, 4)] \cdot 2 \\
 &= (12 + 28 + 13 + 29) \cdot 2 = \boxed{164}.
 \end{aligned}$$



8. Let $f(x, y) = x(1 - y)$, and let R be the closed rectangular region defined by $0 \leq x \leq 3$, $0 \leq y \leq 4$. Partition R into six congruent subrectangles with sides $\Delta x_i = 1$, $i = 1, 2, 3$, and $\Delta y_j = 2$, $j = 1, 2$, each with area $\Delta A_k = 2$.

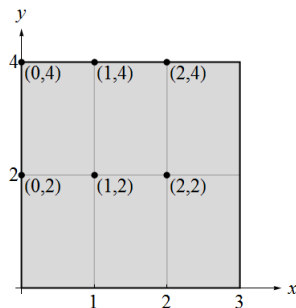
- (a) Choose (u_k, v_k) , $k = 1, 2, \dots, 6$, as the lower left corner of the k th subrectangle: $(u_1, v_1) = (0, 0)$, $(u_2, v_2) = (1, 0)$, $(u_3, v_3) = (2, 0)$, $(u_4, v_4) = (0, 2)$, $(u_5, v_5) = (1, 2)$, and $(u_6, v_6) = (2, 2)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^6 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_6, v_6) \Delta A_6 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_6, v_6)] \cdot 2 \\
 &= [f(0, 0) + f(1, 0) + f(2, 0) + f(0, 2) + f(1, 2) + f(2, 2)] \cdot 2 \\
 &= (0 + 1 + 2 + 0 - 1 - 2) \cdot 2 = \boxed{0}.
 \end{aligned}$$



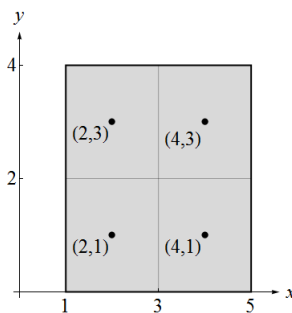
- (b) Choose (u_k, v_k) , $k = 1, 2, \dots, 6$, as the upper left corner of the k th subrectangle: $(u_1, v_1) = (0, 2)$, $(u_2, v_2) = (1, 2)$, $(u_3, v_3) = (2, 2)$, $(u_4, v_4) = (0, 4)$, $(u_5, v_5) = (1, 4)$, and $(u_6, v_6) = (2, 4)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^6 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_6, v_6) \Delta A_6 \\
 &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_6, v_6)] \cdot 2 \\
 &= [f(0, 2) + f(1, 2) + f(2, 2) + f(0, 4) + f(1, 4) + f(2, 4)] \cdot 2 \\
 &= (0 - 1 - 2 + 0 - 3 - 6) \cdot 2 = \boxed{-24}.
 \end{aligned}$$



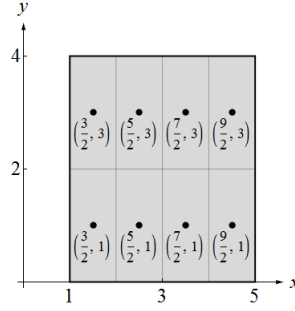
9. Let $f(x, y) = x^2y$, and let R be the closed rectangular region defined by $1 \leq x \leq 5$, $0 \leq y \leq 4$, as given in Example 1. Partition R into four congruent subrectangles with sides $\Delta x_i = 2$, $i = 1, 2$, and $\Delta y_j = 2$, $j = 1, 2$, each with area $\Delta A_k = 4$. Choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the center of the k th subsquare: $(u_1, v_1) = (2, 1)$, $(u_2, v_2) = (4, 1)$, $(u_3, v_3) = (2, 3)$, and $(u_4, v_4) = (4, 3)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned} \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(2, 1) \cdot 4 + f(4, 1) \cdot 4 + f(2, 3) \cdot 4 + f(4, 3) \cdot 4 \\ &= [f(2, 1) + f(4, 1) + f(2, 3) + f(4, 3)] \cdot 4 \\ &= (4 + 16 + 12 + 48) \cdot 4 = \boxed{320}. \end{aligned}$$



10. Let $f(x, y) = x^2y$, and let R be the closed rectangular region defined by $1 \leq x \leq 5$, $0 \leq y \leq 4$, as given in Example 1. Partition R into eight congruent subrectangles with sides $\Delta x_i = 1$, $i = 1, 2, 3, 4$, and $\Delta y_j = 2$, $j = 1, 2$, each with area $\Delta A_k = 2$. Choose (u_k, v_k) , $k = 1, 2, \dots, 8$, as the center of the k th subrectangle: $(u_1, v_1) = \left(\frac{3}{2}, 1\right)$, $(u_2, v_2) = \left(\frac{5}{2}, 1\right)$, $(u_3, v_3) = \left(\frac{7}{2}, 1\right)$, $(u_4, v_4) = \left(\frac{9}{2}, 1\right)$, $(u_5, v_5) = \left(\frac{3}{2}, 3\right)$, $(u_6, v_6) = \left(\frac{5}{2}, 3\right)$, $(u_7, v_7) = \left(\frac{7}{2}, 3\right)$, and $(u_8, v_8) = \left(\frac{9}{2}, 3\right)$. See the figure below. Then the corresponding Riemann sum is:

$$\begin{aligned} \sum_{k=1}^8 f(u_k, v_k) \Delta A_k &= f(u_1, v_1) \Delta A_1 + f(u_2, v_2) \Delta A_2 + \cdots + f(u_8, v_8) \Delta A_8 \\ &= [f(u_1, v_1) + f(u_2, v_2) + \cdots + f(u_8, v_8)] \cdot 2 \\ &= \left[f\left(\frac{3}{2}, 1\right) + f\left(\frac{5}{2}, 1\right) + f\left(\frac{7}{2}, 1\right) + f\left(\frac{9}{2}, 1\right) \right. \\ &\quad \left. + f\left(\frac{3}{2}, 3\right) + f\left(\frac{5}{2}, 3\right) + f\left(\frac{7}{2}, 3\right) + f\left(\frac{9}{2}, 3\right) \right] \cdot 2 \\ &= \left(\frac{9}{4} + \frac{25}{4} + \frac{49}{4} + \frac{81}{4} + \frac{27}{4} + \frac{75}{4} + \frac{147}{4} + \frac{243}{4} \right) \cdot 2 = \boxed{328}. \end{aligned}$$



11. We integrate partially with respect to y , treating x as a constant:

$$\int_1^e \frac{x}{y} dy = x \int_1^e \frac{1}{y} dy = x[\ln |y|]_1^e = x(\ln e - \ln 1) = x(1 - 0) = \boxed{x}.$$

12. We integrate partially with respect to x , treating y as a constant:

$$\int_0^2 \frac{x}{y} dx = \frac{1}{y} \int_0^2 x dx = \frac{1}{y} \left[\frac{x^2}{2} \right]_0^2 = \frac{1}{y}(2 - 0) = \boxed{\frac{2}{y}}.$$

13. We integrate partially with respect to x , treating y as a constant:

$$\int_0^{\pi/2} x \sin y dx = \sin y \int_0^{\pi/2} x dx = \sin y \left[\frac{x^2}{2} \right]_0^{\pi/2} = \sin y \left(\frac{\pi^2}{8} - 0 \right) = \boxed{\frac{\pi^2}{8} \sin y}.$$

14. We integrate partially with respect to y , treating x as a constant:

$$\int_0^{\pi/2} x \sin y dy = x \int_0^{\pi/2} \sin y dy = x[-\cos y]_0^{\pi/2} = x \left(-\cos \frac{\pi}{2} + \cos 0 \right) = x(0 + 1) = \boxed{x}.$$

15. We integrate partially with respect to x , treating y as a constant:

$$\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = \boxed{e - 1}.$$

16. We integrate partially with respect to y , treating x as a constant:

$$\int_0^1 e^x dy = e^x \int_0^1 1 dy = e^x [y]_0^1 = e^x(1 - 0) = \boxed{e^x}.$$

17. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\int_0^1 \left[\int_0^2 x^2 y dy \right] dx = \int_0^1 x^2 \left[\frac{y^2}{2} \right]_0^2 dx = \int_0^1 x^2(2 - 0) dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3} \right]_0^1 = \frac{2}{3} - 0 = \boxed{\frac{2}{3}}.$$

18. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_0^2 \left[\int_0^4 x^2 y dx \right] dy = \int_0^2 y \left[\frac{x^3}{3} \right]_0^4 dy = \int_0^2 y \left(\frac{64}{3} - 0 \right) dy = \int_0^2 \frac{64}{3} y dy = \left[\frac{32y^2}{3} \right]_0^2 = \frac{128}{3} - 0 = \boxed{\frac{128}{3}}.$$

19. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\int_0^3 \left[\int_0^2 3xy \, dy \right] dx = \int_0^3 3x \left[\frac{y^2}{2} \right]_0^2 dx = \int_0^3 3x(2-0) \, dx = \int_0^3 6x \, dx = [3x^2]_0^3 = 27 - 0 = \boxed{27}.$$

20. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_0^2 \left[\int_0^1 3xy \, dx \right] dy = \int_0^2 3y \left[\frac{x^2}{2} \right]_0^1 dy = \int_0^2 3y \left(\frac{1}{2} - 0 \right) dy = \int_0^2 \frac{3}{2}y \, dy = \left[\frac{3y^2}{4} \right]_0^2 = 3 - 0 = \boxed{3}.$$

21. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_{-1}^1 \left[\int_0^1 3x^2y^2 \, dx \right] dy = \int_{-1}^1 y^2 [x^3]_0^1 dy = \int_{-1}^1 y^2(1-0) \, dy = \int_{-1}^1 y^2 \, dy = \left[\frac{y^3}{3} \right]_{-1}^1 = \frac{1}{3} - \left(-\frac{1}{3} \right) = \boxed{\frac{2}{3}}.$$

22. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\int_0^1 \left[\int_0^2 3x^2y^2 \, dy \right] dx = \int_0^1 x^2 [y^3]_0^2 dx = \int_0^1 x^2(8-0) \, dx = \int_0^1 8x^2 \, dx = \left[\frac{8}{3}x^3 \right]_0^1 = \frac{8}{3} - 0 = \boxed{\frac{8}{3}}.$$

23. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_0^2 \left[\int_{-1}^1 2xy^2 \, dx \right] dy = \int_0^2 y^2 [x^2]_{-1}^1 dy = \int_0^2 y^2(1-1) \, dy = \int_0^2 0 \, dy = \boxed{0}.$$

24. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\int_{-1}^1 \left[\int_1^2 2xy^2 \, dy \right] dx = \int_{-1}^1 2x \left[\frac{y^3}{3} \right]_1^2 dx = \int_{-1}^1 2x \left(\frac{8}{3} - \frac{1}{3} \right) dx = \int_{-1}^1 \frac{14}{3}x \, dx = \left[\frac{7}{3}x^2 \right]_{-1}^1 = \frac{7}{3} - \frac{7}{3} = \boxed{0}.$$

25. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\begin{aligned} \int_0^{\pi/4} \left[\int_0^2 x \cos y \, dx \right] dy &= \int_0^{\pi/4} \cos y \left[\frac{x^2}{2} \right]_0^2 dy = \int_0^{\pi/4} \cos y \cdot (2-0) \, dy \\ &= \int_0^{\pi/4} 2 \cos y \, dy = [2 \sin y]_0^{\pi/4} = 2 \sin \frac{\pi}{4} - 2 \sin 0 = 2 \cdot \frac{\sqrt{2}}{2} - 0 = \boxed{\sqrt{2}}. \end{aligned}$$

26. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\begin{aligned} \int_0^2 \left[\int_0^{\pi/3} x \cos y \, dy \right] dx &= \int_0^2 x [\sin y]_0^{\pi/3} dx = \int_0^2 x \left(\sin \frac{\pi}{3} - \sin 0 \right) dx \\ &= \int_0^2 x \left(\frac{\sqrt{3}}{2} - 0 \right) dx = \int_0^2 \frac{\sqrt{3}}{2}x \, dx = \left[\frac{\sqrt{3}}{4}x^2 \right]_0^2 = \sqrt{3} - 0 = \boxed{\sqrt{3}}. \end{aligned}$$

27. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\begin{aligned}\int_0^3 \left[\int_0^{\pi/3} (4x-3)^2 \sin y \, dy \right] dx &= \int_0^3 (4x-3)^2 [-\cos y]_0^{\pi/3} dx = \int_0^3 (4x-3)^2 \left(-\cos \frac{\pi}{3} + \cos 0 \right) dx \\ &= \int_0^3 (4x-3)^2 \left(-\frac{1}{2} + 1 \right) dx = \int_0^3 \frac{1}{2} (4x-3)^2 dx.\end{aligned}$$

In this integral we can make a substitution $u = 4x - 3$; then $du = 4dx$, so $dx = \frac{du}{4}$. We also change the limits of integration: if $x = 0$, then $u = 4 \cdot 0 - 3 = -3$; if $x = 3$, then $u = 4 \cdot 3 - 3 = 9$. So

$$\begin{aligned}\int_0^3 \frac{1}{2} (4x-3)^2 dx &= \int_{-3}^9 \frac{1}{2} u^2 \frac{du}{4} = \frac{1}{8} \int_{-3}^9 u^2 du = \frac{1}{8} \left[\frac{u^3}{3} \right]_{-3}^9 \\ &= \frac{1}{24} (9^3 - (-3)^3) = \frac{1}{24} (729 + 27) = \frac{756}{24} = \boxed{\frac{63}{2}}.\end{aligned}$$

28. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\begin{aligned}\int_0^{\pi/3} \left[\int_0^3 (4x-3) \sin y \, dx \right] dy &= \int_0^{\pi/3} \sin y \cdot [2x^2 - 3x]_0^3 dy = \int_0^{\pi/3} \sin y \cdot [(2 \cdot 3^2 - 3 \cdot 3) - 0] dy \\ &= \int_0^{\pi/3} 9 \sin y \, dy = 9 [-\cos y]_0^{\pi/3} = 9 \left(-\cos \frac{\pi}{3} + \cos 0 \right) \\ &= 9 \left(-\frac{1}{2} + 1 \right) = 9 \cdot \frac{1}{2} = \boxed{\frac{9}{2}}.\end{aligned}$$

29. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\begin{aligned}\int_0^1 \left[\int_0^{\pi/2} e^x \cos y \, dy \right] dx &= \int_0^1 e^x [\sin y]_0^{\pi/2} dx = \int_0^1 e^x \left(\sin \frac{\pi}{2} - \sin 0 \right) dx = \int_0^1 e^x (1 - 0) dx \\ &= \int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = \boxed{e - 1}.\end{aligned}$$

30. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\begin{aligned}\int_0^{\pi/4} \left[\int_0^1 e^x \cos y \, dx \right] dy &= \int_0^{\pi/4} \cos y \cdot [e^x]_0^1 dy = \int_0^{\pi/4} \cos y \cdot (e^1 - e^0) dy \\ &= \int_0^{\pi/4} (e - 1) \cos y \, dy = (e - 1) [\sin y]_0^{\pi/4} = (e - 1) \left(\sin \frac{\pi}{4} - \sin 0 \right) \\ &= (e - 1) \left(\frac{\sqrt{2}}{2} - 0 \right) = \boxed{\frac{(e - 1)\sqrt{2}}{2}}.\end{aligned}$$

31. We begin by using partial integration with respect to y to find the integral within the brackets. Then

we integrate the result with respect to x .

$$\begin{aligned}\int_0^2 \left[\int_{-\pi/2}^{\pi/2} \frac{\sin y}{2x+1} dy \right] dx &= \int_0^2 \frac{1}{2x+1} \left[\int_{-\pi/2}^{\pi/2} \sin y dy \right] dx = \int_0^2 \frac{1}{2x+1} [-\cos y]_{-\pi/2}^{\pi/2} dx \\ &= \int_0^2 \frac{1}{2x+1} \left[-\cos \frac{\pi}{2} + \cos \left(-\frac{\pi}{2} \right) \right] dx = \int_0^2 \frac{1}{2x+1} (-0+0) dx \\ &= \int_0^2 0 dx = \boxed{0}.\end{aligned}$$

32. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_0^{\pi/2} \left[\int_0^2 \frac{\sin y}{2x+1} dx \right] dy = \int_0^{\pi/2} \sin y \left[\int_0^2 \frac{1}{2x+1} dx \right] dy.$$

To find the integral in the brackets, we make a substitution $u = 2x + 1$; then $du = 2dx$, so $dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 2 \cdot 0 + 1 = 1$; if $x = 2$, then $u = 2 \cdot 2 + 1 = 5$. So

$$\int_0^2 \frac{1}{2x+1} dx = \int_1^5 \frac{1}{u} \frac{du}{2} = \frac{1}{2} \int_1^5 \frac{du}{u} = \frac{1}{2} [\ln |u|]_1^5 = \frac{1}{2} (\ln 5 - \ln 1) = \frac{\ln 5}{2}.$$

Going back to the given iterated integral, we find:

$$\begin{aligned}\int_0^{\pi/2} \sin y \left[\int_0^2 \frac{1}{2x+1} dx \right] dy &= \int_0^{\pi/2} \sin y \frac{\ln 5}{2} dy = \frac{\ln 5}{2} [-\cos y]_0^{\pi/2} \\ &= \frac{\ln 5}{2} \left(-\cos \frac{\pi}{2} + \cos 0 \right) = \frac{\ln 5}{2} (-0 + 1) = \boxed{\frac{\ln 5}{2}}.\end{aligned}$$

33. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_0^{\pi/2} \left[\int_0^2 x e^x \sin y dx \right] dy = \int_0^{\pi/2} \sin y \left[\int_0^2 x e^x dx \right] dy.$$

To find the integral in the brackets, we use the method of integration by parts. If we choose $u = x$ and $dv = e^x dx$, then $du = dx$ and $v = \int e^x dx = e^x$. So

$$\int_0^2 x e^x dx = [x e^x]_0^2 - \int_0^2 e^x dx = (2e^2 - 0e^0) - [e^x]_0^2 = 2e^2 - (e^2 - e^0) = 2e^2 - e^2 + 1 = e^2 + 1.$$

Going back to the given iterated integral, we find:

$$\begin{aligned}\int_0^{\pi/2} \sin y \left[\int_0^2 x e^x dx \right] dy &= \int_0^{\pi/2} \sin y \cdot (e^2 + 1) dy = (e^2 + 1) [-\cos y]_0^{\pi/2} \\ &= (e^2 + 1) \left(-\cos \frac{\pi}{2} + \cos 0 \right) = (e^2 + 1) (-0 + 1) = \boxed{e^2 + 1}.\end{aligned}$$

34. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\begin{aligned}\int_0^1 \left[\int_0^{\pi/2} x e^x \sin y dy \right] dx &= \int_0^1 x e^x \left[\int_0^{\pi/2} \sin y dy \right] dx = \int_0^1 x e^x [-\cos y]_0^{\pi/2} dx \\ &= \int_0^1 x e^x \left(-\cos \frac{\pi}{2} + \cos 0 \right) dx = \int_0^1 x e^x (-0 + 1) dx = \int_0^1 x e^x dx.\end{aligned}$$

To find this integral, we use the method of integration by parts. If we choose $u = x$ and $dv = e^x dx$, then $du = dx$ and $v = \int e^x dx = e^x$. So

$$\int_0^1 x e^x dx = [x e^x]_0^1 - \int_0^1 e^x dx = (1e^1 - 0e^0) - [e^x]_0^1 = e - (e^1 - e^0) = e - e + 1 = \boxed{1}.$$

35. We begin by using partial integration with respect to x to find the integral within the brackets. Then we integrate the result with respect to y .

$$\int_1^2 \left[\int_0^{\pi/2} \frac{x \cos x}{y} dx \right] dy = \int_1^2 \frac{1}{y} \left[\int_0^{\pi/2} x \cos x dx \right] dy.$$

To find the integral in the brackets, we use the method of integration by parts. If we choose $u = x$ and $dv = \cos x dx$, then $du = dx$ and $v = \int \cos x dx = \sin x$. So

$$\begin{aligned} \int_0^{\pi/2} x \cos x dx &= [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \\ &= \left(\frac{\pi}{2} \sin \frac{\pi}{2} - 0 \sin 0 \right) + [\cos x]_0^{\pi/2} = \frac{\pi}{2} + \left(\cos \frac{\pi}{2} - \cos 0 \right) = \frac{\pi}{2} - 1. \end{aligned}$$

Going back to the given iterated integral, we find:

$$\begin{aligned} \int_1^2 \frac{1}{y} \left[\int_0^{\pi/2} x \cos x dx \right] dy &= \int_1^2 \frac{1}{y} \left(\frac{\pi}{2} - 1 \right) dy = \left(\frac{\pi}{2} - 1 \right) [\ln |y|]_1^2 = \left(\frac{\pi}{2} - 1 \right) (\ln 2 - \ln 1) \\ &= \left(\frac{\pi}{2} - 1 \right) (\ln 2 - 0) = \left(\frac{\pi}{2} - 1 \right) \ln 2 = \boxed{\frac{(\pi - 2) \ln 2}{2}}. \end{aligned}$$

36. We begin by using partial integration with respect to y to find the integral within the brackets. Then we integrate the result with respect to x .

$$\begin{aligned} \int_0^{\pi/3} \left[\int_1^2 \frac{x \cos x}{y} dy \right] dx &= \int_0^{\pi/3} x \cos x \left[\int_1^2 \frac{1}{y} dy \right] dx = \int_0^{\pi/3} x \cos x \cdot [\ln |y|]_1^2 dx \\ &= \int_0^{\pi/3} x \cos x \cdot (\ln 2 - \ln 1) dx = \ln 2 \int_0^{\pi/3} x \cos x dx. \end{aligned}$$

To find this integral, we use the method of integration by parts. If we choose $u = x$ and $dv = \cos x dx$, then $du = dx$ and $v = \int \cos x dx = \sin x$. So

$$\begin{aligned} \ln 2 \int_0^{\pi/3} x \cos x dx &= \ln 2 \left[[x \sin x]_0^{\pi/3} - \int_0^{\pi/3} \sin x dx \right] \\ &= \ln 2 \left[\left(\frac{\pi}{3} \sin \frac{\pi}{3} - 0 \sin 0 \right) + [\cos x]_0^{\pi/3} \right] = \ln 2 \left[\frac{\pi}{3} \cdot \frac{\sqrt{3}}{2} + \left(\cos \frac{\pi}{3} - \cos 0 \right) \right] \\ &= \ln 2 \left[\frac{\pi\sqrt{3}}{6} + \left(\frac{1}{2} - 1 \right) \right] = \boxed{\ln 2 \left(\frac{\pi\sqrt{3}}{6} - \frac{1}{2} \right)}. \end{aligned}$$

37. Using Fubini's Theorem, the given double integral $\iint_R (2x + y^2) dA$, where R is the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 3$, equals either of the iterated integrals:

$$\int_0^2 \left[\int_0^3 (2x + y^2) dy \right] dx \quad \text{or} \quad \int_0^3 \left[\int_0^2 (2x + y^2) dx \right] dy.$$

Using the latter iterated integral, we begin integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned}\iint_R (2x + y^2) dA &= \int_0^3 \left[\int_0^2 (2x + y^2) dx \right] dy = \int_0^3 [x^2 + xy^2]_0^2 dy = \int_0^3 [(4 + 2y^2) - 0] dy \\ &= \int_0^3 (4 + 2y^2) dy = \left[4y + \frac{2y^3}{3} \right]_0^3 = (12 + 18) - 0 = \boxed{30}.\end{aligned}$$

38. Using Fubini's Theorem, the given double integral $\iint_R (x^2 - 3y) dA$, where R is the closed rectangular region defined by $-1 \leq x \leq 2$, $0 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_{-1}^2 \left[\int_0^1 (x^2 - 3y) dy \right] dx \quad \text{or} \quad \int_0^1 \left[\int_{-1}^2 (x^2 - 3y) dx \right] dy.$$

Using the former iterated integral, we begin by integrating partially with respect to y to find the integral within the brackets, and then we integrate the result with respect to x .

$$\begin{aligned}\iint_R (x^2 - 3y) dA &= \int_{-1}^2 \left[\int_0^1 (x^2 - 3y) dy \right] dx = \int_{-1}^2 \left[x^2 y - \frac{3y^2}{2} \right]_0^1 dx = \int_{-1}^2 \left[\left(x^2 - \frac{3}{2} \right) - 0 \right] dx \\ &= \int_{-1}^2 \left(x^2 - \frac{3}{2} \right) dx = \left[\frac{x^3}{3} - \frac{3x}{2} \right]_{-1}^2 = \left(\frac{8}{3} - 3 \right) - \left(\frac{-1}{3} - \frac{-3}{2} \right) = \boxed{-\frac{3}{2}}.\end{aligned}$$

39. Using Fubini's Theorem, the given double integral $\iint_R x(x^2 + 5) dA = \iint_R (x^3 + 5x) dA$, where R is the closed rectangular region defined by $0 \leq x \leq 2$, $-1 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_0^2 \left[\int_{-1}^1 (x^3 + 5x) dy \right] dx \quad \text{or} \quad \int_{-1}^1 \left[\int_0^2 (x^3 + 5x) dx \right] dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned}\iint_R x(x^2 + 5) dA &= \iint_R (x^3 + 5x) dA = \int_{-1}^1 \left[\int_0^2 (x^3 + 5x) dx \right] dy = \int_{-1}^1 \left[\frac{x^4}{4} + \frac{5x^2}{2} \right]_0^2 dy \\ &= \int_{-1}^1 [(4 + 10) - 0] dy = \int_{-1}^1 14 dy = [14y]_{-1}^1 = 14 - (-14) = \boxed{28}.\end{aligned}$$

40. Using Fubini's Theorem, the given double integral $\iint_R \sqrt{y}(3x^2 + x) dA$, where R is the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 4$, equals either of the iterated integrals:

$$\int_0^2 \left[\int_0^4 \sqrt{y}(3x^2 + x) dy \right] dx \quad \text{or} \quad \int_0^4 \left[\int_0^2 \sqrt{y}(3x^2 + x) dx \right] dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned}\iint_R \sqrt{y}(3x^2 + x) dA &= \int_0^4 \left[\int_0^2 \sqrt{y}(3x^2 + x) dx \right] dy = \int_0^4 \sqrt{y} \left[\int_0^2 (3x^2 + x) dx \right] dy \\ &= \int_0^4 \sqrt{y} \left[x^3 + \frac{x^2}{2} \right]_0^2 dy = \int_0^4 \sqrt{y} [(8 + 2) - 0] dy = \int_0^4 10y^{1/2} dy \\ &= \left[10 \cdot \frac{y^{3/2}}{3/2} \right]_0^4 = \left[\frac{20}{3} (\sqrt{y})^3 \right]_0^4 = \frac{20}{3} (\sqrt{4})^3 - \frac{20}{3} (\sqrt{0})^3 = \frac{160}{3} - 0 = \boxed{\frac{160}{3}}.\end{aligned}$$

41. Using Fubini's Theorem, the given double integral $\iint_R 2xe^y dA$, where R is the closed rectangular region defined by $-3 \leq x \leq 2$, $0 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_{-3}^2 \left[\int_0^1 2xe^y dy \right] dx \quad \text{or} \quad \int_0^1 \left[\int_{-3}^2 2xe^y dx \right] dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned} \iint_R 2xe^y dA &= \int_0^1 \left[\int_{-3}^2 2xe^y dx \right] dy = \int_0^1 e^y \left[\int_{-3}^2 2x dx \right] dy = \int_0^1 e^y [x^2]_{-3}^2 dy = \int_0^1 e^y (4 - 9) dy \\ &= \int_0^1 (-5)e^y dy = -5[e^y]_0^1 = -5(e^1 - e^0) = -5(e - 1) = \boxed{5(1 - e)}. \end{aligned}$$

42. Using Fubini's Theorem, the given double integral $\iint_R e^{2x+y} dA = \iint_R e^{2x} e^y dA$, where R is the closed rectangular region defined by $0 \leq x \leq 1$, $-1 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_0^1 \left[\int_{-1}^1 e^{2x} e^y dy \right] dx \quad \text{or} \quad \int_{-1}^1 \left[\int_0^1 e^{2x} e^y dx \right] dy.$$

Using the former iterated integral, we begin by integrating partially with respect to y to find the integral within the brackets, and then we integrate the result with respect to x .

$$\begin{aligned} \iint_R e^{2x+y} dA &= \iint_R e^{2x} e^y dA = \int_0^1 \left[\int_{-1}^1 e^{2x} e^y dy \right] dx = \int_0^1 e^{2x} \left[\int_{-1}^1 e^y dy \right] dx \\ &= \int_0^1 e^{2x} [e^y]_{-1}^1 dx = \int_0^1 e^{2x} (e^1 - e^{-1}) dx = \left(e - \frac{1}{e} \right) \int_0^1 e^{2x} dx. \end{aligned}$$

To find the remaining integral $\int_0^1 e^{2x} dx$, we make a substitution $u = 2x$; then $du = 2dx$, so $dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 2 \cdot 0 = 0$; if $x = 1$, then $u = 2 \cdot 1 = 2$. So

$$\int_0^1 e^{2x} dx = \int_0^2 e^u \frac{du}{2} = \frac{1}{2} \int_0^2 e^u du = \frac{1}{2} [e^u]_0^2 = \frac{1}{2} (e^2 - e^0) = \frac{1}{2} (e^2 - 1).$$

Then the final answer is

$$\iint_R e^{2x+y} dA = \left(e - \frac{1}{e} \right) \cdot \frac{1}{2} (e^2 - 1) = \frac{e^2 - 1}{e} \cdot \frac{e^2 - 1}{2} = \frac{e^4 - 2e^2 + 1}{2e} = \boxed{\frac{e^3}{2} - e + \frac{1}{2e}}.$$

43. Using Fubini's Theorem, the given double integral $\iint_R x \sec^2 y dA$, where R is the closed rectangular region defined by $0 \leq x \leq 3$, $0 \leq y \leq \frac{\pi}{4}$, equals either of the iterated integrals:

$$\int_0^3 \left[\int_0^{\pi/4} x \sec^2 y dy \right] dx \quad \text{or} \quad \int_0^{\pi/4} \left[\int_0^3 x \sec^2 y dx \right] dy.$$

Using the former iterated integral, we begin by integrating partially with respect to y to find the integral within the brackets, and then we integrate the result with respect to x .

$$\begin{aligned} \int_0^3 \left[\int_0^{\pi/4} x \sec^2 y dy \right] dx &= \int_0^3 x \left[\int_0^{\pi/4} \sec^2 y dy \right] dx = \int_0^3 x [\tan y]_0^{\pi/4} dx \\ &= \int_0^3 x \left(\tan \frac{\pi}{4} - \tan 0 \right) dx = \int_0^3 x(1 - 0) dx = \int_0^3 x dx \\ &= \left[\frac{x^2}{2} \right]_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2} - 0 = \boxed{\frac{9}{2}}. \end{aligned}$$

44. Using Fubini's Theorem, the given double integral $\iint_R y^2 \sec x \tan x \, dA$, where R is the closed rectangular region defined by $0 \leq x \leq \frac{\pi}{3}$, $-1 \leq y \leq 3$, equals either of the iterated integrals:

$$\int_0^{\pi/3} \left[\int_{-1}^3 y^2 \sec x \tan x \, dy \right] dx \quad \text{or} \quad \int_{-1}^3 \left[\int_0^{\pi/3} y^2 \sec x \tan x \, dx \right] dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned} \int_{-1}^3 \left[\int_0^{\pi/3} y^2 \sec x \tan x \, dx \right] dy &= \int_{-1}^3 y^2 \left[\int_0^{\pi/3} \sec x \tan x \, dx \right] dy = \int_{-1}^3 y^2 [\sec x]_0^{\pi/3} dy \\ &= \int_{-1}^3 y^2 \left(\sec \frac{\pi}{3} - \sec 0 \right) dy = \int_{-1}^3 y^2 (2 - 1) dy = \int_{-1}^3 y^2 dy \\ &= \left[\frac{y^3}{3} \right]_{-1}^3 = \frac{3^3}{3} - \frac{(-1)^3}{3} = 9 + \frac{1}{3} = \boxed{\frac{28}{3}}. \end{aligned}$$

45. Using Fubini's Theorem, the given double integral $\iint_R \frac{x}{2y+3} \, dA$, where R is the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_0^2 \left[\int_0^1 \frac{x}{2y+3} \, dy \right] dx \quad \text{or} \quad \int_0^1 \left[\int_0^2 \frac{x}{2y+3} \, dx \right] dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x to find the integral within the brackets, and then we integrate the result with respect to y .

$$\begin{aligned} \int_0^1 \left[\int_0^2 \frac{x}{2y+3} \, dx \right] dy &= \int_0^1 \frac{1}{2y+3} \left[\int_0^2 x \, dx \right] dy \\ &= \int_0^1 \frac{1}{2y+3} \left[\frac{x^2}{2} \right]_0^2 dy = \int_0^1 \frac{1}{2y+3} (2 - 0) dy = \int_0^1 \frac{2}{2y+3} dy. \end{aligned}$$

To find the remaining integral, we make a substitution $u = 2y + 3$; then $du = 2dy$, so $dy = \frac{du}{2}$. We also change the limits of integration: if $y = 0$, then $u = 2 \cdot 0 + 3 = 3$; if $y = 1$, then $u = 2 \cdot 1 + 3 = 5$. So

$$\int_0^1 \frac{2}{2y+3} dy = \int_3^5 \frac{2}{u} \frac{du}{2} = \int_3^5 \frac{du}{u} = [\ln |u|]_3^5 = \ln 5 - \ln 3 = \boxed{\ln \frac{5}{3}}.$$

46. Using Fubini's Theorem, the given double integral $\iint_R \frac{y^2}{x-3} \, dA$, where R is the closed rectangular region defined by $4 \leq x \leq 5$, $0 \leq y \leq 3$, equals either of the iterated integrals:

$$\int_4^5 \left[\int_0^3 \frac{y^2}{x-3} \, dy \right] dx \quad \text{or} \quad \int_0^3 \left[\int_4^5 \frac{y^2}{x-3} \, dx \right] dy.$$

Using the former iterated integral, we begin by integrating partially with respect to y to find the integral within the brackets, and then we integrate the result with respect to x .

$$\begin{aligned} \int_4^5 \left[\int_0^3 \frac{y^2}{x-3} \, dy \right] dx &= \int_4^5 \frac{1}{x-3} \left[\int_0^3 y^2 \, dy \right] dx \\ &= \int_4^5 \frac{1}{x-3} \left[\frac{y^3}{3} \right]_0^3 dx = \int_4^5 \frac{1}{x-3} (9 - 0) dx = \int_4^5 \frac{9}{x-3} dx. \end{aligned}$$

To find the remaining integral, we make a substitution $u = x - 3$; then $du = dx$. We also change the limits of integration: if $x = 4$, then $u = 4 - 3 = 1$; if $x = 5$, then $u = 5 - 3 = 2$. So

$$\int_4^5 \frac{9}{x-3} dx = \int_1^2 \frac{9}{u} du = 9 \int_1^2 \frac{du}{u} = 9[\ln |u|]_1^2 = 9(\ln 2 - \ln 1) = 9(\ln 2 - 0) = \boxed{9 \ln 2}.$$

Applications and Extensions

47. Let R be the closed rectangular region defined by $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 1$. Using Fubini's Theorem,

$$\iint_R x \sin(xy) dA = \int_0^{\pi/2} \left[\int_0^1 x \sin(xy) dy \right] dx.$$

To find the integral in the brackets $\int_0^1 x \sin(xy) dy$, we integrate partially with respect to y treating x as a constant. We make a substitution $u = xy$; then $du = x dy$, so $dy = \frac{du}{x}$. We also change the limits of integration: if $y = 0$, then $u = x \cdot 0 = 0$; if $y = 1$, then $u = x \cdot 1 = x$. So

$$\int_0^1 x \sin(xy) dy = \int_0^x x \sin u \frac{du}{x} = \int_0^x \sin u du = [-\cos u]_0^x = -\cos x + \cos 0 = -\cos x + 1.$$

Then going back to the iterated integral, we obtain:

$$\begin{aligned} \iint_R x \sin(xy) dA &= \int_0^{\pi/2} \left[\int_0^1 x \sin(xy) dy \right] dx = \int_0^{\pi/2} (-\cos x + 1) dx = [-\sin x + x]_0^{\pi/2} \\ &= \left(-\sin \frac{\pi}{2} + \frac{\pi}{2} \right) - (-\sin 0 + 0) = \frac{\pi}{2} - 1 = \boxed{\frac{\pi - 2}{2}}. \end{aligned}$$

48. Let R be the closed rectangular region defined by $0 \leq x \leq 1$, $\frac{\pi}{2} \leq y \leq 2\pi$. Using Fubini's Theorem,

$$\iint_R y \cos(xy) dA = \int_{\pi/2}^{2\pi} \left[\int_0^1 y \cos(xy) dx \right] dy.$$

To find the integral in the brackets $\int_0^1 y \cos(xy) dx$, we integrate partially with respect to x treating y as a constant. We make a substitution $u = xy$; then $du = y dx$, so $dx = \frac{du}{y}$. We also change the limits of integration: if $x = 0$, then $u = 0 \cdot y = 0$; if $x = 1$, then $u = 1 \cdot y = y$. So

$$\int_0^1 y \cos(xy) dx = \int_0^y y \cos u \frac{du}{y} = \int_0^y \cos u du = [\sin u]_0^y = \sin y - \sin 0 = \sin y - 0 = \sin y.$$

Then going back to the iterated integral, we obtain:

$$\begin{aligned} \iint_R y \cos(xy) dA &= \int_{\pi/2}^{2\pi} \left[\int_0^1 y \cos(xy) dx \right] dy = \int_{\pi/2}^{2\pi} \sin y dy \\ &= [-\cos y]_{\pi/2}^{2\pi} = -\cos 2\pi + \cos \frac{\pi}{2} = -1 + 0 = \boxed{-1}. \end{aligned}$$

49. Let R be the closed rectangular region defined by $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 1$. Using Fubini's Theorem,

$$\iint_R x^3 \cos(x^2 y) dA = \int_0^{\pi/2} \left[\int_0^1 x^3 \cos(x^2 y) dy \right] dx.$$

To find the integral in the brackets $\int_0^1 x^3 \cos(x^2 y) dy$, we integrate partially with respect to y treating x as a constant. We make a substitution $u = x^2 y$; then $du = x^2 dy$, so $dy = \frac{du}{x^2}$. We also change the limits of integration: if $y = 0$, then $u = x^2 \cdot 0 = 0$; if $y = 1$, then $u = x^2 \cdot 1 = x^2$. So

$$\begin{aligned} \int_0^1 x^3 \cos(x^2 y) dy &= \int_0^{x^2} x^3 \cos u \frac{du}{x^2} = \int_0^{x^2} x \cos u du = x \int_0^{x^2} \cos u du \\ &= x [\sin u]_0^{x^2} = x [\sin(x^2) - \sin 0] = x [\sin(x^2) - 0] = x \sin(x^2). \end{aligned}$$

Then going back to the iterated integral:

$$\int_0^{\pi/2} \left[\int_0^1 x^3 \cos(x^2 y) dy \right] dx = \int_0^{\pi/2} x \sin(x^2) dx.$$

To find this integral we make a substitution $v = x^2$; then $dv = 2x dx$, so $x dx = \frac{dv}{2}$. We also change the limits of integration: if $x = 0$, then $v = 0^2 = 0$; if $x = \frac{\pi}{2}$, then $v = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$. So

$$\begin{aligned} \int_0^{\pi/2} x \sin(x^2) dx &= \int_0^{\pi^2/4} \sin v \frac{dv}{2} = \frac{1}{2} \int_0^{\pi^2/4} \sin v dv = \frac{1}{2} [-\cos v]_0^{\pi^2/4} \\ &= \frac{1}{2} \left[-\cos \frac{\pi^2}{4} + \cos 0 \right] = \frac{1}{2} \left[-\cos \frac{\pi^2}{4} + 1 \right] = \boxed{\frac{1}{2} - \frac{1}{2} \cos \frac{\pi^2}{4}}. \end{aligned}$$

50. Let R be the closed rectangular region defined by $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 1$. Using Fubini's Theorem,

$$\iint_R x^3 \sin(x^2 y) dA = \int_0^{\pi/2} \left[\int_0^1 x^3 \sin(x^2 y) dy \right] dx.$$

To find the integral in the brackets $\int_0^1 x^3 \sin(x^2 y) dy$, we integrate partially with respect to y treating x as a constant. We make a substitution $u = x^2 y$; then $du = x^2 dy$, so $dy = \frac{du}{x^2}$. We also change the limits of integration: if $y = 0$, then $u = x^2 \cdot 0 = 0$; if $y = 1$, then $u = x^2 \cdot 1 = x^2$. So

$$\begin{aligned} \int_0^1 x^3 \sin(x^2 y) dy &= \int_0^{x^2} x^3 \sin u \frac{du}{x^2} = \int_0^{x^2} x \sin u du = x \int_0^{x^2} \sin u du \\ &= x [-\cos u]_0^{x^2} = x [-\cos(x^2) + \cos 0] = x [-\cos(x^2) + 1] = x - x \cos(x^2). \end{aligned}$$

Then going back to the iterated integral:

$$\int_0^{\pi/2} \left[\int_0^1 x^3 \sin(x^2 y) dy \right] dx = \int_0^{\pi/2} [x - x \cos(x^2)] dx = \int_0^{\pi/2} x dx - \int_0^{\pi/2} x \cos(x^2) dx.$$

The first integral is

$$\int_0^{\pi/2} x dx = \left[\frac{x^2}{2} \right]_0^{\pi/2} = \left[\frac{(\pi/2)^2}{2} - \frac{0^2}{2} \right] = \frac{\pi^2}{8}.$$

To find the second integral we make a substitution $v = x^2$; then $dv = 2x dx$, so $x dx = \frac{dv}{2}$. We also change the limits of integration: if $x = 0$, then $v = 0^2 = 0$; if $x = \frac{\pi}{2}$, then $v = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$. So

$$\begin{aligned} \int_0^{\pi/2} x \cos(x^2) dx &= \int_0^{\pi^2/4} \cos v \frac{dv}{2} = \frac{1}{2} \int_0^{\pi^2/4} \cos v dv = \frac{1}{2} [\sin v]_0^{\pi^2/4} \\ &= \frac{1}{2} \left[\sin \frac{\pi^2}{4} - \sin 0 \right] = \frac{1}{2} \left[\sin \frac{\pi^2}{4} - 0 \right] = \frac{1}{2} \sin \frac{\pi^2}{4}. \end{aligned}$$

So the final answer is

$$\iint_R x^3 \sin(x^2 y) dA = \int_0^{\pi/2} x dx - \int_0^{\pi/2} x \cos(x^2) dx = \boxed{\frac{\pi^2}{8} - \frac{1}{2} \sin \frac{\pi^2}{4}}.$$

51. Let R be the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 3$. Using Fubini's Theorem,

$$\iint_R \frac{y^2}{(1+xy^2)^3} dA = \int_0^3 \left[\int_0^2 \frac{y^2}{(1+xy^2)^3} dx \right] dy.$$

To find the integral in the brackets, we integrate partially with respect to x treating y as a constant. We make a substitution $u = 1 + xy^2$; then $du = y^2 dx$, so $dx = \frac{du}{y^2}$. We also change the limits of integration: if $x = 0$, then $u = 1 + 0y^2 = 1$; if $x = 2$, then $u = 1 + 2y^2$. So

$$\begin{aligned} \int_0^2 \frac{y^2}{(1+xy^2)^3} dx &= \int_1^{1+2y^2} \frac{y^2}{u^3} \frac{du}{y^2} = \int_1^{1+2y^2} \frac{du}{u^3} = \int_1^{1+2y^2} u^{-3} du \\ &= \left[\frac{u^{-2}}{-2} \right]_1^{1+2y^2} = \left[-\frac{1}{2u^2} \right]_1^{1+2y^2} = -\frac{1}{2(1+2y^2)^2} + \frac{1}{2} = \frac{1}{2} - \frac{1}{8(\frac{1}{2} + y^2)^2}. \end{aligned}$$

Then going back to the iterated integral:

$$\int_0^3 \left[\int_0^2 \frac{y^2}{(1+xy^2)^3} dx \right] dy = \int_0^3 \left(\frac{1}{2} - \frac{1}{8(\frac{1}{2} + y^2)^2} \right) dy = \int_0^3 \frac{1}{2} dy - \frac{1}{8} \int_0^3 \frac{1}{(\frac{1}{2} + y^2)^2} dy.$$

The first integral is

$$\int_0^3 \frac{1}{2} dy = \frac{1}{2} [y]_0^3 = \frac{1}{2} \cdot 3 - \frac{1}{2} \cdot 0 = \frac{3}{2}.$$

To find the second integral, we will find an antiderivative using a substitution $y = \frac{1}{\sqrt{2}} \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; then $dy = \frac{1}{\sqrt{2}} \sec^2 \theta d\theta$. Also,

$$\frac{1}{2} + y^2 = \frac{1}{2} + \frac{1}{2} \tan^2 \theta = \frac{1}{2} (1 + \tan^2 \theta) = \frac{1}{2} \sec^2 \theta,$$

and the integral becomes

$$\begin{aligned} \frac{1}{8} \int \frac{1}{(\frac{1}{2} + y^2)^2} dy &= \frac{1}{8} \int \frac{\frac{1}{\sqrt{2}} \sec^2 \theta d\theta}{(\frac{1}{2} \sec^2 \theta)^2} = \frac{1}{8} \int \frac{\frac{1}{\sqrt{2}} \sec^2 \theta d\theta}{\frac{1}{4} \sec^4 \theta} = \frac{1}{8} \int \frac{4}{\sqrt{2}} \frac{1}{\sec^2 \theta} d\theta \\ &= \frac{1}{2\sqrt{2}} \int \cos^2 \theta d\theta = \frac{1}{2\sqrt{2}} \int \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4\sqrt{2}} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4\sqrt{2}} \left(\theta + \frac{1}{2} \sin 2\theta \right) = \frac{1}{4\sqrt{2}} (\theta + \sin \theta \cos \theta). \end{aligned}$$

To express this antiderivative in terms of y , we first observe that since $y = \frac{1}{\sqrt{2}} \tan \theta$, then $\tan \theta = \sqrt{2}y$ and $\theta = \tan^{-1}(\sqrt{2}y)$; we also use trigonometric identities as follows:

$$\sin \theta \cos \theta = \frac{\sin \theta}{\cos \theta} \cdot \cos^2 \theta = \frac{\tan \theta}{\sec^2 \theta} = \frac{\tan \theta}{1 + \tan^2 \theta} = \frac{\sqrt{2}y}{1 + 2y^2}.$$

So,

$$\frac{1}{4\sqrt{2}} (\theta + \sin \theta \cos \theta) = \frac{1}{4\sqrt{2}} \left(\tan^{-1}(\sqrt{2}y) + \frac{\sqrt{2}y}{1 + 2y^2} \right) = \frac{1}{4\sqrt{2}} \tan^{-1}(\sqrt{2}y) + \frac{y}{4(1 + 2y^2)}$$

is an antiderivative, and

$$\begin{aligned}
 \frac{1}{8} \int_0^3 \frac{1}{(\frac{1}{2} + y^2)^2} dy &= \left[\frac{1}{4\sqrt{2}} \tan^{-1}(\sqrt{2}y) + \frac{y}{4(1+2y^2)} \right]_0^3 \\
 &= \left(\frac{1}{4\sqrt{2}} \tan^{-1}(\sqrt{2} \cdot 3) + \frac{3}{4(1+2 \cdot 3^2)} \right) - \left(\frac{1}{4\sqrt{2}} \tan^{-1}(\sqrt{2} \cdot 0) + \frac{0}{4(1+2 \cdot 0^2)} \right) \\
 &= \left(\frac{1}{4\sqrt{2}} \tan^{-1}(3\sqrt{2}) + \frac{3}{76} \right) - \left(\frac{1}{4\sqrt{2}} \tan^{-1}(0) + 0 \right) \\
 &= \left(\frac{1}{4\sqrt{2}} \tan^{-1}(3\sqrt{2}) + \frac{3}{76} \right) - (0 + 0) = \frac{1}{4\sqrt{2}} \tan^{-1}(3\sqrt{2}) + \frac{3}{76}.
 \end{aligned}$$

So the final answer is

$$\frac{3}{2} - \left(\frac{1}{4\sqrt{2}} \tan^{-1}(3\sqrt{2}) + \frac{3}{76} \right) = \frac{3}{2} - \frac{3}{76} - \frac{1}{4\sqrt{2}} \tan^{-1}(3\sqrt{2}) = \boxed{\frac{111}{76} - \frac{\sqrt{2}}{8} \tan^{-1}(3\sqrt{2})}.$$

52. Let R be the closed rectangular region defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$. Using Fubini's Theorem,

$$\begin{aligned}
 \iint_R (x^3 + x)(x^2y + y) dA &= \int_0^1 \left[\int_0^1 (x^3 + x)(x^2y + y) dy \right] dx = \int_0^1 \left[\int_0^1 x(x^2 + 1)y(x^2 + 1) dy \right] dx \\
 &= \int_0^1 \left[\int_0^1 xy(x^2 + 1)^2 dy \right] dx = \int_0^1 x(x^2 + 1)^2 \left[\int_0^1 y dy \right] dx \\
 &= \int_0^1 x(x^2 + 1)^2 \left[\frac{y^2}{2} \right]_0^1 dx = \int_0^1 x(x^2 + 1)^2 \left(\frac{1}{2} - 0 \right) dx \\
 &= \frac{1}{2} \int_0^1 x(x^2 + 1)^2 dx.
 \end{aligned}$$

To find the remaining integral, we make a substitution $u = x^2 + 1$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 + 1 = 1$; if $x = 1$, then $u = 1^2 + 1 = 2$. So

$$\frac{1}{2} \int_0^1 x(x^2 + 1)^2 dx = \frac{1}{2} \int_1^2 u^2 \frac{du}{2} = \frac{1}{4} \int_1^2 u^2 du = \frac{1}{4} \left[\frac{u^3}{3} \right]_1^2 = \frac{1}{4} \left(\frac{8}{3} - \frac{1}{3} \right) = \frac{1}{4} \cdot \frac{7}{3} = \boxed{\frac{7}{12}}.$$

53. (a) Let R be the closed rectangular region defined by $0 \leq x \leq 1$, $0 \leq y \leq 2$. Using Fubini's Theorem,

$$\iint_R y^5 e^{xy^3} dA = \int_0^2 \left[\int_0^1 y^5 e^{xy^3} dx \right] dy = \int_0^2 y^5 \left[\int_0^1 e^{xy^3} dx \right] dy.$$

To find the integral in the brackets, we integrate partially with respect to x treating y as a constant. We make a substitution $u = xy^3$; then $du = y^3 dx$, so $dx = \frac{du}{y^3}$. We also change the limits of integration: if $x = 0$, then $u = 0 \cdot y^3 = 0$; if $x = 1$, then $u = 1 \cdot y^3 = y^3$. So

$$\int_0^1 e^{xy^3} dx = \int_0^{y^3} e^u \frac{du}{y^3} = \frac{1}{y^3} \int_0^{y^3} e^u du = \frac{1}{y^3} [e^u]_0^{y^3} = \frac{1}{y^3} (e^{y^3} - e^0) = \frac{1}{y^3} (e^{y^3} - 1).$$

Going back to the iterated integral,

$$\int_0^2 y^5 \left[\int_0^1 e^{xy^3} dx \right] dy = \int_0^2 y^5 \cdot \frac{1}{y^3} (e^{y^3} - 1) dy = \int_0^2 y^2 (e^{y^3} - 1) dy.$$

Here we make a substitution $v = y^3$; then $dv = 3y^2 dy$, so $y^2 dy = \frac{dv}{3}$. We also change the limits of integration: if $y = 0$, then $v = 0^3 = 0$; if $y = 2$, then $v = 2^3 = 8$. So

$$\begin{aligned} \int_0^2 y^2 (e^{y^3} - 1) dy &= \int_0^8 (e^v - 1) \frac{dv}{3} = \frac{1}{3} \int_0^8 (e^v - 1) dv = \frac{1}{3} [e^v - v]_0^8 \\ &= \frac{1}{3} [(e^8 - 8) - (e^0 - 0)] = \frac{1}{3} (e^8 - 8 - 1 + 0) = \boxed{\frac{1}{3} (e^8 - 9)}. \end{aligned}$$

- (b) Using Fubini's Theorem, we can change the order of integration and also represent the given double integral as

$$\iint_R y^5 e^{xy^3} dA = \int_0^1 \left[\int_0^2 y^5 e^{xy^3} dy \right] dx.$$

Entering this iterated integral into the computer algebra system *Mathematica*, we find that

$$\int_0^1 \left[\int_0^2 y^5 e^{xy^3} dy \right] dx = \boxed{\frac{1}{3} (-9 + e^8)},$$

which agrees with the answer found in part (a).

54. (a) Let R be the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 1$. Using Fubini's Theorem,

$$\iint_R x^5 y e^{x^3 y^2} dA = \int_0^2 \left[\int_0^1 x^5 y e^{x^3 y^2} dy \right] dx = \int_0^2 x^5 \left[\int_0^1 y e^{x^3 y^2} dy \right] dx.$$

To find the integral in the brackets, we integrate partially with respect to y treating x as a constant. We make a substitution $u = x^3 y^2$; then $du = 2x^3 y dy$, so $y dy = \frac{du}{2x^3}$. We also change the limits of integration: if $y = 0$, then $u = x^3 \cdot 0^2 = 0$; if $y = 1$, then $u = x^3 \cdot 1^2 = x^3$. So

$$\int_0^1 y e^{x^3 y^2} dy = \int_0^{x^3} e^u \frac{du}{2x^3} = \frac{1}{2x^3} \int_0^{x^3} e^u du = \frac{1}{2x^3} [e^u]_0^{x^3} = \frac{1}{2x^3} (e^{x^3} - e^0) = \frac{1}{2x^3} (e^{x^3} - 1).$$

Going back to the iterated integral,

$$\int_0^2 x^5 \left[\int_0^1 y e^{x^3 y^2} dy \right] dx = \int_0^2 x^5 \cdot \frac{1}{2x^3} (e^{x^3} - 1) dx = \frac{1}{2} \int_0^2 x^2 (e^{x^3} - 1) dx.$$

Here we make a substitution $v = x^3$; then $dv = 3x^2 dx$, so $x^2 dx = \frac{dv}{3}$. We also change the limits of integration: if $x = 0$, then $v = 0^3 = 0$; if $x = 2$, then $v = 2^3 = 8$. So

$$\begin{aligned} \frac{1}{2} \int_0^2 x^2 (e^{x^3} - 1) dx &= \frac{1}{2} \int_0^8 (e^v - 1) \frac{dv}{3} = \frac{1}{6} \int_0^8 (e^v - 1) dv = \frac{1}{6} [e^v - v]_0^8 \\ &= \frac{1}{6} [(e^8 - 8) - (e^0 - 0)] = \frac{1}{6} (e^8 - 8 - 1 + 0) = \boxed{\frac{1}{6} (e^8 - 9)}. \end{aligned}$$

- (b) Using Fubini's Theorem, we can change the order of integration and also represent the given double integral as

$$\iint_R x^5 y e^{x^3 y^2} dA = \int_0^1 \left[\int_0^2 x^5 y e^{x^3 y^2} dx \right] dy.$$

Entering this iterated integral into the computer algebra system *Mathematica*, we find that

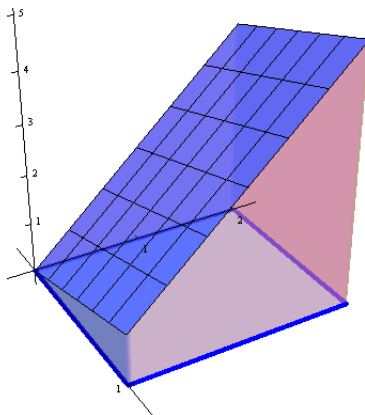
$$\int_0^1 \left[\int_0^2 x^5 y e^{x^3 y^2} dx \right] dy = \boxed{\frac{1}{6} (-9 + e^8)},$$

which agrees with the answer found in part (a).

55. We begin by graphing $z = f(x, y) = x + 2y$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq 2$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (x + 2y) dA$. Using Fubini's

Theorem, we have

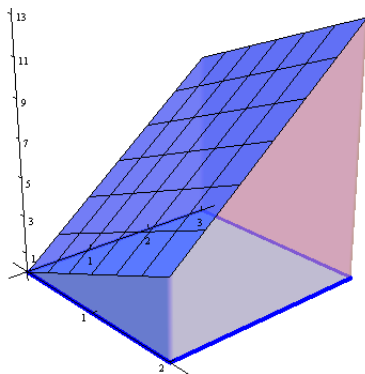
$$\begin{aligned} V &= \int_0^2 \left[\int_0^1 (x + 2y) dx \right] dy = \int_0^2 \left[\frac{x^2}{2} + 2xy \right]_0^1 dy \\ &= \int_0^2 \left(\frac{1}{2} + 2y \right) dy = \left[\frac{1}{2}y + y^2 \right]_0^2 = (1 + 4) - (0 + 0) = \boxed{5 \text{ cubic units}}. \end{aligned}$$



56. We begin by graphing $z = f(x, y) = 2x + 3y$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 2$, $0 \leq y \leq 3$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (2x + 3y) dA$. Using

Fubini's Theorem, we have

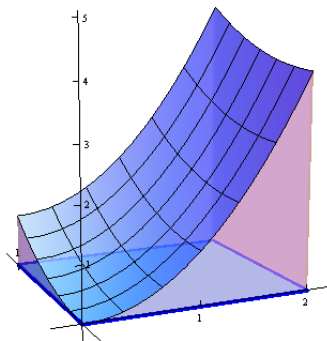
$$\begin{aligned} V &= \int_0^3 \left[\int_0^2 (2x + 3y) dx \right] dy = \int_0^3 [x^2 + 3xy]_0^2 dy \\ &= \int_0^3 (4 + 6y) dy = [4y + 3y^2]_0^3 = (12 + 27) - (0 + 0) = \boxed{39 \text{ cubic units}}. \end{aligned}$$



57. We begin by graphing $z = f(x, y) = x^2 + y^2$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 2$, $0 \leq y \leq 1$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (x^2 + y^2) dA$. Using

Fubini's Theorem, we have

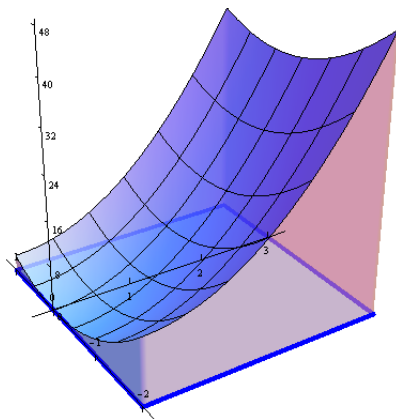
$$\begin{aligned} V &= \int_0^1 \left[\int_0^2 (x^2 + y^2) dx \right] dy = \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_0^2 dy \\ &= \int_0^1 \left(\frac{8}{3} + 2y^2 \right) dy = \left[\frac{8}{3}y + \frac{2}{3}y^3 \right]_0^1 = \left(\frac{8}{3} + \frac{2}{3} \right) - (0 + 0) = \boxed{\frac{10}{3} \text{ cubic units}}. \end{aligned}$$



58. We begin by graphing $z = f(x, y) = 4x^2 + 3y^2$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 3$, $-2 \leq y \leq 1$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (4x^2 + 3y^2) dA$. Using

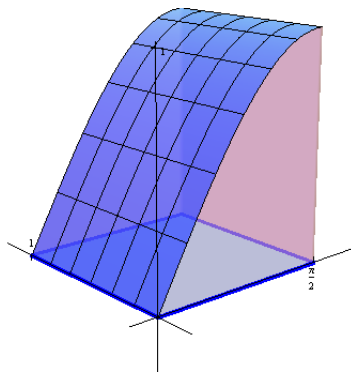
Fubini's Theorem, we have

$$\begin{aligned} V &= \int_0^3 \left[\int_{-2}^1 (4x^2 + 3y^2) dy \right] dx = \int_0^3 [4x^2y + y^3]_{-2}^1 dx = \int_0^3 [(4x^2 + 1) - (-8x^2 - 8)] dx \\ &= \int_0^3 (12x^2 + 9) dx = [4x^3 + 9x]_0^3 = (108 + 27) - (0 + 0) = \boxed{135 \text{ cubic units}}. \end{aligned}$$



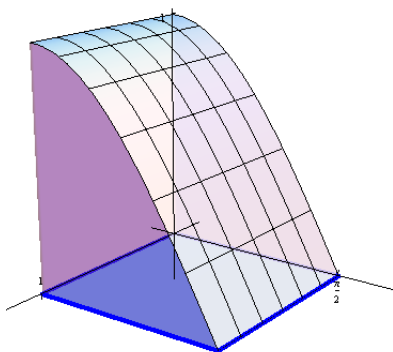
59. We begin by graphing $z = f(x, y) = \sin x$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 1$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R \sin x dA$. Using Fubini's Theorem, we have

$$\begin{aligned} V &= \int_0^1 \left[\int_0^{\pi/2} \sin x dx \right] dy = \int_0^1 [-\cos x]_0^{\pi/2} dy = \int_0^1 \left[-\cos \frac{\pi}{2} + \cos 0 \right] dy \\ &= \int_0^1 (-0 + 1) dy = \int_0^1 1 dy = [y]_0^1 = 1 - 0 = \boxed{1 \text{ cubic unit}}. \end{aligned}$$



60. We begin by graphing $z = f(x, y) = \cos y$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \frac{\pi}{2}$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R \cos y dA$. Using Fubini's Theorem, we have

$$\begin{aligned} V &= \int_0^1 \left[\int_0^{\pi/2} \cos y dy \right] dx = \int_0^1 [\sin y]_0^{\pi/2} dx = \int_0^1 \left[\sin \frac{\pi}{2} - \sin 0 \right] dx \\ &= \int_0^1 (1 - 0) dx = \int_0^1 1 dx = [x]_0^1 = 1 - 0 = \boxed{1 \text{ cubic unit}}. \end{aligned}$$



61. We begin by graphing $z = f(x, y) = \frac{x^2 + y^2}{xy}$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $1 \leq x \leq 2$, $1 \leq y \leq 2$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA$. Using Fubini's Theorem, we have

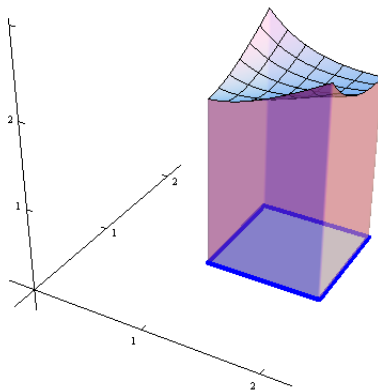
$$V = \iint_R \frac{x^2 + y^2}{xy} dA = \int_1^2 \left[\int_1^2 \frac{x^2 + y^2}{xy} dy \right] dx = \int_1^2 \left[\int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy \right] dx.$$

To find the integral in the brackets, we integrate partially with respect to y treating x as a constant:

$$\begin{aligned} \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy &= \int_1^2 \left(x \cdot \frac{1}{y} + \frac{1}{x} \cdot y \right) dy = \left[x \ln |y| + \frac{1}{x} \cdot \frac{y^2}{2} \right]_1^2 \\ &= \left(x \ln 2 + \frac{2}{x} \right) - \left(x \ln 1 + \frac{1}{2x} \right) = x \ln 2 + \frac{2}{x} - 0 - \frac{1}{2x} = x \ln 2 + \frac{3}{2x}. \end{aligned}$$

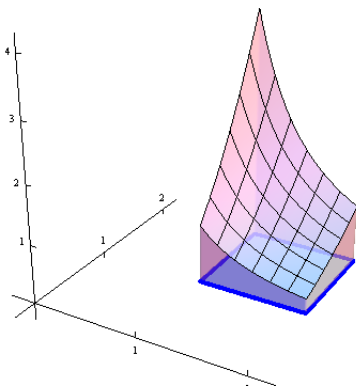
Going back to the iterated integral,

$$\begin{aligned}
 V &= \int_1^2 \left[\int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy \right] dx = \int_1^2 \left(x \ln 2 + \frac{3}{2x} \right) dx = \left[\frac{x^2}{2} \cdot \ln 2 + \frac{3}{2} \ln |x| \right]_1^2 \\
 &= \left(2 \ln 2 + \frac{3}{2} \ln 2 \right) - \left(\frac{1}{2} \ln 2 + \frac{3}{2} \ln 1 \right) = 2 \ln 2 + \frac{3}{2} \ln 2 - \frac{1}{2} \ln 2 - 0 = \boxed{3 \ln 2 \text{ cubic units}}.
 \end{aligned}$$



62. We begin by graphing $z = f(x, y) = \frac{y^2}{x^2}$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $1 \leq x \leq 2$, $1 \leq y \leq 2$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA$. Using Fubini's Theorem, we have

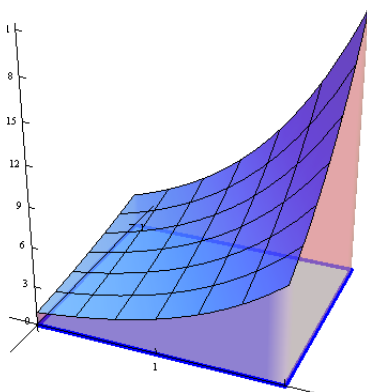
$$\begin{aligned}
 V &= \iint_R \frac{y^2}{x^2} dA = \int_1^2 \left[\int_1^2 \frac{y^2}{x^2} dy \right] dx = \int_1^2 \frac{1}{x^2} \left[\int_1^2 y^2 dy \right] dx = \int_1^2 \frac{1}{x^2} \left[\frac{y^3}{3} \right]_1^2 dx \\
 &= \int_1^2 \frac{1}{x^2} \left(\frac{8}{3} - \frac{1}{3} \right) dx = \frac{7}{3} \int_1^2 x^{-2} dx = \frac{7}{3} \left[\frac{x^{-1}}{-1} \right]_1^2 = \frac{7}{3} \left[-\frac{1}{x} \right]_1^2 \\
 &= \frac{7}{3} \left(-\frac{1}{2} + 1 \right) = \frac{7}{3} \cdot \frac{1}{2} = \boxed{\frac{7}{6} \text{ cubic units}}.
 \end{aligned}$$



63. We begin by graphing $z = f(x, y) = e^{x+y}$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 2$, $0 \leq y \leq 1$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R e^{x+y} dA = \iint_R e^x e^y dA$. Using

Fubini's Theorem, we have

$$\begin{aligned}
 V &= \int_0^2 \left[\int_0^1 e^x e^y dy \right] dx = \int_0^2 e^x \left[\int_0^1 e^y dy \right] dx = \int_0^2 e^x [e^y]_0^1 dx = \int_0^2 e^x (e^1 - e^0) dx \\
 &= (e - 1) \int_0^2 e^x dx = (e - 1) [e^x]_0^2 = (e - 1) (e^2 - e^0) = (e - 1) (e^2 - 1) \\
 &= (e - 1)(e - 1)(e + 1) = \boxed{(e - 1)^2(e + 1) \text{ cubic units}}.
 \end{aligned}$$

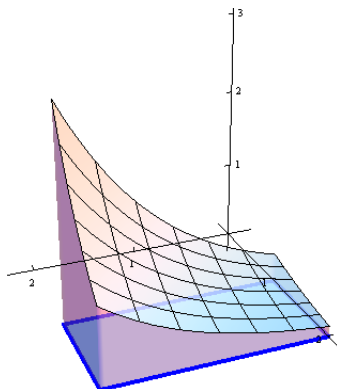


64. We begin by graphing $z = f(x, y) = e^{x-y}$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 2$, $1 \leq y \leq 2$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R e^{x-y} dA = \iint_R e^x e^{-y} dA$. Using Fubini's Theorem, we have

$$\begin{aligned}
 V &= \int_1^2 \left[\int_0^2 e^x e^{-y} dx \right] dy = \int_1^2 e^{-y} \left[\int_0^2 e^x dx \right] dy \\
 &= \int_1^2 e^{-y} [e^x]_0^2 dy = \int_1^2 e^{-y} (e^2 - e^0) dy = (e^2 - 1) \int_1^2 e^{-y} dy.
 \end{aligned}$$

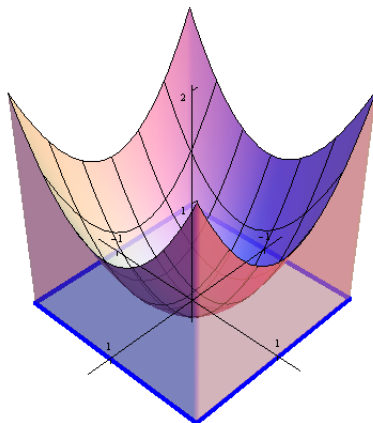
To complete integration, we make a substitution $u = -y$; then $du = -dy$, so $dy = -du$. We also change the limits of integration: if $y = 1$, then $u = -1$; if $y = 2$, then $u = -2$. So

$$\begin{aligned}
 V &= (e^2 - 1) \int_1^2 e^{-y} dy = (e^2 - 1) \int_{-1}^{-2} e^u (-du) = -(e^2 - 1) \int_{-1}^{-2} e^u du = -(e^2 - 1) [e^u]_{-1}^{-2} \\
 &= -(e^2 - 1)(e^{-2} - e^{-1}) = e - 1 - e^{-1} + e^{-2} = \boxed{e - 1 - \frac{1}{e} + \frac{1}{e^2} \text{ cubic units}}.
 \end{aligned}$$



65. The square in the xy -plane enclosed by the lines $x = \pm 1$ and $y = \pm 1$ can in other words be described as the closed rectangular region $R = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. To find the required volume, we begin by graphing the paraboloid $z = f(x, y) = x^2 + y^2$ and the region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the region R , the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (x^2 + y^2) dA$. Using Fubini's Theorem, we have

$$\begin{aligned} V &= \int_{-1}^1 \left[\int_{-1}^1 (x^2 + y^2) dx \right] dy = \int_{-1}^1 \left[\frac{x^3}{3} + xy^2 \right]_{-1}^1 dy = \int_{-1}^1 \left[\left(\frac{1}{3} + y^2 \right) - \left(-\frac{1}{3} - y^2 \right) \right] dy \\ &= \int_{-1}^1 \left(\frac{2}{3} + 2y^2 \right) dy = \left[\frac{2}{3}y + \frac{2}{3}y^3 \right]_{-1}^1 = \left(\frac{2}{3} + \frac{2}{3} \right) - \left(-\frac{2}{3} - \frac{2}{3} \right) = \boxed{\frac{8}{3} \text{ cubic units}}. \end{aligned}$$



66. Let R be the rectangular region in the xy -plane defined by $a \leq x \leq b$, $c \leq y \leq d$. Let $f(x, y) = 1$ be a function of two variables. Since $z = f(x, y) = 1$ is continuous and $z \geq 0$ over the region R , the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R dA$. Using Fubini's Theorem, we can evaluate this double integral as an iterated integral:

$$\iint_R dA = \int_c^d \left[\int_a^b dx \right] dy = \int_c^d [x]_a^b dy = \int_c^d (b - a) dy = (b - a) \int_c^d dy = (b - a)[y]_c^d = (b - a)(d - c).$$

Challenge Problems

67. (a) Let $z = f(x, y)$ be integrable over a closed rectangular region R of area A . If R is partitioned into N subrectangles of equal area ΔA , the area of each subrectangle is $\Delta A = \frac{A}{N}$, from which $N = \frac{A}{\Delta A}$. If (u_k, v_k) is the center of k th subrectangle, $k = 1, 2, \dots, N$, then the average of the values of f at these points is

$$AVG = \frac{1}{N} \sum_{k=1}^N f(u_k, v_k) = \frac{\Delta A}{A} \sum_{k=1}^N f(u_k, v_k) = \frac{1}{A} \sum_{k=1}^N f(u_k, v_k) \Delta A,$$

where at the last step we used the fact that ΔA is a constant independent of k (all subrectangles have equal area), and so it can be distributed into the summation over k .

- (b) Answers will vary.

68. The area of the rectangular region R defined by $0 \leq x \leq \pi$, $1 \leq y \leq 5$ is $A = 5\pi$. Then by the result of Problem 67, the average value of $f(x, y) = y \cos x$ over R is

$$\begin{aligned} \frac{1}{A} \iint_R f(x, y) dA &= \frac{1}{5\pi} \iint_R y \cos x dA = \frac{1}{5\pi} \int_1^5 \left[\int_0^\pi y \cos x dx \right] dy = \frac{1}{5\pi} \int_1^5 [y \sin x]_0^\pi dy \\ &= \frac{1}{5\pi} \int_1^5 (y \sin \pi - y \sin 0) dy = \frac{1}{5\pi} \int_1^5 0 dy = \frac{1}{5\pi} \cdot 0 = \boxed{0}. \end{aligned}$$

69. The area of the rectangular region R defined by $0 \leq x \leq 1$, $0 \leq y \leq 2$ is $A = 2$. Then by the result of Problem 67, the average value of $f(x, y) = \frac{xy}{x^2 + 1}$ over R is

$$\begin{aligned} \frac{1}{A} \iint_R f(x, y) dA &= \frac{1}{2} \iint_R \frac{xy}{x^2 + 1} dA = \frac{1}{2} \int_0^1 \left[\int_0^2 \frac{xy}{x^2 + 1} dy \right] dx = \frac{1}{2} \int_0^1 \frac{x}{x^2 + 1} \left[\int_0^2 y dy \right] dx \\ &= \frac{1}{2} \int_0^1 \frac{x}{x^2 + 1} \left[\frac{y^2}{2} \right]_0^2 dx = \frac{1}{2} \int_0^1 \frac{x}{x^2 + 1} (2 - 0) dx = \int_0^1 \frac{x}{x^2 + 1} dx. \end{aligned}$$

To complete the integration, we make a substitution $u = x^2 + 1$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 + 1 = 1$; if $x = 1$, then $u = 1^2 + 1 = 2$. So

$$\int_0^1 \frac{x}{x^2 + 1} dx = \int_1^2 \frac{1}{u} \frac{du}{2} = \frac{1}{2} \int_1^2 \frac{du}{u} = \frac{1}{2} [\ln |u|]_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} (\ln 2 - 0) = \boxed{\frac{\ln 2}{2}}.$$

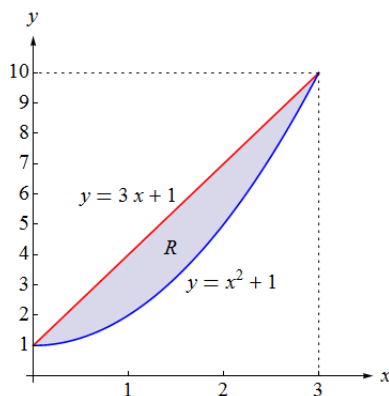
14.2 The Double Integral over Nonrectangular Regions

Concepts and Vocabulary

- By definition, a closed bounded region R that has a boundary consisting of two smooth curves $y = g_1(x)$ and $y = g_2(x)$, both defined on $a \leq x \leq b$ with $g_1(x) \leq g_2(x)$ on $[a, b]$, and possibly portions of the vertical lines $x = a$ and $x = b$, is called an **x -simple** region.
- True**. If R is a closed bounded y -simple region whose boundary consists of two smooth curves $x = h_1(y)$ and $x = h_2(y)$, both defined on $c \leq y \leq d$ with $h_1(y) \leq h_2(y)$ on $[c, d]$, and possibly portions of the horizontal lines $y = c$ and $y = d$, and if $f(x, y)$ is continuous on R , then

$$\iint_R f(x, y) dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy.$$

- True**. See to the figure below. R is x -simple because it is enclosed by the graphs $y = g_1(x) = x^2 + 1$ and $y = g_2(x) = 3x + 1$ on $0 \leq x \leq 3$, and $g_1(x) \leq g_2(x)$ on $[0, 3]$. R is y -simple because it is enclosed by the graphs $x = h_1(y) = \frac{y-1}{3}$ and $x = h_2(y) = \sqrt{y-1}$ on $1 \leq y \leq 10$, and $h_1(y) \leq h_2(y)$ on $[1, 10]$.



- Answers will vary.

Skill Building

- (a) Observe that the given region R is y -simple: it is bounded by the graphs of $x = h_1(y) = 0$ and $x = h_2(y) = 9 - y^2$ between $y = 1$ and $y = 3$. Using Fubini's Theorem for y -simple regions,

$$\iint_R xy^2 dA = \int_1^3 \left[\int_0^{9-y^2} xy^2 dx \right] dy.$$

- (b) Observe that the given region R is x -simple: it is bounded by the graphs of $y = g_1(x) = 1$ and $y = g_2(x) = \sqrt{9-x}$ between $x = 0$ and $x = 8$. Using Fubini's Theorem for x -simple regions,

$$\iint_R xy^2 dA = \int_0^8 \left[\int_1^{\sqrt{9-x}} xy^2 dy \right] dx.$$

(c) To evaluate the iterated integral in part (a), we first integrate partially with respect to x :

$$\begin{aligned}
 \int_1^3 \left[\int_0^{9-y^2} xy^2 dx \right] dy &= \int_1^3 y^2 \left[\int_0^{9-y^2} x dx \right] dy = \int_1^3 y^2 \left[\frac{x^2}{2} \right]_0^{9-y^2} dy \\
 &= \int_1^3 y^2 \left[\frac{(9-y^2)^2}{2} - \frac{0^2}{2} \right] dy = \int_1^3 y^2 \left(\frac{81}{2} - 9y^2 + \frac{1}{2}y^4 \right) dy \\
 &= \int_1^3 \left(\frac{81}{2}y^2 - 9y^4 + \frac{1}{2}y^6 \right) dy = \left[\frac{27}{2}y^3 - \frac{9}{5}y^5 + \frac{1}{14}y^7 \right]_1^3 \\
 &= \left(\frac{27}{2} \cdot 3^3 - \frac{9}{5} \cdot 3^5 + \frac{1}{14} \cdot 3^7 \right) - \left(\frac{27}{2} \cdot 1^3 - \frac{9}{5} \cdot 1^5 + \frac{1}{14} \cdot 1^7 \right) \\
 &= \frac{729}{2} - \frac{2187}{5} + \frac{2187}{14} - \frac{27}{2} + \frac{9}{5} - \frac{1}{14} = \boxed{\frac{2504}{35}}.
 \end{aligned}$$

To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned}
 \int_0^8 \left[\int_1^{\sqrt{9-x}} xy^2 dy \right] dx &= \int_0^8 x \left[\int_1^{\sqrt{9-x}} y^2 dy \right] dx = \int_0^8 x \left[\frac{y^3}{3} \right]_1^{\sqrt{9-x}} dx \\
 &= \int_0^8 x \left(\frac{(\sqrt{9-x})^3}{3} - \frac{1}{3} \right) dx = \frac{1}{3} \int_0^8 x(9-x)^{3/2} dx - \frac{1}{3} \int_0^8 x dx.
 \end{aligned}$$

The second integral can be evaluated directly as

$$\frac{1}{3} \int_0^8 x dx = \frac{1}{3} \left[\frac{x^2}{2} \right]_0^8 = \frac{1}{3}(32 - 0) = \frac{32}{3}.$$

To evaluate the first integral, we make a substitution $u = 9 - x$, so that $x = 9 - u$; then $du = -dx$, so $dx = -du$. We also change the limits of integration: if $x = 0$, then $u = 9 - 0 = 9$; if $x = 8$, then $u = 9 - 8 = 1$. So

$$\begin{aligned}
 \frac{1}{3} \int_0^8 x(9-x)^{3/2} dx &= \frac{1}{3} \int_9^1 (9-u)u^{3/2}(-du) = -\frac{1}{3} \int_9^1 (9-u)u^{3/2} du = \frac{1}{3} \int_1^9 (9-u)u^{3/2} du \\
 &= \frac{1}{3} \int_1^9 (9u^{3/2} - u^{5/2}) du = \frac{1}{3} \left[9 \cdot \frac{u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} \right]_1^9 = \left[\frac{6}{5}(\sqrt{u})^5 - \frac{2}{21}(\sqrt{u})^7 \right]_1^9 \\
 &= \left(\frac{6}{5}(\sqrt{9})^5 - \frac{2}{21}(\sqrt{9})^7 \right) - \left(\frac{6}{5}(\sqrt{1})^5 - \frac{2}{21}(\sqrt{1})^7 \right) \\
 &= \frac{1458}{5} - \frac{1458}{7} - \frac{6}{5} + \frac{2}{21} = \frac{8632}{105}.
 \end{aligned}$$

The final answer is

$$\frac{1}{3} \int_0^8 x(9-x)^{3/2} dx - \frac{1}{3} \int_0^8 x dx = \frac{8632}{105} - \frac{32}{3} = \boxed{\frac{2504}{35}}.$$

6. (a) Observe that the given region R is y -simple: it is bounded by the graphs of $x = h_1(y) = y^2$ and $x = h_2(y) = \sqrt[3]{y}$ between $y = 0$ and $y = 1$. Using Fubini's Theorem for y -simple regions,

$$\iint_R xy^2 dA = \int_0^1 \left[\int_{y^2}^{\sqrt[3]{y}} xy^2 dx \right] dy.$$

- (b) Observe that the given region R is x -simple: it is bounded by the graphs of $y = g_1(x) = x^3$ and $y = g_2(x) = \sqrt{x}$ between $x = 0$ and $x = 1$. Using Fubini's Theorem for x -simple regions,

$$\iint_R xy^2 dA = \boxed{\int_0^1 \left[\int_{x^3}^{\sqrt{x}} xy^2 dy \right] dx}.$$

- (c) To evaluate the iterated integral in part (a), we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_{y^2}^{\sqrt[3]{y}} xy^2 dx \right] dy &= \int_0^1 y^2 \left[\int_{y^2}^{\sqrt[3]{y}} x dx \right] dy = \int_0^1 y^2 \left[\frac{x^2}{2} \right]_{y^2}^{\sqrt[3]{y}} dy \\ &= \int_0^1 y^2 \left[\frac{(\sqrt[3]{y})^2}{2} - \frac{(y^2)^2}{2} \right] dy = \frac{1}{2} \int_0^1 y^2 (y^{2/3} - y^4) dy \\ &= \frac{1}{2} \int_0^1 (y^{8/3} - y^6) dy = \frac{1}{2} \left[\frac{y^{11/3}}{11/3} - \frac{y^7}{7} \right]_0^1 = \frac{1}{2} \left[\frac{3}{11} y^{11/3} - \frac{1}{7} y^7 \right]_0^1 \\ &= \frac{1}{2} \left[\left(\frac{3}{11} - \frac{1}{7} \right) - (0 - 0) \right] = \frac{1}{2} \cdot \frac{10}{77} = \boxed{\frac{5}{77}}. \end{aligned}$$

To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_{x^3}^{\sqrt{x}} xy^2 dy \right] dx &= \int_0^1 x \left[\int_{x^3}^{\sqrt{x}} y^2 dy \right] dx = \int_0^1 x \left[\frac{y^3}{3} \right]_{x^3}^{\sqrt{x}} dx \\ &= \int_0^1 x \left[\frac{(\sqrt{x})^3}{3} - \frac{(x^3)^3}{3} \right] dx = \frac{1}{3} \int_0^1 x (x^{3/2} - x^9) dx \\ &= \frac{1}{3} \int_0^1 (x^{5/2} - x^{10}) dx = \frac{1}{3} \left[\frac{x^{7/2}}{7/2} - \frac{x^{11}}{11} \right]_0^1 = \frac{1}{3} \left[\frac{2}{7} x^{7/2} - \frac{1}{11} x^{11} \right]_0^1 \\ &= \frac{1}{3} \left[\left(\frac{2}{7} - \frac{1}{11} \right) - (0 - 0) \right] = \frac{1}{3} \cdot \frac{15}{77} = \boxed{\frac{5}{77}}. \end{aligned}$$

7. (a) Observe that the given region R is y -simple: it is bounded by the graphs of $x = h_1(y) = 3 - y$ and $x = h_2(y) = \sqrt{9 - y^2}$ between $y = 0$ and $y = 3$. Using Fubini's Theorem for y -simple regions,

$$\iint_R xy^2 dA = \boxed{\int_0^3 \left[\int_{3-y}^{\sqrt{9-y^2}} xy^2 dx \right] dy}.$$

- (b) Observe that the given region R is x -simple: it is bounded by the graphs of $y = g_1(x) = 3 - x$ and $y = g_2(x) = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$. Using Fubini's Theorem for x -simple regions,

$$\iint_R xy^2 dA = \boxed{\int_0^3 \left[\int_{3-x}^{\sqrt{9-x^2}} xy^2 dy \right] dx}.$$

(c) To evaluate the iterated integral in part (a), we first integrate partially with respect to x :

$$\begin{aligned}
 \int_0^3 \left[\int_{3-y}^{\sqrt{9-y^2}} xy^2 dx \right] dy &= \int_0^3 y^2 \left[\int_{3-y}^{\sqrt{9-y^2}} x dx \right] dy = \int_0^3 y^2 \left[\frac{x^2}{2} \right]_{3-y}^{\sqrt{9-y^2}} dy \\
 &= \int_0^3 y^2 \left[\frac{(\sqrt{9-y^2})^2}{2} - \frac{(3-y)^2}{2} \right] dy \\
 &= \int_0^3 y^2 \cdot \frac{1}{2} [(9-y^2) - (9-6y+y^2)] dy \\
 &= \int_0^3 y^2 \cdot \frac{1}{2} (6y-2y^2) dy = \int_0^3 (3y^3 - y^4) dy \\
 &= \left[\frac{3}{4}y^4 - \frac{1}{5}y^5 \right]_0^3 = \left(\frac{3}{4} \cdot 3^4 - \frac{1}{5} \cdot 3^5 \right) - \left(\frac{3}{4} \cdot 0^4 - \frac{1}{5} \cdot 0^5 \right) \\
 &= \left(\frac{243}{4} - \frac{243}{5} \right) - 0 = \boxed{\frac{243}{20}}.
 \end{aligned}$$

To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned}
 \int_0^3 \left[\int_{3-x}^{\sqrt{9-x^2}} xy^2 dy \right] dx &= \int_0^3 x \left[\int_{3-x}^{\sqrt{9-x^2}} y^2 dy \right] dx = \int_0^3 x \left[\frac{y^3}{3} \right]_{3-x}^{\sqrt{9-x^2}} dx \\
 &= \int_0^3 x \left[\frac{(\sqrt{9-x^2})^3}{3} - \frac{(3-x)^3}{3} \right] dx \\
 &= \frac{1}{3} \int_0^3 x (\sqrt{9-x^2})^3 dx - \frac{1}{3} \int_0^3 x(3-x)^3 dx.
 \end{aligned}$$

The second integral can be evaluated directly as

$$\begin{aligned}
 \frac{1}{3} \int_0^3 x(3-x)^3 dx &= \frac{1}{3} \int_0^3 x(27-27x+9x^2-x^3) dx = \int_0^3 \left(9x-9x^2+3x^3-\frac{1}{3}x^4 \right) dx \\
 &= \left[\frac{9}{2}x^2-3x^3+\frac{3}{4}x^4-\frac{1}{15}x^5 \right]_0^3 = \left(\frac{9}{2} \cdot 3^2 - 3 \cdot 3^3 + \frac{3}{4} \cdot 3^4 - \frac{1}{15} \cdot 3^5 \right) - 0 \\
 &= \frac{81}{2} - 81 + \frac{243}{4} - \frac{81}{5} = \frac{81}{20}.
 \end{aligned}$$

To evaluate the first integral, we make a substitution $u = 9 - x^2$; then $du = -2x dx$, so $x dx = \frac{du}{-2}$. We also change the limits of integration: if $x = 0$, then $u = 9 - 0^2 = 9$; if $x = 3$, then $u = 9 - 3^2 = 0$. So

$$\begin{aligned}
 \frac{1}{3} \int_0^3 x (\sqrt{9-x^2})^3 dx &= \frac{1}{3} \int_9^0 (\sqrt{u})^3 \frac{du}{-2} = -\frac{1}{6} \int_9^0 u^{3/2} du = \frac{1}{6} \int_0^9 u^{3/2} du = \frac{1}{6} \left[\frac{u^{5/2}}{5/2} \right]_0^9 \\
 &= \frac{1}{15} [(\sqrt{u})^5]_0^9 = \frac{1}{15} \left((\sqrt{9})^5 - (\sqrt{0})^5 \right) = \frac{1}{15} \cdot 243 = \frac{81}{5}.
 \end{aligned}$$

The final answer is

$$\frac{1}{3} \int_0^3 x (\sqrt{9-x^2})^3 dx - \frac{1}{3} \int_0^3 x(3-x)^3 dx = \frac{81}{5} - \frac{81}{20} = \boxed{\frac{243}{20}}.$$

8. (a) Observe that the given region R is y -simple: it is bounded by the graphs of $x = h_1(y) = \frac{y}{4}$ and $x = h_2(y) = \sqrt[3]{y}$ between $y = 0$ and $y = 8$. Using Fubini's Theorem for y -simple regions,

$$\iint_R xy^2 dA = \int_0^8 \left[\int_{y/4}^{\sqrt[3]{y}} xy^2 dx \right] dy.$$

- (b) Observe that the given region R is x -simple: it is bounded by the graphs of $y = g_1(x) = x^3$ and $y = g_2(x) = 4x$ between $x = 0$ and $x = 2$. Using Fubini's Theorem for x -simple regions,

$$\iint_R xy^2 dA = \int_0^2 \left[\int_{x^3}^{4x} xy^2 dy \right] dx.$$

- (c) To evaluate the iterated integral in part (a), we first integrate partially with respect to x :

$$\begin{aligned} \int_0^8 \left[\int_{y/4}^{\sqrt[3]{y}} xy^2 dx \right] dy &= \int_0^8 y^2 \left[\int_{y/4}^{\sqrt[3]{y}} x dx \right] dy = \int_0^8 y^2 \left[\frac{x^2}{2} \right]_{y/4}^{\sqrt[3]{y}} dy \\ &= \int_0^8 y^2 \left[\frac{(\sqrt[3]{y})^2}{2} - \frac{(y/4)^2}{2} \right] dy = \int_0^8 y^2 \left(\frac{1}{2} y^{2/3} - \frac{1}{32} y^2 \right) dy \\ &= \int_0^8 \left(\frac{1}{2} y^{8/3} - \frac{1}{32} y^4 \right) dy = \left[\frac{1}{2} \cdot \frac{y^{11/3}}{11/3} - \frac{1}{32} \cdot \frac{y^5}{5} \right]_0^8 \\ &= \left[\frac{3}{22} (\sqrt[3]{y})^{11} - \frac{1}{160} y^5 \right]_0^8 \\ &= \left(\frac{3}{22} \cdot (\sqrt[3]{8})^{11} - \frac{1}{160} \cdot 8^5 \right) - \left(\frac{3}{22} \cdot (\sqrt[3]{0})^{11} - \frac{1}{160} \cdot 0^5 \right) \\ &= \left(\frac{3072}{11} - \frac{1024}{5} \right) - 0 = \boxed{\frac{4096}{55}}. \end{aligned}$$

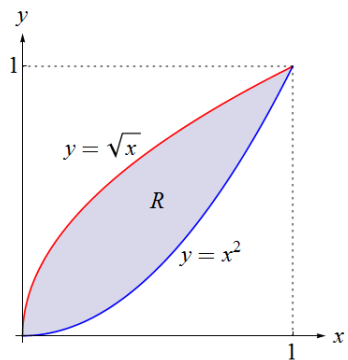
To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned} \int_0^2 \left[\int_{x^3}^{4x} xy^2 dy \right] dx &= \int_0^2 x \left[\int_{x^3}^{4x} y^2 dy \right] dx = \int_0^2 x \left[\frac{y^3}{3} \right]_{x^3}^{4x} dx \\ &= \int_0^2 x \left[\frac{(4x)^3}{3} - \frac{(x^3)^3}{3} \right] dx = \int_0^2 x \left(\frac{64}{3} x^3 - \frac{1}{3} x^9 \right) dx \\ &= \int_0^2 \left(\frac{64}{3} x^4 - \frac{1}{3} x^{10} \right) dx = \left[\frac{64}{3} \cdot \frac{x^5}{5} - \frac{1}{3} \cdot \frac{x^{11}}{11} \right]_0^2 = \left[\frac{64}{15} x^5 - \frac{1}{33} x^{11} \right]_0^2 \\ &= \left(\frac{64}{15} \cdot 2^5 - \frac{1}{33} \cdot 2^{11} \right) - \left(\frac{64}{15} \cdot 0^5 - \frac{1}{33} \cdot 0^{11} \right) \\ &= \left(\frac{2048}{15} - \frac{2048}{33} \right) - 0 = \boxed{\frac{4096}{55}}. \end{aligned}$$

9. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_{x^2}^{\sqrt{x}} dy \right] dx &= \int_0^1 [y]_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^2) dx = \int_0^1 (x^{1/2} - x^2) dx \\ &= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1 = \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1 = \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \boxed{\frac{1}{3}}. \end{aligned}$$

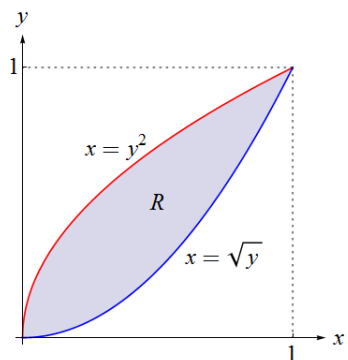
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x^2$ and $y = g_2(x) = \sqrt{x}$ between $x = 0$ and $x = 1$.



10. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_{y^2}^{\sqrt{y}} x \, dx \right] dy &= \int_0^1 \left[\frac{x^2}{2} \right]_{y^2}^{\sqrt{y}} dy = \int_0^1 \left[\frac{(\sqrt{y})^2}{2} - \frac{(y^2)^2}{2} \right] dy = \frac{1}{2} \int_0^1 (y - y^4) dy \\ &= \frac{1}{2} \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1 = \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{5} \right) - (0 - 0) \right] = \frac{1}{2} \cdot \frac{3}{10} = \boxed{\frac{3}{20}}. \end{aligned}$$

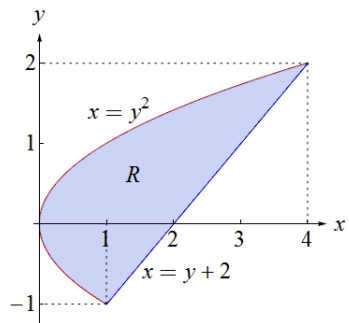
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = y^2$ and $x = h_2(y) = \sqrt{y}$ between $y = 0$ and $y = 1$.



11. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned} \int_{-1}^2 \left[\int_{y^2}^{y+2} dx \right] dy &= \int_{-1}^2 [x]_{y^2}^{y+2} dy = \int_{-1}^2 (y + 2 - y^2) dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 \\ &= \left(\frac{2^2}{2} + 2 \cdot 2 - \frac{2^3}{3} \right) - \left(\frac{(-1)^2}{2} + 2 \cdot (-1) - \frac{(-1)^3}{3} \right) \\ &= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = \boxed{\frac{9}{2}}. \end{aligned}$$

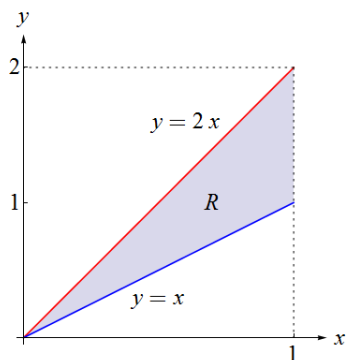
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = y^2$ and $x = h_2(y) = y + 2$ between $y = -1$ and $y = 2$.



12. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_x^{2x} y \, dy \right] dx &= \int_0^1 \left[\frac{y^2}{2} \right]_x^{2x} dx = \int_0^1 \left[\frac{(2x)^2}{2} - \frac{x^2}{2} \right] dx \\ &= \int_0^1 \left(\frac{4x^2}{2} - \frac{x^2}{2} \right) dx = \int_0^1 \frac{3}{2} x^2 \, dx = \left[\frac{1}{2} x^3 \right]_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}. \end{aligned}$$

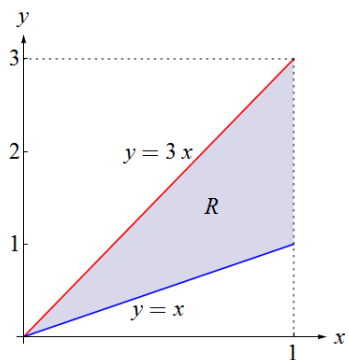
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x$ and $y = g_2(x) = 2x$ between $x = 0$ and $x = 1$.



13. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_x^{3x} xy \, dy \right] dx &= \int_0^1 x \left[\int_x^{3x} y \, dy \right] dx = \int_0^1 x \left[\frac{y^2}{2} \right]_x^{3x} dx = \int_0^1 x \left[\frac{(3x)^2}{2} - \frac{x^2}{2} \right] dx \\ &= \int_0^1 x \left(\frac{9x^2}{2} - \frac{x^2}{2} \right) dx = \int_0^1 x \cdot 4x^2 \, dx = \int_0^1 4x^3 \, dx = \left[x^4 \right]_0^1 = 1 - 0 = \boxed{1}. \end{aligned}$$

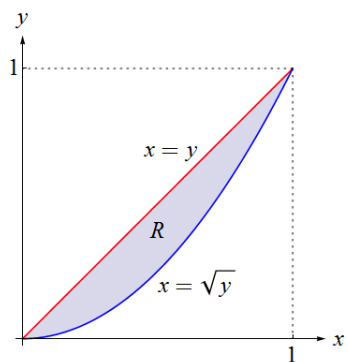
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x$ and $y = g_2(x) = 3x$ between $x = 0$ and $x = 1$.



14. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_y^{\sqrt{y}} x^2 y \, dx \right] dy &= \int_0^1 y \left[\int_y^{\sqrt{y}} x^2 \, dx \right] dy = \int_0^1 y \left[\frac{x^3}{3} \right]_y^{\sqrt{y}} dy = \int_0^1 y \left[\frac{(\sqrt{y})^3}{3} - \frac{y^3}{3} \right] dy \\ &= \frac{1}{3} \int_0^1 y (y^{3/2} - y^3) dy = \frac{1}{3} \int_0^1 (y^{5/2} - y^4) dy = \frac{1}{3} \left[\frac{y^{7/2}}{7/2} - \frac{y^5}{5} \right]_0^1 \\ &= \frac{1}{3} \left[\frac{2}{7} y^{7/2} - \frac{1}{5} y^5 \right]_0^1 = \frac{1}{3} \left[\left(\frac{2}{7} - \frac{1}{5} \right) - (0 - 0) \right] = \frac{1}{3} \cdot \frac{3}{35} = \boxed{\frac{1}{35}}. \end{aligned}$$

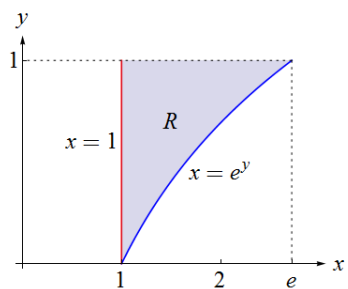
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = y$ and $x = h_2(y) = \sqrt{y}$ between $y = 0$ and $y = 1$.



15. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_1^{e^y} \frac{y}{x} \, dx \right] dy &= \int_0^1 y \left[\int_1^{e^y} \frac{1}{x} \, dx \right] dy = \int_0^1 y [\ln |x|]_1^{e^y} dy \\ &= \int_0^1 y (\ln e^y - \ln 1) dy = \int_0^1 y(y - 0) dy = \int_0^1 y^2 dy = \left[\frac{y^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}. \end{aligned}$$

The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = 1$ and $x = h_2(y) = e^y$ between $y = 0$ and $y = 1$.



16. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_0^{x^2} x e^y \, dy \right] dx &= \int_0^1 x \left[\int_0^{x^2} e^y \, dy \right] dx = \int_0^1 x [e^y]_0^{x^2} dx \\ &= \int_0^1 x (e^{x^2} - e^0) dx = \int_0^1 x (e^{x^2} - 1) dx = \int_0^1 x e^{x^2} dx - \int_0^1 x dx. \end{aligned}$$

The second integral can be evaluated directly as $\int_0^1 x \, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$.

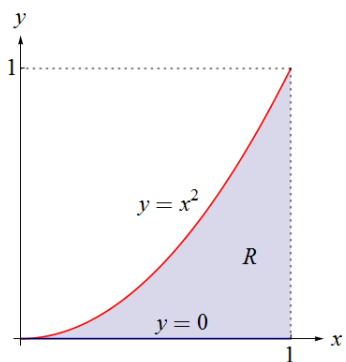
To evaluate the first integral, we make a substitution $u = x^2$; then $du = 2x \, dx$, so $x \, dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 = 0$; if $x = 1$, then $u = 1^2 = 1$. So

$$\int_0^1 x e^{x^2} \, dx = \int_0^1 e^u \frac{du}{2} = \frac{1}{2} \int_0^1 e^u \, du = \frac{1}{2} [e^u]_0^1 = \frac{1}{2} (e^1 - e^0) = \frac{1}{2} (e - 1) = \frac{1}{2}e - \frac{1}{2}.$$

Then the final answer is

$$\int_0^1 x e^{x^2} \, dx - \int_0^1 x \, dx = \frac{1}{2}e - \frac{1}{2} - \frac{1}{2} = \boxed{\frac{1}{2}e - 1}.$$

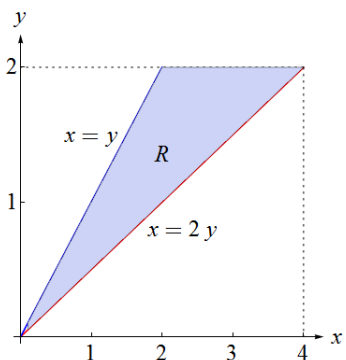
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = 0$ and $y = g_2(x) = x^2$ between $x = 0$ and $x = 1$.



17. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned} \int_0^2 \left[\int_y^{2y} xy \, dx \right] dy &= \int_0^2 y \left[\int_y^{2y} x \, dx \right] dy = \int_0^2 y \left[\frac{x^2}{2} \right]_y^{2y} dy = \int_0^2 y \left(\frac{(2y)^2}{2} - \frac{y^2}{2} \right) dy \\ &= \int_0^2 y \cdot \frac{3}{2} y^2 \, dy = \int_0^2 \frac{3}{2} y^3 \, dy = \left[\frac{3}{8} y^4 \right]_0^2 = \frac{3}{8} \cdot 2^4 - \frac{3}{8} \cdot 0^4 = 6 - 0 = \boxed{6}. \end{aligned}$$

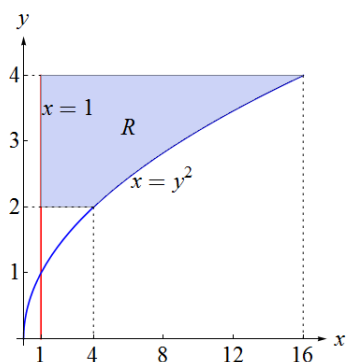
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = y$ and $x = h_2(y) = 2y$ between $y = 0$ and $y = 2$.



18. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned}
 \int_2^4 \left[\int_1^{y^2} \frac{y}{x^2} dx \right] dy &= \int_2^4 y \left[\int_1^{y^2} \frac{1}{x^2} dx \right] dy = \int_2^4 y \left[\int_1^{y^2} x^{-2} dx \right] dy = \int_2^4 y \left[\frac{x^{-1}}{-1} \right]_1^{y^2} dy \\
 &= \int_2^4 y \left[-\frac{1}{x} \right]_1^{y^2} dy = \int_2^4 y \left(-\frac{1}{y^2} + \frac{1}{1} \right) dy = \int_2^4 y \left(1 - \frac{1}{y^2} \right) dy \\
 &= \int_2^4 \left(y - \frac{1}{y} \right) dy = \left[\frac{y^2}{2} - \ln |y| \right]_2^4 = \left(\frac{4^2}{2} - \ln 4 \right) - \left(\frac{2^2}{2} - \ln 2 \right) \\
 &= 8 - \ln 4 - 2 + \ln 2 = \boxed{6 - \ln 2}.
 \end{aligned}$$

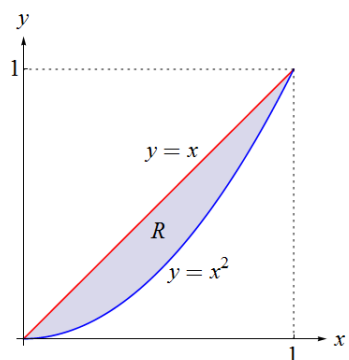
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = 1$ and $x = h_2(y) = y^2$ between $y = 2$ and $y = 4$.



19. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned}
 \int_0^1 \left[\int_{x^2}^x \sqrt{xy} dy \right] dx &= \int_0^1 \left[\int_{x^2}^x \sqrt{x} \sqrt{y} dy \right] dx = \int_0^1 x^{1/2} \left[\int_{x^2}^x y^{1/2} dy \right] dx = \int_0^1 x^{1/2} \left[\frac{y^{3/2}}{3/2} \right]_{x^2}^x dx \\
 &= \frac{2}{3} \int_0^1 x^{1/2} \left(x^{3/2} - (x^2)^{3/2} \right) dx = \frac{2}{3} \int_0^1 x^{1/2} \left(x^{3/2} - x^3 \right) dx \\
 &= \frac{2}{3} \int_0^1 \left(x^2 - x^{7/2} \right) dx = \frac{2}{3} \left[\frac{x^3}{3} - \frac{x^{9/2}}{9/2} \right]_0^1 = \frac{2}{3} \left[\frac{1}{3} x^3 - \frac{2}{9} x^{9/2} \right]_0^1 \\
 &= \frac{2}{3} \left[\left(\frac{1}{3} - \frac{2}{9} \right) - (0 - 0) \right] = \frac{2}{3} \cdot \frac{1}{9} = \boxed{\frac{2}{27}}.
 \end{aligned}$$

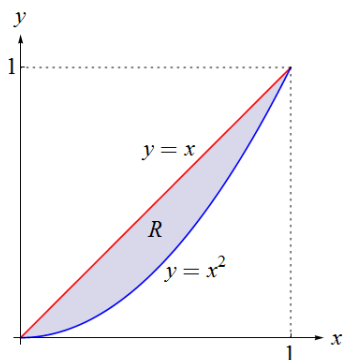
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x^2$ and $y = g_2(x) = x$ between $x = 0$ and $x = 1$.



20. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned}\int_0^1 \left[\int_{x^2}^x \sqrt{x} \, dy \right] dx &= \int_0^1 x^{1/2} \left[\int_{x^2}^x dy \right] dx = \int_0^1 x^{1/2} [y]_{x^2}^x dx = \int_0^1 x^{1/2} (x - x^2) dx \\ &= \int_0^1 (x^{3/2} - x^{5/2}) dx = \left[\frac{x^{5/2}}{5/2} - \frac{x^{7/2}}{7/2} \right]_0^1 = \left[\frac{2}{5} x^{5/2} - \frac{2}{7} x^{7/2} \right]_0^1 \\ &= \left[\left(\frac{2}{5} - \frac{2}{7} \right) - (0 - 0) \right] = \boxed{\frac{4}{35}}.\end{aligned}$$

The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x^2$ and $y = g_2(x) = x$ between $x = 0$ and $x = 1$.



21. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned}\int_0^1 \left[\int_x^1 \frac{1}{x^2 + 1} dy \right] dx &= \int_0^1 \frac{1}{x^2 + 1} \left[\int_x^1 dy \right] dx = \int_0^1 \frac{1}{x^2 + 1} [y]_x^1 dx \\ &= \int_0^1 \frac{1}{x^2 + 1} (1 - x) dx = \int_0^1 \frac{1}{x^2 + 1} dx - \int_0^1 \frac{x}{x^2 + 1} dx.\end{aligned}$$

The first integral can be evaluated directly as

$$\int_0^1 \frac{1}{x^2 + 1} dx = [\tan^{-1}(x)]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

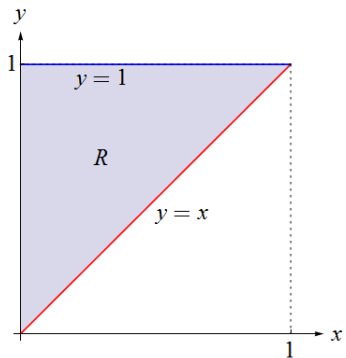
To evaluate the second integral, we make a substitution $u = x^2 + 1$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 + 1 = 1$; if $x = 1$, then $u = 1^2 + 1 = 2$. So

$$\int_0^1 \frac{x}{x^2 + 1} dx = \int_1^2 \frac{1}{u} \frac{du}{2} = \frac{1}{2} \int_1^2 \frac{1}{u} du = \frac{1}{2} [\ln |u|]_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} (\ln 2 - 0) = \frac{\ln 2}{2}.$$

Then the final answer is

$$\int_0^1 \frac{1}{x^2 + 1} dx - \int_0^1 \frac{x}{x^2 + 1} dx = \boxed{\frac{\pi}{4} - \frac{\ln 2}{2}}.$$

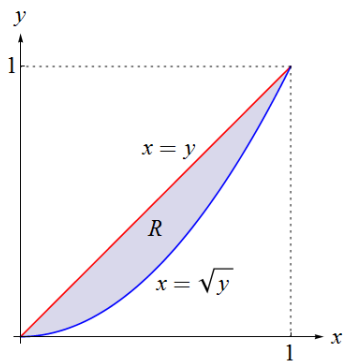
The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = x$ and $y = g_2(x) = 1$ between $x = 0$ and $x = 1$.



22. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\begin{aligned}
 \int_0^1 \left[\int_y^{\sqrt{y}} (x^2 + y^2) dx \right] dy &= \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_y^{\sqrt{y}} dy = \int_0^1 \left[\left(\frac{\sqrt{y}^3}{3} + \sqrt{y}y^2 \right) - \left(\frac{y^3}{3} + yy^2 \right) \right] dy \\
 &= \int_0^1 \left(\frac{1}{3}y^{3/2} + y^{5/2} - \frac{4}{3}y^3 \right) dy = \left[\frac{1}{3} \cdot \frac{y^{5/2}}{5/2} + \frac{y^{7/2}}{7/2} - \frac{4}{3} \cdot \frac{y^4}{4} \right]_0^1 \\
 &= \left[\frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{1}{3}y^4 \right]_0^1 = \left(\frac{2}{15} + \frac{2}{7} - \frac{1}{3} \right) - 0 = \boxed{\frac{3}{35}}.
 \end{aligned}$$

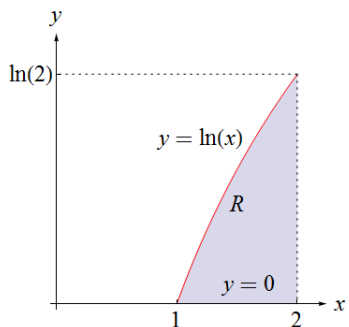
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = y$ and $x = h_2(y) = \sqrt{y}$ between $y = 0$ and $y = 1$.



23. To evaluate this iterated integral, we first integrate partially with respect to y :

$$\begin{aligned}
 \int_1^2 \left[\int_0^{\ln x} xe^y dy \right] dx &= \int_1^2 x \left[\int_0^{\ln x} e^y dy \right] dx = \int_1^2 x [e^y]_0^{\ln x} dx = \int_1^2 x (e^{\ln x} - e^0) dx \\
 &= \int_1^2 x(x - 1) dx = \int_1^2 (x^2 - x) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 \\
 &= \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right) = \boxed{\frac{5}{6}}.
 \end{aligned}$$

The graph of the region R is shown below. It is bounded by the graphs of $y = g_1(x) = 0$ and $y = g_2(x) = \ln x$ between $x = 1$ and $x = 2$.



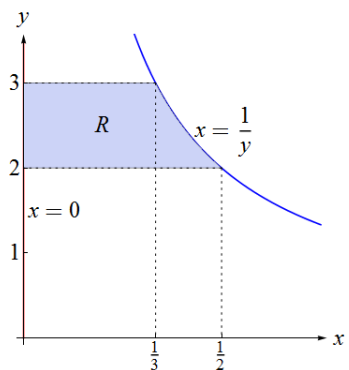
24. To evaluate this iterated integral, we first integrate partially with respect to x :

$$\int_2^3 \left[\int_0^{1/y} \ln y \, dx \right] dy = \int_2^3 \ln y \left[\int_0^{1/y} dx \right] dy = \int_2^3 \ln y [x]_0^{1/y} dy = \int_2^3 \ln y \left(\frac{1}{y} - 0 \right) dy = \int_2^3 \frac{\ln y}{y} dy.$$

To evaluate this integral, we make a substitution $u = \ln y$; then $du = \frac{1}{y} dy$. We also change the limits of integration: if $y = 2$, then $u = \ln 2$; if $y = 3$, then $u = \ln 3$. So

$$\int_2^3 \frac{\ln y}{y} dy = \int_{\ln 2}^{\ln 3} u \, du = \left[\frac{u^2}{2} \right]_{\ln 2}^{\ln 3} = \boxed{\frac{(\ln 3)^2 - (\ln 2)^2}{2}}.$$

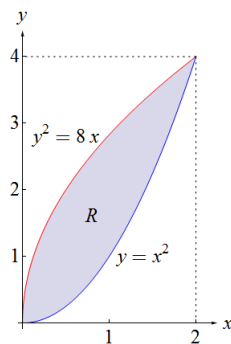
The graph of the region R is shown below. It is bounded by the graphs of $x = h_1(y) = 0$ and $x = h_2(y) = \frac{1}{y}$ between $y = 2$ and $y = 3$.



25. We begin by graphing the region R enclosed by $y = x^2$ and $y^2 = 8x$; see the figure below. Observe that R is a closed bounded region, that is x -simple: it is bounded by the graphs of $y = g_1(x) = x^2$ and

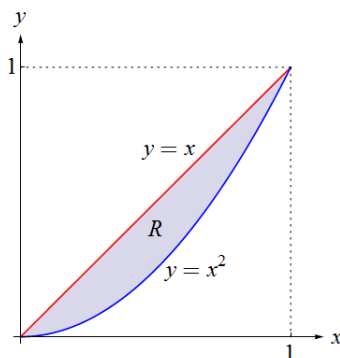
$y = g_2(x) = \sqrt{8x}$ between $x = 0$ and $x = 2$. So we can apply Fubini's Theorem for x -simple regions:

$$\begin{aligned}
 \iint_R (x+y) \, dA &= \int_0^2 \left[\int_{x^2}^{\sqrt{8x}} (x+y) \, dy \right] dx = \int_0^2 \left[xy + \frac{y^2}{2} \right]_{x^2}^{\sqrt{8x}} dx \\
 &= \int_0^2 \left[\left(x\sqrt{8x} + \frac{(\sqrt{8x})^2}{2} \right) - \left(xx^2 + \frac{(x^2)^2}{2} \right) \right] dx \\
 &= \int_0^2 \left(\sqrt{8}x^{3/2} + 4x - x^3 - \frac{1}{2}x^4 \right) dx \\
 &= \left[2\sqrt{2} \cdot \frac{x^{5/2}}{5/2} + 4 \cdot \frac{x^2}{2} - \frac{x^4}{4} - \frac{1}{2} \cdot \frac{x^5}{5} \right]_0^2 = \left[\frac{4\sqrt{2}}{5} (\sqrt{x})^5 + 2x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5 \right]_0^2 \\
 &= \left(\frac{4\sqrt{2}}{5} (\sqrt{2})^5 + 2 \cdot 2^2 - \frac{1}{4} \cdot 2^4 - \frac{1}{10} \cdot 2^5 \right) - 0 = \frac{32}{5} + 8 - 4 - \frac{16}{5} = \boxed{\frac{36}{5}}.
 \end{aligned}$$



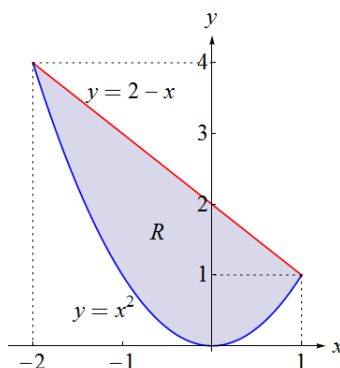
26. We begin by graphing the region R enclosed by $y = x$ and $y = x^2$; see the figure below. Observe that R is a closed bounded region, that is x -simple: it is bounded by the graphs of $y = g_1(x) = x^2$ and $y = g_2(x) = x$ between $x = 0$ and $x = 1$. So we can apply Fubini's Theorem for x -simple regions:

$$\begin{aligned}
 \iint_R (x^2 - y^2) \, dA &= \int_0^1 \left[\int_{x^2}^x (x^2 - y^2) \, dy \right] dx = \int_0^1 \left[x^2y - \frac{y^3}{3} \right]_{x^2}^x dx \\
 &= \int_0^1 \left[\left(x^2x - \frac{x^3}{3} \right) - \left(x^2x^2 - \frac{(x^2)^3}{3} \right) \right] dx = \int_0^1 \left(\frac{2}{3}x^3 - x^4 + \frac{1}{3}x^6 \right) dx \\
 &= \left[\frac{2}{3} \cdot \frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{3} \cdot \frac{x^7}{7} \right]_0^1 = \left[\frac{1}{6}x^4 - \frac{1}{5}x^5 + \frac{1}{21}x^7 \right]_0^1 \\
 &= \left(\frac{1}{6} - \frac{1}{5} + \frac{1}{21} \right) - 0 = \boxed{\frac{1}{70}}.
 \end{aligned}$$



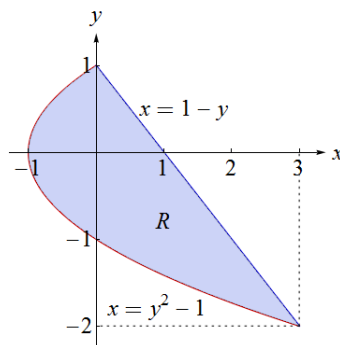
27. We begin by graphing the region R enclosed by $y = 2 - x$ and $y = x^2$; see the figure below. Observe that R is a closed bounded region, that is x -simple: it is bounded by the graphs of $y = g_1(x) = x^2$ and $y = g_2(x) = 2 - x$ between $x = -2$ and $x = 1$. So we can apply Fubini's Theorem for x -simple regions:

$$\begin{aligned}
 \iint_R y^2 dA &= \int_{-2}^1 \left[\int_{x^2}^{2-x} y^2 dy \right] dx = \int_{-2}^1 \left[\frac{y^3}{3} \right]_{x^2}^{2-x} dx = \int_{-2}^1 \frac{1}{3} [(2-x)^3 - (x^2)^3] dx \\
 &= \int_{-2}^1 \frac{1}{3} (8 - 12x + 6x^2 - x^3 - x^6) dx = \int_{-2}^1 \left(\frac{8}{3} - 4x + 2x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^6 \right) dx \\
 &= \left[\frac{8}{3}x - 2x^2 + \frac{2}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{21}x^7 \right]_{-2}^1 \\
 &= \left(\frac{8}{3} - 2 + \frac{2}{3} - \frac{1}{12} - \frac{1}{21} \right) - \left(-\frac{16}{3} - 8 - \frac{16}{3} - \frac{4}{3} + \frac{128}{21} \right) = \boxed{\frac{423}{28}}.
 \end{aligned}$$



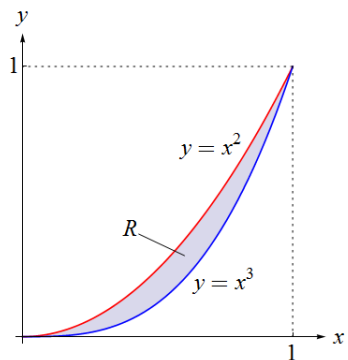
28. We begin by graphing the region R enclosed by $y^2 = x + 1$ and $y = 1 - x$; see the figure below. Observe that R is a closed bounded region, that is y -simple: it is bounded by the graphs of $x = h_1(y) = y^2 - 1$ and $x = h_2(y) = 1 - y$ between $y = -2$ and $y = 1$. So we can apply Fubini's Theorem for y -simple regions:

$$\begin{aligned}
 \iint_R xy dA &= \int_{-2}^1 \left[\int_{y^2-1}^{1-y} xy dx \right] dy = \int_{-2}^1 y \left[\int_{y^2-1}^{1-y} x dx \right] dy = \int_{-2}^1 y \left[\frac{x^2}{2} \right]_{y^2-1}^{1-y} dy \\
 &= \int_{-2}^1 \frac{1}{2} y [(1-y)^2 - (y^2-1)^2] dy = \int_{-2}^1 \frac{1}{2} y [(1-2y+y^2) - (y^4-2y^2+1)] dy \\
 &= \int_{-2}^1 \frac{1}{2} y (-y^4 + 3y^2 - 2y) dy = \int_{-2}^1 \left(-\frac{1}{2}y^5 + \frac{3}{2}y^3 - y^2 \right) dy \\
 &= \left[-\frac{1}{12}y^6 + \frac{3}{8}y^4 - \frac{1}{3}y^3 \right]_{-2}^1 = \left(-\frac{1}{12} + \frac{3}{8} - \frac{1}{3} \right) - \left(-\frac{16}{3} + 6 + \frac{8}{3} \right) = \boxed{-\frac{27}{8}}.
 \end{aligned}$$



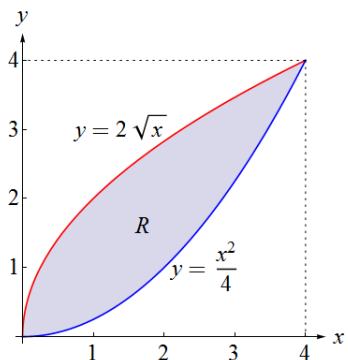
29. We begin by graphing the region R enclosed by the graphs of $y = x^3$ and $y = x^2$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = x^3$ to $y = x^2$, $0 \leq x \leq 1$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_0^1 \left[\int_{x^3}^{x^2} dy \right] dx = \int_0^1 [y]_{x^3}^{x^2} dx \\ &= \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} \right) - 0 = \boxed{\frac{1}{12} \text{ square units}}. \end{aligned}$$



30. We begin by graphing the region R enclosed by the graphs of $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$, $0 \leq x \leq 4$. Then the area A of the region R is

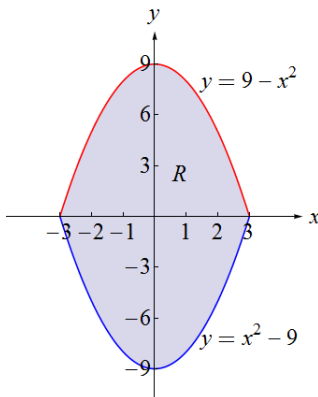
$$\begin{aligned} A &= \iint_R dA = \int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} dy \right] dx = \int_0^4 [y]_{x^2/4}^{2\sqrt{x}} dx \\ &= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx = \int_0^4 \left(2x^{1/2} - \frac{1}{4}x^2 \right) dx = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{1}{4} \cdot \frac{x^3}{3} \right]_0^4 \\ &= \left[\frac{4}{3} (\sqrt{x})^3 - \frac{1}{12} x^3 \right]_0^4 = \left(\frac{4}{3} \cdot (\sqrt{4})^3 - \frac{1}{12} \cdot 4^3 \right) - 0 = \frac{32}{3} - \frac{16}{3} = \boxed{\frac{16}{3} \text{ square units}}. \end{aligned}$$



31. We begin by graphing the region R enclosed by the graphs of $y = x^2 - 9$ and $y = 9 - x^2$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = x^2 - 9$ to

$y = 9 - x^2$, $-3 \leq x \leq 3$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_{-3}^3 \left[\int_{x^2-9}^{9-x^2} dy \right] dx = \int_{-3}^3 [y]_{x^2-9}^{9-x^2} dx = \int_{-3}^3 [(9-x^2) - (x^2-9)] dx \\ &= \int_{-3}^3 (18-2x^2) dx = \left[18x - \frac{2x^3}{3} \right]_{-3}^3 = (54-18) - (-54+18) = \boxed{72 \text{ square units}}. \end{aligned}$$



32. We begin by graphing the region R enclosed by the graphs of $x = \sqrt{16-y^2}$ and $y^2 = 6x$; see the figure below. Observe that R is a closed bounded region, that is y -simple: x varies from $x = \frac{y^2}{6}$ to $x = \sqrt{16-y^2}$, $-2\sqrt{3} \leq y \leq 2\sqrt{3}$. Also note that the region R is symmetric about the x -axis, so we can find the area of its half lying in the first quadrant, that is for $0 \leq y \leq 2\sqrt{3}$, and multiply it by 2. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = 2 \int_0^{2\sqrt{3}} \left[\int_{y^2/6}^{\sqrt{16-y^2}} dx \right] dy = 2 \int_0^{2\sqrt{3}} [x]_{y^2/6}^{\sqrt{16-y^2}} dy \\ &= 2 \int_0^{2\sqrt{3}} \left(\sqrt{16-y^2} - \frac{y^2}{6} \right) dy = 2 \int_0^{2\sqrt{3}} \sqrt{16-y^2} dy - \frac{1}{3} \int_0^{2\sqrt{3}} y^2 dy. \end{aligned}$$

The second integral can be evaluated directly as

$$\frac{1}{3} \int_0^{2\sqrt{3}} y^2 dy = \frac{1}{3} \left[\frac{y^3}{3} \right]_0^{2\sqrt{3}} = \frac{1}{9} [y^3]_0^{2\sqrt{3}} = \frac{1}{9} \left((2\sqrt{3})^3 - 0^3 \right) = \frac{1}{9} \cdot 24\sqrt{3} = \frac{8\sqrt{3}}{3}.$$

To find the first integral, we will find an antiderivative using a substitution $y = 4 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dy = 4 \cos \theta d\theta$. Also,

$$\sqrt{16-y^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16(1-\sin^2\theta)} = \sqrt{16\cos^2\theta} = 4\cos\theta,$$

and the integral becomes

$$\begin{aligned} 2 \int \sqrt{16-y^2} dy &= 2 \int 4\cos\theta \cdot 4\cos\theta d\theta = 32 \int \cos^2\theta d\theta = 32 \int \frac{1}{2}(1+\cos 2\theta) d\theta \\ &= 16 \int (1+\cos 2\theta) d\theta = 16 \left(\theta + \frac{1}{2} \sin 2\theta \right) = 16(\theta + \sin\theta \cos\theta). \end{aligned}$$

To express this antiderivative in terms of y , we first observe that since $y = 4 \sin \theta$, then $\sin \theta = \frac{y}{4}$ and

$\theta = \sin^{-1}\left(\frac{y}{4}\right)$; also, since $\sqrt{16-y^2} = 4\cos\theta$ (as shown above), then $\cos\theta = \frac{\sqrt{16-y^2}}{4}$. So,

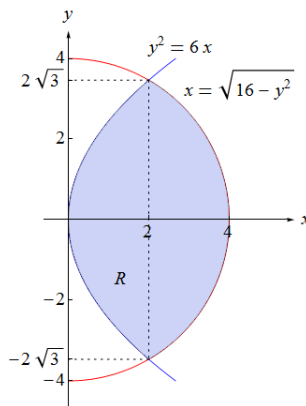
$$16(\theta + \sin\theta\cos\theta) = 16\left(\sin^{-1}\left(\frac{y}{4}\right) + \frac{y}{4} \cdot \frac{\sqrt{16-y^2}}{4}\right) = 16\sin^{-1}\left(\frac{y}{4}\right) + y\sqrt{16-y^2}$$

is an antiderivative, and

$$\begin{aligned} 2 \int_0^{2\sqrt{3}} \sqrt{16-y^2} dy &= \left[16\sin^{-1}\left(\frac{y}{4}\right) + y\sqrt{16-y^2} \right]_0^{2\sqrt{3}} \\ &= \left(16\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2\sqrt{3} \cdot \sqrt{16-12} \right) - \left(16\sin^{-1}(0) + 0 \cdot \sqrt{16} \right) \\ &= 16 \cdot \frac{\pi}{3} + 2\sqrt{3} \cdot 2 - 0 = \frac{16\pi}{3} + 4\sqrt{3}. \end{aligned}$$

So the final answer is

$$A = \frac{16\pi}{3} + 4\sqrt{3} - \frac{8\sqrt{3}}{3} = \frac{16\pi}{3} - \frac{4\sqrt{3}}{3} = \boxed{\frac{16\pi - 4\sqrt{3}}{3} \text{ square units}}.$$

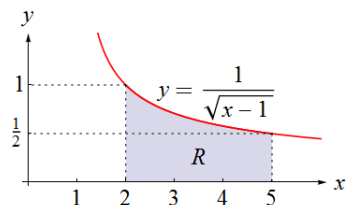


33. We begin by graphing the region R enclosed by the graph $y = \frac{1}{\sqrt{x-1}}$, the x -axis, and the lines $x = 2$ and $x = 5$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = 0$ to $y = \frac{1}{\sqrt{x-1}}$, $2 \leq x \leq 5$. Then the area A of the region R is

$$A = \iint_R dA = \int_2^5 \left[\int_0^{1/\sqrt{x-1}} dy \right] dx = \int_2^5 [y]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}} dx.$$

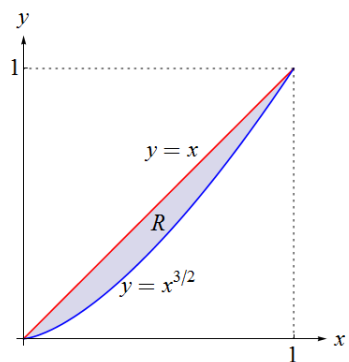
To evaluate this integral, we make a substitution $u = x - 1$; then $du = dx$. We also change the limits of integration: if $x = 2$, then $u = 2 - 1 = 1$; if $x = 5$, then $u = 5 - 1 = 4$. So

$$\begin{aligned} A &= \int_2^5 \frac{1}{\sqrt{x-1}} dx = \int_1^4 \frac{1}{\sqrt{u}} du = \int_1^4 u^{-1/2} du \\ &= \left[\frac{u^{1/2}}{1/2} \right]_1^4 = [2\sqrt{u}]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = \boxed{2 \text{ square units}}. \end{aligned}$$



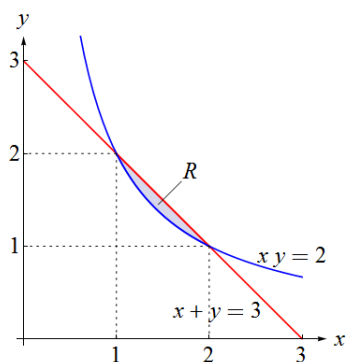
34. We begin by graphing the region R enclosed by the graphs of $y = x^{3/2}$ and $y = x$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = x^{3/2}$ to $y = x$, $0 \leq x \leq 1$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_0^1 \left[\int_{x^{3/2}}^x dy \right] dx = \int_0^1 [y]_{x^{3/2}}^x dx = \int_0^1 (x - x^{3/2}) dx \\ &= \left[\frac{x^2}{2} - \frac{x^{5/2}}{5/2} \right]_0^1 = \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} \right]_0^1 = \left(\frac{1}{2} - \frac{2}{5} \right) - 0 = \boxed{\frac{1}{10} \text{ square units}}. \end{aligned}$$



35. We begin by graphing the region R enclosed by the line $x + y = 3$ and the hyperbola $xy = 2$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = \frac{2}{x}$ to $y = 3 - x$, $1 \leq x \leq 2$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_1^2 \left[\int_{2/x}^{3-x} dy \right] dx = \int_1^2 [y]_{2/x}^{3-x} dx = \int_1^2 \left(3 - x - \frac{2}{x} \right) dx \\ &= \left[3x - \frac{x^2}{2} - 2 \ln x \right]_1^2 = (6 - 2 - 2 \ln 2) - \left(3 - \frac{1}{2} - 2 \ln 1 \right) = \boxed{\frac{3}{2} - 2 \ln 2 \text{ square units}}. \end{aligned}$$



36. We begin by graphing the region R enclosed by the hyperbola $xy = \sqrt{3}$ and the circle $x^2 + y^2 = 4$ in the first quadrant; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = \frac{\sqrt{3}}{x}$ to $y = \sqrt{4 - x^2}$, $1 \leq x \leq \sqrt{3}$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_1^{\sqrt{3}} \left[\int_{\sqrt{3}/x}^{\sqrt{4-x^2}} dy \right] dx = \int_1^{\sqrt{3}} [y]_{\sqrt{3}/x}^{\sqrt{4-x^2}} dx \\ &= \int_1^{\sqrt{3}} \left(\sqrt{4-x^2} - \frac{\sqrt{3}}{x} \right) dx = \int_1^{\sqrt{3}} \sqrt{4-x^2} dx - \int_1^{\sqrt{3}} \frac{\sqrt{3}}{x} dx. \end{aligned}$$

The second integral can be evaluated directly as

$$\int_1^{\sqrt{3}} \frac{\sqrt{3}}{x} dx = \sqrt{3} [\ln x]_1^{\sqrt{3}} = \sqrt{3} (\ln \sqrt{3} - \ln 1) = \sqrt{3} \left(\frac{1}{2} \ln 3 - 0 \right) = \frac{\sqrt{3} \ln 3}{2}.$$

To find the first integral, we will find an antiderivative using a substitution $x = 2 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = 2 \cos \theta d\theta$. Also,

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = \sqrt{4(1-\sin^2 \theta)} = \sqrt{4\cos^2 \theta} = 2 \cos \theta,$$

and the integral becomes

$$\begin{aligned} \int \sqrt{4-x^2} dx &= \int 2 \cos \theta \cdot 2 \cos \theta d\theta = 4 \int \cos^2 \theta d\theta = 4 \int \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= 2 \int (1 + \cos 2\theta) d\theta = 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) = 2(\theta + \sin \theta \cos \theta). \end{aligned}$$

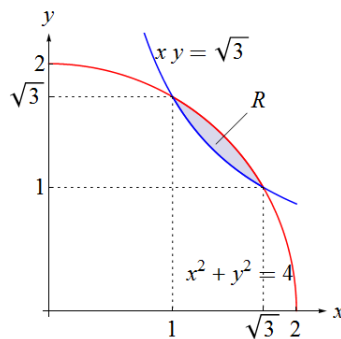
To express this antiderivative in terms of x , we first observe that since $x = 2 \sin \theta$, then $\sin \theta = \frac{x}{2}$ and $\theta = \sin^{-1} \left(\frac{x}{2} \right)$; also, since $\sqrt{4-x^2} = 2 \cos \theta$ (as shown above), then $\cos \theta = \frac{\sqrt{4-x^2}}{2}$. So,

$$2(\theta + \sin \theta \cos \theta) = 2 \left(\sin^{-1} \left(\frac{x}{2} \right) + \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} \right) = 2 \sin^{-1} \left(\frac{x}{2} \right) + \frac{x\sqrt{4-x^2}}{2}$$

is an antiderivative, and

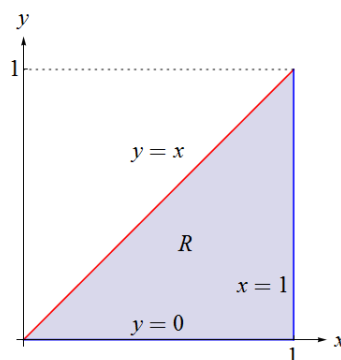
$$\begin{aligned} \int_1^{\sqrt{3}} \sqrt{4-x^2} dx &= \left[2 \sin^{-1} \left(\frac{x}{2} \right) + \frac{x\sqrt{4-x^2}}{2} \right]_1^{\sqrt{3}} \\ &= \left(2 \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) + \frac{\sqrt{3} \cdot \sqrt{4-3}}{2} \right) - \left(2 \sin^{-1} \left(\frac{1}{2} \right) + \frac{1 \cdot \sqrt{4-1}}{2} \right) \\ &= \left(2 \cdot \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) - \left(2 \cdot \frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}. \end{aligned}$$

So the final answer is $A = \boxed{\frac{\pi}{3} - \frac{\sqrt{3} \ln 3}{2} \text{ square units}}.$



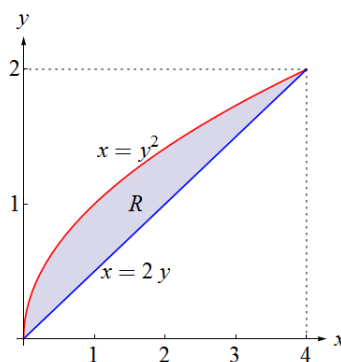
37. We begin by graphing the region R associated with the integral $\int_0^1 \left[\int_0^x f(x, y) dy \right] dx$: it is the region between the graphs of $y = 0$ and $y = x$, where $0 \leq x \leq 1$; see the figure. Observe that R is also a y -simple region: x varies from $x = y$ to $x = 1$, where $0 \leq y \leq 1$. Then according to Fubini's Theorem,

$$\int_0^1 \left[\int_0^x f(x, y) dy \right] dx = \iint_R f(x, y) dA = \boxed{\int_0^1 \left[\int_y^1 f(x, y) dx \right] dy}.$$



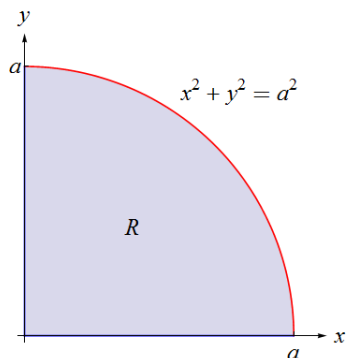
38. We begin by graphing the region R associated with the integral $\int_0^2 \left[\int_{y^2}^{2y} f(x, y) dx \right] dy$: it is the region between the graphs of $x = y^2$ and $x = 2y$, where $0 \leq y \leq 2$; see the figure. Observe that R is also an x -simple region: y varies from $y = \frac{x}{2}$ to $y = \sqrt{x}$, where $0 \leq x \leq 4$. Then according to Fubini's Theorem,

$$\int_0^2 \left[\int_{y^2}^{2y} f(x, y) dx \right] dy = \iint_R f(x, y) dA = \boxed{\int_0^4 \left[\int_{x/2}^{\sqrt{x}} f(x, y) dy \right] dx}.$$



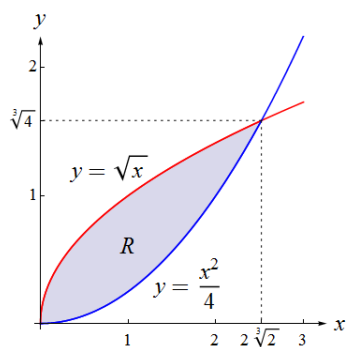
39. We begin by graphing the region R associated with the integral $\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} f(x, y) dx \right] dy$: it is the region between the graphs of $x = 0$ and $x = \sqrt{a^2 - y^2}$, where $0 \leq y \leq a$; see the figure. As we can see, R is the quarter of the disk bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant. Observe that R is also an x -simple region: y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$, where $0 \leq x \leq a$. Then according to Fubini's Theorem,

$$\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} f(x, y) dx \right] dy = \iint_R f(x, y) dA = \boxed{\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} f(x, y) dy \right] dx}.$$



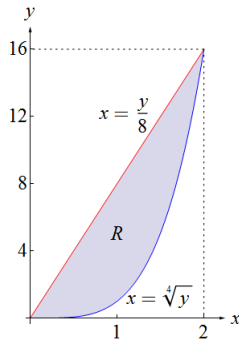
40. We begin by graphing the region R associated with the integral $\int_0^{2\sqrt[3]{2}} \left[\int_{x^2/4}^{\sqrt{x}} f(x, y) dy \right] dx$: it is the region between the graphs of $y = \frac{x^2}{4}$ and $y = \sqrt{x}$, where $0 \leq x \leq 2\sqrt[3]{2}$; see the figure. Observe that R is also a y -simple region: x varies from $x = y^2$ to $x = 2\sqrt{y}$, where $0 \leq y \leq \sqrt[3]{4}$. Then according to Fubini's Theorem,

$$\int_0^{2\sqrt[3]{2}} \left[\int_{x^2/4}^{\sqrt{x}} f(x, y) dy \right] dx = \iint_R f(x, y) dA = \boxed{\int_0^{\sqrt[3]{4}} \left[\int_{y^2}^{2\sqrt{y}} f(x, y) dx \right] dy}.$$



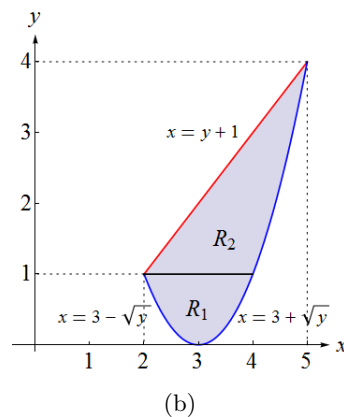
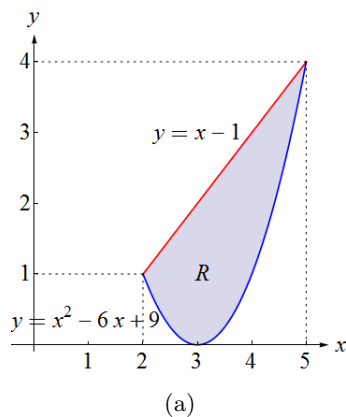
41. We begin by graphing the region R associated with the integral $\int_0^{16} \left[\int_{y/8}^{y^{1/4}} f(x, y) dx \right] dy$: it is the region between the graphs of $x = \frac{y}{8}$ and $x = y^{1/4}$, where $0 \leq y \leq 16$; see the figure. Observe that R is also an x -simple region: y varies from $y = x^4$ to $y = 8x$, where $0 \leq x \leq 2$. Then according to Fubini's Theorem,

$$\int_0^{16} \left[\int_{y/8}^{y^{1/4}} f(x, y) dx \right] dy = \iint_R f(x, y) dA = \boxed{\int_0^2 \left[\int_{x^4}^{8x} f(x, y) dy \right] dx}.$$



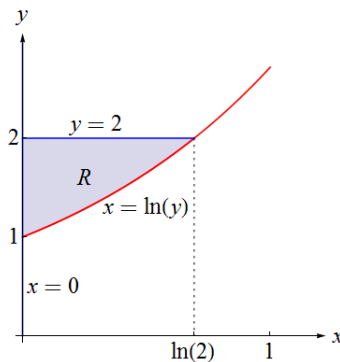
42. We begin by graphing the region R associated with the integral $\int_2^5 \left[\int_{x^2-6x+9}^{x-1} f(x, y) dy \right] dx$: it is the region between the graphs of $y = x^2 - 6x + 9$ and $y = x - 1$, where $2 \leq x \leq 5$; see figure (a). Notice that R is not a y -simple region because of the corner at $(2, 1)$. But the horizontal line through $(2, 1)$ partitions R into two subregions R_1 and R_2 , shown in figure (b), both of which are y -simple. We can solve the equation of the parabola for x : $y = x^2 - 6x + 9 = (x - 3)^2$, so $x - 3 = \pm\sqrt{y}$ and $x = 3 \pm \sqrt{y}$. So R_1 is enclosed between $x = 3 - \sqrt{y}$ and $x = 3 + \sqrt{y}$, where $0 \leq y \leq 1$; and R_2 is enclosed between $x = y + 1$ and $x = 3 + \sqrt{y}$, where $1 \leq y \leq 4$. Then according to Fubini's Theorem,

$$\begin{aligned} \int_2^5 \left[\int_{x^2-6x+9}^{x-1} f(x, y) dy \right] dx &= \iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA \\ &= \int_0^1 \left[\int_{3-\sqrt{y}}^{3+\sqrt{y}} f(x, y) dx \right] dy + \int_1^4 \left[\int_{y+1}^{3+\sqrt{y}} f(x, y) dx \right] dy. \end{aligned}$$



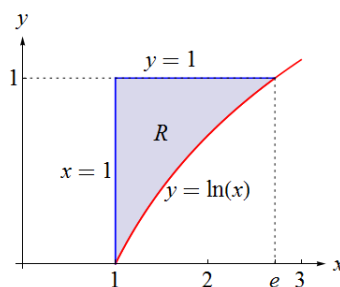
43. We begin by graphing the region R associated with the integral $\int_1^2 \left[\int_0^{\ln y} f(x, y) dx \right] dy$: it is the region between the graphs of $x = 0$ and $x = \ln y$, where $1 \leq y \leq 2$; see the figure. Observe that R is also an x -simple region: y varies from $y = e^x$ to $y = 2$, where $0 \leq x \leq \ln 2$. Then according to Fubini's Theorem,

$$\int_1^2 \left[\int_0^{\ln y} f(x, y) dx \right] dy = \iint_R f(x, y) dA = \int_0^{\ln 2} \left[\int_{e^x}^2 f(x, y) dy \right] dx.$$



44. We begin by graphing the region R associated with the integral $\int_1^e \left[\int_{\ln x}^1 f(x, y) dy \right] dx$: it is the region between the graphs of $y = \ln x$ and $y = 1$, where $1 \leq x \leq e$; see the figure. Observe that R is also an x -simple region: x varies from $x = 1$ to $x = e^y$, where $0 \leq y \leq 1$. Then according to Fubini's Theorem,

$$\int_1^e \left[\int_{\ln x}^1 f(x, y) dy \right] dx = \iint_R f(x, y) dA = \boxed{\int_0^1 \left[\int_1^{e^y} f(x, y) dx \right] dy}.$$



45. Using properties of double integrals,

$$\iint_R [f(x, y) - g(x, y)] dA = \iint_R f(x, y) dA - \iint_R g(x, y) dA = 8 - (-6) = \boxed{14}.$$

46. Using properties of double integrals,

$$\iint_R 4f(x, y) dA = 4 \iint_R f(x, y) dA = 4 \cdot 8 = \boxed{32}.$$

47. Using properties of double integrals,

$$\begin{aligned} \iint_R [3f(x, y) + 4g(x, y)] dA &= \iint_R 3f(x, y) dA + \iint_R 4g(x, y) dA \\ &= 3 \iint_R f(x, y) dA + 4 \iint_R g(x, y) dA = 3 \cdot 8 + 4 \cdot (-6) = \boxed{0}. \end{aligned}$$

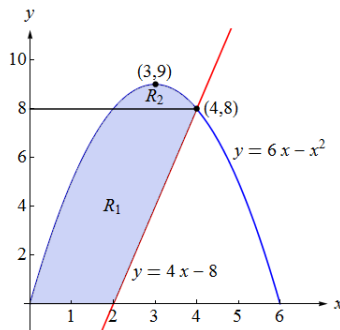
48. Using properties of double integrals,

$$\begin{aligned} \iint_R [2f(x, y) + 5g(x, y)] dA &= \iint_R 2f(x, y) dA + \iint_R 5g(x, y) dA \\ &= 2 \iint_R f(x, y) dA + 5 \iint_R g(x, y) dA = 2 \cdot 8 + 5 \cdot (-6) = \boxed{-14}. \end{aligned}$$

Applications and Extensions

49. We begin by graphing the region R enclosed by the graphs of $y = 6x - x^2$ and $y = 4x - 8$; it is shown in Figure 20. Notice that R is not a y -simple region because of the corner at $(4, 8)$. But the horizontal line through $(4, 8)$ partitions R into two subregions R_1 and R_2 , both of which are y -simple; see the figure below. We can solve the equation of the parabola for x :

$$y = 6x - x^2, \text{ so } -y = x^2 - 6x, \text{ so } 9 - y = x^2 - 6x + 9 = (x - 3)^2, \text{ so } x - 3 = \pm\sqrt{9 - y}, \text{ so } x = 3 \pm \sqrt{9 - y}.$$



So R_1 is enclosed between $x = 3 - \sqrt{9 - y}$ and $x = 2 + \frac{y}{4}$, where $0 \leq y \leq 8$; and R_2 is enclosed between $x = 3 - \sqrt{9 - y}$ and $x = 3 + \sqrt{9 - y}$, where $8 \leq y \leq 9$. Then according to Fubini's Theorem, the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA \\ &= \int_0^8 \left[\int_{3-\sqrt{9-y}}^{2+y/4} dx \right] dy + \int_8^9 \left[\int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx \right] dy = \int_0^8 [x]_{3-\sqrt{9-y}}^{2+y/4} dy + \int_8^9 [x]_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dy \\ &= \int_0^8 \left[\left(2 + \frac{y}{4} \right) - \left(3 - \sqrt{9-y} \right) \right] dy + \int_8^9 \left[\left(3 + \sqrt{9-y} \right) - \left(3 - \sqrt{9-y} \right) \right] dy \\ &= \int_0^8 \left(-1 + \frac{y}{4} + \sqrt{9-y} \right) dy + \int_8^9 2\sqrt{9-y} dy \\ &= \int_0^8 \left(-1 + \frac{y}{4} \right) dy + \int_0^8 \sqrt{9-y} dy + \int_8^9 2\sqrt{9-y} dy. \end{aligned}$$

The first integral $\int_0^8 \left(-1 + \frac{y}{4} \right) dy$ can be evaluated directly as

$$\int_0^8 \left(-1 + \frac{y}{4} \right) dy = \left[-y + \frac{y^2}{8} \right]_0^8 = (-8 + 8) - (-0 + 0) = 0.$$

To evaluate the second integral $\int_0^8 \sqrt{9-y} dy$, we make a substitution $u = 9 - y$; then $du = -dy$, so $dy = -du$. We also change the limits of integration: if $y = 0$, then $u = 9 - 0 = 9$; if $y = 8$, then $u = 9 - 8 = 1$. So

$$\begin{aligned} \int_0^8 \sqrt{9-y} dy &= \int_9^1 \sqrt{u} (-du) = - \int_9^1 \sqrt{u} du = \int_1^9 u^{1/2} du \\ &= \left[\frac{u^{3/2}}{3/2} \right]_1^9 = \left[\frac{2}{3} (\sqrt{u})^3 \right]_1^9 = \frac{2}{3} \left((\sqrt{9})^3 - (\sqrt{1})^3 \right) = \frac{2}{3} (27 - 1) = \frac{52}{3}. \end{aligned}$$

Similarly, we apply the same substitution $u = 9 - y$, $du = -dy$ to evaluate the third integral $\int_8^9 2\sqrt{9-y} dy$. The new limits of integration will be: if $y = 8$, then $u = 9 - 8 = 1$; if $y = 9$,

then $u = 9 - 9 = 0$. So

$$\begin{aligned}\int_8^9 2\sqrt{9-y} dy &= \int_1^0 2\sqrt{u} (-du) = -\int_1^0 2\sqrt{u} du \\ &= \int_0^1 2u^{1/2} du = \left[2 \cdot \frac{u^{3/2}}{3/2} \right]_0^1 = \left[\frac{4}{3} u^{3/2} \right]_0^1 = \frac{4}{3}(1-0) = \frac{4}{3}.\end{aligned}$$

The final answer is

$$A = \int_0^8 \left(-1 + \frac{y}{4}\right) dy + \int_0^8 \sqrt{9-y} dy + \int_8^9 2\sqrt{9-y} dy = 0 + \frac{52}{3} + \frac{4}{3} = \boxed{\frac{56}{3} \text{ square units}}.$$

50. We begin by graphing the region R enclosed by the y -axis, the circle $x^2 + y^2 = 10$, and the parabola $x^2 = 9y$ in the first quadrant; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = \frac{x^2}{9}$ to $y = \sqrt{10-x^2}$, $0 \leq x \leq 3$. So the area A of the region R is

$$\begin{aligned}A &= \iint_R dA = \int_0^3 \left[\int_{x^2/9}^{\sqrt{10-x^2}} dy \right] dx = \int_0^3 [y]_{x^2/9}^{\sqrt{10-x^2}} dx \\ &= \int_0^3 \left(\sqrt{10-x^2} - \frac{x^2}{9} \right) dx = \int_0^3 \sqrt{10-x^2} dx - \frac{1}{9} \int_0^3 x^2 dx.\end{aligned}$$

The second integral can be evaluated directly as

$$\frac{1}{9} \int_0^3 x^2 dx = \frac{1}{9} \left[\frac{x^3}{3} \right]_0^3 = \frac{1}{27} [x^3]_0^3 = \frac{1}{27} (3^3 - 0^3) = \frac{1}{27} \cdot 27 = 1.$$

To find the first integral, we will find an antiderivative using a substitution $x = \sqrt{10} \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = \sqrt{10} \cos \theta d\theta$. Also,

$$\sqrt{10-x^2} = \sqrt{10-10\sin^2 \theta} = \sqrt{10(1-\sin^2 \theta)} = \sqrt{10\cos^2 \theta} = \sqrt{10} \cos \theta,$$

and the integral becomes

$$\begin{aligned}\int \sqrt{10-x^2} dx &= \int \sqrt{10} \cos \theta \cdot \sqrt{10} \cos \theta d\theta = 10 \int \cos^2 \theta d\theta = 10 \int \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= 5 \int (1 + \cos 2\theta) d\theta = 5 \left(\theta + \frac{1}{2} \sin 2\theta \right) = 5 (\theta + \sin \theta \cos \theta).\end{aligned}$$

To express this antiderivative in terms of x , we observe that since $x = \sqrt{10} \sin \theta$, then $\sin \theta = \frac{x}{\sqrt{10}}$ and $\theta = \sin^{-1} \left(\frac{x}{\sqrt{10}} \right)$; also, since $\sqrt{10-x^2} = \sqrt{10} \cos \theta$ (as shown above), then $\cos \theta = \frac{\sqrt{10-x^2}}{\sqrt{10}}$. So,

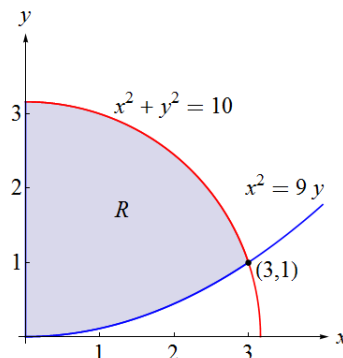
$$5 (\theta + \sin \theta \cos \theta) = 5 \left(\sin^{-1} \left(\frac{x}{\sqrt{10}} \right) + \frac{x}{\sqrt{10}} \cdot \frac{\sqrt{10-x^2}}{\sqrt{10}} \right) = 5 \sin^{-1} \left(\frac{x}{\sqrt{10}} \right) + \frac{1}{2} x \sqrt{10-x^2}$$

is an antiderivative, and

$$\begin{aligned}\int_0^3 \sqrt{10-x^2} dx &= \left[5 \sin^{-1} \left(\frac{x}{\sqrt{10}} \right) + \frac{1}{2} x \sqrt{10-x^2} \right]_0^3 \\ &= \left(5 \sin^{-1} \left(\frac{3}{\sqrt{10}} \right) + \frac{1}{2} \cdot 3 \cdot \sqrt{10-9} \right) - \left(5 \sin^{-1}(0) + \frac{1}{2} \cdot 0 \cdot \sqrt{10-0} \right) \\ &= 5 \sin^{-1} \left(\frac{3}{\sqrt{10}} \right) + \frac{3}{2} - 0 = 5 \sin^{-1} \left(\frac{3}{\sqrt{10}} \right) + \frac{3}{2}.\end{aligned}$$

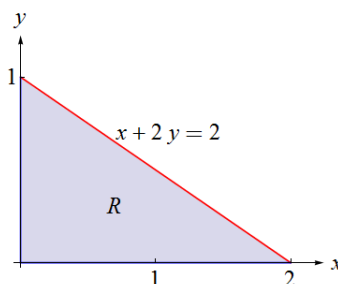
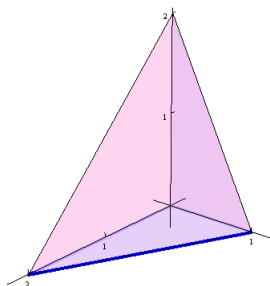
So the final answer is

$$A = 5 \sin^{-1} \left(\frac{3}{\sqrt{10}} \right) + \frac{3}{2} - 1 = \boxed{5 \sin^{-1} \left(\frac{3}{\sqrt{10}} \right) + \frac{1}{2} \text{ square units}}.$$



51. We begin by graphing the solid S enclosed by the plane $x + 2y + z = 2$ and the coordinate planes. This solid lies in the first octant under the graph of $z = f(x, y) = 2 - x - 2y$ and above the region R in the xy -plane bounded by the coordinate axes and the line $x + 2y = 2$. Observe that R is a y -simple region: x varies from $x = 0$ to $x = 2 - 2y$, $0 \leq y \leq 1$; see the figures below. So the volume V of the solid S is

$$\begin{aligned} V &= \iint_R f(x, y) dA = \int_0^1 \left[\int_0^{2-2y} (2 - x - 2y) dx \right] dy = \int_0^1 \left[2x - \frac{x^2}{2} - 2yx \right]_0^{2-2y} dy \\ &= \int_0^1 \left[\left(2(2 - 2y) - \frac{(2 - 2y)^2}{2} - 2y(2 - 2y) \right) - 0 \right] dy \\ &= \int_0^1 [(4 - 4y) - (2 - 4y + 2y^2) - (4y - 4y^2)] dy \\ &= \int_0^1 (2y^2 - 4y + 2) dy = \left[\frac{2y^3}{3} - 2y^2 + 2y \right]_0^1 = \left(\frac{2}{3} - 2 + 2 \right) - 0 = \boxed{\frac{2}{3} \text{ cubic units}}. \end{aligned}$$



52. We begin by graphing the solid S enclosed by the elliptic cylinder $4x^2 + 9y^2 = 36$ and the planes $z = 0$ and $z = 1$. This solid lies under the graph of $z = f(x, y) = 1$ and above the region R in the xy -plane bounded by the ellipse $4x^2 + 9y^2 = 36$; see the figures below. Notice that the solid S is symmetric, consisting of four equal parts over each quadrant. So we can find its volume by integrating over the part of the region R lying in the first quadrant and then multiplying it by four. Observe that the first-quadrant part of R is an x -simple region: y varies from $y = 0$ to $y = \sqrt{4 - \frac{4}{9}x^2} = 2\sqrt{1 - \frac{x^2}{9}}$,

where $0 \leq x \leq 3$. So the volume V of the solid S is

$$\begin{aligned} V &= \iint_R f(x, y) dA = 4 \int_0^3 \left[\int_0^{2\sqrt{1-x^2/9}} 1 dy \right] dx \\ &= 4 \int_0^3 [y]_0^{2\sqrt{1-x^2/9}} dx = 4 \int_0^3 \left[2\sqrt{1 - \frac{x^2}{9}} - 0 \right] dx = 8 \int_0^3 \sqrt{1 - \frac{x^2}{9}} dx. \end{aligned}$$

To evaluate this integral, we will find an antiderivative using a substitution $x = 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = 3 \cos \theta d\theta$. Also,

$$\sqrt{1 - \frac{x^2}{9}} = \sqrt{1 - \frac{9 \sin^2 \theta}{9}} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta,$$

and the integral becomes

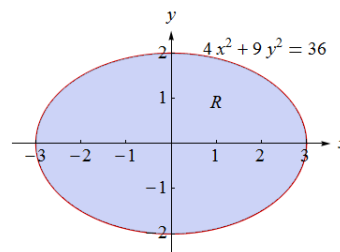
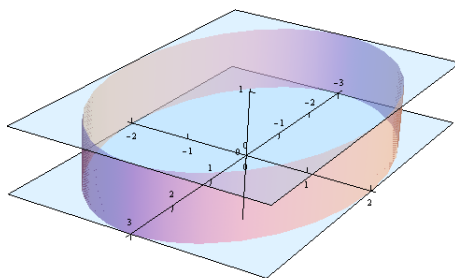
$$\begin{aligned} 8 \int \sqrt{1 - \frac{x^2}{9}} dx &= 8 \int \cos \theta \cdot 3 \cos \theta d\theta = 24 \int \cos^2 \theta d\theta = 24 \int \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= 12 \int (1 + \cos 2\theta) d\theta = 12 \left(\theta + \frac{1}{2} \sin 2\theta \right) = 12 (\theta + \sin \theta \cos \theta). \end{aligned}$$

To express this antiderivative in terms of x , we first observe that since $x = 3 \sin \theta$, then $\sin \theta = \frac{x}{3}$ and $\theta = \sin^{-1} \left(\frac{x}{3} \right)$; also, as shown above, $\cos \theta = \sqrt{1 - \frac{x^2}{9}} = \frac{\sqrt{9 - x^2}}{3}$. So,

$$12 (\theta + \sin \theta \cos \theta) = 12 \left(\sin^{-1} \left(\frac{x}{3} \right) + \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) = 12 \sin^{-1} \left(\frac{x}{3} \right) + \frac{4}{3} x \sqrt{9 - x^2}$$

is an antiderivative, and

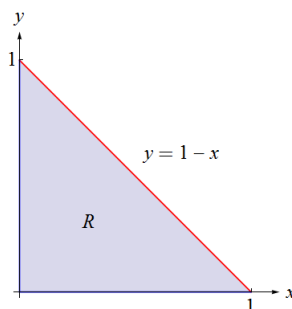
$$\begin{aligned} V &= 8 \int_0^3 \sqrt{1 - \frac{x^2}{9}} dx = \left[12 \sin^{-1} \left(\frac{x}{3} \right) + \frac{4}{3} x \sqrt{9 - x^2} \right]_0^3 \\ &= \left(12 \sin^{-1}(1) + \frac{4}{3} \cdot 3 \cdot \sqrt{9 - 9} \right) - \left(12 \sin^{-1}(0) + \frac{4}{3} \cdot 0 \cdot \sqrt{9 - 0} \right) \\ &= \left(12 \cdot \frac{\pi}{2} + 0 \right) - 0 = \boxed{6\pi \text{ cubic units}}. \end{aligned}$$



53. We begin by graphing the solid S that lies below the circular paraboloid $z = f(x, y) = x^2 + y^2$ and the triangular region R in the xy -plane as shown in the figure in Problem 53. Observe that R is an

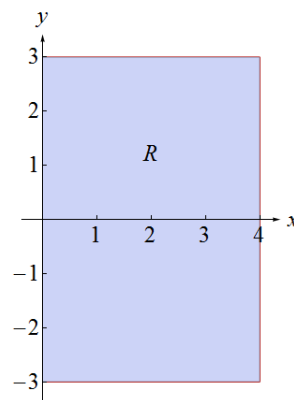
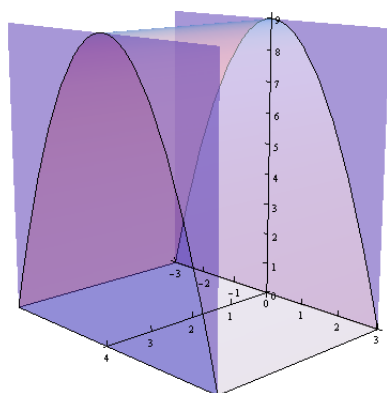
x -simple region: y varies from $y = 0$ to $y = 1 - x$, where $0 \leq x \leq 1$; see the figure below. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^1 \left[\int_0^{1-x} (x^2 + y^2) \, dy \right] dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx \\
 &= \int_0^1 \left[\left(x^2(1-x) + \frac{(1-x)^3}{3} \right) - 0 \right] dx = \int_0^1 \left(x^2 - x^3 + \frac{1-3x+3x^2-x^3}{3} \right) dx \\
 &= \int_0^1 \left(-\frac{4}{3}x^3 + 2x^2 - x + \frac{1}{3} \right) dx = \left[-\frac{1}{3}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x \right]_0^1 \\
 &= \left(-\frac{1}{3} + \frac{2}{3} - \frac{1}{2} + \frac{1}{3} \right) - 0 = \boxed{\frac{1}{6} \text{ cubic units}}.
 \end{aligned}$$



54. We begin by graphing the solid S enclosed by the parabolic cylinder $z = 9 - y^2$ and the planes $z = 0$, $x = 0$, and $x = 4$. This solid lies under the graph of $z = f(x, y) = 9 - y^2$ and above the region R in the xy -plane, which is a rectangle $0 \leq x \leq 4$, $-3 \leq y \leq 3$; see the figures below. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^4 \left[\int_{-3}^3 (9 - y^2) \, dy \right] dx = \int_0^4 \left[9y - \frac{y^3}{3} \right]_{-3}^3 dx \\
 &= \int_0^4 [(27 - 9) - (-27 + 9)] dx = \int_0^4 36 \, dx = [36x]_0^4 = 36 \cdot 4 - 36 \cdot 0 = \boxed{144 \text{ cubic units}}.
 \end{aligned}$$



55. We begin by graphing the solid S enclosed by the circular paraboloid $z = 4 - x^2 - y^2$ and the plane $z = 0$. This solid lies under the graph of $z = f(x, y) = 4 - x^2 - y^2$ and above the region R in the xy -plane, that is the disk $x^2 + y^2 \leq 4$; see the figures below. Notice that the solid S is symmetric, consisting of four equal parts over each quadrant. So we can find its volume by integrating over the part of the region R lying in the first quadrant and then multiplying it by four. Observe that the

first-quadrant part of R is an x -simple region: y varies from $y = 0$ to $y = \sqrt{4 - x^2}$, where $0 \leq x \leq 2$. So the volume V of the solid S is

$$\begin{aligned} V &= \iint_R f(x, y) dA = 4 \int_0^2 \left[\int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy \right] dx = 4 \int_0^2 \left[4y - x^2y - \frac{y^3}{3} \right]_0^{\sqrt{4-x^2}} dx \\ &= 4 \int_0^2 \left[\left(4\sqrt{4-x^2} - x^2\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^3}{3} \right) - 0 \right] dx \\ &= 4 \int_0^2 \left[(4 - x^2)\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^3}{3} \right] dx = 4 \int_0^2 \left[(\sqrt{4-x^2})^3 - \frac{(\sqrt{4-x^2})^3}{3} \right] dx \\ &= 4 \int_0^2 \frac{2}{3} (\sqrt{4-x^2})^3 dx = \frac{8}{3} \int_0^2 (\sqrt{4-x^2})^3 dx. \end{aligned}$$

To evaluate this integral, we will find an antiderivative using a substitution $x = 2 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = 2 \cos \theta d\theta$. Also,

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta,$$

and the integral becomes

$$\begin{aligned} \frac{8}{3} \int (\sqrt{4-x^2})^3 dx &= \frac{8}{3} \int (2\cos\theta)^3 \cdot 2\cos\theta d\theta = \frac{128}{3} \int \cos^4\theta d\theta = \frac{128}{3} \int (\cos^2\theta)^2 d\theta \\ &= \frac{128}{3} \int \left(\frac{1}{2}(1 + \cos 2\theta) \right)^2 d\theta = \frac{128}{3} \int \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) d\theta \\ &= \frac{32}{3} \int \left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right) d\theta = \frac{32}{3} \int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta \\ &= \int \left(16 + \frac{64}{3}\cos 2\theta + \frac{16}{3}\cos 4\theta \right) d\theta = 16\theta + \frac{64}{3} \cdot \frac{1}{2}\sin 2\theta + \frac{16}{3} \cdot \frac{1}{4}\sin 4\theta \\ &= 16\theta + \frac{32}{3}\sin 2\theta + \frac{4}{3}\sin 4\theta. \end{aligned}$$

To express this antiderivative in terms of x , we first observe that since $x = 2 \sin \theta$, then $\sin \theta = \frac{x}{2}$ and $\theta = \sin^{-1}\left(\frac{x}{2}\right)$; also, as shown above, $\sqrt{4-x^2} = 2\cos\theta$, so $\cos\theta = \frac{\sqrt{4-x^2}}{2}$. Then

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} = \frac{1}{2}x\sqrt{4-x^2}$$

and

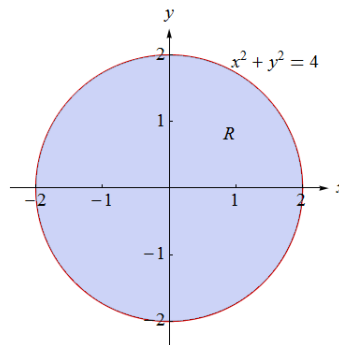
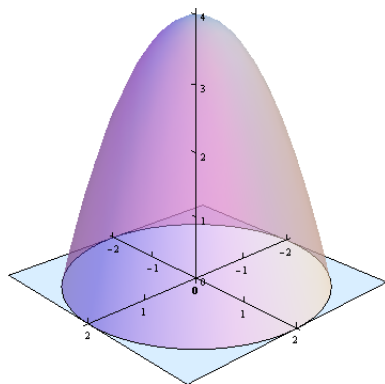
$$\begin{aligned} \sin 4\theta &= 2 \sin 2\theta \cos 2\theta = 2 \cdot \frac{1}{2}x\sqrt{4-x^2} \cdot (\cos^2\theta - \sin^2\theta) \\ &= x\sqrt{4-x^2} \cdot \left(\frac{4-x^2}{4} - \frac{x^2}{4} \right) = x\sqrt{4-x^2} \cdot \frac{4-2x^2}{4} = \frac{1}{2}x\sqrt{4-x^2}(2-x^2), \end{aligned}$$

and so an antiderivative is

$$\begin{aligned} \frac{8}{3} \int (\sqrt{4-x^2})^3 dx &= 16\theta + \frac{32}{3}\sin 2\theta + \frac{4}{3}\sin 4\theta \\ &= 16\sin^{-1}\left(\frac{x}{2}\right) + \frac{32}{3} \cdot \frac{1}{2}x\sqrt{4-x^2} + \frac{4}{3} \cdot \frac{1}{2}x\sqrt{4-x^2}(2-x^2) \\ &= 16\sin^{-1}\left(\frac{x}{2}\right) + \frac{16}{3}x\sqrt{4-x^2} + \frac{2}{3}x\sqrt{4-x^2}(2-x^2) \\ &= 16\sin^{-1}\left(\frac{x}{2}\right) + \frac{2}{3}x\sqrt{4-x^2} \cdot (8 + (2-x^2)) \\ &= 16\sin^{-1}\left(\frac{x}{2}\right) + \frac{2}{3}x(10-x^2)\sqrt{4-x^2}. \end{aligned}$$

Now we can finish evaluating the volume:

$$\begin{aligned}
 V &= \frac{8}{3} \int_0^2 \left(\sqrt{4-x^2} \right)^3 dx = \left[16 \sin^{-1} \left(\frac{x}{2} \right) + \frac{2}{3} x(10-x^2) \sqrt{4-x^2} \right]_0^2 \\
 &= \left(16 \sin^{-1}(1) + \frac{2}{3} \cdot 2 \cdot (10-4) \sqrt{4-4} \right) - \left(16 \sin^{-1}(0) + \frac{2}{3} \cdot 0 \cdot (10-0) \sqrt{4-0} \right) \\
 &= \left(16 \cdot \frac{\pi}{2} + 0 \right) - (0+0) = \boxed{8\pi \text{ cubic units}}.
 \end{aligned}$$



56. We begin by graphing the solid S enclosed by the surfaces $x^2 + y^2 = 1$, $z = x$, and $z = x^2$. This solid lies between the surfaces $z = f_1(x, y) = x^2$ and $z = f_2(x, y) = x$ over the region R in the xy -plane, that is a half-disk $x^2 + y^2 \leq 1$, $x \geq 0$; see the figures below. Notice that the solid S is symmetric, because the xz -plane cuts it into two equal parts. So we can find its volume by integrating over the part of the region R lying in the first quadrant and then multiplying it by two. Observe that the first-quadrant part of R is an x -simple region: y varies from $y = 0$ to $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R [f_2(x, y) - f_1(x, y)] dA = 2 \int_0^1 \left[\int_0^{\sqrt{1-x^2}} (x - x^2) dy \right] dx = 2 \int_0^1 (x - x^2) \left[\int_0^{\sqrt{1-x^2}} dy \right] dx \\
 &= 2 \int_0^1 (x - x^2) [y]_0^{\sqrt{1-x^2}} dx = 2 \int_0^1 (x - x^2) (\sqrt{1-x^2} - 0) dx = 2 \int_0^1 (x - x^2) \sqrt{1-x^2} dx \\
 &= \int_0^1 2x\sqrt{1-x^2} dx - \int_0^1 2x^2\sqrt{1-x^2} dx.
 \end{aligned}$$

To evaluate the first integral, we make a substitution $u = 1 - x^2$; then $du = -2x dx$, so $2x dx = -du$. We also change the limits of integration: if $x = 0$, then $u = 1 - 0^2 = 1$; if $x = 1$, then $u = 1 - 1^2 = 0$. So

$$\int_0^1 2x\sqrt{1-x^2} dx = \int_1^0 \sqrt{u} (-du) = - \int_1^0 \sqrt{u} du = \int_0^1 u^{1/2} du = \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \left[\frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}.$$

To evaluate the second integral, we will find an antiderivative using a substitution $x = \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = \cos \theta d\theta$. Also,

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta,$$

and the integral becomes

$$\begin{aligned}
 \int 2x^2 \sqrt{1-x^2} dx &= \int 2(\sin \theta)^2 \cos \theta \cdot \cos \theta d\theta = 2 \int \sin^2 \theta \cos^2 \theta d\theta \\
 &= 2 \int \left(\frac{1}{2} \sin 2\theta \right)^2 d\theta = \frac{1}{2} \int \sin^2 2\theta d\theta = \frac{1}{2} \int \frac{1}{2} (1 - \cos 4\theta) d\theta \\
 &= \frac{1}{4} \int (1 - \cos 4\theta) d\theta = \frac{1}{4} \left(\theta - \frac{1}{4} \sin 4\theta \right) = \frac{1}{4} \theta - \frac{1}{16} \sin 4\theta.
 \end{aligned}$$

To express this antiderivative in terms of x , we first observe that since $x = \sin \theta$, then $\theta = \sin^{-1}(x)$. Also, as shown above, $\cos \theta = \sqrt{1-x^2}$. Then

$$\begin{aligned}
 \sin 4\theta &= 2 \sin 2\theta \cos 2\theta = 2 \cdot 2 \sin \theta \cos \theta \cdot (\cos^2 \theta - \sin^2 \theta) \\
 &= 4x \sqrt{1-x^2} \cdot (1-x^2-x^2) = 4x(1-2x^2)\sqrt{1-x^2},
 \end{aligned}$$

and so an antiderivative is

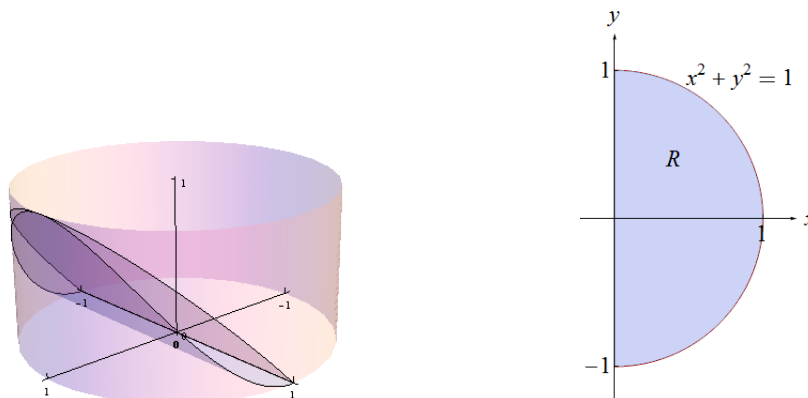
$$\begin{aligned}
 \int 2x^2 \sqrt{1-x^2} dx &= \frac{1}{4} \theta - \frac{1}{16} \sin 4\theta = \frac{1}{4} \sin^{-1}(x) - \frac{1}{16} \cdot 4x(1-2x^2)\sqrt{1-x^2} \\
 &= \frac{1}{4} \sin^{-1}(x) - \frac{1}{4} x(1-2x^2)\sqrt{1-x^2}.
 \end{aligned}$$

Now we can finish evaluating the second integral:

$$\begin{aligned}
 \int_0^1 2x^2 \sqrt{1-x^2} dx &= \left[\frac{1}{4} \sin^{-1}(x) - \frac{1}{4} x(1-2x^2)\sqrt{1-x^2} \right]_0^1 \\
 &= \left(\frac{1}{4} \sin^{-1}(1) - \frac{1}{4} \cdot 1 \cdot (1-2)\sqrt{1-1} \right) - \left(\frac{1}{4} \sin^{-1}(0) - \frac{1}{4} \cdot 0 \cdot (1-0)\sqrt{1-0} \right) \\
 &= \left(\frac{1}{4} \cdot \frac{\pi}{2} - 0 \right) - (0-0) = \frac{\pi}{8}.
 \end{aligned}$$

The final answer is

$$V = \int_0^1 2x \sqrt{1-x^2} dx - \int_0^1 2x^2 \sqrt{1-x^2} dx = \boxed{\frac{2}{3} - \frac{\pi}{8} \text{ cubic units}}.$$



57. The solid S bounded from above by the circular paraboloid $z = f(x, y) = 1 - x^2 - y^2$, from below by the xy -plane, and on the sides by the cylinder $x^2 + y^2 = 1$ is shown in the figure in Problem 57. It lies above the region R in the xy -plane, that is the disk $x^2 + y^2 \leq 1$. Notice that the solid S is symmetric, consisting of four equal parts over each quadrant. So we can find its volume by integrating over the part of the region R lying in the first quadrant and then multiplying it by four. Observe that the

first-quadrant part of R is an x -simple region: y varies from $y = 0$ to $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. So the volume V of the solid S is

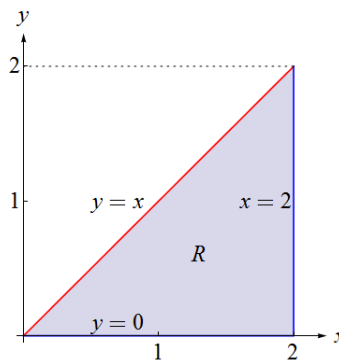
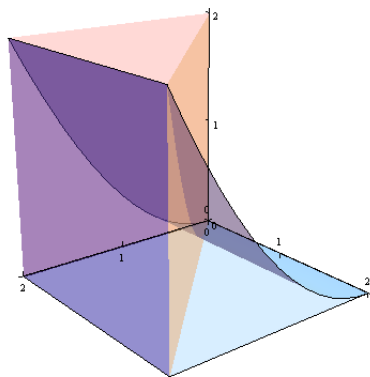
$$\begin{aligned} V &= \iint_R f(x, y) dA = 4 \int_0^1 \left[\int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy \right] dx \\ &= 4 \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{\sqrt{1-x^2}} dx = 4 \int_0^1 \left[\left(x^2 \sqrt{1-x^2} + \frac{(\sqrt{1-x^2})^3}{3} \right) - 0 \right] dx \\ &= \frac{4}{3} \int_0^1 \left[3x^2 \sqrt{1-x^2} + (1-x^2) \sqrt{1-x^2} \right] dx = \frac{4}{3} \int_0^1 (1+2x^2) \sqrt{1-x^2} dx. \end{aligned}$$

In this integral, we make a substitution $x = \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = \cos \theta d\theta$. Also, $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$. The limits of integration can be recalculated as follows: if $x = 0$, then $\sin \theta = 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ imply that $\theta = 0$; if $x = 1$, then $\sin \theta = 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ imply that $\theta = \frac{\pi}{2}$. So the integral becomes

$$\begin{aligned} V &= \frac{4}{3} \int_0^1 (1+2x^2) \sqrt{1-x^2} dx = \frac{4}{3} \int_0^{\pi/2} (1+2\sin^2 \theta) \cos \theta \cdot \cos \theta d\theta \\ &= \int_0^{\pi/2} \left(\frac{4}{3} \cos^2 \theta + \frac{8}{3} \sin^2 \theta \cos^2 \theta \right) d\theta \\ &= \int_0^{\pi/2} \left(\frac{4}{3} \cdot \frac{1}{2} (1 + \cos 2\theta) + \frac{8}{3} \left(\frac{1}{2} \sin 2\theta \right)^2 \right) d\theta \\ &= \int_0^{\pi/2} \left(\frac{2}{3} + \frac{2}{3} \cos 2\theta + \frac{2}{3} \sin^2 2\theta \right) d\theta \\ &= \int_0^{\pi/2} \left(\frac{2}{3} + \frac{2}{3} \cos 2\theta + \frac{2}{3} \cdot \frac{1}{2} (1 - \cos 4\theta) \right) d\theta \\ &= \int_0^{\pi/2} \left(\frac{2}{3} + \frac{2}{3} \cos 2\theta + \frac{1}{3} - \frac{1}{3} \cos 4\theta \right) d\theta \\ &= \int_0^{\pi/2} \left(1 + \frac{2}{3} \cos 2\theta - \frac{1}{3} \cos 4\theta \right) d\theta = \left[\theta + \frac{1}{3} \sin 2\theta - \frac{1}{12} \sin 4\theta \right]_0^{\pi/2} \\ &= \left(\frac{\pi}{2} + \frac{1}{3} \sin \pi - \frac{1}{12} \sin 2\pi \right) - \left(0 + \frac{1}{3} \sin 0 - \frac{1}{12} \sin 0 \right) = \boxed{\frac{\pi}{2} \text{ cubic units}}. \end{aligned}$$

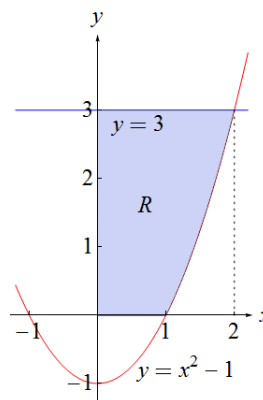
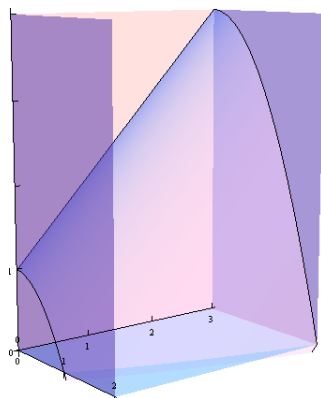
58. We begin by graphing the solid S enclosed by the parabolic cylinder $z = \frac{x^2}{2}$ and the planes $y = 0$, $y = x$, $x = 2$, and $z = 0$. This solid lies under the surface $z = f(x, y) = \frac{x^2}{2}$ over the region R in the xy -plane, that is the triangular region bounded by the lines $y = 0$, $y = x$, and $x = 2$; see the figures below. Observe that R is an x -simple region: y varies from $y = 0$ to $y = x$, where $0 \leq x \leq 2$. So the volume V of the solid S is

$$\begin{aligned} V &= \iint_R f(x, y) dA = \int_0^2 \left[\int_0^x \frac{x^2}{2} dy \right] dx = \int_0^2 \frac{x^2}{2} \left[\int_0^x dy \right] dx = \int_0^2 \frac{x^2}{2} [y]_0^x dx \\ &= \int_0^2 \frac{x^2}{2} (x-0) dx = \int_0^2 \frac{x^3}{2} dx = \left[\frac{x^4}{8} \right]_0^2 = \frac{2^4}{8} - \frac{0^4}{8} = \boxed{2 \text{ cubic units}}. \end{aligned}$$



59. We begin by graphing the solid S enclosed by the coordinate planes, the plane $y = 3$, and the surface $z = y + 1 - x^2$. This solid lies under the surface $z = f(x, y) = y + 1 - x^2$ over the region R in the xy -plane, that is bounded by the coordinate axes, the line $y = 3$, and the curve $y = x^2 - 1$ in the first quadrant; see the figures below. Observe that R is a y -simple region: x varies from $x = 0$ to $x = \sqrt{y+1}$, where $0 \leq y \leq 3$. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) dA = \int_0^3 \left[\int_0^{\sqrt{y+1}} (y + 1 - x^2) dx \right] dy = \int_0^3 \left[(y + 1)x - \frac{x^3}{3} \right]_0^{\sqrt{y+1}} dy \\
 &= \int_0^3 \left[\left((y + 1)\sqrt{y + 1} - \frac{(\sqrt{y + 1})^3}{3} \right) - 0 \right] dy = \int_0^3 \left((y + 1)^{3/2} - \frac{(y + 1)^{3/2}}{3} \right) dy \\
 &= \int_0^3 \frac{2}{3} (y + 1)^{3/2} dy = \left[\frac{2}{3} \cdot \frac{(y + 1)^{5/2}}{5/2} \right]_0^3 = \frac{4}{15} \left[(\sqrt{y + 1})^5 \right]_0^3 \\
 &= \frac{4}{15} \left[(\sqrt{3 + 1})^5 - (\sqrt{0 + 1})^5 \right] = \frac{4}{15} (2^5 - 1^5) = \frac{4}{15} \cdot 31 = \boxed{\frac{124}{15} \text{ cubic units}}.
 \end{aligned}$$



60. We begin by graphing the solid S enclosed by the surfaces $z = xe^y$, $z = 0$, $y = 0$, $x = 1$, and $y = x^2$. This solid lies under the surface $z = f(x, y) = xe^y$ over the region R in the xy -plane, that is bounded by the x -axis, the line $x = 1$, and the curve $y = x^2$; see the figures below. Observe that R is an x -simple region: y varies from $y = 0$ to $y = x^2$, where $0 \leq x \leq 1$. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) dA = \int_0^1 \left[\int_0^{x^2} xe^y dy \right] dx = \int_0^1 x \left[\int_0^{x^2} e^y dy \right] dx = \int_0^1 x [e^y]_0^{x^2} dx \\
 &= \int_0^1 x (e^{x^2} - e^0) dx = \int_0^1 x (e^{x^2} - 1) dx = \int_0^1 xe^{x^2} dx - \int_0^1 x dx.
 \end{aligned}$$

The second integral can be evaluated directly as

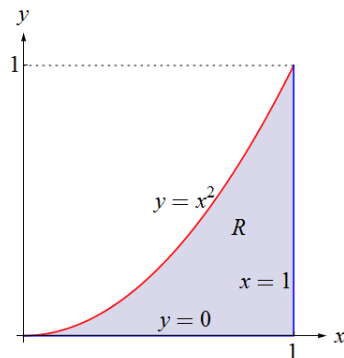
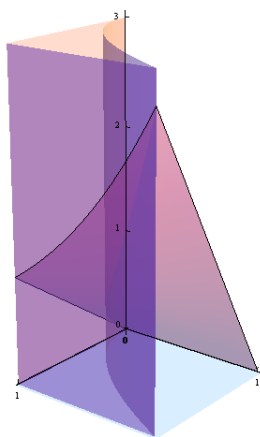
$$\int_0^1 x \, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

To evaluate the first integral, we make a substitution $u = x^2$; then $du = 2x \, dx$, so $x \, dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 = 0$; if $x = 1$, then $u = 1^2 = 1$. So

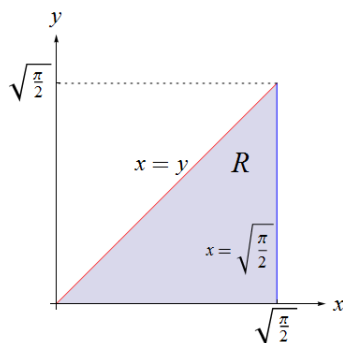
$$\int_0^1 x e^{x^2} \, dx = \int_0^1 e^u \frac{du}{2} = \frac{1}{2} \int_0^1 e^u \, du = \frac{1}{2} [e^u]_0^1 = \frac{1}{2} (e^1 - e^0) = \frac{e}{2} - \frac{1}{2}.$$

So the final answer is

$$V = \left(\frac{e}{2} - \frac{1}{2} \right) - \frac{1}{2} = \boxed{\frac{e}{2} - 1 \text{ cubic units}}.$$



61. (a) The limits of integration of the iterated integral $\int_0^{\sqrt{\pi/2}} \left[\int_y^{\sqrt{\pi/2}} \sin x^2 \, dx \right] dy$ describe the region R bounded between $x = y$ and $x = \sqrt{\pi/2}$, where $0 \leq y \leq \sqrt{\pi/2}$. This is a triangular region, as shown in the figure.



- (b) Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = x$, where $0 \leq x \leq \sqrt{\pi/2}$. So according to Fubini's Theorem,

$$\int_0^{\sqrt{\pi/2}} \left[\int_y^{\sqrt{\pi/2}} \sin x^2 \, dx \right] dy = \iint_R \sin x^2 \, dA = \boxed{\int_0^{\sqrt{\pi/2}} \left[\int_0^x \sin x^2 \, dy \right] dx}.$$

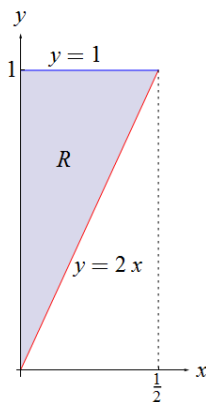
(c) To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} \left[\int_0^x \sin x^2 dy \right] dx &= \int_0^{\sqrt{\pi/2}} \sin x^2 \left[\int_0^x dy \right] dx = \int_0^{\sqrt{\pi/2}} \sin x^2 [y]_0^x dx \\ &= \int_0^{\sqrt{\pi/2}} \sin x^2 \cdot (x - 0) dx = \int_0^{\sqrt{\pi/2}} x \sin x^2 dx.\end{aligned}$$

In this integral we make a substitution $u = x^2$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 = 0$; if $x = \sqrt{\frac{\pi}{2}}$, then $u = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2}$. So

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} x \sin x^2 dx &= \int_0^{\pi/2} \sin u \frac{du}{2} = \frac{1}{2} \int_0^{\pi/2} \sin u du \\ &= \frac{1}{2} [-\cos u]_0^{\pi/2} = \frac{1}{2} \left(-\cos \frac{\pi}{2} + \cos 0 \right) = \frac{1}{2} (-0 + 1) = \boxed{\frac{1}{2}}.\end{aligned}$$

62. (a) The limits of integration of the iterated integral $\int_0^{1/2} \left[\int_{2x}^1 e^{y^2} dy \right] dx$ describe the region R bounded between $y = 2x$ and $y = 1$, where $0 \leq x \leq \frac{1}{2}$. This is a triangular region, as shown in the figure.



- (b) Observe that R is also a y -simple region: it is bounded between $x = 0$ and $x = \frac{y}{2}$, where $0 \leq y \leq 1$. So according to Fubini's Theorem,

$$\int_0^{1/2} \left[\int_{2x}^1 e^{y^2} dy \right] dx = \iint_R e^{y^2} dA = \boxed{\int_0^1 \left[\int_0^{y/2} e^{y^2} dx \right] dy}.$$

(c) To evaluate the iterated integral in part (b), we first integrate partially with respect to x :

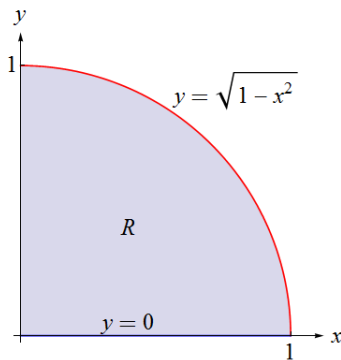
$$\begin{aligned}\int_0^1 \left[\int_0^{y/2} e^{y^2} dx \right] dy &= \int_0^1 e^{y^2} \left[\int_0^{y/2} dx \right] dy = \int_0^1 e^{y^2} [x]_0^{y/2} dy \\ &= \int_0^1 e^{y^2} \cdot \left(\frac{y}{2} - 0 \right) dy = \frac{1}{2} \int_0^1 y e^{y^2} dy.\end{aligned}$$

In this integral we make a substitution $u = y^2$; then $du = 2y dy$, so $y dy = \frac{du}{2}$. We also change

the limits of integration: if $y = 0$, then $u = 0^2 = 0$; if $y = 1$, then $u = 1^2 = 1$. So

$$\begin{aligned} \frac{1}{2} \int_0^1 y e^{y^2} dy &= \frac{1}{2} \int_0^1 e^u \frac{du}{2} = \frac{1}{4} \int_0^1 e^u du \\ &= \frac{1}{4} [e^u]_0^1 = \frac{1}{4} (e^1 - e^0) = \frac{1}{4} (e - 1) = \boxed{\frac{e - 1}{4}}. \end{aligned}$$

63. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-y^2}} dy \right] dx$ describe the region R bounded between $y = 0$ and $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. This is a quarter-disk of radius 1 lying in the first quadrant, as shown in the figure.



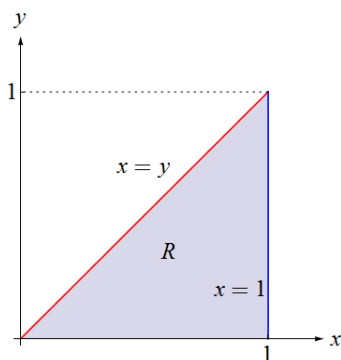
- (b) Observe that R is also a y -simple region: it is bounded between $x = 0$ and $x = \sqrt{1-y^2}$, where $0 \leq y \leq 1$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2}} dx \right] dy = \iint_R \frac{1}{\sqrt{1-y^2}} dA = \boxed{\int_0^1 \left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2}} dx \right] dy}.$$

- (c) To evaluate the iterated integral in part (b), we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2}} dx \right] dy &= \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\int_0^{\sqrt{1-y^2}} dx \right] dy \\ &= \int_0^1 \frac{1}{\sqrt{1-y^2}} [x]_0^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot (\sqrt{1-y^2} - 0) dy \\ &= \int_0^1 1 dy = [y]_0^1 = 1 - 0 = \boxed{1}. \end{aligned}$$

64. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_y^1 \frac{\sin x}{x} dx \right] dy$ describe the region R bounded between $x = y$ and $x = 1$, where $0 \leq y \leq 1$. This is a triangular region, as shown in the figure.



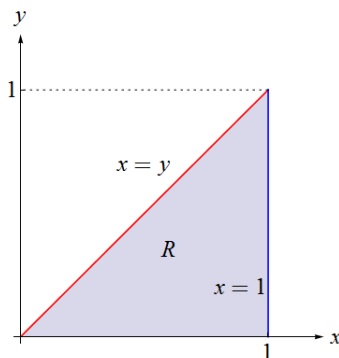
- (b) Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = x$, where $0 \leq x \leq 1$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_y^1 \frac{\sin x}{x} dx \right] dy = \iint_R \frac{\sin x}{x} dA = \boxed{\int_0^1 \left[\int_0^x \frac{\sin x}{x} dy \right] dx}.$$

- (c) To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_0^x \frac{\sin x}{x} dy \right] dx &= \int_0^1 \frac{\sin x}{x} \left[\int_0^x dy \right] dx = \int_0^1 \frac{\sin x}{x} [y]_0^x dx = \int_0^1 \frac{\sin x}{x} \cdot (x - 0) dx \\ &= \int_0^1 \sin x dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = \boxed{1 - \cos 1}. \end{aligned}$$

65. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_y^1 \sqrt{2+x^2} dx \right] dy$ describe the region R bounded between $x = y$ and $x = 1$, where $0 \leq y \leq 1$. This is a triangular region, as shown in the figure.



- (b) Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = x$, where $0 \leq x \leq 1$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_y^1 \sqrt{2+x^2} dx \right] dy = \iint_R \sqrt{2+x^2} dA = \boxed{\int_0^1 \left[\int_0^x \sqrt{2+x^2} dy \right] dx}.$$

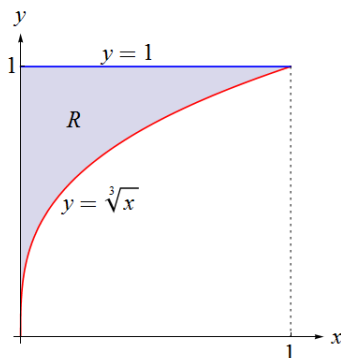
- (c) To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned} \int_0^1 \left[\int_0^x \sqrt{2+x^2} dy \right] dx &= \int_0^1 \sqrt{2+x^2} \left[\int_0^x dy \right] dx = \int_0^1 \sqrt{2+x^2} [y]_0^x dx \\ &= \int_0^1 \sqrt{2+x^2} \cdot (x - 0) dx = \int_0^1 x\sqrt{2+x^2} dx. \end{aligned}$$

In this integral we make a substitution $u = 2 + x^2$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 2 + 0^2 = 2$; if $x = 1$, then $u = 2 + 1^2 = 3$. So

$$\begin{aligned} \int_0^1 x\sqrt{2+x^2} dx &= \int_2^3 \sqrt{u} \frac{du}{2} = \frac{1}{2} \int_2^3 u^{1/2} du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_2^3 \\ &= \frac{1}{3} \left[(\sqrt{u})^3 \right]_2^3 = \frac{1}{3} \left((\sqrt{3})^3 - (\sqrt{2})^3 \right) = \frac{3\sqrt{3} - 2\sqrt{2}}{3} = \boxed{\sqrt{3} - \frac{2\sqrt{2}}{3}}. \end{aligned}$$

66. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_{\sqrt[3]{x}}^1 \sqrt{1+y^4} dy \right] dx$ describe the region R bounded between $y = \sqrt[3]{x}$ and $y = 1$, where $0 \leq x \leq 1$; see the figure.



- (b) Observe that R is also a y -simple region: it is bounded between $x = 0$ and $x = y^3$, where $0 \leq y \leq 1$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_{\sqrt[3]{x}}^1 \sqrt{1+y^4} dy \right] dx = \iint_R \sqrt{1+y^4} dA = \boxed{\int_0^1 \left[\int_0^{y^3} \sqrt{1+y^4} dx \right] dy}.$$

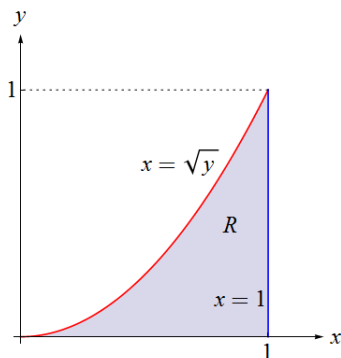
- (c) To evaluate the iterated integral in part (b), we first integrate partially with respect to x :

$$\begin{aligned} \int_0^1 \left[\int_0^{y^3} \sqrt{1+y^4} dx \right] dy &= \int_0^1 \sqrt{1+y^4} \left[\int_0^{y^3} dx \right] dy = \int_0^1 \sqrt{1+y^4} [x]_0^{y^3} dy \\ &= \int_0^1 \sqrt{1+y^4} \cdot (y^3 - 0) dy = \int_0^1 y^3 \sqrt{1+y^4} dy. \end{aligned}$$

In this integral we make a substitution $u = 1 + y^4$; then $du = 4y^3 dy$, so $y^3 dy = \frac{du}{4}$. We also change the limits of integration: if $y = 0$, then $u = 1 + 0^4 = 1$; if $y = 1$, then $u = 1 + 1^4 = 2$. So

$$\begin{aligned} \int_0^1 y^3 \sqrt{1+y^4} dy &= \int_1^2 \sqrt{u} \frac{du}{4} = \frac{1}{4} \int_1^2 u^{1/2} du = \frac{1}{4} \left[\frac{u^{3/2}}{3/2} \right]_1^2 \\ &= \frac{1}{6} \left[(\sqrt{u})^3 \right]_1^2 = \frac{1}{6} \left((\sqrt{2})^3 - (\sqrt{1})^3 \right) = \boxed{\frac{2\sqrt{2}-1}{6}}. \end{aligned}$$

67. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_{\sqrt{y}}^1 e^{y/x} dx \right] dy$ describe the region R bounded between $x = \sqrt{y}$ and $x = 1$, where $0 \leq y \leq 1$; see the figure.



- (b) Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = x^2$, where $0 \leq x \leq 1$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_{\sqrt{y}}^1 e^{y/x} dx \right] dy = \iint_R e^{y/x} dA = \boxed{\int_0^1 \left[\int_0^{x^2} e^{y/x} dy \right] dx}.$$

- (c) To evaluate the iterated integral in part (b), we first find the integral in the brackets by integrating partially with respect to y treating x as a constant. We make a substitution $u = \frac{y}{x}$; then $du = \frac{1}{x} dy$, so $dy = x du$. We also change the limits of integration: if $y = 0$, then $u = \frac{0}{x} = 0$; if $y = x^2$, then $u = \frac{x^2}{x} = x$. So:

$$\int_0^{x^2} e^{y/x} dy = \int_0^x e^u \cdot x du = x \int_0^x e^u du = x[e^u]_0^x = x(e^x - e^0) = x(e^x - 1).$$

Going back to the iterated integral, we obtain

$$\int_0^1 \left[\int_0^{x^2} e^{y/x} dy \right] dx = \int_0^1 x(e^x - 1) dx = \int_0^1 x e^x dx - \int_0^1 x dx.$$

The second integral can be evaluated directly as

$$\int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

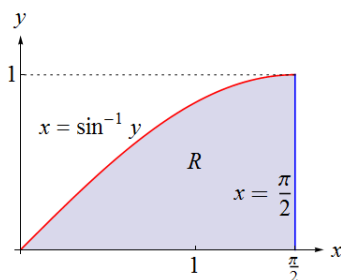
To find the first integral, we use the method of integration by parts. If we choose $u = x$ and $dv = e^x dx$, then $du = dx$ and $v = \int e^x dx = e^x$. So

$$\int_0^1 x e^x dx = [x e^x]_0^1 - \int_0^1 e^x dx = (1e^1 - 0e^0) - [e^x]_0^1 = e - (e^1 - e^0) = e - e + 1 = 1.$$

The final answer is

$$\int_0^1 x e^x dx - \int_0^1 x dx = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}.$$

68. (a) The limits of integration of the iterated integral $\int_0^1 \left[\int_{\sin^{-1} y}^{\pi/2} e^{\cos x} dx \right] dy$ describe the region R bounded between $x = \sin^{-1} y$ and $x = \frac{\pi}{2}$, where $0 \leq y \leq 1$; see the figure.



- (b) Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = \sin x$, where $0 \leq x \leq \frac{\pi}{2}$. So according to Fubini's Theorem,

$$\int_0^1 \left[\int_{\sin^{-1} y}^{\pi/2} e^{\cos x} dx \right] dy = \iint_R e^{\cos x} dA = \boxed{\int_0^{\pi/2} \left[\int_0^{\sin x} e^{\cos x} dy \right] dx}.$$

(c) To evaluate the iterated integral in part (b), we first integrate partially with respect to y :

$$\begin{aligned}\int_0^{\pi/2} \left[\int_0^{\sin x} e^{\cos x} dy \right] dx &= \int_0^{\pi/2} e^{\cos x} \left[\int_0^{\sin x} dy \right] dx = \int_0^{\pi/2} e^{\cos x} [y]_0^{\sin x} dx \\ &= \int_0^{\pi/2} e^{\cos x} \cdot (\sin x - 0) dx = \int_0^{\pi/2} \sin x e^{\cos x} dx.\end{aligned}$$

In this integral we make a substitution $u = \cos x$; then $du = -\sin x dx$, so $\sin x dx = -du$. We also change the limits of integration: if $x = 0$, then $u = \cos 0 = 1$; if $x = \frac{\pi}{2}$, then $u = \cos \frac{\pi}{2} = 0$. So

$$\int_0^{\pi/2} \sin x e^{\cos x} dx = \int_1^0 e^u (-du) = -\int_1^0 e^u du = \int_0^1 e^u du = [e^u]_0^1 = e^1 - e^0 = \boxed{e - 1}.$$

69. Let $B = \iint_R f(x, y) dA$, $C = \iint_{R_1} f(x, y) dA$, and $D = \iint_{R_2} f(x, y) dA$ be the values of these three double integrals. By properties of double integrals, $\iint_R 3f(x, y) dA = 3 \iint_R f(x, y) dA$, and the first given equation can be rewritten as

$$3 \iint_R f(x, y) dA - 2 \iint_{R_1} f(x, y) dA = \iint_{R_2} f(x, y) dA \quad \text{or} \quad 3B - 2C = D.$$

Similarly, $\iint_{R_2} 5f(x, y) dA = 5 \iint_{R_2} f(x, y) dA$, and the second given equation can be rewritten as

$$5 \iint_{R_2} f(x, y) dA - 2 \iint_{R_1} f(x, y) dA = 18 \quad \text{or} \quad 5D - 2C = 18.$$

Also, since the region R is partitioned into subregions R_1 and R_2 , that have no points in common except some portions of their common boundary, by properties of double integrals

$$\iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA = \iint_R f(x, y) dA \quad \text{or} \quad C + D = B.$$

So we need to solve a system of three linear equations with three unknowns (in fact, we are only asked to find the unknown $B = \iint_R f(x, y) dA$):

$$\begin{cases} 3B - 2C = D \\ 5D - 2C = 18 \\ C + D = B \end{cases}$$

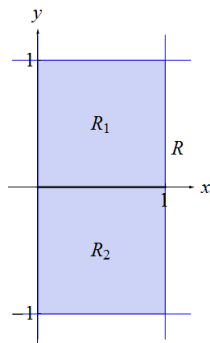
Substituting $B = C + D$ into the first equation yields

$$3B - 2C = D, \text{ so } 3(C + D) - 2C = D, \text{ so } 3C + 3D - 2C = D, \text{ so } C = -2D.$$

Substituting $C = -2D$ into the second equation yields

$$5D - 2C = 18, \text{ so } 5D - 2(-2D) = 18, \text{ so } 9D = 18, \text{ so } D = 2.$$

Now we can find that $C = -2D = -2 \cdot 2 = -4$ and $\iint_R f(x, y) dA = B = C + D = -4 + 2 = \boxed{-2}$.



70. Let $B = \iint_R f(x, y) dA$, $C = \iint_{R_1} f(x, y) dA$, and $D = \iint_{R_2} f(x, y) dA$ be the values of these three double integrals. By properties of double integrals, $\iint_{R_2} 3f(x, y) dA = 3 \iint_{R_2} f(x, y) dA$, and the first given equation can be rewritten as

$$3 \iint_{R_2} f(x, y) dA - \iint_{R_1} f(x, y) dA = 2 \iint_R f(x, y) dA \quad \text{or} \quad 3D - C = 2B.$$

Similarly, $\iint_{R_1} 4f(x, y) dA = 4 \iint_{R_1} f(x, y) dA$ and $\iint_{R_2} 2f(x, y) dA = 2 \iint_{R_2} f(x, y) dA$, and the second given equation can be rewritten as

$$4 \iint_{R_1} f(x, y) dA + 2 \iint_{R_2} f(x, y) dA = 7 \quad \text{or} \quad 4C + 2D = 7.$$

Also, since the region R is partitioned into subregions R_1 and R_2 , that have no points in common except some portions of their common boundary, by properties of double integrals

$$\iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA = \iint_R f(x, y) dA \quad \text{or} \quad C + D = B.$$

So we need to solve a system of three linear equations with three unknowns (in fact, we are only asked to find the unknown $B = \iint_R f(x, y) dA$):

$$\begin{cases} 3D - C = 2B \\ 4C + 2D = 7 \\ C + D = B \end{cases}$$

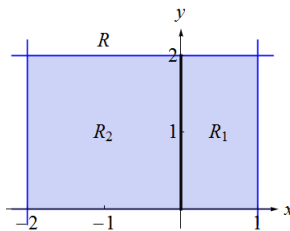
Substituting $B = C + D$ into the first equation yields

$$3D - C = 2B, \text{ so } 3D - C = 2(C + D), \text{ so } 3D - C = 2C + 2D, \text{ so } D = 3C.$$

Substituting $D = 3C$ into the second equation yields

$$4C + 2D = 7, \text{ so } 4C + 2 \cdot 3C = 7, \text{ so } 10C = 7, \text{ so } C = \frac{7}{10}.$$

Now we can find that $D = 3C = 3 \cdot \frac{7}{10} = \frac{21}{10}$ and $\iint_R f(x, y) dA = B = C + D = \frac{7}{10} + \frac{21}{10} = \frac{28}{10} = \boxed{\frac{14}{5}}.$



71. Let $C = \iint_{R_1} f(x, y) dA$, and $D = \iint_{R_2} f(x, y) dA$ be the values of the double integrals over the two subregions. Then the first given equation can be rewritten as

$$2 \iint_{R_2} f(x, y) dA - 7 \iint_{R_1} f(x, y) dA = 17 \quad \text{or} \quad 2D - 7C = 17,$$

and the second given equation can be rewritten as

$$\iint_{R_2} f(x, y) dA - 2 \iint_{R_1} f(x, y) dA = 7 \quad \text{or} \quad D - 2C = 7,$$

giving rise to a system of two linear equations with two unknowns:

$$\begin{cases} 2D - 7C = 17 \\ D - 2C = 7 \end{cases}$$

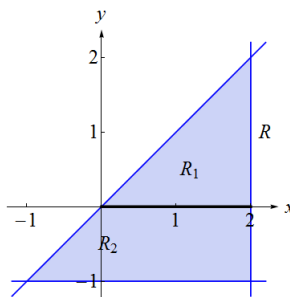
From the second equation we find that $D = 7 + 2C$, and substituting that into the first equation yields

$$2D - 7C = 17, \text{ so } 2(7 + 2C) - 7C = 17, \text{ so } 14 + 4C - 7C = 17, \text{ so } -3C = 3, \text{ so } C = \frac{3}{-3} = -1.$$

Then $D = 7 + 2C = 7 + 2 \cdot (-1) = 5$.

Finally, since the region R is partitioned into subregions R_1 and R_2 , that have no points in common except some portions of their common boundary, by properties of double integrals we find that

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA = C + D = (-1) + 5 = \boxed{4}.$$



72. Let $C = \iint_{R_1} f(x, y) dA$, and $D = \iint_{R_2} f(x, y) dA$ be the values of the double integrals over the two subregions. By properties of double integrals, $\iint_{R_2} 6f(x, y) dA = 6 \iint_{R_2} f(x, y) dA$, and the first given equation can be rewritten as

$$6 \iint_{R_2} f(x, y) dA - 3 \iint_{R_1} f(x, y) dA = -12 \quad \text{or} \quad 6D - 3C = -12.$$

Similarly, $\iint_{R_2} 4f(x, y) dA = 4 \iint_{R_2} f(x, y) dA$, and the second given equation can be rewritten as

$$4 \iint_{R_2} f(x, y) dA + 2 = \iint_{R_1} f(x, y) dA \quad \text{or} \quad 4D + 2 = C.$$

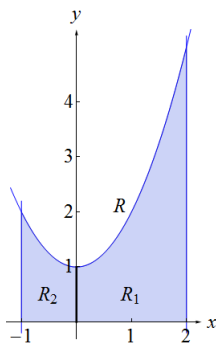
So we have a system of two linear equations with two unknowns:

$$\begin{cases} 6D - 3C = -12 \\ 4D + 2 = C \end{cases}$$

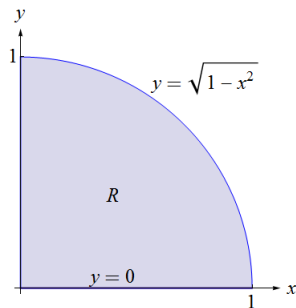
We can simplify the first equation dividing it through by 3 to obtain $2D - C = -4$ or $C = 2D + 4$. Since from the second equation $C = 4D + 2$, we can eliminate C : $2D + 4 = 4D + 2$, so $-2D = -2$, so $D = 1$. Then $C = 4D + 2 = 4 \cdot 1 + 2 = 6$.

Finally, since the region R is partitioned into subregions R_1 and R_2 , that have no points in common except some portions of their common boundary, by properties of double integrals we find that

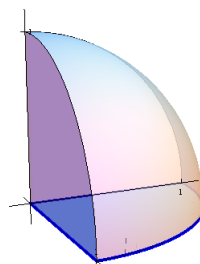
$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA = C + D = 6 + 1 = \boxed{7}.$$



73. The limits of integration of the iterated integral $\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy \right] dx$ describe the region R in the xy -plane defined by $0 \leq y \leq \sqrt{1-x^2}$, $0 \leq x \leq 1$; this region is the first-quadrant quarter of the disk of radius 1 centered at the origin, as shown in figure (a) below. According to Fubini's Theorem, $\int_0^1 \left[\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy \right] dx = \iint_R \sqrt{1-x^2-y^2} dA$. Since the integrand is nonnegative on R , this double integral represents the volume of the solid under the surface $z = f(x, y) = \sqrt{1-x^2-y^2}$ over the region R . This is the solid bounded by the hemisphere $z = \sqrt{1-x^2-y^2}$ and the coordinate planes in the first octant, as shown in figure (b) below.

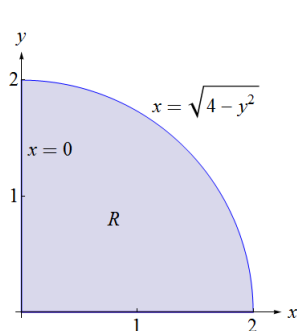


(a)

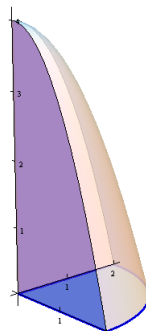


(b)

74. The limits of integration of the iterated integral $\int_0^2 \left[\int_0^{\sqrt{4-y^2}} (4-x^2-y^2) dx \right] dy$ describe the region R in the xy -plane defined by $0 \leq x \leq \sqrt{4-y^2}$, $0 \leq y \leq 2$; this region is the first-quadrant quarter of the disk of radius 2 centered at the origin, as shown in figure (a) below. According to Fubini's Theorem, $\int_0^2 \left[\int_0^{\sqrt{4-y^2}} (4-x^2-y^2) dx \right] dy = \iint_R (4-x^2-y^2) dA$. Since the integrand is nonnegative on R , this double integral represents the volume of the solid under the surface $z = f(x, y) = 4 - x^2 - y^2$ over the region R . This is the solid bounded by the circular paraboloid $z = 4 - x^2 - y^2$ and the coordinate planes in the first octant, as shown in figure (b) below.

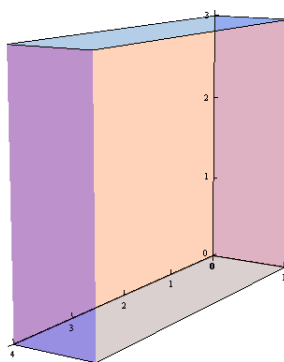


(a)

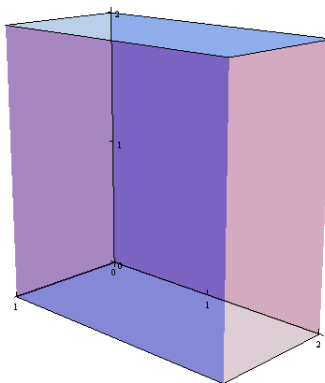


(b)

75. The limits of integration of the iterated integral $\int_0^4 \left[\int_0^1 3 dy \right] dx$ describe the rectangular region R defined by $0 \leq x \leq 4$, $0 \leq y \leq 1$. According to Fubini's Theorem, $\int_0^4 \left[\int_0^1 3 dy \right] dx = \iint_R 3 dA$. Since the integrand is nonnegative on R , this double integral represents the volume of the solid under the surface $z = f(x, y) = 3$ over the region R . This solid is a rectangular box with height $z = 3$ units, with a rectangular base of dimensions $x = 4$ units by $y = 1$ unit, as shown in the figure.



76. The limits of integration of the iterated integral $\int_0^1 \left[\int_0^2 2 dy \right] dx$ describe the rectangular region R defined by $0 \leq x \leq 1$, $0 \leq y \leq 2$. According to Fubini's Theorem, $\int_0^1 \left[\int_0^2 2 dy \right] dx = \iint_R 2 dA$. Since the integrand is nonnegative on R , this double integral represents the volume of the solid under the surface $z = f(x, y) = 2$ over the region R . This solid is a rectangular box with height $z = 2$ units, with a rectangular base of dimensions $x = 1$ unit by $y = 2$ units, as shown in the figure.



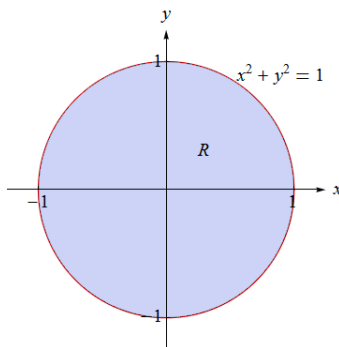
77. (a) Let R be the region enclosed by the circle of radius 1 centered at the origin, that is by the circle $x^2 + y^2 = 1$; see the figure. Notice that it is an x -simple region: y varies from $y = -\sqrt{1-x^2}$ to $y = \sqrt{1-x^2}$, where $-1 \leq x \leq 1$. According to Fubini's Theorem,

$$\begin{aligned} \iint_R x \, dA &= \int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, dy \right] dx = \int_{-1}^1 x \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right] dx = \int_{-1}^1 x [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^1 x \left[\sqrt{1-x^2} - (-\sqrt{1-x^2}) \right] dx = \int_{-1}^1 2x\sqrt{1-x^2} \, dx. \end{aligned}$$

In this integral we make a substitution $u = 1 - x^2$; then $du = -2x dx$, so $2x dx = -du$. We also change the limits of integration: if $x = -1$, then $u = 1 - (-1)^2 = 0$; if $x = 1$, then $u = 1 - 1^2 = 0$. So

$$\int_{-1}^1 2x\sqrt{1-x^2} \, dx = \int_0^0 \sqrt{u} (-du) = \boxed{0}$$

by properties of definite integrals.



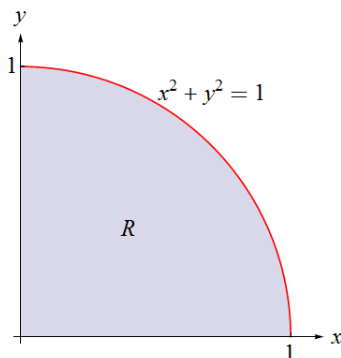
- (b) Let R_1 be the region in the first quadrant enclosed by the circle of radius 1 centered at the origin, that is by the circle $x^2 + y^2 = 1$; see the figure. Notice that it is an x -simple region: y varies from $y = 0$ to $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. According to Fubini's Theorem,

$$\begin{aligned} 4 \iint_{R_1} x \, dA &= 4 \int_0^1 \left[\int_0^{\sqrt{1-x^2}} x \, dy \right] dx = 4 \int_0^1 x \left[\int_0^{\sqrt{1-x^2}} dy \right] dx \\ &= 4 \int_0^1 x [y]_0^{\sqrt{1-x^2}} dx = 4 \int_0^1 x \left[\sqrt{1-x^2} - 0 \right] dx = 4 \int_0^1 x\sqrt{1-x^2} \, dx. \end{aligned}$$

In this integral we make a substitution $u = 1 - x^2$; then $du = -2x dx$, so $x dx = \frac{du}{-2}$. We also

change the limits of integration: if $x = 0$, then $u = 1 - 0^2 = 1$; if $x = 1$, then $u = 1 - 1^2 = 0$. So

$$\begin{aligned} 4 \int_0^1 x \sqrt{1-x^2} dx &= 4 \int_1^0 \sqrt{u} \frac{du}{-2} = -2 \int_1^0 \sqrt{u} du \\ &= 2 \int_0^1 u^{1/2} du = 2 \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \frac{4}{3} \left[u^{3/2} \right]_0^1 = \frac{4}{3}(1-0) = \boxed{\frac{4}{3}}. \end{aligned}$$



78. (a) To evaluate the given iterated integral, we first integrate partially with respect to y :

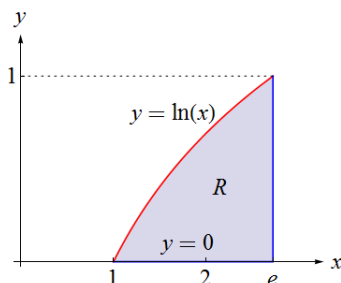
$$\int_1^e \left[\int_0^{\ln x} \frac{dy}{x} \right] dx = \int_1^e \frac{1}{x} \left[\int_0^{\ln x} dy \right] dx = \int_1^e \frac{1}{x} [y]_0^{\ln x} dx = \int_1^e \frac{1}{x} (\ln x - 0) dx = \int_1^e \frac{\ln x}{x} dx.$$

In this integral we make a substitution $u = \ln x$; then $du = \frac{1}{x} dx$. We also change the limits of integration: if $x = 1$, then $u = \ln 1 = 0$; if $x = e$, then $u = \ln e = 1$. So

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u du = \left[\frac{u^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}.$$

- (b) The limits of integration of the iterated integral $\int_1^e \left[\int_0^{\ln x} \frac{dy}{x} \right] dx$ describe the region R bounded between $y = 0$ and $y = \ln x$, where $1 \leq x \leq e$; see the figure below. Observe that R is also a y -simple region: it is bounded between $x = e^y$ and $x = e$, where $0 \leq y \leq 1$. So according to Fubini's Theorem,

$$\begin{aligned} \int_1^e \left[\int_0^{\ln x} \frac{dy}{x} \right] dx &= \iint_R \frac{1}{x} dA = \int_0^1 \left[\int_{e^y}^e \frac{dx}{x} \right] dy \\ &= \int_0^1 [\ln x]_{e^y}^e dy = \int_0^1 (\ln e - \ln e^y) dy \\ &= \int_0^1 (1 - y) dy = \left[y - \frac{y^2}{2} \right]_0^1 = \left(1 - \frac{1}{2} \right) - 0 = \boxed{\frac{1}{2}}. \end{aligned}$$

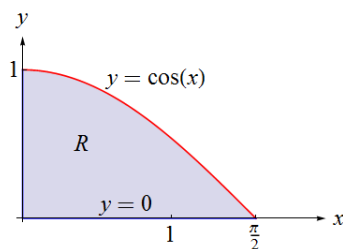


79. (a) To evaluate the given iterated integral, we first integrate partially with respect to y :

$$\begin{aligned}
 \int_0^{\pi/2} \left[\int_0^{\cos x} \sin x \, dy \right] dx &= \int_0^{\pi/2} \sin x \left[\int_0^{\cos x} dy \right] dx = \int_0^{\pi/2} \sin x [y]_0^{\cos x} dx \\
 &= \int_0^{\pi/2} \sin x (\cos x - 0) dx = \int_0^{\pi/2} \sin x \cos x dx \\
 &= \int_0^{\pi/2} \frac{1}{2} \sin 2x dx = \frac{1}{2} \left[-\frac{\cos 2x}{2} \right]_0^{\pi/2} \\
 &= -\frac{1}{4} [\cos 2x]_0^{\pi/2} = -\frac{1}{4} (\cos \pi - \cos 0) = -\frac{1}{4} (-1 - 1) = \boxed{\frac{1}{2}}.
 \end{aligned}$$

- (b) The limits of integration of the iterated integral $\int_0^{\pi/2} [\int_0^{\cos x} \sin x \, dy] dx$ describe the region R bounded between $y = 0$ and $y = \cos x$, where $0 \leq x \leq \frac{\pi}{2}$; see the figure below. Observe that R is also a y -simple region: it is bounded between $x = 0$ and $x = \cos^{-1} y$, where $0 \leq y \leq 1$. So according to Fubini's Theorem,

$$\begin{aligned}
 \int_0^{\pi/2} \left[\int_0^{\cos x} \sin x \, dy \right] dx &= \iint_R \sin x \, dA = \int_0^1 \left[\int_0^{\cos^{-1} y} \sin x \, dx \right] dy \\
 &= \int_0^1 [-\cos x]_0^{\cos^{-1} y} dy = \int_0^1 (-\cos(\cos^{-1} y) + \cos 0) dy \\
 &= \int_0^1 (-y + 1) dy = \left[-\frac{y^2}{2} + y \right]_0^1 = \left(-\frac{1}{2} + 1 \right) - 0 = \boxed{\frac{1}{2}}.
 \end{aligned}$$



80. The volume of the entire half-cylinder tank can be found by multiplying the area of its side face, which is half of a disk of radius $r = 1$ m, by its length $l = 4$ m: $V_{\text{tank}} = \frac{1}{2} \pi r^2 \cdot l = \frac{1}{2} \pi \cdot 1^2 \cdot 4 = 2\pi$. Then we can find the volume of the liquid in the tank, which is filled up to a depth of a meters, as shown in the figure in Problem 80, if we find the volume of the empty cap above the liquid level and subtract it from the volume of the tank.

The side of the tank that lies in the xz -plane is bounded by the semicircle $x^2 + z^2 = 1$, $z \geq 0$, or $z = \sqrt{1 - x^2}$. The horizontal line that represents the water level at $z = a$ meters crosses this semicircle, that is the surface of the tank, at the points with $x = \pm\sqrt{1 - a^2}$; see the figure below. So the empty cap above the water level can be represented as the solid enclosed between $z = f_1(x, y) = a$ and $z = f_2(x, y) = \sqrt{1 - x^2}$ over the region R in the xy -plane, that is a rectangle $-\sqrt{1 - a^2} \leq x \leq \sqrt{1 - a^2}$, $0 \leq y \leq 4$; and we can find its volume as the double integral

$$\iint_R [f_2(x, y) - f_1(x, y)] \, dA = \int_0^4 \left[\int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} (\sqrt{1-x^2} - a) \, dx \right] dy.$$

Let I be the value of the integral in the brackets; we find it by integrating with respect to x . The integral in the brackets can be split into a difference of two integrals:

$$I = \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} (\sqrt{1-x^2} - a) \, dx = \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \sqrt{1-x^2} \, dx - \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} a \, dx.$$

The second integral can be evaluated directly as

$$\int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} a \, dx = [ax]_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} = a \left[\sqrt{1-a^2} - \left(-\sqrt{1-a^2} \right) \right] = 2a\sqrt{1-a^2}.$$

To evaluate the first integral, we will find an antiderivative using a substitution $x = \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = \cos \theta d\theta$. Also, $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$, and the integral becomes

$$\begin{aligned} \int \sqrt{1-x^2} \, dx &= \int \cos \theta \cdot \cos \theta \, d\theta = \int \cos^2 \theta \, d\theta \\ &= \int \frac{1}{2}(1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) = \frac{1}{2}(\theta + \sin \theta \cos \theta). \end{aligned}$$

To express this antiderivative in terms of x , we first observe that since $x = \sin \theta$, then $\theta = \sin^{-1}(x)$. Also, as shown above, $\cos \theta = \sqrt{1-x^2}$. So an antiderivative is

$$\int \sqrt{1-x^2} \, dx = \frac{1}{2}(\theta + \sin \theta \cos \theta) = \frac{1}{2} \left(\sin^{-1} x + x\sqrt{1-x^2} \right).$$

Going back to the integral $\int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \sqrt{1-x^2} \, dx$, notice that the integrand is an even function and the interval of integration is symmetric about the origin, so by properties of definite integrals

$$\begin{aligned} \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \sqrt{1-x^2} \, dx &= 2 \int_0^{\sqrt{1-a^2}} \sqrt{1-x^2} \, dx \\ &= 2 \left[\frac{1}{2} \left(\sin^{-1} x + x\sqrt{1-x^2} \right) \right]_0^{\sqrt{1-a^2}} = \left[\sin^{-1} x + x\sqrt{1-x^2} \right]_0^{\sqrt{1-a^2}} \\ &= \left(\sin^{-1} \sqrt{1-a^2} + \sqrt{1-a^2} \sqrt{1-(1-a^2)} \right) - (\sin^{-1} 0 + 0) \\ &= \sin^{-1} \sqrt{1-a^2} + a\sqrt{1-a^2} - 0 = \cos^{-1} a + a\sqrt{1-a^2}, \end{aligned}$$

where we also used a property of inverse trigonometric functions $\sin^{-1} \sqrt{1-t^2} = \cos^{-1} t$ if $0 \leq t \leq 1$ (see the note at the end of this solution). So the integral in the brackets is

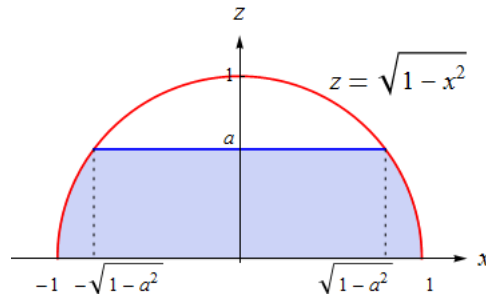
$$I = \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \sqrt{1-x^2} \, dx - \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} a \, dx = \cos^{-1} a + a\sqrt{1-a^2} - 2a\sqrt{1-a^2} = \cos^{-1} a - a\sqrt{1-a^2},$$

the volume of the empty cap is

$$\int_0^4 \left[\int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \left(\sqrt{1-x^2} - a \right) dx \right] dy = \int_0^4 I \, dy = I[y]_0^4 = I \cdot (4-0) = 4I = 4 \left(\cos^{-1} a - a\sqrt{1-a^2} \right),$$

and the volume of the liquid is

$$V_{\text{liquid}} = V_{\text{tank}} - V_{\text{cap}} = 2\pi - 4 \left(\cos^{-1} a - a\sqrt{1-a^2} \right) = \boxed{2\pi - 4 \cos^{-1} a + 4a\sqrt{1-a^2} \, \text{m}^3}.$$



Note: to prove the property $\sin^{-1} \sqrt{1-t^2} = \cos^{-1} t$ if $0 \leq t \leq 1$, let $\alpha = \sin^{-1} \sqrt{1-t^2}$. Since $0 \leq \sqrt{1-t^2} \leq 1$, α lies in the first quadrant, that is $0 \leq \alpha \leq \frac{\pi}{2}$, and therefore $\cos \alpha \geq 0$. Then we can calculate $\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - (\sqrt{1-t^2})^2} = \sqrt{1 - (1-t^2)} = \sqrt{t^2} = t$, since $t \geq 0$. Since $\cos \alpha = t$ and $0 \leq \alpha \leq \frac{\pi}{2}$, we conclude that $\alpha = \cos^{-1} t$, as desired.

81. The side of the tank that lies in the xz -plane is bounded by the semicircle $x^2 + z^2 = 1$, $z \leq 0$, or $z = -\sqrt{1-x^2}$. The horizontal line that represents the water level of depth a meters, that is at $z = -1+a$, crosses this semicircle, that is the surface of the tank, at the points with $x = \pm\sqrt{1-(-1+a)^2} = \pm\sqrt{2a-a^2}$; see the figure below. So the volume of the water can be represented as the solid enclosed between $z = f_1(x, y) = -\sqrt{1-x^2}$ and $z = f_2(x, y) = -1+a$ over the region R in the xy -plane, that is a rectangle $-\sqrt{2a-a^2} \leq x \leq \sqrt{2a-a^2}$, $0 \leq y \leq 4$; and we can find its volume as the double integral

$$\iint_R [f_2(x, y) - f_1(x, y)] dA = \int_0^4 \left[\int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1+a + \sqrt{1-x^2}) dx \right] dy.$$

Let I be the value of the integral in the brackets; we find it by integrating with respect to x . The integral in the brackets can be split into a sum of two integrals:

$$I = \int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1+a + \sqrt{1-x^2}) dx = \int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1+a) dx + \int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} \sqrt{1-x^2} dx.$$

The first integral can be evaluated directly as

$$\int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1+a) dx = [(-1+a)x]_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} = (-1+a) \left[\sqrt{2a-a^2} - (-\sqrt{2a-a^2}) \right] = 2(a-1)\sqrt{2a-a^2}.$$

To evaluate the second integral, we will find an antiderivative using a substitution $x = \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = \cos \theta d\theta$. Also, $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$, and the integral becomes

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int \cos \theta \cdot \cos \theta d\theta = \int \cos^2 \theta d\theta \\ &= \int \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) = \frac{1}{2}(\theta + \sin \theta \cos \theta). \end{aligned}$$

To express this antiderivative in terms of x , we first observe that since $x = \sin \theta$, then $\theta = \sin^{-1}(x)$. Also, as shown above, $\cos \theta = \sqrt{1-x^2}$. So an antiderivative is

$$\int \sqrt{1-x^2} dx = \frac{1}{2}(\theta + \sin \theta \cos \theta) = \frac{1}{2} \left(\sin^{-1} x + x\sqrt{1-x^2} \right).$$

Going back to the integral $\int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} \sqrt{1-x^2} dx$, notice that the integrand is an even function and the interval of integration is symmetric about the origin, so by properties of definite integrals

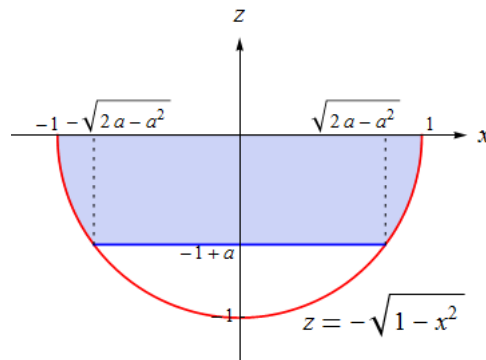
$$\begin{aligned} \int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} \sqrt{1-x^2} dx &= 2 \int_0^{\sqrt{2a-a^2}} \sqrt{1-x^2} dx \\ &= 2 \left[\frac{1}{2} \left(\sin^{-1} x + x\sqrt{1-x^2} \right) \right]_0^{\sqrt{2a-a^2}} = \left[\sin^{-1} x + x\sqrt{1-x^2} \right]_0^{\sqrt{2a-a^2}} \\ &= \left(\sin^{-1} \sqrt{2a-a^2} + \sqrt{2a-a^2} \sqrt{1-(2a-a^2)} \right) - (\sin^{-1} 0 + 0) \\ &= \sin^{-1} \sqrt{2a-a^2} + (1-a)\sqrt{2a-a^2} - 0 = \sin^{-1} \sqrt{2a-a^2} + (1-a)\sqrt{2a-a^2}, \end{aligned}$$

where we simplified $\sqrt{1 - (2a - a^2)} = \sqrt{1 - 2a + a^2} = \sqrt{(1 - a)^2} = 1 - a$ since $a < 1$. So the integral in the brackets is

$$\begin{aligned} I &= \int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1 + a + \sqrt{1-x^2}) dx = 2(a-1)\sqrt{2a-a^2} + \sin^{-1} \sqrt{2a-a^2} - (a-1)\sqrt{2a-a^2} \\ &= (a-1)\sqrt{2a-a^2} + \sin^{-1} \sqrt{2a-a^2}, \end{aligned}$$

and the volume of the liquid is

$$\begin{aligned} V_{\text{liquid}} &= \int_0^4 \left[\int_{-\sqrt{2a-a^2}}^{\sqrt{2a-a^2}} (-1 + a + \sqrt{1-x^2}) dx \right] dy = \int_0^4 I dy \\ &= I \cdot [y]_0^4 = I \cdot (4 - 0) = 4I = \boxed{4 \left[(a-1)\sqrt{2a-a^2} + \sin^{-1} \sqrt{2a-a^2} \right] \text{ m}^3}. \end{aligned}$$



82. Let R be the region in the xy -plane representing the shape of the valley, as shown in the figure in Problem 82. The right side of its V-shape (in the first quadrant) is a line segment through the points $(0,0)$ and $(100,150)$, therefore its slope is $m = \frac{150-0}{100-0} = \frac{3}{2}$, and its equation is $y = \frac{3}{2}x$ or $x = \frac{2}{3}y$; similarly, the equation of the other side (in the second quadrant) passing through $(0,0)$ and $(-100,150)$ is $y = -\frac{3}{2}x$ or $x = -\frac{2}{3}y$. So we can view R as a y -simple region: it is bounded by $x = -\frac{2}{3}y$ and $x = \frac{2}{3}y$, where $0 \leq y \leq 150$.

The surface of the water is at the level of $y = 150$ meters. Partition R into subrectangles of width Δx and height Δy ; assume that after disregarding subrectangles that do not lie entirely in R , we are left with N subrectangles, each of area $\Delta A = \Delta x \Delta y$. In each subrectangle choose an arbitrary point (u_k, v_k) , $k = 1, 2, \dots, N$. If Δy is small, then all points in the k th subrectangle lie at about the same height v_k from the bottom. The pressure due to the water at a depth of $d_k = 150 - v_k$ meters is $P_k = \rho g d_k = \rho g(150 - v_k)$ pascals, so the outward force acting on the k th subrectangle is $F_k = P_k \Delta A = \rho g(150 - v_k) \Delta A$ newtons, and the total force is approximately

$$F \approx \sum_{k=1}^N F_k = \sum_{k=1}^N \rho g(150 - v_k) \Delta A.$$

These are Riemann sums of a continuous function $f(x, y) = \rho g(150 - y)$; and as the number of subrectangles $N \rightarrow \infty$, these sums become better and better approximations to the force due to water

pressure, which can be represented, as the limit of Riemann sums, by the double integral

$$\begin{aligned}
 F &= \iint_R f(x, y) \, dA = \iint_R \rho g(150 - y) \, dA = \int_0^{150} \left[\int_{-2y/3}^{2y/3} \rho g(150 - y) \, dx \right] dy \\
 &= \int_0^{150} \rho g(150 - y) \left[\int_{-2y/3}^{2y/3} dx \right] dy = \int_0^{150} \rho g(150 - y) [x]_{-2y/3}^{2y/3} dy \\
 &= \int_0^{150} \rho g(150 - y) \left[\frac{2y}{3} - \left(-\frac{2y}{3} \right) \right] dy = \int_0^{150} \rho g(150 - y) \cdot \frac{4y}{3} dy \\
 &= \rho g \int_0^{150} \left(200y - \frac{4}{3}y^2 \right) dy = \rho g \left[100y^2 - \frac{4}{9}y^3 \right]_0^{150} = \rho g \left[\left(100 \cdot 150^2 - \frac{4}{9} \cdot 150^3 \right) - (0 - 0) \right] \\
 &= \rho g \cdot 750,000 = 1000 \cdot 9.8 \cdot 750,000 = \boxed{7,350,000,000 \text{ N}}.
 \end{aligned}$$

To convert this value to pounds we use the fact that $1 \text{ N} \approx 0.2248 \text{ lb}$, so

$$F = 7,350,000,000 \text{ N} \times 0.2248 \text{ lb/N} = \boxed{1,652,280,000 \text{ lb}}.$$

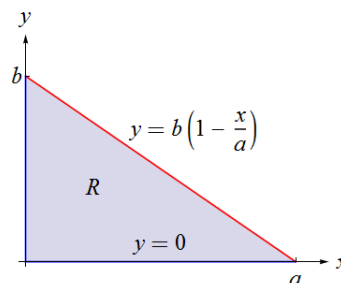
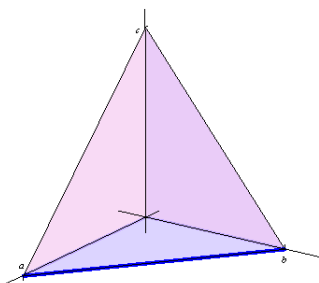
83. Answers will vary.

84. We begin by graphing the solid tetrahedron S enclosed by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (where $a, b, c > 0$). This solid lies under the graph of $z = f(x, y) = c \left(1 - \frac{x}{a} - \frac{y}{b} \right)$ and above the region R in the xy -plane, which is the triangular region in the xy -plane bounded by the coordinate axes and the line $\frac{x}{a} + \frac{y}{b} = 1$. Notice that R is an x -simple region: it is bounded by $y = 0$ and $y = b \left(1 - \frac{x}{a} \right)$, where $0 \leq x \leq a$. See the figures below. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^a \left[\int_0^{b(1-x/a)} c \left(1 - \frac{x}{a} - \frac{y}{b} \right) dy \right] dx \\
 &= \int_0^a c \left[\left(1 - \frac{x}{a} \right) y - \frac{1}{2b} y^2 \right]_0^{b(1-x/a)} dx = \int_0^a c \left[\left(1 - \frac{x}{a} \right) \cdot b \left(1 - \frac{x}{a} \right) - \frac{1}{2b} b^2 \left(1 - \frac{x}{a} \right)^2 \right] dx \\
 &= \int_0^a c \left[b \left(1 - \frac{x}{a} \right)^2 - \frac{b}{2} \left(1 - \frac{x}{a} \right)^2 \right] dx = \int_0^a c \cdot \frac{b}{2} \left(1 - \frac{x}{a} \right)^2 dx = \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a} \right)^2 dx.
 \end{aligned}$$

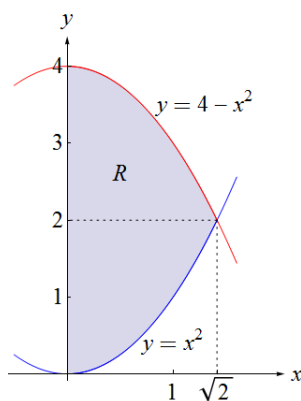
In this integral we make a substitution $u = 1 - \frac{x}{a}$; then $du = -\frac{dx}{a}$, so $dx = -adu$. We also change the limits of integration: if $x = 0$, then $u = 1 - \frac{0}{a} = 1$; if $x = a$, then $u = 1 - \frac{a}{a} = 0$. So

$$\begin{aligned}
 V &= \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a} \right)^2 dx = \frac{bc}{2} \int_1^0 u^2 (-adu) = -\frac{abc}{2} \int_1^0 u^2 du \\
 &= \frac{abc}{2} \int_0^1 u^2 du = \frac{abc}{2} \left[\frac{u^3}{3} \right]_0^1 = \frac{abc}{6} [u^3]_0^1 = \frac{abc}{6} \cdot (1 - 0) = \boxed{\frac{abc}{6} \text{ cubic units}}.
 \end{aligned}$$



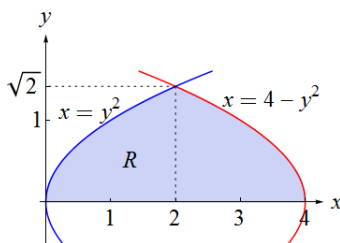
85. We begin by graphing the region R enclosed by the parabolas $y = 4 - x^2$ and $y = x^2$ to the right of the y -axis; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = x^2$ to $y = 4 - x^2$, $0 \leq x \leq \sqrt{2}$. Then

$$\begin{aligned}
 \iint_R xy \, dA &= \int_0^{\sqrt{2}} \left[\int_{x^2}^{4-x^2} xy \, dy \right] dx = \int_0^{\sqrt{2}} x \left[\int_{x^2}^{4-x^2} y \, dy \right] dx = \int_0^{\sqrt{2}} x \left[\frac{y^2}{2} \right]_{x^2}^{4-x^2} dx \\
 &= \int_0^{\sqrt{2}} x \left[\frac{(4-x^2)^2}{2} - \frac{(x^2)^2}{2} \right] dx = \int_0^{\sqrt{2}} x \left(\frac{16-8x^2+x^4}{2} - \frac{x^4}{2} \right) dx \\
 &= \int_0^{\sqrt{2}} x(8-4x^2) \, dx = \int_0^{\sqrt{2}} (8x-4x^3) \, dx = [4x^2 - x^4]_0^{\sqrt{2}} \\
 &= \left[4(\sqrt{2})^2 - (\sqrt{2})^4 \right] - (0-0) = 8-4 = \boxed{4}.
 \end{aligned}$$



86. We begin by graphing the region R enclosed by the parabolas $x = 4 - y^2$ and $x = y^2$ above the x -axis; see the figure below. Observe that R is a closed bounded region, that is y -simple: x varies from $x = y^2$ to $x = 4 - y^2$, $0 \leq y \leq \sqrt{2}$. Then

$$\begin{aligned}
 \iint_R xy \, dA &= \int_0^{\sqrt{2}} \left[\int_{y^2}^{4-y^2} xy \, dx \right] dy = \int_0^{\sqrt{2}} y \left[\int_{y^2}^{4-y^2} x \, dx \right] dy = \int_0^{\sqrt{2}} y \left[\frac{x^2}{2} \right]_{y^2}^{4-y^2} dy \\
 &= \int_0^{\sqrt{2}} y \left[\frac{(4-y^2)^2}{2} - \frac{(y^2)^2}{2} \right] dy = \int_0^{\sqrt{2}} y \left(\frac{16-8y^2+y^4}{2} - \frac{y^4}{2} \right) dy \\
 &= \int_0^{\sqrt{2}} y(8-4y^2) \, dy = \int_0^{\sqrt{2}} (8y-4y^3) \, dy = [4y^2 - y^4]_0^{\sqrt{2}} \\
 &= \left[4(\sqrt{2})^2 - (\sqrt{2})^4 \right] - (0-0) = 8-4 = \boxed{4}.
 \end{aligned}$$



87. We begin by graphing the region R enclosed by the lines $y = x$, $x = 1$, and $y = 0$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = 0$ to $y = x$, $0 \leq x \leq 1$.

Then

$$\begin{aligned}\iint_R e^{\sqrt{x}} dA &= \int_0^1 \left[\int_0^x e^{\sqrt{x}} dy \right] dx = \int_0^1 e^{\sqrt{x}} \left[\int_0^x dy \right] dx \\ &= \int_0^1 e^{\sqrt{x}} [y]_0^x dx = \int_0^1 e^{\sqrt{x}} \cdot (x-0) dx = \int_0^1 x e^{\sqrt{x}} dx.\end{aligned}$$

In this integral we make a substitution $x = t^2$, so that $\sqrt{x} = t$; then $dx = 2t dt$. We also change the limits of integration: if $x = 0$, then $t = \sqrt{0} = 0$; if $x = 1$, then $t = \sqrt{1} = 1$. So

$$\int_0^1 x e^{\sqrt{x}} dx = \int_0^1 t^2 e^t \cdot 2t dt = \int_0^1 2t^3 e^t dt.$$

We evaluate this integral by applying integration by parts three times (because the integrand is a polynomial of degree 3 times e^t). First time, we set $u = 2t^3$ and $dv = e^t dt$, then $du = 6t^2 dt$ and $v = \int e^t dt = e^t$; then

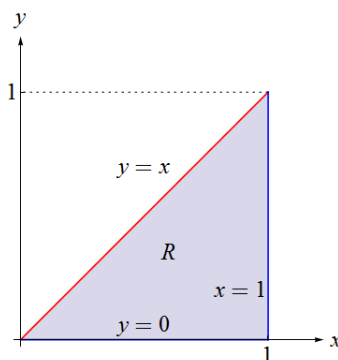
$$\int_0^1 2t^3 e^t dt = [2t^3 \cdot e^t]_0^1 - \int_0^1 e^t \cdot 6t^2 dt = (2e - 0) - \int_0^1 6t^2 e^t dt = 2e - \int_0^1 6t^2 e^t dt.$$

Next, we set $u = 6t^2$ and $dv = e^t dt$, then $du = 12t dt$ and $v = \int e^t dt = e^t$; then

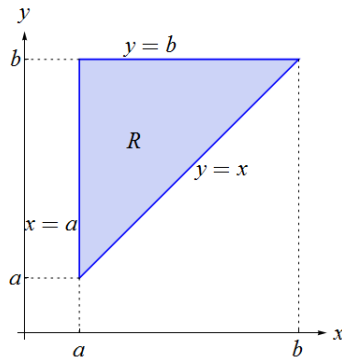
$$\begin{aligned}2e - \int_0^1 6t^2 e^t dt &= 2e - \left([6t^2 \cdot e^t]_0^1 - \int_0^1 e^t \cdot 12t dt \right) \\ &= 2e - \left((6e - 0) - \int_0^1 12te^t dt \right) = -4e + \int_0^1 12te^t dt.\end{aligned}$$

And finally, we set $u = 12t$ and $dv = e^t dt$, then $du = 12 dt$ and $v = \int e^t dt = e^t$; then

$$\begin{aligned}-4e + \int_0^1 12te^t dt &= -4e + \left([12t \cdot e^t]_0^1 - \int_0^1 e^t \cdot 12 dt \right) = -4e + \left((12e - 0) - \int_0^1 12e^t dt \right) \\ &= 8e - \int_0^1 12e^t dt = 8e - 12 [e^t]_0^1 = 8e - 12(e - 1) = \boxed{12 - 4e}.\end{aligned}$$



88. (a) Assume $0 \leq a \leq b$, where a and b are constants. Then the region R defined by $0 \leq a \leq y \leq b$, $0 \leq a \leq x \leq y$ is the triangular region shown in the figure.



- (b) Observe that R is a closed bounded region, that is x -simple: y varies from $y = x$ to $y = b$, $a \leq x \leq b$. Then for a function $f(x)$ continuous on R ,

$$\begin{aligned} \iint_R f(x) dA &= \int_a^b \left[\int_x^b f(x) dy \right] dx = \int_a^b f(x) \left[\int_x^b dy \right] dx \\ &= \int_a^b f(x) [y]_x^b dx = \int_a^b f(x) \cdot (b - x) dx = \int_a^b (b - x)f(x) dx. \end{aligned}$$

89. Let f and g be integrable function on a region R in the xy -plane, and assume that $f(x, y) \leq g(x, y)$ on R . Let $h(x, y) = g(x, y) - f(x, y)$ be a function of two variables defined on R . Since $f(x, y) \leq g(x, y)$ on R , it follows that $h(x, y) \geq 0$ on R . Then the double integral $\iint_R h(x, y) dA$ equals the volume under the surface $z = h(x, y)$ over the region R , and so it is a nonnegative quantity. Now, using properties of double integrals,

$$0 \leq \iint_R h(x, y) dA = \iint_R [g(x, y) - f(x, y)] dA = \iint_R g(x, y) dA - \iint_R f(x, y) dA.$$

Since $\iint_R g(x, y) dA - \iint_R f(x, y) dA \geq 0$, it follows that $\iint_R f(x, y) dA \leq \iint_R g(x, y) dA$.

Challenge Problem

90. We begin by graphing the solid S under the elliptic paraboloid $z = f(x, y) = 8 - 2x^2 - y^2$ and above the first quadrant in the xy -plane; see figure (a). Its base in the xy -plane is the region R bounded by the ellipse $2x^2 + y^2 = 8$ and the coordinate axes in the first quadrant, as shown on figure (b). Observe that R is an x -simple region: y varies from $y = 0$ to $y = \sqrt{8 - 2x^2}$, where $0 \leq x \leq 2$. So the volume V of the solid S is

$$\begin{aligned} V &= \iint_R f(x, y) dA = \int_0^2 \left[\int_0^{\sqrt{8-2x^2}} (8 - 2x^2 - y^2) dy \right] dx = \int_0^2 \left[(8 - 2x^2)y - \frac{1}{3}y^3 \right]_0^{\sqrt{8-2x^2}} dx \\ &= \int_0^2 \left[(8 - 2x^2)\sqrt{8 - 2x^2} - \frac{1}{3}(\sqrt{8 - 2x^2})^3 \right] dx = \int_0^2 \frac{2}{3}(\sqrt{8 - 2x^2})^3 dx. \end{aligned}$$

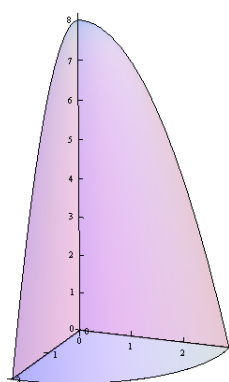
In this integral, we make a substitution $x = 2 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = 2 \cos \theta d\theta$. Also, $\sqrt{8 - 2x^2} = \sqrt{8 - 2(2 \sin \theta)^2} = \sqrt{8 - 8 \sin^2 \theta} = \sqrt{8(1 - \sin^2 \theta)} = \sqrt{8 \cos^2 \theta} = 2\sqrt{2} \cos \theta$. The limits of integration can be recalculated as follows: if $x = 0$, then $\sin \theta = \frac{0}{2} = 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ imply that

$\theta = 0$; if $x = 2$, then $\sin \theta = \frac{2}{2} = 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ imply that $\theta = \frac{\pi}{2}$. So the integral becomes

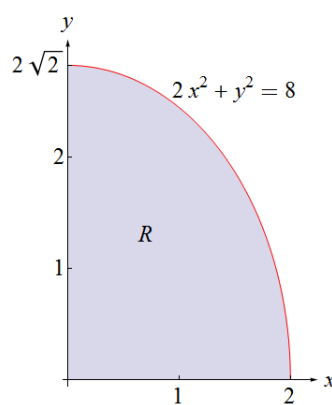
$$\begin{aligned} \int_0^2 \frac{2}{3} \left(\sqrt{8-2x^2} \right)^3 dx &= \int_0^{\pi/2} \frac{2}{3} (2\sqrt{2} \cos \theta)^3 \cdot 2 \cos \theta d\theta \\ &= \int_0^{\pi/2} \frac{2}{3} \cdot 16\sqrt{2} \cos^3 \theta \cdot 2 \cos \theta d\theta = \frac{64\sqrt{2}}{3} \int_0^{\pi/2} \cos^4 \theta d\theta. \end{aligned}$$

To find the value of this integral, we can use Wallis's formula given in Section 7.1, Problem 69. Applying Wallis's formula with $n = 4$, we obtain

$$V = \frac{64\sqrt{2}}{3} \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{64\sqrt{2}}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \boxed{4\sqrt{2}\pi \text{ cubic units}}.$$



(a)



(b)

14.3 Double Integrals Using Polar Coordinates

Concepts and Vocabulary

1. $f(x, y) = \sqrt{4x^2 + 4y^2} = \sqrt{4(x^2 + y^2)} = \sqrt{4r^2} = 2r$, assuming $r \geq 0$, so $\boxed{f(r, \theta) = 2r}$.
2. In polar coordinates, the differential of the area is $dA = \boxed{r \, dr \, d\theta}$.
3. **False**. If $f(x, y) \geq 0$ is continuous on a closed bounded region R , the volume under the surface $z = f(x, y)$ over R can be found as $\iint_R f(x, y) \, dA$. Since in polar coordinates the differential of the area is $dA = r \, dr \, d\theta$, then $\iint_R f(x, y) \, dA = \iint_R f(r \cos \theta, r \sin \theta) r \, dr \, d\theta \neq \iint_R f(r \cos \theta, r \sin \theta) \, dr \, d\theta$.
4. **True**. The area of a closed bounded region R can be found as $A = \iint_R dA = \iint_R r \, dr \, d\theta$.

Skill Building

5. The region R given in the figure in Problem 5 is bounded by the circle of radius 2 centered at the origin. In polar coordinates, this region R is given as $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Then

$$\iint_R f(x, y) \, dA = \boxed{\int_0^{2\pi} \int_0^2 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta}.$$

6. The region R given in the figure in Problem 6 is half of the disk of radius 3 centered at the origin lying to the right of the y -axis. In polar coordinates, this region R is given as $0 \leq r \leq 3$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then

$$\iint_R f(x, y) \, dA = \boxed{\int_{-\pi/2}^{\pi/2} \int_0^3 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta}.$$

7. The region R given in the figure in Problem 7 is the annular region contained between the concentric circles of radii 1 and 3 centered at the origin. In polar coordinates, this region R is given as $1 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$. Then

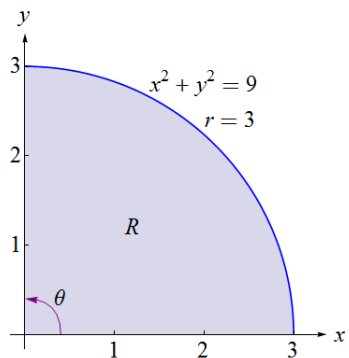
$$\iint_R f(x, y) \, dA = \boxed{\int_0^{2\pi} \int_1^3 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta}.$$

8. The region R given in the figure in Problem 8 is part of the annular region contained between the concentric circles of radii 1 and 2 centered at the origin lying in the first quadrant. In polar coordinates, this region R is given as $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\iint_R f(x, y) \, dA = \boxed{\int_0^{\pi/2} \int_1^2 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta}.$$

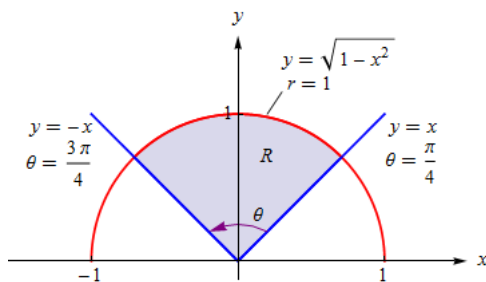
9. The region R enclosed by $x^2 + y^2 = 9$, $x \geq 0$, and $y \geq 0$ is a quarter of the disk of radius 3 centered at the origin lying in the first quadrant, as shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq 3$, $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\iint_R (x + y) \, dA = \int_0^{\pi/2} \int_0^3 (r \cos \theta + r \sin \theta) r \, dr \, d\theta = \boxed{\int_0^{\pi/2} \int_0^3 r^2 (\cos \theta + \sin \theta) \, dr \, d\theta}.$$



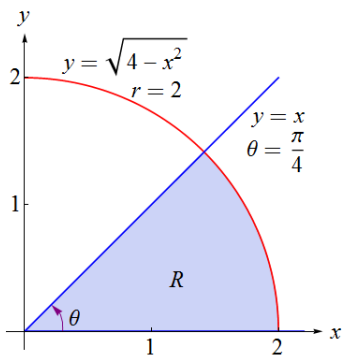
10. The region R enclosed by $y = \sqrt{1 - x^2}$ and the lines $y = x$ and $y = -x$ is part of the disk of radius 1 centered at the origin, as shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq 1$, $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$. Also, the integrand can be simplified as $x^2 + y^2 = r^2$. Then

$$\iint_R (x^2 + y^2) dA = \int_{\pi/4}^{3\pi/4} \int_0^1 r^2 r dr d\theta = \boxed{\int_{\pi/4}^{3\pi/4} \int_0^1 r^3 dr d\theta}.$$



11. The region R enclosed by $x = \sqrt{4 - y^2}$, the positive x -axis, and the line $y = x$ is part of the disk of radius 2 centered at the origin, as shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{4}$. Then

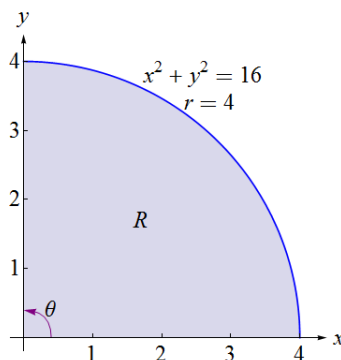
$$\iint_R x dA = \int_0^{\pi/4} \int_0^2 r \cos \theta r dr d\theta = \boxed{\int_0^{\pi/4} \int_0^2 r^2 \cos \theta dr d\theta}.$$



12. The region R enclosed by $y = \sqrt{16 - x^2}$ in the first quadrant is quarter of the disk of radius 4 centered at the origin, as shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq 4$,

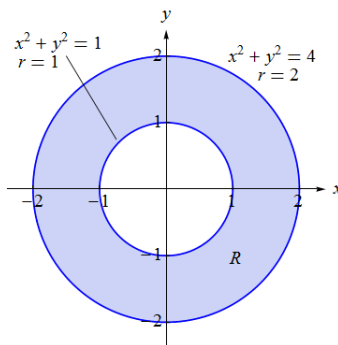
$0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\iint_R y \, dA = \int_0^{\pi/2} \int_0^4 r \sin \theta \, dr \, d\theta = \boxed{\int_0^{\pi/2} \int_0^4 r^2 \sin \theta \, dr \, d\theta}.$$



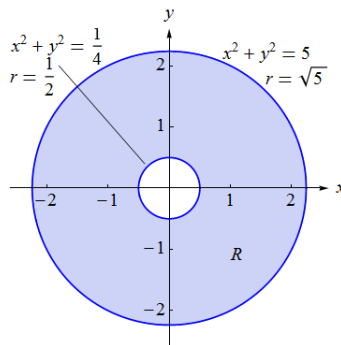
13. The region R enclosed by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ is the annular region contained between the concentric circles of radii 1 and 2 centered at the origin, as shown in the figure below. In polar coordinates, this region R is given as $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Then

$$\iint_R (x + y^2) \, dA = \int_0^{2\pi} \int_1^2 (r \cos \theta + r^2 \sin^2 \theta) r \, dr \, d\theta = \boxed{\int_0^{2\pi} \int_1^2 (r^2 \cos \theta + r^3 \sin^2 \theta) \, dr \, d\theta}.$$



14. The region R enclosed by $x^2 + y^2 = \frac{1}{4}$ and $x^2 + y^2 = 5$ is the annular region contained between the concentric circles of radii $\frac{1}{2}$ and $\sqrt{5}$ centered at the origin, as shown in the figure below. In polar coordinates, this region R is given as $\frac{1}{2} \leq r \leq \sqrt{5}$, $0 \leq \theta \leq 2\pi$. Also, the integrand can be simplified as $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Then

$$\iint_R \sqrt{x^2 + y^2} \, dA = \int_0^{2\pi} \int_{1/2}^{\sqrt{5}} r \, r \, dr \, d\theta = \boxed{\int_0^{2\pi} \int_{1/2}^{\sqrt{5}} r^2 \, dr \, d\theta}.$$



15. To find this iterated integral, we first integrate partially with respect to r .

$$\begin{aligned} \int_0^{\pi/4} \int_0^{4 \sin \theta} r \cos \theta \, dr \, d\theta &= \int_0^{\pi/4} \cos \theta \left[\int_0^{4 \sin \theta} r \, dr \right] d\theta \\ &= \int_0^{\pi/4} \cos \theta \left[\frac{r^2}{2} \right]_0^{4 \sin \theta} d\theta = \int_0^{\pi/4} \cos \theta \left[\frac{(4 \sin \theta)^2}{2} - 0 \right] d\theta \\ &= \int_0^{\pi/4} \cos \theta \cdot 8 \sin^2 \theta \, d\theta = 8 \int_0^{\pi/4} \sin^2 \theta \cos \theta \, d\theta. \end{aligned}$$

In this integral we make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = 0$, then $u = \sin 0 = 0$; if $\theta = \frac{\pi}{4}$, then $u = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. So

$$8 \int_0^{\pi/4} \sin^2 \theta \cos \theta \, d\theta = 8 \int_0^{\sqrt{2}/2} u^2 \, du = 8 \left[\frac{u^3}{3} \right]_0^{\sqrt{2}/2} = \frac{8}{3} \left[\left(\frac{\sqrt{2}}{2} \right)^3 - 0 \right] = \frac{8}{3} \cdot \frac{2\sqrt{2}}{8} = \boxed{\frac{2\sqrt{2}}{3}}.$$

16. To find this iterated integral, we first integrate partially with respect to r .

$$\begin{aligned} \int_0^{5\pi/3} \int_0^{2 \cos \theta} r \sin \theta \, dr \, d\theta &= \int_0^{5\pi/3} \sin \theta \left[\int_0^{2 \cos \theta} r \, dr \right] d\theta \\ &= \int_0^{5\pi/3} \sin \theta \left[\frac{r^2}{2} \right]_0^{2 \cos \theta} d\theta = \int_0^{5\pi/3} \sin \theta \left[\frac{(2 \cos \theta)^2}{2} - 0 \right] d\theta \\ &= \int_0^{5\pi/3} \sin \theta \cdot 2 \cos^2 \theta \, d\theta = 2 \int_0^{5\pi/3} \cos^2 \theta \sin \theta \, d\theta. \end{aligned}$$

In this integral we make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = 0$, then $u = \cos 0 = 1$; if $\theta = \frac{5\pi}{3}$, then $u = \cos \frac{5\pi}{3} = \frac{1}{2}$. So

$$\begin{aligned} 2 \int_0^{5\pi/3} \cos^2 \theta \sin \theta \, d\theta &= 2 \int_1^{1/2} u^2 (-du) = -2 \int_1^{1/2} u^2 \, du \\ &= 2 \int_{1/2}^1 u^2 \, du = 2 \left[\frac{u^3}{3} \right]_{1/2}^1 = \frac{2}{3} \left[1^3 - \left(\frac{1}{2} \right)^3 \right] = \frac{2}{3} \cdot \frac{7}{8} = \boxed{\frac{7}{12}}. \end{aligned}$$

17. To find this iterated integral, we first integrate partially with respect to θ .

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi/3} r \, d\theta \, dr &= \int_0^{\pi/2} r \left[\int_0^{\pi/3} d\theta \right] dr = \int_0^{\pi/2} r [\theta]_0^{\pi/3} dr = \int_0^{\pi/2} r \left[\frac{\pi}{3} - 0 \right] dr \\ &= \frac{\pi}{3} \int_0^{\pi/2} r \, dr = \frac{\pi}{3} \left[\frac{r^2}{2} \right]_0^{\pi/2} = \frac{\pi}{6} \left[\left(\frac{\pi}{2} \right)^2 - 0 \right] = \frac{\pi}{6} \cdot \frac{\pi^2}{4} = \boxed{\frac{\pi^3}{24}}. \end{aligned}$$

18. To find this iterated integral, we first integrate partially with respect to r .

$$\begin{aligned}
 \int_0^{\pi/3} \int_0^{\sin \theta} r \, dr \, d\theta &= \int_0^{\pi/3} \left[\frac{r^2}{2} \right]_0^{\sin \theta} d\theta = \int_0^{\pi/3} \left[\frac{(\sin \theta)^2}{2} - 0 \right] d\theta = \frac{1}{2} \int_0^{\pi/3} \sin^2 \theta \, d\theta \\
 &= \frac{1}{2} \int_0^{\pi/3} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta = \frac{1}{4} \int_0^{\pi/3} (1 - \cos(2\theta)) \, d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin(2\theta) \right]_0^{\pi/3} \\
 &= \left[\frac{1}{4} \theta - \frac{1}{8} \sin(2\theta) \right]_0^{\pi/3} = \left(\frac{1}{4} \cdot \frac{\pi}{3} - \frac{1}{8} \sin \frac{2\pi}{3} \right) - \left(\frac{1}{4} \cdot 0 - \frac{1}{8} \sin 0 \right) \\
 &= \frac{\pi}{12} - \frac{1}{8} \cdot \frac{\sqrt{3}}{2} - 0 = \boxed{\frac{\pi}{12} - \frac{\sqrt{3}}{16}}.
 \end{aligned}$$

19. To find this iterated integral, we first integrate partially with respect to r .

$$\begin{aligned}
 \int_0^{\pi/4} \int_0^{\cos \theta} r \, dr \, d\theta &= \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{\cos \theta} d\theta = \int_0^{\pi/4} \left[\frac{(\cos \theta)^2}{2} - 0 \right] d\theta = \frac{1}{2} \int_0^{\pi/4} \cos^2 \theta \, d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \frac{1}{2} (1 + \cos(2\theta)) \, d\theta = \frac{1}{4} \int_0^{\pi/4} (1 + \cos(2\theta)) \, d\theta = \frac{1}{4} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/4} \\
 &= \left[\frac{1}{4} \theta + \frac{1}{8} \sin(2\theta) \right]_0^{\pi/4} = \left(\frac{1}{4} \cdot \frac{\pi}{4} + \frac{1}{8} \sin \frac{\pi}{2} \right) - \left(\frac{1}{4} \cdot 0 + \frac{1}{8} \sin 0 \right) \\
 &= \frac{\pi}{16} + \frac{1}{8} \cdot 1 - 0 = \frac{\pi}{16} + \frac{1}{8} = \boxed{\frac{\pi + 2}{16}}.
 \end{aligned}$$

20. To find this iterated integral, we first integrate partially with respect to θ .

$$\begin{aligned}
 \int_0^4 \int_{-\pi/4}^{\pi/4} r \cos \theta \, d\theta \, dr &= \int_0^4 r \left[\int_{-\pi/4}^{\pi/4} \cos \theta \, d\theta \right] dr = \int_0^4 r [\sin \theta]_{-\pi/4}^{\pi/4} dr \\
 &= \int_0^4 r \left[\sin \frac{\pi}{4} - \sin \left(-\frac{\pi}{4} \right) \right] dr = \int_0^4 r \left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} \right) \right] dr \\
 &= \sqrt{2} \int_0^4 r \, dr = \sqrt{2} \left[\frac{r^2}{2} \right]_0^4 = \sqrt{2} \cdot (8 - 0) = \boxed{8\sqrt{2}}.
 \end{aligned}$$

21. To find this iterated integral, we first integrate partially with respect to θ .

$$\begin{aligned}
 \int_0^2 \int_{-\pi/3}^{5\pi/3} r \sin \theta \, d\theta \, dr &= \int_0^2 r \left[\int_{-\pi/3}^{5\pi/3} \sin \theta \, d\theta \right] dr = \int_0^2 r [-\cos \theta]_{-\pi/3}^{5\pi/3} dr \\
 &= \int_0^2 r \left[-\cos \frac{5\pi}{3} + \cos \left(-\frac{\pi}{3} \right) \right] dr = \int_0^2 r \left(-\frac{1}{2} + \frac{1}{2} \right) dr = \int_0^2 0 \, dr = \boxed{0}.
 \end{aligned}$$

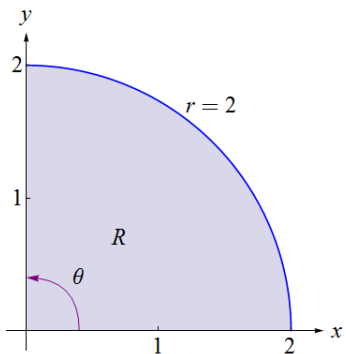
22. To find this iterated integral, we first integrate partially with respect to θ .

$$\begin{aligned}
 \int_0^{\pi} \int_0^{\pi/4} r \, d\theta \, dr &= \int_0^{\pi} r \left[\int_0^{\pi/4} d\theta \right] dr = \int_0^{\pi} r [\theta]_0^{\pi/4} dr \\
 &= \int_0^{\pi} r \left(\frac{\pi}{4} - 0 \right) dr = \frac{\pi}{4} \int_0^{\pi} r \, dr = \frac{\pi}{4} \left[\frac{r^2}{2} \right]_0^{\pi} = \frac{\pi}{4} \left(\frac{\pi^2}{2} - 0 \right) = \boxed{\frac{\pi^3}{8}}.
 \end{aligned}$$

23. The region R described in polar coordinates as $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$ is a quarter of the disk of radius 2 centered at the origin lying in the first quadrant, as shown in the figure below. To find the

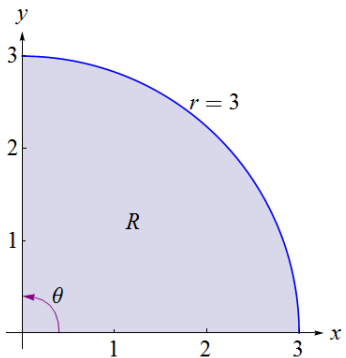
given double integral, we rewrite it as an iterated integral and then evaluate.

$$\begin{aligned}
 \iint_R 3 \sin \theta \, dA &= \int_0^{\pi/2} \int_0^2 3 \sin \theta \cdot r \, dr \, d\theta = \int_0^{\pi/2} 3 \sin \theta \left[\int_0^2 r \, dr \right] d\theta \\
 &= \int_0^{\pi/2} 3 \sin \theta \left[\frac{r^2}{2} \right]_0^2 d\theta = \int_0^{\pi/2} 3 \sin \theta \cdot (2 - 0) \, d\theta = 6 \int_0^{\pi/2} \sin \theta \, d\theta \\
 &= 6[-\cos \theta]_0^{\pi/2} = 6 \left(-\cos \frac{\pi}{2} + \cos 0 \right) = 6 \cdot (-0 + 1) = \boxed{6}.
 \end{aligned}$$



24. The region R described in polar coordinates as $0 \leq r \leq 3$, $0 \leq \theta \leq \frac{\pi}{2}$ is a quarter of the disk of radius 3 centered at the origin lying in the first quadrant, as shown in the figure below. To find the given double integral, we rewrite it as an iterated integral and then evaluate.

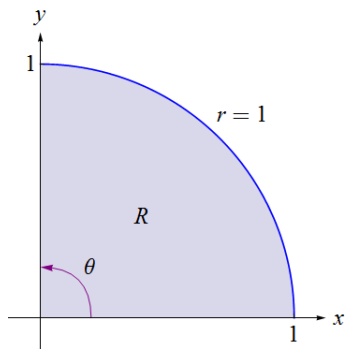
$$\begin{aligned}
 \iint_R 4 \cos \theta \, dA &= \int_0^{\pi/2} \int_0^3 4 \cos \theta \cdot r \, dr \, d\theta = \int_0^{\pi/2} 4 \cos \theta \left[\int_0^3 r \, dr \right] d\theta \\
 &= \int_0^{\pi/2} 4 \cos \theta \left[\frac{r^2}{2} \right]_0^3 d\theta = \int_0^{\pi/2} 4 \cos \theta \cdot \left(\frac{9}{2} - 0 \right) d\theta = 18 \int_0^{\pi/2} \cos \theta \, d\theta \\
 &= 18[\sin \theta]_0^{\pi/2} = 18 \left(\sin \frac{\pi}{2} - \sin 0 \right) = 18 \cdot (1 - 0) = \boxed{18}.
 \end{aligned}$$



25. The region R described in polar coordinates as $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$ is a quarter of the disk of radius 1 centered at the origin lying in the first quadrant, as shown in the figure below. To find the

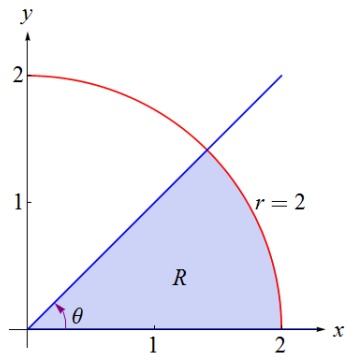
given double integral, we rewrite it as an iterated integral and then evaluate.

$$\begin{aligned}
 \iint_R 2r \sin \theta \, dA &= \int_0^{\pi/2} \int_0^1 2r \sin \theta \cdot r \, dr \, d\theta = \int_0^{\pi/2} 2 \sin \theta \left[\int_0^1 r^2 \, dr \right] d\theta \\
 &= \int_0^{\pi/2} 2 \sin \theta \left[\frac{r^3}{3} \right]_0^1 d\theta = \int_0^{\pi/2} 2 \sin \theta \cdot \left(\frac{1}{3} - 0 \right) d\theta = \frac{2}{3} \int_0^{\pi/2} \sin \theta \, d\theta \\
 &= \frac{2}{3} [-\cos \theta]_0^{\pi/2} = \frac{2}{3} \left(-\cos \frac{\pi}{2} + \cos 0 \right) = \frac{2}{3} \cdot (-0 + 1) = \boxed{\frac{2}{3}}.
 \end{aligned}$$



26. The region R described in polar coordinates as $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{4}$ is part of the disk of radius 2 centered at the origin lying in the first quadrant, as shown in the figure below. To find the given double integral, we rewrite it as an iterated integral and then evaluate.

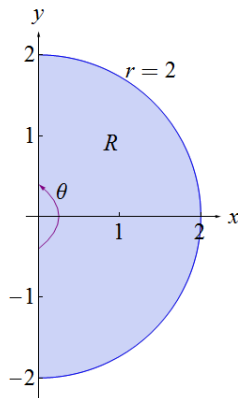
$$\begin{aligned}
 \iint_R 3r \cos \theta \, dA &= \int_0^{\pi/4} \int_0^2 3r \cos \theta \cdot r \, dr \, d\theta = \int_0^{\pi/4} 3 \cos \theta \left[\int_0^2 r^2 \, dr \right] d\theta \\
 &= \int_0^{\pi/4} 3 \cos \theta \left[\frac{r^3}{3} \right]_0^2 d\theta = \int_0^{\pi/4} 3 \cos \theta \cdot \left(\frac{8}{3} - 0 \right) d\theta = 8 \int_0^{\pi/4} \cos \theta \, d\theta \\
 &= 8 [\sin \theta]_0^{\pi/4} = 8 \left(\sin \frac{\pi}{4} - \sin 0 \right) = 8 \cdot \left(\frac{\sqrt{2}}{2} - 0 \right) = \boxed{4\sqrt{2}}.
 \end{aligned}$$



27. The region R described in polar coordinates as $0 \leq r \leq 2$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is half of the disk of radius 2 centered at the origin lying to the right of the y -axis, as shown in the figure below. To find the given

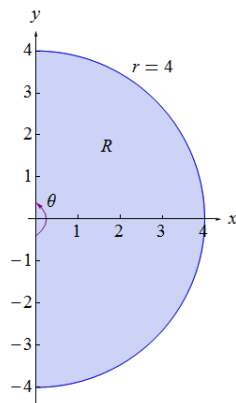
double integral, we rewrite it as an iterated integral and then evaluate.

$$\begin{aligned}
 \iint_R 3r^2 \sin \theta \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^2 3r^2 \sin \theta \cdot r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} 3 \sin \theta \left[\int_0^2 r^3 \, dr \right] d\theta \\
 &= \int_{-\pi/2}^{\pi/2} 3 \sin \theta \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_{-\pi/2}^{\pi/2} 3 \sin \theta \cdot (4 - 0) \, d\theta = 12 \int_{-\pi/2}^{\pi/2} \sin \theta \, d\theta \\
 &= 12[-\cos \theta]_{-\pi/2}^{\pi/2} = 12 \left[-\cos \frac{\pi}{2} + \cos \left(-\frac{\pi}{2} \right) \right] = 12 \cdot (-0 + 0) = \boxed{0}.
 \end{aligned}$$



28. The region R described in polar coordinates as $0 \leq r \leq 4$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is half of the disk of radius 4 centered at the origin lying to the right of the y -axis, as shown in the figure below. To find the given double integral, we rewrite it as an iterated integral and then evaluate.

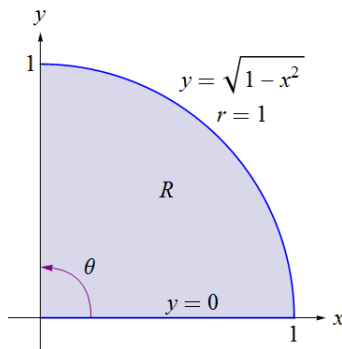
$$\begin{aligned}
 \iint_R 2r^2 \cos \theta \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^4 2r^2 \cos \theta \cdot r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} 2 \cos \theta \left[\int_0^4 r^3 \, dr \right] d\theta \\
 &= \int_{-\pi/2}^{\pi/2} 2 \cos \theta \left[\frac{r^4}{4} \right]_0^4 d\theta = \int_{-\pi/2}^{\pi/2} 2 \cos \theta \cdot (64 - 0) \, d\theta = 128 \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta \\
 &= 128[\sin \theta]_{-\pi/2}^{\pi/2} = 128 \left[\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right] = 128 \cdot [1 - (-1)] = \boxed{256}.
 \end{aligned}$$



29. The region R associated with the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} dy \, dx$ is enclosed between $y = 0$ and $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. This is the first-quadrant quarter of the disk of radius 1 centered at the origin, as

shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} dy dx &= \iint_R dA = \int_0^{\pi/2} \int_0^1 r dr d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^1 d\theta \\ &= \int_0^{\pi/2} \left(\frac{1}{2} - 0 \right) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} [\theta]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi}{4}}. \end{aligned}$$



30. The region R associated with the integral $\int_0^3 \int_0^y \sqrt{x^2 + y^2} dx dy$ is enclosed between $x = 0$ and $x = y$, where $0 \leq y \leq 3$. This is a triangular region shown in the figure below. In polar coordinates, it can be viewed as bounded, between the rays $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$, by the graph of $y = 3$ or $r \sin \theta = 3$, so that $r = \frac{3}{\sin \theta} = 3 \csc \theta$. So in polar coordinates R is given by $0 \leq r \leq 3 \csc \theta$, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. Also, the integrand can be simplified as $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Then

$$\begin{aligned} \int_0^3 \int_0^y \sqrt{x^2 + y^2} dx dy &= \iint_R \sqrt{x^2 + y^2} dA = \int_{\pi/4}^{\pi/2} \int_0^{3 \csc \theta} r \cdot r dr d\theta = \int_{\pi/4}^{\pi/2} \int_0^{3 \csc \theta} r^2 dr d\theta \\ &= \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{3 \csc \theta} d\theta = \int_{\pi/4}^{\pi/2} (9 \csc^3 \theta - 0) d\theta = 9 \int_{\pi/4}^{\pi/2} \csc^3 \theta d\theta. \end{aligned}$$

To evaluate this integral, we find an antiderivative of $\csc^3 \theta = \csc \theta \csc^2 \theta$ using the method of integration by parts: let $u = \csc \theta$ and $dv = \csc^2 \theta d\theta$; then $du = -\cot \theta \csc \theta d\theta$ and $v = \int \csc^2 \theta d\theta = -\cot \theta$. Then

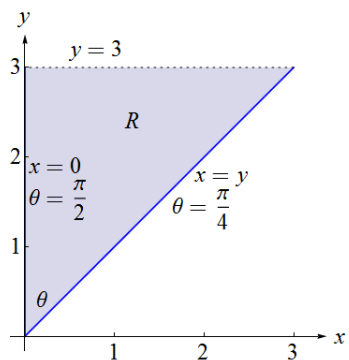
$$\begin{aligned} \int \csc^3 \theta d\theta &= \int \csc \theta \csc^2 \theta d\theta = \csc \theta \cdot (-\cot \theta) - \int (-\cot \theta) \cdot (-\cot \theta \csc \theta d\theta) \\ &= -\cot \theta \csc \theta - \int \csc \theta \cot^2 \theta d\theta = -\cot \theta \csc \theta - \int \csc \theta (\csc^2 \theta - 1) d\theta \\ &= -\cot \theta \csc \theta - \int \csc^3 \theta d\theta + \int \csc \theta d\theta = -\cot \theta \csc \theta - \int \csc^3 \theta d\theta + \ln |\cot \theta - \csc \theta|. \end{aligned}$$

This relation allows us to solve for the desired antiderivative:

$$\begin{aligned} \int \csc^3 \theta d\theta &= -\cot \theta \csc \theta - \int \csc^3 \theta d\theta + \ln |\cot \theta - \csc \theta|; \\ 2 \int \csc^3 \theta d\theta &= \ln |\cot \theta - \csc \theta| - \cot \theta \csc \theta; \\ \int \csc^3 \theta d\theta &= \frac{1}{2} (\ln |\cot \theta - \csc \theta| - \cot \theta \csc \theta). \end{aligned}$$

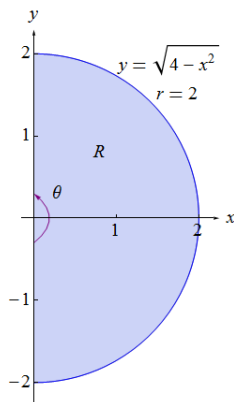
Now we can finish evaluating the original integral:

$$\begin{aligned}
 9 \int_{\pi/4}^{\pi/2} \csc^3 \theta \, d\theta &= \left[\frac{9}{2} (\ln |\cot \theta - \csc \theta| - \cot \theta \csc \theta) \right]_{\pi/4}^{\pi/2} \\
 &= \frac{9}{2} \left(\ln \left| \cot \frac{\pi}{2} - \csc \frac{\pi}{2} \right| - \cot \frac{\pi}{2} \csc \frac{\pi}{2} \right) - \frac{9}{2} \left(\ln \left| \cot \frac{\pi}{4} - \csc \frac{\pi}{4} \right| - \cot \frac{\pi}{4} \csc \frac{\pi}{4} \right) \\
 &= \frac{9}{2} (\ln |0 - 1| - 0 \cdot 1) - \frac{9}{2} (\ln |1 - \sqrt{2}| - 1 \cdot \sqrt{2}) \\
 &= \frac{9}{2} (\ln 1 - 0) - \frac{9}{2} [\ln (\sqrt{2} - 1) - \sqrt{2}] \\
 &= \frac{9}{2} \cdot 0 + \frac{9}{2} [\sqrt{2} - \ln (\sqrt{2} - 1)] = \boxed{\frac{9 [\sqrt{2} - \ln (\sqrt{2} - 1)]}{2}}.
 \end{aligned}$$



31. The region R associated with the integral $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) \, dx \, dy$ is enclosed between $x = 0$ and $x = \sqrt{4 - y^2}$, where $-2 \leq y \leq 2$. This is half of the disk of radius 2 centered at the origin lying to the right of the y -axis, as shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 2$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Also, the integrand can be simplified as $x^2 + y^2 = r^2$. Then

$$\begin{aligned}
 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) \, dx \, dy &= \iint_R (x^2 + y^2) \, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 r^2 \cdot r \, dr \, d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \int_0^2 r^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_0^2 \, d\theta = \int_{-\pi/2}^{\pi/2} (4 - 0) \, d\theta \\
 &= 4 \int_{-\pi/2}^{\pi/2} d\theta = 4 [\theta]_{-\pi/2}^{\pi/2} = 4 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \boxed{4\pi}.
 \end{aligned}$$



32. The region R associated with the integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2} dy dx$ is enclosed between $y = 0$ and $y = \sqrt{4-x^2}$, where $0 \leq x \leq 2$. This is the first-quadrant quarter of the disk of radius 2 centered at the origin, as shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, the integrand can be simplified as $\sqrt{4-x^2-y^2} = \sqrt{4-(x^2+y^2)} = \sqrt{4-r^2}$. Then

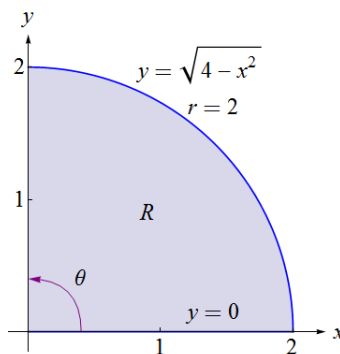
$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2} dy dx = \iint_R \sqrt{4-x^2-y^2} dA = \int_0^{\pi/2} \int_0^2 \sqrt{4-r^2} \cdot r dr d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^2 \sqrt{4-r^2} \cdot r dr$. In this integral we make a substitution $u = 4 - r^2$; then $du = -2rdr$, so $rdr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = 4 - 0^2 = 4$; if $r = 2$, then $u = 4 - 2^2 = 0$. So

$$\begin{aligned} \int_0^2 \sqrt{4-r^2} \cdot r dr &= \int_4^0 \sqrt{u} \left(-\frac{du}{2}\right) = -\frac{1}{2} \int_4^0 u^{1/2} du = \frac{1}{2} \int_0^4 u^{1/2} du \\ &= \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_0^4 = \frac{1}{3} [(\sqrt{u})^3]_0^4 = \frac{1}{3} [(\sqrt{4})^3 - (\sqrt{0})^3] = \frac{1}{3} \cdot (8 - 0) = \frac{8}{3}. \end{aligned}$$

Going back to the iterated integral,

$$\int_0^{\pi/2} \int_0^2 \sqrt{4-r^2} \cdot r dr d\theta = \int_0^{\pi/2} \frac{8}{3} d\theta = \frac{8}{3} [\theta]_0^{\pi/2} = \frac{8}{3} \cdot \left(\frac{\pi}{2} - 0\right) = \boxed{\frac{4\pi}{3}}.$$



33. The region R associated with the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(x^2 + y^2) dy dx$ is enclosed between $y = 0$ and $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. This is the first-quadrant quarter of the disk of radius 1 centered at the origin, as shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, in the integrand we can simplify $x^2 + y^2 = r^2$. Then

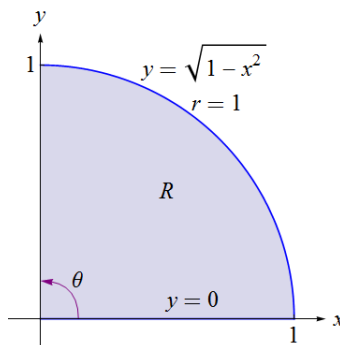
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(x^2 + y^2) dy dx = \iint_R \cos(x^2 + y^2) dA = \int_0^{\pi/2} \int_0^1 \cos(r^2) \cdot r dr d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^1 \cos(r^2) \cdot r dr$. In this integral we make a substitution $u = r^2$; then $du = 2rdr$, so $rdr = \frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = 0^2 = 0$; if $r = 1$, then $u = 1^2 = 1$. So

$$\int_0^1 \cos(r^2) \cdot r dr = \int_0^1 \cos u \cdot \frac{du}{2} = \frac{1}{2} \int_0^1 \cos u du = \frac{1}{2} [\sin u]_0^1 = \frac{1}{2} (\sin 1 - \sin 0) = \frac{\sin 1}{2}.$$

Going back to the iterated integral,

$$\int_0^{\pi/2} \int_0^1 \cos(r^2) \cdot r \, dr \, d\theta = \int_0^{\pi/2} \frac{\sin 1}{2} \, d\theta = \frac{\sin 1}{2} [\theta]_0^{\pi/2} = \frac{\sin 1}{2} \cdot \left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\pi \sin 1}{4}}.$$



34. The region R associated with the integral $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy$ is enclosed between $x = 0$ and $x = \sqrt{1-y^2}$, where $0 \leq y \leq 1$. This is the first-quadrant quarter of the disk of radius 1 centered at the origin, as shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, in the integrand we can simplify $\sqrt{x^2+y^2} = r$. Then

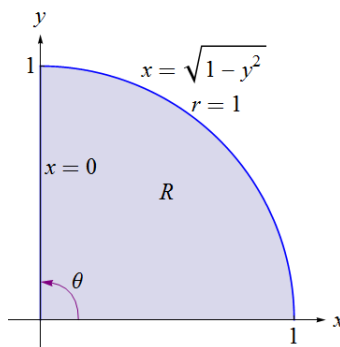
$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy = \iint_R e^{\sqrt{x^2+y^2}} \, dA = \int_0^{\pi/2} \int_0^1 e^r r \, dr \, d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^1 e^r r \, dr$. We find this integral using the method of integration by parts. If we choose $u = r$ and $dv = e^r dr$, then $du = dr$ and $v = \int e^r dr = e^r$. So

$$\int_0^1 e^r r \, dr = [re^r]_0^1 - \int_0^1 e^r \, dr = (1e^1 - 0e^0) - [e^r]_0^1 = e - (e^1 - e^0) = e - e + 1 = 1.$$

Going back to the iterated integral,

$$\int_0^{\pi/2} \int_0^1 e^r r \, dr \, d\theta = \int_0^{\pi/2} 1 \, d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}.$$



35. The region R enclosed in the first quadrant by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ is shown in the figure below. In polar coordinates, this region R is given as $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, in the integrand we can simplify $x^2 + y^2 = r^2$. Then

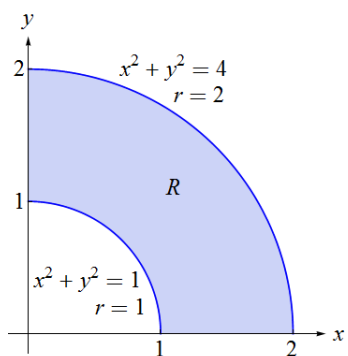
$$\iint_R e^{-(x^2+y^2)} \, dA = \int_0^{\pi/2} \int_1^2 e^{-r^2} r \, dr \, d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_1^2 e^{-r^2} r \, dr$. In this integral we make a substitution $u = -r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{du}{2}$. We also change the limits of integration: if $r = 1$, then $u = -1^2 = -1$; if $r = 2$, then $u = -2^2 = -4$. So

$$\begin{aligned} \int_1^2 e^{-r^2} r \, dr &= \int_{-1}^{-4} e^u \left(-\frac{du}{2} \right) = -\frac{1}{2} \int_{-1}^{-4} e^u \, du = \frac{1}{2} \int_{-4}^{-1} e^u \, du \\ &= \frac{1}{2} [e^u]_{-4}^{-1} = \frac{1}{2} (e^{-1} - e^{-4}) = \frac{1}{2} \left(\frac{1}{e} - \frac{1}{e^4} \right) = \frac{1}{2} \cdot \frac{e^3 - 1}{e^4} = \frac{e^3 - 1}{2e^4}. \end{aligned}$$

Going back to the iterated integral,

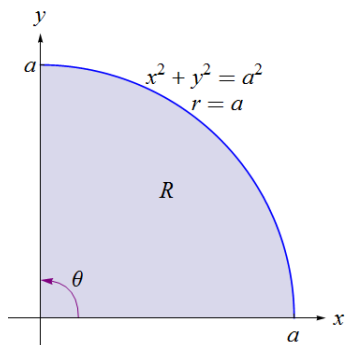
$$\int_0^{\pi/2} \int_1^2 e^{-r^2} r \, dr \, d\theta = \int_0^{\pi/2} \frac{e^3 - 1}{2e^4} \, d\theta = \frac{e^3 - 1}{2e^4} [\theta]_0^{\pi/2} = \frac{e^3 - 1}{2e^4} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi(e^3 - 1)}{4e^4}}.$$



36. The region R enclosed in the first quadrant by the circle $x^2 + y^2 = a^2$ is shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, the integrand simplifies as

$$\frac{y}{\sqrt{x^2 + y^2}} = \frac{r \sin \theta}{r} = \sin \theta. \text{ Then}$$

$$\begin{aligned} \iint_R \frac{y}{\sqrt{x^2 + y^2}} \, dA &= \int_0^{\pi/2} \int_0^a \sin \theta \cdot r \, dr \, d\theta = \int_0^{\pi/2} \sin \theta \left[\int_0^a r \, dr \right] d\theta \\ &= \int_0^{\pi/2} \sin \theta \left[\frac{r^2}{2} \right]_0^a d\theta = \int_0^{\pi/2} \sin \theta \left(\frac{a^2}{2} - 0 \right) d\theta = \frac{a^2}{2} \int_0^{\pi/2} \sin \theta \, d\theta \\ &= \frac{a^2}{2} [-\cos \theta]_0^{\pi/2} = \frac{a^2}{2} \left(-\cos \frac{\pi}{2} + \cos 0 \right) = \frac{a^2}{2} (-0 + 1) = \boxed{\frac{a^2}{2}}. \end{aligned}$$

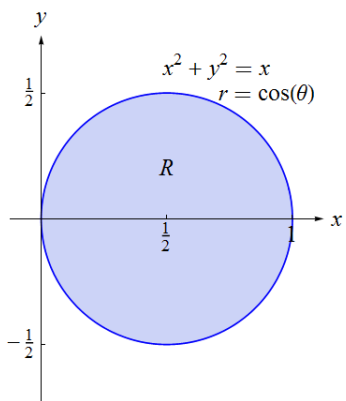


37. First, we convert the equation of the boundary of the given region R to polar coordinates:

$$\begin{aligned}x^2 + y^2 &= x \\r^2 &= r \cos \theta \\r &= \cos \theta\end{aligned}$$

So R is enclosed by the circle $r = \cos \theta$, as shown in the figure below. In polar coordinates, this region R is given as $0 \leq r \leq \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Also, in the integrand we substitute $x = r \cos \theta$. Then

$$\begin{aligned}\iint_R x \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} r \cos \theta \cdot r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \cos \theta \left[\int_0^{\cos \theta} r^2 \, dr \right] d\theta \\&= \int_{-\pi/2}^{\pi/2} \cos \theta \left[\frac{r^3}{3} \right]_0^{\cos \theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3} \cos \theta [(\cos \theta)^3 - 0] d\theta = \frac{1}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta \\&= \frac{1}{3} \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}(1 + \cos(2\theta)) \right]^2 d\theta = \frac{1}{12} \int_{-\pi/2}^{\pi/2} (1 + 2\cos(2\theta) + \cos^2(2\theta)) d\theta \\&= \frac{1}{12} \int_{-\pi/2}^{\pi/2} \left[1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta)) \right] d\theta = \frac{1}{12} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) \right) d\theta \\&= \frac{1}{12} \left[\frac{3}{2}\theta + 2 \cdot \frac{\sin(2\theta)}{2} + \frac{1}{2} \cdot \frac{\sin(4\theta)}{4} \right]_{-\pi/2}^{\pi/2} = \frac{1}{12} \left[\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta) \right]_{-\pi/2}^{\pi/2} \\&= \frac{1}{12} \left[\left(\frac{3}{2} \cdot \frac{\pi}{2} + \sin \pi + \frac{1}{8}\sin(2\pi) \right) - \left(\frac{3}{2} \cdot \left(-\frac{\pi}{2} \right) + \sin(-\pi) + \frac{1}{8}\sin(-2\pi) \right) \right] \\&= \frac{1}{12} \left[\left(\frac{3\pi}{4} + 0 + 0 \right) - \left(-\frac{3\pi}{4} + 0 + 0 \right) \right] = \frac{1}{12} \cdot \frac{3\pi}{2} = \boxed{\frac{\pi}{8}}.\end{aligned}$$



38. First, we convert the equation of the boundary of the given region R to polar coordinates:

$$\begin{aligned}x^2 + y^2 &= 2y \\r^2 &= 2r \sin \theta \\r &= 2 \sin \theta\end{aligned}$$

So R is enclosed by the circle $r = 2 \sin \theta$, as shown in the figure below. In polar coordinates, this region

R is given as $0 \leq r \leq 2 \sin \theta$, $0 \leq \theta \leq \pi$. Also, in the integrand we substitute $y = r \sin \theta$. Then

$$\begin{aligned}
 \iint_R y^2 dA &= \int_0^\pi \int_0^{2 \sin \theta} (r \sin \theta)^2 \cdot r dr d\theta = \int_0^\pi \sin^2 \theta \left[\int_0^{2 \sin \theta} r^3 dr \right] d\theta \\
 &= \int_0^\pi \sin^2 \theta \left[\frac{r^4}{4} \right]_0^{2 \sin \theta} d\theta = \int_0^\pi \sin^2 \theta \left[\frac{(2 \sin \theta)^4}{4} - 0 \right] d\theta = \int_0^\pi 4 \sin^6 \theta d\theta \\
 &= 4 \int_0^\pi \left[\frac{1}{2} (1 - \cos(2\theta)) \right]^3 d\theta = \frac{1}{2} \int_0^\pi (1 - 3 \cos(2\theta) + 3 \cos^2(2\theta) - \cos^3(2\theta)) d\theta \\
 &= \frac{1}{2} \int_0^\pi \left[1 - 3 \cos(2\theta) + 3 \cdot \frac{1}{2} (1 + \cos(4\theta)) - \cos^3(2\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^\pi \left(\frac{5}{2} - 3 \cos(2\theta) + \frac{3}{2} \cos(4\theta) - \cos^3(2\theta) \right) d\theta \\
 &= \frac{5}{4} \int_0^\pi d\theta - \frac{3}{2} \int_0^\pi \cos(2\theta) d\theta + \frac{3}{4} \int_0^\pi \cos(4\theta) d\theta - \frac{1}{2} \int_0^\pi \cos^3(2\theta) d\theta.
 \end{aligned}$$

The first three integrals can be evaluated directly:

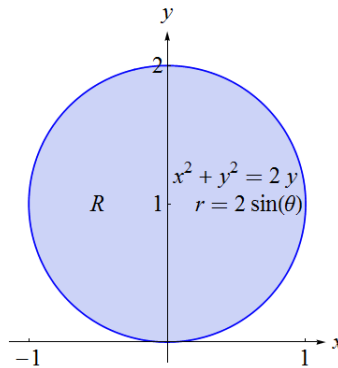
$$\begin{aligned}
 \frac{5}{4} \int_0^\pi d\theta &= \frac{5}{4} [\theta]_0^\pi = \frac{5}{4} (\pi - 0) = \frac{5\pi}{4}; \\
 \frac{3}{2} \int_0^\pi \cos(2\theta) d\theta &= \frac{3}{2} \left[\frac{\sin(2\theta)}{2} \right]_0^\pi = \frac{3}{4} [\sin(2\theta)]_0^\pi = \frac{3}{4} (\sin(2\pi) - \sin 0) = \frac{3}{4} (0 - 0) = 0; \\
 \frac{3}{4} \int_0^\pi \cos(4\theta) d\theta &= \frac{3}{4} \left[\frac{\sin(4\theta)}{4} \right]_0^\pi = \frac{3}{16} [\sin(4\theta)]_0^\pi = \frac{3}{16} (\sin(4\pi) - \sin 0) = \frac{3}{16} (0 - 0) = 0.
 \end{aligned}$$

To evaluate the last integral, we use a trigonometric identity to transform the integrand $\cos^3(2\theta) = \cos^2(2\theta) \cdot \cos(2\theta) = (1 - \sin^2(2\theta)) \cos(2\theta)$, and then make a substitution $u = \sin(2\theta)$; then $du = 2 \cos(2\theta) d\theta$, so $\cos(2\theta) d\theta = \frac{du}{2}$. We also change the limits of integration: if $\theta = 0$, then $u = \sin 0 = 0$; if $\theta = \pi$, then $u = \sin(2\pi) = 0$. So

$$\frac{1}{2} \int_0^\pi \cos^3(2\theta) d\theta = \frac{1}{2} \int_0^\pi (1 - \sin^2(2\theta)) \cos(2\theta) d\theta = \frac{1}{2} \int_0^0 (1 - u^2) \frac{du}{2} = \frac{1}{4} \int_0^0 (1 - u^2) du = 0.$$

Then the final answer is

$$\frac{5}{4} \int_0^\pi d\theta - \frac{3}{2} \int_0^\pi \cos(2\theta) d\theta + \frac{3}{4} \int_0^\pi \cos(4\theta) d\theta - \frac{1}{2} \int_0^\pi \cos^3(2\theta) d\theta = \frac{5\pi}{4} - 0 + 0 - 0 = \boxed{\frac{5\pi}{4}}.$$



39. The region R enclosed by the cardioid $r = 3 + \cos \theta$ is shown in the figure below. In polar coordinates,

it is given as $0 \leq r \leq 3 + \cos \theta$, $0 \leq \theta \leq 2\pi$. Also, the integrand simplifies as $\sqrt{x^2 + y^2} = r$. Then

$$\begin{aligned}
 \iint_R \sqrt{x^2 + y^2} dA &= \int_0^{2\pi} \int_0^{3+\cos \theta} r \cdot r dr d\theta = \int_0^{2\pi} \int_0^{3+\cos \theta} r^2 dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^{3+\cos \theta} d\theta = \int_0^{2\pi} \left[\frac{(3 + \cos \theta)^3}{3} - 0 \right] d\theta \\
 &= \int_0^{2\pi} \frac{27 + 27 \cos \theta + 9 \cos^2 \theta + \cos^3 \theta}{3} d\theta \\
 &= \int_0^{2\pi} \left(9 + 9 \cos \theta + 3 \cos^2 \theta + \frac{1}{3} \cos^3 \theta \right) d\theta \\
 &= \int_0^{2\pi} \left[9 + 9 \cos \theta + 3 \cdot \frac{1}{2} (1 + \cos(2\theta)) + \frac{1}{3} \cos^3 \theta \right] d\theta \\
 &= \int_0^{2\pi} \left(\frac{21}{2} + 9 \cos \theta + \frac{3}{2} \cos(2\theta) + \frac{1}{3} \cos^3 \theta \right) d\theta \\
 &= \frac{21}{2} \int_0^{2\pi} d\theta + 9 \int_0^{2\pi} \cos \theta d\theta + \frac{3}{2} \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{3} \int_0^{2\pi} \cos^3 \theta d\theta.
 \end{aligned}$$

The first three integrals can be evaluated directly:

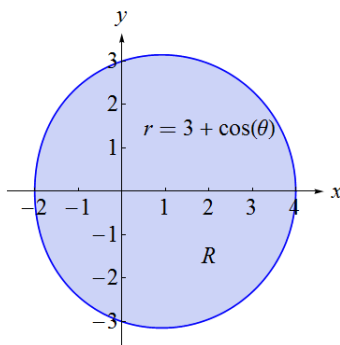
$$\begin{aligned}
 \frac{21}{2} \int_0^{2\pi} d\theta &= \frac{21}{2} [\theta]_0^{2\pi} = \frac{21}{2} (2\pi - 0) = 21\pi; \\
 9 \int_0^{2\pi} \cos \theta d\theta &= 9 [\sin \theta]_0^{2\pi} = 9 (\sin 2\pi - \sin 0) = 9(0 - 0) = 0; \\
 \frac{3}{2} \int_0^{2\pi} \cos(2\theta) d\theta &= \frac{3}{2} \left[\frac{\sin(2\theta)}{2} \right]_0^{2\pi} = \frac{3}{4} [\sin(2\theta)]_0^{2\pi} = \frac{3}{4} (\sin(4\pi) - \sin 0) = \frac{3}{4} (0 - 0) = 0.
 \end{aligned}$$

To evaluate the last integral, we use a trigonometric identity to transform the integrand $\cos^3 \theta = \cos^2 \theta \cdot \cos \theta = (1 - \sin^2 \theta) \cos \theta$, and then make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = 0$, then $u = \sin 0 = 0$; if $\theta = 2\pi$, then $u = \sin(2\pi) = 0$. So

$$\frac{1}{3} \int_0^{2\pi} \cos^3 \theta d\theta = \frac{1}{3} \int_0^{2\pi} (1 - \sin^2 \theta) \cos \theta d\theta = \frac{1}{2} \int_0^0 (1 - u^2) du = 0.$$

Then the final answer is

$$\frac{21}{2} \int_0^{2\pi} d\theta + 9 \int_0^{2\pi} \cos \theta d\theta + \frac{3}{2} \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{3} \int_0^{2\pi} \cos^3 \theta d\theta = 21\pi + 0 + 0 + 0 = \boxed{21\pi}.$$



40. The region R enclosed by the cardioid $r = 2(1 + \sin \theta)$ is shown in the figure below. In polar coordinates, it is given as $0 \leq r \leq 2(1 + \sin \theta)$, $0 \leq \theta \leq 2\pi$. Also, the integrand simplifies as $x^2 + y^2 = r^2$. Then

$$\begin{aligned}
 \iint_R (x^2 + y^2) dA &= \int_0^{2\pi} \int_0^{2(1+\sin \theta)} r^2 \cdot r dr d\theta = \int_0^{2\pi} \int_0^{2(1+\sin \theta)} r^3 dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^{2(1+\sin \theta)} d\theta = \int_0^{2\pi} \left[\frac{2^4(1+\sin \theta)^4}{4} - 0 \right] d\theta \\
 &= \int_0^{2\pi} 4(1 + 4\sin \theta + 6\sin^2 \theta + 4\sin^3 \theta + \sin^4 \theta) d\theta \\
 &= \int_0^{2\pi} (4 + 16\sin \theta + 24\sin^2 \theta + 16\sin^3 \theta + 4\sin^4 \theta) d\theta \\
 &= \int_0^{2\pi} \left[4 + 16\sin \theta + 24 \cdot \frac{1}{2}(1 - \cos(2\theta)) + 16\sin^3 \theta + 4 \left[\frac{1}{2}(1 - \cos(2\theta)) \right]^2 \right] d\theta \\
 &= \int_0^{2\pi} (4 + 16\sin \theta + 12 - 12\cos(2\theta) + 16\sin^3 \theta + 1 - 2\cos(2\theta) + \cos^2(2\theta)) d\theta \\
 &= \int_0^{2\pi} \left[17 + 16\sin \theta - 14\cos(2\theta) + 16\sin^3 \theta + \frac{1}{2}(1 + \cos(4\theta)) \right] d\theta \\
 &= \int_0^{2\pi} \left(\frac{35}{2} + 16\sin \theta - 14\cos(2\theta) + \frac{1}{2}\cos(4\theta) + 16\sin^3 \theta \right) d\theta \\
 &= \frac{35}{2} \int_0^{2\pi} d\theta + 16 \int_0^{2\pi} \sin \theta d\theta - 14 \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{2} \int_0^{2\pi} \cos(4\theta) d\theta + 16 \int_0^{2\pi} \sin^3 \theta d\theta.
 \end{aligned}$$

The first four integrals can be evaluated directly:

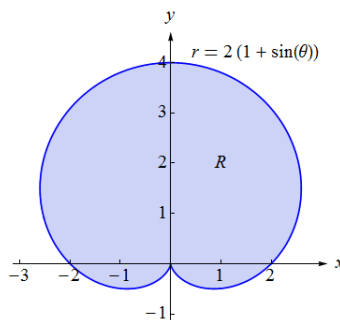
$$\begin{aligned}
 \frac{35}{2} \int_0^{2\pi} d\theta &= \frac{35}{2} [\theta]_0^{2\pi} = \frac{35}{2} (2\pi - 0) = 35\pi; \\
 16 \int_0^{2\pi} \sin \theta d\theta &= 16[-\cos \theta]_0^{2\pi} = 16(-\cos(2\pi) + \cos 0) = 16(-1 + 1) = 0; \\
 14 \int_0^{2\pi} \cos(2\theta) d\theta &= 14 \left[\frac{\sin(2\theta)}{2} \right]_0^{2\pi} = 7[\sin(2\theta)]_0^{2\pi} = 7(\sin(4\pi) - \sin 0) = 7(0 - 0) = 0; \\
 \frac{1}{2} \int_0^{2\pi} \cos(4\theta) d\theta &= \frac{1}{2} \left[\frac{\sin(4\theta)}{4} \right]_0^{2\pi} = \frac{1}{8}[\sin(4\theta)]_0^{2\pi} = \frac{1}{8}(\sin(8\pi) - \sin 0) = \frac{1}{8}(0 - 0) = 0.
 \end{aligned}$$

To evaluate the last integral, we use a trigonometric identity to transform the integrand $\sin^3 \theta = \sin^2 \theta \cdot \sin \theta = (1 - \cos^2 \theta) \sin \theta$, and then make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = 0$, then $u = \cos 0 = 1$; if $\theta = 2\pi$, then $u = \cos 2\pi = 1$. So

$$16 \int_0^{2\pi} \sin^3 \theta d\theta = 16 \int_0^{2\pi} (1 - \cos^2 \theta) \sin \theta d\theta = 16 \int_1^1 (1 - u^2) (-du) = -16 \int_1^1 (1 - u^2) du = 0.$$

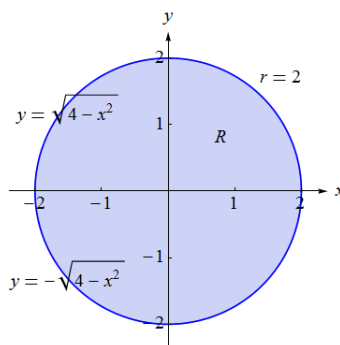
Then the final answer is

$$\frac{35}{2} \int_0^{2\pi} d\theta + 16 \int_0^{2\pi} \sin \theta d\theta - 14 \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{2} \int_0^{2\pi} \cos(4\theta) d\theta + 16 \int_0^{2\pi} \sin^3 \theta d\theta = 35\pi + 0 - 0 + 0 + 0 = \boxed{35\pi}.$$



41. The region R associated with the integral $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)^2 dx dy$ is enclosed between $x = -\sqrt{4-y^2}$ and $x = \sqrt{4-y^2}$, where $-2 \leq y \leq 2$. This is the disk of radius 2 centered at the origin, as shown in the figure below. In polar coordinates R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Also, the integrand simplifies as $(x^2 + y^2)^2 = (r^2)^2 = r^4$. Then

$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)^2 dx dy &= \iint_R (x^2 + y^2)^2 dA = \int_0^{2\pi} \int_0^2 r^4 \cdot r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 r^5 dr d\theta = \int_0^{2\pi} \left[\frac{r^6}{6} \right]_0^2 d\theta = \int_0^{2\pi} \left(\frac{2^6}{6} - 0 \right) d\theta \\ &= \int_0^{2\pi} \frac{64}{6} d\theta = \frac{32}{3} \int_0^{2\pi} d\theta = \frac{32}{3} [\theta]_0^{2\pi} = \frac{32}{3} (2\pi - 0) = \boxed{\frac{64\pi}{3}}. \end{aligned}$$



42. Since in both integrals the integrand is the same, $f(x, y) = 1$, we can treat the given sum of double integrals

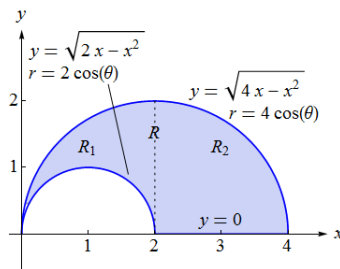
$$\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4x-x^2}} dy dx + \int_2^4 \int_0^{\sqrt{4x-x^2}} dy dx$$

as integration over a region R that has been split into two subregions R_1 and R_2 : the region R_1 is bounded between $y = \sqrt{2x-x^2}$ and $y = \sqrt{4x-x^2}$, where $0 \leq x \leq 2$; and the region R_2 is bounded between $y = 0$ and $y = \sqrt{4x-x^2}$, where $2 \leq x \leq 4$. See the figure below. The two curved boundaries are parts of circles, whose equations can be converted to polar coordinates as follows:

$$\begin{array}{ll} y = \sqrt{2x-x^2} & y = \sqrt{4x-x^2} \\ y^2 = 2x-x^2 & y^2 = 4x-x^2 \\ x^2 + y^2 = 2x & x^2 + y^2 = 4x \\ r^2 = 2r \cos \theta & r^2 = 4r \cos \theta \\ r = 2 \cos \theta & r = 4 \cos \theta \end{array}$$

In polar coordinates R is given by $2 \cos \theta \leq r \leq 4 \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\begin{aligned}
 \int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4x-x^2}} dy dx &+ \int_2^4 \int_0^{\sqrt{4x-x^2}} dy dx = \iint_{R_1} dA + \iint_{R_2} dA = \iint_R dA = \int_0^{\pi/2} \int_{2 \cos \theta}^{4 \cos \theta} r dr d\theta \\
 &= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{2 \cos \theta}^{4 \cos \theta} d\theta = \int_0^{\pi/2} \left(\frac{16 \cos^2 \theta}{2} - \frac{4 \cos^2 \theta}{2} \right) d\theta = \int_0^{\pi/2} 6 \cos^2 \theta d\theta \\
 &= \int_0^{\pi/2} 6 \cdot \frac{1}{2} (1 + \cos(2\theta)) d\theta = \int_0^{\pi/2} (3 + 3 \cos(2\theta)) d\theta = \left[3\theta + 3 \cdot \frac{\sin(2\theta)}{2} \right]_0^{\pi/2} \\
 &= \left[3\theta + \frac{3}{2} \sin(2\theta) \right]_0^{\pi/2} = \left(3 \cdot \frac{\pi}{2} + \frac{3}{2} \sin \pi \right) - \left(3 \cdot 0 + \frac{3}{2} \sin 0 \right) \\
 &= \left(\frac{3\pi}{2} + 0 \right) - (0 + 0) = \boxed{\frac{3\pi}{2}}.
 \end{aligned}$$



Applications and Extensions

43. We begin by graphing the ellipsoid $x^2 + y^2 + 4z^2 = 4$; see figure (a) below. Since it is symmetric about the xy -plane, to find the volume of the entire ellipsoid we can find the volume of its upper half lying above the xy -plane and multiply it by 2. We solve for z assuming $z \geq 0$ to find the equation of the upper half of the ellipsoid: $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}}$. To determine the region R that is the intersection of the ellipsoid with the xy -plane, we let $z = 0$ and find that $x^2 + y^2 = 4$, so R is a disk of radius 2 centered at the origin; see figure (b) below. In polar coordinates it is given by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Also, the equation of the surface of the ellipsoid simplifies as $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}} = \sqrt{1 - \frac{x^2 + y^2}{4}} = \sqrt{1 - \frac{r^2}{4}}$. Then the volume is

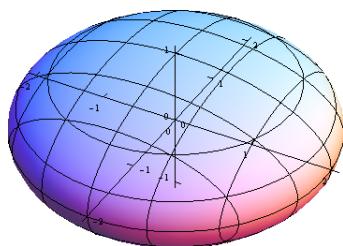
$$V = 2 \iint_R \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}} dA = 2 \int_0^{2\pi} \int_0^2 \sqrt{1 - \frac{r^2}{4}} \cdot r dr d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^2 \sqrt{1 - r^2/4} \cdot r dr$. In this integral we make a substitution $u = 1 - \frac{r^2}{4}$; then $du = -\frac{1}{2}rdr$, so $rdr = -2du$. We also change the limits of integration: if $r = 0$, then $u = 1 - \frac{0^2}{4} = 1$; if $r = 2$, then $u = 1 - \frac{2^2}{4} = 0$. So

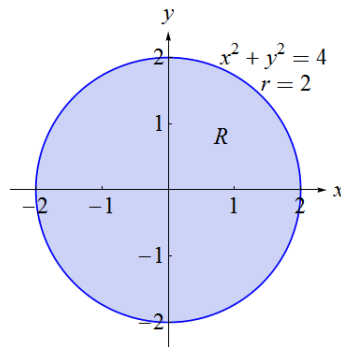
$$\begin{aligned}
 \int_0^2 \sqrt{1 - \frac{r^2}{4}} \cdot r dr &= \int_1^0 \sqrt{u} (-2du) = -2 \int_1^0 u^{1/2} du \\
 &= 2 \int_0^1 u^{1/2} du = 2 \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \frac{4}{3} \left[u^{3/2} \right]_0^1 = \frac{4}{3} (1 - 0) = \frac{4}{3}.
 \end{aligned}$$

Going back to the iterated integral,

$$V = 2 \int_0^{2\pi} \int_0^2 \sqrt{1 - \frac{r^2}{4}} \cdot r \, dr \, d\theta = 2 \int_0^{2\pi} \frac{4}{3} d\theta = \frac{8}{3} [\theta]_0^{2\pi} = \frac{8}{3} (2\pi - 0) = \boxed{\frac{16\pi}{3} \text{ cubic units}}.$$



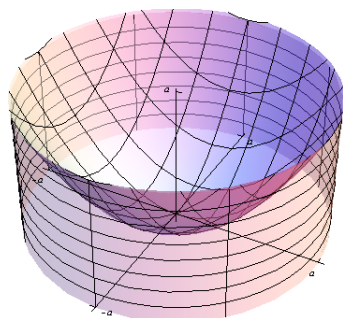
(a)



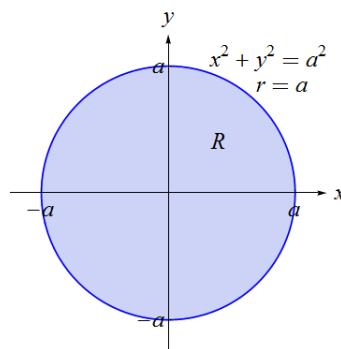
(b)

44. We begin by graphing the solid enclosed by the paraboloid $x^2 + y^2 = az$, the xy -plane, and the cylinder $x^2 + y^2 = a^2$; see figure (a) below. It is bounded from below by the plane $z = 0$, from above by the paraboloid $z = \frac{1}{a}(x^2 + y^2)$, and it lies over the region R in the xy -plane that is the base of the cylinder $x^2 + y^2 = a^2$. In polar coordinates, R is given by $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$; see figure (b) below. Also, the equation of the surface of the paraboloid simplifies as $z = \frac{1}{a}(x^2 + y^2) = \frac{r^2}{a}$. Then the volume is

$$\begin{aligned} V &= \iint_R \frac{1}{a}(x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^a \frac{r^2}{a} \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \frac{1}{a} r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{1}{a} \left[\frac{r^4}{4} \right]_0^a d\theta \\ &= \int_0^{2\pi} \frac{1}{a} \left(\frac{a^4}{4} - 0 \right) d\theta = \int_0^{2\pi} \frac{a^3}{4} d\theta = \frac{a^3}{4} [\theta]_0^{2\pi} = \frac{a^3}{4} (2\pi - 0) = \boxed{\frac{\pi a^3}{2} \text{ cubic units}}. \end{aligned}$$



(a)



(b)

45. The solid shown in the figure in Problem 45 is bounded from below by the plane $z = 0$, from above by the upper hemisphere $z = \sqrt{16 - x^2 - y^2}$, and it lies over the region R in the xy -plane bounded by the circle $r = 4 \sin \theta$. Observe that this solid (and the region R) is symmetric about the yz -plane, so we can find the volume of its half lying to the right of the yz -plane, that is over the half of R to the right of the y -axis, and multiply it by 2. The half of the region R to the right of the y -axis is given

by $0 \leq r \leq 4 \sin \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, in polar coordinates, the equation of the hemisphere simplifies as $z = \sqrt{16 - x^2 - y^2} = \sqrt{16 - (x^2 + y^2)} = \sqrt{16 - r^2}$. Then the volume is

$$V = \iint_R \sqrt{16 - x^2 - y^2} \, dA = 2 \int_0^{\pi/2} \int_0^{4 \sin \theta} \sqrt{16 - r^2} \cdot r \, dr \, d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^{4 \sin \theta} \sqrt{16 - r^2} \cdot r \, dr$. In this integral we make a substitution $u = 16 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = 16 - 0^2 = 16$; if $r = 4 \sin \theta$, then $u = 16 - (4 \sin \theta)^2 = 16 - 16 \sin^2 \theta = 16(1 - \sin^2 \theta) = 16 \cos^2 \theta$. So

$$\begin{aligned} \int_0^{4 \sin \theta} \sqrt{16 - r^2} \cdot r \, dr &= \int_{16}^{16 \cos^2 \theta} \sqrt{u} \left(-\frac{du}{2} \right) = -\frac{1}{2} \int_{16}^{16 \cos^2 \theta} u^{1/2} \, du \\ &= \frac{1}{2} \int_{16 \cos^2 \theta}^{16} u^{1/2} \, du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_{16 \cos^2 \theta}^{16} = \frac{1}{3} \left[(\sqrt{u})^3 \right]_{16 \cos^2 \theta}^{16} \\ &= \frac{1}{3} \left[(\sqrt{16})^3 - (\sqrt{16 \cos^2 \theta})^3 \right] = \frac{1}{3} (64 - 64 \cos^3 \theta) = \frac{64}{3} (1 - \cos^3 \theta), \end{aligned}$$

where we used the fact that $\sqrt{\cos^2 \theta} = \cos \theta$, because $0 \leq \theta \leq \frac{\pi}{2}$ and so $\cos \theta \geq 0$. Going back to the iterated integral,

$$\begin{aligned} V &= 2 \int_0^{\pi/2} \int_0^{4 \sin \theta} \sqrt{16 - r^2} \cdot r \, dr \, d\theta = 2 \int_0^{\pi/2} \frac{64}{3} (1 - \cos^3 \theta) \, d\theta \\ &= \frac{128}{3} \int_0^{\pi/2} (1 - \cos^3 \theta) \, d\theta = \frac{128}{3} \int_0^{\pi/2} d\theta - \frac{128}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta. \end{aligned}$$

The first integral can be evaluated directly:

$$\frac{128}{3} \int_0^{\pi/2} d\theta = \frac{128}{3} [\theta]_0^{\pi/2} = \frac{128}{3} \left(\frac{\pi}{2} - 0 \right) = \frac{64\pi}{3}.$$

To evaluate the second integral, we use a trigonometric identity to transform the integrand $\cos^3 \theta = \cos^2 \theta \cdot \cos \theta = (1 - \sin^2 \theta) \cos \theta$, and then make a substitution $v = \sin \theta$; then $dv = \cos \theta \, d\theta$. We also change the limits of integration: if $\theta = 0$, then $v = \sin 0 = 0$; if $\theta = \frac{\pi}{2}$, then $v = \sin \frac{\pi}{2} = 1$. So

$$\begin{aligned} \frac{128}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta &= \frac{128}{3} \int_0^{\pi/2} (1 - \sin^2 \theta) \cos \theta \, d\theta = \frac{128}{3} \int_0^1 (1 - v^2) \, dv \\ &= \frac{128}{3} \left[v - \frac{v^3}{3} \right]_0^1 = \frac{128}{3} \left[\left(1 - \frac{1}{3} \right) - (0 - 0) \right] = \frac{128}{3} \cdot \frac{2}{3} = \frac{256}{9}. \end{aligned}$$

Then the final answer is

$$V = \frac{128}{3} \int_0^{\pi/2} d\theta - \frac{128}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta = \boxed{\frac{64\pi}{3} - \frac{256}{9} \text{ cubic units}}.$$

46. We begin by graphing the solid cut from the ellipsoid $x^2 + y^2 + 4z^2 = 4$ by the cylinder $x^2 + y^2 = 1$; see figure (a) below. Since it is symmetric about the xy -plane, to find the volume of the entire solid we can find the volume of its upper half lying above the xy -plane and multiply it by 2. We solve for z assuming $z \geq 0$ to find the equation of the upper half of the ellipsoid: $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}}$.

The equation of the trace of the cylinder in the xy -plane is $x^2 + y^2 = 1$, so it encloses the region R that is the disk of radius 1 centered at the origin; see figure (b) below. In polar coordinates it is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Also, the equation of the surface of the ellipsoid simplifies as $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}} = \sqrt{1 - \frac{x^2 + y^2}{4}} = \sqrt{1 - \frac{r^2}{4}}$. Then the volume is

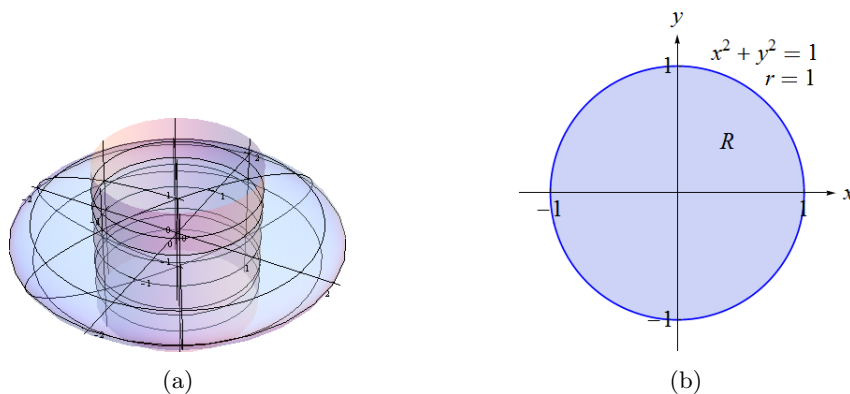
$$V = 2 \iint_R \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{4}} dA = 2 \int_0^{2\pi} \int_0^1 \sqrt{1 - \frac{r^2}{4}} \cdot r dr d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^1 \sqrt{1 - r^2/4} \cdot r dr$. In this integral we make a substitution $u = 1 - \frac{r^2}{4}$; then $du = -\frac{1}{2}rdr$, so $rdr = -2du$. We also change the limits of integration: if $r = 0$, then $u = 1 - \frac{0^2}{4} = 1$; if $r = 1$, then $u = 1 - \frac{1^2}{4} = \frac{3}{4}$. So

$$\begin{aligned} \int_0^1 \sqrt{1 - \frac{r^2}{4}} \cdot r dr &= \int_1^{3/4} \sqrt{u} (-2du) = -2 \int_1^{3/4} u^{1/2} du = 2 \int_{3/4}^1 u^{1/2} du = 2 \left[\frac{u^{3/2}}{3/2} \right]_{3/4}^1 \\ &= \frac{4}{3} \left[(\sqrt{u})^3 \right]_{3/4}^1 = \frac{4}{3} \left[(\sqrt{1})^3 - \left(\sqrt{\frac{3}{4}} \right)^3 \right] = \frac{4}{3} \left(1 - \frac{3\sqrt{3}}{8} \right) = \frac{4}{3} - \frac{\sqrt{3}}{2}. \end{aligned}$$

Going back to the iterated integral,

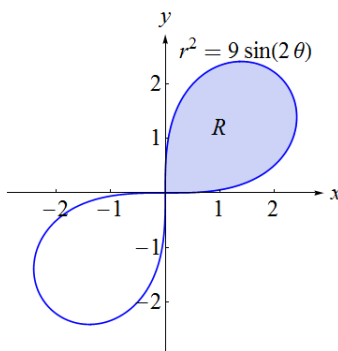
$$\begin{aligned} V &= 2 \int_0^{2\pi} \int_0^1 \sqrt{1 - \frac{r^2}{4}} \cdot r dr d\theta = 2 \int_0^{2\pi} \left(\frac{4}{3} - \frac{\sqrt{3}}{2} \right) d\theta \\ &= \left(\frac{8}{3} - \sqrt{3} \right) [\theta]_0^{2\pi} = \left(\frac{8}{3} - \sqrt{3} \right) (2\pi - 0) = \boxed{\frac{(16 - 6\sqrt{3})\pi}{3} \text{ cubic units}}. \end{aligned}$$



47. We begin by graphing the polar curve $r^2 = 9 \sin(2\theta)$, shown in the figure below. The graph consists of two similar loops. The loop lying in the first quadrant encloses the region R that can be defined in

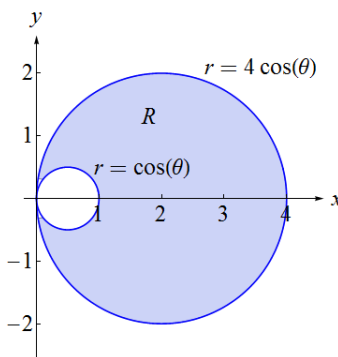
polar coordinates as $0 \leq r \leq 3\sqrt{\sin(2\theta)}$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the area of this region can be found as

$$\begin{aligned}
 A &= \iint_R dA = \int_0^{\pi/2} \int_0^{3\sqrt{\sin(2\theta)}} r \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{3\sqrt{\sin(2\theta)}} d\theta \\
 &= \int_0^{\pi/2} \left[\frac{9\sin(2\theta)}{2} - 0 \right] d\theta = \frac{9}{2} \int_0^{\pi/2} \sin(2\theta) \, d\theta = \frac{9}{2} \left[\frac{-\cos(2\theta)}{2} \right]_0^{\pi/2} \\
 &= -\frac{9}{4} [\cos(2\theta)]_0^{\pi/2} = -\frac{9}{4} (\cos \pi - \cos 0) = -\frac{9}{4} (-1 - 1) = \boxed{\frac{9}{2} \text{ square units}}.
 \end{aligned}$$



48. We begin by graphing the region R lying inside the circle $r = 4 \cos \theta$ but outside the circle $r = \cos \theta$; see the figure below. Notice that R is symmetric about the x -axis, so its area can be found by finding the area of its half lying in the upper half-plane and multiplying it by 2. This upper half of the region R can be defined in polar coordinates as $\cos \theta \leq r \leq 4 \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the area of R can be found as

$$\begin{aligned}
 A &= \iint_R dA = 2 \int_0^{\pi/2} \int_{\cos \theta}^{4 \cos \theta} r \, dr \, d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{\cos \theta}^{4 \cos \theta} d\theta = 2 \int_0^{\pi/2} \left[\frac{16 \cos^2 \theta}{2} - \frac{\cos^2 \theta}{2} \right] d\theta \\
 &= 2 \cdot \frac{15}{2} \int_0^{\pi/2} \cos^2 \theta \, d\theta = 15 \int_0^{\pi/2} \frac{1}{2} (1 + \cos(2\theta)) \, d\theta = \frac{15}{2} \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{\pi/2} \\
 &= \frac{15}{2} \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right] = \frac{15}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right] = \boxed{\frac{15\pi}{4} \text{ square units}}.
 \end{aligned}$$



49. The region R that lies in the first quadrant inside the limaçon $r = 3 + 2 \sin \theta$ but outside the circle $r = 4 \sin \theta$, as shown in the figure in Problem 49, can be defined in polar coordinates as $4 \sin \theta \leq r \leq$

$3 + 2 \sin \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the area of R can be found as

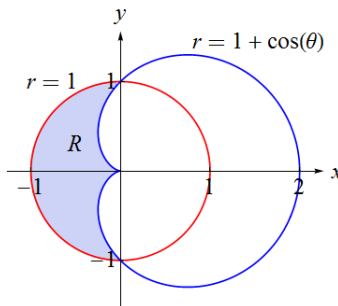
$$\begin{aligned}
 A &= \iint_R dA = \int_0^{\pi/2} \int_{4 \sin \theta}^{3+2 \sin \theta} r \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{4 \sin \theta}^{3+2 \sin \theta} d\theta \\
 &= \int_0^{\pi/2} \left[\frac{(3+2 \sin \theta)^2}{2} - \frac{(4 \sin \theta)^2}{2} \right] d\theta = \int_0^{\pi/2} \left(\frac{9+12 \sin \theta+4 \sin^2 \theta}{2} - \frac{16 \sin^2 \theta}{2} \right) d\theta \\
 &= \int_0^{\pi/2} \left(\frac{9}{2} + 6 \sin \theta - 6 \sin^2 \theta \right) d\theta = \int_0^{\pi/2} \left[\frac{9}{2} + 6 \sin \theta - 6 \cdot \frac{1}{2}(1 - \cos(2\theta)) \right] d\theta \\
 &= \int_0^{\pi/2} \left(\frac{9}{2} + 6 \sin \theta - 3 + 3 \cos(2\theta) \right) d\theta = \int_0^{\pi/2} \left(\frac{3}{2} + 6 \sin \theta + 3 \cos(2\theta) \right) d\theta \\
 &= \left[\frac{3}{2}\theta - 6 \cos \theta + \frac{3}{2} \sin(2\theta) \right]_0^{\pi/2} = \left(\frac{3}{2} \cdot \frac{\pi}{2} - 6 \cos \frac{\pi}{2} + \frac{3}{2} \sin \pi \right) - \left(\frac{3}{2} \cdot 0 - 6 \cos 0 + \frac{3}{2} \sin 0 \right) \\
 &= \left[\left(\frac{3\pi}{4} - 0 + 0 \right) - (0 - 6 + 0) \right] = \boxed{\frac{3\pi}{4} + 6 \text{ square units}}.
 \end{aligned}$$

50. We begin by graphing the region R lying inside the circle $r = 1$ but outside the cardioid $r = 1 + \cos \theta$; see the figure below. The graphs intersect at the points where

$$1 + \cos \theta = 1, \quad \text{so} \quad \cos \theta = 0, \quad \text{so} \quad \theta = \pm \frac{\pi}{2}.$$

Notice that R is symmetric about the x -axis, so its area can be found by finding the area of its half lying in the upper half-plane (that is, in the third quadrant) and multiplying it by 2. This upper half of the region R can be defined in polar coordinates as $1 + \cos \theta \leq r \leq 1$, $\frac{\pi}{2} \leq \theta \leq \pi$. Then the area of R can be found as

$$\begin{aligned}
 A &= \iint_R dA = 2 \int_{\pi/2}^{\pi} \int_{1+\cos \theta}^1 r \, dr \, d\theta = 2 \int_{\pi/2}^{\pi} \left[\frac{r^2}{2} \right]_{1+\cos \theta}^1 d\theta = \int_{\pi/2}^{\pi} [r^2]_{1+\cos \theta}^1 d\theta \\
 &= \int_{\pi/2}^{\pi} [1^2 - (1 + \cos \theta)^2] d\theta = \int_{\pi/2}^{\pi} [1 - (1 + 2 \cos \theta + \cos^2 \theta)] d\theta = \int_{\pi/2}^{\pi} (-2 \cos \theta - \cos^2 \theta) d\theta \\
 &= \int_{\pi/2}^{\pi} \left[-2 \cos \theta - \frac{1}{2}(1 + \cos(2\theta)) \right] d\theta = \int_{\pi/2}^{\pi} \left(-\frac{1}{2} - 2 \cos \theta - \frac{1}{2} \cos(2\theta) \right) d\theta \\
 &= \left[-\frac{1}{2}\theta - 2 \sin \theta - \frac{1}{4} \sin(2\theta) \right]_{\pi/2}^{\pi} = \left(-\frac{1}{2} \cdot \pi - 2 \sin \pi - \frac{1}{4} \sin(2\pi) \right) - \left(-\frac{1}{2} \cdot \frac{\pi}{2} - 2 \sin \frac{\pi}{2} - \frac{1}{4} \sin \pi \right) \\
 &= \left(-\frac{\pi}{2} - 0 - 0 \right) - \left(-\frac{\pi}{4} - 2 - 0 \right) = \boxed{2 - \frac{\pi}{4} \text{ square units}}.
 \end{aligned}$$

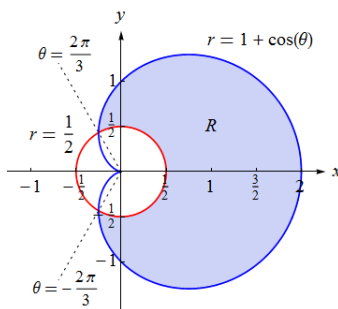


51. We begin by graphing the region R lying inside the cardioid $r = 1 + \cos \theta$ but outside the circle $r = \frac{1}{2}$; see the figure below. The graphs intersect at the points where

$$1 + \cos \theta = \frac{1}{2}, \quad \text{so} \quad \cos \theta = -\frac{1}{2}, \quad \text{so} \quad \theta = \pm \frac{2\pi}{3}.$$

Notice that R is symmetric about the x -axis, so its area can be found by finding the area of its half lying in the upper half-plane and multiplying it by 2. This upper half of the region R can be defined in polar coordinates as $\frac{1}{2} \leq r \leq 1 + \cos \theta$, $0 \leq \theta \leq \frac{2\pi}{3}$. Then the area of R can be found as

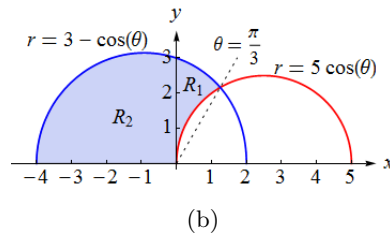
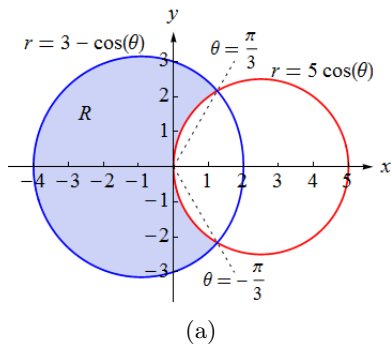
$$\begin{aligned} A &= \iint_R dA = 2 \int_0^{2\pi/3} \int_{1/2}^{1+\cos \theta} r \, dr \, d\theta = 2 \int_0^{2\pi/3} \left[\frac{r^2}{2} \right]_{1/2}^{1+\cos \theta} d\theta = \int_0^{2\pi/3} [r^2]_{1/2}^{1+\cos \theta} d\theta \\ &= \int_0^{2\pi/3} \left[(1 + \cos \theta)^2 - \left(\frac{1}{2} \right)^2 \right] d\theta = \int_0^{2\pi/3} \left[1 + 2 \cos \theta + \cos^2 \theta - \frac{1}{4} \right] d\theta \\ &= \int_0^{2\pi/3} \left(\frac{3}{4} + 2 \cos \theta + \cos^2 \theta \right) d\theta = \int_0^{2\pi/3} \left[\frac{3}{4} + 2 \cos \theta + \frac{1}{2}(1 + \cos(2\theta)) \right] d\theta \\ &= \int_0^{2\pi/3} \left(\frac{5}{4} + 2 \cos \theta + \frac{1}{2} \cos(2\theta) \right) d\theta = \left[\frac{5}{4}\theta + 2 \sin \theta + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi/3} \\ &= \left(\frac{5}{4} \cdot \frac{2\pi}{3} + 2 \sin \frac{2\pi}{3} + \frac{1}{4} \sin \frac{4\pi}{3} \right) - \left(\frac{5}{4} \cdot 0 + 2 \sin 0 + \frac{1}{4} \sin 0 \right) \\ &= \left[\frac{5\pi}{6} + 2 \cdot \frac{\sqrt{3}}{2} + \frac{1}{4} \cdot \left(-\frac{\sqrt{3}}{2} \right) \right] - (0 + 0 + 0) \\ &= \frac{5\pi}{6} + \sqrt{3} - \frac{\sqrt{3}}{8} = \boxed{\frac{5\pi}{6} + \frac{7\sqrt{3}}{8} \text{ square units}}. \end{aligned}$$



52. We begin by graphing the region R lying inside the limaçon $r = 3 - \cos \theta$ but outside the circle $r = 5 \cos \theta$; see figure (a) below. The graphs intersect at the points where

$$3 - \cos \theta = 5 \cos \theta, \quad \text{so} \quad 6 \cos \theta = 3, \quad \text{so} \quad \cos \theta = \frac{1}{2}, \quad \text{so} \quad \theta = \pm \frac{\pi}{3}.$$

Notice that R is symmetric about the x -axis, so its area can be found by finding the area of its half lying in the upper half-plane and multiplying it by 2. Moreover, it is convenient to split this upper half of the region R into two subregions: subregion R_1 lying in the first quadrant can be defined in polar coordinates as $5 \cos \theta \leq r \leq 3 - \cos \theta$, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$; and subregion R_2 lying in the second quadrant can be defined in polar coordinates as $0 \leq r \leq 3 - \cos \theta$, $\frac{\pi}{2} \leq \theta \leq \pi$; see figure (b) below.



The area of the subregion R_1 is

$$\begin{aligned}
 A_{R_1} &= \iint_{R_1} dA = \int_{\pi/3}^{\pi/2} \int_{5 \cos \theta}^{3 - \cos \theta} r \, dr \, d\theta = \int_{\pi/3}^{\pi/2} \left[\frac{r^2}{2} \right]_{5 \cos \theta}^{3 - \cos \theta} d\theta \\
 &= \int_{\pi/3}^{\pi/2} \left[\frac{(3 - \cos \theta)^2}{2} - \frac{(5 \cos \theta)^2}{2} \right] d\theta = \int_{\pi/3}^{\pi/2} \left(\frac{9 - 6 \cos \theta + \cos^2 \theta}{2} - \frac{25 \cos^2 \theta}{2} \right) d\theta \\
 &= \int_{\pi/3}^{\pi/2} \left(\frac{9}{2} - 3 \cos \theta - 12 \cos^2 \theta \right) d\theta = \int_{\pi/3}^{\pi/2} \left[\frac{9}{2} - 3 \cos \theta - 12 \cdot \frac{1}{2} (1 + \cos(2\theta)) \right] d\theta \\
 &= \int_{\pi/3}^{\pi/2} \left(-\frac{3}{2} - 3 \cos \theta - 6 \cos(2\theta) \right) d\theta = \left[-\frac{3}{2} \theta - 3 \sin \theta - 3 \sin(2\theta) \right]_{\pi/3}^{\pi/2} \\
 &= \left(-\frac{3}{2} \cdot \frac{\pi}{2} - 3 \sin \frac{\pi}{2} - 3 \sin \pi \right) - \left(-\frac{3}{2} \cdot \frac{\pi}{3} - 3 \sin \frac{\pi}{3} - 3 \sin \frac{2\pi}{3} \right) \\
 &= \left(-\frac{3\pi}{4} - 3 \cdot 1 - 3 \cdot 0 \right) - \left(-\frac{\pi}{2} - 3 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{\sqrt{3}}{2} \right) \\
 &= -\frac{3\pi}{4} - 3 - 0 + \frac{\pi}{2} + \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 3\sqrt{3} - 3 - \frac{\pi}{4}.
 \end{aligned}$$

The area of the subregion R_2 is

$$\begin{aligned}
 A_{R_2} &= \iint_{R_2} dA = \int_{\pi/2}^{\pi} \int_0^{3 - \cos \theta} r \, dr \, d\theta = \int_{\pi/2}^{\pi} \left[\frac{r^2}{2} \right]_0^{3 - \cos \theta} d\theta \\
 &= \int_{\pi/2}^{\pi} \left[\frac{(3 - \cos \theta)^2}{2} - 0 \right] d\theta = \int_{\pi/2}^{\pi} \frac{9 - 6 \cos \theta + \cos^2 \theta}{2} d\theta \\
 &= \int_{\pi/2}^{\pi} \left(\frac{9}{2} - 3 \cos \theta + \frac{1}{2} \cos^2 \theta \right) d\theta = \int_{\pi/2}^{\pi} \left[\frac{9}{2} - 3 \cos \theta + \frac{1}{2} \cdot \frac{1}{2} (1 + \cos(2\theta)) \right] d\theta \\
 &= \int_{\pi/2}^{\pi} \left(\frac{19}{4} - 3 \cos \theta + \frac{1}{4} \cos(2\theta) \right) d\theta = \left[\frac{19}{4} \theta - 3 \sin \theta + \frac{1}{8} \sin(2\theta) \right]_{\pi/2}^{\pi} \\
 &= \left(\frac{19}{4} \cdot \pi - 3 \sin \pi + \frac{1}{8} \sin(2\pi) \right) - \left(\frac{19}{4} \cdot \frac{\pi}{2} - 3 \sin \frac{\pi}{2} + \frac{1}{8} \sin \pi \right) \\
 &= \left(\frac{19\pi}{4} - 3 \cdot 0 + \frac{1}{8} \cdot 0 \right) - \left(\frac{19\pi}{8} - 3 \cdot 1 + \frac{1}{8} \cdot 0 \right) = \frac{19\pi}{8} + 3.
 \end{aligned}$$

Finally, the area of the entire region R is

$$\begin{aligned} A_R &= 2(A_{R_1} + A_{R_2}) = 2 \left[\left(3\sqrt{3} - 3 - \frac{\pi}{4} \right) + \left(\frac{19\pi}{8} + 3 \right) \right] \\ &= 2 \left(3\sqrt{3} + \frac{17\pi}{8} \right) = \boxed{6\sqrt{3} + \frac{17\pi}{4} \text{ square units}}. \end{aligned}$$

53. The region R associated with the integral $\int_0^{1/\sqrt{2}} \int_0^{\sin^{-1} r} r \, d\theta \, dr$ is enclosed between $\theta = 0$ (the positive x -axis) and $\theta = \sin^{-1} r$, where $0 \leq r \leq \frac{1}{\sqrt{2}}$. In other words, for each value of the polar radius r , $0 \leq r \leq \frac{1}{\sqrt{2}}$, the polar angle θ ranges from $\theta = 0$ to $\theta = \sin^{-1} r$. This region R is shown in the figure below. Notice that R is a y -simple region. To obtain equations of the bounding curves, we convert them from polar to cartesian coordinates:

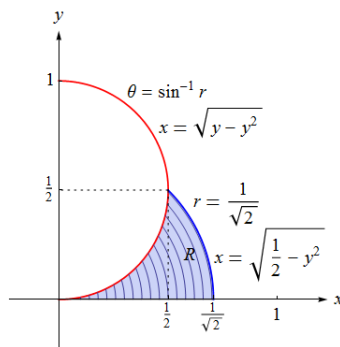
$$\begin{aligned} \theta &= \sin^{-1} r & r &= \frac{1}{\sqrt{2}} \\ r &= \sin \theta & r^2 &= \frac{1}{2} \\ r^2 &= r \sin \theta & x^2 + y^2 &= \frac{1}{2} \\ x^2 + y^2 &= y & x^2 &= \frac{1}{2} - y^2 \\ x^2 &= y - y^2 & x &= \sqrt{\frac{1}{2} - y^2} \\ x &= \sqrt{y - y^2} \end{aligned}$$

The point of intersection of the two graphs can be found by setting their equations equal to each other:

$$\sqrt{y - y^2} = \sqrt{\frac{1}{2} - y^2}, \quad \text{so } y - y^2 = \frac{1}{2} - y^2, \quad \text{so } y = \frac{1}{2}, \quad \text{and } x = \sqrt{y - y^2} = \sqrt{\frac{1}{2} - \frac{1}{4}} = \frac{1}{2}.$$

Finally, since $r \, d\theta \, dr = dA$, the given integral can be rewritten as

$$\int_0^{1/\sqrt{2}} \int_0^{\sin^{-1} r} r \, d\theta \, dr = \iint_R dA = \boxed{\int_0^{1/2} \int_{\sqrt{y-y^2}}^{\sqrt{\frac{1}{2}-y^2}} dx \, dy}.$$



54. The region R associated with the integral $\int_0^1 \int_{\cos^{-1} r}^{\pi/2} r^2 \sin \theta \, d\theta \, dr$ is enclosed between $\theta = \cos^{-1} r$ and $\theta = \frac{\pi}{2}$ (the positive y -axis), where $0 \leq r \leq 1$. In other words, for each value of the polar radius r , $0 \leq r \leq 1$, the polar angle θ ranges from $\theta = \cos^{-1} r$ to $\theta = \frac{\pi}{2}$. This region R is shown in the figure

below. Notice that R is an x -simple region. To obtain equations of the bounding curves, we convert them from polar to cartesian coordinates:

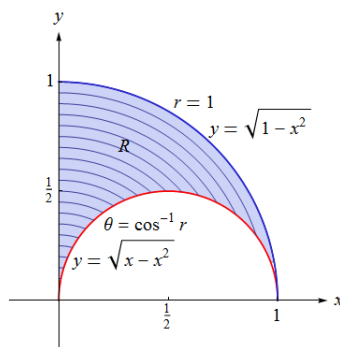
$$\begin{array}{ll} \theta = \cos^{-1} r & r = 1 \\ r = \cos \theta & r^2 = 1 \\ r^2 = r \cos \theta & x^2 + y^2 = 1 \\ x^2 + y^2 = x & y^2 = 1 - x^2 \\ y^2 = x - x^2 & y = \sqrt{1 - x^2} \\ y = \sqrt{x - x^2} & \end{array}$$

The point of intersection of the two graphs can be found by setting their equations equal to each other:

$$\sqrt{x - x^2} = \sqrt{1 - x^2}, \quad \text{so } x - x^2 = 1 - x^2, \quad \text{so } x = 1, \quad \text{and } y = \sqrt{x - x^2} = \sqrt{1 - 1} = 0.$$

Finally, since $r^2 \sin \theta \, d\theta \, dr = r \sin \theta \cdot r \, d\theta \, dr = y \, dA$, the given integral can be rewritten as

$$\int_0^1 \int_{\cos^{-1} r}^{\pi/2} r^2 \sin \theta \, d\theta \, dr = \iint_R y \, dA = \boxed{\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} y \, dy \, dx}.$$



55. The volume of the entire hemisphere tank is: $V_{\text{tank}} = \frac{1}{2} V_{\text{sphere}} = \frac{1}{2} \cdot \frac{4}{3} \pi r^3 = \frac{1}{2} \cdot \frac{4}{3} \pi \cdot 1^3 = \frac{2\pi}{3} \text{ m}^3$. Then we can find the volume of the liquid in the tank, which is filled up to a depth of a meters, $0 \leq a \leq 1$, as shown in the figure in Problem 55, if we find the volume of the empty cap above the liquid level and subtract it from the volume of the tank.

The empty cap above the water level can be represented as the solid enclosed between $z = f_1(x, y) = a$ and $z = f_2(x, y) = \sqrt{1 - x^2 - y^2}$ over the region R in the xy -plane, that is bounded by the trace obtained as the intersection of the hemisphere with the plane $z = a$; the equation of this trace is $x^2 + y^2 + a^2 = 1$ or $x^2 + y^2 = 1 - a^2$. So we can find the volume of the empty cap as the double integral

$$V_{\text{cap}} = \iint_R [f_2(x, y) - f_1(x, y)] \, dA = \iint_R (\sqrt{1 - x^2 - y^2} - a) \, dA.$$

In polar coordinates, the region R can be described as $0 \leq r \leq \sqrt{1 - a^2}$, $0 \leq \theta \leq 2\pi$. Also, in the integrand we can simplify $\sqrt{1 - x^2 - y^2} = \sqrt{1 - (x^2 + y^2)} = \sqrt{1 - r^2}$. So this double integral can be converted to polar coordinates as

$$V_{\text{cap}} = \iint_R (\sqrt{1 - x^2 - y^2} - a) \, dA = \int_0^{2\pi} \int_0^{\sqrt{1-a^2}} (\sqrt{1 - r^2} - a) \cdot r \, dr \, d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^{\sqrt{1-a^2}} (\sqrt{1-r^2} - a) \cdot r \, dr$. In this integral we make a substitution $u = 1 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = 1 - 0^2 = 1$; if $r = \sqrt{1 - a^2}$, then $u = 1 - (\sqrt{1 - a^2})^2 = 1 - (1 - a^2) = a^2$. So

$$\begin{aligned} \int_0^{\sqrt{1-a^2}} (\sqrt{1-r^2} - a) \cdot r \, dr &= \int_1^{a^2} (\sqrt{u} - a) \cdot \frac{du}{-2} = -\frac{1}{2} \int_1^{a^2} (\sqrt{u} - a) \, du \\ &= \frac{1}{2} \int_{a^2}^1 (u^{1/2} - a) \, du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} - au \right]_{a^2}^1 = \left[\frac{1}{3} (\sqrt{u})^3 - \frac{1}{2} au \right]_{a^2}^1 \\ &= \left[\frac{1}{3} (\sqrt{1})^3 - \frac{1}{2} a \cdot 1 \right] - \left[\frac{1}{3} (\sqrt{a^2})^3 - \frac{1}{2} a \cdot a^2 \right] \\ &= \left(\frac{1}{3} - \frac{a}{2} \right) - \left(\frac{a^3}{3} - \frac{a^3}{2} \right) = \frac{1}{3} - \frac{a}{2} + \frac{a^3}{6} = \frac{2 - 3a + a^3}{6}. \end{aligned}$$

Going back to the iterated integral, the volume of the empty cap is

$$\begin{aligned} V_{\text{cap}} &= \int_0^{2\pi} \int_0^{\sqrt{1-a^2}} (\sqrt{1-r^2} - a) \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{2 - 3a + a^3}{6} \, d\theta \\ &= \frac{2 - 3a + a^3}{6} \cdot [\theta]_0^{2\pi} = \frac{2 - 3a + a^3}{6} \cdot (2\pi - 0) = \frac{(2 - 3a + a^3)\pi}{3} \, \text{m}^3, \end{aligned}$$

and the volume of the liquid is

$$V_{\text{liquid}} = V_{\text{tank}} - V_{\text{cap}} = \frac{2\pi}{3} - \frac{(2 - 3a + a^3)\pi}{3} = \frac{(3a - a^3)\pi}{3} = \boxed{\pi a \left(1 - \frac{a^2}{3} \right) \, \text{m}^3}.$$

56. The volume of the liquid in the tank, which is filled up to a depth of a meters, $-1 \leq a \leq 0$, as shown in the figure in Problem 56, can be represented as the volume of the solid enclosed between $z = f_1(x, y) = -\sqrt{1 - x^2 - y^2}$ and $z = f_2(x, y) = a$ over the region R in the xy -plane, that is bounded by the trace obtained as the intersection of the hemisphere with the plane $z = a$; the equation of this trace is $x^2 + y^2 + a^2 = 1$ or $x^2 + y^2 = 1 - a^2$. So we can find this volume as the double integral

$$V = \iint_R [f_2(x, y) - f_1(x, y)] \, dA = \iint_R (a + \sqrt{1 - x^2 - y^2}) \, dA.$$

In polar coordinates, the region R can be described as $0 \leq r \leq \sqrt{1 - a^2}$, $0 \leq \theta \leq 2\pi$. Also, in the integrand we can simplify $\sqrt{1 - x^2 - y^2} = \sqrt{1 - (x^2 + y^2)} = \sqrt{1 - r^2}$. So this double integral can be converted to polar coordinates as

$$V = \iint_R (a + \sqrt{1 - x^2 - y^2}) \, dA = \int_0^{2\pi} \int_0^{\sqrt{1-a^2}} (a + \sqrt{1-r^2}) \cdot r \, dr \, d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^{\sqrt{1-a^2}} (a + \sqrt{1-r^2}) \cdot r \, dr$. In this integral we make a substitution $u = 1 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = 1 - 0^2 = 1$; if $r = \sqrt{1 - a^2}$, then $u = 1 - (\sqrt{1 - a^2})^2 = 1 - (1 - a^2) = a^2$.

So

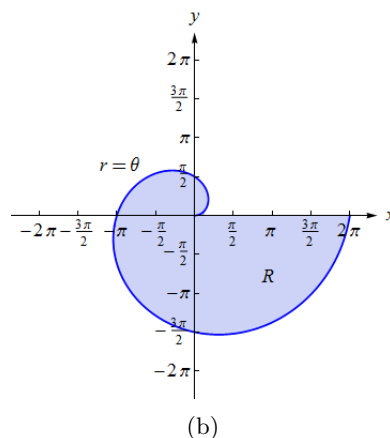
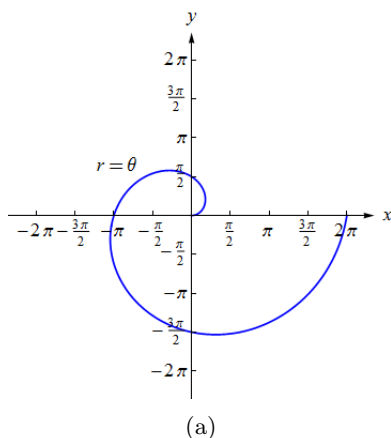
$$\begin{aligned}
 \int_0^{\sqrt{1-a^2}} (a + \sqrt{1-r^2}) \cdot r \, dr &= \int_1^{a^2} (a + \sqrt{u}) \cdot \frac{du}{-2} = -\frac{1}{2} \int_1^{a^2} (a + \sqrt{u}) \, du \\
 &= \frac{1}{2} \int_{a^2}^1 (a + u^{1/2}) \, du = \frac{1}{2} \left[au + \frac{u^{3/2}}{3/2} \right]_{a^2}^1 = \left[\frac{1}{2} au + \frac{1}{3} (\sqrt{u})^3 \right]_{a^2}^1 \\
 &= \left[\frac{1}{2} a \cdot 1 + \frac{1}{3} (\sqrt{1})^3 \right] - \left[\frac{1}{2} a \cdot a^2 + \frac{1}{3} (\sqrt{a^2})^3 \right] \\
 &= \left(\frac{a}{2} + \frac{1}{3} \right) - \left(\frac{a^3}{2} + \frac{a^3}{3} \right) = \frac{a}{2} + \frac{1}{3} - \frac{a^3}{6} = \frac{2 + 3a - a^3}{6}.
 \end{aligned}$$

(Note that $\sqrt{a^2} = |a| = -a$, because $a \leq 0$.)

Going back to the iterated integral, the volume of the liquid is

$$\begin{aligned}
 V &= \int_0^{2\pi} \int_0^{\sqrt{1-a^2}} (a + \sqrt{1-r^2}) \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{2 + 3a - a^3}{6} \, d\theta \\
 &= \frac{2 + 3a - a^3}{6} \cdot [\theta]_0^{2\pi} = \frac{2 + 3a - a^3}{6} \cdot (2\pi - 0) = \boxed{\frac{(2 + 3a - a^3)\pi}{3} \text{ m}^3}.
 \end{aligned}$$

57. (a) Using the computer algebra system *Mathematica*, we can generate the graph of $r = \theta$, $0 \leq \theta \leq 2\pi$, shown in figure (a) below.



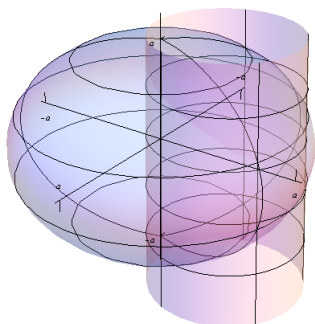
- (b) The region R bounded by the graph of $r = \theta$, $0 \leq \theta \leq 2\pi$, and the positive x -axis is shown in figure (b) above. In polar coordinates it can be described as $0 \leq r \leq \theta$, $0 \leq \theta \leq 2\pi$. So its area is

$$\begin{aligned}
 A &= \iint_R dA = \int_0^{2\pi} \int_0^\theta r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^\theta d\theta = \int_0^{2\pi} \left(\frac{\theta^2}{2} - \frac{0^2}{2} \right) d\theta \\
 &= \int_0^{2\pi} \frac{\theta^2}{2} d\theta = \left[\frac{\theta^3}{6} \right]_0^{2\pi} = \frac{(2\pi)^3}{6} - \frac{0^3}{6} = \frac{8\pi^3}{6} = \boxed{\frac{4\pi^3}{3} \text{ square units}}.
 \end{aligned}$$

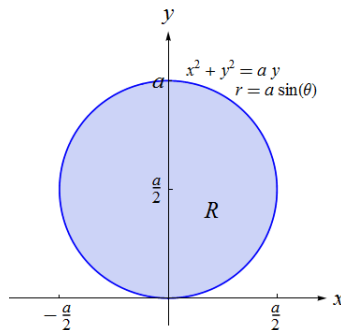
58. The solid S cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ay$, $a > 0$, is shown in figure (a) below. Notice that it is symmetric about the xy -plane, so we can find its volume if we find the volume of its upper half and multiply that by 2. This upper half is bounded from below by the plane $z = 0$, from above by the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$, and it lies over the region R in the xy -plane bounded by the circle $x^2 + y^2 = ay$ or $r = a \sin \theta$; see figure (b) below.

Moreover, this region R and the part of the solid S above it are symmetric about the yz -plane, so we can find the volume of the quarter of the solid S lying over R to the right of the yz -plane and multiply that by 4. In polar coordinates, the equation of the upper hemisphere simplifies as $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$. The part of the region R to the right of the y -axis is given by $0 \leq r \leq a \sin \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the volume is

$$V = 2 \iint_R \sqrt{a^2 - x^2 - y^2} dA = 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r dr d\theta.$$



(a)



(b)

In this iterated integral, first we evaluate the “inside” integral $\int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r dr$. In this integral we make a substitution $u = a^2 - r^2$; then $du = -2r dr$, so $r dr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = a^2 - 0^2 = a^2$; if $r = a \sin \theta$, then $u = a^2 - (a \sin \theta)^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$. So

$$\begin{aligned} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r dr &= \int_{a^2}^{a^2 \cos^2 \theta} \sqrt{u} \left(-\frac{du}{2} \right) = -\frac{1}{2} \int_{a^2}^{a^2 \cos^2 \theta} u^{1/2} du \\ &= \frac{1}{2} \int_{a^2 \cos^2 \theta}^{a^2} u^{1/2} du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_{a^2 \cos^2 \theta}^{a^2} = \frac{1}{3} \left[(\sqrt{u})^3 \right]_{a^2 \cos^2 \theta}^{a^2} \\ &= \frac{1}{3} \left[(\sqrt{a^2})^3 - (\sqrt{a^2 \cos^2 \theta})^3 \right] = \frac{1}{3} (a^3 - a^3 \cos^3 \theta) = \frac{a^3}{3} (1 - \cos^3 \theta), \end{aligned}$$

where we used the facts that $\sqrt{a^2} = a$ (because $a > 0$) and $\sqrt{\cos^2 \theta} = \cos \theta$ (because $0 \leq \theta \leq \frac{\pi}{2}$ and so $\cos \theta \geq 0$). Going back to the iterated integral,

$$\begin{aligned} V &= 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \sqrt{a^2 - r^2} \cdot r dr d\theta = 4 \int_0^{\pi/2} \frac{a^3}{3} (1 - \cos^3 \theta) d\theta \\ &= \frac{4a^3}{3} \int_0^{\pi/2} (1 - \cos^3 \theta) d\theta = \frac{4a^3}{3} \int_0^{\pi/2} d\theta - \frac{4a^3}{3} \int_0^{\pi/2} \cos^3 \theta d\theta. \end{aligned}$$

The first integral can be evaluated directly:

$$\frac{4a^3}{3} \int_0^{\pi/2} d\theta = \frac{4a^3}{3} [\theta]_0^{\pi/2} = \frac{4a^3}{3} \left(\frac{\pi}{2} - 0 \right) = \frac{2a^3\pi}{3}.$$

To evaluate the second integral, we use a trigonometric identity to transform the integrand $\cos^3 \theta = \cos^2 \theta \cdot \cos \theta = (1 - \sin^2 \theta) \cos \theta$, and then make a substitution $v = \sin \theta$; then $dv = \cos \theta d\theta$. We also

change the limits of integration: if $\theta = 0$, then $v = \sin 0 = 0$; if $\theta = \frac{\pi}{2}$, then $v = \sin \frac{\pi}{2} = 1$. So

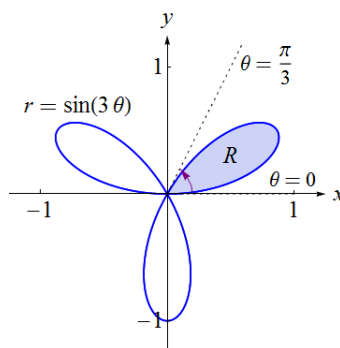
$$\begin{aligned} \frac{4a^3}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta &= \frac{4a^3}{3} \int_0^{\pi/2} (1 - \sin^2 \theta) \cos \theta \, d\theta = \frac{4a^3}{3} \int_0^1 (1 - v^2) \, dv \\ &= \frac{4a^3}{3} \left[v - \frac{v^3}{3} \right]_0^1 = \frac{4a^3}{3} \left[\left(1 - \frac{1}{3}\right) - (0 - 0) \right] = \frac{4a^3}{3} \cdot \frac{2}{3} = \frac{8a^3}{9}. \end{aligned}$$

Then the final answer is

$$V = \frac{4a^3}{3} \int_0^{\pi/2} d\theta - \frac{4a^3}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta = \frac{2a^3\pi}{3} - \frac{8a^3}{9} = \boxed{\frac{2a^3(3\pi - 4)}{9} \text{ cubic units}}.$$

59. We begin by graphing the rose curve $r = \sin(3\theta)$; see the figure below. Its leaf lying in the first quadrant, that is the region R shaded in the figure, is defined in polar coordinates by $0 \leq r \leq \sin(3\theta)$, $0 \leq \theta \leq \frac{\pi}{3}$. Then the area of this leaf can be found as

$$\begin{aligned} A &= \iint_R dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r \, dr \, d\theta = \int_0^{\pi/3} \left[\frac{r^2}{2} \right]_0^{\sin(3\theta)} d\theta = \int_0^{\pi/3} \left(\frac{\sin^2(3\theta)}{2} - 0 \right) d\theta \\ &= \int_0^{\pi/3} \frac{1}{2} \sin^2(3\theta) \, d\theta = \int_0^{\pi/3} \frac{1}{2} \cdot \frac{1}{2} (1 - \cos(6\theta)) \, d\theta = \frac{1}{4} \left[\theta - \frac{1}{6} \sin(6\theta) \right]_0^{\pi/3} \\ &= \frac{1}{4} \left(\frac{\pi}{3} - \frac{1}{6} \sin(2\pi) \right) - \frac{1}{4} \left(0 - \frac{1}{6} \sin 0 \right) = \frac{1}{4} \left(\frac{\pi}{3} - 0 \right) - \frac{1}{4} (0 - 0) = \boxed{\frac{\pi}{12} \text{ square units}}. \end{aligned}$$



Challenge Problems

60. (a) In polar coordinates, $x = r \cos \theta$ and $\sqrt{x^2 + y^2} = r$, so

$$f(x, y) = \frac{1}{x^2 \sqrt{x^2 + y^2}} = \frac{1}{(r \cos \theta)^2 \cdot r} = \frac{1}{r^3 \cos^2 \theta} = \boxed{\frac{\sec^2 \theta}{r^3}}.$$

- (b) The region R defined by $1 \leq x \leq 2$, $0 \leq y \leq 1$ is a square region shown in figure (a) below. To represent it in polar coordinates, we draw three rays from the origin: along the positive x -axis — its equation is $\theta = 0$; through the corner of the square at $(2, 1)$ — since $\tan \theta = \frac{y}{x} = \frac{1}{2}$, its equation is $\theta = \tan^{-1} \left(\frac{1}{2} \right)$; and through the corner of the square at $(1, 1)$ — since $\tan \theta = \frac{y}{x} = \frac{1}{1} = 1$, its equation is $\theta = \tan^{-1} 1 = \frac{\pi}{4}$. These rays cut the region R into two subregions R_1 and R_2 ,

as shown in figure (b) below. The equations of the sides of the square can be converted to polar coordinates as follows:

$$\begin{array}{lll}
 x = 1 & x = 2 & y = 1 \\
 r \cos \theta = 1 & r \cos \theta = 2 & r \sin \theta = 1 \\
 r = \frac{1}{\cos \theta} & r = \frac{2}{\cos \theta} & r = \frac{1}{\sin \theta} \\
 r = \sec \theta & r = 2 \sec \theta & r = \csc \theta
 \end{array}$$

Then in polar coordinates the subregion R_1 can be described as $\sec \theta \leq r \leq 2 \sec \theta$, $0 \leq \theta \leq \tan^{-1} \left(\frac{1}{2} \right)$, and the subregion R_2 as $\sec \theta \leq r \leq \csc \theta$, $\tan^{-1} \left(\frac{1}{2} \right) \leq \theta \leq \frac{\pi}{4}$. Then, using the result of part (a), the given double integral can be converted to polar coordinates as follows:

$$\begin{aligned}
 \int_1^2 \int_0^1 \frac{1}{x^2 \sqrt{x^2 + y^2}} dy dx &= \iint_R \frac{1}{x^2 \sqrt{x^2 + y^2}} dA \\
 &= \iint_{R_1} \frac{1}{x^2 \sqrt{x^2 + y^2}} dA + \iint_{R_2} \frac{1}{x^2 \sqrt{x^2 + y^2}} dA \\
 &= \int_0^{\tan^{-1}(1/2)} \int_{\sec \theta}^{2 \sec \theta} \frac{\sec^2 \theta}{r^3} r dr d\theta + \int_{\tan^{-1}(1/2)}^{\pi/4} \int_{\sec \theta}^{\csc \theta} \frac{\sec^2 \theta}{r^3} r dr d\theta \\
 &= \int_0^{\tan^{-1}(1/2)} \int_{\sec \theta}^{2 \sec \theta} \frac{\sec^2 \theta}{r^2} dr d\theta + \int_{\tan^{-1}(1/2)}^{\pi/4} \int_{\sec \theta}^{\csc \theta} \frac{\sec^2 \theta}{r^2} dr d\theta \\
 &= \int_0^{\alpha} \int_{\sec \theta}^{2 \sec \theta} \frac{\sec^2 \theta}{r^2} dr d\theta + \int_{\alpha}^{\pi/4} \int_{\sec \theta}^{\csc \theta} \frac{\sec^2 \theta}{r^2} dr d\theta,
 \end{aligned}$$

where we denoted $\alpha = \tan^{-1} \left(\frac{1}{2} \right)$ for brevity.

The first integral evaluates as follows:

$$\begin{aligned}
 \int_0^{\alpha} \int_{\sec \theta}^{2 \sec \theta} \frac{\sec^2 \theta}{r^2} dr d\theta &= \int_0^{\alpha} \sec^2 \theta \left[\int_{\sec \theta}^{2 \sec \theta} r^{-2} dr \right] d\theta \\
 &= \int_0^{\alpha} \sec^2 \theta \left[\frac{r^{-1}}{-1} \right]_{\sec \theta}^{2 \sec \theta} d\theta = \int_0^{\alpha} \sec^2 \theta \left[-\frac{1}{r} \right]_{\sec \theta}^{2 \sec \theta} d\theta \\
 &= \int_0^{\alpha} \sec^2 \theta \left(-\frac{1}{2 \sec \theta} + \frac{1}{\sec \theta} \right) d\theta = \int_0^{\alpha} \sec^2 \theta \cdot \frac{1}{2 \sec \theta} d\theta \\
 &= \int_0^{\alpha} \frac{1}{2} \sec \theta d\theta = \frac{1}{2} [\ln |\sec \theta + \tan \theta|]_0^{\alpha} \\
 &= \frac{1}{2} \ln |\sec \alpha + \tan \alpha| - \frac{1}{2} \ln |\sec 0 + \tan 0| \\
 &= \frac{1}{2} \ln \left| \frac{\sqrt{5}}{2} + \frac{1}{2} \right| - \frac{1}{2} \ln |1 + 0| = \frac{1}{2} \ln \frac{\sqrt{5} + 1}{2} - \frac{1}{2} \cdot 0 = \frac{1}{2} \ln \frac{\sqrt{5} + 1}{2},
 \end{aligned}$$

where in the last line we substituted back $\alpha = \tan^{-1} \left(\frac{1}{2} \right)$, so $\tan \alpha = \frac{1}{2}$ and $\sec \alpha = \sqrt{1 + \tan^2 \alpha} =$

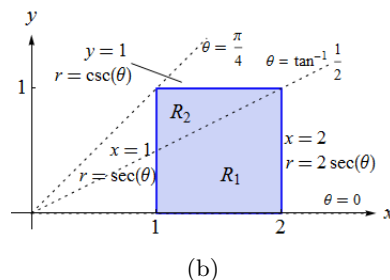
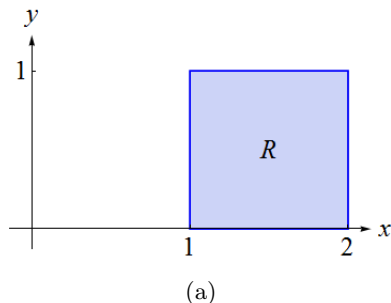
$$\sqrt{1 + \left(\frac{1}{2} \right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \text{ (sec } \alpha \text{ is positive because } \alpha \text{ lies in the first quadrant).}$$

The second integral evaluates as follows:

$$\begin{aligned}
 \int_{\alpha}^{\pi/4} \int_{\sec \theta}^{\csc \theta} \frac{\sec^2 \theta}{r^2} dr d\theta &= \int_{\alpha}^{\pi/4} \sec^2 \theta \left[\int_{\sec \theta}^{\csc \theta} r^{-2} dr \right] d\theta \\
 &= \int_{\alpha}^{\pi/4} \sec^2 \theta \left[\frac{r^{-1}}{-1} \right]_{\sec \theta}^{\csc \theta} d\theta = \int_{\alpha}^{\pi/4} \sec^2 \theta \left[-\frac{1}{r} \right]_{\sec \theta}^{\csc \theta} d\theta \\
 &= \int_{\alpha}^{\pi/4} \sec^2 \theta \left(-\frac{1}{\csc \theta} + \frac{1}{\sec \theta} \right) d\theta = \int_{\alpha}^{\pi/4} \left(-\frac{\sec^2 \theta}{\csc \theta} + \sec \theta \right) d\theta \\
 &= \int_{\alpha}^{\pi/4} (\sec \theta - \tan \theta \sec \theta) d\theta = [\ln |\sec \theta + \tan \theta| - \sec \theta]_{\alpha}^{\pi/4} \\
 &= \left(\ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \sec \frac{\pi}{4} \right) - (\ln |\sec \alpha + \tan \alpha| - \sec \alpha) \\
 &= \left(\ln |\sqrt{2} + 1| - \sqrt{2} \right) - \left(\ln \left| \frac{\sqrt{5}}{2} + \frac{1}{2} \right| - \frac{\sqrt{5}}{2} \right) \\
 &= \ln(\sqrt{2} + 1) - \sqrt{2} - \ln \frac{\sqrt{5} + 1}{2} + \frac{\sqrt{5}}{2}.
 \end{aligned}$$

Adding the values of the two integrals, we obtain the final answer:

$$\begin{aligned}
 \int_1^2 \int_0^1 \frac{1}{x^2 \sqrt{x^2 + y^2}} dy dx &= \int_0^{\alpha} \int_{\sec \theta}^{2 \sec \theta} \frac{\sec^2 \theta}{r^2} dr d\theta + \int_{\alpha}^{\pi/4} \int_{\sec \theta}^{\csc \theta} \frac{\sec^2 \theta}{r^2} dr d\theta \\
 &= \left(\frac{1}{2} \ln \frac{\sqrt{5} + 1}{2} \right) + \left(\ln(\sqrt{2} + 1) - \sqrt{2} - \ln \frac{\sqrt{5} + 1}{2} + \frac{\sqrt{5}}{2} \right) \\
 &= \boxed{\frac{\sqrt{5}}{2} - \sqrt{2} - \frac{1}{2} \ln \frac{\sqrt{5} + 1}{2} + \ln(\sqrt{2} + 1)}.
 \end{aligned}$$



61. (a) Let $I_a = \int_0^a e^{-x^2} dx$. Since a definite integral is independent of the symbol used as the variable of integration, we can also write that $I_a = \int_0^a e^{-y^2} dy$. Then

$$\begin{aligned}
 I_a^2 &= \int_0^a e^{-x^2} dx \cdot \int_0^a e^{-y^2} dy = I_a \int_0^a e^{-y^2} dy = \int_0^a I_a e^{-y^2} dy = \int_0^a \left[\int_0^a e^{-x^2} dx \right] e^{-y^2} dy \\
 &= \int_0^a \left[\int_0^a e^{-x^2} e^{-y^2} dx \right] dy = \int_0^a \int_0^a e^{-x^2} e^{-y^2} dx dy = \int_0^a \int_0^a e^{-(x^2+y^2)} dx dy,
 \end{aligned}$$

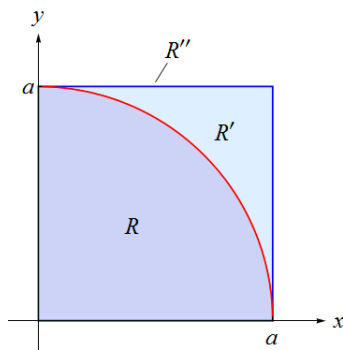
where we used Fubini's Theorem for rectangular regions, the fact that I_a is a constant (it is a number), and that e^{-y^2} is constant with respect to x .

- (b) Let $J_a = \iint_R e^{-(x^2+y^2)} dA$, where R is the quarter circle $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$. If R'' is the square $0 \leq x \leq a$, $0 \leq y \leq a$, and R' is the region lying inside the square R'' but outside the

quarter-circle R (see the figure below), then $I_a^2 = \iint_{R''} e^{-(x^2+y^2)} dA$ according to part (a), and by properties of double integrals

$$\begin{aligned} \iint_R e^{-(x^2+y^2)} dA + \iint_{R'} e^{-(x^2+y^2)} dA &= \iint_{R''} e^{-(x^2+y^2)} dA \\ J_a + \iint_{R'} e^{-(x^2+y^2)} dA &= I_a^2 \\ \iint_{R'} e^{-(x^2+y^2)} dA &= I_a^2 - J_a \end{aligned}$$

Notice that $I_a^2 - J_a$ is positive since it is equal to a double integral of a positive function.



Since the region R' lies outside the circle $x^2 + y^2 = a^2$, it follows that $x^2 + y^2 \geq a^2$ on R' , and therefore $e^{-(x^2+y^2)} \leq e^{-a^2}$ on R' . If we denote $f(x, y) = e^{-(x^2+y^2)}$ and $g(x, y) = e^{-a^2}$, then we have just demonstrated that $f(x, y) \leq g(x, y)$ on R' , and by the property proven in Problem 89, Section 14.2, $\iint_{R'} f(x, y) dA \leq \iint_{R'} g(x, y) dA$; that is

$$\begin{aligned} |I_a^2 - J_a| &= I_a^2 - J_a = \iint_{R'} e^{-(x^2+y^2)} dA \leq \iint_{R'} e^{-a^2} dA = e^{-a^2} \iint_{R'} dA \\ &= e^{-a^2} \cdot (\text{area of } R') = e^{-a^2} \cdot \left(a^2 - \frac{1}{4}\pi a^2 \right) = \frac{(4-\pi)a^2}{4} e^{-a^2}, \end{aligned}$$

where we found the area of R' by subtracting the area of the quarter-circle R from the area of the square R'' . So we have shown that

$$|I_a^2 - J_a| \leq \frac{(4-\pi)a^2}{4} e^{-a^2}.$$

(c) To find $J_a = \iint_R e^{-(x^2+y^2)} dA$, we convert it to polar coordinates:

$$J_a = \iint_R e^{-(x^2+y^2)} dA = \iint_R e^{-r^2} \cdot r dr d\theta = \int_0^{\pi/2} \int_0^a e^{-r^2} \cdot r dr d\theta.$$

In this iterated integral, first we evaluate the “inside” integral $\int_0^a e^{-r^2} \cdot r dr$. In this integral we make a substitution $u = -r^2$; then $du = -2rdr$, so $rdr = -\frac{du}{2}$. We also change the limits of integration: if $r = 0$, then $u = -0^2 = 0$; if $r = a$, then $u = -a^2$. So

$$\begin{aligned} \int_0^a e^{-r^2} \cdot r dr &= \int_0^{-a^2} e^u \left(-\frac{du}{2} \right) = -\frac{1}{2} \int_0^{-a^2} e^u du \\ &= \frac{1}{2} \int_{-a^2}^0 e^u du = \frac{1}{2} [e^u]_{-a^2}^0 = \frac{1}{2} (e^0 - e^{-a^2}) = \frac{1}{2} (1 - e^{-a^2}). \end{aligned}$$

Going back to the iterated integral,

$$\begin{aligned}\int_0^{\pi/2} \int_0^a e^{-r^2} \cdot r \, dr \, d\theta &= \int_0^{\pi/2} \frac{1}{2} (1 - e^{-a^2}) \, d\theta = \frac{1}{2} (1 - e^{-a^2}) \int_0^{\pi/2} d\theta \\ &= \frac{1}{2} (1 - e^{-a^2}) \cdot [\theta]_0^{\pi/2} = \frac{1}{2} (1 - e^{-a^2}) \cdot \left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\pi}{4} (1 - e^{-a^2})}.\end{aligned}$$

- (d) Let $t = -a^2$. If $a \rightarrow \infty$, then $t = -a^2 \rightarrow -\infty$, and $\lim_{a \rightarrow \infty} e^{-a^2} = \lim_{t \rightarrow -\infty} e^t = 0$ by properties of the exponential function. Then

$$\lim_{a \rightarrow \infty} J_a = \lim_{a \rightarrow \infty} \frac{\pi}{4} (1 - e^{-a^2}) = \frac{\pi}{4} \lim_{a \rightarrow \infty} (1 - e^{-a^2}) = \frac{\pi}{4} \left(1 - \lim_{a \rightarrow \infty} e^{-a^2}\right) = \frac{\pi}{4} (1 - 0) = \frac{\pi}{4},$$

proving the first limit. Also,

$$\lim_{a \rightarrow \infty} \frac{(4 - \pi)a^2}{4} e^{-a^2} = \frac{4 - \pi}{4} \lim_{a \rightarrow \infty} \frac{a^2}{e^{a^2}} = \frac{4 - \pi}{4} \lim_{a \rightarrow \infty} \frac{2a}{2ae^{a^2}} = \frac{4 - \pi}{4} \lim_{a \rightarrow \infty} \frac{1}{e^{a^2}} = 0,$$

where we applied L'Hospital's rule at the second step. Since according to part (b), $0 \leq |I_a^2 - J_a| \leq \frac{(4 - \pi)a^2}{4} e^{-a^2}$, and $\lim_{a \rightarrow \infty} 0 = \lim_{a \rightarrow \infty} \frac{(4 - \pi)a^2}{4} e^{-a^2} = 0$ as shown above, then by the Squeeze Theorem, $\lim_{a \rightarrow \infty} |I_a^2 - J_a| = 0$, proving the second limit.

- (e) First, using the results of part (d), we can find the limit of I_a as $a \rightarrow \infty$: $\lim_{a \rightarrow \infty} |I_a^2 - J_a| = 0$ is equivalent to $\lim_{a \rightarrow \infty} (I_a^2 - J_a) = 0$, and so

$$\lim_{a \rightarrow \infty} I_a^2 = \lim_{a \rightarrow \infty} [J_a + (I_a^2 - J_a)] = \lim_{a \rightarrow \infty} J_a + \lim_{a \rightarrow \infty} (I_a^2 - J_a) = \frac{\pi}{4} + 0 = \frac{\pi}{4},$$

which implies that $\lim_{a \rightarrow \infty} I_a = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

Now we can find the improper integral $\int_0^\infty e^{-x^2} dx$:

$$\int_0^\infty e^{-x^2} dx = \lim_{a \rightarrow \infty} \int_0^a e^{-x^2} dx = \lim_{a \rightarrow \infty} I_a = \frac{\sqrt{\pi}}{2}.$$

Since e^{-x^2} is an even function, for any $a > 0$, $\int_{-a}^0 e^{-x^2} dx = \int_0^a e^{-x^2} dx$ by properties of definite integrals, and therefore

$$\int_{-\infty}^0 e^{-x^2} dx = \lim_{a \rightarrow \infty} \int_{-a}^0 e^{-x^2} dx = \lim_{a \rightarrow \infty} \int_0^a e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Adding these together, we obtain that

$$\int_{-\infty}^\infty e^{-x^2} dx = \int_{-\infty}^0 e^{-x^2} dx + \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

14.4 Center of Mass; Moment of Inertia

Concepts and Vocabulary

1. If a lamina is represented by a closed bounded region R and has a continuous mass density function $\rho(x, y)$, then the double integral $\iint_R \rho(x, y) dA$ gives the mass of the lamina.
2. False. If a lamina is represented by a closed bounded region R and has a continuous mass density function $\rho(x, y)$, then $\iint_R y\rho(x, y) dA = M_x$, the moment of mass about the x -axis. The moment of mass about the y -axis is given by $M_y = \iint_R x\rho(x, y) dA$.
3. If a lamina is represented by a closed bounded region R and has a continuous mass density function $\rho(x, y)$, then the moment of mass about the y -axis is given by the double integral $M_y = \iint_R x\rho(x, y) dA$.
4. False. The centroid is a geometrical property of a lamina; it coincides with the center of mass in the case of a homogeneous lamina (when the mass density is constant). The center of mass is a physical property of a lamina, and if the mass density is variable, it may not be at the geometrical center (centroid) of the lamina.
5. True. If a particle of mass m is located a distance d from a fixed axis a , the moment of mass of the particle about the a -axis is $M_a = md$, and it measures the tendency of the particle to rotate about this axis.
6. False. The moment of inertia about the origin equals the sum of the moment of inertia about the x -axis and the moment of inertia about the y -axis.

Skill Building

7. The lamina occupies the triangular region R shown in the figure in Problem 7, see also the figure below. The side of the triangle joining the points $(0, 0)$ and $(3, 6)$ has the equation $y = 2x$. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2x$, $0 \leq x \leq 3$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R x^2 y dA = \int_0^3 \int_0^{2x} x^2 y dy dx = \int_0^3 x^2 \left[\frac{y^2}{2} \right]_0^{2x} dx \\ &= \int_0^3 x^2 (2x^2 - 0) dx = \int_0^3 2x^4 dx = 2 \left[\frac{x^5}{5} \right]_0^3 = 2 \left(\frac{243}{5} - 0 \right) = \frac{486}{5}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x \cdot x^2 y dA = \int_0^3 \int_0^{2x} x^3 y dy dx = \int_0^3 x^3 \left[\frac{y^2}{2} \right]_0^{2x} dx \\ &= \int_0^3 x^3 (2x^2 - 0) dx = \int_0^3 2x^5 dx = \left[\frac{x^6}{3} \right]_0^3 = \frac{3^6}{3} - 0 = 243, \end{aligned}$$

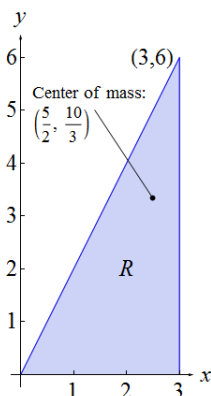
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{243}{486/5} = \frac{5}{2}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y \cdot x^2 y dA = \int_0^3 \int_0^{2x} x^2 y^2 dy dx = \int_0^3 x^2 \left[\frac{y^3}{3} \right]_0^{2x} dx \\ &= \int_0^3 x^2 \left(\frac{8x^3}{3} - 0 \right) dx = \int_0^3 \frac{8x^5}{3} dx = \left[\frac{4x^6}{9} \right]_0^3 = \frac{4 \cdot 3^6}{9} - 0 = 324, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{324}{486/5} = \frac{10}{3}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{5}{2}, \frac{10}{3} \right)$.



8. The lamina occupies the square region R shown in the figure in Problem 8; see also the figure below. Notice that R is an x -simple region: it is bounded between $y = 1$ and $y = 4$, $1 \leq x \leq 4$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R xy^2 dA = \int_1^4 \int_1^4 xy^2 dy dx = \int_1^4 x \left[\frac{y^3}{3} \right]_1^4 dx \\ &= \int_1^4 x \left(\frac{64}{3} - \frac{1}{3} \right) dx = \int_1^4 21x dx = 21 \left[\frac{x^2}{2} \right]_1^4 = 21 \left(8 - \frac{1}{2} \right) = 21 \cdot \frac{15}{2} = \boxed{\frac{315}{2}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x \cdot xy^2 dA = \int_1^4 \int_1^4 x^2 y^2 dy dx = \int_1^4 x^2 \left[\frac{y^3}{3} \right]_1^4 dx \\ &= \int_1^4 x^2 \left(\frac{64}{3} - \frac{1}{3} \right) dx = \int_1^4 21x^2 dx = [7x^3]_1^4 = 7(64 - 1) = 441, \end{aligned}$$

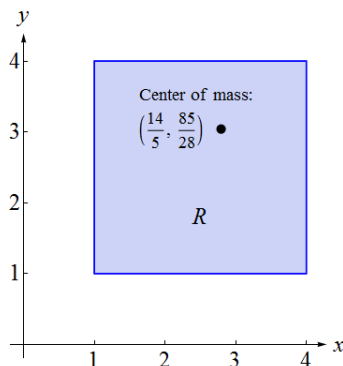
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{441}{315/2} = \frac{14}{5}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y \cdot xy^2 dA = \int_1^4 \int_1^4 xy^3 dy dx = \int_1^4 x \left[\frac{y^4}{4} \right]_1^4 dx \\ &= \int_1^4 x \left(64 - \frac{1}{4} \right) dx = \int_1^4 \frac{255x}{4} dx = \left[\frac{255x^2}{8} \right]_1^4 = 510 - \frac{255}{8} = \frac{3825}{8}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{3825/8}{315/2} = \frac{85}{28}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{14}{5}, \frac{85}{28} \right)$.



9. The region R enclosed by the lines $x = 2$ and $y = 4$ and the coordinate axes is a rectangle, as shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 4$, $0 \leq x \leq 2$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R 3x^2 y dA = \int_0^2 \int_0^4 3x^2 y dy dx = \int_0^2 3x^2 \left[\frac{y^2}{2} \right]_0^4 dx \\ &= \int_0^2 3x^2 (8 - 0) dx = \int_0^2 24x^2 dx = [8x^3]_0^2 = 8(8 - 0) = \boxed{64}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x \rho(x, y) dA = \iint_R x \cdot 3x^2 y dA = \int_0^2 \int_0^4 3x^3 y dy dx = \int_0^2 3x^3 \left[\frac{y^2}{2} \right]_0^4 dx \\ &= \int_0^2 3x^3 (8 - 0) dx = \int_0^2 24x^3 dx = [6x^4]_0^2 = 6(16 - 0) = 96, \end{aligned}$$

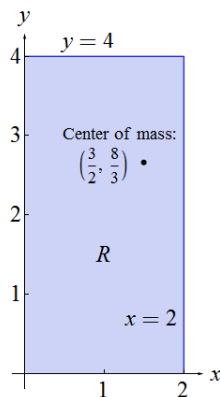
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{96}{64} = \frac{3}{2}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y \rho(x, y) dA = \iint_R y \cdot 3x^2 y dA = \int_0^2 \int_0^4 3x^2 y^2 dy dx = \int_0^2 x^2 [y^3]_0^4 dx \\ &= \int_0^2 x^2 (64 - 0) dx = \int_0^2 64x^2 dx = \left[\frac{64x^3}{3} \right]_0^2 = \frac{512}{3} - 0 = \frac{512}{3}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{512/3}{64} = \frac{8}{3}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{8}{3} \right)$.



10. The region R enclosed by the lines $x = 1$ and $y = 2$ and the coordinate axes is a rectangle, as shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2$, $0 \leq x \leq 1$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R 2x^2 y^2 dA = \int_0^1 \int_0^2 2x^2 y^2 dy dx = \int_0^1 2x^2 \left[\frac{y^3}{3} \right]_0^2 dx \\ &= \int_0^1 2x^2 \left(\frac{8}{3} - 0 \right) dx = \int_0^1 \frac{16x^2}{3} dx = \left[\frac{16x^3}{9} \right]_0^1 = \frac{16}{9} - 0 = \boxed{\frac{16}{9}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x \rho(x, y) dA = \iint_R x \cdot 2x^2 y^2 dA = \int_0^1 \int_0^2 2x^3 y^2 dy dx = \int_0^1 2x^3 \left[\frac{y^3}{3} \right]_0^2 dx \\ &= \int_0^1 2x^3 \left(\frac{8}{3} - 0 \right) dx = \int_0^1 \frac{16x^3}{3} dx = \left[\frac{4x^4}{3} \right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}, \end{aligned}$$

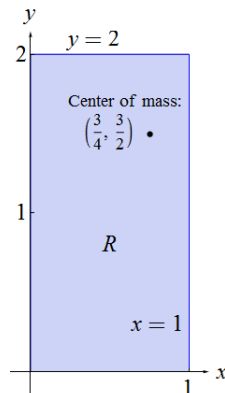
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{4/3}{16/9} = \frac{3}{4}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y \rho(x, y) dA = \iint_R y \cdot 2x^2 y^2 dA = \int_0^1 \int_0^2 2x^2 y^3 dy dx = \int_0^1 2x^2 \left[\frac{y^4}{4} \right]_0^2 dx \\ &= \int_0^1 2x^2 (4 - 0) dx = \int_0^1 8x^2 dx = \left[\frac{8x^3}{3} \right]_0^1 = \frac{8}{3} - 0 = \frac{8}{3}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{8/3}{16/9} = \frac{3}{2}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \boxed{\left(\frac{3}{4}, \frac{3}{2} \right)}$.



11. The region R enclosed in the first quadrant by $y^2 = x$, $x = 1$, and the x -axis is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = \sqrt{x}$, $0 \leq x \leq 1$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R (2x + 3y) dA = \int_0^1 \int_0^{\sqrt{x}} (2x + 3y) dy dx = \int_0^1 \left[2xy + \frac{3y^2}{2} \right]_0^{\sqrt{x}} dx \\
 &= \int_0^1 \left[2x\sqrt{x} + \frac{3(\sqrt{x})^2}{2} \right] - (0 + 0) dx = \int_0^1 \left(2x^{3/2} + \frac{3}{2}x \right) dx = \left[2 \cdot \frac{x^{5/2}}{5/2} + \frac{3}{2} \cdot \frac{x^2}{2} \right]_0^1 \\
 &= \left[\frac{4}{5}x^{5/2} + \frac{3}{4}x^2 \right]_0^1 = \left(\frac{4}{5} + \frac{3}{4} \right) - (0 + 0) = \boxed{\frac{31}{20}}.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R x(2x + 3y) dA = \int_0^1 \int_0^{\sqrt{x}} (2x^2 + 3xy) dy dx \\
 &= \int_0^1 \left[2x^2y + \frac{3xy^2}{2} \right]_0^{\sqrt{x}} dx = \int_0^1 \left[2x^2\sqrt{x} + \frac{3x(\sqrt{x})^2}{2} \right] - (0 + 0) dx \\
 &= \int_0^1 \left(2x^{5/2} + \frac{3}{2}x^2 \right) dx = \left[2 \cdot \frac{x^{7/2}}{7/2} + \frac{3}{2} \cdot \frac{x^3}{3} \right]_0^1 \\
 &= \left[\frac{4}{7}x^{7/2} + \frac{1}{2}x^3 \right]_0^1 = \left(\frac{4}{7} + \frac{1}{2} \right) - (0 + 0) = \frac{15}{14},
 \end{aligned}$$

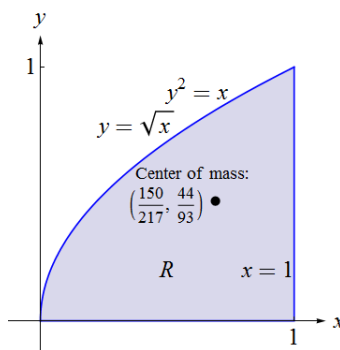
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{15/14}{31/20} = \frac{150}{217}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R y(2x + 3y) dA = \int_0^1 \int_0^{\sqrt{x}} (2xy + 3y^2) dy dx \\
 &= \int_0^1 \left[xy^2 + y^3 \right]_0^{\sqrt{x}} dx = \int_0^1 \left[(x(\sqrt{x})^2 + (\sqrt{x})^3) - (0 + 0) \right] dx = \int_0^1 (x^2 + x^{3/2}) dx \\
 &= \left[\frac{x^3}{3} + \frac{x^{5/2}}{5/2} \right]_0^1 = \left[\frac{1}{3}x^3 + \frac{2}{5}x^{5/2} \right]_0^1 = \left(\frac{1}{3} + \frac{2}{5} \right) - (0 + 0) = \frac{11}{15},
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{11/15}{31/20} = \frac{44}{93}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{150}{217}, \frac{44}{93} \right)$.



12. The region R enclosed in the first quadrant by $y^2 = 4x$, $x = 1$, and the x -axis is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2\sqrt{x}$, $0 \leq x \leq 1$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R (x+1) dA = \int_0^1 \int_0^{2\sqrt{x}} (x+1) dy dx = \int_0^1 [(x+1)y]_0^{2\sqrt{x}} dx \\ &= \int_0^1 [(x+1) \cdot 2\sqrt{x} - (x+1) \cdot 0] dx = \int_0^1 (2x^{3/2} + 2x^{1/2}) dx = \left[2 \cdot \frac{x^{5/2}}{5/2} + 2 \cdot \frac{x^{3/2}}{3/2} \right]_0^1 \\ &= \left[\frac{4}{5}x^{5/2} + \frac{4}{3}x^{3/2} \right]_0^1 = \left(\frac{4}{5} + \frac{4}{3} \right) - (0+0) = \boxed{\frac{32}{15}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x(x+1) dA = \int_0^1 \int_0^{2\sqrt{x}} (x^2 + x) dy dx = \int_0^1 [(x^2 + x)y]_0^{2\sqrt{x}} dx \\ &= \int_0^1 [(x^2 + x) \cdot 2\sqrt{x} - (x+1) \cdot 0] dx = \int_0^1 (2x^{5/2} + 2x^{3/2}) dx = \left[2 \cdot \frac{x^{7/2}}{7/2} + 2 \cdot \frac{x^{5/2}}{5/2} \right]_0^1 \\ &= \left[\frac{4}{7}x^{7/2} + \frac{4}{5}x^{5/2} \right]_0^1 = \left(\frac{4}{7} + \frac{4}{5} \right) - (0+0) = \frac{48}{35}, \end{aligned}$$

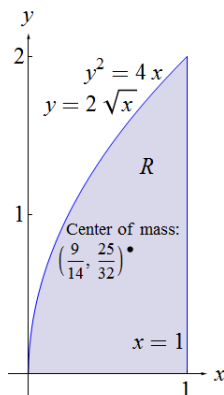
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{48/35}{32/15} = \frac{9}{14}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y(x+1) dA = \int_0^1 \int_0^{2\sqrt{x}} (x+1)y dy dx = \int_0^1 \left[(x+1)\frac{y^2}{2} \right]_0^{2\sqrt{x}} dx \\ &= \int_0^1 [(x+1) \cdot 2x - (x+1) \cdot 0] dx = \int_0^1 (2x^2 + 2x) dx = \left[\frac{2}{3}x^3 + x^2 \right]_0^1 \\ &= \left(\frac{2}{3} + 1 \right) - (0+0) = \frac{5}{3}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{5/3}{32/15} = \frac{25}{32}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{9}{14}, \frac{25}{32} \right)$.



13. The region R enclosed by $y^2 = x$ and $y = x$ is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = x$ and $y = \sqrt{x}$, $0 \leq x \leq 1$. The mass density of the lamina is proportional to the distance from the y -axis; that is, $\rho(x, y) = kx$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R kx dA = k \int_0^1 \int_x^{\sqrt{x}} x dy dx = k \int_0^1 [xy]_x^{\sqrt{x}} dx \\
 &= k \int_0^1 [x \cdot \sqrt{x} - x \cdot x] dx = k \int_0^1 (x^{3/2} - x^2) dx = k \left[\frac{x^{5/2}}{5/2} - \frac{x^3}{3} \right]_0^1 \\
 &= k \left[\frac{2}{5} x^{5/2} - \frac{1}{3} x^3 \right]_0^1 = k \left[\left(\frac{2}{5} - \frac{1}{3} \right) - (0 - 0) \right] = \frac{1}{15} k \\
 &= \boxed{\frac{k}{15}}, \text{ where } k \text{ is the constant of proportionality.}
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x \rho(x, y) dA = \iint_R x \cdot kx dA = k \int_0^1 \int_x^{\sqrt{x}} x^2 dy dx = k \int_0^1 [x^2 y]_x^{\sqrt{x}} dx \\
 &= k \int_0^1 [x^2 \cdot \sqrt{x} - x^2 \cdot x] dx = k \int_0^1 (x^{5/2} - x^3) dx = k \left[\frac{x^{7/2}}{7/2} - \frac{x^4}{4} \right]_0^1 \\
 &= k \left[\frac{2}{7} x^{7/2} - \frac{1}{4} x^4 \right]_0^1 = k \left[\left(\frac{2}{7} - \frac{1}{4} \right) - (0 - 0) \right] = \frac{1}{28} k,
 \end{aligned}$$

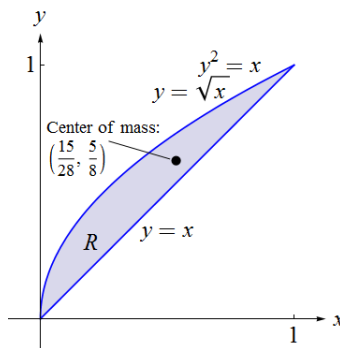
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(1/28)k}{(1/15)k} = \frac{15}{28}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y \rho(x, y) dA = \iint_R y \cdot kx dA = k \int_0^1 \int_x^{\sqrt{x}} xy dy dx = k \int_0^1 \left[x \frac{y^2}{2} \right]_x^{\sqrt{x}} dx \\
 &= k \int_0^1 \left[x \cdot \frac{x}{2} - x \cdot \frac{x^2}{2} \right] dx = k \int_0^1 \left(\frac{1}{2} x^2 - \frac{1}{2} x^3 \right) dx = k \left[\frac{1}{6} x^3 - \frac{1}{8} x^4 \right]_0^1 \\
 &= k \left[\left(\frac{1}{6} - \frac{1}{8} \right) - (0 - 0) \right] = \frac{1}{24} k,
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{(1/24)k}{(1/15)k} = \frac{5}{8}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{15}{28}, \frac{5}{8} \right)$.



14. The region R enclosed by $y = \sin x$ and the x -axis, $0 \leq x \leq \pi$, is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = \sin x$, $0 \leq x \leq \pi$. The mass density of the lamina is proportional to the distance from the x -axis; that is, $\rho(x, y) = ky$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R ky dA = k \int_0^\pi \int_0^{\sin x} y dy dx = k \int_0^\pi \left[\frac{y^2}{2} \right]_0^{\sin x} dx \\ &= k \int_0^\pi \left[\frac{(\sin x)^2}{2} - 0 \right] dx = \frac{k}{2} \int_0^\pi \sin^2 x dx = \frac{k}{2} \int_0^\pi \frac{1}{2} (1 - \cos(2x)) dx = \frac{k}{4} \left[x - \frac{1}{2} \sin(2x) \right]_0^\pi \\ &= \frac{k}{4} \left[\left(\pi - \frac{1}{2} \sin(2\pi) \right) - \left(0 - \frac{1}{2} \sin(0) \right) \right] = \frac{k}{4} [(\pi - 0) - (0 - 0)] \\ &= \boxed{\frac{k\pi}{4}, \text{ where } k \text{ is the constant of proportionality.}} \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x \cdot ky dA = k \int_0^\pi \int_0^{\sin x} xy dy dx = k \int_0^\pi x \left[\frac{y^2}{2} \right]_0^{\sin x} dx \\ &= k \int_0^\pi x \left[\frac{(\sin x)^2}{2} - 0 \right] dx = \frac{k}{2} \int_0^\pi x \sin^2 x dx = \frac{k}{2} \int_0^\pi x \cdot \frac{1}{2} (1 - \cos(2x)) dx \\ &= \frac{k}{4} \int_0^\pi (x - x \cos(2x)) dx = \frac{k}{4} \left[\int_0^\pi x dx - \int_0^\pi x \cos(2x) dx \right]. \end{aligned}$$

The first integral in the brackets can be evaluated directly:

$$\int_0^\pi x dx = \left[\frac{x^2}{2} \right]_0^\pi = \frac{\pi^2}{2} - \frac{0^2}{2} = \frac{\pi^2}{2}.$$

To evaluate the second integral in the brackets, we use the method of integration by parts. If we choose $u = x$ and $dv = \cos(2x) dx$, then $du = dx$ and $v = \int \cos(2x) dx = \frac{1}{2} \sin(2x)$. So

$$\begin{aligned} \int_0^\pi x \cos(2x) dx &= \left[x \cdot \frac{1}{2} \sin(2x) \right]_0^\pi - \int_0^\pi \frac{1}{2} \sin(2x) dx = \left(\frac{1}{2} \cdot \pi \sin(2\pi) - \frac{1}{2} \cdot 0 \sin(0) \right) + \left[\frac{1}{4} \cos(2x) \right]_0^\pi \\ &= (0 - 0) + \left(\frac{1}{4} \cos(2\pi) - \frac{1}{4} \cos(0) \right) = 0 + \left(\frac{1}{4} - \frac{1}{4} \right) = 0. \end{aligned}$$

Substituting these values back into the computation of M_y , we obtain

$$M_y = \frac{k}{4} \left[\int_0^\pi x \, dx - \int_0^\pi x \cos(2x) \, dx \right] = \frac{k}{4} \left[\frac{\pi^2}{2} - 0 \right] = \frac{k\pi^2}{8},$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{k\pi^2/8}{k\pi/4} = \frac{\pi}{2}.$$

Similarly, the moment of mass about the x -axis is

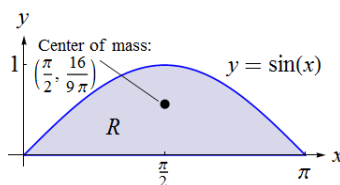
$$\begin{aligned} M_x &= \iint_R y \rho(x, y) \, dA = \iint_R y \cdot ky \, dA = k \int_0^\pi \int_0^{\sin x} y^2 \, dy \, dx = k \int_0^\pi \left[\frac{y^3}{3} \right]_0^{\sin x} dx \\ &= k \int_0^\pi \left[\frac{(\sin x)^3}{3} - 0 \right] dx = \frac{k}{3} \int_0^\pi \sin^3 x \, dx = \frac{k}{3} \int_0^\pi \sin^2 x \sin x \, dx = \frac{k}{3} \int_0^\pi (1 - \cos^2 x) \sin x \, dx. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos x$; then $du = -\sin x \, dx$, so $\sin x \, dx = -du$. We also change the limits of integration: if $x = 0$, then $u = \cos 0 = 1$; if $x = \pi$, then $u = \cos \pi = -1$. So

$$\begin{aligned} M_x &= \frac{k}{3} \int_0^\pi (1 - \cos^2 x) \sin x \, dx = \frac{k}{3} \int_1^{-1} (1 - u^2)(-du) = -\frac{k}{3} \int_1^{-1} (1 - u^2) \, du \\ &= \frac{k}{3} \int_{-1}^1 (1 - u^2) \, du = \frac{k}{3} \left[u - \frac{u^3}{3} \right]_{-1}^1 = \frac{k}{3} \left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) \right] \\ &= \frac{k}{3} \left[\frac{2}{3} - \left(-\frac{2}{3}\right) \right] = \frac{k}{3} \cdot \frac{4}{3} = \frac{4k}{9}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{4k/9}{k\pi/4} = \frac{16}{9\pi}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{16}{9\pi} \right)$.



15. The region R enclosed by the lines $2x + 3y = 6$, $x = 0$ (the y -axis), and $y = 0$ (the x -axis) is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2 - \frac{2x}{3}$, $0 \leq x \leq 3$. The mass density of the lamina is proportional to the sum of the distances from the coordinate axes; that is, $\rho(x, y) = k(x + y)$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) \, dA = \iint_R k(x + y) \, dA = k \int_0^3 \int_0^{2-2x/3} (x + y) \, dy \, dx = k \int_0^3 \left[xy + \frac{y^2}{2} \right]_0^{2-2x/3} dx \\ &= k \int_0^3 \left\{ x \left(2 - \frac{2x}{3} \right) + \frac{\left(2 - \frac{2x}{3} \right)^2}{2} \right\} - (0 + 0) \, dx = k \int_0^3 \left(2x - \frac{2x^2}{3} + 2 - \frac{4x}{3} + \frac{2x^2}{9} \right) dx \\ &= k \int_0^3 \left(2 + \frac{2x}{3} - \frac{4x^2}{9} \right) dx = k \left[2x + \frac{x^2}{3} - \frac{4x^3}{27} \right]_0^3 = k[(6 + 3 - 4) - (0 + 0 - 0)] \\ &= \boxed{5k, \text{ where } k \text{ is the constant of proportionality}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x,y) dA = \iint_R x \cdot k(x+y) dA = k \int_0^3 \int_0^{2-2x/3} (x^2 + xy) dy dx \\
 &= k \int_0^3 \left[x^2 y + x \frac{y^2}{2} \right]_0^{2-2x/3} dx = k \int_0^3 \left\{ \left[x^2 \left(2 - \frac{2x}{3} \right) + x \frac{\left(2 - \frac{2x}{3} \right)^2}{2} \right] - (0+0) \right\} dx \\
 &= k \int_0^3 \left(2x^2 - \frac{2x^3}{3} + 2x - \frac{4x^2}{3} + \frac{2x^3}{9} \right) dx = k \int_0^3 \left(2x + \frac{2x^2}{3} - \frac{4x^3}{9} \right) dx \\
 &= k \left[x^2 + \frac{2x^3}{9} - \frac{x^4}{9} \right]_0^3 = k [(9+6-9) - (0+0-0)] = 6k,
 \end{aligned}$$

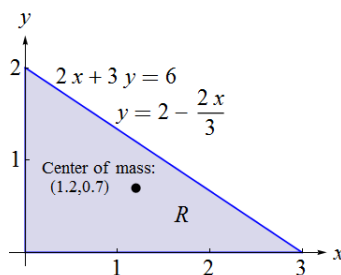
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{6k}{5k} = \frac{6}{5} = 1.2.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x,y) dA = \iint_R y \cdot k(x+y) dA = k \int_0^3 \int_0^{2-2x/3} (xy + y^2) dy dx \\
 &= k \int_0^3 \left[x \frac{y^2}{2} + \frac{y^3}{3} \right]_0^{2-2x/3} dx = k \int_0^3 \left\{ \left[x \frac{\left(2 - \frac{2x}{3} \right)^2}{2} + \frac{\left(2 - \frac{2x}{3} \right)^3}{3} \right] - (0+0) \right\} dx \\
 &= k \int_0^3 \left(2x - \frac{4x^2}{3} + \frac{2x^3}{9} + \frac{8}{3} - \frac{8x}{3} + \frac{8x^2}{9} - \frac{8x^3}{81} \right) dx = k \int_0^3 \left(\frac{8}{3} - \frac{2x}{3} - \frac{4x^2}{9} + \frac{10x^3}{81} \right) dx \\
 &= k \left[\frac{8x}{3} - \frac{x^2}{3} - \frac{4x^3}{27} + \frac{5x^4}{162} \right]_0^3 = k \left[\left(8 - 3 - 4 + \frac{5}{2} \right) - (0 - 0 - 0 + 0) \right] = \frac{7}{2}k,
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{(7/2)k}{5k} = \frac{7}{10} = 0.7.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{6}{5}, \frac{7}{10} \right) = (1.2, 0.7)$.



16. The region R enclosed by the lines $3x + 4y = 12$, $x = 0$ (the y -axis), and $y = 0$ (the x -axis) is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 3 - \frac{3x}{4}$, $0 \leq x \leq 4$. The mass density of the lamina is proportional to the product of the distances from the coordinate axes; that is, $\rho(x,y) = kxy$, where k is a constant coefficient of proportionality. Then the

mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R kxy dA = k \int_0^4 \int_0^{3-3x/4} xy dy dx = k \int_0^4 \left[x \frac{y^2}{2} \right]_0^{3-3x/4} dx \\
 &= k \int_0^4 \left[x \frac{\left(3 - \frac{3x}{4}\right)^2}{2} - 0 \right] dx = k \int_0^4 \left(\frac{9x}{2} - \frac{9x^2}{4} + \frac{9x^3}{32} \right) dx \\
 &= k \left[\frac{9x^2}{4} - \frac{3x^3}{4} + \frac{9x^4}{128} \right]_0^4 = k [(36 - 48 + 18) - (0 - 0 + 0)] = \\
 &= \boxed{6k, \text{ where } k \text{ is the constant of proportionality}}.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R x \cdot kxy dA = k \int_0^4 \int_0^{3-3x/4} x^2 y dy dx = k \int_0^4 \left[x^2 \frac{y^2}{2} \right]_0^{3-3x/4} dx \\
 &= k \int_0^4 \left[x^2 \frac{\left(3 - \frac{3x}{4}\right)^2}{2} - 0 \right] dx = k \int_0^4 \left(\frac{9x^2}{2} - \frac{9x^3}{4} + \frac{9x^4}{32} \right) dx \\
 &= k \left[\frac{3x^3}{2} - \frac{9x^4}{16} + \frac{9x^5}{160} \right]_0^4 = k \left[\left(96 - 144 + \frac{288}{5} \right) - (0 - 0 + 0) \right] = \frac{48}{5}k,
 \end{aligned}$$

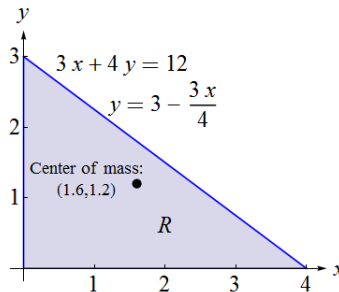
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(48/5)k}{6k} = \frac{8}{5} = 1.6.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R y \cdot kxy dA = k \int_0^4 \int_0^{3-3x/4} xy^2 dy dx = k \int_0^4 \left[x \frac{y^3}{3} \right]_0^{3-3x/4} dx \\
 &= k \int_0^4 \left[x \frac{\left(3 - \frac{3x}{4}\right)^3}{3} - 0 \right] dx = k \int_0^4 \left(9x - \frac{27x^2}{4} + \frac{27x^3}{16} - \frac{9x^4}{64} \right) dx \\
 &= k \left[\frac{9x^2}{2} - \frac{9x^3}{4} + \frac{27x^4}{64} - \frac{9x^5}{320} \right]_0^4 = k \left[\left(72 - 144 + 108 - \frac{144}{5} \right) - (0 - 0 + 0 - 0) \right] = \frac{36}{5}k,
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{(36/5)k}{6k} = \frac{6}{5} = 1.2.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \boxed{\left(\frac{8}{5}, \frac{6}{5} \right) = (1.6, 1.2)}$.



17. The region R enclosed by the cardioid $r = 1 + \sin \theta$ is shown in the figure below. In polar coordinates, it is given by $0 \leq r \leq 1 + \sin \theta$, $0 \leq \theta \leq 2\pi$. The mass density of the lamina is proportional to the distance from the pole; that is, $\rho(r, \theta) = kr$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R kr \cdot r dr d\theta = k \int_0^{2\pi} \int_0^{1+\sin \theta} r^2 dr d\theta = k \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^{1+\sin \theta} d\theta \\
 &= k \int_0^{2\pi} \left[\frac{1}{3}(1 + \sin \theta)^3 - 0 \right] d\theta = k \int_0^{2\pi} \left(\frac{1}{3} + \sin \theta + \sin^2 \theta + \frac{1}{3} \sin^3 \theta \right) d\theta \\
 &= k \int_0^{2\pi} \left(\frac{1}{3} + \sin \theta + \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{3} \sin^3 \theta \right) d\theta = k \int_0^{2\pi} \left(\frac{5}{6} + \sin \theta - \frac{1}{2} \cos(2\theta) + \frac{1}{3} \sin^3 \theta \right) d\theta \\
 &= k \left(\frac{5}{6} \int_0^{2\pi} d\theta + \int_0^{2\pi} \sin \theta d\theta - \frac{1}{2} \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{3} \int_0^{2\pi} \sin^3 \theta d\theta \right).
 \end{aligned}$$

The first three integrals in parentheses can be evaluated directly:

$$\begin{aligned}
 \frac{5}{6} \int_0^{2\pi} d\theta &= \frac{5}{6} [\theta]_0^{2\pi} = \frac{5}{6} (2\pi - 0) = \frac{5\pi}{3}; \\
 \int_0^{2\pi} \sin \theta d\theta &= [-\cos \theta]_0^{2\pi} = -\cos 2\pi + \cos 0 = -1 + 1 = 0; \\
 \frac{1}{2} \int_0^{2\pi} \cos(2\theta) d\theta &= \frac{1}{2} \left[\frac{\sin(2\theta)}{2} \right]_0^{2\pi} = \frac{1}{4} [\sin(2\theta)]_0^{2\pi} = \frac{1}{4} (\sin(4\pi) - \sin 0) = \frac{1}{4} (0 - 0) = 0.
 \end{aligned}$$

To evaluate the last integral in parentheses, we use a trigonometric identity to transform the integrand $\sin^3 \theta = \sin^2 \theta \cdot \sin \theta = (1 - \cos^2 \theta) \sin \theta$, and then make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = 0$, then $u = \cos 0 = 1$; if $\theta = 2\pi$, then $u = \cos 2\pi = 1$. So

$$\frac{1}{3} \int_0^{2\pi} \sin^3 \theta d\theta = \frac{1}{3} \int_0^{2\pi} (1 - \cos^2 \theta) \sin \theta d\theta = \frac{1}{3} \int_1^1 (1 - u^2)(-du) = -\frac{1}{3} \int_1^1 (1 - u^2) du = 0.$$

Substituting these values back into the computation of M , we obtain

$$\begin{aligned}
 M &= k \left(\frac{5}{6} \int_0^{2\pi} d\theta + \int_0^{2\pi} \sin \theta d\theta - \frac{1}{2} \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{3} \int_0^{2\pi} \sin^3 \theta d\theta \right) = k \left(\frac{5\pi}{3} + 0 - 0 + 0 \right) \\
 &= \boxed{\frac{5\pi k}{3}}, \text{ where } k \text{ is the constant of proportionality.}
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R r \cos \theta \cdot kr \cdot r dr d\theta = k \int_0^{2\pi} \int_0^{1+\sin \theta} r^3 \cos \theta dr d\theta \\
 &= k \int_0^{2\pi} \cos \theta \left[\frac{r^4}{4} \right]_0^{1+\sin \theta} d\theta = \frac{k}{4} \int_0^{2\pi} \cos \theta [(1 + \sin \theta)^4 - 0] d\theta = \frac{k}{4} \int_0^{2\pi} (1 + \sin \theta)^4 \cos \theta d\theta.
 \end{aligned}$$

To evaluate this integral, we make a substitution $u = 1 + \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = 0$, then $u = 1 + \sin 0 = 1 + 0 = 1$; if $\theta = 2\pi$, then $u = 1 + \sin 2\pi = 1 + 0 = 1$. So

$$M_y = \frac{k}{4} \int_0^{2\pi} (1 + \sin \theta)^4 \cos \theta d\theta = \frac{k}{4} \int_1^1 u^4 du = 0,$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{0}{5\pi k/3} = 0.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y \rho(x, y) dA = \iint_R r \sin \theta \cdot kr \cdot r dr d\theta = k \int_0^{2\pi} \int_0^{1+\sin \theta} r^3 \sin \theta dr d\theta \\
 &= k \int_0^{2\pi} \sin \theta \left[\frac{r^4}{4} \right]_0^{1+\sin \theta} d\theta = k \int_0^{2\pi} \sin \theta \left[\frac{1}{4}(1 + \sin \theta)^4 - 0 \right] d\theta \\
 &= k \int_0^{2\pi} \left(\frac{1}{4} \sin \theta + \sin^2 \theta + \frac{3}{2} \sin^3 \theta + \sin^4 \theta + \frac{1}{4} \sin^5 \theta \right) d\theta \\
 &= k \int_0^{2\pi} \left[\frac{1}{4} \sin \theta + \frac{1}{2}(1 - \cos(2\theta)) + \frac{3}{2} \sin^3 \theta + \left[\frac{1}{2}(1 - \cos(2\theta)) \right]^2 + \frac{1}{4} \sin^5 \theta \right] d\theta \\
 &= k \int_0^{2\pi} \left(\frac{1}{4} \sin \theta + \frac{1}{2} - \frac{1}{2} \cos(2\theta) + \frac{3}{2} \sin^3 \theta + \frac{1}{4} - \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos^2(2\theta) + \frac{1}{4} \sin^5 \theta \right) d\theta \\
 &= k \int_0^{2\pi} \left(\frac{3}{4} + \frac{1}{4} \sin \theta - \cos(2\theta) + \frac{1}{4} \cdot \frac{1}{2}(1 + \cos(4\theta)) + \frac{3}{2} \sin^3 \theta + \frac{1}{4} \sin^5 \theta \right) d\theta \\
 &= k \int_0^{2\pi} \left(\frac{7}{8} + \frac{1}{4} \sin \theta - \cos(2\theta) + \frac{1}{8} \cos(4\theta) + \frac{3}{2} \sin^3 \theta + \frac{1}{4} \sin^5 \theta \right) d\theta \\
 &= k \left[\frac{7}{8} \int_0^{2\pi} d\theta + \frac{1}{4} \int_0^{2\pi} \sin \theta d\theta - \int_0^{2\pi} \cos(2\theta) d\theta + \frac{1}{8} \int_0^{2\pi} \cos(4\theta) d\theta + \frac{3}{2} \int_0^{2\pi} \sin^3 \theta d\theta + \frac{1}{4} \int_0^{2\pi} \sin^5 \theta d\theta \right].
 \end{aligned}$$

The first four integrals in parentheses can be evaluated directly:

$$\begin{aligned}
 \frac{7}{8} \int_0^{2\pi} d\theta &= \frac{7}{8} [\theta]_0^{2\pi} = \frac{7}{8}(2\pi - 0) = \frac{7\pi}{4}; \\
 \frac{1}{4} \int_0^{2\pi} \sin \theta d\theta &= \frac{1}{4} [-\cos \theta]_0^{2\pi} = \frac{1}{4}(-\cos 2\pi + \cos 0) = \frac{1}{4}(-1 + 1) = 0; \\
 \int_0^{2\pi} \cos(2\theta) d\theta &= \left[\frac{\sin(2\theta)}{2} \right]_0^{2\pi} = \frac{1}{2}[\sin(2\theta)]_0^{2\pi} = \frac{1}{2}(\sin(4\pi) - \sin 0) = \frac{1}{2}(0 - 0) = 0; \\
 \frac{1}{8} \int_0^{2\pi} \cos(4\theta) d\theta &= \frac{1}{8} \left[\frac{\sin(4\theta)}{4} \right]_0^{2\pi} = \frac{1}{32}[\sin(4\theta)]_0^{2\pi} = \frac{1}{32}(\sin(8\pi) - \sin 0) = \frac{1}{32}(0 - 0) = 0.
 \end{aligned}$$

To evaluate the last two integrals in parentheses, we use a trigonometric identity to transform their integrands: $\sin^3 \theta = \sin^2 \theta \cdot \sin \theta = (1 - \cos^2 \theta) \sin \theta$ and $\sin^5 \theta = \sin^4 \theta \cdot \sin \theta = (1 - \cos^2 \theta)^2 \sin \theta$. Then in both integrals we can make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = 0$, then $u = \cos 0 = 1$; if $\theta = 2\pi$, then $u = \cos 2\pi = 1$. So

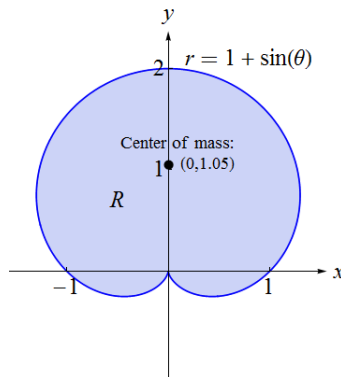
$$\begin{aligned}
 \frac{3}{2} \int_0^{2\pi} \sin^3 \theta d\theta &= \frac{3}{2} \int_0^{2\pi} (1 - \cos^2 \theta) \sin \theta d\theta = \frac{3}{2} \int_1^1 (1 - u^2)(-du) = -\frac{3}{2} \int_1^1 (1 - u^2) du = 0; \\
 \frac{1}{4} \int_0^{2\pi} \sin^5 \theta d\theta &= \frac{1}{4} \int_0^{2\pi} (1 - \cos^2 \theta)^2 \sin \theta d\theta = \frac{1}{4} \int_1^1 (1 - u^2)^2(-du) = -\frac{1}{4} \int_1^1 (1 - u^2)^2 du = 0.
 \end{aligned}$$

Substituting these values back into the computation of M_x , we obtain

$$M_x = k \left(\frac{7\pi}{4} + 0 - 0 + 0 + 0 + 0 \right) = \frac{7\pi k}{4},$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{7\pi k/4}{5\pi k/3} = \frac{21}{20} = 1.05.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(0, \frac{21}{20} \right) = (0, 1.05)$.



18. The region R enclosed by $r = \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, is shown in the figure below. In polar coordinates, it is given by $0 \leq r \leq \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The mass density of the lamina is proportional to the distance from the pole; that is, $\rho(r, \theta) = kr$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R kr \cdot r dr d\theta = k \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} r^2 dr d\theta = k \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{\cos \theta} d\theta \\ &= \frac{k}{3} \int_{-\pi/2}^{\pi/2} (\cos^3 \theta - 0) d\theta = \frac{k}{3} \int_{-\pi/2}^{\pi/2} \cos^2 \theta \cos \theta d\theta = \frac{k}{3} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta) \cos \theta d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = -\frac{\pi}{2}$, then $u = \sin\left(-\frac{\pi}{2}\right) = -1$; if $\theta = \frac{\pi}{2}$, then $u = \sin \frac{\pi}{2} = 1$. So

$$\begin{aligned} M &= \frac{k}{3} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta) \cos \theta d\theta = \frac{k}{3} \int_{-1}^1 (1 - u^2) du = \frac{k}{3} \left[u - \frac{u^3}{3} \right]_{-1}^1 \\ &= \frac{k}{3} \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] = \frac{k}{3} \left[\frac{2}{3} - \left(-\frac{2}{3}\right) \right] = \frac{k}{3} \cdot \frac{4}{3} \\ &= \boxed{\frac{4k}{9}, \text{ where } k \text{ is the constant of proportionality}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x \rho(x, y) dA = \iint_R r \cos \theta \cdot kr \cdot r dr d\theta = k \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} r^3 \cos \theta dr d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \cos \theta \left[\frac{r^4}{4} \right]_0^{\cos \theta} d\theta = \frac{k}{4} \int_{-\pi/2}^{\pi/2} \cos \theta (\cos^4 \theta - 0) d\theta = \frac{k}{4} \int_{-\pi/2}^{\pi/2} \cos^5 \theta d\theta \\ &= \frac{k}{4} \int_{-\pi/2}^{\pi/2} (\cos^2 \theta)^2 \cos \theta d\theta = \frac{k}{4} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta)^2 \cos \theta d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the

limits of integration: if $\theta = -\frac{\pi}{2}$, then $u = \sin\left(-\frac{\pi}{2}\right) = -1$; if $\theta = \frac{\pi}{2}$, then $u = \sin\frac{\pi}{2} = 1$. So

$$\begin{aligned} M_y &= \frac{k}{4} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \theta)^2 \cos \theta \, d\theta = \frac{k}{4} \int_{-1}^1 (1 - u^2)^2 \, du \\ &= \frac{k}{4} \int_{-1}^1 (1 - 2u^2 + u^4) \, du = \frac{k}{4} \left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_{-1}^1 \\ &= \frac{k}{4} \left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) \right] = \frac{k}{4} \left[\frac{8}{15} - \left(-\frac{8}{15}\right) \right] = \frac{k}{4} \cdot \frac{16}{15} = \frac{4k}{15}, \end{aligned}$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{4k/15}{4k/9} = \frac{3}{5} = 0.6.$$

Similarly, the moment of mass about the x -axis is

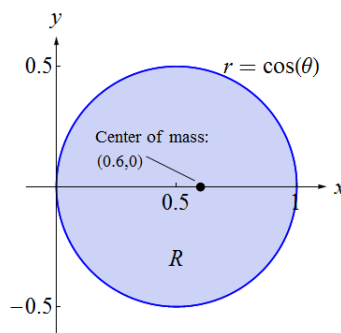
$$\begin{aligned} M_x &= \iint_R y \rho(x, y) \, dA = \iint_R r \sin \theta \cdot k r \cdot r \, dr \, d\theta = k \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} r^3 \sin \theta \, dr \, d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \sin \theta \left[\frac{r^4}{4} \right]_0^{\cos \theta} d\theta = \frac{k}{4} \int_{-\pi/2}^{\pi/2} \sin \theta (\cos^4 \theta - 0) \, d\theta = \frac{k}{4} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \sin \theta \, d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta \, d\theta$, so $\sin \theta \, d\theta = -du$. We also change the limits of integration: if $\theta = -\frac{\pi}{2}$, then $u = \cos\left(-\frac{\pi}{2}\right) = 0$; if $\theta = \frac{\pi}{2}$, then $u = \cos\frac{\pi}{2} = 0$. So

$$M_x = \frac{k}{4} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \sin \theta \, d\theta = \frac{k}{4} \int_0^0 u^4 (-du) = -\frac{k}{4} \int_0^0 u^4 \, du = 0,$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{0}{4k/9} = 0.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{3}{5}, 0\right) = (0.6, 0)$.



19. The region R lying inside the graph of $r = 2a \sin \theta$ and outside the graph of $r = a$ is shown in the figure below. The graphs intersect at the points where

$$2a \sin \theta = a, \quad \text{so } 2 \sin \theta = 1, \quad \text{so } \sin \theta = \frac{1}{2}, \quad \text{so } \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

So in polar coordinates, R is given by $a \leq r \leq 2a \sin \theta$, $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$. The mass density of the lamina is inversely proportional to the distance from the pole; that is, $\rho(r, \theta) = \frac{k}{r}$, where k is a constant

coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R \frac{k}{r} \cdot r dr d\theta = k \int_{\pi/6}^{5\pi/6} \int_a^{2a \sin \theta} dr d\theta = k \int_{\pi/6}^{5\pi/6} [r]_a^{2a \sin \theta} d\theta \\
 &= k \int_{\pi/6}^{5\pi/6} (2a \sin \theta - a) d\theta = ka \int_{\pi/6}^{5\pi/6} (2 \sin \theta - 1) d\theta = ka [-2 \cos \theta - \theta]_{\pi/6}^{5\pi/6} \\
 &= ka \left[\left(-2 \cos \frac{5\pi}{6} - \frac{5\pi}{6} \right) - \left(-2 \cos \frac{\pi}{6} - \frac{\pi}{6} \right) \right] = ka \left[\sqrt{3} - \frac{5\pi}{6} + \sqrt{3} + \frac{\pi}{6} \right] = \\
 &= ka \left(2\sqrt{3} - \frac{2\pi}{3} \right) = \boxed{2ka \left(\sqrt{3} - \frac{\pi}{3} \right) \approx 1.370ka}.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x \rho(x, y) dA = \iint_R r \cos \theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{\pi/6}^{5\pi/6} \int_a^{2a \sin \theta} r \cos \theta dr d\theta \\
 &= k \int_{\pi/6}^{5\pi/6} \cos \theta \left[\frac{r^2}{2} \right]_a^{2a \sin \theta} d\theta = \frac{k}{2} \int_{\pi/6}^{5\pi/6} \cos \theta [(2a \sin \theta)^2 - a^2] d\theta \\
 &= \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - 1) \cos \theta d\theta.
 \end{aligned}$$

To evaluate this integral, we make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = \frac{\pi}{6}$, then $u = \sin \frac{\pi}{6} = \frac{1}{2}$; if $\theta = \frac{5\pi}{6}$, then $u = \sin \frac{5\pi}{6} = \frac{1}{2}$. So

$$M_y = \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - 1) \cos \theta d\theta = \frac{ka^2}{2} \int_{1/2}^{1/2} (4u^2 - 1) du = 0,$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{0}{ka \left(2\sqrt{3} - \frac{2\pi}{3} \right)} = 0.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y \rho(x, y) dA = \iint_R r \sin \theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{\pi/6}^{5\pi/6} \int_a^{2a \sin \theta} r \sin \theta dr d\theta \\
 &= k \int_{\pi/6}^{5\pi/6} \sin \theta \left[\frac{r^2}{2} \right]_a^{2a \sin \theta} d\theta = \frac{k}{2} \int_{\pi/6}^{5\pi/6} \sin \theta [(2a \sin \theta)^2 - a^2] d\theta \\
 &= \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - 1) \sin \theta d\theta = \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} [4(1 - \cos^2 \theta) - 1] \sin \theta d\theta \\
 &= \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} (3 - 4 \cos^2 \theta) \sin \theta d\theta.
 \end{aligned}$$

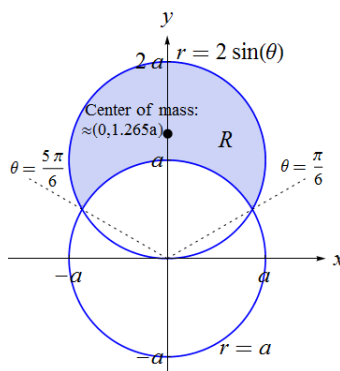
To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = \frac{\pi}{6}$, then $u = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$; if $\theta = \frac{5\pi}{6}$, then $u = \cos \frac{5\pi}{6} =$

$-\frac{\sqrt{3}}{2}$. So

$$\begin{aligned}
 M_x &= \frac{ka^2}{2} \int_{\pi/6}^{5\pi/6} (3 - 4\cos^2 \theta) \sin \theta \, d\theta = \frac{ka^2}{2} \int_{\sqrt{3}/2}^{-\sqrt{3}/2} (3 - 4u^2) (-du) \\
 &= -\frac{ka^2}{2} \int_{\sqrt{3}/2}^{-\sqrt{3}/2} (3 - 4u^2) \, du = \frac{ka^2}{2} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (3 - 4u^2) \, du = \frac{ka^2}{2} \cdot 2 \int_0^{\sqrt{3}/2} (3 - 4u^2) \, du \\
 &= ka^2 \int_0^{\sqrt{3}/2} (3 - 4u^2) \, du = ka^2 \left[3u - \frac{4}{3}u^3 \right]_0^{\sqrt{3}/2} = ka^2 \left\{ \left[3 \cdot \frac{\sqrt{3}}{2} - \frac{4}{3} \cdot \left(\frac{\sqrt{3}}{2} \right)^3 \right] - (0 - 0) \right\} \\
 &= ka^2 \left(\frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) = ka^2 \sqrt{3},
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{ka^2 \sqrt{3}}{ka(2\sqrt{3} - \frac{2\pi}{3})} = \frac{a\sqrt{3}}{2\sqrt{3} - \frac{2\pi}{3}} = \frac{3a\sqrt{3}}{6\sqrt{3} - 2\pi} = \frac{3\sqrt{3}a}{2(3\sqrt{3} - \pi)}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(0, \frac{3\sqrt{3}a}{2(3\sqrt{3} - \pi)} \right) \approx (0, 1.265a)$.



20. The region R lying outside the limaçon $r = 2 - \cos \theta$ and inside the circle $r = 4 \cos \theta$ is shown in the figure below. The graphs intersect at the points where

$$2 - \cos \theta = 4 \cos \theta, \quad \text{so} \quad 5 \cos \theta = 2, \quad \text{so} \quad \cos \theta = \frac{2}{5}, \quad \text{so} \quad \theta = \pm \cos^{-1} \left(\frac{2}{5} \right).$$

So in polar coordinates, R is given by $2 - \cos \theta \leq r \leq 4 \cos \theta$, $-\alpha \leq \theta \leq \alpha$, where we denoted $\alpha = \cos^{-1} \left(\frac{2}{5} \right)$ for brevity. The mass density of the lamina is inversely proportional to the distance

from the pole; that is, $\rho(r, \theta) = \frac{k}{r}$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) \, dA = \iint_R \frac{k}{r} \cdot r \, dr \, d\theta = k \int_{-\alpha}^{\alpha} \int_{2-\cos \theta}^{4 \cos \theta} dr \, d\theta = k \int_{-\alpha}^{\alpha} [r]_{2-\cos \theta}^{4 \cos \theta} d\theta \\
 &= k \int_{-\alpha}^{\alpha} [4 \cos \theta - (2 - \cos \theta)] \, d\theta = k \int_{-\alpha}^{\alpha} (5 \cos \theta - 2) \, d\theta = k [5 \sin \theta - 2\theta]_{-\alpha}^{\alpha} \\
 &= k [(5 \sin \alpha - 2\alpha) - (5 \sin(-\alpha) - 2(-\alpha))] = k [(5 \sin \alpha - 2\alpha) - (-5 \sin \alpha + 2\alpha)] \\
 &= k(10 \sin \alpha - 4\alpha) = k \left[10 \cdot \frac{\sqrt{21}}{5} - 4 \cos^{-1} \left(\frac{2}{5} \right) \right] = \boxed{k \left[2\sqrt{21} - 4 \cos^{-1} \left(\frac{2}{5} \right) \right] \approx 4.528k},
 \end{aligned}$$

where we used the value $\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{2}{5}\right)^2} = \sqrt{1 - \frac{4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$ (positive since $0 < \alpha < \frac{\pi}{2}$).

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x \rho(x, y) dA = \iint_R r \cos \theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{-\alpha}^{\alpha} \int_{2-\cos \theta}^{4 \cos \theta} r \cos \theta dr d\theta \\
 &= k \int_{-\alpha}^{\alpha} \cos \theta \left[\frac{r^2}{2} \right]_{2-\cos \theta}^{4 \cos \theta} d\theta = k \int_{-\alpha}^{\alpha} \frac{1}{2} \cos \theta [(4 \cos \theta)^2 - (2 - \cos \theta)^2] d\theta \\
 &= k \int_{-\alpha}^{\alpha} \frac{1}{2} \cos \theta (16 \cos^2 \theta - 4 + 4 \cos \theta - \cos^2 \theta) d\theta = k \int_{-\alpha}^{\alpha} \left(\frac{15}{2} \cos^3 \theta + 2 \cos^2 \theta - 2 \cos \theta \right) d\theta \\
 &= k \int_{-\alpha}^{\alpha} \left[\frac{15}{2} \cos^2 \theta \cos \theta + 2 \cdot \frac{1}{2} (1 + \cos(2\theta)) - 2 \cos \theta \right] d\theta \\
 &= k \int_{-\alpha}^{\alpha} \left[\frac{15}{2} (1 - \sin^2 \theta) \cos \theta + 1 + \cos(2\theta) - 2 \cos \theta \right] d\theta \\
 &= k \int_{-\alpha}^{\alpha} \left(\frac{15}{2} \cos \theta - \frac{15}{2} \sin^2 \theta \cos \theta + 1 + \cos(2\theta) - 2 \cos \theta \right) d\theta \\
 &= k \int_{-\alpha}^{\alpha} \left(1 + \frac{11}{2} \cos \theta + \cos(2\theta) - \frac{15}{2} \sin^2 \theta \cos \theta \right) d\theta \\
 &= k \left(\int_{-\alpha}^{\alpha} d\theta + \frac{11}{2} \int_{-\alpha}^{\alpha} \cos \theta d\theta + \int_{-\alpha}^{\alpha} \cos(2\theta) d\theta - \frac{15}{2} \int_{-\alpha}^{\alpha} \sin^2 \theta \cos \theta d\theta \right).
 \end{aligned}$$

The first three integrals in parentheses can be evaluated directly:

$$\begin{aligned}
 \int_{-\alpha}^{\alpha} d\theta &= [\theta]_{-\alpha}^{\alpha} = \alpha - (-\alpha) = 2\alpha = 2 \cos^{-1} \left(\frac{2}{5} \right); \\
 \frac{11}{2} \int_{-\alpha}^{\alpha} \cos \theta d\theta &= \frac{11}{2} [\sin \theta]_{-\alpha}^{\alpha} = \frac{11}{2} [\sin \alpha - \sin(-\alpha)] = \frac{11}{2} \cdot 2 \sin \alpha = 11 \sin \alpha = 11 \cdot \frac{\sqrt{21}}{5} = \frac{11\sqrt{21}}{5}; \\
 \int_{-\alpha}^{\alpha} \cos(2\theta) d\theta &= \left[\frac{\sin(2\theta)}{2} \right]_{-\alpha}^{\alpha} = \frac{1}{2} [\sin(2\alpha) - \sin(-2\alpha)] = \frac{1}{2} \cdot 2 \sin(2\alpha) = \sin(2\alpha) = \frac{4\sqrt{21}}{25},
 \end{aligned}$$

where in the last computation we used the value $\sin(2\alpha) = 2 \sin \alpha \cos \alpha = 2 \cdot \frac{\sqrt{21}}{5} \cdot \frac{2}{5} = \frac{4\sqrt{21}}{25}$.

To evaluate the last integral in parentheses, we make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$.

We also change the limits of integration: if $\theta = -\alpha$, then $u = \sin(-\alpha) = -\frac{\sqrt{21}}{5}$; if $\theta = \alpha$, then

$u = \sin \alpha = \frac{\sqrt{21}}{5}$. So

$$\begin{aligned}
 \frac{15}{2} \int_{-\alpha}^{\alpha} \sin^2 \theta \cos \theta d\theta &= \frac{15}{2} \int_{-\sqrt{21}/5}^{\sqrt{21}/5} u^2 du = \frac{15}{2} \cdot 2 \int_0^{\sqrt{21}/5} u^2 du = 15 \int_0^{\sqrt{21}/5} u^2 du \\
 &= 15 \left[\frac{u^3}{3} \right]_0^{\sqrt{21}/5} = 5 \left[\left(\frac{\sqrt{21}}{5} \right)^3 - 0^3 \right] = 5 \cdot \frac{21\sqrt{21}}{125} = \frac{21\sqrt{21}}{25}.
 \end{aligned}$$

Substituting these values back into the computation of M_y , we obtain

$$M_y = k \left[2 \cos^{-1} \left(\frac{2}{5} \right) + \frac{11\sqrt{21}}{5} + \frac{4\sqrt{21}}{25} - \frac{21\sqrt{21}}{25} \right] = k \left[2 \cos^{-1} \left(\frac{2}{5} \right) + \frac{38\sqrt{21}}{25} \right],$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{k \left[2 \cos^{-1} \left(\frac{2}{5} \right) + \frac{38\sqrt{21}}{25} \right]}{k \left[2\sqrt{21} - 4 \cos^{-1} \left(\frac{2}{5} \right) \right]} = \frac{2 \cos^{-1} \left(\frac{2}{5} \right) + \frac{38\sqrt{21}}{25}}{2\sqrt{21} - 4 \cos^{-1} \left(\frac{2}{5} \right)} = \frac{25 \cos^{-1} \left(\frac{2}{5} \right) + 19\sqrt{21}}{25\sqrt{21} - 50 \cos^{-1} \left(\frac{2}{5} \right)} \approx 2.050.$$

Similarly, the moment of mass about the x -axis is

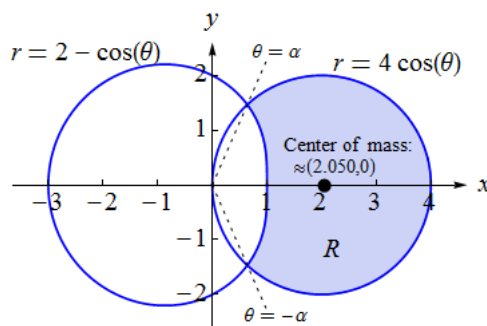
$$\begin{aligned} M_x &= \iint_R y \rho(x, y) dA = \iint_R r \sin \theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{-\alpha}^{\alpha} \int_{2-\cos \theta}^{4 \cos \theta} r \sin \theta dr d\theta \\ &= k \int_{-\alpha}^{\alpha} \sin \theta \left[\frac{r^2}{2} \right]_{2-\cos \theta}^{4 \cos \theta} d\theta = k \int_{-\alpha}^{\alpha} \frac{1}{2} \sin \theta [(4 \cos \theta)^2 - (2 - \cos \theta)^2] d\theta \\ &= k \int_{-\alpha}^{\alpha} \frac{1}{2} \sin \theta (16 \cos^2 \theta - 4 + 4 \cos \theta - \cos^2 \theta) d\theta = k \int_{-\alpha}^{\alpha} \left(\frac{15}{2} \cos^2 \theta + 2 \cos \theta - 2 \right) \sin \theta d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = -\alpha$, then $u = \cos(-\alpha) = \frac{2}{5}$; if $\theta = \alpha$, then $u = \cos \alpha = \frac{2}{5}$. So

$$\begin{aligned} M_x &= k \int_{-\alpha}^{\alpha} \left(\frac{15}{2} \cos^2 \theta + 2 \cos \theta - 2 \right) \sin \theta d\theta \\ &= k \int_{2/5}^{2/5} \left(\frac{15}{2} u^2 + 2u - 2 \right) (-du) = -k \int_{2/5}^{2/5} \left(\frac{15}{2} u^2 + 2u - 2 \right) du = 0, \end{aligned}$$

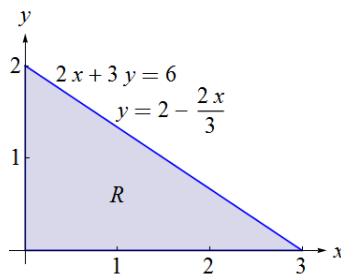
$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{0}{k [2\sqrt{21} - 4 \cos^{-1} (\frac{2}{5})]} = 0.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{25 \cos^{-1} (\frac{2}{5}) + 19\sqrt{21}}{25\sqrt{21} - 50 \cos^{-1} (\frac{2}{5})}, 0 \right) \approx (2.050, 0)$.



21. The region R enclosed by the lines $2x + 3y = 6$, $x = 0$ (the y -axis), and $y = 0$ (the x -axis) is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2 - \frac{2x}{3}$, $0 \leq x \leq 3$. Assuming that the lamina is homogeneous of constant mass density ρ , its moment of inertia about the x -axis can be found as a double integral:

$$\begin{aligned} I_x &= \iint_R y^2 \rho(x, y) dA = \iint_R y^2 \rho dA = \rho \int_0^3 \int_0^{2-2x/3} y^2 dy dx = \rho \int_0^3 \left[\frac{y^3}{3} \right]_0^{2-2x/3} dx \\ &= \rho \int_0^3 \left[\frac{(2 - \frac{2x}{3})^3}{3} - 0 \right] dx = \rho \int_0^3 \left(\frac{8}{3} - \frac{8x}{3} + \frac{8x^2}{9} - \frac{8x^3}{81} \right) dx \\ &= \rho \left[\frac{8x}{3} - \frac{4x^2}{3} + \frac{8x^3}{27} - \frac{2x^4}{81} \right]_0^3 = \rho [(8 - 12 + 8 - 2) - 0] = \boxed{2\rho}. \end{aligned}$$

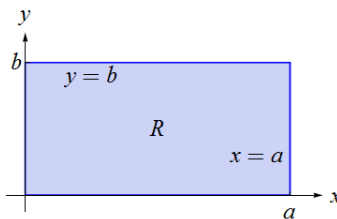


22. For the lamina from Problem 21 (see the figure above), its moment of inertia about the y -axis can be found as a double integral:

$$\begin{aligned}
 I_y &= \iint_R x^2 \rho(x, y) dA = \iint_R x^2 \rho dA = \rho \int_0^3 \int_0^{2-2x/3} x^2 dy dx = \rho \int_0^3 x^2 [y]_0^{2-2x/3} dx \\
 &= \rho \int_0^3 x^2 \left[\left(2 - \frac{2x}{3} \right) - 0 \right] dx = \rho \int_0^3 \left(2x^2 - \frac{2x^3}{3} \right) dx = \rho \left[\frac{2x^3}{3} - \frac{x^4}{6} \right]_0^3 \\
 &= \rho \left[\left(18 - \frac{27}{2} \right) - 0 \right] = \boxed{\frac{9\rho}{2}}.
 \end{aligned}$$

23. The rectangular region R enclosed in the first quadrant by the lines $x = a$ ($a > 0$) and $y = b$ ($b > 0$) is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = b$, $0 \leq x \leq a$. Assuming that the lamina is homogeneous of constant mass density ρ , its moment of inertia about the y -axis can be found as a double integral:

$$\begin{aligned}
 I_y &= \iint_R x^2 \rho(x, y) dA = \iint_R x^2 \rho dA = \rho \int_0^a \int_0^b x^2 dy dx = \rho \int_0^a x^2 [y]_0^b dx \\
 &= \rho \int_0^a x^2 (b - 0) dx = b\rho \int_0^a x^2 dx = b\rho \left[\frac{x^3}{3} \right]_0^a = b\rho \left(\frac{a^3}{3} - 0 \right) = \boxed{\frac{a^3 b \rho}{3}}.
 \end{aligned}$$



24. For the lamina from Problem 23 (see the figure above), its moment of inertia about the x -axis can be found as a double integral:

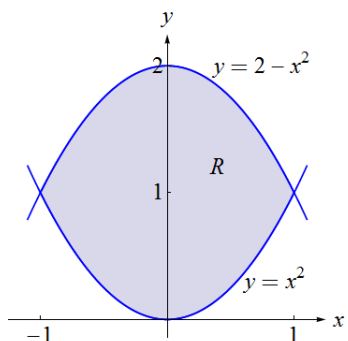
$$\begin{aligned}
 I_x &= \iint_R y^2 \rho(x, y) dA = \iint_R y^2 \rho dA = \rho \int_0^a \int_0^b y^2 dy dx = \rho \int_0^a \left[\frac{y^3}{3} \right]_0^b dx \\
 &= \rho \int_0^a \frac{b^3}{3} dx = \frac{b^3 \rho}{3} \int_0^a dx = \frac{b^3 \rho}{3} [x]_0^a = \frac{b^3 \rho}{3} (a - 0) = \boxed{\frac{ab^3 \rho}{3}}.
 \end{aligned}$$

25. The region R enclosed by the parabolas $y = x^2$ and $y = 2 - x^2$ is shown in the figure below. The graphs intersect at the points where

$$x^2 = 2 - x^2, \quad \text{so} \quad 2x^2 = 2, \quad \text{so} \quad x^2 = 1, \quad \text{so} \quad x = \pm 1.$$

Notice that R is an x -simple region: it is bounded between $y = x^2$ and $y = 2 - x^2$, $-1 \leq x \leq 1$. Assuming that the lamina is homogeneous of constant mass density ρ , its moment of inertia about the y -axis can be found as a double integral:

$$\begin{aligned} I_y &= \iint_R x^2 \rho(x, y) dA = \iint_R x^2 \rho dA = \rho \int_{-1}^1 \int_{x^2}^{2-x^2} x^2 dy dx = \rho \int_{-1}^1 x^2 [y]_{x^2}^{2-x^2} dx \\ &= \rho \int_{-1}^1 x^2 [(2 - x^2) - x^2] dx = \rho \int_{-1}^1 (2x^2 - 2x^4) dx = \rho \left[\frac{2x^3}{3} - \frac{2x^5}{5} \right]_{-1}^1 \\ &= \rho \left[\left(\frac{2}{3} - \frac{2}{5} \right) - \left(-\frac{2}{3} + \frac{2}{5} \right) \right] = \rho \left[\frac{4}{15} - \left(-\frac{4}{15} \right) \right] = \boxed{\frac{8\rho}{15}}. \end{aligned}$$

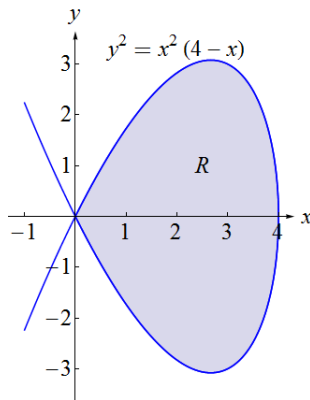


26. The graph of $y^2 = x^2(4 - x)$ and the region R it encloses in its loop are shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = -x\sqrt{4-x}$ and $y = x\sqrt{4-x}$, $0 \leq x \leq 4$. Assuming that the lamina is homogeneous of constant mass density ρ , its moment of inertia about the y -axis can be found as a double integral:

$$\begin{aligned} I_y &= \iint_R x^2 \rho(x, y) dA = \iint_R x^2 \rho dA = \rho \int_0^4 \int_{-x\sqrt{4-x}}^{x\sqrt{4-x}} x^2 dy dx = \rho \int_0^4 x^2 [y]_{-x\sqrt{4-x}}^{x\sqrt{4-x}} dx \\ &= \rho \int_0^4 x^2 [x\sqrt{4-x} - (-x\sqrt{4-x})] dx = \rho \int_0^4 x^2 \cdot 2x\sqrt{4-x} dx = 2\rho \int_0^4 x^3 \sqrt{4-x} dx. \end{aligned}$$

To evaluate this integral, we make a substitution $u = 4 - x$; then $du = -dx$, so $dx = -du$. We also change the limits of integration: if $x = 0$, then $u = 4 - 0 = 4$; if $x = 4$, then $u = 4 - 4 = 0$. So

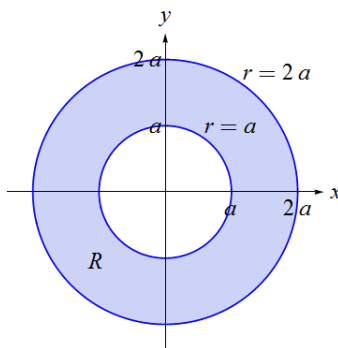
$$\begin{aligned} I_y &= 2\rho \int_0^4 x^3 \sqrt{4-x} dx = 2\rho \int_4^0 (4-u)^3 \sqrt{u} (-du) = -2\rho \int_4^0 (64 - 48u + 12u^2 - u^3) u^{1/2} du \\ &= 2\rho \int_0^4 (64u^{1/2} - 48u^{3/2} + 12u^{5/2} - u^{7/2}) du = 2\rho \left[64 \cdot \frac{u^{3/2}}{3/2} - 48 \cdot \frac{u^{5/2}}{5/2} + 12 \cdot \frac{u^{7/2}}{7/2} - \frac{u^{9/2}}{9/2} \right]_0^4 \\ &= 2\rho \left[\frac{128}{3} (\sqrt{u})^3 - \frac{96}{5} (\sqrt{u})^5 + \frac{24}{7} (\sqrt{u})^7 - \frac{2}{9} (\sqrt{u})^9 \right]_0^4 \\ &= 2\rho \left\{ \left[\frac{128}{3} (\sqrt{4})^3 - \frac{96}{5} (\sqrt{4})^5 + \frac{24}{7} (\sqrt{4})^7 - \frac{2}{9} (\sqrt{4})^9 \right] - (0 - 0 + 0 - 0) \right\} \\ &= 2\rho \left(\frac{128}{3} \cdot 8 - \frac{96}{5} \cdot 32 + \frac{24}{7} \cdot 128 - \frac{2}{9} \cdot 512 \right) = 2\rho \cdot \frac{16384}{315} = \boxed{\frac{32768\rho}{315}}. \end{aligned}$$



Applications and Extensions

27. If we place this lamina, which has the shape of a flat circular washer with inner radius a and outer radius $2a$, in the polar coordinate system so that its center is at the pole, then it occupies the region R given by $a \leq r \leq 2a$, $0 \leq \theta \leq 2\pi$; see the figure below. The mass density of the lamina is inversely proportional to the square of the distance from the center; that is, $\rho(r, \theta) = \frac{k}{r^2}$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) \, dA = \iint_R \frac{k}{r^2} \cdot r \, dr \, d\theta = k \int_0^{2\pi} \int_a^{2a} \frac{1}{r} \, dr \, d\theta \\
 &= k \int_0^{2\pi} [\ln |r|]_a^{2a} \, d\theta = k \int_0^{2\pi} (\ln 2a - \ln a) \, d\theta = k \int_0^{2\pi} \ln \left(\frac{2a}{a} \right) \, d\theta \\
 &= k \ln 2 \int_0^{2\pi} d\theta = k \ln 2 \cdot [\theta]_0^{2\pi} = k \ln 2 \cdot (2\pi - 0) = \boxed{2\pi k \ln 2}.
 \end{aligned}$$



28. For the lamina from Problem 27 (see the figure above), if the mass density is inversely proportional to the distance from the center, then it is $\rho(r, \theta) = \frac{k}{r}$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) \, dA = \iint_R \frac{k}{r} \cdot r \, dr \, d\theta = k \int_0^{2\pi} \int_a^{2a} dr \, d\theta = k \int_0^{2\pi} [r]_a^{2a} \, d\theta \\
 &= k \int_0^{2\pi} (2a - a) \, d\theta = ka \int_0^{2\pi} d\theta = ka [\theta]_0^{2\pi} = ka \cdot (2\pi - 0) = \boxed{2\pi ka}.
 \end{aligned}$$

29. The region R enclosed on the left by the line $x = a$ ($a > 0$) and on the right by the circle $r = 2a \cos \theta$ is shown in the figure below. The equation of the line can be converted to polar coordinates as

$$x = a, \quad \text{so} \quad r \cos \theta = a, \quad \text{so} \quad r = \frac{a}{\cos \theta}, \quad \text{or} \quad r = a \sec \theta.$$

To find the coordinates of their points of intersection, we set the equations of the line and the circle equal to each other:

$$a = 2a \cos \theta, \quad \text{so} \quad \cos \theta = \frac{1}{2}, \quad \text{so} \quad \theta = \pm \frac{\pi}{4}.$$

So in polar coordinates R is given by $a \sec \theta \leq r \leq 2a \cos \theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. The mass density of the lamina is inversely proportional to the distance from the y -axis; that is, $\rho(r, \theta) = \frac{k}{x} = \frac{k}{r \cos \theta} = \frac{k \sec \theta}{r}$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R \frac{k \sec \theta}{r} \cdot r dr d\theta = k \int_{-\pi/4}^{\pi/4} \int_{a \sec \theta}^{2a \cos \theta} \sec \theta dr d\theta = k \int_{-\pi/4}^{\pi/4} \sec \theta [r]_{a \sec \theta}^{2a \cos \theta} d\theta \\ &= k \int_{-\pi/4}^{\pi/4} \sec \theta (2a \cos \theta - a \sec \theta) d\theta = ka \int_{-\pi/4}^{\pi/4} (2 - \sec^2 \theta) d\theta = ka [2\theta - \tan \theta]_{-\pi/4}^{\pi/4} \\ &= ka \left\{ \left[2 \cdot \frac{\pi}{4} - \tan \frac{\pi}{4} \right] - \left[2 \cdot \left(-\frac{\pi}{4} \right) - \tan \left(-\frac{\pi}{4} \right) \right] \right\} \\ &= ka \left[\left(\frac{\pi}{2} - 1 \right) - \left(-\frac{\pi}{2} + 1 \right) \right] = \boxed{ka(\pi - 2) \approx 1.142ka}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x \rho(x, y) dA = \iint_R r \cos \theta \cdot \frac{k \sec \theta}{r} \cdot r dr d\theta = k \int_{-\pi/4}^{\pi/4} \int_{a \sec \theta}^{2a \cos \theta} r dr d\theta \\ &= k \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2} \right]_{a \sec \theta}^{2a \cos \theta} d\theta = k \int_{-\pi/4}^{\pi/4} \left(2a^2 \cos^2 \theta - \frac{1}{2} a^2 \sec^2 \theta \right) d\theta \\ &= ka^2 \int_{-\pi/4}^{\pi/4} \left[2 \cdot \frac{1}{2} (1 + \cos(2\theta)) - \frac{1}{2} \sec^2 \theta \right] d\theta = ka^2 \int_{-\pi/4}^{\pi/4} \left(1 + \cos(2\theta) - \frac{1}{2} \sec^2 \theta \right) d\theta \\ &= ka^2 \left[\theta + \frac{1}{2} \sin(2\theta) - \frac{1}{2} \tan \theta \right]_{-\pi/4}^{\pi/4} \\ &= ka^2 \left\{ \left[\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \tan \frac{\pi}{4} \right] - \left[-\frac{\pi}{4} + \frac{1}{2} \sin \left(-\frac{\pi}{2} \right) - \frac{1}{2} \tan \left(-\frac{\pi}{4} \right) \right] \right\} \\ &= ka^2 \left[\left(\frac{\pi}{4} + \frac{1}{2} - \frac{1}{2} \right) - \left(-\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2} \right) \right] = \frac{ka^2 \pi}{2}, \end{aligned}$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{ka^2 \pi / 2}{ka(\pi - 2)} = \frac{a\pi}{2(\pi - 2)} \approx 1.376a.$$

Similarly, the moment of mass about the x -axis is

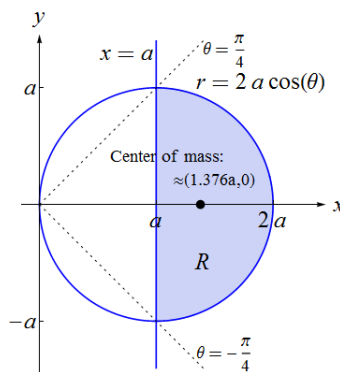
$$\begin{aligned} M_x &= \iint_R y \rho(x, y) dA = \iint_R r \sin \theta \cdot \frac{k \sec \theta}{r} \cdot r dr d\theta = k \int_{-\pi/4}^{\pi/4} \int_{a \sec \theta}^{2a \cos \theta} r \tan \theta dr d\theta \\ &= k \int_{-\pi/4}^{\pi/4} \tan \theta \left[\frac{r^2}{2} \right]_{a \sec \theta}^{2a \cos \theta} d\theta = k \int_{-\pi/4}^{\pi/4} \tan \theta \left(2a^2 \cos^2 \theta - \frac{1}{2} a^2 \sec^2 \theta \right) d\theta \\ &= ka^2 \int_{-\pi/4}^{\pi/4} \left(2 \sin \theta \cos \theta - \frac{\sin \theta}{2 \cos^3 \theta} \right) d\theta = ka^2 \int_{-\pi/4}^{\pi/4} \left(2 \cos \theta - \frac{1}{2 \cos^3 \theta} \right) \sin \theta d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = -\frac{\pi}{4}$, then $u = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; if $\theta = \frac{\pi}{4}$, then $u = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. So

$$\begin{aligned} M_x &= ka^2 \int_{-\pi/4}^{\pi/4} \left(2 \cos \theta - \frac{1}{2 \cos^3 \theta}\right) \sin \theta d\theta \\ &= ka^2 \int_{\sqrt{2}/2}^{\sqrt{2}/2} \left(2u - \frac{1}{2u^3}\right) (-du) = -ka^2 \int_{\sqrt{2}/2}^{\sqrt{2}/2} \left(2u - \frac{1}{2u^3}\right) du = 0, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{0}{ka(\pi - 2)} = 0.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{a\pi}{2(\pi - 2)}, 0\right) \approx (1.376a, 0)$.



30. The smaller region R cut from the circle $r = 6$ by the line $r \cos \theta = 3$, which is equivalent to $r = 3 \sec \theta$, is shown in the figure below. The graphs intersect at the points where

$$3 \sec \theta = 6, \quad \text{so} \quad \sec \theta = 2, \quad \text{so} \quad \cos \theta = \frac{1}{2}, \quad \text{so} \quad \theta = \pm \frac{\pi}{3}.$$

So in polar coordinates R is given by $3 \sec \theta \leq r \leq 6$, $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R \cos^2 \theta \cdot r dr d\theta = \int_{-\pi/3}^{\pi/3} \int_{3 \sec \theta}^6 \cos^2 \theta \cdot r dr d\theta \\ &= \int_{-\pi/3}^{\pi/3} \cos^2 \theta \left[\frac{r^2}{2} \right]_{3 \sec \theta}^6 d\theta = \int_{-\pi/3}^{\pi/3} \cos^2 \theta \left(18 - \frac{9}{2} \sec^2 \theta \right) d\theta = 9 \int_{-\pi/3}^{\pi/3} \left(2 \cos^2 \theta - \frac{1}{2} \right) d\theta \\ &= 9 \int_{-\pi/3}^{\pi/3} \left[2 \cdot \frac{1}{2} (1 + \cos(2\theta)) - \frac{1}{2} \right] d\theta = 9 \int_{-\pi/3}^{\pi/3} \left(1 + \cos(2\theta) - \frac{1}{2} \right) d\theta \\ &= 9 \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} + \cos(2\theta) \right) d\theta = 9 \left[\frac{1}{2} \theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/3}^{\pi/3} = \frac{9}{2} [\theta + \sin(2\theta)]_{-\pi/3}^{\pi/3} \\ &= \frac{9}{2} \left\{ \left[\frac{\pi}{3} + \sin \frac{2\pi}{3} \right] - \left[-\frac{\pi}{3} + \sin \left(-\frac{2\pi}{3} \right) \right] \right\} = \frac{9}{2} \left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \right] \\ &= \frac{9}{2} \left(\frac{2\pi}{3} + \sqrt{3} \right) = \boxed{3\pi + \frac{9\sqrt{3}}{2}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R r \cos \theta \cdot \cos^2 \theta \cdot r dr d\theta = \int_{-\pi/3}^{\pi/3} \int_{3 \sec \theta}^6 \cos^3 \theta \cdot r^2 dr d\theta \\ &= \int_{-\pi/3}^{\pi/3} \cos^3 \theta \left[\frac{r^3}{3} \right]_{3 \sec \theta}^6 d\theta = \int_{-\pi/3}^{\pi/3} \cos^3 \theta \cdot (72 - 9 \sec^3 \theta) d\theta \\ &= \int_{-\pi/3}^{\pi/3} (72 \cos^3 \theta - 9) d\theta = 72 \int_{-\pi/3}^{\pi/3} \cos^3 \theta d\theta - 9 \int_{-\pi/3}^{\pi/3} d\theta. \end{aligned}$$

The second integral can be evaluated directly:

$$9 \int_{-\pi/3}^{\pi/3} d\theta = 9 [\theta]_{-\pi/3}^{\pi/3} = 9 \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = 9 \cdot \frac{2\pi}{3} = 6\pi.$$

To evaluate the first integral, we use a trigonometric identity to transform the integrand: $\cos^3 \theta = \cos^2 \theta \cdot \cos \theta = (1 - \sin^2 \theta) \cos \theta$, and then make a substitution $u = \sin \theta$; then $du = \cos \theta d\theta$. We also change the limits of integration: if $\theta = -\frac{\pi}{3}$, then $u = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$; if $\theta = \frac{\pi}{3}$, then $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. So

$$\begin{aligned} 72 \int_{-\pi/3}^{\pi/3} \cos^3 \theta d\theta &= 72 \int_{-\pi/3}^{\pi/3} (1 - \sin^2 \theta) \cos \theta d\theta = 72 \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (1 - u^2) du = 72 \left[u - \frac{u^3}{3} \right]_{-\sqrt{3}/2}^{\sqrt{3}/2} \\ &= 72 \left[\left(\frac{\sqrt{3}}{2} - \frac{(\sqrt{3}/2)^3}{3} \right) - \left(-\frac{\sqrt{3}}{2} - \frac{(-\sqrt{3}/2)^3}{3} \right) \right] \\ &= 72 \left[\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} \right) - \left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} \right) \right] = 72 \cdot \frac{3\sqrt{3}}{4} = 54\sqrt{3}. \end{aligned}$$

Substituting these values back into the computation of M_y , we obtain

$$M_y = 72 \int_{-\pi/3}^{\pi/3} \cos^3 \theta d\theta - 9 \int_{-\pi/3}^{\pi/3} d\theta = 54\sqrt{3} - 6\pi,$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{54\sqrt{3} - 6\pi}{3\pi + \frac{9\sqrt{3}}{2}} = \frac{36\sqrt{3} - 4\pi}{2\pi + 3\sqrt{3}} \approx 4.337.$$

Similarly, the moment of mass about the x -axis is

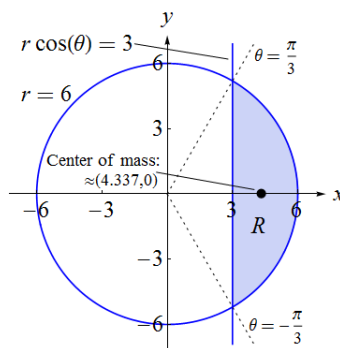
$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R r \sin \theta \cdot \cos^2 \theta \cdot r dr d\theta = \int_{-\pi/3}^{\pi/3} \int_{3 \sec \theta}^6 \sin \theta \cos^2 \theta \cdot r^2 dr d\theta \\ &= \int_{-\pi/3}^{\pi/3} \sin \theta \cos^2 \theta \left[\frac{r^3}{3} \right]_{3 \sec \theta}^6 d\theta = \int_{-\pi/3}^{\pi/3} \sin \theta \cos^2 \theta \cdot (72 - 9 \sec^3 \theta) d\theta \\ &= \int_{-\pi/3}^{\pi/3} (72 \cos^2 \theta - 9 \sec \theta) \sin \theta d\theta = 9 \int_{-\pi/3}^{\pi/3} \left(8 \cos^2 \theta - \frac{1}{\cos \theta} \right) \sin \theta d\theta. \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta d\theta$, so $\sin \theta d\theta = -du$. We also change the limits of integration: if $\theta = -\frac{\pi}{3}$, then $u = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$; if $\theta = \frac{\pi}{3}$, then $u = \cos\frac{\pi}{3} = \frac{1}{2}$. So

$$M_x = 9 \int_{-\pi/3}^{\pi/3} \left(8 \cos^2 \theta - \frac{1}{\cos \theta} \right) \sin \theta d\theta = 9 \int_{1/2}^{1/2} \left(8u^2 - \frac{1}{u} \right) (-du) = -9 \int_{1/2}^{1/2} \left(8u^2 - \frac{1}{u} \right) du = 0,$$

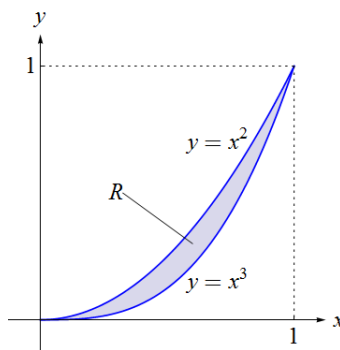
$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{0}{3\pi + \frac{9\sqrt{3}}{2}} = 0.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{36\sqrt{3} - 4\pi}{2\pi + 3\sqrt{3}}, 0 \right) \approx (4.337, 0)$.



31. The region R enclosed by $y = x^2$ and $y = x^3$ is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = x^3$ and $y = x^2$, $0 \leq x \leq 1$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R \sqrt{xy} dA = \int_0^1 \int_{x^3}^{x^2} x^{1/2} y^{1/2} dy dx = \int_0^1 x^{1/2} \left[\frac{y^{3/2}}{3/2} \right]_{x^3}^{x^2} dx \\ &= \int_0^1 \frac{2}{3} x^{1/2} \left[(x^2)^{3/2} - (x^3)^{3/2} \right] dx = \frac{2}{3} \int_0^1 x^{1/2} (x^3 - x^{9/2}) dx = \frac{2}{3} \int_0^1 (x^{7/2} - x^5) dx \\ &= \frac{2}{3} \left[\frac{x^{9/2}}{9/2} - \frac{x^6}{6} \right]_0^1 = \frac{2}{3} \left[\frac{2}{9} x^{9/2} - \frac{1}{6} x^6 \right]_0^1 = \frac{2}{3} \left[\left(\frac{2}{9} - \frac{1}{6} \right) - (0 - 0) \right] = \frac{2}{3} \cdot \frac{1}{18} = \boxed{\frac{1}{27}}. \end{aligned}$$



32. The region R lying inside the circle $r = 4 \cos \theta$ and outside the circle $r = 2\sqrt{3}$ is shown in the figure below. The graphs intersect at the points where

$$4 \cos \theta = 2\sqrt{3}, \quad \text{so} \quad \cos \theta = \frac{\sqrt{3}}{2}, \quad \text{so} \quad \theta = \pm \frac{\pi}{6}.$$

So in polar coordinates, R is given by $2\sqrt{3} \leq r \leq 4 \cos \theta$, $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. The mass density of the lamina is inversely proportional to the distance from the origin; that is, $\rho(r, \theta) = \frac{k}{r}$, where k is a constant

coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R \frac{k}{r} \cdot r dr d\theta = k \int_{-\pi/6}^{\pi/6} \int_{2\sqrt{3}}^{4\cos\theta} dr d\theta = k \int_{-\pi/6}^{\pi/6} [r]_{2\sqrt{3}}^{4\cos\theta} d\theta \\
 &= k \int_{-\pi/6}^{\pi/6} (4\cos\theta - 2\sqrt{3}) d\theta = 2k \int_{-\pi/6}^{\pi/6} (2\cos\theta - \sqrt{3}) d\theta = 2k \left[2\sin\theta - \sqrt{3}\theta \right]_{-\pi/6}^{\pi/6} \\
 &= 2k \left\{ \left[2\sin\frac{\pi}{6} - \sqrt{3} \cdot \frac{\pi}{6} \right] - \left[2\sin\left(-\frac{\pi}{6}\right) - \sqrt{3} \cdot \left(-\frac{\pi}{6}\right) \right] \right\} \\
 &= 2k \left[\left(1 - \frac{\pi\sqrt{3}}{6} \right) - \left(-1 + \frac{\pi\sqrt{3}}{6} \right) \right] = 2k \left(2 - \frac{\pi\sqrt{3}}{3} \right) = \frac{2k}{3} (6 - \pi\sqrt{3}).
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R r\cos\theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{-\pi/6}^{\pi/6} \int_{2\sqrt{3}}^{4\cos\theta} r\cos\theta dr d\theta \\
 &= k \int_{-\pi/6}^{\pi/6} \cos\theta \left[\frac{r^2}{2} \right]_{2\sqrt{3}}^{4\cos\theta} d\theta = k \int_{-\pi/6}^{\pi/6} \cos\theta (8\cos^2\theta - 6) d\theta \\
 &= 2k \int_{-\pi/6}^{\pi/6} [4(1 - \sin^2\theta) - 3] \cos\theta d\theta = 2k \int_{-\pi/6}^{\pi/6} (1 - 4\sin^2\theta) \cos\theta d\theta.
 \end{aligned}$$

To evaluate this integral, we make a substitution $u = \sin\theta$; then $du = \cos\theta d\theta$. We also change the limits of integration: if $\theta = -\frac{\pi}{6}$, then $u = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$; if $\theta = \frac{\pi}{6}$, then $u = \sin\frac{\pi}{6} = \frac{1}{2}$. So

$$\begin{aligned}
 M_y &= 2k \int_{-\pi/6}^{\pi/6} (1 - 4\sin^2\theta) \cos\theta d\theta = 2k \int_{-1/2}^{1/2} (1 - 4u^2) du \\
 &= 2k \left[u - \frac{4u^3}{3} \right]_{-1/2}^{1/2} = 2k \left[\left(\frac{1}{2} - \frac{1}{6} \right) - \left(-\frac{1}{2} + \frac{1}{6} \right) \right] = 2k \cdot \frac{2}{3} = \frac{4k}{3},
 \end{aligned}$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{4k/3}{(2k/3)(6 - \pi\sqrt{3})} = \frac{2}{6 - \pi\sqrt{3}} \approx 3.580.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R r\sin\theta \cdot \frac{k}{r} \cdot r dr d\theta = k \int_{-\pi/6}^{\pi/6} \int_{2\sqrt{3}}^{4\cos\theta} r\sin\theta dr d\theta \\
 &= k \int_{-\pi/6}^{\pi/6} \sin\theta \left[\frac{r^2}{2} \right]_{2\sqrt{3}}^{4\cos\theta} d\theta = k \int_{-\pi/6}^{\pi/6} \sin\theta (8\cos^2\theta - 6) d\theta = 2k \int_{-\pi/6}^{\pi/6} (4\cos^2\theta - 3) \sin\theta d\theta.
 \end{aligned}$$

To evaluate this integral, we make a substitution $u = \cos\theta$; then $du = -\sin\theta d\theta$, so $\sin\theta d\theta = -du$.

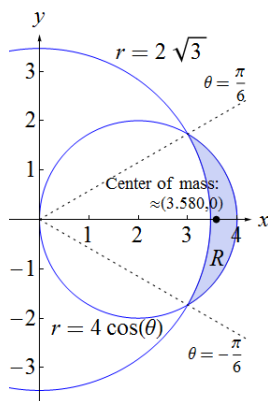
We also change the limits of integration: if $\theta = -\frac{\pi}{6}$, then $u = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$; if $\theta = \frac{\pi}{6}$, then

$$u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}. \text{ So}$$

$$M_x = 2k \int_{-\pi/6}^{\pi/6} (4\cos^2\theta - 3) \sin\theta d\theta = 2k \int_{\sqrt{3}/2}^{\sqrt{3}/2} (4u^2 - 3) (-du) = -2k \int_{\sqrt{3}/2}^{\sqrt{3}/2} (4u^2 - 3) du = 0,$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{0}{(2k/3)(6 - \pi\sqrt{3})} = 0.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{2}{6 - \pi\sqrt{3}}, 0 \right) \approx (3.580, 0)$.



33. The region R enclosed by the line $x = y - 2$ and the parabola $x = -y^2$ is shown in the figure below. The graphs intersect at the points where

$$y - 2 = -y^2, \quad \text{so} \quad y^2 + y - 2 = 0, \quad \text{so} \quad (y + 2)(y - 1) = 0, \quad \text{so} \quad y = -2, 1.$$

Notice that R is a y -simple region: it is bounded between $x = y - 2$ and $x = -y^2$, $-2 \leq y \leq 1$. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R x^2 dA = \int_{-2}^1 \int_{y-2}^{-y^2} x^2 dx dy = \int_{-2}^1 \left[\frac{x^3}{3} \right]_{y-2}^{-y^2} dy \\ &= \int_{-2}^1 \frac{1}{3} [(-y^2)^3 - (y-2)^3] dy = \int_{-2}^1 \frac{1}{3} (-y^6 - y^3 + 6y^2 - 12y + 8) dy \\ &= \int_{-2}^1 \left(-\frac{1}{3}y^6 - \frac{1}{3}y^3 + 2y^2 - 4y + \frac{8}{3} \right) dy = \left[-\frac{1}{21}y^7 - \frac{1}{12}y^4 + \frac{2}{3}y^3 - 2y^2 + \frac{8}{3}y \right]_{-2}^1 \\ &= \left[-\frac{1}{21} - \frac{1}{12} + \frac{2}{3} - 2 + \frac{8}{3} \right] - \left[-\frac{1}{21} \cdot (-128) - \frac{1}{12} \cdot 16 + \frac{2}{3} \cdot (-8) - 2 \cdot 4 + \frac{8}{3} \cdot (-2) \right] \\ &= -\frac{1}{21} - \frac{1}{12} + \frac{2}{3} - 2 + \frac{8}{3} - \frac{128}{21} + \frac{4}{3} + \frac{16}{3} + 8 + \frac{16}{3} = \boxed{\frac{423}{28}}. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x \cdot x^2 dA = \int_{-2}^1 \int_{y-2}^{-y^2} x^3 dx dy = \int_{-2}^1 \left[\frac{x^4}{4} \right]_{y-2}^{-y^2} dy \\ &= \int_{-2}^1 \frac{1}{4} [(-y^2)^4 - (y-2)^4] dy = \int_{-2}^1 \frac{1}{4} (y^8 - y^4 + 8y^3 - 24y^2 + 32y - 16) dy \\ &= \int_{-2}^1 \left(\frac{1}{4}y^8 - \frac{1}{4}y^4 + 2y^3 - 6y^2 + 8y - 4 \right) dy = \left[\frac{1}{36}y^9 - \frac{1}{20}y^5 + \frac{1}{2}y^4 - 2y^3 + 4y^2 - 4y \right]_{-2}^1 \\ &= \left[\frac{1}{36} - \frac{1}{20} + \frac{1}{2} - 2 + 4 - 4 \right] - \left[\frac{1}{36} \cdot (-512) - \frac{1}{20} \cdot (-32) + \frac{1}{2} \cdot 16 - 2 \cdot (-8) + 4 \cdot 4 - 4 \cdot (-2) \right] \\ &= \frac{1}{36} - \frac{1}{20} + \frac{1}{2} - 2 + 4 - 4 + \frac{128}{9} - \frac{8}{5} - 8 - 16 - 16 - 8 = -\frac{369}{10}, \end{aligned}$$

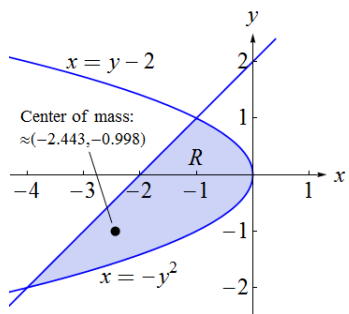
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{-369/10}{423/28} = -\frac{574}{235} \approx -2.443.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y \cdot x^2 dA = \int_{-2}^1 \int_{y-2}^{-y^2} yx^2 dx dy = \int_{-2}^1 y \left[\frac{x^3}{3} \right]_{y-2}^{-y^2} dy \\ &= \int_{-2}^1 \frac{1}{3} y [(-y^2)^3 - (y-2)^3] dy = \int_{-2}^1 \frac{1}{3} y (-y^6 - y^3 + 6y^2 - 12y + 8) dy \\ &= \int_{-2}^1 \left(-\frac{1}{3}y^7 - \frac{1}{3}y^4 + 2y^3 - 4y^2 + \frac{8}{3}y \right) dy = \left[-\frac{1}{24}y^8 - \frac{1}{15}y^5 + \frac{1}{2}y^4 - \frac{4}{3}y^3 + \frac{4}{3}y^2 \right]_{-2}^1 \\ &= \left[-\frac{1}{24} - \frac{1}{15} + \frac{1}{2} - \frac{4}{3} + \frac{4}{3} \right] - \left[-\frac{1}{24} \cdot 256 - \frac{1}{15} \cdot (-32) + \frac{1}{2} \cdot 16 - \frac{4}{3} \cdot (-8) + \frac{4}{3} \cdot 4 \right] \\ &= -\frac{1}{24} - \frac{1}{15} + \frac{1}{2} - \frac{4}{3} + \frac{4}{3} + \frac{32}{3} - \frac{32}{15} - 8 - \frac{32}{3} - \frac{16}{3} = -\frac{603}{40}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{-603/40}{423/28} = -\frac{469}{470} \approx -0.998.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \boxed{\left(-\frac{574}{235}, -\frac{469}{470} \right)}$.



34. The region R enclosed by the line $x - 2y + 1 = 0$, which is equivalent to $y = \frac{x+1}{2}$, and the parabola $y = x^2$ is shown in the figure below. The graphs intersect at the points where

$$x^2 = \frac{x+1}{2}, \quad \text{so } 2x^2 - x - 1 = 0, \quad \text{so } (2x+1)(x-1) = 0, \quad \text{so } x = -\frac{1}{2}, 1.$$

Notice that R is an x -simple region: it is bounded between $y = x^2$ and $y = \frac{1}{2}(x+1)$, $-\frac{1}{2} \leq x \leq 1$.

Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R (2x + 8y + 2) dA \\
 &= \int_{-1/2}^1 \int_{x^2}^{(x+1)/2} (2x + 8y + 2) dy dx = \int_{-1/2}^1 [2xy + 4y^2 + 2y]_{x^2}^{(x+1)/2} dx \\
 &= \int_{-1/2}^1 \left\{ \left[2x \cdot \frac{x+1}{2} + 4 \left(\frac{x+1}{2} \right)^2 + 2 \cdot \frac{x+1}{2} \right] - \left[2x \cdot x^2 + 4(x^2)^2 + 2 \cdot x^2 \right] \right\} dx \\
 &= \int_{-1/2}^1 [(x^2 + x + x^2 + 2x + 1 + x + 1) - (2x^3 + 4x^4 + 2x^2)] dx \\
 &= \int_{-1/2}^1 (-4x^4 - 2x^3 + 4x + 2) dx = \left[-\frac{4}{5}x^5 - \frac{1}{2}x^4 + 2x^2 + 2x \right]_{-1/2}^1 \\
 &= \left[-\frac{4}{5} - \frac{1}{2} + 2 + 2 \right] - \left[-\frac{4}{5} \cdot \left(-\frac{1}{32} \right) - \frac{1}{2} \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 2 \cdot \left(-\frac{1}{2} \right) \right] \\
 &= -\frac{4}{5} - \frac{1}{2} + 2 + 2 - \frac{1}{40} + \frac{1}{32} - \frac{1}{2} + 1 = \frac{513}{160}.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R x(2x + 8y + 2) dA \\
 &= \int_{-1/2}^1 \int_{x^2}^{(x+1)/2} (2x^2 + 8xy + 2x) dy dx = \int_{-1/2}^1 [2x^2y + 4xy^2 + 2xy]_{x^2}^{(x+1)/2} dx \\
 &= \int_{-1/2}^1 \left\{ \left[2x^2 \cdot \frac{x+1}{2} + 4x \left(\frac{x+1}{2} \right)^2 + 2x \cdot \frac{x+1}{2} \right] - \left[2x^2 \cdot x^2 + 4x(x^2)^2 + 2x \cdot x^2 \right] \right\} dx \\
 &= \int_{-1/2}^1 [(x^3 + x^2 + x^3 + 2x^2 + x + x^2 + x) - (2x^4 + 4x^5 + 2x^3)] dx \\
 &= \int_{-1/2}^1 (-4x^5 - 2x^4 + 4x^2 + 2x) dx = \left[-\frac{2}{3}x^6 - \frac{2}{5}x^5 + \frac{4}{3}x^3 + x^2 \right]_{-1/2}^1 \\
 &= \left[-\frac{2}{3} - \frac{2}{5} + \frac{4}{3} + 1 \right] - \left[-\frac{2}{3} \cdot \frac{1}{64} - \frac{2}{5} \cdot \left(-\frac{1}{32} \right) + \frac{4}{3} \cdot \left(-\frac{1}{8} \right) + \frac{1}{4} \right] \\
 &= -\frac{2}{3} - \frac{2}{5} + \frac{4}{3} + 1 + \frac{1}{96} - \frac{1}{80} + \frac{1}{6} - \frac{1}{4} = \frac{189}{160},
 \end{aligned}$$

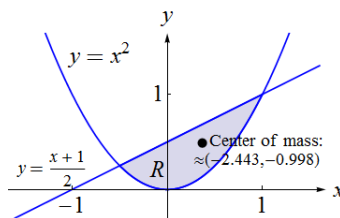
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{189/160}{513/160} = \frac{7}{19} \approx 0.368.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R y(2x + 8y + 2) dA \\
 &= \int_{-1/2}^1 \int_{x^2}^{(x+1)/2} (2xy + 8y^2 + 2y) dy dx = \int_{-1/2}^1 \left[xy^2 + \frac{8}{3}y^3 + y^2 \right]_{x^2}^{(x+1)/2} dx \\
 &= \int_{-1/2}^1 \left\{ \left[x \cdot \left(\frac{x+1}{2} \right)^2 + \frac{8}{3} \left(\frac{x+1}{2} \right)^3 + \left(\frac{x+1}{2} \right)^2 \right] - \left[x \cdot (x^2)^2 + \frac{8}{3} (x^2)^3 + (x^2)^2 \right] \right\} dx \\
 &= \int_{-1/2}^1 \left[\left(\frac{1}{4}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4} \right) - \left(x^5 + \frac{8}{3}x^6 + x^4 \right) \right] dx \\
 &= \int_{-1/2}^1 \left(-\frac{8}{3}x^6 - x^5 - x^4 + \frac{7}{12}x^3 + \frac{7}{4}x^2 + \frac{7}{4}x + \frac{7}{12} \right) dx \\
 &= \left[-\frac{8}{21}x^7 - \frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{7}{48}x^4 + \frac{7}{12}x^3 + \frac{7}{8}x^2 + \frac{7}{12}x \right]_{-1/2}^1 \\
 &= \left[-\frac{8}{21} - \frac{1}{6} - \frac{1}{5} + \frac{7}{48} + \frac{7}{12} + \frac{7}{8} + \frac{7}{12} \right] \\
 &\quad - \left[-\frac{8}{21} \cdot \left(-\frac{1}{128} \right) - \frac{1}{6} \cdot \frac{1}{64} - \frac{1}{5} \cdot \left(-\frac{1}{32} \right) + \frac{7}{48} \cdot \frac{1}{16} + \frac{7}{12} \cdot \left(-\frac{1}{8} \right) + \frac{7}{8} \cdot \frac{1}{4} + \frac{7}{12} \cdot \left(-\frac{1}{2} \right) \right] \\
 &= -\frac{8}{21} - \frac{1}{6} - \frac{1}{5} + \frac{7}{48} + \frac{7}{12} + \frac{7}{8} + \frac{7}{12} - \frac{1}{336} + \frac{1}{384} - \frac{1}{160} - \frac{7}{768} + \frac{7}{96} - \frac{7}{32} + \frac{7}{24} = \frac{14067}{8960},
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{14067/8960}{513/160} = \frac{521}{1064} \approx 0.490.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{7}{19}, \frac{521}{1064} \right) \approx (0.368, 0.490)$.



35. The region R enclosed by the parabola $bx^2 = a^2y$, or $y = \frac{b}{a^2}x^2$, and the line $ay = bx$, or $y = \frac{b}{a}x$, is shown in the figure below. The graphs intersect at the points where

$$\frac{b}{a^2}x^2 = \frac{b}{a}x \quad \text{so} \quad \frac{b}{a^2}x^2 - \frac{b}{a}x = 0, \quad \text{so} \quad \frac{b}{a}x \left(\frac{1}{a}x - 1 \right) = 0, \quad \text{so} \quad x = 0, a.$$

Notice that R is an x -simple region: it is bounded between $y = \frac{b}{a^2}x^2$ and $y = \frac{b}{a}x$, $0 \leq x \leq a$. Assuming that the lamina is homogeneous with constant mass density ρ , its mass M can be found as a double

integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R \rho dA = \rho \int_0^a \int_{bx^2/a^2}^{bx/a} dy dx = \rho \int_0^a [y]_{bx^2/a^2}^{bx/a} dx \\
 &= \rho \int_0^a \left(\frac{bx}{a} - \frac{bx^2}{a^2} \right) dx = \rho \int_0^a \frac{b}{a} \left(x - \frac{x^2}{a} \right) dx = \frac{b\rho}{a} \left[\frac{x^2}{2} - \frac{x^3}{3a} \right]_0^a \\
 &= \frac{b\rho}{a} \left[\left(\frac{a^2}{2} - \frac{a^3}{3a} \right) - (0 - 0) \right] = \frac{b\rho}{a} \cdot a^2 \left(\frac{1}{2} - \frac{1}{3} \right) = ab\rho \cdot \frac{1}{6} = \frac{ab\rho}{6}.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R x\rho dA = \rho \int_0^a \int_{bx^2/a^2}^{bx/a} x dy dx = \rho \int_0^a [xy]_{bx^2/a^2}^{bx/a} dx \\
 &= \rho \int_0^a x \left(\frac{bx}{a} - \frac{bx^2}{a^2} \right) dx = \rho \int_0^a \frac{b}{a} \left(x^2 - \frac{x^3}{a} \right) dx = \frac{b\rho}{a} \left[\frac{x^3}{3} - \frac{x^4}{4a} \right]_0^a \\
 &= \frac{b\rho}{a} \left[\left(\frac{a^3}{3} - \frac{a^4}{4a} \right) - (0 - 0) \right] = \frac{b\rho}{a} \cdot a^3 \left(\frac{1}{3} - \frac{1}{4} \right) = a^2 b\rho \cdot \frac{1}{12} = \frac{a^2 b\rho}{12},
 \end{aligned}$$

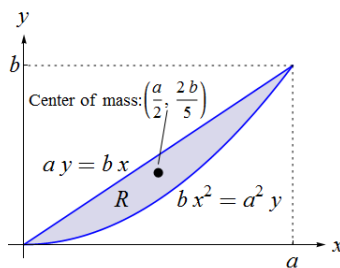
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{a^2 b\rho/12}{ab\rho/6} = \frac{a}{2}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R y\rho dA = \rho \int_0^a \int_{bx^2/a^2}^{bx/a} y dy dx = \rho \int_0^a \left[\frac{y^2}{2} \right]_{bx^2/a^2}^{bx/a} dx \\
 &= \rho \int_0^a \left(\frac{(bx/a)^2}{2} - \frac{(bx^2/a^2)^2}{2} \right) dx = \frac{\rho}{2} \int_0^a \left(\frac{b^2 x^2}{a^2} - \frac{b^2 x^4}{a^4} \right) dx \\
 &= \frac{\rho}{2} \int_0^a \frac{b^2}{a^2} \left(x^2 - \frac{x^4}{a^2} \right) dx = \frac{b^2 \rho}{2a^2} \left[\frac{x^3}{3} - \frac{x^5}{5a^2} \right]_0^a \\
 &= \frac{b^2 \rho}{2a^2} \left[\left(\frac{a^3}{3} - \frac{a^5}{5a^2} \right) - (0 - 0) \right] = \frac{b^2 \rho}{2a^2} \cdot a^3 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{ab^2 \rho}{2} \cdot \frac{2}{15} = \frac{ab^2 \rho}{15},
 \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{ab^2 \rho/15}{ab\rho/6} = \frac{2b}{5}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{2b}{5} \right)$.



36. To find the mass and the moments of mass of the lamina, we will need to set up certain double integrals, and we can evaluate each of them using the computer algebra system *Mathematica*.

The region R enclosed by the graph of $y = \ln x$, the x -axis, and the vertical line $x = e^2$ is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = \ln x$, $1 \leq x \leq e^2$. Assuming that the lamina is homogeneous with constant mass density ρ , its mass M can be found as a double integral:

$$M = \iint_R \rho(x, y) dA = \iint_R \rho dA = \rho \int_1^{e^2} \int_0^{\ln x} dy dx = (1 + e^2)\rho.$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$M_y = \iint_R x\rho(x, y) dA = \iint_R x\rho dA = \rho \int_1^{e^2} \int_0^{\ln x} x dy dx = \frac{1}{4}(1 + 3e^4)\rho.$$

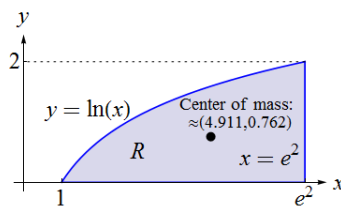
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{\frac{1}{4}(1 + 3e^4)\rho}{(1 + e^2)\rho} = \frac{1 + 3e^4}{4(1 + e^2)} \approx 4.911.$$

Similarly, the moment of mass about the x -axis is

$$M_x = \iint_R y\rho(x, y) dA = \iint_R y\rho dA = \rho \int_1^{e^2} \int_0^{\ln x} y dy dx = (-1 + e^2)\rho.$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(-1 + e^2)\rho}{(1 + e^2)\rho} = \frac{-1 + e^2}{1 + e^2} \approx 0.762.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{1 + 3e^4}{4(1 + e^2)}, \frac{-1 + e^2}{1 + e^2} \right) \approx (4.911, 0.762)$.



37. (a) If we place this washer, whose inner radius is a and outer radius is b , in the polar coordinate system so that its center is at the pole, then it occupies the region R given by $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$; see the figure below. The mass density of the washer is inversely proportional to the square of the distance from the center; that is, $\rho(r, \theta) = \frac{k}{r^2}$, where k is a constant coefficient of proportionality. Then the mass M of the washer can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R \frac{k}{r^2} \cdot r dr d\theta = k \int_0^{2\pi} \int_a^b \frac{1}{r} dr d\theta \\ &= k \int_0^{2\pi} [\ln |r|]_a^b d\theta = k \int_0^{2\pi} (\ln b - \ln a) d\theta = k \int_0^{2\pi} \ln \left(\frac{b}{a} \right) d\theta \\ &= k \ln \left(\frac{b}{a} \right) \int_0^{2\pi} d\theta = k \ln \left(\frac{b}{a} \right) [\theta]_0^{2\pi} = k \ln \left(\frac{b}{a} \right) \cdot (2\pi - 0) = \boxed{2\pi k \ln \left(\frac{b}{a} \right)}. \end{aligned}$$

If we keep b fixed, but let $a \rightarrow 0^+$, then $\frac{b}{a} \rightarrow \infty$, and so

$$\lim_{a \rightarrow 0^+} M = \lim_{a \rightarrow 0^+} 2\pi k \ln \left(\frac{b}{a} \right) = 2\pi k \lim_{a \rightarrow 0^+} \ln \left(\frac{b}{a} \right) = \infty.$$

In other words, if the radius a of this washer's hole decreases getting closer and closer to 0, its mass will be growing arbitrarily large.

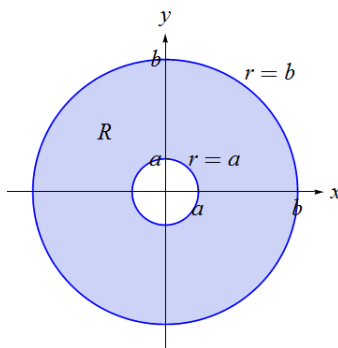
- (b) The moment of inertia of this washer about its center, that is about the origin, is

$$\begin{aligned} I_O &= \iint_R (x^2 + y^2) \rho(x, y) dA = \iint_R r^2 \cdot \frac{k}{r^2} \cdot r dr d\theta = k \int_0^{2\pi} \int_a^b r dr d\theta \\ &= k \int_0^{2\pi} \left[\frac{r^2}{2} \right]_a^b d\theta = \frac{k}{2} \int_0^{2\pi} (b^2 - a^2) d\theta = \frac{k(b^2 - a^2)}{2} \int_0^{2\pi} d\theta \\ &= \frac{k(b^2 - a^2)}{2} [\theta]_0^{2\pi} = \frac{k(b^2 - a^2)}{2} \cdot (2\pi - 0) = \boxed{\pi k(b^2 - a^2)}. \end{aligned}$$

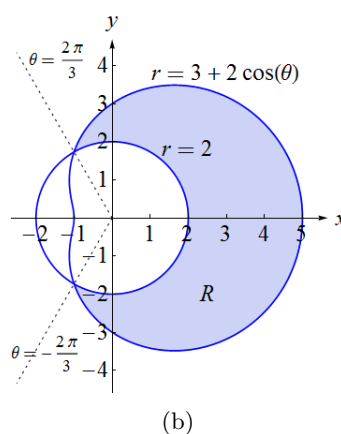
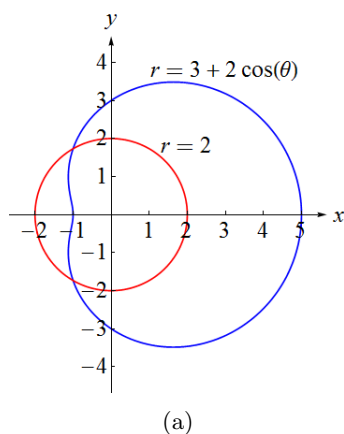
If we keep b fixed, but let $a \rightarrow 0^+$, then

$$\lim_{a \rightarrow 0^+} I_O = \lim_{a \rightarrow 0^+} \pi k(b^2 - a^2) = \pi k b^2.$$

In other words, if the radius a of this washer's hole decreases getting closer and closer to 0, the moment of inertia of the washer about its center will remain finite approaching the value of $\pi k b^2$.



38. (a) The graphs of the limaçon $r = 3 + 2 \cos \theta$ and the circle $r = 2$ are shown in the figure (a) below.



- (b) The region R lying inside the limaçon $r = 3 + 2 \cos \theta$ and outside the circle $r = 2$ is shown in the figure (b) above. The graphs intersect at the points where

$$3 + 2 \cos \theta = 2, \quad \text{so} \quad 2 \cos \theta = -1, \quad \text{so} \quad \cos \theta = -\frac{1}{2}, \quad \text{so} \quad \theta = \pm \frac{2\pi}{3}.$$

So R is given by $2 \leq r \leq 3 + 2 \cos \theta$, $-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}$. The mass density of the lamina is inversely proportional to the square of the distance from the origin (pole); that is, $\rho(r, \theta) = \frac{k}{r^2}$, where k is a constant coefficient of proportionality. Then the moment of inertia of the lamina about the x -axis can be found as a double integral:

$$I_x = \iint_R y^2 \rho(x, y) dA = \iint_R r^2 \sin^2 \theta \cdot \frac{k}{r^2} \cdot r dr d\theta = \boxed{k \int_{-2\pi/3}^{2\pi/3} \int_2^{3+2\cos\theta} r \sin^2 \theta dr d\theta}.$$

(c) Using the computer algebra system *Mathematica*, we find that the value of this double integral is

$$I_x = k \int_{-2\pi/3}^{2\pi/3} \int_2^{3+2\cos\theta} r \sin^2 \theta dr d\theta = \boxed{k \left(\frac{33\sqrt{3}}{16} + 2\pi \right)}.$$

39. Place this rectangular lamina on the coordinate plane so that one of its vertices is at the origin and two of its sides lie on the positive x - and y -axes. If the width of the lamina is a and the height is b , the region R it occupies is given by $0 \leq x \leq a$, $0 \leq y \leq b$; see the figure below. The area of this rectangle is $A = ab$. Assuming the lamina is homogeneous, its mass density is ρ , a constant, and the mass of the lamina is $M = \rho A = \rho \cdot ab = ab\rho$.

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x \rho(x, y) dA = \iint_R x \rho dy dx = \rho \int_0^a \int_0^b x dy dx = \rho \int_0^a x [y]_0^b dx \\ &= \rho \int_0^a x(b-0) dx = b\rho \int_0^a x dx = b\rho \left[\frac{x^2}{2} \right]_0^a = b\rho \left(\frac{a^2}{2} - 0 \right) = \frac{a^2 b \rho}{2}, \end{aligned}$$

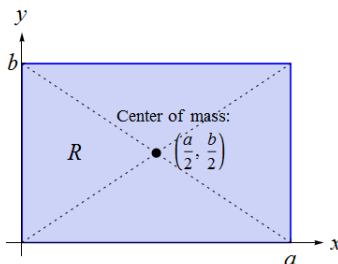
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{a^2 b \rho / 2}{ab\rho} = \frac{a}{2}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y \rho(x, y) dA = \iint_R y \rho dy dx = \rho \int_0^a \int_0^b y dy dx = \rho \int_0^a \left[\frac{y^2}{2} \right]_0^b dx \\ &= \rho \int_0^a \left(\frac{b^2}{2} - 0 \right) dx = \frac{b^2 \rho}{2} \int_0^a dx = \frac{b^2 \rho}{2} [x]_0^a = \frac{b^2 \rho}{2} (a - 0) = \frac{ab^2 \rho}{2}, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{ab^2 \rho / 2}{ab\rho} = \frac{b}{2}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{2} \right)$, which is precisely the geometrical center of the rectangle, that is the point where its diagonals intersect.

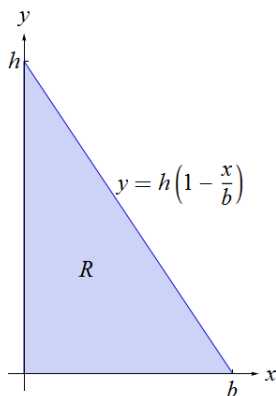


40. Place this lamina on the coordinate plane so that its right-angle vertex is at the origin, the base of length b lies on the positive x -axis, and the height of length h lies on the positive y -axis. The hypotenuse of the triangle has $(b, 0)$ and $(0, h)$ as its intercepts, so its equation is $\frac{x}{b} + \frac{y}{h} = 1$ or $y = h\left(1 - \frac{x}{b}\right)$. So the region R that the lamina occupies is given by $0 \leq y \leq h\left(1 - \frac{x}{b}\right)$, $0 \leq x \leq b$; see the figure below. The area of the triangle is $A = \frac{1}{2}bh$. Assuming the lamina is homogeneous, its mass density is ρ , a constant, and the mass of the lamina is $m = \rho A = \rho \cdot \frac{1}{2}bh = \frac{bh\rho}{2}$.

The moment of inertia of the lamina about the base, that is about the x -axis, can be found as a double integral:

$$\begin{aligned}
 I_x &= \iint_R y^2 \rho(x, y) dA = \iint_R y^2 \rho dy dx = \rho \int_0^b \int_0^{h(1-x/b)} y^2 dy dx = \rho \int_0^b \left[\frac{y^3}{3} \right]_0^{h(1-x/b)} dx \\
 &= \frac{\rho}{3} \int_0^b \left\{ \left[h\left(1 - \frac{x}{b}\right) \right]^3 - 0 \right\} dx = \frac{\rho}{3} \int_0^b h^3 \left(1 - \frac{x}{b}\right)^3 dx = \frac{h^3 \rho}{3} \int_0^b \left(1 - \frac{3x}{b} + \frac{3x^2}{b^2} - \frac{x^3}{b^3}\right) dx \\
 &= \frac{h^3 \rho}{3} \left[x - \frac{3x^2}{2b} + \frac{x^3}{b^2} - \frac{x^4}{4b^3} \right]_0^b = \frac{h^3 \rho}{3} \left[\left(b - \frac{3b^2}{2b} + \frac{b^3}{b^2} - \frac{b^4}{4b^3} \right) - 0 \right] \\
 &= \frac{h^3 \rho}{3} \cdot b \left(1 - \frac{3}{2} + 1 - \frac{1}{4} \right) = \frac{bh^3 \rho}{3} \cdot \frac{1}{4} = \frac{bh^3 \rho}{12} = \frac{bh\rho}{2} \cdot \frac{h^2}{6} = m \cdot \frac{h^2}{6} = \frac{1}{6}mh^2,
 \end{aligned}$$

as desired.



Challenge Problems

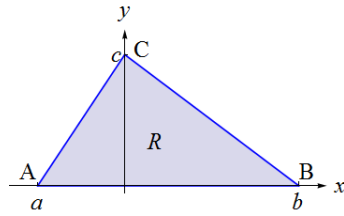
41. Consider a triangular lamina occupying the region R positioned as shown in the figure (a) below. The vertices of the triangle are $A(a, 0)$, $B(b, 0)$, and $C(0, c)$.

First, let us find the point $M(x_M, y_M)$ of intersection of the medians of the triangle. The midpoint of the base AB is at the point $K\left(\frac{a+b}{2}, 0\right)$. We know from geometry that all three medians in a triangle intersect at a single point, which divides each median in the ratio of 2 : 1 counting from the vertex. So the point $M(x_M, y_M)$ divides the segment CK in the ratio of 2 : 1; see figure (b) below. In other words, the point M lies on the segment CK so that MK is one third of the entire segment CK ,

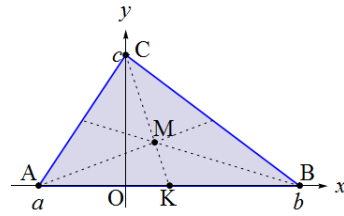
and so the coordinates of M satisfy

$$\begin{aligned}
 x_M - x_K &= \frac{1}{3}(x_C - x_K) & y_M - y_K &= \frac{1}{3}(y_C - y_K) \\
 x_M - \frac{a+b}{2} &= \frac{1}{3}\left(0 - \frac{a+b}{2}\right) & y_M - 0 &= \frac{1}{3}(c - 0) \\
 x_M &= \frac{a+b}{2} - \frac{1}{3} \cdot \frac{a+b}{2} & y_M &= \frac{1}{3} \cdot c \\
 x_M &= \frac{a+b}{2} \cdot \frac{2}{3} & y_M &= \frac{c}{3} \\
 x_M &= \frac{a+b}{3}
 \end{aligned}$$

The medians of this triangle intersect at the point $M\left(\frac{a+b}{3}, \frac{c}{3}\right)$.



(a)



(b)

Now we proceed to finding the center of mass of this triangular lamina. Notice that R is a y -simple region. The side AC of the triangle has $(a, 0)$ and $(0, c)$ as its intercepts, so its equation is $\frac{x}{a} + \frac{y}{c} = 1$ or $x = a\left(1 - \frac{y}{c}\right)$. The side BC of the triangle has $(b, 0)$ and $(0, c)$ as its intercepts, so its equation is $\frac{x}{b} + \frac{y}{c} = 1$ or $x = b\left(1 - \frac{y}{c}\right)$. So the region R that the lamina occupies is given by $a\left(1 - \frac{y}{c}\right) \leq x \leq b\left(1 - \frac{y}{c}\right)$, $0 \leq y \leq c$. The area of the triangle is $A = \frac{1}{2} \cdot AB \cdot CO = \frac{1}{2}(b-a)c = \frac{(b-a)c}{2}$. Assuming the lamina is homogeneous, its mass density is ρ , a constant, and the mass of the lamina is $M = \rho A = \rho \cdot \frac{(b-a)c}{2} = \frac{(b-a)c\rho}{2}$.

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned}
 M_y &= \iint_R x\rho(x, y) dA = \iint_R x\rho dx dy = \rho \int_0^c \int_{a(1-y/c)}^{b(1-y/c)} x dx dy = \rho \int_0^c \left[\frac{x^2}{2} \right]_{a(1-y/c)}^{b(1-y/c)} dy \\
 &= \frac{\rho}{2} \int_0^c \left[b^2 \left(1 - \frac{y}{c}\right)^2 - a^2 \left(1 - \frac{y}{c}\right)^2 \right] dy = \frac{\rho}{2} \int_0^c (b^2 - a^2) \left(1 - \frac{y}{c}\right)^2 dy \\
 &= \frac{(b^2 - a^2)\rho}{2} \int_0^c \left(1 - \frac{2y}{c} + \frac{y^2}{c^2}\right) dy = \frac{(b^2 - a^2)\rho}{2} \left[y - \frac{y^2}{c} + \frac{y^3}{3c^2} \right]_0^c \\
 &= \frac{(b^2 - a^2)\rho}{2} \left[\left(c - \frac{c^2}{c} + \frac{c^3}{3c^2} \right) - 0 \right] = \frac{(b^2 - a^2)\rho}{2} \cdot \frac{c}{3} = \frac{(b^2 - a^2)c\rho}{6}.
 \end{aligned}$$

$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(b^2 - a^2)c\rho/6}{(b-a)c\rho/2} = \frac{b^2 - a^2}{3(b-a)} = \frac{(b-a)(b+a)}{3(b-a)} = \frac{b+a}{3} = \frac{a+b}{3}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned}
 M_x &= \iint_R y\rho(x, y) dA = \iint_R y\rho dx dy = \rho \int_0^c \int_{a(1-y/c)}^{b(1-y/c)} y dx dy = \rho \int_0^c y [x]_{a(1-y/c)}^{b(1-y/c)} dy \\
 &= \rho \int_0^c y \left[b \left(1 - \frac{y}{c} \right) - a \left(1 - \frac{y}{c} \right) \right] dy = \rho \int_0^c y(b-a) \left(1 - \frac{y}{c} \right) dy = (b-a)\rho \int_0^c \left(y - \frac{y^2}{c} \right) dy \\
 &= (b-a)\rho \left[\frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = (b-a)\rho \left[\left(\frac{c^2}{2} - \frac{c^3}{3c} \right) - 0 \right] = (b-a)\rho \cdot \frac{c^2}{6} = \frac{(b-a)c^2\rho}{6}.
 \end{aligned}$$

so $\bar{y} = \frac{M_x}{M} = \frac{(b-a)c^2\rho/6}{(b-a)c\rho/2} = \frac{c}{3}$.

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{a+b}{3}, \frac{c}{3} \right)$, which is precisely the point M of intersection of the medians of the triangle.

42. Suppose a homogeneous lamina of constant mass density ρ occupies a region R in the xy -plane, and let $A = \iint_R dA$ be the area of the region R . Then the mass of the lamina is

$$M = \iint_R \rho(x, y) dA = \iint_R \rho dA = \rho \iint_R dA = \rho A.$$

The moment of mass of the lamina about the y -axis is

$$M_y = \iint_R x\rho(x, y) dA = \iint_R x\rho dA = \rho \iint_R x dA,$$

and so the first coordinate of the center of mass of the lamina is $\bar{x} = \frac{M_y}{M} = \frac{\rho \iint_R x dA}{\rho A} = \frac{1}{A} \iint_R x dA$.

On the other hand, the average value of the function $f(x, y) = x$ over the region R is $\frac{1}{A} \iint_R f(x, y) dA = \frac{1}{A} \iint_R x dA = \bar{x}$, as desired.

Similarly, the moment of mass of the lamina about the x -axis is

$$M_x = \iint_R y\rho(x, y) dA = \iint_R y\rho dA = \rho \iint_R y dA,$$

and so the second coordinate of the center of mass of the lamina is $\bar{y} = \frac{M_x}{M} = \frac{\rho \iint_R y dA}{\rho A} = \frac{1}{A} \iint_R y dA$.

But the average value of the function $g(x, y) = y$ over R is $\frac{1}{A} \iint_R g(x, y) dA = \frac{1}{A} \iint_R y dA = \boxed{\bar{y}}$.

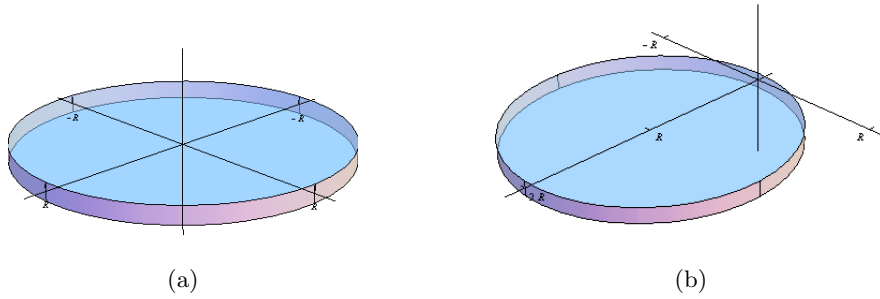
43. (a) Consider the thin flat plate shown in the figure in Problem 43; let D be the region in the xy -plane that it occupies. Note that for any point in the xy -plane, in particular for any point on the plate, the distance from this point to the z -axis is r , the distance from the origin, so

$$\begin{aligned}
 I_z &= I_O = \iint_D r^2 \rho(x, y) dA = \iint_D (x^2 + y^2) \rho(x, y) dA \\
 &= \iint_D x^2 \rho(x, y) dA + \iint_D y^2 \rho(x, y) dA = I_y + I_x = I_x + I_y,
 \end{aligned}$$

as desired.

- (b) Place the disk in the coordinate space so that it lies in the xy -plane with its center at the origin; see figure (a) below. Then the axis through its center and perpendicular to its plane is the z -axis, and two of its diameters lie on the x - and y -axes. Assuming the disk is homogeneous, due to symmetry its moment of inertia I_d about any diameter is the same. Then using part (a), we find that

$$2I_d = I_x + I_y = I_z = \frac{mR^2}{2}, \quad \text{so} \quad I_d = \frac{1}{2} \cdot \frac{mR^2}{2} = \boxed{\frac{mR^2}{4}}.$$



- (c) Place the disk in the coordinate space so that it lies in the xy -plane, its diameter lies on the x -axis, and its edge contains the origin; see figure (b) above. This way the y -axis is an axis tangent to the disk's edge. In the xy -plane, the equation of the disk's boundary in polar coordinates is $r = 2R \cos \theta$, and the region D that the disk occupies in the xy -plane is given by $0 \leq r \leq 2R \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then the moment of inertia of the disk about the y -axis is

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x, y) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2R \cos \theta} (r \cos \theta)^2 \cdot \rho \cdot r dr d\theta \\ &= \rho \int_{-\pi/2}^{\pi/2} \int_0^{2R \cos \theta} r^3 \cos^2 \theta dr d\theta = \rho \int_{-\pi/2}^{\pi/2} \cos^2 \theta \cdot \left[\frac{r^4}{4} \right]_0^{2R \cos \theta} d\theta \\ &= \rho \int_{-\pi/2}^{\pi/2} \cos^2 \theta \cdot (4R^4 \cos^4 \theta - 0) d\theta = 4R^4 \rho \int_{-\pi/2}^{\pi/2} \cos^6 \theta d\theta \\ &= 4R^4 \rho \int_{-\pi/2}^{\pi/2} (\cos^2 \theta)^3 d\theta = 4R^4 \rho \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}(1 + \cos(2\theta)) \right]^3 d\theta \\ &= 4R^4 \rho \int_{-\pi/2}^{\pi/2} \frac{1}{8} (1 + 3 \cos(2\theta) + 3 \cos^2(2\theta) + \cos^3(2\theta)) d\theta \\ &= \frac{R^4 \rho}{2} \int_{-\pi/2}^{\pi/2} \left[1 + 3 \cos(2\theta) + 3 \cdot \frac{1}{2}(1 + \cos(4\theta)) + \cos^3(2\theta) \right] d\theta \\ &= \frac{R^4 \rho}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{5}{2} + 3 \cos(2\theta) + \frac{3}{2} \cos(4\theta) + \cos^3(2\theta) \right) d\theta \\ &= \frac{R^4 \rho}{2} \left(\frac{5}{2} \int_{-\pi/2}^{\pi/2} d\theta + 3 \int_{-\pi/2}^{\pi/2} \cos(2\theta) d\theta + \frac{3}{2} \int_{-\pi/2}^{\pi/2} \cos(4\theta) d\theta + \int_{-\pi/2}^{\pi/2} \cos^3(2\theta) d\theta \right). \end{aligned}$$

The first three integrals in parentheses can be evaluated directly:

$$\begin{aligned}\frac{5}{2} \int_{-\pi/2}^{\pi/2} d\theta &= \frac{5}{2} [\theta]_{-\pi/2}^{\pi/2} = \frac{5}{2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = \frac{5}{2} \cdot \pi = \frac{5\pi}{2}; \\ 3 \int_{-\pi/2}^{\pi/2} \cos(2\theta) d\theta &= 3 \left[\frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} = \frac{3}{2} [\sin \pi - \sin(-\pi)] = \frac{3}{2} \cdot (0 - 0) = 0; \\ \frac{3}{2} \int_{-\pi/2}^{\pi/2} \cos(4\theta) d\theta &= \frac{3}{2} \left[\frac{\sin(4\theta)}{4} \right]_{-\pi/2}^{\pi/2} = \frac{3}{8} [\sin(2\pi) - \sin(-2\pi)] = \frac{3}{8} \cdot (0 - 0) = 0.\end{aligned}$$

To evaluate the last integral in parentheses, we rewrite it in the form

$$\int_{-\pi/2}^{\pi/2} \cos^3(2\theta) d\theta = \int_{-\pi/2}^{\pi/2} \cos^2(2\theta) \cdot \cos(2\theta) d\theta = \int_{-\pi/2}^{\pi/2} (1 - \sin^2(2\theta)) \cdot \cos(2\theta) d\theta,$$

and make a substitution $u = \sin(2\theta)$; then $du = 2 \cos(2\theta) d\theta$, so $\cos(2\theta) d\theta = \frac{du}{2}$. We also change the limits of integration: if $\theta = -\frac{\pi}{2}$, then $u = \sin(-\pi) = 0$; if $\theta = \frac{\pi}{2}$, then $u = \sin \pi = 0$. So

$$\int_{-\pi/2}^{\pi/2} \cos^3(2\theta) d\theta = \int_{-\pi/2}^{\pi/2} (1 - \sin^2(2\theta)) \cdot \cos(2\theta) d\theta = \int_0^0 (1 - u^2) \frac{du}{2} = \frac{1}{2} \int_0^0 (1 - u^2) du = 0.$$

Substituting these values back into the computation of I_y , we obtain

$$I_y = \frac{R^4 \rho}{2} \left(\frac{5\pi}{2} + 0 + 0 + 0 \right) = \frac{5\pi R^4 \rho}{4}.$$

Let A be the area of the disk: $A = \pi R^2$. Since the disk is homogeneous of mass density ρ , its mass is $m = \rho A = \rho \pi R^2$. Then the moment of inertia about the y -axis can be expressed as

$$I_y = \frac{5\pi R^4 \rho}{4} = \frac{5R^2 \cdot \pi R^2 \rho}{4} = \frac{5R^2 \cdot m}{4} = \boxed{\frac{5mR^2}{4}}.$$

14.5 Surface Area

Concepts and Vocabulary

1. True, according to formula (1) on page 936 of the textbook.
2. False. The area of the part of the surface $z = f(x, y)$ lying over the closed bounded region R is given by $S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA$.

Skill Building

3. The equation of the plane $2x + 2y + z = 6$ can be rewritten as $z = f(x, y) = 6 - 2x - 2y$. The region R is bounded in the first quadrant by the graph of $x^2 + y^2 = 4$ and the coordinate axes; see the figure in Problem 3. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -2$ and $f_y(x, y) = -2$. Since both f_x and f_y are continuous on R , the surface area S of the part of the plane that lies above R is

$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{(-2)^2 + (-2)^2 + 1} dA = \iint_R 3 dA.$$

Since the region R is bounded by a part of a circle, we switch to polar coordinates. In polar coordinates, R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$\begin{aligned} S &= \iint_R 3 dA = \int_0^{\pi/2} \int_0^2 3 \cdot r dr d\theta = 3 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^2 d\theta \\ &= 3 \int_0^{\pi/2} (2 - 0) d\theta = 6 \int_0^{\pi/2} d\theta = 6 [\theta]_0^{\pi/2} = 6 \left(\frac{\pi}{2} - 0 \right) = \boxed{3\pi}. \end{aligned}$$

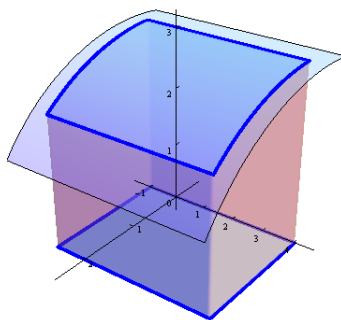
4. The equation of the plane $4z - 2x - y = 8$ can be rewritten as $z = f(x, y) = 2 + \frac{x}{2} + \frac{y}{4}$. The region R in the xy -plane is the triangle with the vertices $(0, 0, 0)$, $(1, 0, 0)$, and $(1, 1, 0)$; see the figure in Problem 4. In the xy -plane, R is given by $0 \leq y \leq x$, $0 \leq x \leq 1$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = \frac{1}{2}$ and $f_y(x, y) = \frac{1}{4}$. Since both f_x and f_y are continuous on R , the surface area S of the part of the plane that lies above R is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + 1} dA = \iint_R \frac{\sqrt{21}}{4} dA \\ &= \int_0^1 \int_0^x \frac{\sqrt{21}}{4} dy dx = \frac{\sqrt{21}}{4} \int_0^1 [y]_0^x dx = \frac{\sqrt{21}}{4} \int_0^1 (x - 0) dx = \frac{\sqrt{21}}{4} \int_0^1 x dx \\ &= \frac{\sqrt{21}}{4} \left[\frac{x^2}{2} \right]_0^1 = \frac{\sqrt{21}}{4} \left(\frac{1}{2} - 0 \right) = \boxed{\frac{\sqrt{21}}{8}}. \end{aligned}$$

5. The surface $z = f(x, y) = \sqrt{9 - x^2}$ and the rectangular region R in the xy -plane enclosed by the lines $x = 0$, $x = 2$, $y = -1$, and $y = 4$ are shown in the figure below. In the xy -plane, R is given by $0 \leq x \leq 2$, $-1 \leq y \leq 4$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{9 - x^2}}$ and

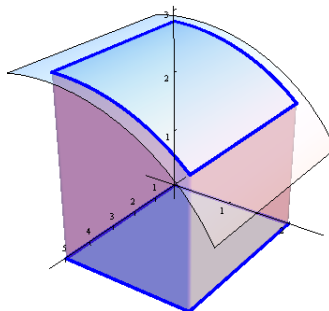
$f_y(x, y) = 0$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(-\frac{x}{\sqrt{9-x^2}}\right)^2 + 0^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{x^2}{9-x^2} + 1} \, dA = \iint_R \sqrt{\frac{9}{9-x^2}} \, dA = \iint_R \frac{3}{\sqrt{9-x^2}} \, dA = 3 \int_{-1}^4 \int_0^2 \frac{1}{\sqrt{9-x^2}} \, dx dy \\
 &= 3 \int_{-1}^4 \left[\sin^{-1} \left(\frac{x}{3} \right) \right]_0^2 dy = 3 \int_{-1}^4 \left[\sin^{-1} \left(\frac{2}{3} \right) - \sin^{-1} 0 \right] dy = 3 \int_{-1}^4 \left[\sin^{-1} \left(\frac{2}{3} \right) - 0 \right] dy \\
 &= 3 \sin^{-1} \left(\frac{2}{3} \right) \int_{-1}^4 dy = 3 \sin^{-1} \left(\frac{2}{3} \right) [y]_{-1}^4 = 3 \sin^{-1} \left(\frac{2}{3} \right) \cdot [4 - (-1)] = \boxed{15 \sin^{-1} \left(\frac{2}{3} \right)}.
 \end{aligned}$$



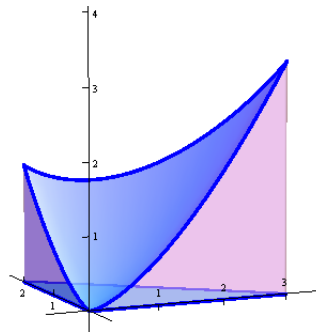
6. The surface $z = f(x, y) = \sqrt{8-y^2}$ and the rectangular region R in the xy -plane enclosed by the lines $x = 0$, $x = 5$, $y = 0$, and $y = 2$ are shown in the figure below. In the xy -plane, R is given by $0 \leq x \leq 5$, $0 \leq y \leq 2$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = 0$ and $f_y(x, y) = -\frac{y}{\sqrt{8-y^2}}$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{0^2 + \left(-\frac{y}{\sqrt{8-y^2}}\right)^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{y^2}{8-y^2} + 1} \, dA = \iint_R \sqrt{\frac{8}{8-y^2}} \, dA = \iint_R \frac{2\sqrt{2}}{\sqrt{8-y^2}} \, dA = 2\sqrt{2} \int_0^5 \int_0^2 \frac{1}{\sqrt{8-y^2}} \, dy dx \\
 &= 2\sqrt{2} \int_0^5 \left[\sin^{-1} \left(\frac{y}{2\sqrt{2}} \right) \right]_0^2 dx = 2\sqrt{2} \int_0^5 \left[\sin^{-1} \left(\frac{1}{\sqrt{2}} \right) - \sin^{-1} 0 \right] dx = 2\sqrt{2} \int_0^5 \left(\frac{\pi}{4} - 0 \right) dx \\
 &= 2\sqrt{2} \cdot \frac{\pi}{4} \int_0^5 dx = \frac{\pi\sqrt{2}}{2} [x]_0^5 = \frac{\pi\sqrt{2}}{2} \cdot (5 - 0) = \boxed{\frac{5\pi\sqrt{2}}{2}}.
 \end{aligned}$$

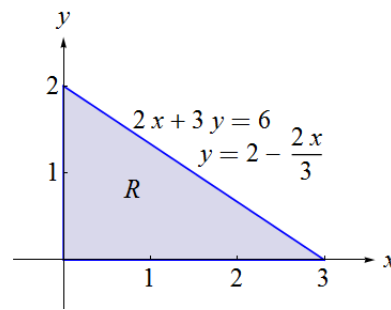


7. The surface $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and the triangular region R in the xy -plane enclosed by the lines $x = 0$, $y = 0$, and $2x + 3y = 6$ are shown in the figures below. In the xy -plane, R is given by $0 \leq y \leq 2 - \frac{2x}{3}$, $0 \leq x \leq 3$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = x^{1/2}$ and $f_y(x, y) = y^{1/2}$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1} dA = \iint_R \sqrt{x + y + 1} dA \\
 &= \int_0^3 \int_0^{2-2x/3} (x + y + 1)^{1/2} dy dx = \int_0^3 \left[\frac{2}{3}(x + y + 1)^{3/2} \right]_0^{2-2x/3} dx \\
 &= \frac{2}{3} \int_0^3 \left[\left(x + 2 - \frac{2x}{3} + 1 \right)^{3/2} - (x + 0 + 1)^{3/2} \right] dx = \frac{2}{3} \int_0^3 \left[\left(\frac{x}{3} + 3 \right)^{3/2} - (x + 1)^{3/2} \right] dx \\
 &= \frac{2}{3} \left[3 \cdot \frac{2}{5} \left(\frac{x}{3} + 3 \right)^{5/2} - \frac{2}{5} (x + 1)^{5/2} \right]_0^3 = \frac{2}{3} \cdot \frac{2}{5} \left[3 \left(\sqrt{\frac{x}{3} + 3} \right)^5 - (\sqrt{x + 1})^5 \right]_0^3 \\
 &= \frac{4}{15} \left\{ \left[3(\sqrt{4})^5 - (\sqrt{4})^5 \right] - \left[3(\sqrt{3})^5 - (\sqrt{1})^5 \right] \right\} \\
 &= \frac{4}{15} (96 - 32 - 27\sqrt{3} + 1) = \frac{4}{15} (65 - 27\sqrt{3}) = \boxed{\frac{52}{3} - \frac{36\sqrt{3}}{5}}.
 \end{aligned}$$



(a)

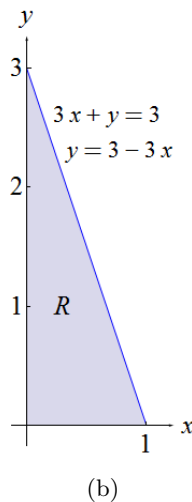
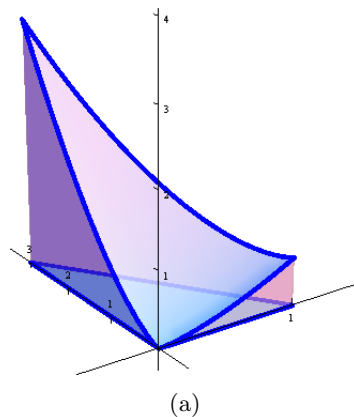


(b)

8. The surface $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and the triangular region R in the xy -plane enclosed by the lines $x = 0$, $y = 0$, and $3x + y = 3$ are shown in the figures below. In the xy -plane, R is given by $0 \leq y \leq 3 - 3x$, $0 \leq x \leq 1$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = x^{1/2}$ and

$f_y(x, y) = y^{1/2}$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1} dA = \iint_R \sqrt{x + y + 1} dA \\
 &= \int_0^1 \int_0^{3-3x} (x + y + 1)^{1/2} dy dx = \int_0^1 \left[\frac{2}{3} (x + y + 1)^{3/2} \right]_0^{3-3x} dx \\
 &= \frac{2}{3} \int_0^1 \left[(x + 3 - 3x + 1)^{3/2} - (x + 0 + 1)^{3/2} \right] dx = \frac{2}{3} \int_0^1 \left[(-2x + 4)^{3/2} - (x + 1)^{3/2} \right] dx \\
 &= \frac{2}{3} \left[-\frac{1}{2} \cdot \frac{2}{5} (-2x + 4)^{5/2} - \frac{2}{5} (x + 1)^{5/2} \right]_0^1 = \frac{2}{3} \cdot \frac{2}{5} \left[-\frac{1}{2} (\sqrt{-2x + 4})^5 - (\sqrt{x + 1})^5 \right]_0^1 \\
 &= \frac{4}{15} \left\{ \left[-\frac{1}{2} (\sqrt{2})^5 - (\sqrt{2})^5 \right] - \left[-\frac{1}{2} (\sqrt{4})^5 - (\sqrt{1})^5 \right] \right\} \\
 &= \frac{4}{15} (-2\sqrt{2} - 4\sqrt{2} + 16 + 1) = \frac{4}{15} (17 - 6\sqrt{2}) = \boxed{\frac{4(17 - 6\sqrt{2})}{15}}.
 \end{aligned}$$



9. The part of the surface $z = f(x, y) = 4 - x^2 - y^2$ that lies above the xy plane is shown in the figure below. Its trace in the xy -plane is the disk R given by $x^2 + y^2 \leq 4$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -2x$ and $f_y(x, y) = -2y$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{(-2x)^2 + (-2y)^2 + 1} dA \\
 &= \iint_R \sqrt{4x^2 + 4y^2 + 1} dA = \iint_R \sqrt{4(x^2 + y^2) + 1} dA.
 \end{aligned}$$

Since the region R is a disk and the integrand involves $x^2 + y^2$, we switch to polar coordinates. In polar coordinates, R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$; and we also use the identity $r^2 = x^2 + y^2$ to transform the integrand. Then

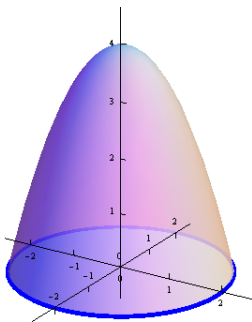
$$S = \iint_R \sqrt{4(x^2 + y^2) + 1} dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} \cdot r dr d\theta.$$

To evaluate the “inside” integral $\int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr$, we make a substitution $u = 4r^2 + 1$; then $du = 8r \, dr$, so $r \, dr = \frac{1}{8} du$. We also change the limits of integration: if $r = 0$, then $u = 4 \cdot 0^2 + 1 = 1$; if $r = 2$, then $u = 4 \cdot 2^2 + 1 = 17$. So

$$\int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr = \int_1^{17} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \int_1^{17} u^{1/2} du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^{17} = \frac{1}{12} \left[(\sqrt{u})^3 \right]_1^{17} = \frac{17\sqrt{17} - 1}{12}.$$

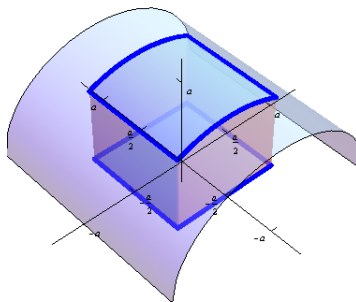
Going back to the calculation of S ,

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{17\sqrt{17} - 1}{12} d\theta \\ &= \frac{17\sqrt{17} - 1}{12} \cdot [\theta]_0^{2\pi} = \frac{17\sqrt{17} - 1}{12} \cdot (2\pi - 0) = \boxed{\frac{(17\sqrt{17} - 1)\pi}{6}}. \end{aligned}$$



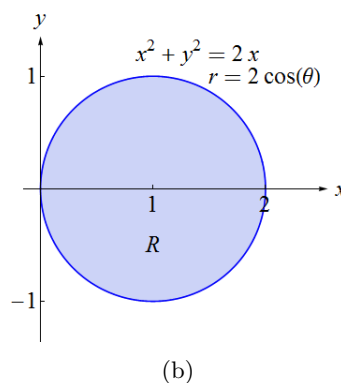
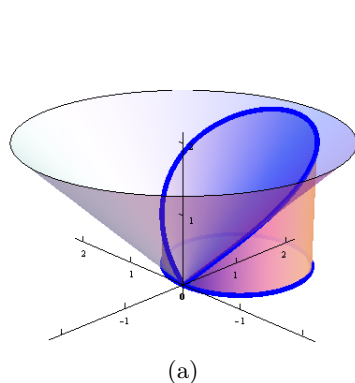
10. The part of the cylinder $z = f(x, y) = \sqrt{a^2 - x^2}$ that lies above the square region R in the xy -plane defined by $-\frac{a}{2} \leq x \leq \frac{a}{2}$, $-\frac{a}{2} \leq y \leq \frac{a}{2}$ is shown in the figure below. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{a^2 - x^2}}$ and $f_y(x, y) = 0$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(-\frac{x}{\sqrt{a^2 - x^2}}\right)^2 + 0^2 + 1} \, dA \\ &= \iint_R \sqrt{\frac{x^2}{a^2 - x^2} + 1} \, dA = \iint_R \sqrt{\frac{a^2}{a^2 - x^2}} \, dA = a \iint_R \frac{1}{\sqrt{a^2 - x^2}} \, dA \\ &= a \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{1}{\sqrt{a^2 - x^2}} \, dx \, dy = a \int_{-a/2}^{a/2} \left[\sin^{-1} \left(\frac{x}{a} \right) \right]_{-a/2}^{a/2} dy \\ &= a \int_{-a/2}^{a/2} \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(-\frac{1}{2} \right) \right] dy = a \int_{-a/2}^{a/2} \left[\frac{\pi}{6} - \left(-\frac{\pi}{6} \right) \right] dy = a \cdot \frac{\pi}{3} \int_{-a/2}^{a/2} dy \\ &= \frac{a\pi}{3} [y]_{-a/2}^{a/2} = \frac{a\pi}{3} \cdot \left[\frac{a}{2} - \left(-\frac{a}{2} \right) \right] = \frac{a\pi}{3} \cdot a = \boxed{\frac{a^2\pi}{3}}. \end{aligned}$$



11. The part of the cone $z = f(x, y) = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$ is shown in figure (a) below. It lies over the region R in the xy -plane that is the trace of the cylinder in this plane: R is defined by $x^2 + y^2 \leq 2x$ in Cartesian coordinates, and it is a disk of radius 1 centered at the point $(1, 0)$; see figure (b) below. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ and $f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$. Then the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} \, dA = \iint_R \sqrt{2} \, dA = \sqrt{2} \iint_R dA \\
 &= \sqrt{2} \cdot (\text{Area of the disk } R) = \sqrt{2} \cdot (\pi \cdot 1^2) = \boxed{\pi\sqrt{2}}.
 \end{aligned}$$



12. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $x^2 + y^2 = 2z$ is shown in the figure below. The equation of the sphere can be rewritten as $x^2 + y^2 + (z - 2)^2 = 4$, so it is a sphere of radius 2 centered at the point $(0, 0, 2)$. The curve of intersection of the sphere and the paraboloid is defined by the system of equations

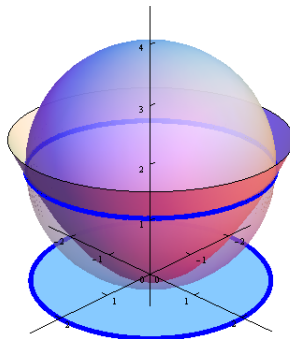
$$\begin{cases} x^2 + y^2 + z^2 = 4z \\ x^2 + y^2 = 2z \end{cases} \quad \text{or} \quad \begin{cases} x^2 + y^2 = 4z - z^2 \\ x^2 + y^2 = 2z \end{cases}$$

By setting the right-hand sides equal to each other, we find

$$4z - z^2 = 2z, \quad \text{so} \quad 2z - z^2 = 0, \quad \text{so} \quad z(2 - z) = 0, \quad \text{so} \quad z = 0, 2.$$

$z = 0$ is an extraneous solution (the vertex of the paraboloid, which coincides with the lowest point of the sphere). If $z = 2$, then $x^2 + y^2 = 4$, so the curve of intersection is the circle $x^2 + y^2 = 4$, $z = 2$ at the “equator” of the sphere. So the part of the sphere that lies within the paraboloid is the upper hemisphere and its surface area is

$$S = \frac{1}{2} \cdot (\text{Surface area of a sphere of radius } 2) = \frac{1}{2} \cdot (4\pi \cdot 2^2) = \boxed{8\pi}.$$



13. The part of the surface $z = f(x, y) = xy$ that lies in the first octant within the cylinder $x^2 + y^2 = a^2$ ($a > 0$) is shown in figure (a) below. It lies over the region R in the xy -plane that is bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes in the first quadrant; see figure (b) below. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = y$ and $f_y(x, y) = x$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{y^2 + x^2 + 1} \, dA = \iint_R \sqrt{x^2 + y^2 + 1} \, dA.$$

Since the region R is bounded by a part of a circle and the integrand involves $x^2 + y^2$, we switch to polar coordinates. In polar coordinates, R is given by $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$; and we also use the identity $r^2 = x^2 + y^2$ to transform the integrand. Then

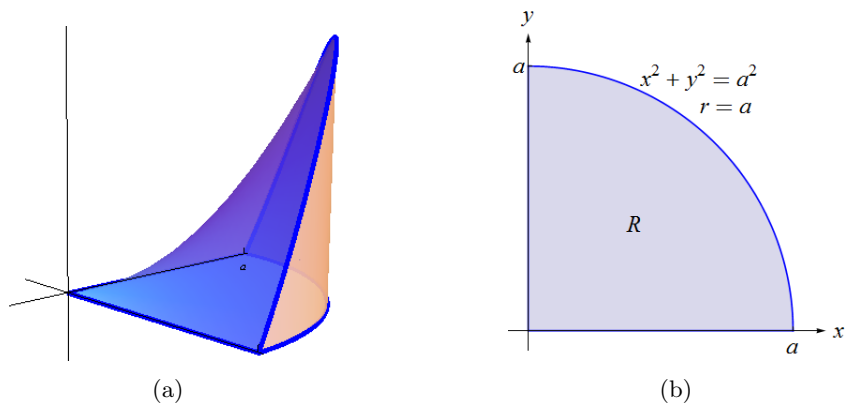
$$S = \iint_R \sqrt{x^2 + y^2 + 1} \, dA = \int_0^{\pi/2} \int_0^a \sqrt{r^2 + 1} \cdot r \, dr \, d\theta.$$

To evaluate the “inside” integral $\int_0^a \sqrt{r^2 + 1} \cdot r \, dr$, we make a substitution $u = r^2 + 1$; then $du = 2r \, dr$, so $r \, dr = \frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = 0^2 + 1 = 1$; if $r = a$, then $u = a^2 + 1$. So

$$\begin{aligned} \int_0^a \sqrt{r^2 + 1} \cdot r \, dr &= \int_1^{a^2+1} \sqrt{u} \cdot \frac{1}{2} \, du = \frac{1}{2} \int_1^{a^2+1} u^{1/2} \, du \\ &= \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_1^{a^2+1} = \frac{1}{3} \left[u^{3/2} \right]_1^{a^2+1} = \frac{(a^2 + 1)^{3/2} - 1}{3}. \end{aligned}$$

Going back to the calculation of S ,

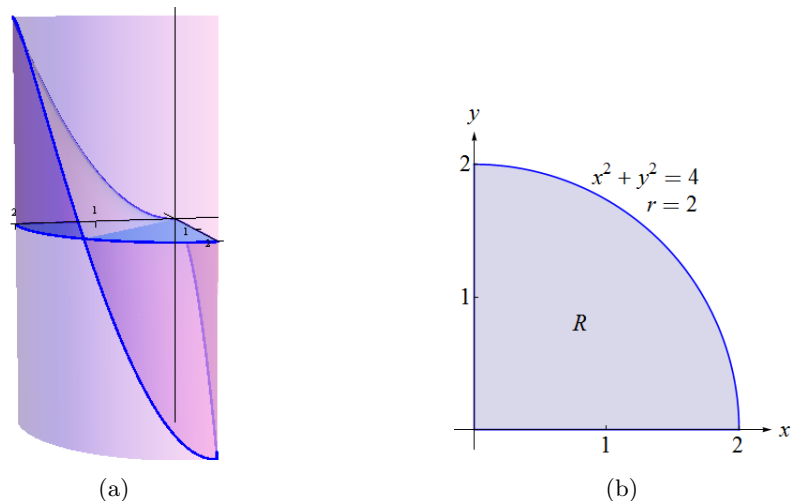
$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^a \sqrt{r^2 + 1} \cdot r \, dr \, d\theta = \int_0^{\pi/2} \frac{(a^2 + 1)^{3/2} - 1}{3} \, d\theta \\ &= \frac{(a^2 + 1)^{3/2} - 1}{3} \cdot [\theta]_0^{\pi/2} = \frac{(a^2 + 1)^{3/2} - 1}{3} \cdot \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi [(a^2 + 1)^{3/2} - 1]}{6}}. \end{aligned}$$



14. The part of the surface $z = f(x, y) = x^2 - y^2$ that lies in the first octant within the cylinder $x^2 + y^2 = 4$ is shown in figure (a) below. It lies over the region R in the xy -plane that is bounded by the circle $x^2 + y^2 = 4$ and the coordinate axes in the first quadrant; see figure (b) below. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = 2x$ and $f_y(x, y) = -2y$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{(2x)^2 + (-2y)^2 + 1} \, dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA = \iint_R \sqrt{4(x^2 + y^2) + 1} \, dA. \end{aligned}$$

Since the region R is part of a disk and the integrand involves $x^2 + y^2$, we switch to polar coordinates.



In polar coordinates, R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$; and we also use the identity $r^2 = x^2 + y^2$ to transform the integrand. Then

$$S = \iint_R \sqrt{4(x^2 + y^2) + 1} \, dA = \int_0^{\pi/2} \int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta.$$

To evaluate the “inside” integral $\int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr$, we make a substitution $u = 4r^2 + 1$; then $du = 8r \, dr$, so $r \, dr = \frac{1}{8} du$. We also change the limits of integration: if $r = 0$, then $u = 4 \cdot 0^2 + 1 = 1$; if $r = 2$, then

$u = 4 \cdot 2^2 + 1 = 17$. So

$$\int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr = \int_1^{17} \sqrt{u} \cdot \frac{1}{8} \, du = \frac{1}{8} \int_1^{17} u^{1/2} \, du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^{17} = \frac{1}{12} \left[(\sqrt{u})^3 \right]_1^{17} = \frac{17\sqrt{17} - 1}{12}.$$

Going back to the calculation of S ,

$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^2 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta = \int_0^{\pi/2} \frac{17\sqrt{17} - 1}{12} \, d\theta \\ &= \frac{17\sqrt{17} - 1}{12} \cdot [\theta]_0^{\pi/2} = \frac{17\sqrt{17} - 1}{12} \cdot \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{(17\sqrt{17} - 1)\pi}{24}}. \end{aligned}$$

15. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies between the planes $z = 1$ and $z = 3$ is shown in figure (a) below. The equation of the sphere can be rewritten as $x^2 + y^2 + (z - 2)^2 = 4$, so it is a sphere of radius 2 centered at the point $(0, 0, 2)$. If we substitute either $z = 1$ or $z = 3$ into the equation of the sphere, in both cases we obtain $x^2 + y^2 = 3$, so the curve of intersection of the sphere with either plane is a circle $x^2 + y^2 = 3$ lying in the respective plane. The projection of this part of the sphere onto the xy -plane is the annular region R shown in figure (b) below; in polar coordinates R is given by $\sqrt{3} \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Notice that this part of the sphere is symmetric about the sphere's "equator": the part of the surface between $z = 1$ and $z = 2$ is symmetric to the part of the surface between $z = 2$ and $z = 3$; so we can find the surface area of the latter and multiply it by 2. The equation of the upper hemisphere above $z = 2$ can be written as $z = f(x, y) = 2 + \sqrt{4 - x^2 - y^2}$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{4 - x^2 - y^2}}$ and $f_y(x, y) = -\frac{y}{\sqrt{4 - x^2 - y^2}}$. Then the surface area S we seek is

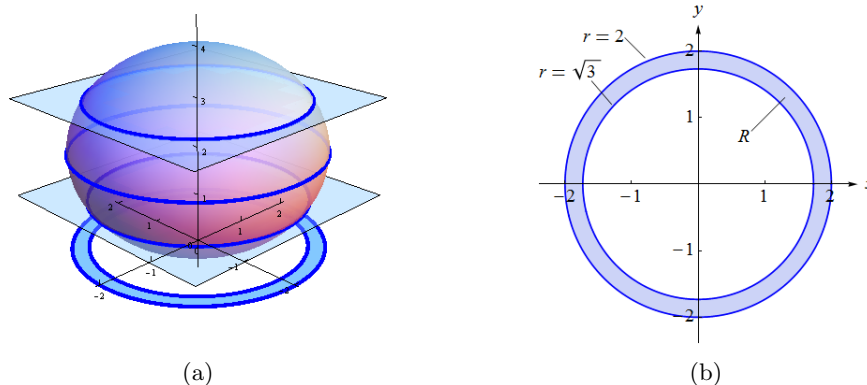
$$\begin{aligned} S &= 2 \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\ &= 2 \iint_R \sqrt{\left(-\frac{x}{\sqrt{4 - x^2 - y^2}} \right)^2 + \left(-\frac{y}{\sqrt{4 - x^2 - y^2}} \right)^2 + 1} \, dA \\ &= 2 \iint_R \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} \, dA = 2 \iint_R \sqrt{\frac{4}{4 - x^2 - y^2}} \, dA \\ &= 2 \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}} \, dA = 4 \int_0^{2\pi} \int_{\sqrt{3}}^2 \frac{1}{\sqrt{4 - r^2}} \cdot r \, dr \, d\theta = 4 \int_0^{2\pi} \int_{\sqrt{3}}^2 (4 - r^2)^{-1/2} \cdot r \, dr \, d\theta. \end{aligned}$$

To evaluate the "inside" integral $\int_{\sqrt{3}}^2 (4 - r^2)^{-1/2} \cdot r \, dr$, we make a substitution $u = 4 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = \sqrt{3}$, then $u = 4 - (\sqrt{3})^2 = 1$; if $r = 2$, then $u = 4 - 2^2 = 0$. So

$$\begin{aligned} \int_{\sqrt{3}}^2 (4 - r^2)^{-1/2} \cdot r \, dr &= \int_1^0 u^{-1/2} \cdot \left(-\frac{1}{2} du \right) = -\frac{1}{2} \int_1^0 u^{-1/2} \, du \\ &= \frac{1}{2} \int_0^1 u^{-1/2} \, du = \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_0^1 = [\sqrt{u}]_0^1 = 1 - 0 = 1. \end{aligned}$$

Going back to the calculation of S ,

$$S = 4 \int_0^{2\pi} \int_{\sqrt{3}}^2 (4 - r^2)^{-1/2} \cdot r \, dr \, d\theta = 4 \int_0^{2\pi} 1 \, d\theta = 4 [\theta]_0^{2\pi} = 4(2\pi - 0) = \boxed{8\pi}.$$



16. The part of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$ (where $a > 0$) is shown in figure (a) below. Notice that it consists of two symmetric parts in the upper and lower hemispheres; so to find the total required surface area, we can find the surface area of the part lying in the upper hemisphere and multiply it by 2. The projection of this part onto the xy -plane is the disk R bounded by $x^2 + y^2 = 2ax$ that is the trace of the cylinder in this plane. In polar coordinates R is given by $0 \leq r \leq 2a \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; see figure (b) below.

The equation of the upper hemisphere can be written as $z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{4a^2 - x^2 - y^2}}$ and $f_y(x, y) = -\frac{y}{\sqrt{4a^2 - x^2 - y^2}}$. Then the surface area S we seek is

$$\begin{aligned}
 S &= 2 \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA \\
 &= 2 \iint_R \sqrt{\left(-\frac{x}{\sqrt{4a^2 - x^2 - y^2}}\right)^2 + \left(-\frac{y}{\sqrt{4a^2 - x^2 - y^2}}\right)^2 + 1} dA \\
 &= 2 \iint_R \sqrt{\frac{x^2}{4a^2 - x^2 - y^2} + \frac{y^2}{4a^2 - x^2 - y^2} + 1} dA = 2 \iint_R \sqrt{\frac{4a^2}{4a^2 - x^2 - y^2}} dA \\
 &= 2 \iint_R \frac{2a}{\sqrt{4a^2 - x^2 - y^2}} dA = 4a \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} \frac{1}{\sqrt{4a^2 - r^2}} \cdot r dr d\theta \\
 &= 4a \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} (4a^2 - r^2)^{-1/2} \cdot r dr d\theta.
 \end{aligned}$$

To evaluate the “inside” integral $\int_0^{2a \cos \theta} (4a^2 - r^2)^{-1/2} \cdot r dr$, we make a substitution $u = 4a^2 - r^2$; then $du = -2r dr$, so $r dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = 4a^2 - 0^2 = 4a^2$; if $r = 2a \cos \theta$, then $u = 4a^2 - (2a \cos \theta)^2 = 4a^2 - 4a^2 \cos^2 \theta = 4a^2(1 - \cos^2 \theta) = 4a^2 \sin^2 \theta$. So

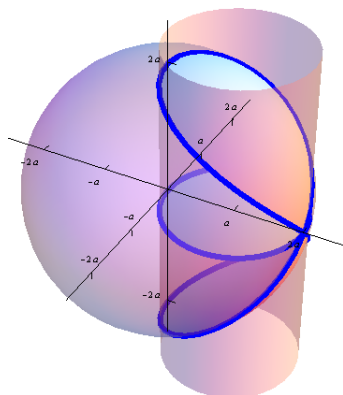
$$\begin{aligned}
 \int_0^{2a \cos \theta} (4a^2 - r^2)^{-1/2} \cdot r dr &= \int_{4a^2}^{4a^2 \sin^2 \theta} u^{-1/2} \cdot \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_{4a^2}^{4a^2 \sin^2 \theta} u^{-1/2} du \\
 &= \frac{1}{2} \int_{4a^2 \sin^2 \theta}^{4a^2} u^{-1/2} du = \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_{4a^2 \sin^2 \theta}^{4a^2} = [\sqrt{u}]_{4a^2 \sin^2 \theta}^{4a^2} \\
 &= \sqrt{4a^2} - \sqrt{4a^2 \sin^2 \theta} = 2a - 2a |\sin \theta| = 2a(1 - |\sin \theta|).
 \end{aligned}$$

Going back to the calculation of S ,

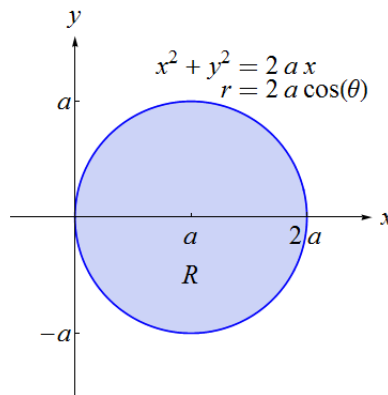
$$S = 4a \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} (4a^2 - r^2)^{-1/2} \cdot r \, dr \, d\theta = 4a \int_{-\pi/2}^{\pi/2} 2a (1 - |\sin \theta|) \, d\theta = 8a^2 \int_{-\pi/2}^{\pi/2} (1 - |\sin \theta|) \, d\theta.$$

Notice that the integrand $(1 - |\sin \theta|)$ is an even function of θ . So using the property of definite integrals of even functions and the fact that $\sin \theta$ is non-negative when $0 \leq \theta \leq \frac{\pi}{2}$, we can proceed as follows:

$$\begin{aligned} S &= 8a^2 \int_{-\pi/2}^{\pi/2} (1 - |\sin \theta|) \, d\theta = 8a^2 \cdot 2 \int_0^{\pi/2} (1 - |\sin \theta|) \, d\theta = 16a^2 \int_0^{\pi/2} (1 - \sin \theta) \, d\theta \\ &= 16a^2 [\theta + \cos \theta]_0^{\pi/2} = 16a^2 \left[\left(\frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 + \cos 0) \right] = 16a^2 \left(\frac{\pi}{2} + 0 - 0 - 1 \right) = \boxed{8a^2(\pi - 2)}. \end{aligned}$$



(a)



(b)

Applications and Extensions

17. The part of the surface $y^2 - x^2 = 6z$, or equivalently $z = f(x, y) = \frac{1}{6}(y^2 - x^2)$, that lies within the cylinder $x^2 + y^2 = 36$ is shown in figure (a) below. Its projection onto the xy -plane is the region R that is bounded by the circle $x^2 + y^2 = 36$; see figure (b) below. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{3}$ and $f_y(x, y) = \frac{y}{3}$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(-\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + 1} \, dA \\ &= \iint_R \sqrt{\frac{x^2}{9} + \frac{y^2}{9} + 1} \, dA = \iint_R \sqrt{\frac{x^2 + y^2 + 9}{9}} \, dA = \frac{1}{3} \iint_R \sqrt{x^2 + y^2 + 9} \, dA. \end{aligned}$$

Since the region R is a disk and the integrand involves $x^2 + y^2$, we switch to polar coordinates. In polar coordinates, R is given by $0 \leq r \leq 6$, $0 \leq \theta \leq 2\pi$; and we also use the identity $r^2 = x^2 + y^2$ to transform the integrand. Then

$$S = \frac{1}{3} \iint_R \sqrt{x^2 + y^2 + 9} \, dA = \frac{1}{3} \int_0^{2\pi} \int_0^6 \sqrt{r^2 + 9} \cdot r \, dr \, d\theta.$$

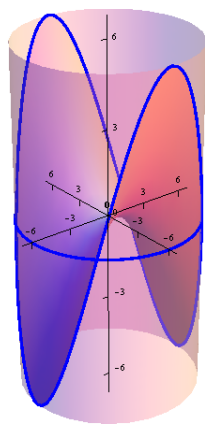
To evaluate the “inside” integral $\int_0^6 \sqrt{r^2 + 9} \cdot r \, dr$, we make a substitution $u = r^2 + 9$; then $du = 2r \, dr$, so $r \, dr = \frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = 0^2 + 9 = 9$; if $r = 6$, then

$u = 6^2 + 9 = 45$. So

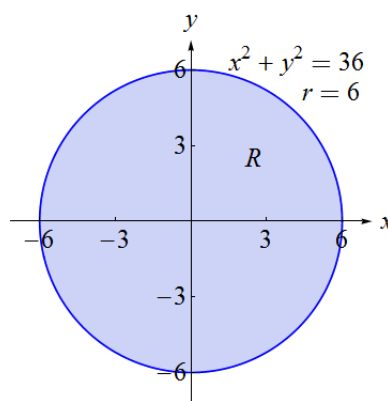
$$\begin{aligned}\int_0^6 \sqrt{r^2 + 9} \cdot r \, dr &= \int_9^{45} \sqrt{u} \cdot \frac{1}{2} \, du = \frac{1}{2} \int_9^{45} u^{1/2} \, du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_9^{45} \\ &= \frac{1}{3} \left[(\sqrt{u})^3 \right]_9^{45} = \frac{1}{3} \left[(\sqrt{45})^3 - (\sqrt{9})^3 \right] = \frac{1}{3} (135\sqrt{5} - 27) = 45\sqrt{5} - 9.\end{aligned}$$

Going back to the calculation of S ,

$$\begin{aligned}S &= \frac{1}{3} \int_0^{2\pi} \int_0^6 \sqrt{r^2 + 9} \cdot r \, dr \, d\theta = \frac{1}{3} \int_0^{2\pi} (45\sqrt{5} - 9) \, d\theta \\ &= \frac{45\sqrt{5} - 9}{3} \cdot [\theta]_0^{2\pi} = (15\sqrt{5} - 3) \cdot (2\pi - 0) = 3(5\sqrt{5} - 1) \cdot 2\pi = \boxed{6\pi(5\sqrt{5} - 1)}.\end{aligned}$$



(a)



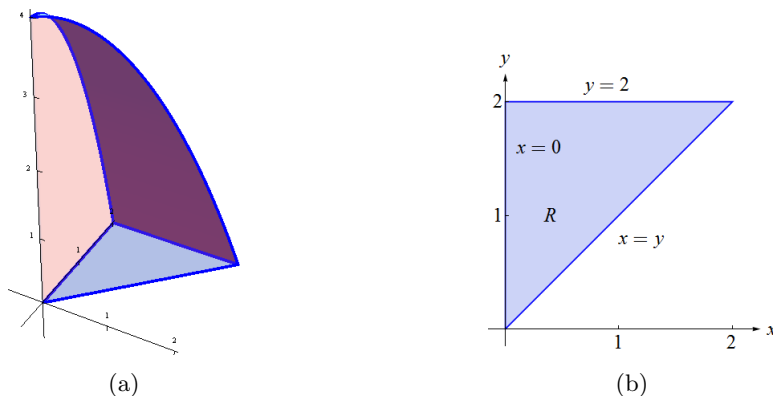
(b)

18. The part of the surface $z = f(x, y) = 4 - y^2$ that lies in the first octant and is enclosed by $z = 0$ (the xy -plane), $x = 0$, $x = y$, and $y = 2$ is shown in figure (a) below. It lies over the region R in the xy -plane that is the triangle enclosed by the lines $x = 0$, $x = y$, and $y = 2$; see figure (b) below. In the xy -plane, R is given by $0 \leq x \leq y$, $0 \leq y \leq 2$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = 0$ and $f_y(x, y) = -2y$. Then the surface area S we seek is

$$\begin{aligned}S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{0^2 + (-2y)^2 + 1} \, dA \\ &= \iint_R \sqrt{4y^2 + 1} \, dA = \int_0^2 \int_0^y \sqrt{4y^2 + 1} \, dx \, dy = \int_0^2 \sqrt{4y^2 + 1} \cdot [x]_0^y \, dy \\ &= \int_0^2 \sqrt{4y^2 + 1} \cdot (y - 0) \, dy = \int_0^2 \sqrt{4y^2 + 1} \cdot y \, dy.\end{aligned}$$

To evaluate this integral, we make a substitution $u = 4y^2 + 1$; then $du = 8y \, dy$, so $y \, dy = \frac{1}{8} du$. We also change the limits of integration: if $y = 0$, then $u = 4 \cdot 0^2 + 1 = 1$; if $y = 2$, then $u = 4 \cdot 2^2 + 1 = 17$. So

$$\int_0^2 \sqrt{4y^2 + 1} \cdot y \, dy = \int_1^{17} \sqrt{u} \cdot \frac{1}{8} \, du = \frac{1}{8} \int_1^{17} u^{1/2} \, du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^{17} = \frac{1}{12} \left[(\sqrt{u})^3 \right]_1^{17} = \boxed{\frac{17\sqrt{17} - 1}{12}}.$$



19. The part of the surface $x^2 = y$ that lies in the first octant under the plane $x + z = 3$ is shown in figure (a) below. This surface can be interpreted as the graph of a function $y = f(x, z) = x^2$, whose domain is the region R in the xz -plane that is the triangular region enclosed by the line $x + z = 3$ and the coordinate axes; see figure (b) below. In the xz -plane, R is given by $0 \leq z \leq 3 - x$, $0 \leq x \leq 3$. The partial derivatives of $y = f(x, z)$ are $f_x(x, z) = 2x$ and $f_z(x, z) = 0$. Since both f_x and f_z are continuous on R , the surface area S we seek is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1} \, dA = \iint_R \sqrt{(2x)^2 + 0^2 + 1} \, dA = \iint_R \sqrt{4x^2 + 1} \, dA \\ &= \int_0^3 \int_0^{3-x} \sqrt{4x^2 + 1} \, dz \, dx = \int_0^3 \sqrt{4x^2 + 1} \cdot [z]_0^{3-x} \, dx = \int_0^3 \sqrt{4x^2 + 1} \cdot [(3-x) - 0] \, dx \\ &= \int_0^3 \sqrt{4x^2 + 1} \cdot (3-x) \, dx = 3 \int_0^3 \sqrt{4x^2 + 1} \, dx - \int_0^3 \sqrt{4x^2 + 1} \cdot x \, dx. \end{aligned}$$

To evaluate the second integral, we make a substitution $u = 4x^2 + 1$; then $du = 8x \, dx$, so $x \, dx = \frac{1}{8} du$. We also change the limits of integration: if $x = 0$, then $u = 4 \cdot 0^2 + 1 = 1$; if $x = 3$, then $u = 4 \cdot 3^2 + 1 = 37$. So

$$\int_0^3 \sqrt{4x^2 + 1} \cdot x \, dx = \int_1^{37} \sqrt{u} \cdot \frac{1}{8} \, du = \frac{1}{8} \int_1^{37} u^{1/2} \, du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^{37} = \frac{1}{12} \left[(\sqrt{u})^3 \right]_1^{37} = \frac{37\sqrt{37} - 1}{12}.$$

To find the first integral, we will find an antiderivative using a substitution $2x = \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; then $x = \frac{1}{2} \tan \theta$ and $dx = \frac{1}{2} \sec^2 \theta \, d\theta$. Also,

$$\sqrt{4x^2 + 1} = \sqrt{4 \cdot \frac{1}{4} \tan^2 \theta + 1} = \sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = \sec \theta,$$

and the integral becomes

$$3 \int \sqrt{4x^2 + 1} \, dx = 3 \int \sec \theta \cdot \frac{1}{2} \sec^2 \theta \, d\theta = \frac{3}{2} \int \sec^3 \theta \, d\theta.$$

To find the integral $\int \sec^3 \theta \, d\theta$, we use the method of integration by parts. If we choose $u = \sec \theta$ and $dv = \sec^2 \theta \, d\theta$, then $du = \sec \theta \tan \theta \, d\theta$ and $v = \int \sec^2 \theta \, d\theta = \tan \theta$. So

$$\begin{aligned} \int \sec^3 \theta \, d\theta &= \sec \theta \cdot \tan \theta - \int \tan \theta \cdot \sec \theta \tan \theta \, d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta \, d\theta \\ &= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta \\ &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| - \int \sec^3 \theta \, d\theta. \end{aligned}$$

So we obtained an equation

$$\int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| - \int \sec^3 \theta \, d\theta,$$

from which we find that

$$\begin{aligned} 2 \int \sec^3 \theta \, d\theta &= \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|, \\ \int \sec^3 \theta \, d\theta &= \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|), \\ \frac{3}{2} \int \sec^3 \theta \, d\theta &= \frac{3}{4} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|). \end{aligned}$$

To express this antiderivative in terms of x , recall that we substituted $2x = \tan \theta$ and $\sqrt{4x^2 + 1} = \sec \theta$. So an antiderivative is

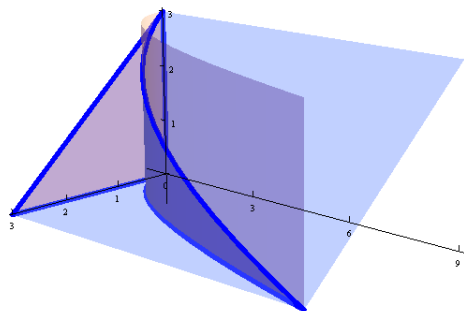
$$\begin{aligned} 3 \int \sqrt{4x^2 + 1} \, dx &= \frac{3}{2} \int \sec^3 \theta \, d\theta = \frac{3}{4} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \\ &= \frac{3}{4} (\sqrt{4x^2 + 1} \cdot 2x + \ln |\sqrt{4x^2 + 1} + 2x|) \\ &= \frac{3}{4} (2x\sqrt{4x^2 + 1} + \ln |\sqrt{4x^2 + 1} + 2x|), \end{aligned}$$

and the value of the first integral is

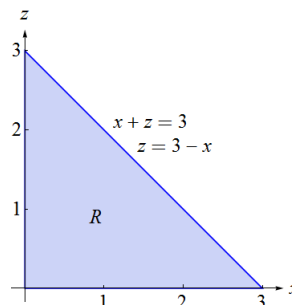
$$\begin{aligned} 3 \int_0^3 \sqrt{4x^2 + 1} \, dx &= \frac{3}{4} \left[(2x\sqrt{4x^2 + 1} + \ln |\sqrt{4x^2 + 1} + 2x|) \right]_0^3 \\ &= \frac{3}{4} (6\sqrt{37} + \ln |\sqrt{37} + 6|) - \frac{3}{4} (0\sqrt{1} + \ln |\sqrt{1} + 0|) \\ &= \frac{3}{4} [6\sqrt{37} + \ln (\sqrt{37} + 6)] - \frac{3}{4} (0 + \ln 1) = \frac{18\sqrt{37} + 3 \ln (\sqrt{37} + 6)}{4}. \end{aligned}$$

Going back to the calculation of the surface area,

$$\begin{aligned} S &= 3 \int_0^3 \sqrt{4x^2 + 1} \, dx - \int_0^3 \sqrt{4x^2 + 1} \cdot x \, dx = \frac{18\sqrt{37} + 3 \ln (\sqrt{37} + 6)}{4} - \frac{37\sqrt{37} - 1}{12} \\ &= \frac{[54\sqrt{37} + 9 \ln (\sqrt{37} + 6)] - (37\sqrt{37} - 1)}{12} = \boxed{\frac{1 + 17\sqrt{37} + 9 \ln (\sqrt{37} + 6)}{12}}. \end{aligned}$$



(a)



(b)

20. The part of the paraboloid $z = 9 - x^2 - y^2$ that lies between the planes $z = 0$ and $z = 8$ is shown in figure (a) below. If we substitute $z = 0$ into the equation of the paraboloid, we obtain $0 = 9 - x^2 - y^2$ or $x^2 + y^2 = 9$; if we substitute $z = 8$ into the equation of the paraboloid, we obtain $8 = 9 - x^2 - y^2$ or $x^2 + y^2 = 1$. So the projection of this part of the paraboloid onto the xy -plane is the annular region R shown in figure (b) below; in polar coordinates R is given by $1 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.

The partial derivatives of $z = f(x, y) = 9 - x^2 - y^2$ are $f_x(x, y) = -2x$ and $f_y(x, y) = -2y$. Since both f_x and f_y are continuous on R , the surface area S we seek is

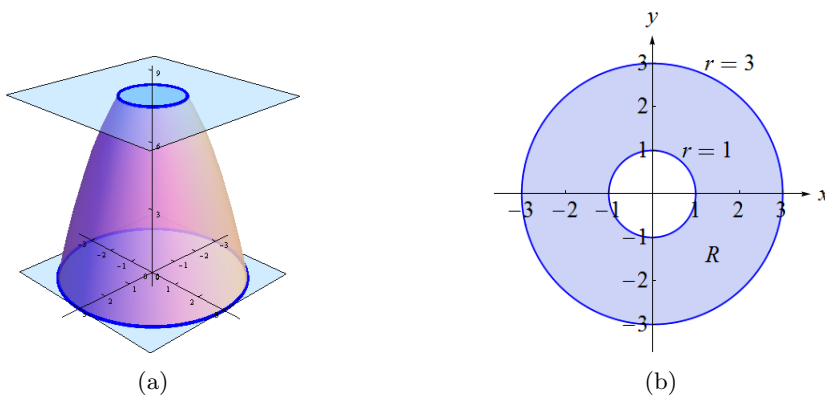
$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{(-2x)^2 + (-2y)^2 + 1} \, dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA = \iint_R \sqrt{4(x^2 + y^2) + 1} \, dA = \int_0^{2\pi} \int_1^3 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta. \end{aligned}$$

To evaluate the “inside” integral $\int_1^3 \sqrt{4r^2 + 1} \cdot r \, dr$, we make a substitution $u = 4r^2 + 1$; then $du = 8r \, dr$, so $r \, dr = \frac{1}{8} du$. We also change the limits of integration: if $r = 1$, then $u = 4 \cdot 1^2 + 1 = 5$; if $r = 3$, then $u = 4 \cdot 3^2 + 1 = 37$. So

$$\int_1^3 \sqrt{4r^2 + 1} \cdot r \, dr = \int_5^{37} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \int_5^{37} u^{1/2} du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_5^{37} = \frac{1}{12} \left[(\sqrt{u})^3 \right]_5^{37} = \frac{37\sqrt{37} - 5\sqrt{5}}{12}.$$

Going back to the calculation of S ,

$$\begin{aligned} S &= \int_0^{2\pi} \int_1^3 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{37\sqrt{37} - 5\sqrt{5}}{12} d\theta \\ &= \frac{37\sqrt{37} - 5\sqrt{5}}{12} \cdot [\theta]_0^{2\pi} = \frac{37\sqrt{37} - 5\sqrt{5}}{12} \cdot (2\pi - 0) = \boxed{\frac{(37\sqrt{37} - 5\sqrt{5})\pi}{6}}. \end{aligned}$$



21. The part of the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$ that lies inside the cylinder $x^2 + y^2 = 4$ is shown in the figure in Problem 21. The equation of the hemisphere can be rewritten as $z = f(x, y) = \sqrt{16 - x^2 - y^2}$. The projection of this part of the hemisphere onto the xy -plane is the region R that is a disk in the xy -plane bounded by the circle $x^2 + y^2 = 4$ (the base of the cylinder, as shown in the same figure). In polar coordinates R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{16 - x^2 - y^2}}$ and $f_y(x, y) = -\frac{y}{\sqrt{16 - x^2 - y^2}}$.

Then the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\
 &= \iint_R \sqrt{\left(-\frac{x}{\sqrt{16-x^2-y^2}}\right)^2 + \left(-\frac{y}{\sqrt{16-x^2-y^2}}\right)^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{x^2}{16-x^2-y^2} + \frac{y^2}{16-x^2-y^2} + 1} \, dA = \iint_R \sqrt{\frac{16}{16-x^2-y^2}} \, dA \\
 &= \iint_R \frac{4}{\sqrt{16-x^2-y^2}} \, dA = 4 \int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{16-r^2}} \cdot r \, dr \, d\theta = 4 \int_0^{2\pi} \int_0^2 (16-r^2)^{-1/2} \cdot r \, dr \, d\theta.
 \end{aligned}$$

To evaluate the “inside” integral $\int_0^2 (16-r^2)^{-1/2} \cdot r \, dr$, we make a substitution $u = 16-r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = 16-0^2 = 16$; if $r = 2$, then $u = 16-2^2 = 12$. So

$$\begin{aligned}
 \int_0^2 (16-r^2)^{-1/2} \cdot r \, dr &= \int_{16}^{12} u^{-1/2} \cdot \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_{16}^{12} u^{-1/2} \, du = \frac{1}{2} \int_{12}^{16} u^{-1/2} \, du \\
 &= \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_{12}^{16} = [\sqrt{u}]_{12}^{16} = \sqrt{16} - \sqrt{12} = 4 - 2\sqrt{3} = 2(2 - \sqrt{3}).
 \end{aligned}$$

Going back to the calculation of S ,

$$\begin{aligned}
 S &= 4 \int_0^{2\pi} \int_0^2 (16-r^2)^{-1/2} \cdot r \, dr \, d\theta = 4 \int_0^{2\pi} 2(2 - \sqrt{3}) \, d\theta \\
 &= 4 \cdot 2(2 - \sqrt{3}) \cdot [\theta]_0^{2\pi} = 8(2 - \sqrt{3}) \cdot (2\pi - 0) = \boxed{16\pi(2 - \sqrt{3})}.
 \end{aligned}$$

22. Let's introduce a rectangular coordinate system such that its xy -plane is the floor of the Pantheon, the origin is at the circular floor's center, and the z -axis goes vertically up. Then the inner surface of the Pantheon, which is a hemisphere of diameter 43.3 m, or equivalently or radius 21.65 m, can be represented by the equation $x^2 + y^2 + z^2 = 21.65^2$, $z \geq 0$, or $z = f(x, y) = \sqrt{21.65^2 - x^2 - y^2}$. The open circular ocular cut from its apex (see the figure in Problem 22) has a diameter of 9.1 m, or equivalently a radius of 4.55 m. The projection of this opening onto the xy -plane is the interior of the circle $x^2 + y^2 = 4.55^2$. A geometrical representation of the inner surface of the Pantheon is shown in figure (a) below. The projection of this surface onto the xy -plane is the annular region R shown in figure (b) below; in polar coordinates R is given by $4.55 \leq r \leq 21.65$, $0 \leq \theta \leq 2\pi$.

The partial derivatives of $z = f(x, y)$ are

$$f_x(x, y) = -\frac{x}{\sqrt{21.65^2 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = -\frac{y}{\sqrt{21.65^2 - x^2 - y^2}}.$$

Then the surface area S we seek is

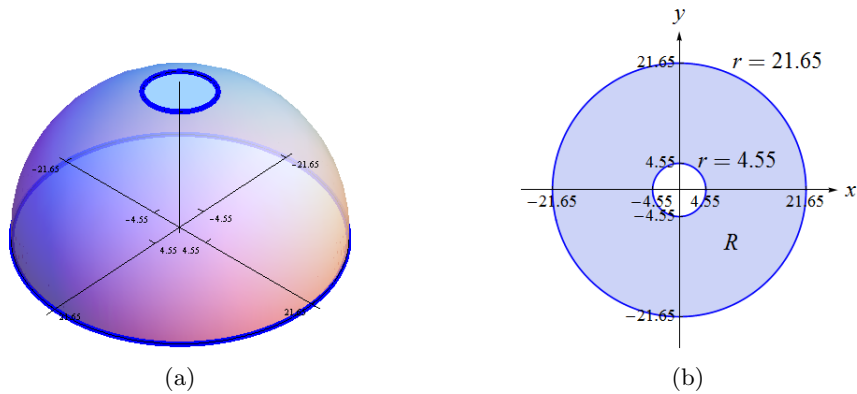
$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA \\
 &= \iint_R \sqrt{\left(-\frac{x}{\sqrt{21.65^2 - x^2 - y^2}}\right)^2 + \left(-\frac{y}{\sqrt{21.65^2 - x^2 - y^2}}\right)^2 + 1} dA \\
 &= \iint_R \sqrt{\frac{x^2}{21.65^2 - x^2 - y^2} + \frac{y^2}{21.65^2 - x^2 - y^2} + 1} dA = \iint_R \sqrt{\frac{21.65^2}{21.65^2 - x^2 - y^2}} dA \\
 &= \iint_R \frac{21.65}{\sqrt{21.65^2 - x^2 - y^2}} dA = 21.65 \int_0^{2\pi} \int_{4.55}^{21.65} \frac{1}{\sqrt{21.65^2 - r^2}} \cdot r dr d\theta \\
 &= 21.65 \int_0^{2\pi} \int_{4.55}^{21.65} (21.65^2 - r^2)^{-1/2} \cdot r dr d\theta.
 \end{aligned}$$

To evaluate the “inside” integral $\int_{4.55}^{21.65} (21.65^2 - r^2)^{-1/2} \cdot r dr$, we make a substitution $u = 21.65^2 - r^2$; then $du = -2r dr$, so $r dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = 4.55$, then $u = 21.65^2 - 4.55^2 = 448.02$; if $r = 21.65$, then $u = 21.65^2 - 21.65^2 = 0$. So

$$\begin{aligned}
 \int_{4.55}^{21.65} (21.65^2 - r^2)^{-1/2} \cdot r dr &= \int_{448.02}^0 u^{-1/2} \cdot \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_{448.02}^0 u^{-1/2} du \\
 &= \frac{1}{2} \int_0^{448.02} u^{-1/2} du = \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_0^{448.02} = [\sqrt{u}]_0^{448.02} \\
 &= \sqrt{448.02} - \sqrt{0} = \sqrt{448.02}.
 \end{aligned}$$

Going back to the calculation of S ,

$$\begin{aligned}
 S &= 21.65 \int_0^{2\pi} \int_{4.55}^{21.65} (21.65^2 - r^2)^{-1/2} \cdot r dr d\theta = 21.65 \int_0^{2\pi} \sqrt{448.02} d\theta \\
 &= 21.65 \cdot \sqrt{448.02} \cdot [\theta]_0^{2\pi} = 21.65 \sqrt{448.02} \cdot (2\pi - 0) = \boxed{43.3 \sqrt{448.02} \pi \approx 2879.297 \text{ m}^2}.
 \end{aligned}$$



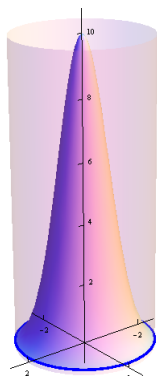
23. (a) Using the computer algebra system *Mathematica*, we can obtain a graph of the part of the surface $z = f(x, y) = 10e^{-(x^2+y^2)}$ that lies inside the cylinder $x^2 + y^2 = 4$ as shown in figure (a) below.

- (b) The projection of this surface onto the xy -plane is the region R that is the disk bounded by the circle $x^2 + y^2 = 4$; see figure (b) below. In polar coordinates R is given by $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -20xe^{-(x^2+y^2)}$ and $f_y(x, y) = -20ye^{-(x^2+y^2)}$. Then the surface area S we seek is

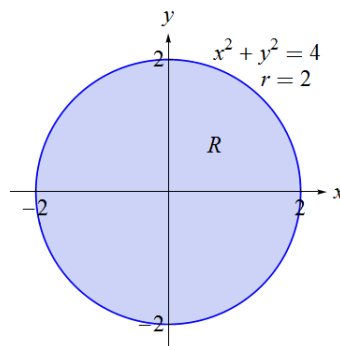
$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\
 &= \iint_R \sqrt{(-20xe^{-(x^2+y^2)})^2 + (-20ye^{-(x^2+y^2)})^2 + 1} \, dA \\
 &= \iint_R \sqrt{400x^2e^{-2(x^2+y^2)} + 400y^2e^{-2(x^2+y^2)} + 1} \, dA \\
 &= \iint_R \sqrt{400(x^2 + y^2)e^{-2(x^2+y^2)} + 1} \, dA \\
 &= \int_0^{2\pi} \int_0^2 \sqrt{400r^2e^{-2r^2} + 1} \cdot r \, dr \, d\theta.
 \end{aligned}$$

Entering this double integral into *Mathematica*, we find that its value is approximately

$$S = \int_0^{2\pi} \int_0^2 \sqrt{400r^2e^{-2r^2} + 1} \cdot r \, dr \, d\theta \approx \boxed{55.436}.$$



(a)

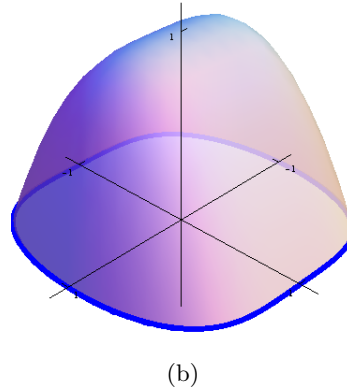
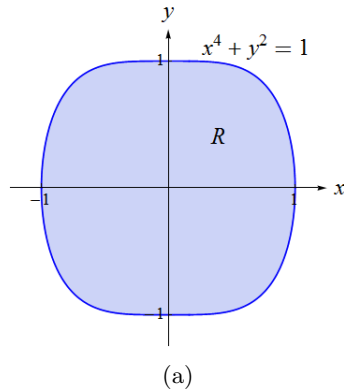


(b)

24. (a) If we substitute $z = 0$ into the equation of the surface $z = 1 - x^4 - y^2$, we obtain $0 = 1 - x^4 - y^2$ or $x^4 + y^2 = 1$. So the projection of the part of this surface that lies above the xy -plane onto the xy -plane is the region R enclosed by the graph of $x^4 + y^2 = 1$, as shown in figure (a) below. The partial derivatives of $z = f(x, y) = 1 - x^4 - y^2$ are $f_x(x, y) = -4x^3$ and $f_y(x, y) = -2y$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\
 &= \iint_R \sqrt{(-4x^3)^2 + (-2y)^2 + 1} \, dA = \boxed{\iint_R \sqrt{16x^6 + 4y^2 + 1} \, dA}.
 \end{aligned}$$

- (b) Using the computer algebra system *Mathematica*, we can generate the graph of the part of the surface $z = 1 - x^4 - y^2$ that lies above the xy -plane as shown in figure (b) below.

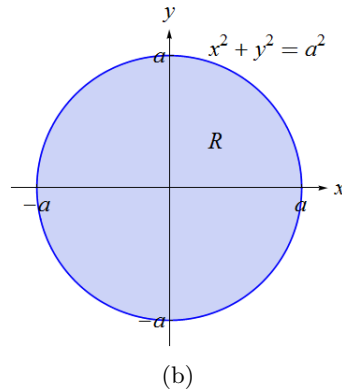
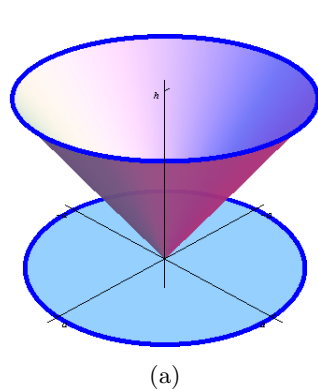


25. If we place a right circular cone of base radius a and altitude h in the rectangular coordinate system “upside down” with its vertex at the origin and its axis of symmetry up along the z -axis, as shown in figure (a) below, then this cone is given by the equation $\frac{z^2}{h^2} = \frac{x^2}{a^2} + \frac{y^2}{a^2}$, $0 \leq z \leq h$. This equation can be rewritten as $z = f(x, y) = \frac{h}{a} \sqrt{x^2 + y^2}$, $0 \leq z \leq h$. Its projection onto the xy -plane is the region R that is a disk bounded by the circle $x^2 + y^2 = a^2$, as shown in figure (b) below.

The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = \frac{hx}{a\sqrt{x^2 + y^2}}$ and $f_y(x, y) = \frac{hy}{a\sqrt{x^2 + y^2}}$. Then the lateral surface area S of this cone is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(\frac{hx}{a\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{hy}{a\sqrt{x^2 + y^2}}\right)^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{h^2 x^2}{a^2(x^2 + y^2)} + \frac{h^2 y^2}{a^2(x^2 + y^2)} + 1} \, dA = \iint_R \sqrt{\frac{h^2(x^2 + y^2)}{a^2(x^2 + y^2)} + 1} \, dA \\
 &= \iint_R \sqrt{\frac{h^2}{a^2} + 1} \, dA = \sqrt{\frac{h^2}{a^2} + 1} \iint_R dA = \sqrt{\frac{h^2 + a^2}{a^2}} \iint_R dA = \frac{1}{a} \sqrt{h^2 + a^2} \iint_R dA \\
 &= \frac{1}{a} \sqrt{a^2 + h^2} \cdot (\text{area of } R) = \frac{1}{a} \sqrt{a^2 + h^2} \cdot \pi a^2 = \pi a \sqrt{a^2 + h^2},
 \end{aligned}$$

as desired.



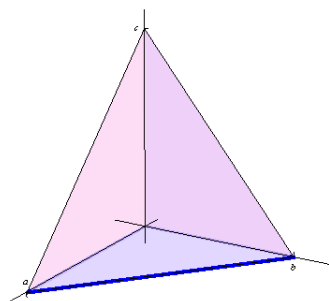
26. The first-octant portion of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (where a , b , and c are positive) is shown in

figure (a) below. This equation can be rewritten as $z = f(x, y) = c \left(1 - \frac{x}{a} - \frac{y}{b}\right)$. If substitute $z = 0$ into the equation of the plane, we obtain $\frac{x}{a} + \frac{y}{b} = 1$. So the projection of this part of the plane onto the xy -plane is the triangular region R in the xy -plane bounded by the coordinate axes and the line $\frac{x}{a} + \frac{y}{b} = 1$; see figure (b) below.

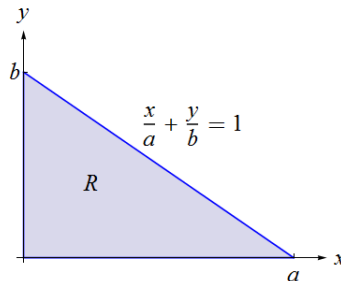
The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{c}{a}$ and $f_y(x, y) = -\frac{c}{b}$. Since both f_x and f_y are continuous on R , the surface area S we seek is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(-\frac{c}{a}\right)^2 + \left(-\frac{c}{b}\right)^2 + 1} \, dA \\
 &= \iint_R \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2} + 1} \, dA = \iint_R \sqrt{\frac{c^2 b^2}{a^2 b^2} + \frac{c^2 a^2}{a^2 b^2} + \frac{a^2 b^2}{a^2 b^2}} \, dA = \iint_R \sqrt{\frac{c^2 b^2 + c^2 a^2 + a^2 b^2}{a^2 b^2}} \, dA \\
 &= \sqrt{\frac{c^2 b^2 + c^2 a^2 + a^2 b^2}{a^2 b^2}} \iint_R dA = \frac{\sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}}{ab} \iint_R dA \\
 &= \frac{\sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}}{ab} \cdot (\text{area of } R) = \frac{\sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}}{ab} \cdot \frac{1}{2} ab = \frac{1}{2} \sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2},
 \end{aligned}$$

as desired.



(a)



(b)

27. If we place the sphere of radius R in the rectangular coordinate system so that its center is at the origin, then the equation of the sphere is $x^2 + y^2 + z^2 = R^2$; see figure (a) below. Since the sphere is symmetric about the xy -plane, we can find the surface area of the upper hemisphere $x^2 + y^2 + z^2 = R^2$, $z \geq 0$ and multiply it by 2.

The equation of the upper hemisphere can be rewritten as $z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$. If we substitute $z = 0$ into the equation of the sphere, we obtain $x^2 + y^2 = R^2$, so the projection of the sphere onto the xy -plane is the region T that is a disk in the xy -plane bounded by the circle $x^2 + y^2 = R^2$; see figure (b) below. In polar coordinates T is given by $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$.

The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$ and $f_y(x, y) = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$.

So the surface area S we seek is

$$\begin{aligned}
 S &= 2 \iint_T \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\
 &= 2 \iint_T \sqrt{\left(-\frac{x}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + \left(-\frac{y}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + 1} \, dA \\
 &= 2 \iint_T \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \, dA = 2 \iint_T \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} \, dA \\
 &= 2 \iint_T \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA = 2R \int_0^{2\pi} \int_0^R \frac{1}{\sqrt{R^2 - r^2}} \cdot r \, dr \, d\theta \\
 &= 2R \int_0^{2\pi} \int_0^R (R^2 - r^2)^{-1/2} \cdot r \, dr \, d\theta.
 \end{aligned}$$

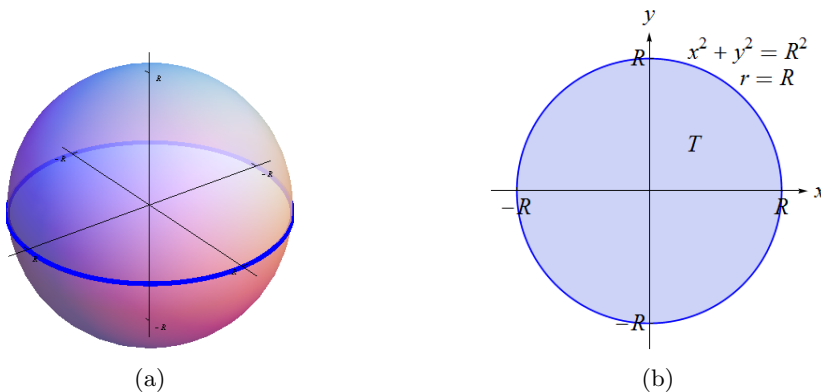
To evaluate the “inside” integral $\int_0^R (R^2 - r^2)^{-1/2} \cdot r \, dr$, we make a substitution $u = R^2 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = R^2 - 0^2 = R^2$; if $r = R$, then $u = R^2 - R^2 = 0$. So

$$\begin{aligned}
 \int_0^R (R^2 - r^2)^{-1/2} \cdot r \, dr &= \int_{R^2}^0 u^{-1/2} \cdot \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_{R^2}^0 u^{-1/2} du \\
 &= \frac{1}{2} \int_0^{R^2} u^{-1/2} du = \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_0^{R^2} = [\sqrt{u}]_0^{R^2} = \sqrt{R^2} - \sqrt{0} = R - 0 = R.
 \end{aligned}$$

Going back to the calculation of S ,

$$S = 2R \int_0^{2\pi} \int_0^R (R^2 - r^2)^{-1/2} \cdot r \, dr \, d\theta = 2R \int_0^{2\pi} R \, d\theta = 2R \cdot R \cdot [\theta]_0^{2\pi} = 2R^2 \cdot (2\pi - 0) = 4\pi R^2,$$

which is precisely the well-known formula for the surface area of a sphere.



Challenge Problems

28. Since it is given that $F_z(x, y, z) \neq 0$ at all points of the domain, we can apply the Implicit Differentiation Formulas II (Section 12.5, pg. 855): the equation $F(x, y, z) = 0$ defines $z = f(x, y)$ as an implicit function of x and y , and its partial derivatives are

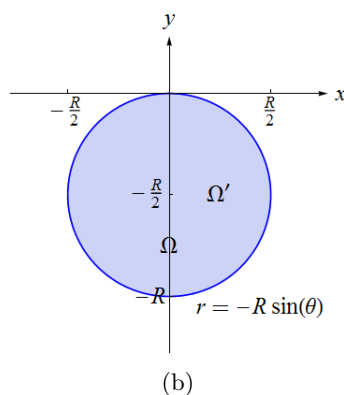
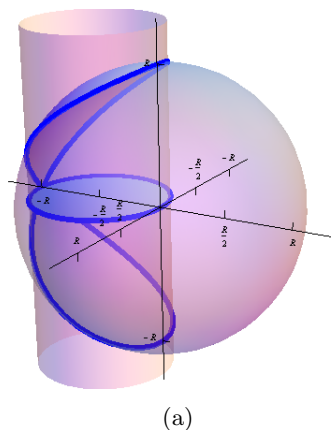
$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}.$$

Then the area S of the part of the surface $F(x, y, z) = 0$ that lies over a closed bounded region R is

$$\begin{aligned}
 S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA \\
 &= \iint_R \sqrt{\left[-\frac{F_x(x, y, z)}{F_z(x, y, z)}\right]^2 + \left[-\frac{F_y(x, y, z)}{F_z(x, y, z)}\right]^2 + 1} dA \\
 &= \iint_R \sqrt{\frac{[F_x(x, y, z)]^2}{[F_z(x, y, z)]^2} + \frac{[F_y(x, y, z)]^2}{[F_z(x, y, z)]^2} + 1} dA \\
 &= \iint_R \sqrt{\frac{[F_x(x, y, z)]^2}{[F_z(x, y, z)]^2} + \frac{[F_y(x, y, z)]^2}{[F_z(x, y, z)]^2} + \frac{[F_z(x, y, z)]^2}{[F_z(x, y, z)]^2}} dA \\
 &= \iint_R \sqrt{\frac{[F_x(x, y, z)]^2 + [F_y(x, y, z)]^2 + [F_z(x, y, z)]^2}{[F_z(x, y, z)]^2}} dA \\
 &= \iint_R \frac{\sqrt{[F_x(x, y, z)]^2 + [F_y(x, y, z)]^2 + [F_z(x, y, z)]^2}}{|F_z(x, y, z)|} dA,
 \end{aligned}$$

as desired.

29. Consider the sphere, the cylinder, and the surface that is cut from the sphere by the cylinder, as shown in the figure in Problem 29. If we shift the entire configuration to the left along the y -axis by $\frac{R}{2}$ units, so that the center of the sphere moves to the origin, then the area of the surface does not change; see figure (a) below. Now the sphere with its center at the origin and radius of R can be described by the equation $x^2 + y^2 + z^2 = R^2$. The cylinder after the shift contains the z -axis, and its trace in the xy -plane is the circle of radius $\frac{R}{2}$ centered at the point $\left(0, -\frac{R}{2}, 0\right)$. In Cartesian coordinates, the equation of the cylinder, as well as of its trace in the xy -plane, is $x^2 + \left(y + \frac{R}{2}\right)^2 = \frac{R^2}{4}$. This trace, which is a circle, encloses a region Ω in the xy -plane, that in polar coordinates can be described by $0 \leq r \leq -R \sin \theta$, where $\pi \leq \theta \leq 2\pi$; see figure (b) below.



Notice that the part of the sphere that lies inside the cylinder consists of two symmetric parts in the upper and lower hemispheres; moreover, either half of the surface is also symmetric about the yz -plane, and the region Ω is symmetric about the y -axis. So to find the total required surface area, we can find the surface area of the part lying in the upper hemisphere above the fourth-quadrant

half of Ω and multiply that by 4. The equation of the upper hemisphere can be written as $z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$ and $f_y(x, y) = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$. The region Ω' , that is the fourth-quadrant half of Ω , is defined by limiting the values of the polar angle to $\frac{3\pi}{2} \leq \theta \leq 2\pi$. Then the surface area S we seek is

$$\begin{aligned}
 S &= 4 \iint_{\Omega'} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA \\
 &= 4 \iint_{\Omega'} \sqrt{\left(-\frac{x}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + \left(-\frac{y}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + 1} \, dA \\
 &= 4 \iint_{\Omega'} \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1} \, dA = 4 \iint_{\Omega'} \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} \, dA \\
 &= 4 \iint_{\Omega'} \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, dA = 4R \int_{3\pi/2}^{2\pi} \int_0^{-R \sin \theta} \frac{1}{\sqrt{R^2 - r^2}} \cdot r \, dr \, d\theta \\
 &= 4R \int_{3\pi/2}^{2\pi} \int_0^{-R \sin \theta} (R^2 - r^2)^{-1/2} \cdot r \, dr \, d\theta.
 \end{aligned}$$

To evaluate the “inside” integral $\int_0^{-R \sin \theta} (R^2 - r^2)^{-1/2} \cdot r \, dr$, we make a substitution $u = R^2 - r^2$; then $du = -2r \, dr$, so $r \, dr = -\frac{1}{2} du$. We also change the limits of integration: if $r = 0$, then $u = R^2 - 0^2 = R^2$; if $r = -R \sin \theta$, then $u = R^2 - (-R \sin \theta)^2 = R^2 - R^2 \sin^2 \theta = R^2(1 - \sin^2 \theta) = R^2 \cos^2 \theta$. So

$$\begin{aligned}
 \int_0^{-R \sin \theta} (R^2 - r^2)^{-1/2} \cdot r \, dr &= \int_{R^2}^{R^2 \cos^2 \theta} u^{-1/2} \cdot \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_{R^2}^{R^2 \cos^2 \theta} u^{-1/2} \, du \\
 &= \frac{1}{2} \int_{R^2 \cos^2 \theta}^{R^2} u^{-1/2} \, du = \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_{R^2 \cos^2 \theta}^{R^2} = [\sqrt{u}]_{R^2 \cos^2 \theta}^{R^2} \\
 &= \sqrt{R^2} - \sqrt{R^2 \cos^2 \theta} = R - R \cos \theta = R(1 - \cos \theta),
 \end{aligned}$$

where we used the fact that $\cos \theta \geq 0$ because $\frac{3\pi}{2} \leq \theta \leq 2\pi$. Going back to the calculation of S ,

$$\begin{aligned}
 S &= 4R \int_{3\pi/2}^{2\pi} \int_0^{-R \sin \theta} (R^2 - r^2)^{-1/2} \cdot r \, dr \, d\theta = 4R \int_{3\pi/2}^{2\pi} R(1 - \cos \theta) \, d\theta \\
 &= 4R^2 \int_{3\pi/2}^{2\pi} (1 - \cos \theta) \, d\theta = 4R^2 [\theta - \sin \theta]_{3\pi/2}^{2\pi} = 4R^2 \left[(2\pi - \sin(2\pi)) - \left(\frac{3\pi}{2} - \sin \frac{3\pi}{2} \right) \right] \\
 &= 4R^2 \left[(2\pi - 0) - \left(\frac{3\pi}{2} - (-1) \right) \right] = 4R^2 \left(\frac{\pi}{2} - 1 \right) = \boxed{2R^2(\pi - 2) \text{ square units}}.
 \end{aligned}$$

30. To find the positive non-obtuse angle θ between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, we can use the formula $\cos \theta = \frac{|\mathbf{v}_1 \cdot \mathbf{v}_2|}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|}$, where $0 \leq \theta \leq \frac{\pi}{2}$ (see the discussion for Problems 71-76 in Section 10.6, pg. 742).

In particular, if γ is the positive acute angle between the normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, where $C \neq 0$, and the unit vector $\mathbf{k}$, then

$$\cos \gamma = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\| \|\mathbf{k}\|} = \frac{|(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot \mathbf{k}|}{\sqrt{A^2 + B^2 + C^2} \cdot 1} = \frac{|C|}{\sqrt{A^2 + B^2 + C^2}},$$

$$\text{so } \sec \gamma = \frac{1}{\cos \gamma} = \frac{\sqrt{A^2 + B^2 + C^2}}{|C|}.$$

Now, consider the plane surface given by the equation $F(x, y, z) = Ax + By + Cz - D = 0$, where $C \neq 0$. The normal vector to this plane is precisely the vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ considered above.

The equation of the plane can be rewritten as $z = f(x, y) = \frac{D - Ax - By}{C}$. The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{A}{C}$ and $f_y(x, y) = -\frac{B}{C}$. So the surface area S of the part of the plane lying above a closed bounded region R in the xy -plane is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{\left(-\frac{A}{C}\right)^2 + \left(-\frac{B}{C}\right)^2 + 1} \, dA \\ &= \iint_R \sqrt{\frac{A^2}{C^2} + \frac{B^2}{C^2} + 1} \, dA = \iint_R \sqrt{\frac{A^2}{C^2} + \frac{B^2}{C^2} + \frac{C^2}{C^2}} \, dA \\ &= \iint_R \sqrt{\frac{A^2 + B^2 + C^2}{C^2}} \, dA = \iint_R \frac{\sqrt{A^2 + B^2 + C^2}}{|C|} \, dA = \iint_R \sec \gamma \, dA, \end{aligned}$$

as desired.

14.6 The Triple Integral

Concepts and Vocabulary

1. True by the statement of an xy -simple solid in subsection 2.
2. False. The projection of an xy -simple solid onto the xy -plane is a closed, bounded region R , but this region R does not have to be bounded by smooth curves.
3. True by the statement of the mass M of a solid E of volume V with mass density $\rho = \rho(x, y, z)$ from subsection 4.
4. False. The triple integral of $\iiint_E dV$ represents the volume of the solid E .

Skill Building

5. Since $f(x, y, z) = xy^2$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned}\iiint_E xy^2 dV &= \int_0^1 \int_0^1 \int_0^4 xy^2 dz dy dx = \int_0^1 \int_0^1 xy^2 [z]_0^4 dy dx = \int_0^1 \int_0^1 4xy^2 dy dx \\ &= \int_0^1 4x \left[\frac{1}{3}y^3 \right]_0^1 dx = \int_0^1 \frac{4}{3}x dx = \left[\frac{2}{3}x^2 \right]_0^1 = \boxed{\frac{2}{3}}\end{aligned}$$

6. Since $f(x, y, z) = x^2yz$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned}\iiint_E x^2yz dV &= \int_0^2 \int_0^1 \int_0^2 x^2yz dz dy dx = \int_0^2 \int_0^1 x^2y \left[\frac{1}{2}z^2 \right]_0^2 dy dx = \int_0^2 \int_0^1 2x^2y dy dx \\ &= \int_0^2 2x^2 \left[\frac{1}{2}y^2 \right]_0^1 dx = \int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \boxed{\frac{8}{3}}\end{aligned}$$

7. Since $f(x, y, z) = x^2 + y^2 + z^2$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned}\iiint_E (x^2 + y^2 + z^2) dV &= \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx = \int_0^1 \int_0^1 \left[x^2z + y^2z + \frac{1}{3}z^3 \right]_0^1 dy dx \\ &= \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3} \right) dy dx = \int_0^1 \left[x^2y + \frac{1}{3}y^3 + \frac{1}{3}y \right]_0^1 dx \\ &= \int_0^1 \left(x^2 + \frac{2}{3} \right) dx = \left[\frac{1}{3}x^3 + \frac{2}{3}x \right]_0^1 = \boxed{1}\end{aligned}$$

8. Since $f(x, y, z) = x^2 - y^2 + z^2$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned}\iiint_E (x^2 - y^2 + z^2) dV &= \int_0^1 \int_0^2 \int_0^1 (x^2 - y^2 + z^2) dz dy dx = \int_0^1 \int_0^2 \left[x^2z - y^2z + \frac{1}{3}z^3 \right]_0^1 dy dx \\ &= \int_0^1 \int_0^2 \left(x^2 - y^2 + \frac{1}{3} \right) dy dx = \int_0^1 \left[x^2y - \frac{1}{3}y^3 + \frac{1}{3}y \right]_0^2 dx \\ &= \int_0^1 (2x^2 - 2) dx = \left[\frac{2}{3}x^3 - 2x \right]_0^1 = \boxed{-\frac{4}{3}}\end{aligned}$$

9. Since $f(x, y, z) = e^z \sin x \cos y$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \iiint_E e^z \sin x \cos y \, dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^z \sin x \cos y \, dz \, dy \, dx = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y [e^z]_0^1 \, dy \, dx \\ &= \int_0^{\pi/2} \int_0^{\pi/2} (e - 1) \sin x \cos y \, dy \, dx = \int_0^{\pi/2} (e - 1) \sin x [\sin y]_0^{\pi/2} \, dx \\ &= \int_0^{\pi/2} (e - 1) \sin x \, dx = (e - 1) [(-\cos x)]_0^{\pi/2} = \boxed{e - 1} \end{aligned}$$

10. Since $f(x, y, z) = e^{-z} \cos x \cos y$ is continuous on E , we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \iiint_E e^{-z} \cos x \cos y \, dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{-z} \cos x \cos y \, dz \, dy \, dx = \int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos y [-e^{-z}]_0^1 \, dy \, dx \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \left(1 - \frac{1}{e}\right) \cos x \cos y \, dy \, dx = \int_0^{\pi/2} \left(1 - \frac{1}{e}\right) \cos x [\sin y]_0^{\pi/2} \, dx \\ &= \int_0^{\pi/2} \left(1 - \frac{1}{e}\right) \cos x \, dx = \left(1 - \frac{1}{e}\right) [\sin x]_0^{\pi/2} = \boxed{1 - \frac{1}{e}} \end{aligned}$$

11. Since $f(x, y, z) = x$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \int_0^2 \int_0^{2-3x} \int_0^{x+y} x \, dz \, dy \, dx &= \int_0^2 \int_0^{2-3x} x [z]_0^{x+y} \, dy \, dx = \int_0^2 \int_0^{2-3x} (x^2 + xy) \, dy \, dx \\ &= \int_0^2 \left[x^2 y + \frac{1}{2} x y^2 \right]_0^{2-3x} \, dx = \int_0^2 [(2-3x)x^2 + \frac{1}{2} x(2-3x)^2] \, dx \\ &= \int_0^2 \left[2x^2 - 3x^3 + \frac{1}{2} x(4 - 12x + 9x^2) \right] \, dx = \int_0^2 \left(\frac{3}{2} x^3 - 4x^2 + 2x \right) \, dx \\ &= \left[\frac{3}{8} x^4 - \frac{4}{3} x^3 + x^2 \right]_0^2 = \boxed{-\frac{2}{3}} \end{aligned}$$

12. Since $f(x, y, z) = z$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \int_0^1 \int_0^{-x} \int_0^{2x+y} z \, dz \, dy \, dx &= \int_0^1 \int_0^{-x} \left[\frac{1}{2} z^2 \right]_0^{2x+y} \, dy \, dx = \int_0^1 \int_0^{-x} \left(2x^2 + 2xy + \frac{1}{2} y^2 \right) \, dy \, dx \\ &= \int_0^1 \left[2x^2 y + xy^2 + \frac{1}{6} y^3 \right]_0^{-x} \, dx = \int_0^1 \left(-2x^3 + x^3 - \frac{1}{6} x^3 \right) \, dx \\ &= \int_0^1 -\frac{7}{6} x^3 \, dx = \left[-\frac{7}{24} x^4 \right]_0^1 = \boxed{-\frac{7}{24}} \end{aligned}$$

13. Since $f(x, y, z) = 2x + 1$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \int_0^3 \int_z^{z+2} \int_y^{y+z} (2x + 1) \, dx \, dy \, dz &= \int_0^3 \int_z^{z+2} [x^2 + x]_y^{y+z} \, dy \, dz \\ &= \int_0^3 \int_z^{z+2} [y^2 + 2yz + z^2 + y + z - (y^2 + y)] \, dy \, dz \\ &= \int_0^3 \int_z^{z+2} (2yz + z^2 + z) \, dy \, dz = \int_0^3 [y^2 z + yz^2 + yz]_z^{z+2} \, dz \\ &= \int_0^3 [z(z^2 + 4z + 4) + z^2(z + 2) + z(z + 2) - (z^3 + z^3 + z^2)] \, dz \\ &= \int_0^3 (6z^2 + 6z) \, dz = [2z^3 + 3z^2]_0^3 = \boxed{81} \end{aligned}$$

14. Since $f(x, y, z) = 4z - 1$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals.

$$\begin{aligned}
 \int_0^2 \int_y^{y+2} \int_y^{x+y} (4z - 1) dz dx dy &= \int_0^2 \int_y^{y+2} [2z^2 - z]_y^{x+y} dx dy \\
 &= \int_0^2 \int_y^{y+2} [2x^2 + 4xy + 2y^2 - (x + y) - (2y^2 - y)] dx dy \\
 &= \int_0^2 \int_y^{y+2} (2x^2 + 4xy - x) dx dy = \int_0^2 \left[\frac{2}{3}x^3 + 2x^2y - \frac{1}{2}x^2 \right]_y^{y+2} dy \\
 &= \int_0^2 \left(12y^2 + 14y + \frac{10}{3} \right) dy = \left[4y^3 + 7y^2 + \frac{10}{3}y \right]_0^2 = \boxed{\frac{200}{3}}
 \end{aligned}$$

15. Since $f(x, y, z) = e^x$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals.

$$\begin{aligned}
 \int_0^2 \int_0^1 \int_0^y e^x dx dy dz &= \int_0^2 \int_0^1 [e^x]_0^y dy dz = \int_0^2 \int_0^1 (e^y - 1) dy dz \\
 &= \int_0^2 [e^y - y]_0^1 dz = \int_0^2 (e - 2) dz = (e - 2)[z]_0^2 = \boxed{2(e - 2)}
 \end{aligned}$$

16. Since $f(x, y, z) = xe^{x^2}$ is continuous for all real numbers, we use Fubini's Theorem for triple integrals. To integrate with respect to x , we use the substitution $u = x^2$. Then $du = 2x dx$ or $\frac{1}{2}du = x dx$, and the limits of integration become $u = 0$ to $u = y$.

$$\begin{aligned}
 \int_1^2 \int_0^z \int_0^{\sqrt{y}} xe^{x^2} dx dy dz &= \frac{1}{2} \int_1^2 \int_0^z \int_0^y e^u du dy dz = \frac{1}{2} \int_1^2 \int_0^z [e^u]_0^y dy dz \\
 &= \frac{1}{2} \int_1^2 \int_0^z (e^y - 1) dy dz = \frac{1}{2} \int_1^2 [e^y - y]_0^z dz = \frac{1}{2} \int_1^2 (e^z - z - 1) dz \\
 &= \frac{1}{2} \left[e^z - \frac{1}{2}z^2 - z \right]_1^2 = \frac{1}{2} \left[e^2 - 2 - 2 - \left(e - \frac{1}{2} - 1 \right) \right] = \frac{1}{2} \left(e^2 - e - \frac{5}{2} \right) \\
 &= \boxed{\frac{e^2}{2} - \frac{e}{2} - \frac{5}{4}}
 \end{aligned}$$

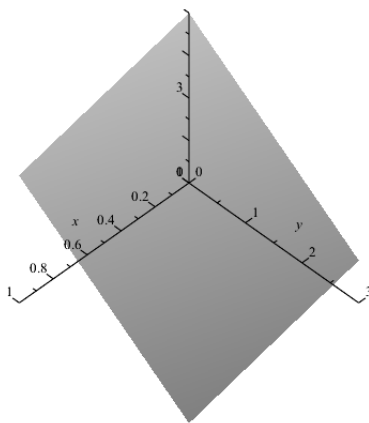
17. To evaluate $f(x, y, z) = \sin\left(\frac{z}{y}\right)$, we use Fubini's Theorem for triple integrals.

$$\begin{aligned}
 \int_0^{\pi/2} \int_y^{\pi/2} \int_0^{xy} \sin\left(\frac{z}{y}\right) dz dx dy &= \int_0^{\pi/2} \int_y^{\pi/2} \left[-y \cos\left(\frac{z}{y}\right) \right]_0^{xy} dx dy \\
 &= \int_0^{\pi/2} \int_y^{\pi/2} (-y \cos x + y) dx dy = \int_0^{\pi/2} [-y \sin x + xy]_y^{\pi/2} dy \\
 &= \int_0^{\pi/2} \left(-y + \frac{\pi}{2}y + y \sin y - y^2 \right) dy \\
 &= \left[-\frac{1}{2}y^2 + \frac{\pi}{4}y^2 - y \cos y + \sin y - \frac{1}{3}y^3 \right]_0^{\pi/2} \\
 &= -\frac{\pi^2}{8} + \frac{\pi^3}{16} + 1 - \frac{\pi^3}{24} = \boxed{1 - \frac{\pi^2}{8} + \frac{\pi^3}{48}}
 \end{aligned}$$

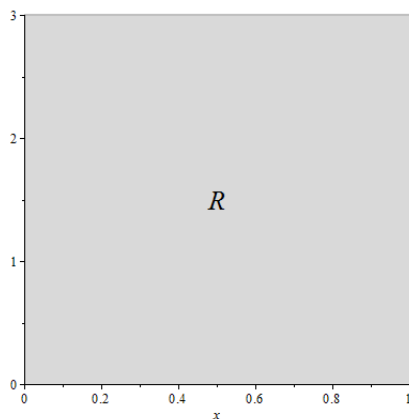
18. To evaluate $f(x, y, z) = \cos\left(\frac{z}{x}\right)$, we use Fubini's Theorem for triple integrals.

$$\begin{aligned} \int_0^{\pi/2} \int_x^{\pi/2} \int_0^{xy} \cos\left(\frac{z}{x}\right) dz dy dx &= \int_0^{\pi/2} \int_x^{\pi/2} \left[x \sin\left(\frac{z}{x}\right) \right]_0^{xy} dy dx \\ &= \int_0^{\pi/2} \int_x^{\pi/2} x \sin y dy dx = \int_0^{\pi/2} [-x \cos y]_x^{\pi/2} dx \\ &= \int_0^{\pi/2} x \cos x dx = [x \sin x + \cos x]_0^{\pi/2} = \boxed{\frac{\pi}{2} - 1} \end{aligned}$$

19. Since $0 \leq z \leq 5 - x - y$, E is xy -simple.



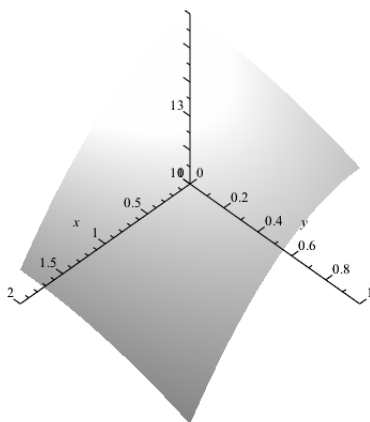
The region R in the xy -plane is enclosed by the x -axis, the y -axis, and the lines $x = 1$ and $y = 3$, so R is both x - and y -simple.



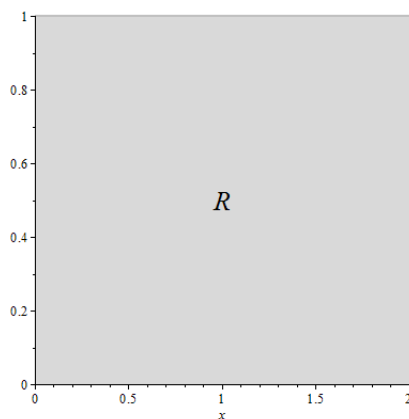
Treating R as x -simple,

$$\begin{aligned} \iiint_E xy dV &= \int_0^1 \int_0^3 \int_0^{5-x-y} xy dz dy dx = \int_0^1 \int_0^3 xy [z]_0^{5-x-y} dy dx = \int_0^1 \int_0^3 (5xy - x^2y - xy^2) dy dx \\ &= \int_0^1 \left[\frac{5}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3 \right]_0^3 dx = \int_0^1 \left(\frac{45}{2}x - \frac{9}{2}x^2 - 9x \right) dx = \left[\frac{45}{4}x^2 - \frac{3}{2}x^3 - \frac{9}{2}x^2 \right]_0^1 \\ &= \frac{45}{4} - \frac{3}{2} - \frac{9}{2} = \boxed{\frac{21}{4}} \end{aligned}$$

20. Since $0 \leq z \leq 16 - x^2 - y^2$, E is xy -simple.



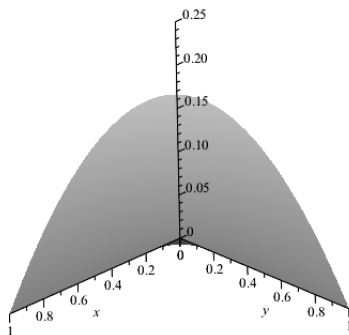
The region R in the xy -plane is enclosed by the x -axis, the y -axis, and the lines $x = 2$ and $y = 1$, so R is both x - and y -simple.



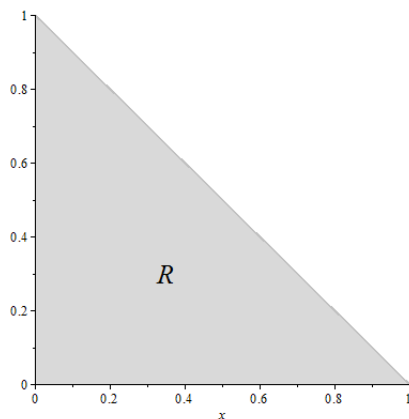
Treating R as x -simple,

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^2 \int_0^1 \int_0^{16-x^2-y^2} xy \, dz \, dy \, dx = \int_0^2 \int_0^1 xy [z]_0^{16-x^2-y^2} \, dy \, dx \\
 &= \int_0^2 \int_0^1 (16xy - x^3y - xy^3) \, dy \, dx = \int_0^2 \left[8xy^2 - \frac{1}{2}x^3y^2 - \frac{1}{4}xy^4 \right]_0^1 \, dx \\
 &= \int_0^2 \left(8x - \frac{1}{2}x^3 - \frac{1}{4}x \right) \, dx = \left[4x^2 - \frac{1}{8}x^4 - \frac{1}{8}x^2 \right]_0^2 = \boxed{\frac{27}{2}}
 \end{aligned}$$

21. Since $0 \leq z \leq xy$, E is xy -simple.



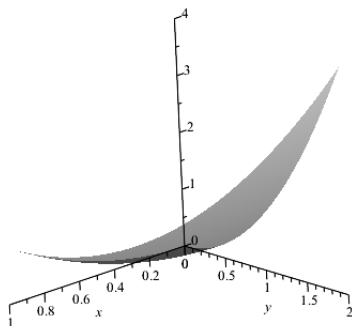
The region R in the xy -plane is the triangle enclosed by the x -axis, the y -axis, and the line $y = 1 - x$, so R is both x - and y -simple.



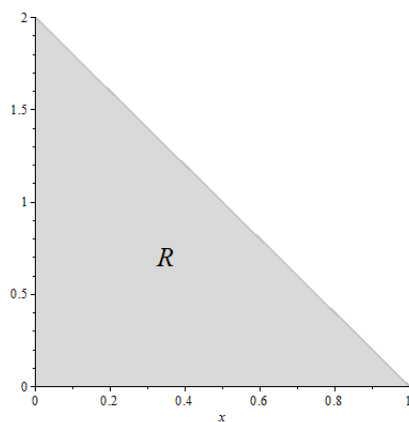
Treating R as x -simple,

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^1 \int_0^{1-x} \int_0^{xy} xy \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} xy [z]_0^{xy} \, dy \, dx = \int_0^1 \int_0^{1-x} x^2 y^2 \, dy \, dx \\
 &= \int_0^1 \frac{1}{3} x^2 [y^3]_0^{1-x} \, dx = \int_0^1 \frac{1}{3} x^2 (1 - 3x + 3x^2 - x^3) \, dx = \int_0^1 \left(\frac{1}{3} x^2 - x^3 + x^4 - \frac{1}{3} x^5 \right) \, dx \\
 &= \left[\frac{1}{9} x^3 - \frac{1}{4} x^4 + \frac{1}{5} x^5 - \frac{1}{18} x^6 \right]_0^1 = \frac{1}{9} - \frac{1}{4} + \frac{1}{5} - \frac{1}{18} = \boxed{\frac{1}{180}}
 \end{aligned}$$

22. Since $0 \leq z \leq x^2 + y^2$, E is xy -simple.



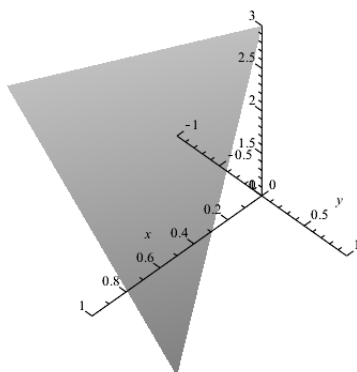
The region R in the xy -plane is the triangle enclosed by the x -axis, the y -axis, and the line $y = 2 - 2x$, so R is both x - and y -simple.



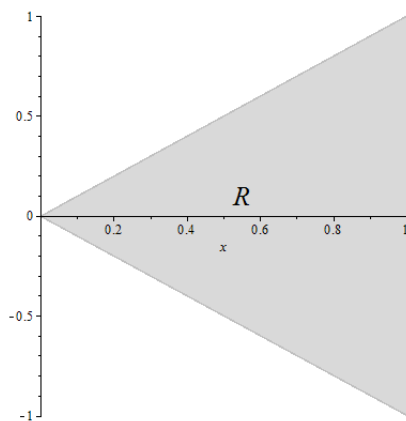
Treating R as x -simple,

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^1 \int_0^{2-2x} \int_0^{x^2+y^2} xy \, dz \, dy \, dx = \int_0^1 \int_0^{2-2x} xy [z]_0^{x^2+y^2} \, dy \, dx \\
 &= \int_0^1 \int_0^{2-2x} (x^3y + xy^3) \, dy \, dx = \int_0^1 \left[\frac{1}{2}x^3y^2 + \frac{1}{4}xy^4 \right]_0^{2-2x} \, dx \\
 &= \int_0^1 \left[\frac{1}{2}x^3(4-8x+4x^2) + \frac{1}{4}x(16-64x+96x^2-64x^3+16x^4) \right] \, dx \\
 &= \int_0^1 (6x^5 - 20x^4 + 26x^3 - 16x^2 + 4x) \, dx = \left[x^6 - 4x^5 + \frac{13}{2}x^4 - \frac{16}{3}x^3 + 2x^2 \right]_0^1 = \boxed{\frac{1}{6}}
 \end{aligned}$$

23. Since $0 \leq z \leq 3 - x - y$, E is xy -simple.



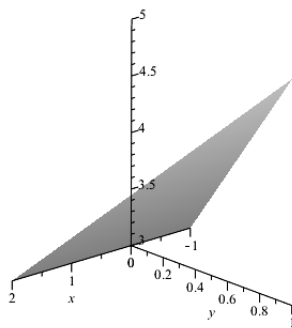
The region R in the xy -plane is the triangle enclosed by the lines $y = x$ and $y = -x$, so R is x -simple.



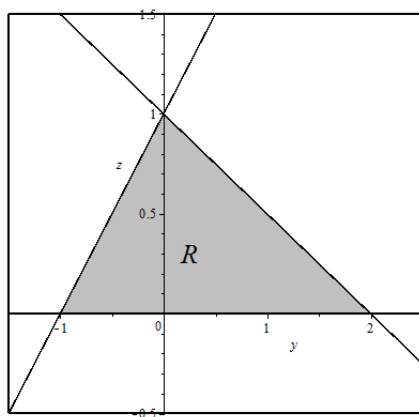
Then

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^1 \int_{-x}^x \int_0^{3-x-y} xy \, dz \, dy \, dx = \int_0^1 \int_{-x}^x xy [z]_0^{3-x-y} \, dy \, dx \\
 &= \int_0^1 \int_{-x}^x (3xy - x^2y - xy^2) \, dy \, dx = \int_0^1 \left[\frac{3}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3 \right]_{-x}^x \, dx \\
 &= \int_0^1 -\frac{2}{3}x^4 \, dx = \left[-\frac{2}{15}x^5 \right]_0^1 = \boxed{-\frac{2}{15}}
 \end{aligned}$$

24. Since $0 \leq z \leq 3 + 2y$, E is xy -simple.



The region R in the xy -plane is the triangle enclosed by the x -axis and the lines $x = y - 1$ and $x = 2 - 2y$, so R is y -simple.



Then

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^1 \int_{y-1}^{2-2y} \int_0^{3+2y} xy \, dz \, dx \, dy = \int_0^1 \int_{y-1}^{2-2y} xy [z]_0^{3+2y} \, dx \, dy \\
 &= \int_0^1 \int_{y-1}^{2-2y} (3xy + 2xy^2) \, dx \, dy = \int_0^1 \left[\frac{3}{2}x^2y + x^2y^2 \right]_{y-1}^{2-2y} dy \\
 &= \int_0^1 \left[6y - 12y^2 + 6y^3 + 4y^2 - 8y^3 + 4y^4 - \left(\frac{3}{2}y^3 - 3y^2 + \frac{3}{2}y + y^4 - 2y^3 + y^2 \right) \right] dy \\
 &= \int_0^1 \left(3y^4 - \frac{3}{2}y^3 - 6y^2 + \frac{9}{2}y \right) dy = \left[\frac{3}{5}y^5 - \frac{3}{8}y^4 - 2y^3 + \frac{9}{4}y^2 \right]_0^1 = \boxed{\frac{19}{40}}
 \end{aligned}$$

25. Using MapleTM,

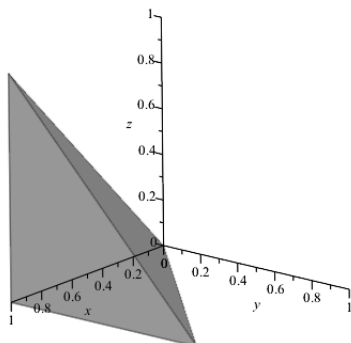
$$\int_0^1 \int_1^{x^2} \int_0^{\sqrt{x}} ye^x \, dz \, dy \, dx \approx \boxed{-0.4142}.$$

26. Using MapleTM,

$$\int_3^4 \int_2^3 \int_0^1 \ln(x^2 + y^2 + z^2) \, dx \, dy \, dz \approx \boxed{2.9356}.$$

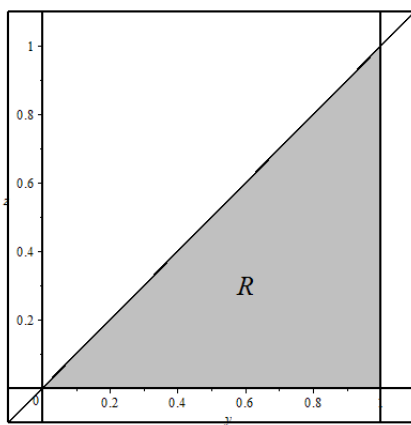
Applications and Extensions

27. The tetrahedron is bounded by four planes.



The points $(0, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 0)$ all have a z -coordinate of 0, which means that the tetrahedron is bounded by the plane $z = 0$, or the xy -plane. Similarly, the tetrahedron is bounded by the xz -plane and the plane $x = 1$. Using the points $(0, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 1)$, we find that the vectors $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + \mathbf{k}$ lie in the fourth plane, so the vector $\mathbf{N} = \mathbf{v} \times \mathbf{w} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane. Using the point $(0, 0, 0)$, the equation of the plane is $x - y - z = 0$ or $z = x - y$. Then $0 \leq z \leq x - y$, and E is xy -simple.

The region R in the xy -plane is the triangle enclosed by the x -axis, the y -axis, and the line $y = x$, so R is both x - and y -simple.



Treating R as x -simple,

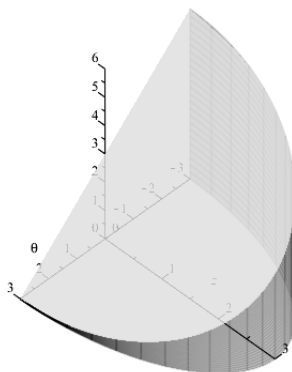
$$\begin{aligned} \iiint_E x \, dV &= \int_0^1 \int_0^x \int_0^{x-y} x \, dz \, dy \, dx = \int_0^1 \int_0^x x [z]_0^{x-y} \, dy \, dx = \int_0^1 \int_0^x (x^2 - xy) \, dy \, dx \\ &= \int_0^1 \left[x^2 y - \frac{1}{2} xy^2 \right]_0^x \, dx = \int_0^1 \left(x^3 - \frac{1}{2} x^3 \right) \, dx = \int_0^1 \frac{1}{2} x^3 \, dx = \left[\frac{1}{8} x^4 \right]_0^1 = \boxed{\frac{1}{8}} \end{aligned}$$

28. See the figures and explanation of the equations for the planes of the tetrahedron in Problem 27.

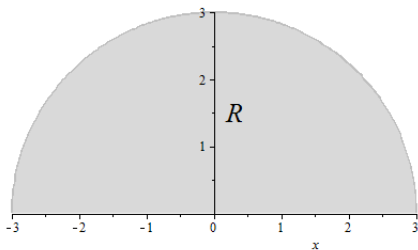
Treating R as x -simple,

$$\begin{aligned}
 \iiint_E (x^2 + z^2) dV &= \int_0^1 \int_0^x \int_0^{x-y} (x^2 + z^2) dz dy dx = \int_0^1 \int_0^x \left[x^2 z + \frac{1}{3} z^3 \right]_0^{x-y} dy dx \\
 &= \int_0^1 \int_0^x \left[x^3 - x^2 y + \frac{1}{3} (x^3 - 3x^2 y + 3xy^2 - y^3) \right] dy dx \\
 &= \int_0^1 \int_0^x \left(\frac{4}{3} x^3 - 2x^2 y + xy^2 - \frac{1}{3} y^3 \right) dy dx \\
 &= \int_0^1 \left[\frac{4}{3} x^3 y - x^2 y^2 + \frac{1}{3} xy^3 - \frac{1}{12} y^4 \right]_0^x dx = \int_0^1 \left(\frac{4}{3} x^4 - x^4 + \frac{1}{3} x^4 - \frac{1}{12} x^4 \right) dx \\
 &= \int_0^1 \frac{7}{12} x^4 dx = \left[\frac{7}{60} x^5 \right]_0^1 = \boxed{\frac{7}{60}}
 \end{aligned}$$

29. Since $0 \leq z \leq 3 - x$, E is xy -simple.



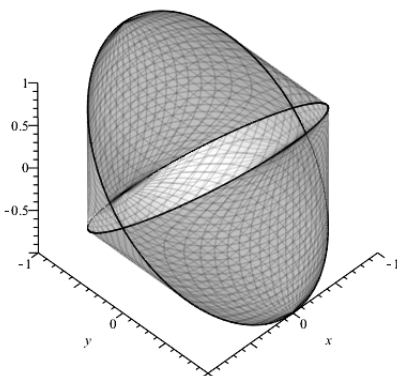
The region R in the xy -plane is the region enclosed by x -axis and the semicircle $y = \sqrt{9 - x^2}$, so R is x -simple.



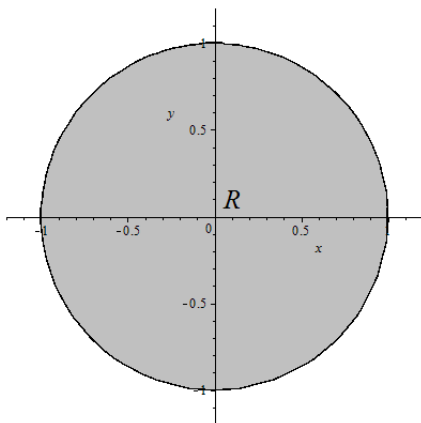
Then

$$\begin{aligned}
 \iiint_E (xy + 3y) dV &= \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^{3-x} (xy + 3y) dz dy dx = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} [xyz + 3yz]_0^{3-x} dy dx \\
 &= \int_{-3}^3 \int_0^{\sqrt{9-x^2}} (3xy - x^2y + 9y - 3xy) dy dx = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} (9y - x^2y) dy dx \\
 &= \int_{-3}^3 \left[\frac{9}{2}y^2 - \frac{1}{2}x^2y^2 \right]_0^{\sqrt{9-x^2}} dx = \int_{-3}^3 \left[\frac{9}{2}(9-x^2) - \frac{1}{2}x^2(9-x^2) \right] dx \\
 &= \int_{-3}^3 \left(\frac{81}{2} - 9x^2 + \frac{1}{2}x^4 \right) dx = \left[\frac{81}{2}x - 3x^3 + \frac{1}{10}x^5 \right]_{-3}^3 \\
 &= \left(\frac{243}{2} - 81 + \frac{243}{10} \right) - \left(-\frac{243}{2} + 81 - \frac{243}{10} \right) = \boxed{\frac{648}{5}}
 \end{aligned}$$

30. Since $x^2 + z^2 = 1$, $-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}$, so E is xy -simple.



The region R in the xy -plane is bounded by the unit circle, so R is both x - and y -simple.

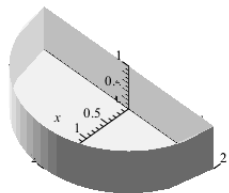


Treating R as x -simple,

$$\begin{aligned}
 \iiint_E xyz dV &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xyz dz dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy \left[\frac{1}{2}z^2 \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx \\
 &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}xy[(1-x^2) - (1-x^2)] dy dx = \boxed{0}
 \end{aligned}$$

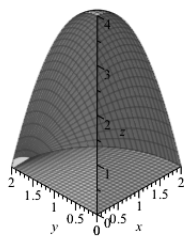
31. (a) From the limits of integration, $0 \leq z \leq 1$, $0 \leq x \leq \sqrt{4 - y^2}$, and $-2 \leq y \leq 2$. The region R in the xy -plane is bounded by the right side of a semicircle of radius 2. Since $0 \leq z \leq 1$, the solid is the semicircle bounded by $x^2 + y^2 = 4$ and the planes $x = 0$, $z = 0$, and $z = 1$.

(b) The solid was generated using MapleTM.



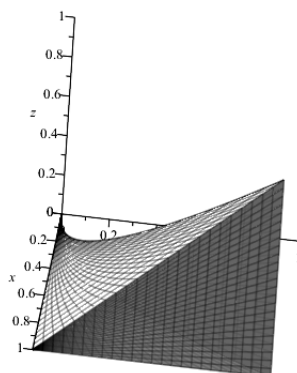
32. (a) From the limits of integration, $0 \leq z \leq 4 - x^2 - y^2$, $0 \leq x \leq \sqrt{4 - y^2}$, and $0 \leq y \leq 2$. The graph of $z = 4 - x^2 - y^2$ is a paraboloid of revolution. Since $x = \sqrt{4 - y^2}$ is a semicircle of radius 2 and both y and z are restricted to positive values, the solid is a paraboloid of revolution in the first octant.

(b) The solid was generated using MapleTM.

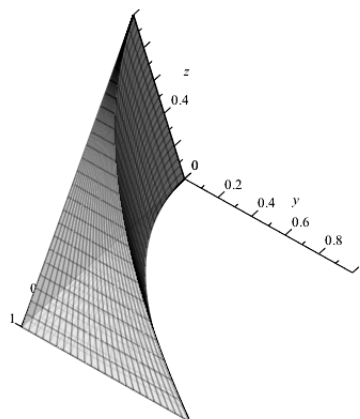


33. (a) From the limits of integration, $0 \leq z \leq y$, $0 \leq y \leq x^2$, and $0 \leq x \leq 1$. The solid is bounded by $y = x^2$ and the planes $z = 0$, $x = 1$, and $z = y$.

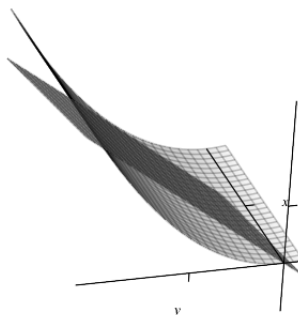
(b) The solid was generated using MapleTM.



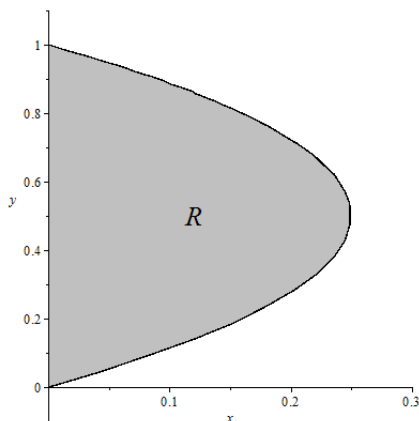
34. (a) From the limits of integration, $0 \leq y \leq 1$, $y^2 \leq x \leq 1$, and $0 \leq z \leq 1 - x$. The solid is bounded by $x = y^2$ and the planes $y = 0$, $y = 1$, $z = 0$, and $z = 1 - x$.
 (b) The solid was generated using Maple™.



35. The solid E is enclosed by the parabolic cylinder $z = y^2$ and the planes $x = 0$ and $x = y - z$, which is xy -simple. The upper surface is the plane $z = y - x$, and the lower surface is the parabolic cylinder $z = y^2$.



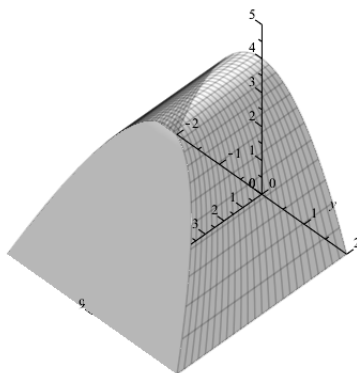
The parabolic cylinder and the plane intersect at $y^2 = y - x$ or $x = y - y^2$. Then the region R is described by $0 \leq x \leq y - y^2$ and $0 \leq y \leq 1$ and is x -simple.



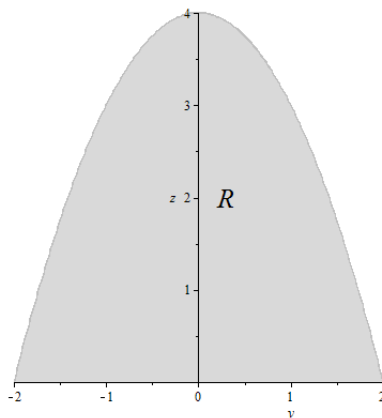
Then

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^1 \int_0^{y-y^2} \int_{y^2}^{y-x} dz \, dx \, dy = \int_0^1 \int_0^{y-y^2} [z]_{y^2}^{y-x} \, dx \, dy = \int_0^1 \int_0^{y-y^2} (y-x-y^2) \, dx \, dy \\
 &= \int_0^1 \left[xy - \frac{1}{2}x^2 - xy^2 \right]_0^{y-y^2} dy = \int_0^1 \left[y^2 - y^3 - \frac{1}{2}(y^2 - 2y^3 + y^4) - (y^3 - y^4) \right] dy \\
 &= \int_0^1 \left(\frac{1}{2}y^4 - y^3 + \frac{1}{2}y^2 \right) dy = \left[\frac{1}{10}y^4 - \frac{1}{4}y^4 + \frac{1}{6}y^3 \right]_0^1 = \boxed{\frac{1}{60}}
 \end{aligned}$$

36. The solid E is yz -simple.



The front surface is the plane $x = 9 - z$, and the back surface is $x = 0$. The region R in the yz -plane is enclosed by $z = 0$ and the parabola $z = 4 - y^2$ and is z -simple.



Then the volume of E is

$$\begin{aligned}
 V &= \iiint_E dV = \int_{-2}^2 \int_0^{4-y^2} \int_0^{9-z} dx \, dz \, dy = \int_{-2}^2 \int_0^{4-y^2} [x]_0^{9-z} \, dz \, dy = \int_{-2}^2 \int_0^{4-y^2} (9-z) \, dz \, dy \\
 &= \int_{-2}^2 \left[9z - \frac{1}{2}z^2 \right]_0^{4-y^2} dy = \int_{-2}^2 \left[36 - 9y^2 - \frac{1}{2}(16 - 8y^2 + y^4) \right] dy = \int_{-2}^2 \left(-\frac{1}{2}y^4 - 5y^2 + 28 \right) dy \\
 &= 2 \int_0^2 \left(-\frac{1}{2}y^4 - 5y^2 + 28 \right) dy = 2 \left[-\frac{1}{10}y^5 - \frac{5}{3}y^3 + 28y \right]_0^2 = 2 \left(-\frac{32}{10} - \frac{40}{3} + 56 \right) = \boxed{\frac{1184}{15}}
 \end{aligned}$$

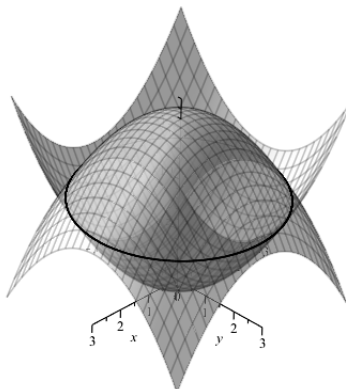
37. The region R is the projection in the xy -plane of the region enclosed by the circle $x^2 + y^2 = 1$ which can also be expressed by $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $-1 \leq x \leq 1$. The solid E is bounded by $z = 3 - x - y$ above and $z = 1 - x - y$ below and is xy -simple. Then

$$\iiint_E xy \, dV = \boxed{\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-x-y}^{3-x-y} xy \, dz \, dy \, dx}$$

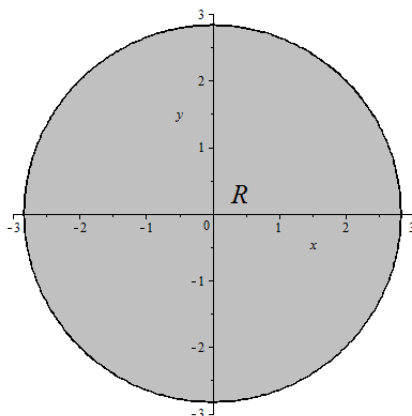
38. The region R is the projection in the xy -plane of the region enclosed by the circle $x^2 + y^2 = 4$ which can also be expressed by $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ and $-2 \leq x \leq 2$. The solid E is bounded by $z = x^2 + y$ above and $z = 0$ below and is xy -simple. Then

$$\iiint_E xy \, dV = \boxed{\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2+y} xy \, dz \, dy \, dx}$$

39. The solid E is enclosed by the paraboloids of revolution $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.



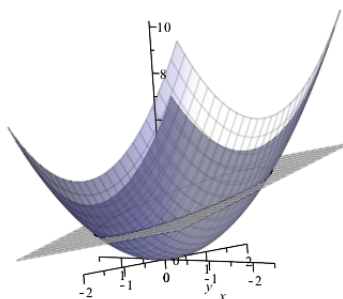
The region R is bounded by the projection of the intersection of these paraboloids of revolution. Simplifying $x^2 + y^2 = 16 - x^2 - y^2$ gives the boundary for R as $x^2 + y^2 = 8$ which can also be expressed as $-\sqrt{8-x^2} \leq y \leq \sqrt{8-x^2}$, $-2\sqrt{2} \leq x \leq 2\sqrt{2}$.



Then using symmetry,

$$V = \iiint_E dV = \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{x^2+y^2}^{16-x^2-y^2} dz \, dy \, dx = \boxed{4 \int_0^{\sqrt{8}} \int_0^{\sqrt{8-x^2}} \int_{x^2+y^2}^{16-x^2-y^2} dz \, dy \, dx}.$$

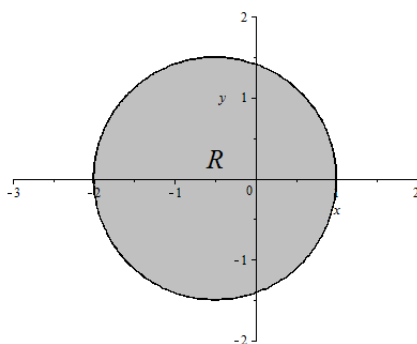
40. The solid E is enclosed by the paraboloid of revolution $z = x^2 + y^2$ and the plane $z = 2 - x$, and the region R is bounded by the projection of the intersection of these functions.



Simplifying $x^2 + y^2 = 2 - x$ gives $x^2 + x + y^2 = 2$. We complete the square.

$$\begin{aligned} x^2 + x + \left(\frac{1}{2}\right)^2 + y^2 &= 2 + \left(\frac{1}{2}\right)^2 \\ \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{9}{4} \end{aligned}$$

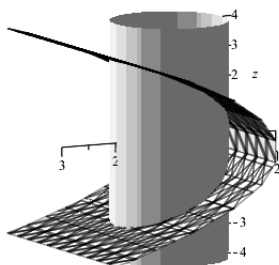
The boundary for the region R is a circle centered at $(-\frac{1}{2}, 0)$ with radius $r = \frac{3}{2}$, which can also be expressed as $-\sqrt{\frac{9}{4} - (x + \frac{1}{2})^2} \leq y \leq \sqrt{\frac{9}{4} - (x + \frac{1}{2})^2}$, $-2 \leq x \leq 1$.



Then

$$V = \iiint_E dV = \int_{-2}^1 \int_{-\sqrt{\frac{9}{4} - (x + \frac{1}{2})^2}}^{\sqrt{\frac{9}{4} - (x + \frac{1}{2})^2}} \int_{x^2 + y^2}^{2-x} dz \, dy \, dx$$

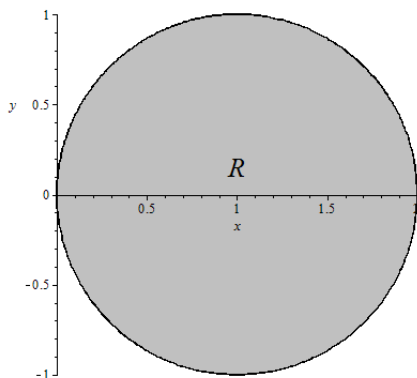
41. The solid E is enclosed by the parabolic cylinder $z^2 = 4x$ and the right circular cylinder $x^2 + y^2 = 2x$. Because $z^2 = 4x$, $-2\sqrt{x} \leq z \leq 2\sqrt{x}$. We note the symmetry with respect to the xy -plane.



We can use the projection on to the xy -plane as the region R . We complete the square in the equation $x^2 + y^2 = 2x$.

$$\begin{aligned} x^2 - 2x + \left(\frac{-2}{2}\right)^2 + y^2 &= \left(\frac{-2}{2}\right)^2 \\ (x - 1)^2 + y^2 &= 1 \end{aligned}$$

The boundary for the region R is a circle centered at $(1, 0)$ with radius $r = 1$, which can also be expressed as $-\sqrt{1 - (x - 1)^2} \leq y \leq \sqrt{1 - (x - 1)^2}$, $0 \leq x \leq 2$.



Then

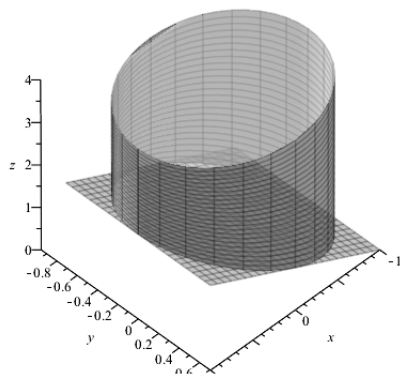
$$V = \iiint_E dV = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \int_{-2\sqrt{x}}^{2\sqrt{x}} dz dy dx = \boxed{4 \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \int_0^{2\sqrt{x}} dz dy dx}$$

42. (a) The solid E is enclosed by the elliptic cylinder $x^2 + 2y^2 = 1$ and the planes $z = x + 1$ and $z = 4$.

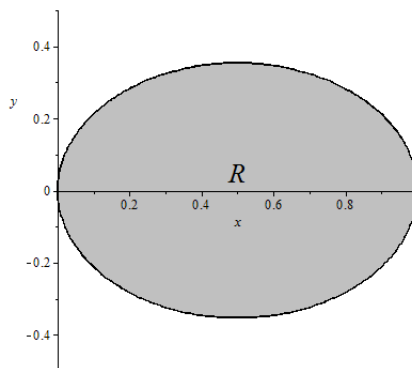
The region R is bounded by the ellipse $x^2 + 2y^2 = 1$ in the xy -plane, which can also be expressed as $-\sqrt{\frac{1-x^2}{2}} \leq y \leq \sqrt{\frac{1-x^2}{2}}$, $-1 \leq x \leq 1$. Then using symmetry,

$$V = \iiint_E dV = 4 \int_0^1 \int_0^{\sqrt{\frac{1-x^2}{2}}} \int_{x+1}^4 dz dy dx = \boxed{\frac{3\sqrt{2}}{2}\pi - \frac{2\sqrt{2}}{3}}.$$

- (b) The solid E



and the region R



43. Let a corner of the cube be located at the origin so that the distance between it and any point is $d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$. The cube is xy -, yz -, and xz -simple, and any region R will be bounded by a square with side length a . We treat the cube as xy -simple and the region R as x -simple. Since the mass density is proportional to the square of the distance from one corner, $\rho = \rho(x, y, z) = kd^2 = k(x^2 + y^2 + z^2)$ where k is the constant of proportionality. We know the length of each side of the cube is a , so the mass M is

$$\begin{aligned}
 M &= \iiint_E \rho(x, y, z) dV = \iiint_E k(x^2 + y^2 + z^2) dV = \int_0^a \int_0^a \int_0^a k(x^2 + y^2 + z^2) dz dy dx \\
 &= \int_0^a \int_0^a k \left[x^2 z + y^2 z + \frac{1}{3} z^3 \right]_0^a dy dx = \int_0^a \int_0^a k \left(ax^2 + ay^2 + \frac{1}{3} a^3 \right) dy dx \\
 &= \int_0^a k \left[ax^2 y + \frac{1}{3} ay^3 + \frac{1}{3} a^3 y \right]_0^a dx = \int_0^a k \left(a^2 x^2 + \frac{1}{3} a^4 + \frac{1}{3} a^4 \right) dx = \int_0^a k \left(a^2 x^2 + \frac{2}{3} a^4 \right) dx \\
 &= k \left[\frac{1}{3} a^2 x^3 + \frac{2}{3} a^4 x \right]_0^a = k \left(\frac{1}{3} a^5 + \frac{2}{3} a^5 \right) = \boxed{ka^5}.
 \end{aligned}$$

44. Let the axis of the cylinder be the z -axis. For any point (x, y, z) in the cylinder, the closest point on the axis will be $(0, 0, z)$, so the distance between a point and the axis is

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2} = \sqrt{x^2 + y^2}.$$

The cylinder is bounded by $z = 0$ below and $z = h$ above, so it is xy -simple. The region R in the xy -plane is bounded by a circle of radius a and is both x - and y -simple. We treat R as x -simple. Since the mass density is proportional to the square of the distance from the axis to the cylinder, $\rho = \rho(x, y, z) = kd^2 = k(x^2 + y^2)$ where k is the constant of proportionality. Then the mass M is

$$\begin{aligned}
 M &= \iiint_E \rho(x, y, z) dV = \iiint_E k(x^2 + y^2) dz dy dx = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h dz dy dx \\
 &= \boxed{4 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h dz dy dx}.
 \end{aligned}$$

45. For any point (x, y, z) in the tetrahedron, the closest point on the xy -plane will be $(x, y, 0)$, so the distance between a point and the xy -plane is $d_{xy} = \sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2} = z$. Similarly, the distance from the point to the other coordinate planes will be $d_{yz} = x$ and $d_{xz} = y$. The tetrahedron is bounded by $z = 0$ below and $z = 1 - x - y$ above, so it is xy -simple. The region R in the xy -plane is bounded by the x -axis, the y -axis, and the line $y = 1 - x$ and is both x - and y -simple. We treat R as x -simple. Since the mass density is proportional to the product of the distances from the three coordinate

planes, $\rho = \rho(x, y, z) = k(d_{xy})(d_{yz})(d_{xz}) = k(xyz)$ where k is the constant of proportionality. Then the mass M is

$$M = \iiint_E \rho(x, y, z) dV = \iiint_E k(xyz) dz dy dx = \boxed{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} k(xyz) dz dy dx}.$$

46. The cylinder is yz -simple. It is bounded by $x = L$ above and $x = -L$ below. The region R is bounded by a circle with equation $y^2 + z^2 = R^2$. Then since the mass density is $\rho(x, y, z) = kz^2$,

$$\begin{aligned} M &= \iiint_E \rho(x, y, z) dV = \iiint_E kz^2 dV = \int_{-R}^R \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-L}^L kz^2 dx dy dz \\ &= 8 \int_0^R \int_0^{\sqrt{R^2-z^2}} \int_0^L kz^2 dx dy dz = 8 \int_0^R \int_0^{\sqrt{R^2-z^2}} kz^2 [x]_0^L dy dz \\ &= 8 \int_0^R \int_0^{\sqrt{R^2-z^2}} kLz^2 dy dz = 8 \int_0^R kLz^2 [y]_0^{\sqrt{R^2-z^2}} dz = 8 \int_0^R kLz^2 \sqrt{R^2 - z^2} dz. \end{aligned}$$

Using MapleTM with the assumptions that R and L are positive,

$$8 \int_0^R kLz^2 \sqrt{R^2 - z^2} dz = \boxed{\frac{\pi}{2} kR^4 L}.$$

47. The moment of inertia about the x -axis is

$$I_x = \iiint_E r^2 \rho dV$$

where $r = \sqrt{y^2 + z^2}$ is the distance of the point (x, y, z) from the x -axis. The hemisphere is xy -simple, and the region R is enclosed by the circle $x^2 + y^2 = 9$. R is both x - and y -simple. Since the mass density $\rho = \rho(x, y, z)$ is proportional to the distance from the xy -plane, $\rho(x, y, z) = kz$, where k is the constant of proportionality. Then

$$I_x = \iiint_E r^2 \rho(x, y, z) dV = \iiint_E (y^2 + z^2)(kz) dV = \boxed{\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} k(y^2 + z^2)z dz dy dx}.$$

Similarly, the moment of inertia about the y -axis is

$$I_y = \iiint_E r^2 \rho dV$$

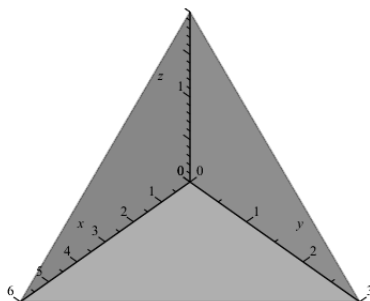
where $r = \sqrt{x^2 + z^2}$ is the distance of the point (x, y, z) from the y -axis, so

$$I_y = \iiint_E r^2 \rho(x, y, z) dV = \iiint_E (x^2 + z^2)(kz) dV = \boxed{\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} k(x^2 + z^2)z dz dy dx}.$$

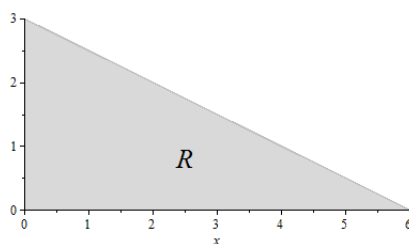
48. The solid is xy -simple. The top surface is the plane $z = 3$ and the bottom surface is the cone $z^2 = 3x^2 + 3y^2$. The region R in the xy -plane is enclosed by the circle $x^2 + y^2 = 3$, and it is both x - and y -simple. Then

$$\iiint_E x^2 yz dV = \boxed{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{3x^2+3y^2}}^3 x^2 yz dz dy dx}.$$

49. The tetrahedron E is xy -, yz -, and xz -simple.



First we consider the tetrahedron as an xy -simple solid so that $0 \leq z \leq 2 - \frac{x}{3} - \frac{2y}{3}$. The region R is bounded by the x - and y -axes and the line $x + 2y = 6$.



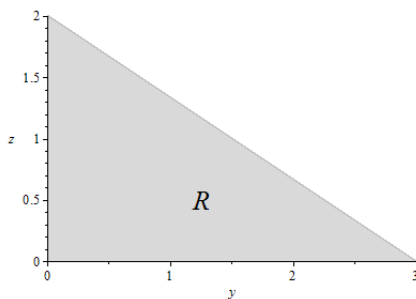
Treating R as x -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{2-\frac{x}{3}-\frac{2y}{3}} f(x, y, z) dz dy dx$$

while treating R as y -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{6-2y} \int_0^{2-\frac{x}{3}-\frac{2y}{3}} f(x, y, z) dz dx dy.$$

Next we consider the tetrahedron as a yz -simple solid so $0 \leq x \leq 6 - 2y - 3z$. The region R is bounded by the y - and z -axes and the line $2y + 3z = 6$.



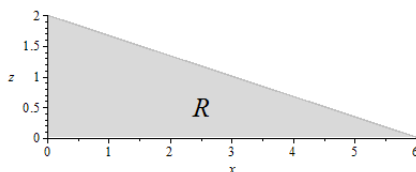
Treating R as y -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{2-\frac{2y}{3}} \int_0^{6-2y-3z} f(x, y, z) dx dz dy$$

while treating R as z -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^2 \int_0^{3-\frac{3z}{2}} \int_0^{6-2y-3z} f(x, y, z) dx dy dz.$$

Finally we consider the tetrahedron as an xz -simple solid so that $0 \leq y \leq 3 - \frac{x}{2} - \frac{3z}{2}$. The region R is bounded by the x - and z -axes and the line $x + 3z = 6$.



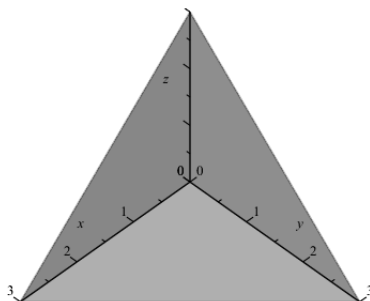
Treating R as x -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^6 \int_0^{2-\frac{x}{3}} \int_0^{3-\frac{x}{2}-\frac{3z}{2}} f(x, y, z) dy dz dx$$

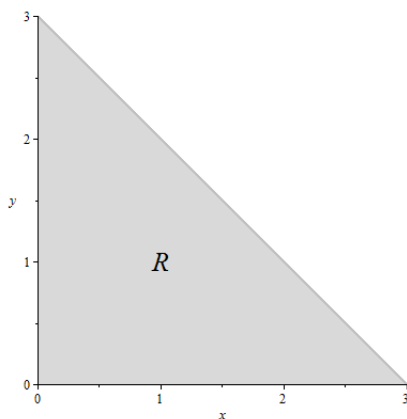
while treating R as z -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^2 \int_0^{6-3z} \int_0^{3-\frac{x}{2}-\frac{3z}{2}} f(x, y, z) dy dx dz.$$

50. The tetrahedron E is xy -, yz -, and xz -simple.



First we consider the tetrahedron as an xy -simple solid so that $0 \leq z \leq 3 - x - y$. The region R is bounded by the x - and y -axes and the line $x + y = 3$.



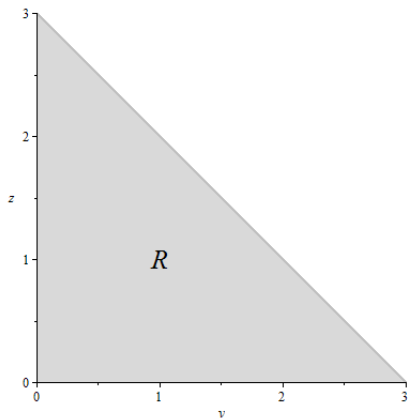
Treating R as x -simple gives

$$\iiint_E f(x, y, z) dV = \boxed{\int_0^3 \int_0^{3-x} \int_0^{3-x-y} f(x, y, z) dz dy dx}$$

while treating R as y -simple gives

$$\iiint_E f(x, y, z) dV = \boxed{\int_0^3 \int_0^{3-y} \int_0^{3-x-y} f(x, y, z) dz dx dy}.$$

Next we consider the tetrahedron as a yz -simple solid so $0 \leq x \leq 3 - y - z$. The region R is bounded by the y - and z -axes and the line $y + z = 3$.



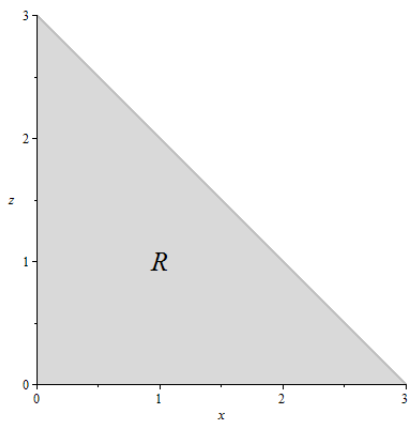
Treating R as y -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{3-y} \int_0^{3-y-z} f(x, y, z) dx dz dy$$

while treating R as z -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{3-z} \int_0^{3-y-z} f(x, y, z) dx dy dz.$$

Finally we consider the tetrahedron as an xz -simple solid so that $0 \leq y \leq 3 - x - z$. The region R is bounded by the x - and z -axes and the line $x + z = 3$.



Treating R as x -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{3-x} \int_0^{3-x-z} f(x, y, z) dy dz dx$$

while treating R as z -simple gives

$$\iiint_E f(x, y, z) dV = \int_0^3 \int_0^{3-z} \int_0^{3-x-z} f(x, y, z) dy dx dz.$$

51. We begin by solving the equation for the ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, for z to find

$$-c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \leq z \leq c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

The region R in the xy -plane is bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so

$$-b\sqrt{1 - \frac{x^2}{a^2}} \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}$$

and $-a \leq x \leq a$. Then using symmetry, the volume of the ellipsoid is

$$\begin{aligned} V &= \iiint_E dV = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz dy dx \\ &= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy dx = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy dx. \end{aligned}$$

To evaluate $\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy$, we use the substitution $u = \frac{y}{b}$ so $du = \frac{dy}{b}$ or $b du = dy$. The limits of integration become $u = 0$ to $u = \sqrt{1 - \frac{x^2}{a^2}}$.

$$\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy = \int_0^{\sqrt{1-\frac{x^2}{a^2}}} bc\sqrt{1 - \frac{x^2}{a^2} - u^2} du$$

We can make the integral look more like formula #62 from the Table of Integrals by letting $A^2 = 1 - \frac{x^2}{a^2}$.

$$\begin{aligned} \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy &= \int_0^A bc\sqrt{A^2 - u^2} du \stackrel{\#62}{=} bc \left[\frac{u}{2}\sqrt{A^2 - u^2} + \frac{A^2}{2}\sin^{-1}\left(\frac{u}{A}\right) \right]_0^A \\ &= bc \left[\frac{A}{2}\sqrt{A^2 - A^2} + \frac{A^2}{2}\sin^{-1}\left(\frac{A}{A}\right) - (0 + 0) \right] = \frac{A^2 bc \pi}{4} \\ &= \frac{bc\pi}{4} \left(1 - \frac{x^2}{a^2} \right) \end{aligned}$$

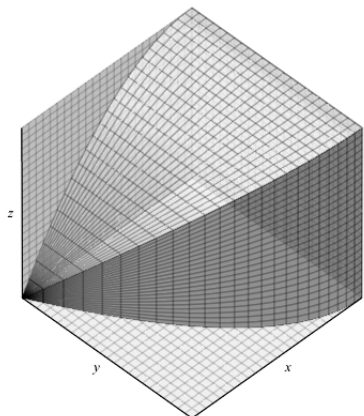
Then

$$\begin{aligned} V &= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy dx = 8 \left(\frac{bc\pi}{4} \right) \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = 2bc\pi \left[x - \frac{x^3}{3a^2} \right]_0^a \\ &= 2bc\pi \left(a - \frac{a^3}{3a^2} \right) = \boxed{\frac{4}{3}\pi abc}. \end{aligned}$$

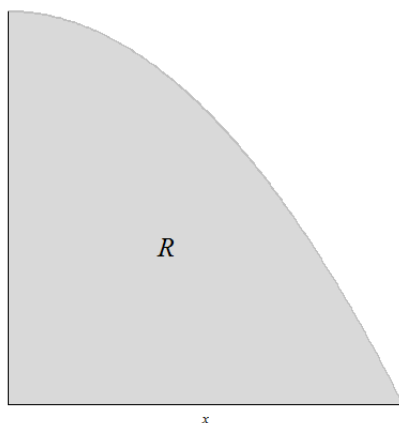
If $a = b = c$, then the volume becomes $V = \frac{4}{3}\pi a^3$, which is the volume of a sphere of radius a .

Challenge Problems

52. The region E is enclosed by the coordinate planes and the parabolic cylinders $a^2y = b(a^2 - x^2)$ and $a^2z = c(a^2 - x^2)$, so $0 \leq z \leq c\left(1 - \frac{x^2}{a^2}\right)$.

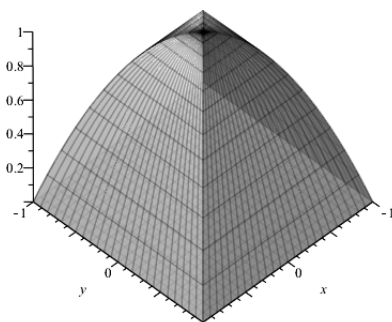


The region R in the xy -plane is $0 \leq y \leq b\left(1 - \frac{x^2}{a^2}\right)$, and $0 \leq x \leq a$.

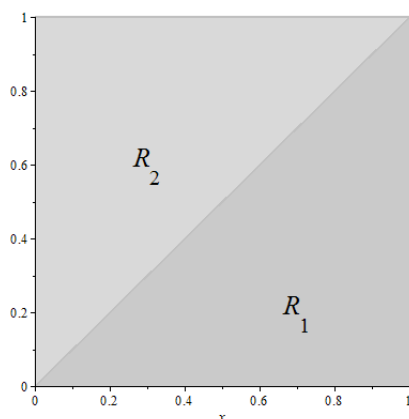


$$\begin{aligned}
 V &= \iiint_E dV = \int_0^a \int_0^{b\left(1 - \frac{x^2}{a^2}\right)} \int_0^{c\left(1 - \frac{x^2}{a^2}\right)} dz \, dy \, dx = \int_0^a \int_0^{b\left(1 - \frac{x^2}{a^2}\right)} [z]_0^{c\left(1 - \frac{x^2}{a^2}\right)} dy \, dx \\
 &= \int_0^a \int_0^{b\left(1 - \frac{x^2}{a^2}\right)} c\left(1 - \frac{x^2}{a^2}\right) dy \, dx = \int_0^a c\left(1 - \frac{x^2}{a^2}\right) [y]_0^{b\left(1 - \frac{x^2}{a^2}\right)} dx = \int_0^a bc\left(1 - \frac{x^2}{a^2}\right)^2 dx \\
 &= \int_0^a bc\left(1 - 2\frac{x^2}{a^2} + \frac{x^4}{a^4}\right) dx = bc\left[x - \frac{2x^3}{3a^2} + \frac{x^5}{5a^4}\right]_0^a = bc\left(a - \frac{2a}{3} + \frac{a}{5}\right) = \boxed{\frac{8}{15}abc}
 \end{aligned}$$

53. We will find the volume in the first octant and then use symmetry to get the total volume. The two parabolic cylinders intersect when $1 - x^2 = 1 - y^2$ or at $y = \pm x$.



We split the region R in the xy -plane into two subregions, R_1 and R_2 .



R_1 is bounded by $0 \leq y \leq x$, $0 \leq x \leq 1$. Since $y \leq x$ in R_1 , $y^2 \leq x^2$. Then $-y^2 \geq -x^2$ and $1 - y^2 \geq 1 - x^2$ which means that $0 \leq z \leq 1 - x^2$ in R_1 .

Similarly, R_2 is bounded by $x \leq y \leq 1$, $0 \leq x \leq 1$. Since $x \leq y$ in R_2 , $x^2 \leq y^2$. Then $-x^2 \geq -y^2$ and $1 - x^2 \geq 1 - y^2$ which means that $0 \leq z \leq 1 - y^2$ in R_2 . The volume in the first octant is

$$\begin{aligned}
 V_1 &= \iiint_E dV = \iiint_{E_1} dV + \iiint_{E_2} dV = \int_0^1 \int_0^x \int_0^{1-x^2} dz dy dx + \int_0^1 \int_x^1 \int_0^{1-y^2} dz dy dx \\
 &= \int_0^1 \int_0^x [z]_0^{1-x^2} dy dx + \int_0^1 \int_x^1 [z]_0^{1-y^2} dy dx = \int_0^1 \int_0^x (1-x^2) dy dx + \int_0^1 \int_x^1 (1-y^2) dy dx \\
 &= \int_0^1 (1-x^2) [y]_0^x dx + \int_0^1 \left[y - \frac{1}{3}y^3 \right]_x^1 dx = \int_0^1 (x-x^3) dx + \int_0^1 \left[\left(1 - \frac{1}{3}\right) - \left(x - \frac{1}{3}x^3\right) \right] dx \\
 &= \int_0^1 (x-x^3) dx + \int_0^1 \left(\frac{2}{3} - x + \frac{1}{3}x^3 \right) dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 + \left[\frac{2}{3}x - \frac{1}{2}x^2 + \frac{1}{12}x^4 \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{4} + \frac{2}{3} - \frac{1}{2} + \frac{1}{12} = \frac{1}{2}.
 \end{aligned}$$

The total volume of the solid is then $V = 4V_1 = 4\left(\frac{1}{2}\right) = \boxed{2}$.

54. (a) In both cases, we are integrating the function over an interval or a region, and then we are dividing the result by the size of that interval or region. Suppose R is a rectangular region bounded by

$a \leq x \leq b$ and $c \leq y \leq d$ and $f(x, y)$ does not depend on y . In this special case, $A = (b-a)(d-c)$ and $f(x, y) = f(x)$.

$$\begin{aligned}\bar{f} &= \frac{1}{A} \iint_R f(x, y) dA = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x) dy dx = \frac{1}{(b-a)(d-c)} \int_a^b [y]_c^d f(x) dx \\ &= \frac{1}{(b-a)(d-c)} \int_a^b (d-c)f(x) dx = \frac{1}{b-a} \int_a^b f(x) dx\end{aligned}$$

(b) The average value of f over a solid E in space would be the number $\boxed{\frac{1}{V} \iiint_E f(x, y, z) dV}$.

55. (a) The limits of integration here mean that $\frac{x^2+y^2}{4} \leq z \leq 4$, $0 \leq y \leq \sqrt{16-x^2}$, and $0 \leq x \leq 4$. The solid is in the first octant and is bounded below by the paraboloid of revolution $\frac{x^2+y^2}{4}$ and above by the plane $z = 4$. The region R in the xy -plane is enclosed by the quarter of a circle of radius 4 in the first quadrant. In other words, this integral represents the volume over the first quadrant between the paraboloid of revolution and the plane $z = 4$.
- (b) The limits of integration here mean that $0 \leq y \leq \sqrt{4z-x^2}$, $0 \leq x \leq 2\sqrt{z}$, and $0 \leq z \leq 4$. Since $y \leq \sqrt{4z-x^2}$, $y^2 \leq 4z-x^2$ which implies that $z \geq \frac{1}{4}(x^2+y^2)$. This is the same paraboloid of revolution found in part (a). The region R in the xz -plane is enclosed by $x^2 = 4z$, $x = 0$, $z = 0$, and $z = 4$ which is the projection of the paraboloid of revolution onto the xz -plane.
- (c) The limits of integration here mean that $0 \leq x \leq \sqrt{4z-y^2}$, $\frac{y^2}{4} \leq z \leq 4$, and $0 \leq y \leq 4$. Since $x \leq \sqrt{4z-y^2}$, $x^2 \leq 4z-y^2$ which implies that $z \geq \frac{1}{4}(x^2+y^2)$. This is the same paraboloid of revolution found in part (a). The region R in the yz -plane is enclosed by $\frac{y^2}{4} = z$, $y = 0$, $z = 0$, and $z = 4$ which is the projection of the paraboloid of revolution onto the xz -plane.
56. We change the order of integration. From the limits of integration, we know $a \leq x \leq y$, $a \leq y \leq z$, and $a \leq z \leq b$. We can put these inequalities into a single inequality.

$$a \leq x \leq y \leq z \leq b.$$

We choose new limits that satisfy that inequality, noting that we need $a \leq x \leq b$. We see $a \leq x \leq b$, $x \leq y \leq b$, and $y \leq z \leq b$. We can rewrite the integral in a different order with these new limits.

$$\begin{aligned}\int_a^b \int_a^z \int_a^y f(x) dx dy dz &= \int_a^b \int_x^b \int_y^b f(x) dz dy dx = \int_a^b \int_x^b f(x) [z]_y^b dy dx = \int_a^b \int_x^b f(x)(b-y) dy dx \\ &= \int_a^b f(x) \left[\left(by - \frac{1}{2}y^2 \right) \right]_x^b dx = \int_a^b f(x) \left[\left(b^2 - \frac{1}{2}b^2 \right) - \left(bx - \frac{1}{2}x^2 \right) \right] dx \\ &= \int_a^b f(x) \frac{1}{2} (b^2 - 2bx + x^2) dx = \int_a^b f(x) \frac{(b-x)^2}{2} dx = \int_a^b \frac{(b-x)^2}{2} f(x) dx\end{aligned}$$

14.7 Triple Integrals Using Cylindrical Coordinates

Concepts and Vocabulary

1. To convert points in rectangular coordinates to cylindrical coordinates use the equations $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$.
2. According to Table 1, the circular cone $x^2 + y^2 = 4z^2$ has the form $r = 2z$ in cylindrical coordinates.
3. The differential $dV = r dr d\theta dz$.
4. **True**. We know that $\int \int \int_E r dr d\theta dz = \int \int \int_E dV$ which we have seen in previous sections is equal to the volume of E .

Skill Building

5. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$. Then $\cos \theta = \frac{x}{r} = \frac{-1}{2}$ and $\sin \theta = \frac{y}{r} = \frac{-1}{2}$ so $\theta = \frac{7\pi}{6}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{7\pi}{6}, -5\right)$.
6. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$. Then $\cos \theta = \frac{x}{r} = \frac{-1}{2}$ and $\sin \theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$ so $\theta = \frac{5\pi}{6}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(2, \frac{5\pi}{6}, 4\right)$.
7. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2}$. Then $\cos \theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$ and $\sin \theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$ so $\theta = \frac{\pi}{4}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(\sqrt{2}, \frac{\pi}{4}, \sqrt{2}\right)$.
8. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = 2\sqrt{2}$. Then $\cos \theta = \frac{x}{r} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $\sin \theta = \frac{y}{r} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$ so $\theta = \frac{7\pi}{4}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(2\sqrt{2}, \frac{7\pi}{4}, 4\right)$.
9. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{4 + 0} = 2$. Then $\cos \theta = \frac{x}{r} = \frac{2}{2} = 1$ and $\sin \theta = \frac{y}{r} = \frac{0}{2} = 0$ so $\theta = 0$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = (2, 0, 4)$.
10. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{1 + 0} = 1$. Then $\cos \theta = \frac{x}{r} = \frac{-1}{1} = -1$ and $\sin \theta = \frac{y}{r} = \frac{0}{1} = 0$ so $\theta = \pi$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(1, \pi, \frac{1}{2}\right)$.
11. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{0 + 9} = 3$. Then $\cos \theta = \frac{x}{r} = \frac{0}{3} = 0$ and $\sin \theta = \frac{y}{r} = \frac{3}{3} = 1$ so $\theta = \frac{\pi}{2}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(3, \frac{\pi}{2}, 4\right)$.

12. In cylindrical coordinates $r = \sqrt{x^2 + y^2} = \sqrt{0 + 1} = 1$. Then $\cos \theta = \frac{x}{r} = \frac{0}{1} = 0$ and $\sin \theta = \frac{y}{r} = \frac{1}{1} = 1$ so $\theta = \frac{\pi}{2}$. Since z remains the same, the point P in cylindrical coordinates is $(r, \theta, z) = \left(1, \frac{\pi}{2}, -3\right)$.

13. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = 2$ and $\theta = \frac{\pi}{6}$. Then

$$x = 2 \cos \frac{\pi}{6} = \sqrt{3} \quad y = 2 \sin \frac{\pi}{6} = 1 \quad z = -5$$

In rectangular coordinates, $P = (\sqrt{3}, 1, -5)$.

14. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = 4$ and $\theta = \frac{\pi}{3}$. Then

$$x = 4 \cos \frac{\pi}{3} = 2 \quad y = 4 \sin \frac{\pi}{3} = 2\sqrt{3} \quad z = 3$$

In rectangular coordinates, $P = (2, 2\sqrt{3}, 3)$.

15. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = 1$ and $\theta = 0$. Then

$$x = 1 \cos 0 = 1 \quad y = 1 \sin 0 = 0 \quad z = 8$$

In rectangular coordinates, $P = (1, 0, 8)$.

16. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = 4$ and $\theta = \frac{\pi}{6}$. Then

$$x = 4 \cos \frac{\pi}{6} = 2\sqrt{3} \quad y = 4 \sin \frac{\pi}{6} = 2 \quad z = 2$$

In rectangular coordinates, $P = (2\sqrt{3}, 2, 2)$.

17. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = 2$ and $\theta = \frac{\pi}{2}$. Then

$$x = 2 \cos \frac{\pi}{2} = 0 \quad y = 2 \sin \frac{\pi}{2} = 2 \quad z = 0$$

In rectangular coordinates, $P = (0, 2, 0)$.

18. We use the equations $x = r \cos \theta$ and $y = r \sin \theta$ with $r = -3$ and $\theta = \frac{\pi}{2}$. Then

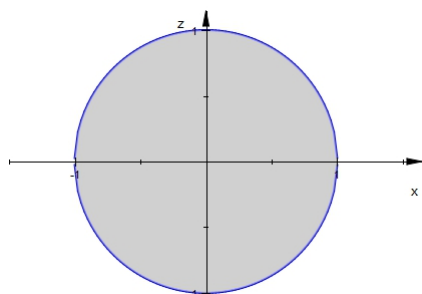
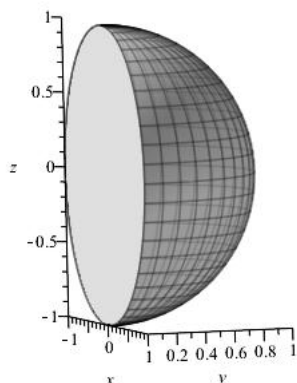
$$x = -3 \cos \frac{\pi}{2} = 0 \quad y = -3 \sin \frac{\pi}{2} = -3 \quad z = 1$$

In rectangular coordinates, $P = (0, -3, 1)$.

19. The solid E of integration and its projection onto the xz -plane can be described by the inequalities

$$0 \leq y \leq \sqrt{1 - x^2 - z^2} \quad -\sqrt{1 - x^2} \leq z \leq \sqrt{1 - x^2} \quad -1 \leq x \leq 1$$

The graph of this solid and its projection onto the xz -plane are given below.

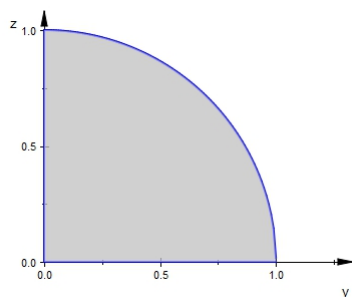


The limits of integration on y are $y = 0$ and $y = \sqrt{1 - x^2 - z^2}$ or, equivalently, $y^2 = 1 - x^2 - z^2$, $y \geq 0$. We can interpret the integral as the volume of a solid E that is a hemisphere of radius 1 situated toward the positive side of the xz -plane, that is $y \geq 0$.

20. The solid E of integration and its projection onto the yz -plane can be described by the inequalities

$$0 \leq x \leq \sqrt{1 - y^2 - z^2} \quad 0 \leq y \leq \sqrt{1 - z^2} \quad 0 \leq z \leq 1$$

The graph of this solid and its projection onto the yz -plane are given below.

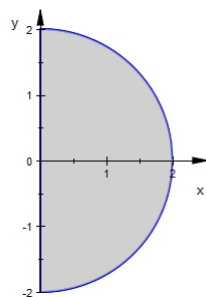
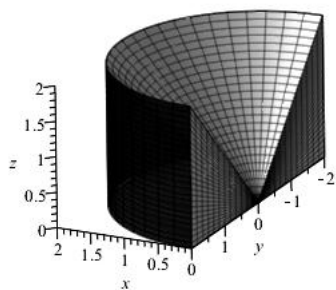


The limits of integration on x are $x = 0$ and $x = \sqrt{1 - y^2 - z^2}$ or, equivalently, $x^2 = 1 - y^2 - z^2$, $x \geq 0$. We can interpret the integral as the volume of a solid E that is the portion of a sphere of radius 1 found in the first octant.

21. The solid E of integration and its projection onto the xy -plane can be described by the inequalities

$$0 \leq z \leq \sqrt{x^2 + y^2} \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2} \quad 0 \leq x \leq 2$$

The graph of this solid and its projection onto the xy -plane are given below.

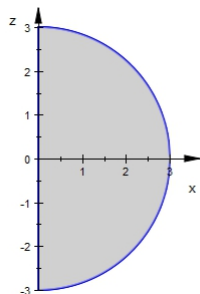
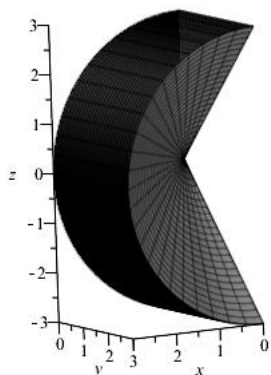


The limits of integration on z are $z = 0$ and $z = \sqrt{x^2 + y^2}$ or, equivalently, $z^2 = x^2 + y^2$, $z \geq 0$. We can see that the volume being computed lies below the cone $z^2 = x^2 + y^2$ and above a portion of the xy -plane. Since $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ while $0 \leq x \leq 2$, the volume is being restricted to the vertical cylinder $x^2 + y^2 = 4$, $x \geq 0$ (i.e., a “semicircular” cylinder). So we may interpret the integral as the volume of a solid E that is inside the vertical cylinder $x^2 + y^2 = 4$ with $x \geq 0$ and is bounded below by the xy -plane and bounded above by the cone $z^2 = x^2 + y^2$.

22. The solid E of integration and its projection onto the xz -plane can be described by the inequalities

$$0 \leq y \leq \sqrt{x^2 + z^2} \quad -\sqrt{9-x^2} \leq z \leq \sqrt{9-x^2} \quad 0 \leq x \leq 3$$

The graph of this solid and its projection onto the xz -plane are given below.



The limits of integration on y are $y = 0$ and $y = \sqrt{x^2 + z^2}$ or, equivalently, $y^2 = x^2 + z^2$, $y \geq 0$. We can see that the volume being computed lies between the cone $y^2 = x^2 + z^2$, $y \geq 0$ and the xz -plane. Since $-\sqrt{9 - x^2} \leq z \leq \sqrt{9 - x^2}$ while $0 \leq x \leq 3$, the volume is being restricted to the cylinder $x^2 + z^2 = 9$, $x \geq 0$ (i.e., a “semicircular” cylinder). So we may interpret the integral as the volume of a solid E that is inside the cylinder $x^2 + z^2 = 9$ with $x \geq 0$ and is bounded by the xz -plane and by the cone $y^2 = x^2 + z^2$, $y \geq 0$.

23. The triple integral given is expressed in cylindrical coordinates, so we integrate with respect to the variables as given to obtain

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \int_0^3 \int_0^{r \sin \theta} r \csc^3 \theta \, dz \, dr \, d\theta &= \int_{\pi/6}^{\pi/2} \int_0^3 [r \csc^3 \theta z]_0^{r \sin \theta} \, dr \, d\theta = \int_{\pi/6}^{\pi/2} \int_0^3 [r \csc^3 \theta \cdot r \sin \theta] \, dr \, d\theta \\ &= \int_{\pi/6}^{\pi/2} \int_0^3 r^2 \csc^2 \theta \, dr \, d\theta = \int_{\pi/6}^{\pi/2} \left[\frac{1}{3} r^3 \csc^2 \theta \right]_0^3 \, d\theta = \int_{\pi/6}^{\pi/2} 9 \csc^2 \theta \, d\theta \\ &= [-9 \cot \theta]_{\pi/6}^{\pi/2} = \boxed{9\sqrt{3}} \end{aligned}$$

24. The triple integral given is expressed in cylindrical coordinates, so we integrate with respect to the variables as given to obtain

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \int_0^1 \int_0^{\sin \theta} r \cos \theta \sin \theta \, dz \, dr \, d\theta &= \int_{\pi/6}^{\pi/2} \int_0^1 [r \cos \theta \sin \theta z]_0^{\sin \theta} \, dr \, d\theta = \int_{\pi/6}^{\pi/2} \int_0^1 r \cos \theta \sin^2 \theta \, dr \, d\theta \\ &= \int_{\pi/6}^{\pi/2} \left[\frac{1}{2} r^2 \cos \theta \sin^2 \theta \right]_0^1 \, d\theta = \int_{\pi/6}^{\pi/2} \frac{1}{2} \cos \theta \sin^2 \theta \, d\theta \end{aligned}$$

To evaluate this last integral, we let $u = \sin \theta$ so that $du = \cos \theta d\theta$. Also, $u = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ and $u = 1$ when $\theta = \frac{\pi}{2}$. We then have

$$\int_{\pi/6}^{\pi/2} \frac{1}{2} \cos \theta \sin^2 \theta d\theta = \int_{1/2}^1 \frac{1}{2} u^2 du = \left[\frac{1}{6} u^3 \right]_{1/2}^1 = \boxed{\frac{7}{48}}$$

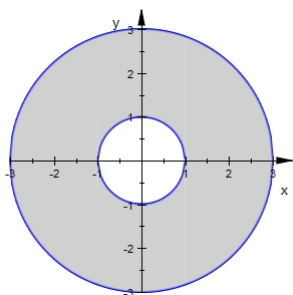
25. The triple integral given is expressed in cylindrical coordinates, so we integrate with respect to the variables as given to obtain

$$\begin{aligned} \int_0^{\pi/3} \int_0^1 \int_0^{e^{-1}} r dz dr d\theta &= \int_0^{\pi/3} \int_0^1 [rz]_0^{e^{-1}} dr d\theta = e^{-1} \int_0^{\pi/3} \int_0^1 r dr d\theta \\ &= e^{-1} \int_0^{\pi/3} \left[\frac{1}{2} r^2 \right]_0^1 d\theta = \frac{1}{2e} \int_0^{\pi/3} d\theta = \boxed{\frac{\pi}{6e}} \end{aligned}$$

26. The triple integral given is expressed in cylindrical coordinates, so we integrate with respect to the variables as given to obtain

$$\begin{aligned} \int_0^{\pi/3} \int_0^{\sin \theta} \int_0^{r \sin \theta} r dz dr d\theta &= \int_0^{\pi/3} \int_0^{\sin \theta} [rz]_0^{r \sin \theta} dr d\theta = \int_0^{\pi/3} \int_0^{\sin \theta} r^2 \sin \theta dr d\theta \\ &= \int_0^{\pi/3} \left[\frac{1}{3} r^3 \sin \theta \right]_0^{\sin \theta} d\theta = \int_0^{\pi/3} \frac{1}{3} \sin^4 \theta d\theta = \int_0^{\pi/3} \frac{1}{3} (\sin^2 \theta)^2 d\theta \\ &= \int_0^{\pi/3} \frac{1}{12} (1 - 2 \cos(2\theta) + \cos^2(2\theta)) d\theta \\ &= \int_0^{\pi/3} \frac{1}{12} \left(1 - 2 \cos(2\theta) + \frac{1}{2} (1 + \cos(4\theta)) \right) d\theta \\ &= \int_0^{\pi/3} \frac{1}{24} (3 - 4 \cos(2\theta) + \cos(4\theta)) d\theta = \frac{1}{24} \left[3\theta - 2 \sin(2\theta) + \frac{1}{4} \sin(4\theta) \right]_0^{\pi/3} \\ &= \frac{1}{24} \left[\pi - \sqrt{3} - \frac{\sqrt{3}}{8} \right] = \boxed{\frac{\pi}{24} - \frac{3\sqrt{3}}{64}} \end{aligned}$$

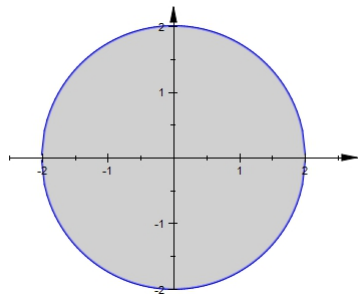
27. We see that $1 \leq z \leq 4$, so we will integrate with respect to z first. The projection onto the xy -plane is the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ as shown in the figure.



We use cylindrical coordinates to describe the solid of integration: $1 \leq z \leq 4$, $1 \leq r \leq 3$, and $0 \leq \theta \leq 2\pi$. We compute

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_1^3 \int_1^4 r dz dr d\theta = \int_0^{2\pi} \int_1^3 [rz]_1^4 dr d\theta = \int_0^{2\pi} \int_1^3 3r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{3}{2} r^2 \right]_1^3 d\theta = \int_0^{2\pi} 12 d\theta = \boxed{24\pi} \end{aligned}$$

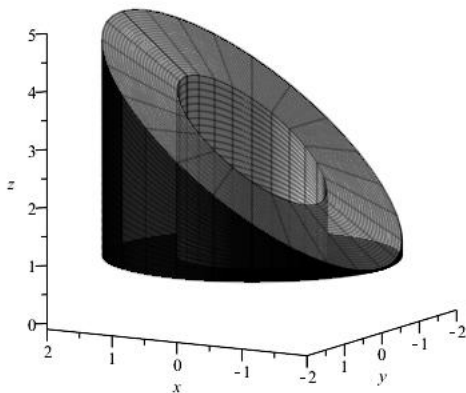
28. We see that $0 \leq z \leq 3$, so we will integrate with respect to z first. The projection onto the xy -plane is the region bounded by the circle $x^2 + y^2 = 4$ as shown in the figure.



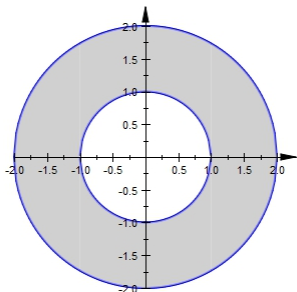
We use cylindrical coordinates to describe the solid of integration: $0 \leq z \leq 3$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. We compute

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_0^2 \int_0^3 r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [rz]_0^3 \, dr \, d\theta = \int_0^{2\pi} \int_0^2 3r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{3}{2} r^2 \right]_0^2 \, d\theta = \int_0^{2\pi} 6 \, d\theta = \boxed{12\pi} \end{aligned}$$

29. We see that the z -coordinate lies between $z = 1$ and $z = x + 3$ and, because the smallest value of x in this solid is $x = -2$, we see that $1 \leq z \leq x + 3$.



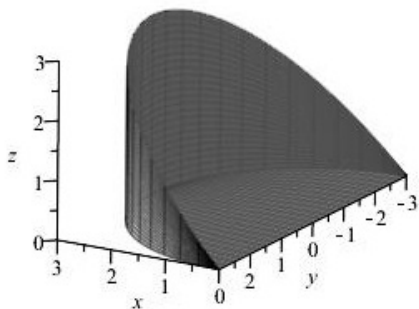
We will integrate with respect to z first. The projection onto the xy -plane is the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ as shown in the figure.



We use cylindrical coordinates to describe the solid of integration: $1 \leq z \leq r \cos \theta + 3$, $1 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. We compute

$$\begin{aligned} \int \int \int_E y \, dV &= \int_0^{2\pi} \int_1^2 \int_1^{r \cos \theta + 3} r \sin \theta \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_1^2 [r^2 \sin \theta z]_1^{r \cos \theta + 3} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_1^2 (r^3 \sin \theta \cos \theta + 3r^2 \sin \theta) \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \sin \theta \cos \theta + r^3 \sin \theta \right]_1^2 \, d\theta \\ &= \int_0^{2\pi} \left(\frac{15}{4} \sin \theta \cos \theta + 7 \sin \theta \right) \, d\theta = \left[\frac{15}{8} \sin^2 \theta - 7 \cos \theta \right]_0^{2\pi} = \boxed{0} \end{aligned}$$

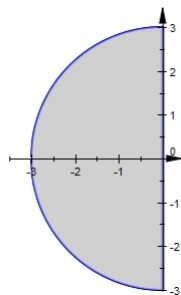
30. We see that the solid can be broken into two solids S_1 where $x \leq z \leq 0$ for $x \leq 0$ and S_2 where $0 \leq z \leq x$ for $x \geq 0$.



We will set up a triple integral for S_1 and S_2 integrating with respect to z first in each case. That is,

$$\int \int \int_E x \, dV = \int \int \int_{S_1} x \, dV + \int \int \int_{S_2} x \, dV$$

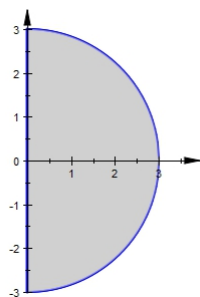
For S_1 , the projection onto the xy -plane is the region bounded by the semicircle $x^2 + y^2 = 9$ where $x \leq 0$ and $y = 0$ as shown in the figure.



We use cylindrical coordinates to describe the solid of integration: $r \cos \theta \leq z \leq 0$, $0 \leq r \leq 3$, and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. We compute

$$\begin{aligned}
\int \int \int_{S_1} x \, dV &= \int_{\pi/2}^{3\pi/2} \int_0^3 \int_{r \cos \theta}^0 r \cos \theta \cdot r \, dz \, dr \, d\theta = \int_{\pi/2}^{3\pi/2} \int_0^3 [r^2 \cos \theta z]_{r \cos \theta}^0 \, dr \, d\theta \\
&= - \int_{\pi/2}^{3\pi/2} \int_0^3 (r^3 \cos^2 \theta) \, dr \, d\theta = - \int_{\pi/2}^{3\pi/2} \left[\frac{r^4}{4} \cos^2 \theta \right]_0^3 \, d\theta \\
&= - \int_{\pi/2}^{3\pi/2} \left(\frac{81}{4} \cos^2 \theta \right) \, d\theta = - \int_{\pi/2}^{3\pi/2} \left(\frac{81}{8} (1 + \cos(2\theta)) \right) \, d\theta \\
&= - \frac{81}{8} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{\pi/2}^{3\pi/2} = - \frac{81}{8} \pi
\end{aligned}$$

For S_2 , the projection onto the xy -plane is the region bounded by the semicircle $x^2 + y^2 = 9$ where $x \geq 0$ and $y = 0$ as shown in the figure.



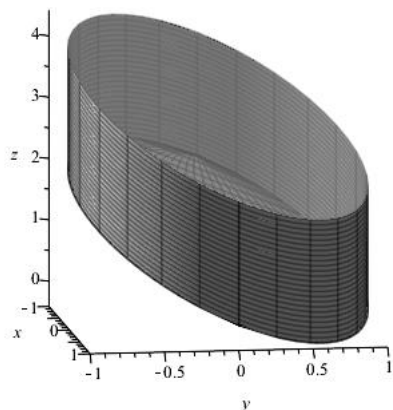
We use cylindrical coordinates to describe the solid of integration: $0 \leq z \leq r \cos \theta$, $0 \leq r \leq 3$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We compute

$$\begin{aligned}
\int \int \int_{S_2} x \, dV &= \int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{r \cos \theta} r \cos \theta \cdot r \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^3 [r^2 \cos \theta z]_0^{r \cos \theta} \, dr \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^3 (r^3 \cos^2 \theta) \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \cos^2 \theta \right]_0^3 \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left(\frac{81}{4} \cos^2 \theta \right) \, d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{81}{8} (1 + \cos(2\theta)) \right) \, d\theta \\
&= \frac{81}{8} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/2}^{\pi/2} = \frac{81}{8} \pi
\end{aligned}$$

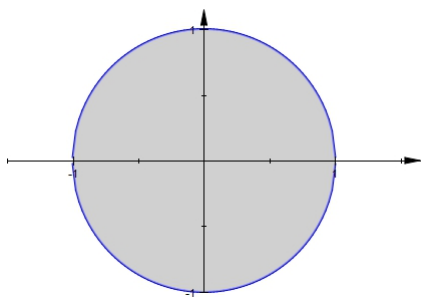
We have

$$\int \int \int_E x \, dV = \boxed{0}$$

31. We see that the z -coordinate lies between $z = 1 - x - y$ and $z = 3 - x - y$, so we will integrate with respect to z first.



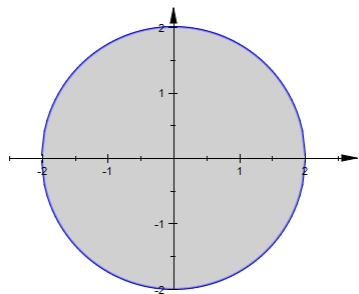
The projection onto the xy -plane is the region bounded by the circle $x^2 + y^2 = 1$ as shown in the figure.



We use cylindrical coordinates to describe the solid of integration: $1 - r \cos \theta - r \sin \theta \leq z \leq 3 - r \cos \theta - r \sin \theta$, $0 \leq r \leq 1$, and $0 \leq \theta \leq 2\pi$. We compute

$$\begin{aligned}
 \iiint_E xy \, dV &= \int_0^{2\pi} \int_0^1 \int_{1-r \cos \theta - r \sin \theta}^{3-r \cos \theta - r \sin \theta} r \cos \theta \cdot r \sin \theta \cdot r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 [r^3 \sin \theta \cos \theta z]_{1-r \cos \theta - r \sin \theta}^{3-r \cos \theta - r \sin \theta} \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (2r^3 \sin \theta \cos \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 (r^3 \sin(2\theta)) \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \sin(2\theta) \right]_0^1 \, d\theta \\
 &= \int_0^{2\pi} \frac{1}{4} \sin(2\theta) \, d\theta = \left[-\frac{1}{8} \cos(2\theta) \right]_0^{2\pi} = \boxed{0}
 \end{aligned}$$

32. We see that the z -coordinate lies between $z = 0$ and $z = x^2 + y^2$, so we will integrate with respect to z first. The projection onto the xy -plane is the region bounded by the circle $x^2 + y^2 = 4$ as shown in the figure.

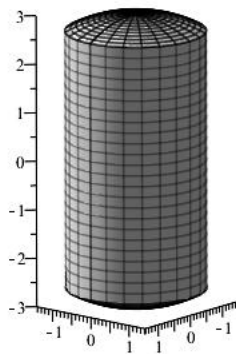


We use cylindrical coordinates to describe the solid of integration: $0 \leq z \leq x^2 + y^2$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. We compute

$$\begin{aligned} \int \int \int_E xy \, dV &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} r \cos \theta \cdot r \sin \theta \cdot r \, dz dr d\theta = \int_0^{2\pi} \int_0^2 [r^3 \sin \theta \cos \theta z]_0^{r^2} dr d\theta \\ &= \int_0^{2\pi} \int_0^2 r^5 \sin \theta \cos \theta \, dr d\theta = \int_0^{2\pi} \left[\frac{r^6}{6} \sin \theta \cos \theta \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{32}{3} \sin \theta \cos \theta \, d\theta = \left[\frac{16}{3} \sin^2 \theta \right]_0^{2\pi} = \boxed{0} \end{aligned}$$

Applications and Extensions

33. We are seeking the volume of the solid that lies inside the given cylinder and the given sphere simultaneously (see figure).



The solid E whose volume we seek lies beneath the surface $x^2 + y^2 + z^2 = 9$ over the region $x^2 + y^2 = 2$. Because $x^2 + y^2$ appears in both the sphere and the cylinder, we use cylindrical coordinates. The rectangular equation $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$ in cylindrical coordinates. That is, $-\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}$. In cylindrical coordinates, the rectangular region bounded by $x^2 + y^2 = 2$

in the xy -plane may be described by $0 \leq r \leq \sqrt{2}$ where $0 \leq \theta \leq 2\pi$. The volume V of the solid is

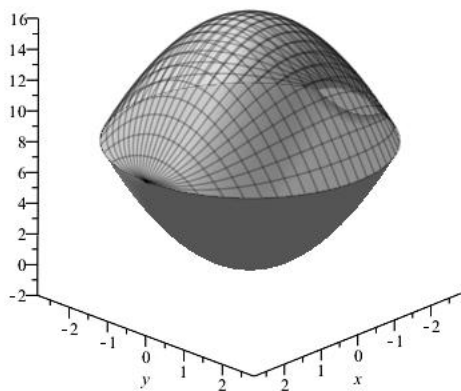
$$\begin{aligned} V &= \int \int \int_E dV = \int \int \int_E r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} [rz]_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} 2r\sqrt{9-r^2} \, dr \, d\theta = \int_0^{\sqrt{2}} 2r\sqrt{9-r^2} \, dr \int_0^{2\pi} d\theta = \left[-(9-r^2)^{3/2} \cdot \frac{2}{3} \right]_0^{\sqrt{2}} (2\pi) \\ &= -\frac{4\pi}{3} [7^{3/2} - 9^{3/2}] = -\frac{4\pi}{3} (7\sqrt{7} - 27) = 4\pi \left(9 - \frac{7\sqrt{7}}{3} \right) \end{aligned}$$

The volume is $\boxed{4\pi \left(9 - \frac{7\sqrt{7}}{3} \right) \text{ cubic units}}.$

34. We see that for $z \geq 0$, this solid is identical to the solid discussed in Example 3. Therefore the volume of this portion of the solid is $\frac{8\pi}{3} - \frac{32}{9}$ cubic units. By symmetry, the total volume of the solid described

in this problem is $\boxed{2 \left(\frac{8\pi}{3} - \frac{32}{9} \right) = \frac{16\pi}{3} - \frac{64}{9} \text{ cubic units}}.$

35. We are seeking the volume of the solid that lies inside the given paraboloids of revolution simultaneously (see figure).

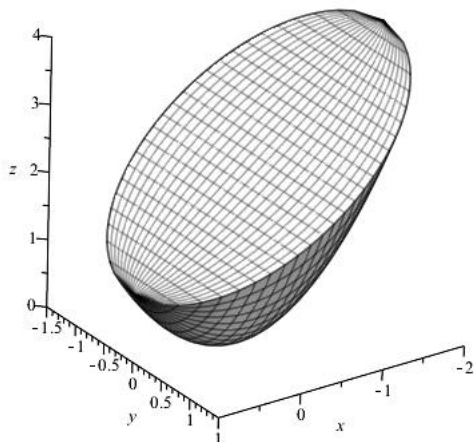


The solid E whose volume we seek lies beneath the surface $z = 16 - x^2 - y^2$ and above the surface $z = x^2 + y^2$. To find the projection of this solid onto the xy -plane, we find the intersection of the two surfaces. That is, we set $16 - x^2 - y^2 = x^2 + y^2$ to obtain $x^2 + y^2 = 8$. Because $x^2 + y^2$ that appears throughout these expressions, we use cylindrical coordinates. The rectangular equation $z = 16 - x^2 - y^2$ becomes $z = 16 - r^2$, $z = x^2 + y^2$ becomes $z = r^2$, and $x^2 + y^2 = 8$ may be described by $0 \leq r \leq 2\sqrt{2}$ where $0 \leq \theta \leq 2\pi$. The volume V of the solid is

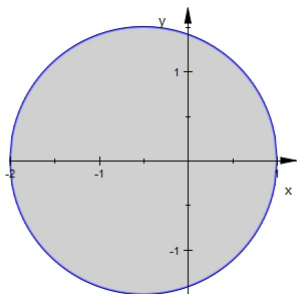
$$\begin{aligned} V &= \int \int \int_E dV = \int \int \int_E r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{2\sqrt{2}} \int_{r^2}^{16-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{2\sqrt{2}} [rz]_{r^2}^{16-r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{2\sqrt{2}} (16r - 2r^3) \, dr \, d\theta = \int_0^{2\sqrt{2}} (16r - 2r^3) \, dr \int_0^{2\pi} d\theta = \left[8r^2 - \frac{2}{4}r^4 \right]_0^{2\sqrt{2}} (2\pi) \\ &= (32)(2\pi) = 64\pi \end{aligned}$$

The volume is $\boxed{64\pi \text{ cubic units}}$.

36. We are seeking the volume of the solid that lies inside the given paraboloid of revolution and under the given plane simultaneously (see figure).



The solid E whose volume we seek lies beneath the surface $z = 2 - x$ and above the surface $z = x^2 + y^2$. To find the projection of this solid onto the xy plane, we find the intersection of the two surfaces. That is, we set $2 - x = x^2 + y^2$ to obtain $x^2 + x + y^2 = 2$. Completing the square, we may write $\left(x + \frac{1}{2}\right)^2 + y^2 = \frac{9}{4}$, so that we see that the projection onto the xy -plane is a region bounded by a circle.



This region in the plane is not well described by polar coordinates, so we will use rectangular coordinates to find this volume. The volume of the solid is

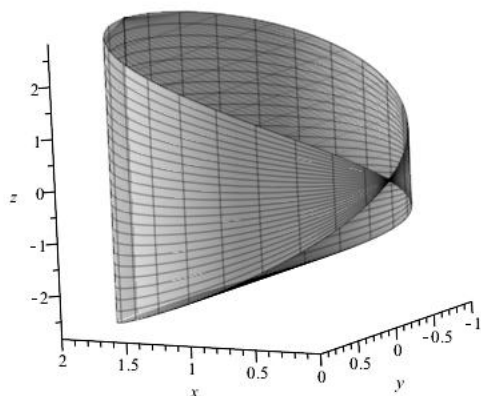
$$\begin{aligned} V &= \int \int \int_E dV = \int_{-2}^1 \int_{-\sqrt{9/4-(x+1/2)^2}}^{\sqrt{9/4-(x+1/2)^2}} \int_{x^2+y^2}^{2-x} dz dy dx = \int_{-2}^1 \int_{-\sqrt{9/4-(x+1/2)^2}}^{\sqrt{9/4-(x+1/2)^2}} (2-x-x^2-y^2) dy dx \\ &= \int_{-2}^1 \left[2y - xy - x^2y - \frac{1}{3}y^3 \right]_{-\sqrt{9/4-(x+1/2)^2}}^{\sqrt{9/4-(x+1/2)^2}} dx \\ &= \int_{-2}^1 -2\sqrt{9/4-(x+1/2)^2} \left(x - \frac{1}{3} \left(x + \frac{1}{2} \right)^2 + x^2 - \frac{5}{4} \right) dx = \int_{-2}^1 \frac{4}{3} \left(\frac{9}{4} - \left(x + \frac{1}{2} \right)^2 \right)^{3/2} dx \end{aligned}$$

We now make the substitution $\frac{3}{2} \sin \theta = x + \frac{1}{2}$ so that $\frac{3}{2} \cos \theta d\theta = dx$ and $\theta = -\frac{\pi}{2}$ when $x = -2$ and $\theta = \frac{\pi}{2}$ when $x = 1$. We may compute

$$\begin{aligned}
\int_{-2}^1 \frac{4}{3} \left(\frac{9}{4} - \left(x + \frac{1}{2} \right)^2 \right)^{3/2} dx &= \frac{4}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{9}{4} - \frac{9}{4} \sin^2 \theta \right)^{3/2} \frac{3}{2} \cos \theta d\theta = \frac{27}{4} \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta \\
&= \frac{27}{4} \int_{-\pi/2}^{\pi/2} (\cos^2 \theta)^2 d\theta = \frac{27}{4} \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} (1 + \cos(2\theta)) \right)^2 d\theta \\
&= \frac{27}{16} \int_{-\pi/2}^{\pi/2} (1 + 2\cos(2\theta) + \cos^2(2\theta)) d\theta \\
&= \frac{27}{16} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta)) \right) d\theta \\
&= \frac{27}{16} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) \right) d\theta \\
&= \frac{27}{16} \left[\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta) \right]_{-\pi/2}^{\pi/2} = \frac{27}{16} \left[\frac{3}{2} \cdot \frac{\pi}{2} - \frac{3}{2} \left(-\frac{\pi}{2} \right) \right] \\
&= \frac{81\pi}{32}
\end{aligned}$$

The volume is $\boxed{\frac{81\pi}{32} \text{ cubic units}}.$

37. We are seeking the volume of the solid that lies inside the cylinder $x^2 + y^2 = 2x$ and inside $z^2 = 4x$ simultaneously (see figure).



By symmetry, we will find the volume between $z = 0$ and $z^2 = 4x, z \geq 0$ over the region in the xy -plane bounded by $x^2 + y^2 = 2x$ and double it. As was shown in Example 3, $x^2 + y^2 = 2x$ can be written as $r = 2 \cos \theta$ in cylindrical coordinates while $z = \sqrt{4x} = 2\sqrt{r \cos \theta}$. This solid may be described by

$0 \leq z \leq 2\sqrt{r \cos \theta}$, $0 \leq r \leq 2 \cos \theta$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The total volume V of the original solid is

$$\begin{aligned} V &= \int \int \int_E dV = \int \int \int_E r \, dz \, dr \, d\theta = 2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_0^{2\sqrt{r \cos \theta}} r \, dz \, dr \, d\theta \\ &= 2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} [rz]_0^{2\sqrt{r \cos \theta}} \, dr \, d\theta = 4 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^{3/2} \cos^{1/2} \theta \, dr \, d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \left[\frac{2}{5} r^{5/2} \cos^{1/2} \theta \right]_0^{2 \cos \theta} d\theta = \frac{8}{5} \int_{-\pi/2}^{\pi/2} \left(2^{5/2} \cos^{5/2} \theta \cos^{1/2} \theta \right) d\theta = \frac{32\sqrt{2}}{5} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta \\ &= \frac{32\sqrt{2}}{5} \int_{-\pi/2}^{\pi/2} \cos \theta (1 - \sin^2 \theta) \, d\theta = \frac{32\sqrt{2}}{5} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_{-\pi/2}^{\pi/2} = \frac{32\sqrt{2}}{5} \left[\frac{4}{3} \right] = \frac{128\sqrt{2}}{15} \end{aligned}$$

The volume is $\boxed{\frac{128\sqrt{2}}{15} \text{ cubic units}}.$

38. Since ρ is constant and the solid E is a sphere of radius a , the mass M of the solid is

$$M = \int \int \int_E \rho \, dV = \rho \int \int \int_E dV = \rho \frac{4}{3} \pi a^3 = \frac{4\rho\pi a^3}{3}$$

The mass is $\boxed{\frac{4\rho\pi a^3}{3}}.$

39. We will center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = a^2$. Then the density at each point (x, y, z) in this solid is given by $\rho(x, y, z) = k(x^2 + y^2 + z^2)$ where k is the constant of proportionality. Using cylindrical coordinates, the sphere can be expressed as $r^2 + z^2 = a^2$, so that the solid E whose mass we seek is $-\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$, $0 \leq r \leq a$, and $0 \leq \theta \leq 2\pi$. By symmetry, we can double the mass of the upper hemisphere of this solid to obtain the total mass. The total mass M of the solid is

$$\begin{aligned} M &= \int \int \int_E \rho(x, y, z) \, dV = 2k \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2 - r^2}} (r^2 + z^2) r \, dz \, dr \, d\theta \\ &= 2k \int_0^{2\pi} \int_0^a \left[r^3 z + \frac{1}{3} r z^3 \right]_0^{\sqrt{a^2 - r^2}} \, dr \, d\theta = 2k \int_0^{2\pi} \int_0^a \left(r^3 \sqrt{a^2 - r^2} + \frac{1}{3} r (\sqrt{a^2 - r^2})^3 \right) \, dr \, d\theta \\ &= 2k \int_0^{2\pi} d\theta \frac{1}{3} \int_0^a \left(3r^3 \sqrt{a^2 - r^2} + r (\sqrt{a^2 - r^2})^3 \right) \, dr \end{aligned}$$

The first of these integrals is 2π . To evaluate the second integral, we use the substitution $u = a^2 - r^2$ with $du = -2r \, dr$. We see that $u = a^2$ when $r = 0$, $u = 0$ when $r = a$, and $r^2 = a^2 - u$.

$$\begin{aligned} \frac{1}{3} \int_0^a \left(3r^3 \sqrt{a^2 - r^2} + r (\sqrt{a^2 - r^2})^3 \right) \, dr &= -\frac{1}{6} \int_0^a \left(3r^2 (-2r) \sqrt{a^2 - r^2} + (-2r) (\sqrt{a^2 - r^2})^3 \right) \, dr \\ &= -\frac{1}{6} \int_{a^2}^0 \left(3(a^2 - u) \sqrt{u} + (\sqrt{u})^3 \right) \, du = \frac{1}{6} \int_0^{a^2} \left(3(a^2 - u) u^{1/2} + u^{3/2} \right) \, du \\ &= \frac{1}{6} \int_0^{a^2} \left(3a^2 u^{1/2} - 2u^{3/2} \right) \, du = \frac{1}{6} \left[2a^2 u^{3/2} - \frac{4}{5} u^{5/2} \right]_0^{a^2} = \frac{1}{6} \left(2a^5 - \frac{4}{5} a^5 \right) = \frac{a^5}{5} \end{aligned}$$

So we have the total mass of the solid given by

$$M = 2k \int_0^{2\pi} d\theta \frac{1}{3} \int_0^a \left(3r^3 \sqrt{a^2 - r^2} + r (\sqrt{a^2 - r^2})^3 \right) \, dr = 2k(2\pi) \frac{a^5}{5} = \frac{4k\pi a^5}{5}$$

The mass is $\boxed{\frac{4k\pi a^5}{5}}$.

40. We will position the cylinder with its base on the xy plane centered at the origin, so that the cylinder E has equation $x^2 + y^2 = a^2$ with $0 \leq z \leq h$. Then the density at each point (x, y, z) in this solid is given by $\rho(x, y, z) = k(x^2 + y^2)$ where k is the constant of proportionality. Using cylindrical coordinates, the solid can be expressed as $0 \leq z \leq h$, $0 \leq r \leq a$, and $0 \leq \theta \leq 2\pi$. The mass M of the solid is

$$\begin{aligned} M &= \int \int \int_E \rho(x, y, z) \, dV = \int_0^{2\pi} \int_0^a \int_0^h kr^2 \cdot r \, dzdrd\theta = k \int_0^{2\pi} \int_0^a \int_0^h r^3 \, dzdrd\theta \\ &= k \int_0^{2\pi} \int_0^a [r^3 z]_0^h \, drd\theta = kh \int_0^{2\pi} \int_0^a r^3 \, drd\theta = kh \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^a \, d\theta = \frac{kha^4}{4} \int_0^{2\pi} d\theta = \frac{kha^4\pi}{2} \end{aligned}$$

The mass of the solid is $\boxed{\frac{kha^4\pi}{2}}$

41. The projection of the hemisphere onto the xy plane is the region bounded by $x^2 + y^2 = 9$. The solid E can be described in polar coordinates as $0 \leq z \leq \sqrt{9 - r^2}$, $0 \leq r \leq 3$, and $0 \leq \theta \leq 2\pi$. We are given $\rho(x, y, z) = kz$ where k is the constant of proportionality. We will compute the moment of inertia about the z -axis and then use the fact that $I_x = \frac{1}{2}I_z$ to find the moment of inertia about the x -axis. The moment of inertia about the z -axis is

$$\begin{aligned} I_z &= \int \int \int_E (\sqrt{x^2 + y^2})^2 \rho(x, y, z) \, dV = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{9-r^2}} r^2 kz \cdot r \, dzdrd\theta \\ &= k \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{9-r^2}} r^3 z \, dzdrd\theta = k \int_0^{2\pi} \int_0^3 \left[\frac{1}{2} r^3 z^2 \right]_0^{\sqrt{9-r^2}} \, drd\theta = \frac{k}{2} \int_0^{2\pi} \int_0^3 (9r^3 - r^5) \, drd\theta \\ &= \frac{k}{2} \int_0^{2\pi} \left[\frac{9}{4} r^4 - \frac{1}{6} r^6 \right]_0^3 \, d\theta = \frac{k}{2} \int_0^{2\pi} \frac{243}{4} \, d\theta = \frac{243k}{8} \int_0^{2\pi} d\theta = \frac{243k\pi}{4} \end{aligned}$$

We see that $I_x = \frac{1}{2}I_z = \frac{243k\pi}{8}$. By symmetry the moment of inertia about x -axis and the moment of

inertia about the y -axis are equal. That is, $\boxed{I_y = I_x = \frac{243k\pi}{8}}$.

42. Using cylindrical coordinates we have $xy = r^2 \cos \theta \sin \theta$, so the solid E is given by $0 \leq z \leq r^2 \cos \theta \sin \theta$, $0 \leq r \leq 2$, and $0 \leq \theta \leq \frac{\pi}{2}$. Using ρ to represent the density, the mass M of E is

$$\begin{aligned} M &= \int \int \int_E \rho \, dV = \rho \int_0^{\pi/2} \int_0^2 \int_0^{r^2 \cos \theta \sin \theta} r \, dzdrd\theta = \rho \int_0^{\pi/2} \int_0^2 [rz]_0^{r^2 \cos \theta \sin \theta} \, drd\theta \\ &= \rho \int_0^{\pi/2} \int_0^2 r^3 \cos \theta \sin \theta \, drd\theta = \rho \int_0^{\pi/2} \left[\frac{1}{4} r^4 \cos \theta \sin \theta \right]_0^2 \, d\theta = \rho \int_0^{\pi/2} 4 \cos \theta \sin \theta \, d\theta \\ &= 4\rho \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = 2\rho \end{aligned}$$

We now compute M_{yz}

$$\begin{aligned} M_{yz} &= \int \int \int_E x \rho \, dV = \rho \int_0^{\pi/2} \int_0^2 \int_0^{r^2 \cos \theta \sin \theta} r^2 \cos \theta \, dz \, dr \, d\theta = \rho \int_0^{\pi/2} \int_0^2 [r^2 \cos \theta z]_0^{r^2 \cos \theta \sin \theta} \, dr \, d\theta \\ &= \rho \int_0^{\pi/2} \int_0^2 r^4 \cos^2 \theta \sin \theta \, dr \, d\theta = \rho \int_0^{\pi/2} \left[\frac{1}{5} r^5 \cos^2 \theta \sin \theta \right]_0^2 \, d\theta = \frac{32\rho}{5} \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \\ &= \frac{32\rho}{5} \left[-\frac{1}{3} \cos^3 \theta \right]_0^{\pi/2} = \frac{32\rho}{15} \end{aligned}$$

We see that x and y are interchangeable throughout this problem. This symmetry allows us to conclude that $M_{xz} = \frac{32\rho}{15}$.

We now compute M_{xy} .

$$\begin{aligned} M_{xy} &= \int \int \int_E z \rho \, dV = \rho \int_0^{\pi/2} \int_0^2 \int_0^{r^2 \cos \theta \sin \theta} r z \, dz \, dr \, d\theta = \rho \int_0^{\pi/2} \int_0^2 \left[\frac{1}{2} r z^2 \right]_0^{r^2 \cos \theta \sin \theta} \, dr \, d\theta \\ &= \frac{\rho}{2} \int_0^{\pi/2} \int_0^2 r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta = \frac{\rho}{2} \int_0^{\pi/2} \left[\frac{1}{6} r^6 \cos^2 \theta \sin^2 \theta \right]_0^2 \, d\theta = \frac{16\rho}{3} \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta \, d\theta \\ &= \frac{16\rho}{3} \int_0^{\pi/2} \frac{1}{2} (1 + \cos(2\theta)) \frac{1}{2} (1 - \cos(2\theta)) \, d\theta = \frac{4\rho}{3} \int_0^{\pi/2} (1 - \cos^2(2\theta)) \, d\theta \\ &= \frac{4\rho}{3} \int_0^{\pi/2} \left(1 - \frac{1}{2} (1 + \cos(4\theta)) \right) \, d\theta = \frac{4\rho}{3} \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2} \cos(4\theta) \right) \, d\theta = \frac{2\rho}{3} \left[\theta - \frac{1}{4} \sin(4\theta) \right]_0^{\pi/2} \\ &= \frac{\pi\rho}{3} \end{aligned}$$

We may now compute

$$\bar{x} = \frac{M_{yz}}{M} = \frac{\frac{32\rho}{15}}{\frac{32\rho}{15}} = \frac{16}{15} \quad \bar{y} = \frac{M_{xz}}{M} = \frac{\frac{32\rho}{15}}{\frac{32\rho}{15}} = \frac{16}{15} \quad \bar{z} = \frac{M_{xy}}{M} = \frac{\frac{\pi\rho}{3}}{\frac{32\rho}{15}} = \frac{\pi}{6}$$

The center of mass is $\left(\bar{x}, \bar{y}, \bar{z} \right) = \left(\frac{16}{15}, \frac{16}{15}, \frac{\pi}{6} \right)$.

43. Because the z -axis is a line of symmetry for the solid, we know that the center of mass must lie on the z -axis. That is $\bar{x} = 0$ and $\bar{y} = 0$. The projection of the solid E onto the xy -plane is bounded by $x^2 + y^2 = 8$. Using cylindrical coordinates we have $\frac{1}{4}(x^2 + y^2) = \frac{1}{4}r^2$, so the solid is given by $\frac{1}{4}r^2 \leq z \leq 2$, $0 \leq r \leq 2\sqrt{2}$, and $0 \leq \theta \leq 2\pi$. Using ρ to represent the density, the mass M of E is

$$\begin{aligned} M &= \int \int \int_E \rho \, dV = \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} \int_{\frac{1}{4}r^2}^2 r \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} [rz]_{\frac{1}{4}r^2}^2 \, dr \, d\theta \\ &= \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(2r - \frac{1}{4}r^3 \right) \, dr \, d\theta = \frac{\rho}{4} \int_0^{2\pi} \int_0^{2\sqrt{2}} (8r - r^3) \, dr \, d\theta = \frac{\rho}{4} \int_0^{2\pi} \left[4r^2 - \frac{1}{4}r^4 \right]_0^{2\sqrt{2}} \, d\theta \\ &= \frac{\rho}{4} \int_0^{2\pi} 16 \, d\theta = 8\rho\pi \end{aligned}$$

We now compute M_{xy} .

$$\begin{aligned}
 M_{xy} &= \int \int_E z \rho \, dV = \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} \int_{\frac{1}{4}r^2}^2 rz \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} \left[\frac{1}{2} r z^2 \right]_{\frac{1}{4}r^2}^2 dr \, d\theta \\
 &= \rho \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(2r - \frac{1}{32} r^5 \right) dr \, d\theta = \frac{\rho}{32} \int_0^{2\pi} \int_0^{2\sqrt{2}} (64r - r^5) dr \, d\theta = \frac{\rho}{32} \int_0^{2\pi} \left[32r^2 - \frac{1}{6} r^6 \right]_0^{2\sqrt{2}} d\theta \\
 &= \frac{\rho}{32} \int_0^{2\pi} \frac{512}{3} d\theta = \frac{32\rho\pi}{3}
 \end{aligned}$$

We may now compute

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{32\rho\pi}{3}}{8\rho\pi} = \frac{4}{3}$$

The center of mass is $\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{4}{3}\right)}$.

44. Because the z -axis is a line of symmetry for the solid, we know that the center of mass must lie on the z -axis. That is $\bar{x} = 0$ and $\bar{y} = 0$. The projection of the solid E onto the xy -plane is found by finding the intersection of the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 12$. Substituting $z = x^2 + y^2$ into $x^2 + y^2 + z^2 = 12$, we obtain $z + z^2 = 12$ which means $z^2 + z - 12 = 0$ or $(z + 4)(z - 3) = 0$. We see that this implies $z = -4$ or $z = 3$, but we must have $z \geq 0$ since $z = x^2 + y^2$, so $z = 3$. Substituting $z = 3$ into $z = x^2 + y^2$ we get the curve of intersection to be $x^2 + y^2 = 3$, which is the bounding curve for the projection of E onto the xy -plane. Using cylindrical coordinates $x^2 + y^2 = r^2$ and $\sqrt{12 - x^2 - y^2} = \sqrt{12 - r^2}$, so the solid is given by $r^2 \leq z \leq \sqrt{12 - r^2}$, $0 \leq r \leq \sqrt{3}$, and $0 \leq \theta \leq 2\pi$. Using ρ to represent the density, the mass M of E is

$$\begin{aligned}
 M &= \int \int_E \rho \, dV = \rho \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} r \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^{\sqrt{3}} [rz]_{r^2}^{\sqrt{12-r^2}} dr \, d\theta \\
 &= \rho \int_0^{2\pi} \int_0^{\sqrt{3}} \left(r\sqrt{12-r^2} - r^3 \right) dr \, d\theta = \rho \int_0^{2\pi} \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (12-r^2)^{3/2} - \frac{1}{4} r^4 \right]_0^{\sqrt{3}} d\theta \\
 &= \rho \int_0^{2\pi} \left(-\frac{1}{3}(27) - \frac{9}{4} + \frac{1}{3}(12)^{3/2} \right) d\theta = \rho \int_0^{2\pi} \left(8\sqrt{3} - \frac{45}{4} \right) d\theta \\
 &= \rho \frac{32\sqrt{3} - 45}{4} (2\pi) = \frac{32\sqrt{3} - 45}{2} \rho\pi
 \end{aligned}$$

We now compute M_{xy} .

$$\begin{aligned}
 M_{xy} &= \int \int_E z \rho \, dV = \rho \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} rz \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^{\sqrt{3}} \left[\frac{1}{2} r z^2 \right]_{r^2}^{\sqrt{12-r^2}} dr \, d\theta \\
 &= \rho \int_0^{2\pi} \int_0^{\sqrt{3}} \left(6r - \frac{1}{2} r^3 - \frac{1}{2} r^5 \right) dr \, d\theta = \frac{\rho}{2} \int_0^{2\pi} \int_0^{\sqrt{3}} (12r - r^3 - r^5) dr \, d\theta \\
 &= \frac{\rho}{2} \int_0^{2\pi} \left[6r^2 - \frac{1}{4} r^4 - \frac{1}{6} r^6 \right]_0^{\sqrt{3}} d\theta = \frac{\rho}{2} \int_0^{2\pi} \left(6(3) - \frac{1}{4}(9) - \frac{1}{6}(27) \right) d\theta = \frac{\rho}{2} \int_0^{2\pi} \left(\frac{45}{4} \right) d\theta \\
 &= \frac{45\rho}{8} (2\pi) = \frac{45}{4} \rho\pi
 \end{aligned}$$

We may now compute

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{45}{4} \rho\pi}{\frac{32\sqrt{3}-45}{2} \rho\pi} = \frac{45}{2(32\sqrt{3} - 45)}$$

The center of mass is $\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{45}{2(32\sqrt{3} - 45)}\right)}$.

45. Because the z -axis is a line of symmetry for the solid, we know that the center of mass must lie on the z -axis. That is $\bar{x} = 0$ and $\bar{y} = 0$. The projection of the solid E onto the xy -plane is bounded by $x^2 + y^2 = 4$. Using cylindrical coordinates we have $x^2 + y^2 = r^2$, so the solid is given by $r^2 \leq z \leq 4$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. Using ρ to represent the density, the mass M of E is

$$\begin{aligned} M &= \int \int \int_E \rho \, dV = \rho \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^2 [rz]_{r^2}^4 \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \rho \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 d\theta = \rho \int_0^{2\pi} (4) \, d\theta = 4\rho(2\pi) = 8\rho\pi \end{aligned}$$

We now compute M_{xy} .

$$\begin{aligned} M_{xy} &= \int \int \int_E z\rho \, dV = \rho \int_0^{2\pi} \int_0^2 \int_{r^2}^4 rz \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^2 \left[\frac{1}{2}rz^2 \right]_{r^2}^4 \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^2 \left(8r - \frac{1}{2}r^5 \right) \, dr \, d\theta \\ &= \rho \int_0^{2\pi} \left[4r^2 - \frac{1}{12}r^6 \right]_0^2 d\theta = \rho \int_0^{2\pi} \left(16 - \frac{16}{3} \right) d\theta = \frac{32}{3}\rho(2\pi) = \frac{64}{3}\rho\pi \end{aligned}$$

We may now compute

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{64}{3}\rho\pi}{8\rho\pi} = \frac{8}{3}$$

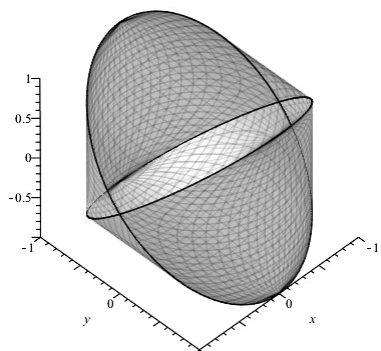
The center of mass is $\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{8}{3}\right)}$.

46. The projection of the solid E onto the xy -plane is bounded by $x^2 + y^2 = 1$. Using cylindrical coordinates $x^2 + y^2 + z^2 = 2$ becomes $r^2 + z^2 = 2$ so that $z = \sqrt{2 - r^2}$, $z \geq 0$. The solid is given by $0 \leq z \leq \sqrt{2 - r^2}$, $0 \leq r \leq 1$, and $0 \leq \theta \leq 2\pi$. Using ρ to represent the density, the mass M of E is

$$\begin{aligned} M &= \int \int \int_E \rho \, dV = \rho \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta = \rho \int_0^{2\pi} \int_0^1 [rz]_0^{\sqrt{2-r^2}} \, dr \, d\theta \\ &= \rho \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} \right) \, dr \, d\theta = \rho \int_0^{2\pi} \left[-\frac{1}{2}(2-r^2)^{3/2} \frac{2}{3} \right]_0^1 d\theta = \rho \int_0^{2\pi} \left(-\frac{1}{3} + \frac{2\sqrt{2}}{3} \right) d\theta \\ &= \frac{2\sqrt{2}-1}{3}\rho(2\pi) = \frac{2(2\sqrt{2}-1)}{3}\rho\pi \end{aligned}$$

The mass is $\boxed{\frac{2(2\sqrt{2}-1)}{3}\rho\pi}$.

47. The solid E that is formed by the intersection of the two cylinders can be described as $-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}$ where $x^2 + y^2 \leq 1$ (see figure).

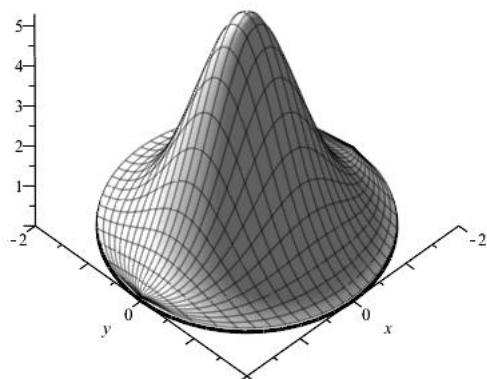


Though the solid involves the intersection of cylinders, its volume can be computed using rectangular coordinates. We compute the mass M of the solid as

$$\begin{aligned}
 M &= \iiint_E \rho \, dV = \rho \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \, dy \, dx = \rho \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [z]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dy \, dx \\
 &= \rho \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2\sqrt{1-x^2}) \, dy \, dx = \rho \int_{-1}^1 \left[2\sqrt{1-x^2} y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx \\
 &= \rho \int_{-1}^1 \left[2\sqrt{1-x^2} \cdot 2\sqrt{1-x^2} \right] \, dx = \rho \int_{-1}^1 4(1-x^2) \, dx = 4\rho \left[x - \frac{1}{3}x^3 \right]_{-1}^1 = \frac{16\rho}{3}
 \end{aligned}$$

The mass of the solid is $\boxed{\frac{16\rho}{3}}$.

48. For part (a), we use a CAS to plot the surface (see figure).



For part (b), we see that the mountain can be described in cylindrical coordinates as $0 \leq z \leq 5.3e^{-r^2}$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. We may compute the volume V as

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{5.3e^{-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [rz]_0^{5.3e^{-r^2}} \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left(r(5.3e^{-r^2}) \right) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left[-2.65e^{-r^2} \right]_0^2 \, d\theta = \int_0^{2\pi} (-2.65(e^{-4} - 1)) \, d\theta = \left(2.65 \left(1 - \frac{1}{e^4} \right) \right) (2\pi) \approx 16.345
 \end{aligned}$$

The volume is approximately $\boxed{16.345 \text{ km}^3}$.

49. From the limits of integration on z , we have $-r \leq z \leq \sqrt{16-r^2}$. In rectangular coordinates, we see that this corresponds to $-\sqrt{x^2+y^2} \leq z \leq \sqrt{16-x^2-y^2}$. The limits on r and θ indicate that the projection of the solid on the xy -plane is bounded by the circle centered at the origin with radius 4. That is, $x^2 + y^2 \leq 16$. We may describe this region as $-\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$, $-4 \leq x \leq 4$. We may now express the given triple integral in rectangular coordinates.

$$\begin{aligned} \int_0^{2\pi} \int_0^4 \int_{-r}^{\sqrt{16-r^2}} z^2 r^5 \cos^4 \theta \, dz dr d\theta &= \int_0^{2\pi} \int_0^4 \int_{-r}^{\sqrt{16-r^2}} z^2 (r \cos \theta)^4 \cdot r \, dz dr d\theta \\ &= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} z^2 x^4 \, dz dy dx \end{aligned}$$

In rectangular coordinates, we have

$$\boxed{\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} z^2 x^4 \, dz dy dx}.$$

50. From the limits of integration on z , we have $-r^2 \leq z \leq 9$. In rectangular coordinates, we see that this corresponds to $-(x^2+y^2) \leq z \leq 9$. The limits on r and θ indicate that the projection of the solid on the xy -plane is bounded by the portion of the circle centered at the origin with radius 2 in the first quadrant. That is, $x^2 + y^2 \leq 4$, $x \geq 0$, $y \geq 0$. We may describe this region as $0 \leq y \leq \sqrt{4-x^2}$, $0 \leq x \leq 2$. Note that $y = r \sin \theta$ implies that $\sin \theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$. We may now express the given triple integral in rectangular coordinates.

$$\begin{aligned} \int_0^{\pi/2} \int_0^2 \int_{-r^2}^9 z^2 r^4 \sin^4 \theta \, dz dr d\theta &= \int_0^{\pi/2} \int_0^2 \int_{-r^2}^9 z^2 (r \sin \theta)^3 \sin \theta \cdot r \, dz dr d\theta \\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-(x^2+y^2)}^9 z^2 y^3 \frac{y}{\sqrt{x^2+y^2}} \, dz dy dx \\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-(x^2+y^2)}^9 \frac{z^2 y^4}{\sqrt{x^2+y^2}} \, dz dy dx \end{aligned}$$

In rectangular coordinates, we have

$$\boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-(x^2+y^2)}^9 \frac{z^2 y^4}{\sqrt{x^2+y^2}} \, dz dy dx}.$$

51. (a) For ease of notation below, we will refer to the radius R of the sphere as R_2 and the radius r of the hole as R_1 . By symmetry we may solve this problem by considering the hemisphere $z = \sqrt{R_2^2 - x^2 - y^2}$ and determining R_1 so that the hole removes half the volume of the hemisphere. We see that the projection onto the xy -plane is the region bounded by the circles $x^2 + y^2 = R_1^2$ and $x^2 + y^2 = R_2^2$. Converting to cylindrical coordinates, the hemisphere with a hole drilled through it is given by $0 \leq z \leq \sqrt{R_2^2 - r^2}$, $R_1 \leq r \leq R_2$, and $0 \leq \theta \leq 2\pi$. The volume V of this solid E is given by

$$\begin{aligned} V &= \iiint_E dV = \int_0^{2\pi} \int_{R_1}^{R_2} \int_0^{\sqrt{R_2^2-r^2}} r \, dz dr d\theta = \int_0^{2\pi} \int_{R_1}^{R_2} [rz]_0^{\sqrt{R_2^2-r^2}} dr d\theta \\ &= \int_0^{2\pi} \int_{R_1}^{R_2} \left(r \sqrt{R_2^2 - r^2} \right) dr d\theta = \int_0^{2\pi} \left[-\frac{1}{2} (R_2^2 - r^2)^{3/2} \frac{2}{3} \right]_{R_1}^{R_2} d\theta = \int_0^{2\pi} \left(\frac{1}{3} (R_2^2 - R_1^2)^{3/2} \right) d\theta \\ &= \frac{2\pi}{3} (R_2^2 - R_1^2)^{3/2} \end{aligned}$$

From this calculation, we see that the total volume of the sphere described in the problem statement is $\frac{4\pi}{3}(R_2^2 - R_1^2)^{3/2}$. Now we wish to find R_1 such that this volume equals half the volume of a sphere with radius R_2 with no hole. That is, we will solve for R_1 in the equation

$$\begin{aligned}\frac{4\pi}{3}(R_2^2 - R_1^2)^{3/2} &= \frac{1}{2} \left(\frac{4\pi}{3} R_2^3 \right) \\ (R_2^2 - R_1^2)^{3/2} &= \frac{1}{2} R_2^3 \\ R_2^2 - R_1^2 &= \frac{1}{2^{2/3}} R_2^2 \\ R_1^2 &= R_2^2 \left(1 - \frac{1}{2^{2/3}} \right) \\ R_1 &= \sqrt{R_2^2 \frac{2^{2/3} - 1}{2^{2/3}}} \\ R_1 &= R_2 \sqrt{\frac{2^{2/3} - 1}{2^{2/3}}}\end{aligned}$$

To remove half the volume of a sphere of radius R , drill a hole of radius $R\sqrt{\frac{2^{2/3} - 1}{2^{2/3}}}$ cm through the center of the sphere.

For part (b), we substitute $R = 10$ into the formula found in part (a) to find $r \approx 6.083$ cm.

52. The projection of the solid E onto the xy -plane is found by finding the intersection of the surfaces $4z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 5$. Substituting $4z = x^2 + y^2$ into $x^2 + y^2 + z^2 = 5$, we obtain $4z + z^2 = 5$ which means $z^2 + 4z - 5 = 0$ or $(z + 5)(z - 1) = 0$. We see that this implies $z = -5$ or $z = 1$, but we must have $z \geq 0$ since $z = \frac{1}{4}(x^2 + y^2)$, so $z = 1$. Substituting $z = 1$ into $4z = x^2 + y^2$ we get the curve of intersection to be $x^2 + y^2 = 4$, which is the bounding curve for the projection of E onto the xy plane. Using cylindrical coordinates $x^2 + y^2 = r^2$ and $\sqrt{5 - x^2 - y^2} = \sqrt{5 - r^2}$, so the solid is given by $\frac{1}{4}r^2 \leq z \leq \sqrt{5 - r^2}$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$. We compute the volume V as

$$\begin{aligned}V &= \int \int \int_E dV = \int_0^{2\pi} \int_0^2 \int_{\frac{1}{4}r^2}^{\sqrt{5-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [rz]_{\frac{1}{4}r^2}^{\sqrt{5-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left(r\sqrt{5-r^2} - \frac{1}{4}r^3 \right) \, dr \, d\theta = \int_0^{2\pi} \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (5-r^2)^{3/2} - \frac{1}{16}r^4 \right]_0^2 \, d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{3} - 1 + \frac{1}{3}(5)^{3/2} \right) \, d\theta = \int_0^{2\pi} \left(\frac{5\sqrt{5}}{3} - \frac{4}{3} \right) \, d\theta = \frac{5\sqrt{5}-4}{3}(2\pi) = \frac{2\pi(5\sqrt{5}-4)}{3}\end{aligned}$$

The volume of the solid is $\frac{2\pi(5\sqrt{5}-4)}{3}$ cubic units.

53. We will define the two pipes using the equations $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$. The solid E that is formed by the intersection of the two cylinders can be described as $-\sqrt{r^2 - x^2} \leq z \leq \sqrt{r^2 - x^2}$ where $x^2 + y^2 \leq r^2$. Though the solid involves the intersection of cylinders, its volume is can be computed

using rectangular coordinates. We compute the volume V of the solid as

$$\begin{aligned}
 V &= \int \int \int_E dV = \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dz dy dx = \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} [z]_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy dx \\
 &= \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} (2\sqrt{r^2-x^2}) dy dx = \int_{-r}^r [2\sqrt{r^2-x^2} y]_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dx \\
 &= \int_{-r}^r [2\sqrt{r^2-x^2} \cdot 2\sqrt{r^2-x^2}] dx = \int_{-r}^r 4(r^2-x^2) dx = 4 \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^r = \frac{16r^3}{3}
 \end{aligned}$$

To determine r such that the volume is 10 m^3 , we substitute $V = 10$ into the equation $V = \frac{16r^3}{3}$ and

solve for r . That is, we solve $10 = \frac{16r^3}{3}$ to get $r = \frac{\sqrt[3]{15}}{2} \text{ m}$.

14.8 Triple Integrals Using Spherical Coordinates

Concepts and Vocabulary

1. In spherical coordinates, the number ρ equals the distance from the *origin* to the point P .
2. In spherical coordinates, ϕ is the angle between the positive z -axis and the line segment OP .
3. False. The correct equations are $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$.
4. True. See Equation (2) in Objective 1.
5. $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$. See Objective 2.
6. In spherical coordinates, the surface described by the equation $\rho = a$, where $a > 0$ is a constant, is a *sphere of radius a* .

Skill Building

7. Since the z -coordinate in cylindrical coordinates is $-\frac{2\sqrt{3}}{3}$, that is the z -coordinate in rectangular coordinates as well. Next, we know that $\tan \theta = \frac{y}{x}$, so that $\tan \frac{\pi}{6} = \frac{y}{1} = y$, or $y = \frac{\sqrt{3}}{3}$. So the rectangular coordinates of the point are $\left(1, \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(-\frac{2\sqrt{3}}{3}\right)^2} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}.$$

Next, $\cos \phi = \frac{z}{\rho}$, so that

$$\cos \phi = \frac{-(2\sqrt{3}/3)}{(2\sqrt{6}/3)} = -\frac{\sqrt{2}}{2},$$

so that $\phi = \frac{3\pi}{4}$. Finally, θ is the same in cylindrical and spherical coordinates. So the three sets of coordinates are:

rectangular : $(x, y, z) = \left(1, \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$ cylindrical : $(r, \theta, z) = \left(\frac{2\sqrt{3}}{3}, \frac{\pi}{6}, -\frac{2\sqrt{3}}{3}\right)$ spherical : $(\rho, \theta, \phi) = \left(\frac{2\sqrt{6}}{3}, \frac{\pi}{6}, \frac{3\pi}{4}\right)$
--

8. Since the z -coordinate in rectangular coordinates is $2\sqrt{2}$, that is the z -coordinate in cylindrical coordinates as well. Since θ is the same in cylindrical and spherical coordinates, we get for the cylindrical coordinates of the point $(\sqrt{8}, \frac{\pi}{6}, 2\sqrt{2})$. Next, $\tan \theta = \frac{y}{x}$, so that $\tan \frac{\pi}{6} = \frac{y}{x}$, or $\frac{y}{x} = \frac{\sqrt{3}}{3}$, which can be rewritten as $y^2 = \frac{1}{3}x^2$. Also $r^2 = x^2 + y^2 = 8$; substituting in the value for y^2 gives $x^2 + \frac{1}{3}x^2 = \frac{4}{3}x^2 = 8$, so that $x = \pm\sqrt{6}$ and $y = \pm\sqrt{2}$. Since $\theta = \frac{\pi}{6}$, the point $(x, y, 0)$ lies in the first quadrant, so we take both plus signs. Therefore the rectangular coordinates of the point are $(\sqrt{6}, \sqrt{2}, 2\sqrt{2})$. It follows that

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2 + (2\sqrt{2})^2} = 4.$$

Finally, $\cos \phi = \frac{z}{\rho}$, so that

$$\cos \phi = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2},$$

and therefore $\phi = \frac{\pi}{4}$. So we get

rectangular : $(x, y, z) = (\sqrt{6}, \sqrt{2}, 2\sqrt{2})$ cylindrical : $(r, \theta, z) = (\sqrt{8}, \frac{\pi}{6}, 2\sqrt{2})$ spherical : $(\rho, \theta, \phi) = (4, \frac{\pi}{6}, \frac{\pi}{4})$.
--

9. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2 + (2\sqrt{3})^2} = 4$, and $\tan \theta = \frac{y}{x} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1$. Since both x and y are negative, θ is in the third quadrant, so we get $\theta = \frac{5\pi}{4}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, so that

$\phi = \frac{\pi}{6}$. The spherical coordinates are $\left(4, \frac{5\pi}{4}, \frac{\pi}{6}\right)$.

10. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + (\sqrt{3})^2 + 2^2} = 2\sqrt{2}$, and $\tan \theta = \frac{y}{x} = -\sqrt{3}$. Since x is negative and y positive, this is a second quadrant angle, so that $\theta = \frac{2\pi}{3}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$, so that

$\phi = \frac{\pi}{4}$. The spherical coordinates are $\left(2\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$.

11. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2$, and $\tan \theta = \frac{y}{x} = 1$. Since x and y are both positive, this is a first quadrant angle, so that $\theta = \frac{\pi}{4}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2}$, so that $\phi = \frac{\pi}{4}$. The spherical

coordinates are $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$.

12. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-\sqrt{3})^2 + (-2)^2} = 2\sqrt{2}$, and $\tan \theta = \frac{y}{x} = -\sqrt{3}$. Since x is positive and y negative, this is a fourth quadrant angle, so that $\theta = \frac{5\pi}{3}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{-2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$, so that

$\phi = \frac{3\pi}{4}$. The spherical coordinates are $\left(2\sqrt{2}, \frac{5\pi}{3}, \frac{3\pi}{4}\right)$.

13. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, and $\tan \theta = \frac{y}{x} = 2$. Since x and y are both positive, this is a first quadrant angle, so that $\theta = \tan^{-1}(2)$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{3}{\sqrt{14}}$, so that $\phi = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$.

The spherical coordinates are $\left(\sqrt{14}, \tan^{-1}(2), \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)\right)$.

14. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} = 2$, and $\tan \theta = \frac{y}{x} = -1$. Since x is positive and y negative, this is a fourth quadrant angle, so that $\theta = \frac{7\pi}{4}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2}$, so that $\phi = \frac{\pi}{4}$. The

spherical coordinates are $\left(2, \frac{7\pi}{4}, \frac{\pi}{4}\right)$.

15. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (3\sqrt{3})^2 + 3^2} = 6$. Since $x = 0$, $\tan \theta$ is undefined, so that $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Since y is positive, we see that $\theta = \frac{\pi}{2}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{3}{6} = \frac{1}{2}$, so that $\phi = \frac{\pi}{3}$. The spherical

coordinates are $\left(6, \frac{\pi}{2}, \frac{\pi}{3}\right)$.

16. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-5\sqrt{3})^2 + 5^2 + 0^2} = 10$, and $\tan \theta = \frac{y}{x} = -\frac{\sqrt{3}}{3}$. Since x is negative and y positive, this is a second quadrant angle, so that $\theta = \frac{5\pi}{6}$. Finally, $\cos \phi = \frac{z}{\rho} = 0$, so that $\phi = \frac{\pi}{2}$. The spherical coordinates are $\boxed{\left(10, \frac{5\pi}{6}, \frac{\pi}{2}\right)}$.

17. Computing gives

$$\begin{aligned}
 \int_0^{\pi/2} \int_0^{\sin \phi} \int_0^{\pi/4} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi &= \int_0^{\pi/2} \int_0^{\sin \phi} [\rho^2 \theta \sin \phi]_0^{\pi/4} \, d\rho \, d\phi \\
 &= \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \\
 &= \frac{\pi}{4} \int_0^{\pi/2} \left[\frac{1}{3} \rho^3 \sin \phi \right]_0^{\sin \phi} \, d\phi \\
 &= \frac{\pi}{12} \int_0^{\pi/2} \sin^4 \phi \, d\phi.
 \end{aligned}$$

Using Formulas 84 and 90 in the Table of Integrals, we get

$$\begin{aligned}
 \frac{\pi}{12} \int_0^{\pi/2} \sin^4 \phi \, d\phi &= \frac{\pi}{12} \left(\left[-\frac{1}{4} \sin^3 \phi \cos \phi \right]_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2 \phi \, d\phi \right) \\
 &= \frac{\pi}{16} \int_0^{\pi/2} \sin^2 \phi \, d\phi \\
 &= \frac{\pi}{16} \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} \right]_0^{\pi/2} = \boxed{\frac{\pi^2}{64}}.
 \end{aligned}$$

18. Computing gives

$$\begin{aligned}
 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sin \phi} \rho^2 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{1}{3} \rho^3 \sin \phi \cos \phi \right]_0^{\sin \phi} \, d\theta \, d\phi \\
 &= \frac{1}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4 \phi \cos \phi \, d\theta \, d\phi \\
 &= \frac{1}{3} \int_0^{\pi/2} [\theta \sin^4 \phi \cos \phi]_0^{\pi/2} \, d\phi \\
 &= \frac{\pi}{6} \int_0^{\pi/2} \sin^4 \phi \cos \phi \, d\phi \\
 &= \frac{\pi}{6} \left[\frac{1}{5} \sin^5 \phi \right]_0^{\pi/2} = \boxed{\frac{\pi}{30}}.
 \end{aligned}$$

19. Computing gives

$$\begin{aligned}
 \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin^2 \phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{1}{3} \rho^3 \sin^2 \phi \right]_0^{\sec \phi} d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \frac{\sin^2 \phi}{\cos^3 \phi} d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \tan^2 \phi \sec \phi \, d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec^2 \phi - 1) \sec \phi \, d\phi \, d\theta \\
 &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec^3 \phi - \sec \phi) \, d\phi \, d\theta.
 \end{aligned}$$

Using Formula 88 from the Table of Integrals (or see Example 7 in Section 7.2), we get

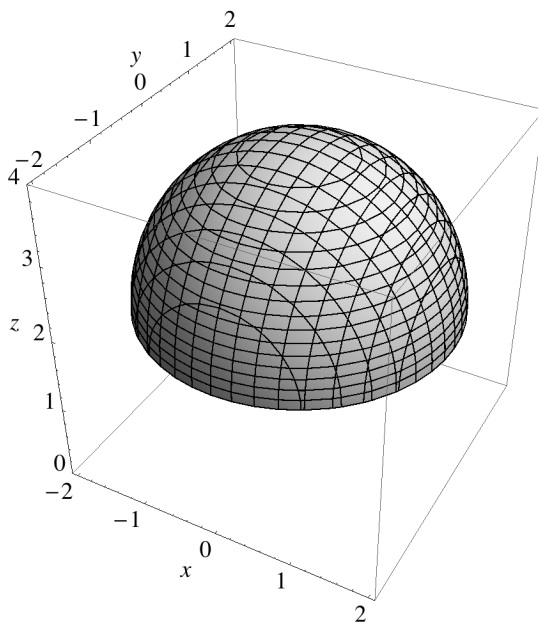
$$\begin{aligned}
 \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} (\sec^3 \phi - \sec \phi) \, d\phi \, d\theta &= \frac{1}{3} \int_0^{2\pi} \left[\frac{1}{2} (\sec \phi \tan \phi + \ln(\sec \phi + \tan \phi)) - \ln(\sec \phi + \tan \phi) \right]_0^{\pi/4} d\theta \\
 &= \frac{1}{6} \int_0^{2\pi} [\sec \phi \tan \phi - \ln(\sec \phi + \tan \phi)]_0^{\pi/4} d\theta \\
 &= \frac{1}{6} \int_0^{2\pi} \left((\sqrt{2} \cdot 1 - \ln(\sqrt{2} + 1)) - (1 \cdot 0 - \ln(1 + 0)) \right) d\theta \\
 &= \frac{1}{6} \int_0^{2\pi} (\sqrt{2} - \ln(1 + \sqrt{2})) \, d\theta \\
 &= \frac{\pi}{3} (\sqrt{2} - \ln(1 + \sqrt{2})).
 \end{aligned}$$

This can be simplified to $\boxed{\frac{\pi}{3}(\sqrt{2} + \ln(\sqrt{2} - 1))}$.

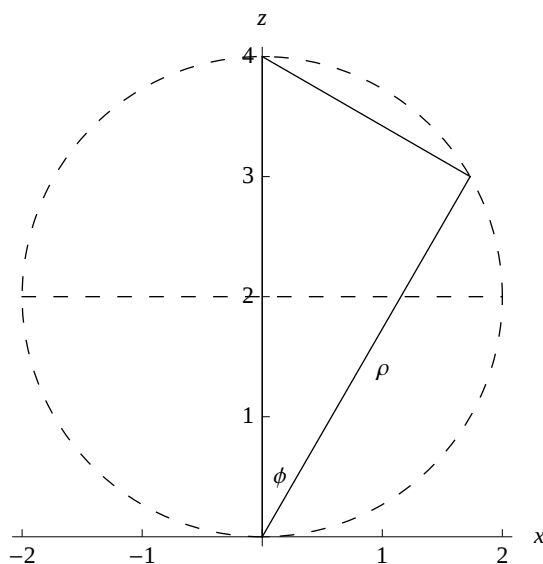
20. Computing gives

$$\begin{aligned}
 \int_0^{\pi/4} \int_0^{\cos \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi &= 2\pi \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \\
 &= 2\pi \int_0^{\pi/4} \left[\frac{1}{3} \rho^3 \sin \phi \right]_0^{\cos \phi} d\phi \\
 &= \frac{2\pi}{3} \int_0^{\pi/4} \cos^3 \phi \sin \phi \, d\phi \\
 &= \frac{2\pi}{3} \left[-\frac{1}{4} \cos^4 \phi \right]_0^{\pi/4} = \frac{2\pi}{3} \left(-\frac{1}{16} + \frac{1}{4} \right) = \boxed{\frac{\pi}{8}}.
 \end{aligned}$$

21. The region is:



This region is the upper half of a sphere of radius 2 centered at $(0, 0, 2)$. In spherical coordinates, θ is unrestricted, so the range is $0 \leq \theta \leq 2\pi$. The lower edge of the hemisphere has a boundary along a circle of radius 2 in the plane $z = 2$, so that a point on that circle is for example $(2, 0, 2)$. Then $\cos \phi = \frac{z}{\rho} = \frac{2}{\sqrt{2^2 + 2^2}} = \frac{\sqrt{2}}{2}$, so that $\phi = \frac{\pi}{4}$ at the lower edge of the hemisphere. Therefore the range of ϕ is $0 \leq \phi \leq \frac{\pi}{4}$. For a given value of ϕ , the corresponding maximum value of ρ can be seen from the following projection of the figure above in the xz -plane:

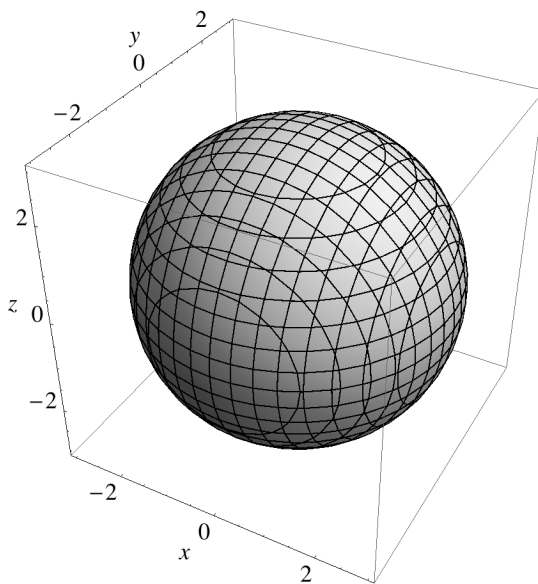


Since the triangle shown is inscribed in a semicircle, it is a right triangle, so that $\cos \phi = \frac{\rho}{4}$ and therefore

$\rho = 4 \cos \phi$. Putting this all together, we get

$$\begin{aligned}
 \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_2^{2+\sqrt{4-x^2-y^2}} y \, dz \, dx \, dy &= \int_0^{\pi/4} \int_0^{4 \cos \phi} \int_0^{2\pi} \rho \sin \phi \sin \theta \cdot \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi \\
 &= \int_0^{\pi/4} \int_0^{4 \cos \phi} \int_0^{2\pi} \rho^3 \sin^2 \phi \sin \theta \, d\theta \, d\rho \, d\phi \\
 &= \int_0^{\pi/4} \int_0^{4 \cos \phi} \left[-\rho^3 \sin^2 \phi \cos \theta \right]_0^{2\pi} d\rho \, d\phi \\
 &= \boxed{0}.
 \end{aligned}$$

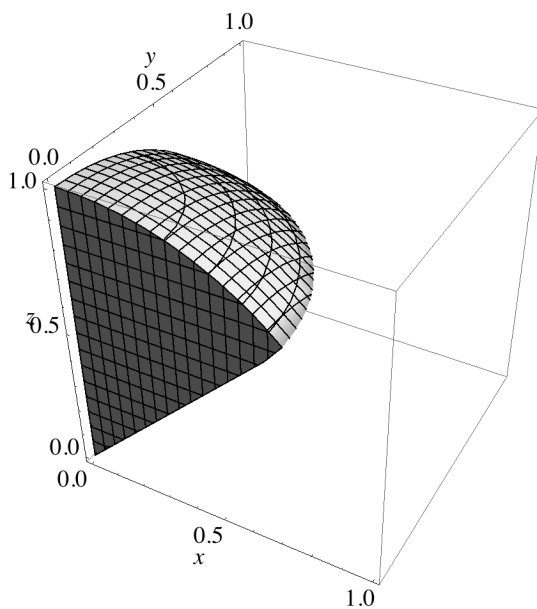
22. The region is:



This is a sphere of radius 3 centered at the origin, so converting to spherical coordinates and using the appropriate equations for x , y , and z in the integrand gives

$$\begin{aligned}
 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} (x^2 z + y^2 z + z^3) \, dz \, dy \, dx \\
 &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} z (x^2 + y^2 + z^2) \, dz \, dy \, dx \\
 &= \int_0^3 \int_0^{\pi} \int_0^{2\pi} \rho \cos \phi \cdot \rho^2 \cdot \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho \\
 &= 2\pi \int_0^3 \int_0^{\pi} \rho^5 \cos \phi \sin \phi \, d\phi \, d\rho \\
 &= 2\pi \int_0^3 \left[\frac{1}{2} \rho^5 \sin^2 \phi \right]_0^{\pi} d\rho \\
 &= \boxed{0}.
 \end{aligned}$$

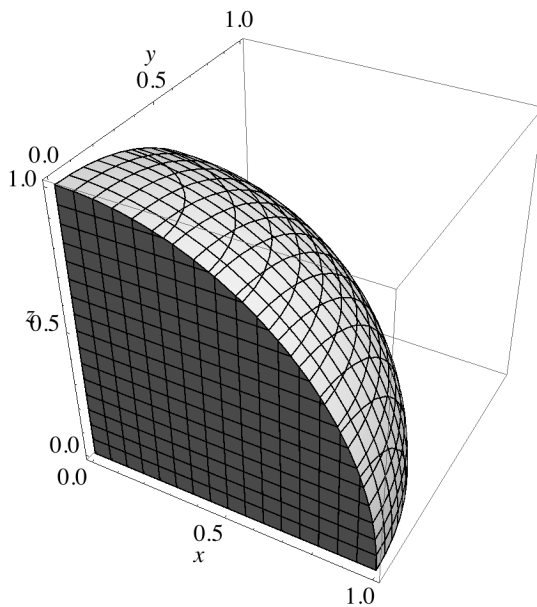
23. The region is:



The bounds of the region in the z direction are from $z = \sqrt{x^2 + y^2}$, which is the cone $\phi = \frac{\pi}{4}$ to $z = \sqrt{1 - x^2 - y^2}$, which is a sphere of radius 1. Since the region lies over the first quadrant in the xy -plane, $0 \leq \theta \leq \frac{\pi}{2}$. So the integral becomes

$$\begin{aligned}
 \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^{3/2} dz dx dy &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/4} (\rho^2)^{3/2} \cdot \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/4} \rho^5 \sin \phi d\phi d\theta d\rho \\
 &= \int_0^1 \int_0^{\pi/2} [-\rho^5 \cos \phi]_0^{\pi/4} d\theta d\rho \\
 &= \left(\frac{2 - \sqrt{2}}{2} \right) \int_0^1 \int_0^{\pi/2} \rho^5 d\theta d\rho \\
 &= \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) \int_0^1 \rho^5 d\rho \\
 &= \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) \left[\frac{1}{6} \rho^6 \right]_0^1 \\
 &= \frac{\pi}{12} \left(1 - \frac{\sqrt{2}}{2} \right) \\
 &= \boxed{\frac{\pi}{24} (2 - \sqrt{2})}.
 \end{aligned}$$

24. The region is:



This is the portion of the sphere of radius 1 where $x, y, z \geq 0$; this can also be seen from the bounds of the integration; since $0 \leq z \leq \sqrt{1 - x^2 - y^2}$, we get $0 \leq z^2 \leq 1 - x^2 - y^2$, so that $x^2 + y^2 + z^2$ ranges from 0 to 1. Similarly, $x^2 + y^2$ ranges from 0 to 1 when $z = 0$. Therefore $0 \leq \rho \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \phi \leq \frac{\pi}{2}$, so the integral is

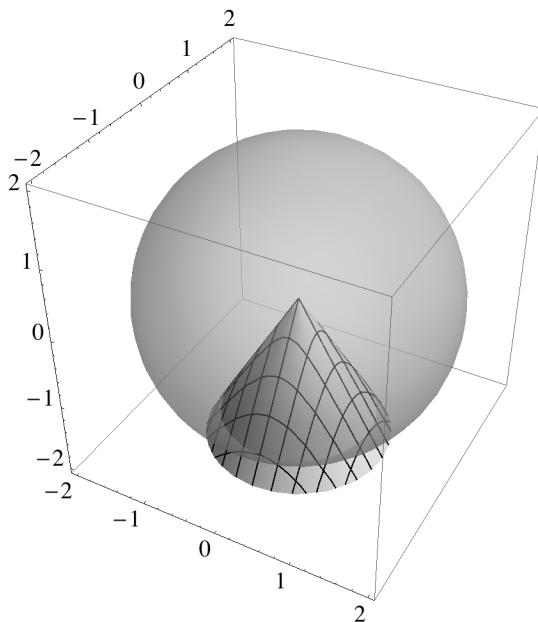
$$\begin{aligned}
 \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} dz dx dy &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} e^{(\rho^2)^{3/2}} \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 e^{\rho^3} \sin \phi d\phi d\theta d\rho \\
 &= \int_0^1 \int_0^{\pi/2} \left[-\rho^2 e^{\rho^3} \cos \phi \right]_0^{\pi/2} d\theta d\rho \\
 &= \int_0^1 \int_0^{\pi/2} \rho^2 e^{\rho^3} d\theta d\rho \\
 &= \frac{\pi}{2} \int_0^1 \rho^2 e^{\rho^3} d\rho.
 \end{aligned}$$

Now use the substitution $u = \rho^3$, so that $du = 3\rho^2 d\rho$. Then $\rho = 0$ corresponds to $u = 0$, and $\rho = 1$ to $u = 1$, and we get

$$\frac{\pi}{2} \int_0^1 \rho^2 e^{\rho^3} d\rho = \frac{\pi}{2} \cdot \frac{1}{3} \int_0^1 e^u du = \frac{\pi}{6} [e^u]_0^1 = \boxed{\frac{\pi}{6}(e - 1)}.$$

Applications and Extensions

25. The region is:



Setting $x = 1$ and $y = 0$ in the equation of the cone gives $z = -\sqrt{3}$, so that the angle ϕ corresponding to this point satisfies $\cos \phi = \frac{z}{\rho} = \frac{-\sqrt{3}}{\sqrt{1^2+0^2+(-\sqrt{3})^2}} = -\frac{\sqrt{3}}{2}$. Therefore $\phi = \frac{5\pi}{6}$. The integrand is $\sqrt{x^2 + y^2 + z^2} = \rho$, so the integral is

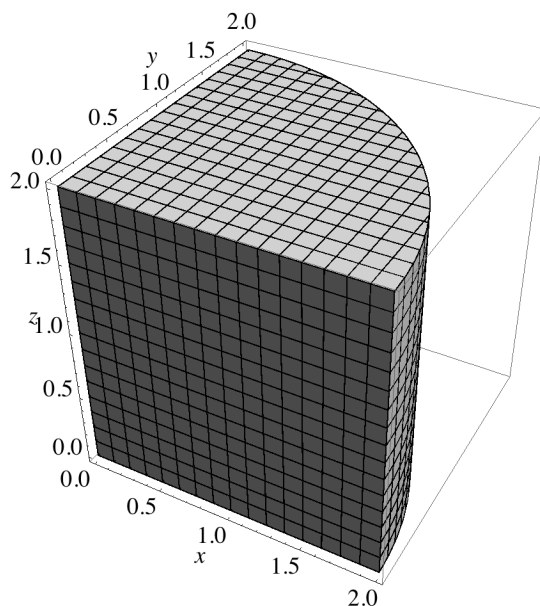
$$\begin{aligned}
 \int_0^2 \int_0^{2\pi} \int_0^{5\pi/6} \rho \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= \int_0^2 \int_0^{2\pi} \int_0^{5\pi/6} \rho^3 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= \int_0^2 \int_0^{2\pi} [-\rho^3 \cos \phi]_0^{5\pi/6} \, d\theta \, d\rho \\
 &= \int_0^2 \int_0^{2\pi} \rho^3 \left(1 + \frac{\sqrt{3}}{2}\right) \, d\theta \, d\rho \\
 &= \left(1 + \frac{\sqrt{3}}{2}\right) \int_0^2 \int_0^{2\pi} \rho^3 \, d\theta \, d\rho \\
 &= 2\pi \left(1 + \frac{\sqrt{3}}{2}\right) \int_0^2 \rho^3 \, d\rho \\
 &= \pi (2 + \sqrt{3}) \left[\frac{1}{4}\rho^4\right]_0^2 \\
 &= \boxed{4\pi(2 + \sqrt{3})}.
 \end{aligned}$$

26. The two concentric spheres have radii 1 and 2, and the integrand is $\sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$, so the

integral is

$$\begin{aligned}
 \int_0^{2\pi} \int_0^\pi \int_1^2 \rho \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\pi \left[\frac{1}{4} \rho^4 \sin \phi \right]_1^2 \, d\phi \, d\theta \\
 &= \frac{15}{4} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta \\
 &= \frac{15}{4} \int_0^{2\pi} [-\cos \phi]_0^\pi \, d\theta \\
 &= \frac{15}{2} \int_0^{2\pi} 1 \, d\theta \\
 &= \boxed{15\pi}.
 \end{aligned}$$

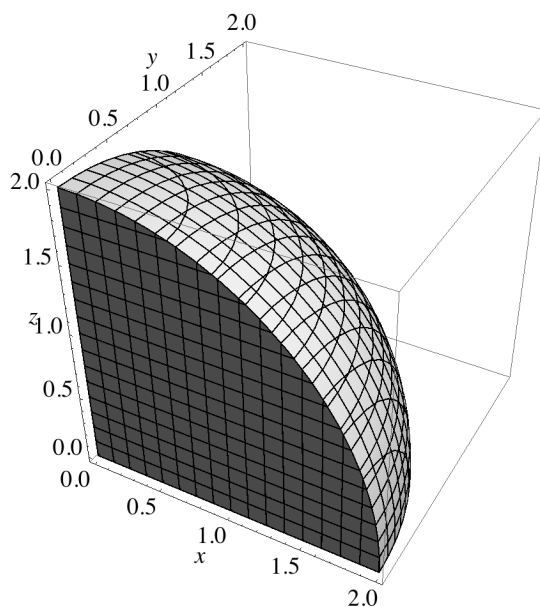
27. The region is:



This region has rotational symmetry about the z -axis, so cylindrical coordinates are appropriate. The integrand is $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$, so that $0 \leq r \leq 2$ and $0 \leq z \leq 2$. From the diagram, $0 \leq \theta \leq \frac{\pi}{2}$. The integral is

$$\begin{aligned}
 \int_0^2 \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \, dz &= \int_0^2 \int_0^2 \int_0^{\pi/2} r \cdot r \, d\theta \, dz \, dr \\
 &= \int_0^2 \int_0^2 [r^2 \theta]_0^{\pi/2} \, dz \, dr \\
 &= \frac{\pi}{2} \int_0^2 \int_0^2 r^2 \, dz \, dr \\
 &= \pi \int_0^2 r^2 \, dr \\
 &= \pi \left[\frac{1}{3} r^3 \right]_0^2 = \boxed{\frac{8}{3}\pi}.
 \end{aligned}$$

28. The region is:



This is the portion of a sphere of radius 2 lying in the first octant, so the spherical coordinate integration limits are $0 \leq \rho \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$. The integrand is

$$\begin{aligned} \frac{2z}{\sqrt{x^2 + y^2}} &= \frac{2\rho \cos \phi}{\sqrt{\rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi}} \\ &= \frac{2\rho \cos \phi}{\rho \sin \phi \sqrt{\cos^2 \theta + \sin^2 \theta}} \\ &= 2 \cot \phi. \end{aligned}$$

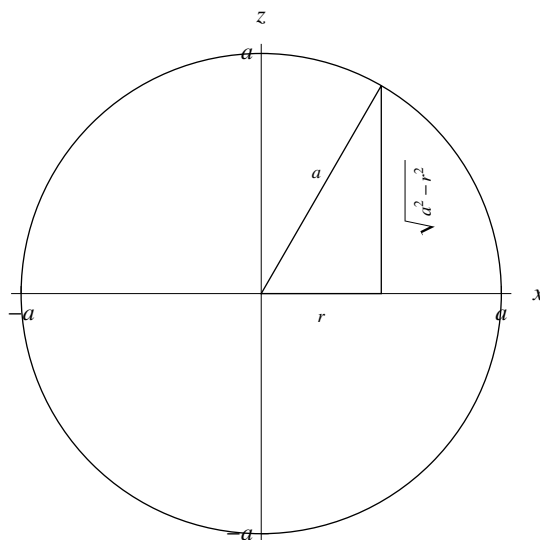
Therefore the integral is

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{2z}{\sqrt{x^2 + y^2}} dz dx dy &= \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} 2 \cot \phi \cdot \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} 2\rho^2 \cos \phi d\phi d\theta d\rho \\ &= 2 \int_0^2 \int_0^{\pi/2} [\rho^2 \sin \phi]_0^{\pi/2} d\theta d\rho \\ &= 2 \int_0^2 \int_0^{\pi/2} \rho^2 d\theta d\rho \\ &= \pi \int_0^2 \rho^2 d\rho \\ &= \pi \left[\frac{1}{3} \rho^3 \right]_0^2 = \boxed{\frac{8}{3} \pi}. \end{aligned}$$

29. In rectangular coordinates, x ranges from $-a$ to a . For each value of x , values of y that stay within the sphere range from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. Finally, for given values of y and x , values of z range from $-\sqrt{a^2 - x^2 - y^2}$ to $\sqrt{a^2 - x^2 - y^2}$. Therefore the integral in rectangular coordinates is

$$\int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_{-\sqrt{a^2 - x^2 - y^2}}^{\sqrt{a^2 - x^2 - y^2}} f(x, y, z) dz dy dx.$$

In cylindrical coordinates, r ranges from 0 to a , and θ ranges from 0 to 2π . To determine the range of z , consider the projection of the sphere onto the xz -plane:



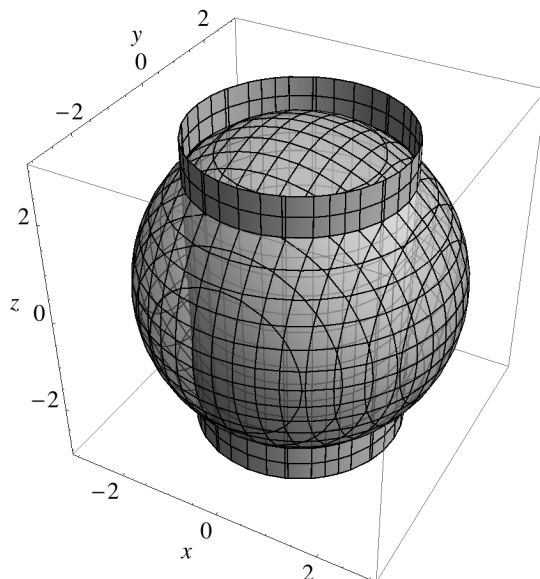
For a given value of r , z ranges from $-\sqrt{a^2 - r^2}$ to $\sqrt{a^2 - r^2}$, and the integral is

$$\int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} f(x(r, \theta), y(r, \theta), z) r \, dr \, d\theta \, dz.$$

In spherical coordinates, the sphere has ranges $0 \leq \rho \leq a$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$, so the integral is

$$\int_0^{2\pi} \int_0^\pi \int_0^a f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

30. The region is:



In rectangular coordinates, we let x go from -2 for 2 ; for each value of x , values of y range from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$ to remain within the cylinder. For each pair of values of x and y , z ranges from

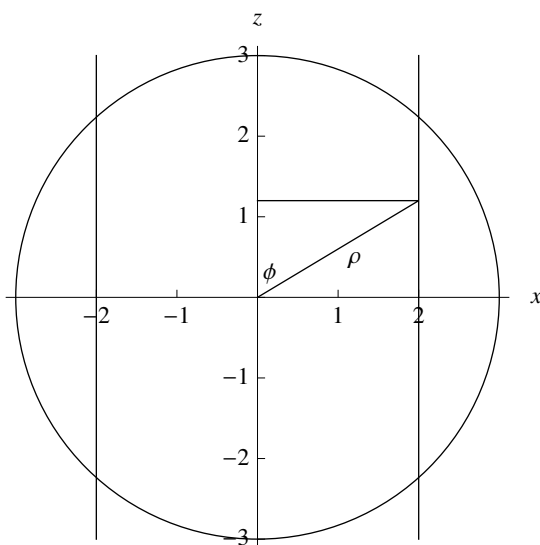
$-\sqrt{9-x^2-y^2}$ to $\sqrt{9-x^2-y^2}$, remaining within the sphere. So the integral is

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} f(x, y, z) dz dy dx.$$

In cylindrical coordinates, r ranges from 0 to 2, and θ from 0 to 2π . Since $r = \sqrt{x^2 + y^2}$, we must have $r^2 + z^2 \leq 9$, so that z ranges from $-\sqrt{9-r^2}$ to $\sqrt{9-r^2}$, and the integral is

$$\int_0^2 \int_0^{2\pi} \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} f(x(r, \theta), y(r, \theta), z) r dz d\theta dr.$$

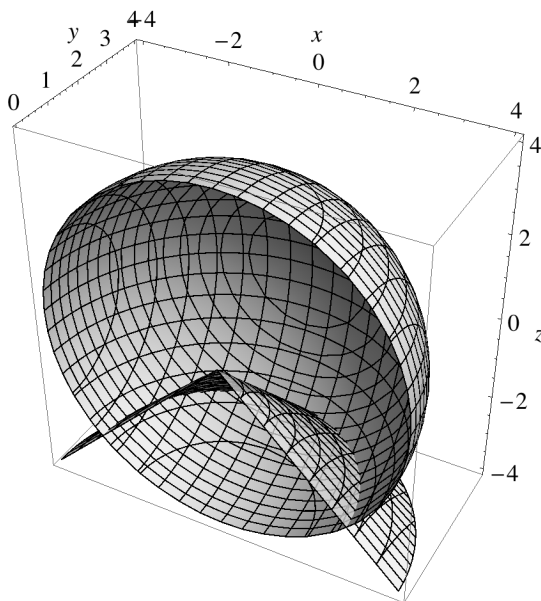
In spherical coordinates, θ ranges from 0 to 2π . To determine the other two ranges, look at a projection of the region onto the xz -plane:



The point of intersection of the projections of the cylinder and the sphere in the first quadrant is the intersection of the line $x = 2$ with the circle $x^2 + z^2 = 9$, so it is the point $(2, \sqrt{5})$. This point corresponds to an angle $\phi = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$. The equivalent point in the fourth quadrant corresponds to $\phi = \pi + \tan^{-1}\left(-\frac{2}{\sqrt{5}}\right)$. For $0 \leq \phi \leq \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$ and $\pi + \tan^{-1}\left(-\frac{2}{\sqrt{5}}\right) \leq \phi \leq \pi$, we can integrate from $\rho = 0$ to $\rho = 3$. In between those two values, however, the radius of integration changes as ϕ changes. From the diagram, for a given value of ϕ , ρ ranges from 0 to $2 \csc \phi$. Therefore the integral is a sum of three integrals:

$$\begin{aligned} & \int_0^{2\pi} \int_0^{\tan^{-1}(2/\sqrt{5})} \int_0^3 f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) d\rho d\phi d\theta + \\ & \int_0^{2\pi} \int_{\pi + \tan^{-1}(-2/\sqrt{5})}^{\pi} \int_0^3 f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) d\rho d\phi d\theta + \\ & \int_0^{2\pi} \int_{\tan^{-1}(2/\sqrt{5})}^{\pi + \tan^{-1}(-2/\sqrt{5})} \int_0^{2 \csc \phi} f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) d\rho d\phi d\theta. \end{aligned}$$

31. The region is the volume below the conical surface and above the surface of the sphere:



Letting ϕ range from $\frac{3\pi}{4}$ to π with $0 \leq \rho \leq 4$ slices out a conical piece from the bottom of the sphere of radius 4. Since θ is constrained to $0 \leq \theta \leq \pi$, we want only the half of that piece that lies in the range $y \geq 0$. The cone, which has rectangular equation $z = -\sqrt{x^2 + y^2}$, meets the sphere when

$$x^2 + y^2 + z^2 = x^2 + y^2 + (x^2 + y^2) = 2(x^2 + y^2) = 16,$$

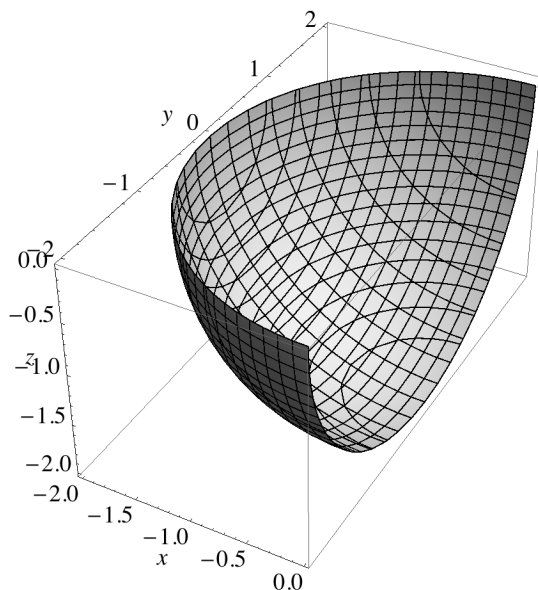
so it meets the sphere on a circle of radius $2\sqrt{2}$ in the plane $z = -2\sqrt{2}$. Then if we allow x to range from $-2\sqrt{2}$ to $2\sqrt{2}$, we can let y range from 0 to $\sqrt{8 - x^2}$ (since we want $y \geq 0$), and then z ranges from $-\sqrt{16 - x^2 - y^2}$ to $-\sqrt{x^2 + y^2}$. The integrand is $\rho^5 \cos \phi \sin \phi = \rho^3 \cos \phi \cdot \rho^2 \sin \phi$, so the integrand in rectangular coordinates is

$$\rho^3 \cos \phi = \rho^2 \cdot \rho \cos \phi = (x^2 + y^2 + z^2)z.$$

Therefore the integral becomes

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{-\sqrt{x^2+y^2}} (x^2 + y^2 + z^2)z \, dz \, dy \, dx.$$

32. The region is the volume enclosed by the surface shown together with its bounding planes:



Since $0 \leq \rho \leq 2$ and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, the projection of this region onto the xy -plane will lie inside the semicircle of radius 2 in the second and third quadrants. Then $\frac{\pi}{2} \leq \phi \leq \pi$, so the region is the portion of the sphere of radius 2 lying below the second and third quadrants. In terms of rectangular coordinates, we get $-2 \leq x \leq 0$, and then $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$, and finally $-\sqrt{4-x^2-y^2} \leq z \leq 0$. The integrand is

$$\frac{\rho^4 \sin \phi \cos \phi}{\rho^2 + 3} = \frac{\rho^2 \cos \phi}{\rho^2 + 3} \cdot \rho^2 \sin \phi.$$

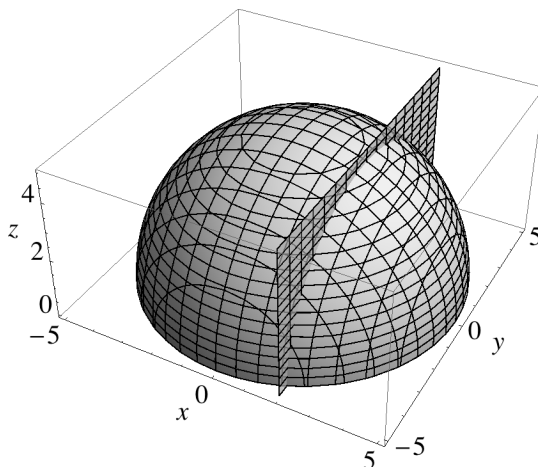
So in rectangular coordinates, the integrand is

$$\frac{\rho^2 \cos \phi}{\rho^2 + 3} = \frac{\rho \cdot \rho \cos \phi}{\rho^2 + 3} = \frac{z \sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2 + 3}.$$

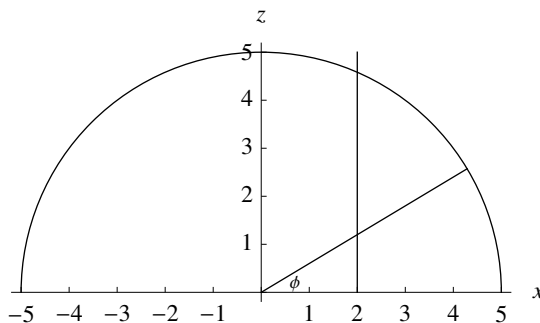
Finally, the integral becomes

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^0 \frac{z \sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2 + 3} dz dy dx.$$

33. The situation is:



We use a variant of the standard spherical coordinates in which x is parametrized with $x = \rho \cos \phi$; then $y = \rho \cos \theta \sin \phi$ and $z = \rho \sin \theta \sin \phi$. We do this so that the part of the sphere to the right of $x = 2$ is at the “top” of the re-parametrized sphere. Considering the portion to the right of $x = 2$, it is described first by $0 \leq \theta \leq \pi$. For any ϕ , we can determine the range of ρ from the following projection of the region onto the xz -plane:



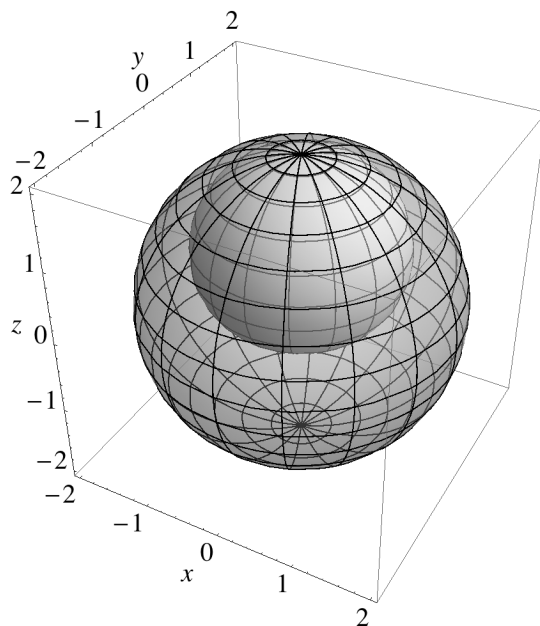
For a given ϕ , the length of the diagonal line from the origin to the intersection with $x = 2$ is $2 \sec \phi$, so the range of integration (since we want the region to the right of $x = 2$) is $2 \sec \phi \leq \rho \leq 5$. The range for ϕ is from 0 to the point where the plane intersects the sphere; this is the point where $2 \sec \phi = 5$, or $\cos \phi = \frac{2}{5}$. So the integral is

$$\begin{aligned}
 \int_0^{\cos^{-1}(2/5)} \int_{2 \sec \phi}^5 \int_0^\pi \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi &= \pi \int_0^{\cos^{-1}(2/5)} \int_{2 \sec \phi}^5 \rho^2 \sin \phi \, d\rho \, d\phi \\
 &= \pi \int_0^{\cos^{-1}(2/5)} \left[\frac{1}{3} \rho^3 \sin \phi \right]_{2 \sec \phi}^5 d\phi \\
 &= \frac{\pi}{3} \int_0^{\cos^{-1}(2/5)} (125 \sin \phi - 8 \sec^3 \phi \sin \phi) \, d\phi \\
 &= \frac{\pi}{3} \int_0^{\cos^{-1}(2/5)} (125 \sin \phi - 8 \sec^2 \phi \tan \phi) \, d\phi \\
 &= \frac{\pi}{3} [-125 \cos \phi - 4 \sec^2 \phi]_0^{\cos^{-1}(2/5)}.
 \end{aligned}$$

Now, since $\sec^2 \phi = \frac{1}{\cos^2 \phi}$, evaluating this final result is not difficult:

$$\frac{\pi}{3} [-125 \cos \phi - 4 \sec^2 \phi]_0^{\cos^{-1}(2/5)} = \frac{\pi}{3} \left(\left(-125 \cdot \left(\frac{2}{5} \right) - 4 \left(\frac{5}{2} \right)^2 \right) - (-125 - 4) \right) = \boxed{18\pi}.$$

Since the volume of the entire hemisphere is $\frac{1}{2} \cdot \frac{4}{3} \pi r^3 = \frac{2}{3} \pi \cdot 5^3 = \frac{250}{3} \pi$, the volume of the piece to the left of $x = 2$ is $\frac{250}{3} \pi - 18\pi = \boxed{\frac{196}{3} \pi}$.



The volume we are looking for is the volume inside the large sphere but outside the inner surface; this is described by $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $1 + \cos \phi \leq \rho \leq 2$, so the volume it

$$\begin{aligned}
 V &= \int_0^\pi \int_{1+\cos \phi}^2 \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi \\
 &= 2\pi \int_0^\pi \int_{1+\cos \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \\
 &= 2\pi \int_0^\pi \left[\frac{1}{3} \rho^3 \sin \phi \right]_{1+\cos \phi}^2 d\phi \\
 &= \frac{2}{3} \pi \int_0^\pi (8 \sin \phi - (1 + \cos \phi)^3 \sin \phi) \, d\phi \\
 &= \frac{2}{3} \pi \left[-8 \cos \phi + \frac{1}{4} (1 + \cos \phi)^4 \right]_0^\pi \\
 &= \frac{2}{3} \pi (8 - (-8 + 4)) = \boxed{8\pi}.
 \end{aligned}$$

35. In spherical coordinates, the solid lying above the cone $\phi = \alpha$ and inside the sphere $\rho = a$ is defined by $0 \leq \rho \leq a$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \alpha$, so its volume is

$$\begin{aligned}
 \int_0^a \int_0^{2\pi} \int_0^\alpha \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= \int_0^a \int_0^{2\pi} [-\rho^2 \cos \phi]_0^\alpha \, d\theta \, d\rho \\
 &= (1 - \cos \alpha) \int_0^a \int_0^{2\pi} \rho^2 \, d\theta \, d\rho \\
 &= 2\pi(1 - \cos \alpha) \int_0^a \rho^2 \, d\rho \\
 &= 2\pi(1 - \cos \alpha) \left[\frac{1}{3} \rho^3 \right]_0^a = \boxed{\frac{2\pi}{3} a^3 (1 - \cos \alpha)}.
 \end{aligned}$$

36. This solid, since it lies in the first octant, is described by $b \leq \rho \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$. The mass density at any point is

$$\frac{k}{\sqrt{x^2 + y^2 + z^2}} = \frac{k}{\rho},$$

where k is the constant of proportionality. Therefore the mass of the solid is

$$\begin{aligned}
 \int_b^a \int_0^{\pi/2} \int_0^{\pi/2} \frac{k}{\rho} \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= k \int_b^a \int_0^{\pi/2} \int_0^{\pi/2} \rho \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_b^a \int_0^{\pi/2} [-\rho \cos \phi]_0^{\pi/2} \, d\theta \, d\rho \\
 &= k \int_b^a \int_0^{\pi/2} \rho \, d\theta \, d\rho \\
 &= \frac{k\pi}{2} \int_b^a \rho \, d\rho \\
 &= \frac{k\pi}{2} \left[\frac{1}{2} \rho^2 \right]_b^a = \boxed{\frac{k\pi}{4} (a^2 - b^2)}.
 \end{aligned}$$

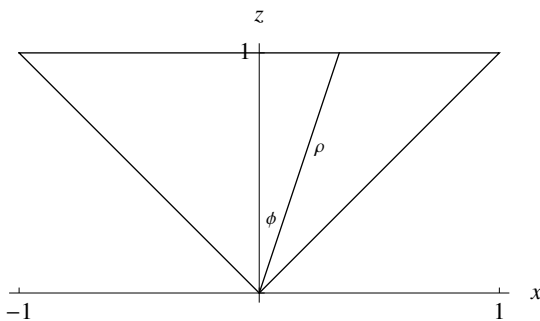
37. The mass density is $k(x^2 + y^2 + z^2) = k\rho^2$, where k is the constant of proportionality, so the mass of the ball is

$$\begin{aligned}
 \int_0^a \int_0^{2\pi} \int_0^\pi k\rho^2 \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= k \int_0^a \int_0^{2\pi} \int_0^\pi \rho^4 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} [-\rho^4 \cos \phi]_0^\pi \, d\theta \, d\rho \\
 &= 2k \int_0^a \int_0^{2\pi} \rho^4 \, d\theta \, d\rho \\
 &= 4\pi k \int_0^a \rho^4 \, d\rho \\
 &= 4\pi k \left[\frac{1}{5} \rho^5 \right]_0^a = \boxed{\frac{4\pi}{5} ka^5}.
 \end{aligned}$$

38. The given solid lies below the plane $z = 1$ and above the half-cone $z = \sqrt{x^2 + y^2}$, which in spherical coordinates is

$$\rho \cos \phi = \sqrt{\rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi} = \rho \sin \phi \sqrt{\cos^2 \theta + \sin^2 \theta} = \rho \sin \phi.$$

This is just the equation $\tan \phi = 1$, or $\phi = \frac{\pi}{4}$. So the integral to find the mass has bounds $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$. For a given value of ϕ , the point of intersection with $z = 1$ is found by looking at the cross-section



The length ρ is such that $\cos \phi = \frac{1}{\rho}$, so that $\rho = \sec \phi$. The mass density function is $z\sqrt{x^2 + y^2 + z^2} =$

$\rho \cos \phi \cdot \rho = \rho^2 \cos \phi$. Therefore the mass of the solid is

$$\begin{aligned}
 \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec \phi} \rho^2 \cos \phi \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec \phi} \rho^4 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi \\
 &= \int_0^{\pi/4} \int_0^{2\pi} \left[\frac{1}{5} \rho^5 \sin \phi \cos \phi \right]_0^{\sec \phi} d\theta \, d\phi \\
 &= \frac{1}{5} \int_0^{\pi/4} \int_0^{2\pi} \sec^4 \phi \sin \phi \, d\theta \, d\phi \\
 &= \frac{2\pi}{5} \int_0^{\pi/4} \sec^2 \phi \cdot \sec \phi \tan \phi \, d\phi \\
 &= \frac{2\pi}{5} \left[\frac{1}{3} \sec^3 \phi \right]_0^{\pi/4} = \boxed{\frac{2\pi}{15}(2\sqrt{2} - 1)}.
 \end{aligned}$$

39. Position the hemisphere so that its base is on the xy -plane and lies in the region $z \geq 0$. Then the mass density function is $k\sqrt{x^2 + y^2 + z^2} = k\rho$, where k is the constant of proportionality. Therefore both the solid and the mass density function are symmetric about the z -axis, so that by symmetry $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$. The mass M of the hemisphere is

$$\begin{aligned}
 \int_0^a \int_0^{2\pi} \int_0^{\pi/2} k\rho \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} [-\rho^3 \cos \phi]_0^{\pi/2} d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \rho^3 \, d\theta \, d\rho \\
 &= 2\pi k \int_0^a \rho^3 \, d\rho \\
 &= 2\pi k \left[\frac{1}{4} \rho^4 \right]_0^a = \frac{\pi}{2} k a^4.
 \end{aligned}$$

From Objective 4 in Section 14.6, the moment about the xy -plane, M_{xy} , is

$$\begin{aligned}
 \iiint_E z \delta(x, y, z) \, dV &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho \cos \phi \cdot k\rho \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^4 \sin \phi \cos \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \left[\frac{1}{2} \rho^4 \sin^2 \phi \right]_0^{\pi/2} d\theta \, d\rho \\
 &= \frac{1}{2} k \int_0^a \int_0^{2\pi} \rho^4 \, d\theta \, d\rho \\
 &= \pi k \int_0^a \rho^4 \, d\rho \\
 &= \pi k \left[\frac{1}{5} \rho^5 \right]_0^a = \frac{\pi}{5} k a^5.
 \end{aligned}$$

Then the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{M_{xy}}{M} \right) = \boxed{\left(0, 0, \frac{2a}{5} \right)}.$$

40. Position the hemisphere so that its base is on the xy -plane and lies in the region $z \geq 0$. Then the distance of (x, y, z) from the axis of symmetry, which is the z -axis, is $\sqrt{x^2 + y^2} = \rho \sin \phi$, so that the

mass density function is $k\rho \sin \phi$, where k is the constant of proportionality. Therefore both the solid and the mass density function are symmetric about the z -axis (since the mass density function does not depend on θ), so that by symmetry $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$. The mass M of the hemisphere is

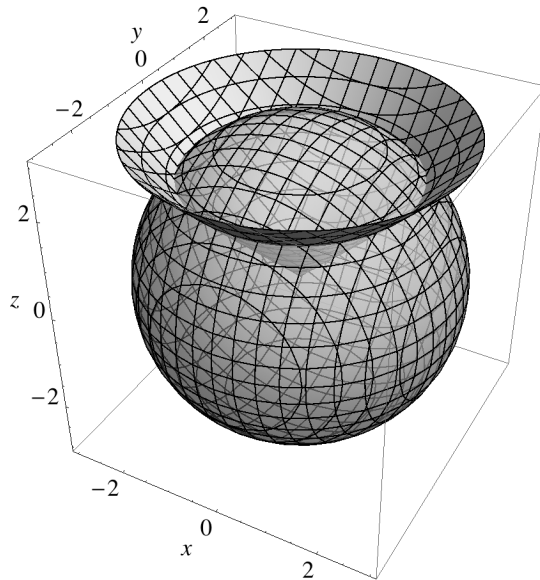
$$\begin{aligned}
 \int_0^a \int_0^{2\pi} \int_0^{\pi/2} k\rho \sin \phi \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \sin^2 \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \left(\frac{1}{2} - \frac{1}{2} \cos(2\phi) \right) d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \left[\rho^3 \left(\frac{1}{2}\phi - \frac{1}{4} \sin(2\phi) \right) \right]_0^{\pi/2} d\theta \, d\rho \\
 &= \frac{\pi}{4} k \int_0^a \int_0^{2\pi} \rho^3 \, d\theta \, d\rho \\
 &= \frac{\pi^2}{2} k \int_0^a \rho^3 \, d\rho \\
 &= \frac{\pi^2}{2} k \left[\frac{1}{4} \rho^4 \right]_0^a = \frac{\pi^2}{8} k a^4.
 \end{aligned}$$

From Objective 4 in Section 14.6, the moment about the xy -plane, M_{xy} , is

$$\begin{aligned}
 \iiint_E z \delta(x, y, z) \, dV &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho \cos \phi \cdot k\rho \sin \phi \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^4 \sin^2 \phi \cos \phi \, d\phi \, d\theta \, d\rho \\
 &= k \int_0^a \int_0^{2\pi} \left[\frac{1}{3} \rho^4 \sin^3 \phi \right]_0^{\pi/2} d\theta \, d\rho \\
 &= \frac{1}{3} k \int_0^a \int_0^{2\pi} \rho^4 \, d\theta \, d\rho \\
 &= \frac{2\pi}{3} k \int_0^a \rho^4 \, d\rho \\
 &= \frac{2\pi}{3} k \left[\frac{1}{5} \rho^5 \right]_0^a = \frac{2\pi}{15} k a^5.
 \end{aligned}$$

Then the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{M_{xy}}{M} \right) = \boxed{\left(0, 0, \frac{16}{15\pi} a \right)}.$$



The half-cone $z = \sqrt{x^2 + y^2}$ is defined in spherical coordinates by

$$\rho \cos \phi = \sqrt{\rho^2 \cos^2 \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi} = \rho \sin \phi,$$

or $\tan \phi = 1$. Therefore the cone is defined by $\phi = \frac{\pi}{4}$. So the parametrization of the solid in spherical coordinates is $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$, $0 \leq \rho \leq 3$. By symmetry, $\bar{x} = \bar{y} = 0$, so it remains only to find $\bar{z}$. The solid is homogeneous, so we take its mass density to be a constant k . The mass M of the solid is then

$$\begin{aligned} \int_0^{\pi/4} \int_0^{2\pi} \int_0^3 k \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi &= k \int_0^{\pi/4} \int_0^{2\pi} \left[\frac{1}{3} \rho^3 \sin \phi \right]_0^3 d\theta \, d\phi \\ &= 9k \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \, d\theta \, d\phi \\ &= 18\pi k \int_0^{\pi/4} \sin \phi \, d\phi \\ &= 18\pi k [-\cos \phi]_0^{\pi/4} = 9\pi k(2 - \sqrt{2}). \end{aligned}$$

From Objective 4 in Section 14.6, the moment about the xy -plane, M_{xy} , is

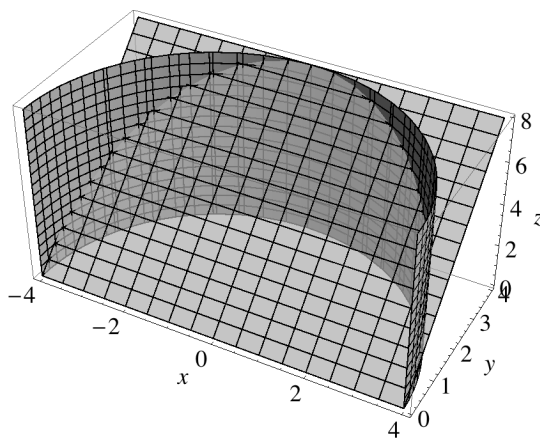
$$\begin{aligned} \iiint_E z \delta(x, y, z) \, dV &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^3 k \rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= k \int_0^{\pi/4} \int_0^{2\pi} \int_0^3 \rho^3 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi \\ &= k \int_0^{\pi/4} \int_0^{2\pi} \left[\frac{1}{4} \rho^4 \sin \phi \cos \phi \right]_0^3 d\theta \, d\phi \\ &= \frac{81}{4} k \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \cos \phi \, d\theta \, d\phi \\ &= \frac{81}{2} \pi k \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi \\ &= \frac{81}{2} \pi k \left[\frac{1}{2} \sin^2 \phi \right]_0^{\pi/4} = \frac{81}{8} \pi k. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{(81/8)\pi k}{9\pi k(2 - \sqrt{2})}\right) = \left(0, 0, \frac{9}{8(2 - \sqrt{2})}\right).$$

This can be simplified to $\boxed{\left(0, 0, \frac{9}{16}(2 + \sqrt{2})\right)}$ by rationalizing the denominator.

42. The solid is:



This solid is most easily modeled using cylindrical coordinates. We have $0 \leq \theta \leq \pi$ and $0 \leq r \leq 4$; for each $y = r \sin \theta$, z varies from 0 to $2y = 2r \sin \theta$. If the mass density of the solid is a constant k , then its mass M is

$$\begin{aligned} \int_0^\pi \int_0^4 \int_0^{2r \sin \theta} r \, dz \, dr \, d\theta &= \int_0^\pi \int_0^4 [rz]_0^{2r \sin \theta} \, dr \, d\theta \\ &= \int_0^\pi \int_0^4 2r^2 \sin \theta \, dr \, d\theta \\ &= \int_0^\pi \left[\frac{2}{3} r^3 \sin \theta \right]_0^4 \, d\theta \\ &= \frac{128}{3} \int_0^\pi \sin \theta \, d\theta \\ &= \frac{128}{3} [-\cos \theta]_0^\pi = \frac{256}{3}. \end{aligned}$$

By symmetry, $\bar{x} = 0$. From Objective 4 in Section 14.6, the remaining moments are

$$\begin{aligned}
 M_{xz} &= \iiint_E y \delta(x, y, z) \, dV \\
 &= \int_0^\pi \int_0^4 \int_0^{2r \sin \theta} r \sin \theta \cdot k \cdot r \, dz \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 \int_0^{2r \sin \theta} r^2 \sin \theta \, dz \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 [r^2 z \sin \theta]_0^{2r \sin \theta} \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 2r^3 \sin^2 \theta \, dr \, d\theta \\
 &= k \int_0^\pi \left[\frac{1}{2} r^4 \sin^2 \theta \right]_0^4 \, d\theta \\
 &= k \int_0^\pi 128 \sin^2 \theta \, d\theta \\
 &= k \int_0^\pi (64 - 64 \cos(2\theta)) \, d\theta \\
 &= k [64\theta - 32 \sin(2\theta)]_0^\pi = 64k\pi
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \iiint_E z \delta(x, y, z) \, dV \\
 &= \int_0^\pi \int_0^4 \int_0^{2r \sin \theta} z \cdot k \cdot r \, dz \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 \int_0^{2r \sin \theta} rz \, dz \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 \left[\frac{1}{2} r z^2 \right]_0^{2r \sin \theta} \, dr \, d\theta \\
 &= k \int_0^\pi \int_0^4 2r^3 \sin^2 \theta \, dr \, d\theta \\
 &= 64k\pi,
 \end{aligned}$$

since the final integral is the same one as in the computation of M_{xz} . Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) = \boxed{\left(0, \frac{3\pi}{4}, \frac{3\pi}{4} \right)}.$$

43. Parametrize the sphere using spherical coordinates, and compute the moment of inertia about the diameter that lies along the z -axis. The distance of any point in the solid from the z -axis is $\sqrt{x^2 + y^2} =$

$\rho \sin \phi$; letting the mass density be a constant δ , the moment of inertia is

$$\begin{aligned}
 I_z &= \iiint_E r^2 \delta(x, y, z) dV \\
 &= \int_0^\pi \int_0^2 \int_0^{2\pi} (\rho \sin \phi)^2 \cdot \delta \cdot \rho^2 \sin \phi d\theta d\rho d\phi \\
 &= \delta \int_0^\pi \int_0^2 \int_0^{2\pi} \rho^4 \sin^3 \phi d\theta d\rho d\phi \\
 &= 2\pi\delta \int_0^\pi \int_0^2 \rho^4 \sin^3 \phi d\rho d\phi \\
 &= 2\pi\delta \int_0^\pi \left[\frac{1}{5} \rho^5 \sin^3 \phi \right]_0^2 d\phi \\
 &= \frac{64}{5} \pi \delta \int_0^\pi \sin^3 \phi d\phi \\
 &= \frac{64}{5} \pi \delta \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi \\
 &= \frac{64}{5} \pi \delta \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^\pi = \boxed{\frac{256}{15} \pi \delta}.
 \end{aligned}$$

44. Parametrize the sphere using spherical coordinates, and compute the moment of inertia about the diameter that lies along the z -axis. The distance of any point in the solid from the z -axis is $\sqrt{x^2 + y^2} = \rho \sin \phi$; letting the mass density be a constant δ , the moment of inertia is

$$\begin{aligned}
 I_z &= \iiint_E r^2 \delta(x, y, z) dV \\
 &= \int_0^\pi \int_0^a \int_0^{2\pi} (\rho \sin \phi)^2 \cdot \delta \cdot \rho^2 \sin \phi d\theta d\rho d\phi \\
 &= \delta \int_0^\pi \int_0^a \int_0^{2\pi} \rho^4 \sin^3 \phi d\theta d\rho d\phi \\
 &= 2\pi\delta \int_0^\pi \int_0^a \rho^4 \sin^3 \phi d\rho d\phi \\
 &= 2\pi\delta \int_0^\pi \left[\frac{1}{5} \rho^5 \sin^3 \phi \right]_0^a d\phi \\
 &= \frac{2}{5} a^5 \pi \delta \int_0^\pi \sin^3 \phi d\phi \\
 &= \frac{2}{5} a^5 \pi \delta \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi \\
 &= \frac{2}{5} a^5 \pi \delta \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^\pi = \boxed{\frac{8}{15} a^5 \pi \delta}.
 \end{aligned}$$

45. The solid is symmetric about the z -axis, so that by symmetry $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$. The mass M of the hemisphere (which is assumed to have mass density k for some constant k) is half the volume of a sphere of radius a with mass density k , which is

$$M = \frac{1}{2} \cdot \frac{4}{3} \pi a^3 \cdot k = \frac{2}{3} \pi k a^3.$$

From Objective 4 in Section 14.6, the moment about the xy -plane, M_{xy} , is

$$\begin{aligned}
 \iiint_E z\delta(x, y, z) dV &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho \cos \phi \cdot k \cdot \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= k \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \sin \phi \cos \phi d\phi d\theta d\rho \\
 &= k \int_0^a \int_0^{2\pi} \left[\frac{1}{2} \rho^3 \sin^2 \phi \right]_0^{\pi/2} d\theta d\rho \\
 &= \frac{1}{2} k \int_0^a \int_0^{2\pi} \rho^3 d\theta d\rho \\
 &= \pi k \int_0^a \rho^3 d\rho \\
 &= \pi k \left[\frac{1}{4} \rho^4 \right]_0^a = \frac{\pi}{4} k a^4.
 \end{aligned}$$

Then the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{M_{xy}}{M} \right) = \boxed{\left(0, 0, \frac{3a}{8} \right)}.$$

46. Converting to spherical coordinates, we get

$$\begin{aligned}
 F &= k\delta \iiint_E \frac{\cos \phi}{\rho^2} dV = k\delta \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \frac{\cos \phi}{\rho^2} \cdot \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= k\delta \int_0^a \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos \phi d\phi d\theta d\rho \\
 &= k\delta \int_0^a \int_0^{2\pi} \left[\frac{1}{2} \sin^2 \phi \right]_0^{\pi/2} d\theta d\rho \\
 &= k\delta \int_0^a \int_0^{2\pi} \frac{1}{2} d\theta d\rho \\
 &= \boxed{\pi a k \delta}.
 \end{aligned}$$

47. (a) With $D(\rho) = D_0 e^{-k\rho}$, we know that $D(0) = D_0 e^{-k \cdot 0} = D_0 = 2500 \text{ kg/m}^3$, and it follows that $D(\rho) = 2500 e^{-k\rho}$. Next, the radius of Dione is 560 km, which is $5.6 \times 10^5 \text{ m}$; then we get $D(5.6 \times 10^5) = 2500 e^{-5.6 \times 10^5 k} = 1250$, so that $e^{-5.6 \times 10^5 k} = \frac{1}{2}$ and solving for k gives $k = \frac{-\ln 2}{5.6 \times 10^5} \approx 1.238 \times 10^{-6} \text{ m}^{-1}$. So $D(\rho) = 2500 e^{-1.238 \times 10^{-6} \rho}$.

(b) Modeling Dione as a sphere, the mass density is $D(\rho)$, so the mass of Dione is approximately

$$\begin{aligned}
 \int_0^{5.6 \times 10^5} \int_0^{2\pi} \int_0^\pi D(\rho) \rho^2 \sin \phi d\phi d\theta d\rho &= 2500 \int_0^{5.6 \times 10^5} \int_0^{2\pi} \int_0^\pi \rho^2 e^{-1.24 \times 10^{-6} \rho} \sin \phi d\phi d\theta d\rho \\
 &= 2500 \int_0^{5.6 \times 10^5} \int_0^{2\pi} \left[-\rho^2 e^{-1.24 \times 10^{-6} \rho} \cos \phi \right]_0^\pi d\theta d\rho \\
 &= 5000 \int_0^{5.6 \times 10^5} \int_0^{2\pi} \rho^2 e^{-1.24 \times 10^{-6} \rho} d\theta d\rho \\
 &= 10000\pi \int_0^{5.6 \times 10^5} \rho^2 e^{-1.24 \times 10^{-6} \rho} d\rho.
 \end{aligned}$$

Using Formulas 115 and 116 in the Table of Integrals, we get

$$\begin{aligned}
 & 10000\pi \int_0^{5.6 \times 10^5} \rho^2 e^{-1.24 \times 10^{-6} \rho} d\rho \\
 &= 10000\pi \left(\left[\frac{1}{-1.24 \times 10^{-6}} \rho^2 e^{-1.24 \times 10^{-6} \rho} \right]_0^{5.6 \times 10^5} \right. \\
 &\quad \left. - \frac{2}{-1.24 \times 10^{-6}} \int_0^{5.6 \times 10^5} \rho e^{-1.24 \times 10^{-6} \rho} d\rho \right) \\
 &\approx 10000\pi \left(-1.263 \times 10^{17} \right. \\
 &\quad \left. + \frac{2}{(1.24 \times 10^{-6})^3} \left[(-1.24 \times 10^{-6} \rho - 1) e^{-1.24 \times 10^{-6} \rho} \right]_0^{5.6 \times 10^5} \right) \\
 &\approx 10000\pi \left(-1.263 \times 10^{17} + 1.614 \times 10^{17} \right) \approx \boxed{1.103 \times 10^{21} \text{ kg}}.
 \end{aligned}$$

This is remarkably close to the measured mass. It differs from the measured mass by only 4.76%. The density model for the asteroid is reasonable.

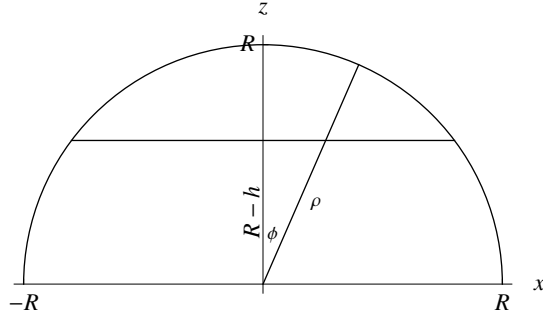
48. If the density decrease is linear, then the density function is $D(\rho) = 2500 - \frac{1250}{5.6 \times 10^5} \rho$ kg/m³, and the mass is computed to be

$$\begin{aligned}
 & \int_0^{5.6 \times 10^5} \int_0^{2\pi} \int_0^\pi D(\rho) \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= \int_0^{5.6 \times 10^5} \int_0^{2\pi} \int_0^\pi \left(2500 - \frac{1250}{5.6 \times 10^5} \rho \right) \rho^2 \sin \phi d\phi d\theta d\rho \\
 &= \int_0^{5.6 \times 10^5} \int_0^{2\pi} \left[- \left(2500 - \frac{1250}{5.6 \times 10^5} \rho \right) \rho^2 \cos \phi \right]_0^\pi d\theta d\rho \\
 &= 2 \int_0^{5.6 \times 10^5} \int_0^{2\pi} \left(2500 - \frac{1250}{5.6 \times 10^5} \rho \right) \rho^2 d\theta d\rho \\
 &= 4\pi \int_0^{5.6 \times 10^5} \left(2500 \rho^2 - \frac{1250}{5.6 \times 10^5} \rho^3 \right) d\rho \\
 &= 4\pi \left[\frac{2500}{3} \rho^3 - \frac{312.5}{5.6 \times 10^5} \rho^4 \right]_0^{5.6 \times 10^5} \\
 &\approx \boxed{1.149 \times 10^{21} \text{ kg}}.
 \end{aligned}$$

This is still reasonably close to the measured mass, though not as good an approximation as in the previous problem.

Challenge Problems

49. Parametrize the sphere using spherical coordinates, and consider the spherical cap shown. The flat bottom of the cap is in the plane $z = R - h$. For an angle ϕ , we can compute the bounds on ρ by considering the diagram



For a given ϕ , the length of the diagonal line from the origin to the intersection with $z = R - h$ is $(R - h) \sec \phi$, so the range of integration (since we want the region above $z = R - h$) is $(R - h) \sec \phi \leq \rho \leq R$. The range for ϕ is from 0 to the point where the plane intersects the sphere; this is the point where $(R - h) \sec \phi = R$, or $\cos \phi = \frac{R-h}{R}$. So the integral is

$$\begin{aligned}
 & \int_0^{\cos^{-1}((R-h)/R)} \int_{(R-h) \sec \phi}^R \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi \\
 &= 2\pi \int_0^{\cos^{-1}((R-h)/R)} \int_{(R-h) \sec \phi}^R \rho^2 \sin \phi \, d\rho \, d\phi \\
 &= 2\pi \int_0^{\cos^{-1}((R-h)/R)} \left[\frac{1}{3} \rho^3 \sin \phi \right]_{(R-h) \sec \phi}^R d\phi \\
 &= \frac{2\pi}{3} \int_0^{\cos^{-1}((R-h)/R)} (R^3 \sin \phi - (R-h)^3 \sec^3 \phi \sin \phi) \, d\phi \\
 &= \frac{2\pi}{3} \int_0^{\cos^{-1}((R-h)/R)} (R^3 \sin \phi - (R-h)^3 \sec^2 \phi \tan \phi) \, d\phi \\
 &= \frac{2\pi}{3} \left[-R^3 \cos \phi - \frac{(R-h)^3}{2} \sec^2 \phi \right]_0^{\cos^{-1}((R-h)/R)}.
 \end{aligned}$$

Now, since $\sec^2 \phi = \frac{1}{\cos^2 \phi}$, we evaluate the final result as follows:

$$\begin{aligned}
 & \frac{2\pi}{3} \left[-R^3 \cos \phi - \frac{(R-h)^3}{2} \sec^2 \phi \right]_0^{\cos^{-1}((R-h)/R)} \\
 &= \frac{2\pi}{3} \left(\left(-R^3 \cdot \left(\frac{R-h}{R} \right) - \frac{(R-h)^3}{2} \left(\frac{R}{R-h} \right)^2 \right) - \left(-R^3 - \frac{(R-h)^3}{2} \right) \right) \\
 &= \frac{1}{3} \pi h^2 (3R - h).
 \end{aligned}$$

14.9 Change of Variables Using Jacobians

Concepts and Vocabulary

1. The Jacobian is the determinant $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.
2. True. See the theorems Change in the Variables in a Double (and Triple) Integral.

Skill Building

3. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \boxed{-2}.$$

4. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \boxed{1}.$$

5. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 3 & \frac{1}{2} \end{vmatrix} = \boxed{5}.$$

6. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = \boxed{8uv}.$$

7. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{v} + \frac{u}{v^2} = \boxed{\frac{u+v}{v^2}}.$$

8. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u} \cos^2 v + e^{2u} \sin^2 v = \boxed{e^{2u}}.$$

9. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} ve^u & e^u \\ e^{-v} & -ue^{-v} \end{vmatrix} = -uve^{u-v} - e^{u-v} = \boxed{-(uv+1)e^{u-v}}.$$

10. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u \sin v \\ v \cos u & \sin u \end{vmatrix} = \boxed{\sin u \cos v + uv \cos u \sin v}.$$

11. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \boxed{-4}.$$

12. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \boxed{-5}.$$

13. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} = \boxed{-2}.$$

14. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{0}.$$

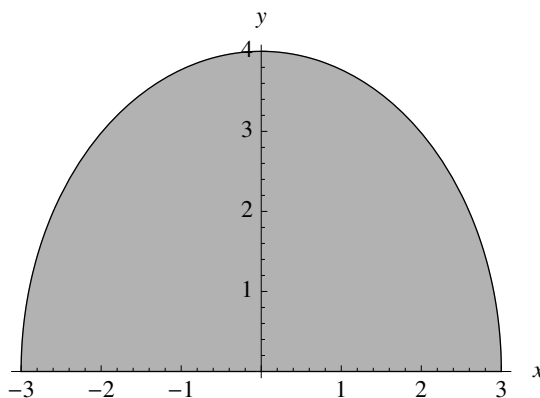
15. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 3w^2 \end{vmatrix} = \boxed{6vw^2}.$$

16. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 \\ 1 & 0 & -1 \\ 2u & 0 & -2w \end{vmatrix} = \boxed{6(w - u)}.$$

17. The region R is:



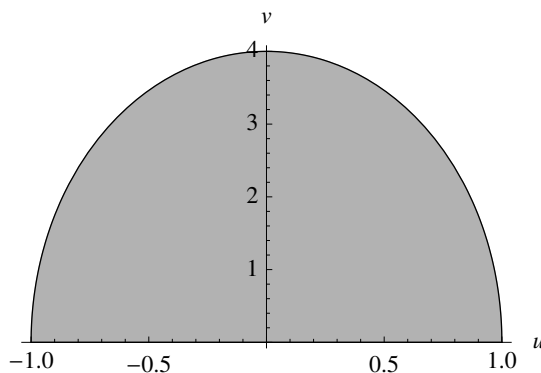
The area of this region is

$$\iint_R 1 \, dA.$$

Using the indicated change of variables and solving for x and y gives $x = 3u$ and $y = 2v$, so that the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

The line $y = 0$ gives the line $v = 0$ in the uv -plane, and $y = 4\sqrt{1 - \frac{x^2}{9}}$ gives the equation in the uv -plane $2v = 4\sqrt{1 - \frac{(3u)^2}{9}} = 4\sqrt{1 - u^2}$, or $v = 2\sqrt{1 - u^2}$. The region in the uv -plane is



Therefore the area is

$$\iint_R 1 \, dA = \int_{-1}^1 \int_0^{2\sqrt{1-u^2}} |6| \, dv \, du = 12 \int_{-1}^1 \sqrt{1-u^2} \, du.$$

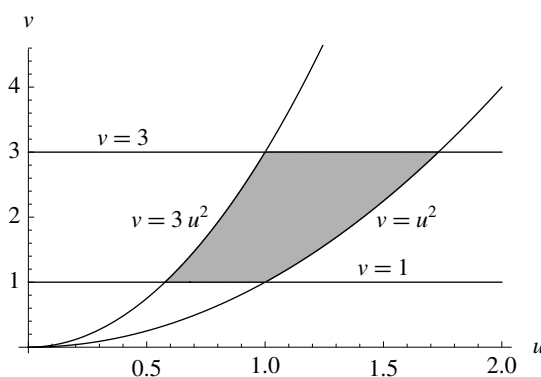
From Formula 62 in the Table of Integrals, we get

$$12 \int_{-1}^1 \sqrt{1-u^2} \, du = 12 \left[\frac{u}{2} \sqrt{1-u^2} + \frac{1}{2} \sin^{-1} u \right]_{-1}^1 = \boxed{6\pi}.$$

18. Using the indicated change of variables and solving for x and y gives $x = u$ and $y = \frac{v}{x} = \frac{v}{u}$, so that the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}.$$

The line $y = x$ gives the line $\frac{v}{u} = u$, or $v = u^2$ in the uv -plane, and $y = 3x$ gives $v = 3u^2$. The curves $xy = 1$ and $xy = 3$ correspond to $v = 1$ and $v = 3$. The region in the uv -plane is



To find the area, we integrate first with respect to u and then with respect to v :

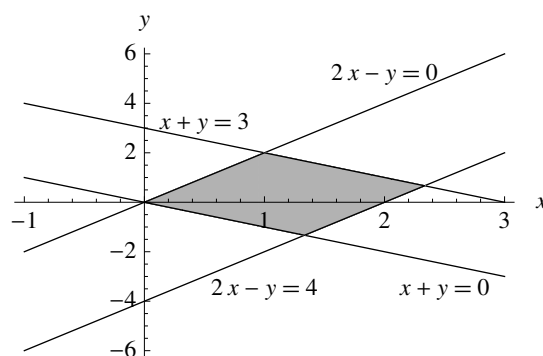
$$\iint_R 1 \, dA = \int_1^3 \int_{\sqrt{v/3}}^{\sqrt{v}} \left| \frac{1}{u} \right| \, du \, dv.$$

The integration bounds on the inner integral are both positive, so we can drop the absolute value signs,

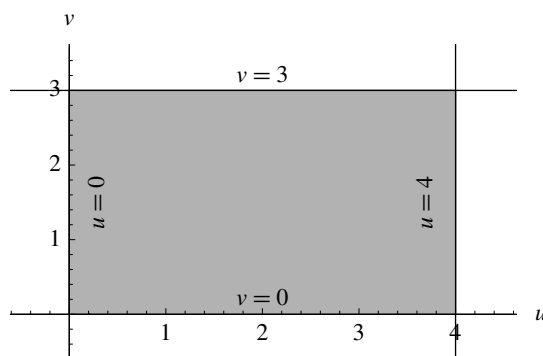
giving

$$\begin{aligned}
 \int_1^3 \int_{\sqrt{v/3}}^{\sqrt{v}} \left| \frac{1}{u} \right| du dv &= \int_1^3 [\ln u]_{\sqrt{v/3}}^{\sqrt{v}} dv \\
 &= \int_1^3 \left(\frac{1}{2} \ln v - \frac{1}{2} \ln \frac{v}{3} \right) dv \\
 &= \int_1^3 \left(\frac{1}{2} \ln v - \frac{1}{2} \ln v + \frac{1}{2} \ln 3 \right) dv \\
 &= \left[\frac{1}{2} v \ln 3 \right]_1^3 = \boxed{\ln 3}.
 \end{aligned}$$

19. (a) The region R is



- (b) With the indicated change of variables, $2x - y = 0$ and $2x - y = 4$ become $u = 0$ and $u = 4$, while $x + y = 0$ and $x + y = 3$ become $v = 0$ and $v = 3$ respectively. The region $R^\#$ is



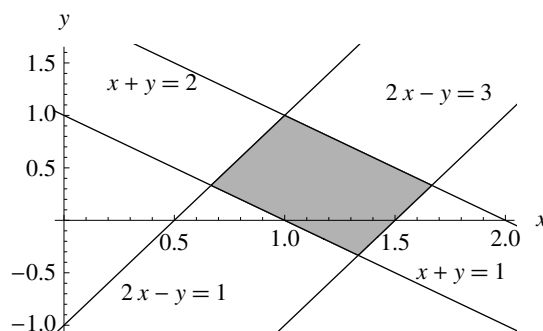
- (c) Solving the equations for x and y gives $x = \frac{v+u}{3}$ and $y = \frac{2v-u}{3}$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}.$$

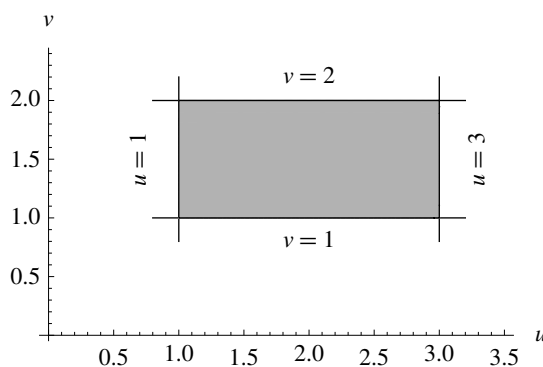
(d) The integral becomes

$$\begin{aligned}
 \iint_R x \, dA &= \iint_{R^\#} \frac{u+v}{3} \cdot \left| \frac{1}{3} \right| \, du \, dv \\
 &= \frac{1}{9} \int_0^3 \int_0^4 (u+v) \, du \, dv \\
 &= \frac{1}{9} \int_0^3 \left[\frac{1}{2}u^2 + uv \right]_0^4 \, dv \\
 &= \frac{1}{9} \int_0^3 (8 + 4v) \, dv \\
 &= \frac{1}{9} [8v + 2v^2]_0^3 = \boxed{\frac{14}{3}}.
 \end{aligned}$$

20. (a) The region R is



(b) With the indicated change of variables, $2x - y = 1$ and $2x - y = 3$ become $u = 1$ and $u = 3$, while $x + y = 1$ and $x + y = 2$ becomes $v = 1$ and $v = 2$ respectively. The region $R^\#$ is



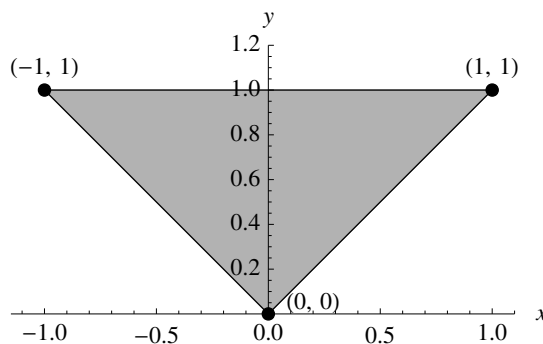
(c) Solving the equations for x and y gives $x = \frac{v+u}{3}$ and $y = \frac{2v-u}{3}$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}.$$

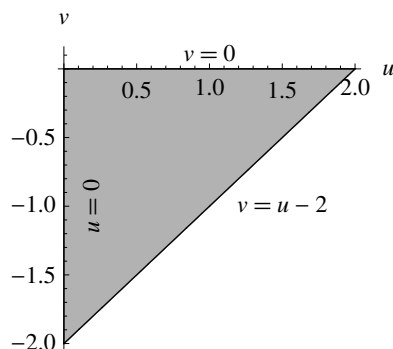
(d) The integral becomes

$$\begin{aligned}
 \iint_R (x^2 + y^2) dA &= \iint_{R^\#} \left(\left(\frac{u+v}{3} \right)^2 + \left(\frac{2v-u}{3} \right)^2 \right) \cdot \left| \frac{1}{3} \right| du dv \\
 &= \frac{1}{27} \int_1^2 \int_1^3 ((u+v)^2 + (2v-u)^2) du dv \\
 &= \frac{1}{27} \int_1^2 \int_1^3 (2u^2 - 2uv + 5v^2) du dv \\
 &= \frac{1}{27} \int_1^2 \left[\frac{2}{3}u^3 - u^2v + 5uv^2 \right]_1^3 dv \\
 &= \frac{1}{27} \int_1^2 \left(\frac{52}{3} - 8v + 10v^2 \right) dv \\
 &= \frac{1}{27} \left[\frac{52}{3}v - 4v^2 + \frac{10}{3}v^3 \right]_1^2 \\
 &= \boxed{\frac{86}{81}}.
 \end{aligned}$$

21. (a) The region R is



(b) The three edges of the triangle are the lines $y = 1$, $x + y = 0$, and $x - y = 0$. Solving the given equations for x and y gives $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Then $y = 1$ becomes $u - v = 2$, or $v = u - 2$, while $x + y = 0$ and $x - y = 0$ become $u = 0$ and $v = 0$ respectively. The region $R^\#$ is



(c) From the equations for x and y in part (b), the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

(d) The integral becomes

$$\begin{aligned}
 \iint_R xy \, dA &= \iint_{R^\#} \left(\frac{u+v}{2} \right) \left(\frac{u-v}{2} \right) \cdot \left| -\frac{1}{2} \right| \, dv \, du \\
 &= \frac{1}{8} \int_0^2 \int_{u-2}^0 (u^2 - v^2) \, dv \, du \\
 &= \frac{1}{8} \int_0^2 \left[u^2 v - \frac{1}{3} v^3 \right]_{u-2}^0 \, du \\
 &= \frac{1}{8} \int_0^2 \left(-u^2(u-2) + \frac{1}{3}(u-2)^3 \right) \, du \\
 &= \frac{1}{8} \int_0^2 \left(2u^2 - u^3 + \frac{1}{3}(u-2)^3 \right) \, du \\
 &= \frac{1}{8} \left[\frac{2}{3} u^3 - \frac{1}{4} u^4 + \frac{1}{12} (u-2)^4 \right]_0^2 \\
 &= \frac{1}{8} \left(\frac{16}{3} - 4 - \frac{4}{3} \right) = \boxed{0}.
 \end{aligned}$$

22. (a) The region R is the same as the region R in Problem 21.
 (b) The change of variables is the same as that in Problem 21, so the region $R^\#$ is the same as well, and the boundaries of the region are as well.
 (c) As in Problem 21, the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

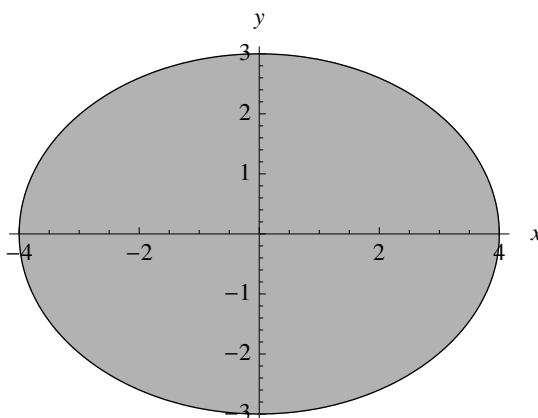
(d) The integral becomes

$$\begin{aligned}
 \iint_R (x+y) \sin(x-y) \, dA &= \iint_{R^\#} u \sin v \cdot \left| -\frac{1}{2} \right| \, dv \, du \\
 &= \frac{1}{2} \int_0^2 \int_{u-2}^0 u \sin v \, dv \, du \\
 &= \frac{1}{2} \int_0^2 [-u \cos v]_{u-2}^0 \, du \\
 &= \frac{1}{2} \int_0^2 (u \cos(u-2) - u) \, du \\
 &= \frac{1}{2} \int_0^2 u \cos(u-2) \, du - \frac{1}{2} \int_0^2 u \, du.
 \end{aligned}$$

For the first integral, use integration by parts, with $t = u$ and $ds = \cos(u-2) \, du$. Then $dt = du$ and $s = \sin(u-2)$. Proceeding,

$$\begin{aligned}
 &= \frac{1}{2} \left([u \sin(u-2)]_0^2 - \int_0^2 \sin(u-2) \, du \right) - \frac{1}{2} \left[\frac{1}{2} u^2 \right]_0^2 \\
 &= \frac{1}{2} \left(0 - [-\cos(u-2)]_0^2 \right) - 1 \\
 &= -\frac{1}{2} (-1 + \cos(-2)) - 1 \\
 &= \boxed{-\frac{1}{2}(1 + \cos 2)}.
 \end{aligned}$$

23. The region R is



Solving the given change of variables for x and y gives $x = 4u$ and $y = 3v$. Then $9x^2 + 16y^2$ becomes $9(4u)^2 + 16(3v)^2 = 144u^2 + 144v^2 = 144$, which is just $u^2 + v^2 = 1$, so that $R^\#$ is the circle of radius 1 centered at the origin together with its interior. The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} = 12.$$

Then the integral is

$$\iint_R (3x - 2y) dA = \iint_{R^\#} (12u - 6v) \cdot |12| dA = 72 \iint_{R^\#} (2u - v) dA.$$

To perform the integration, use polar coordinates in the uv -plane:

$$\begin{aligned} 72 \iint_{R^\#} (2u - v) dA &= 72 \int_0^1 \int_0^{2\pi} (2r \cos \theta - r \sin \theta) \cdot r dr d\theta \\ &= 72 \int_0^1 \int_0^{2\pi} r^2 (2 \cos \theta - \sin \theta) dr d\theta \\ &= 72 \int_0^2 [r^2 (2 \sin \theta + \cos \theta)]_0^{2\pi} dr \\ &= \boxed{0}. \end{aligned}$$

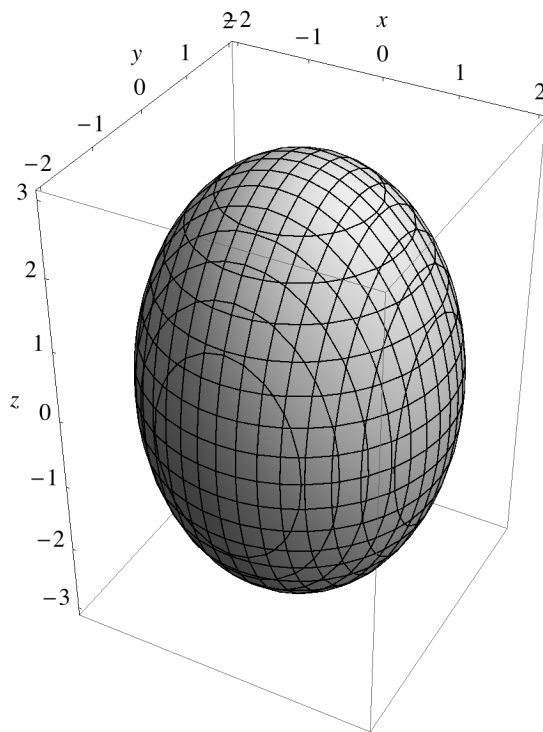
24. The region R , change of variables formula, and Jacobian are as in Problem 23. Also as in Problem 23, the region $R^\#$ is the circle of radius 1 centered at the origin, together with its interior. The integral is

$$\iint_R (x^2 - y) dA = \iint_{R^\#} (16u^2 - 3v) \cdot |12| dA = 12 \iint_{R^\#} (16u^2 - 3v) dA.$$

To perform the integration, use polar coordinates in the uv -plane:

$$\begin{aligned}
 12 \iint_{R^\#} (16u^2 - 3v) \, dA &= 12 \int_0^1 \int_0^{2\pi} (16r^2 \cos^2 \theta - 3r \sin \theta) \cdot r \, d\theta \, dr \\
 &= 12 \int_0^1 \int_0^{2\pi} \left(16r^3 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) - 3r^2 \sin \theta \right) d\theta \, dr \\
 &= 12 \int_0^1 \int_0^{2\pi} (8r^3 + 8r^3 \cos 2\theta - 3r^2 \sin \theta) \, d\theta \, dr \\
 &= 12 \int_0^1 [8r^3 \theta + 4r^3 \sin 2\theta + 3r^2 \cos \theta]_0^{2\pi} \, dr \\
 &= 12 \int_0^1 16\pi r^3 \, dr \\
 &= 12 [4\pi r^4]_0^1 = \boxed{48\pi}.
 \end{aligned}$$

25. The solid E is



Solving the given change of variables for x , y , and z gives $x = 2u$, $y = 2v$, and $z = 3w$. Then $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$ becomes $\frac{(2u)^2}{4} + \frac{(2v)^2}{4} + \frac{(3w)^2}{9} = 1$, which is just $u^2 + v^2 + w^2 = 1$, so that $E^\#$ is the sphere of radius 1 centered at the origin, together with its interior. The Jacobian of the transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 12.$$

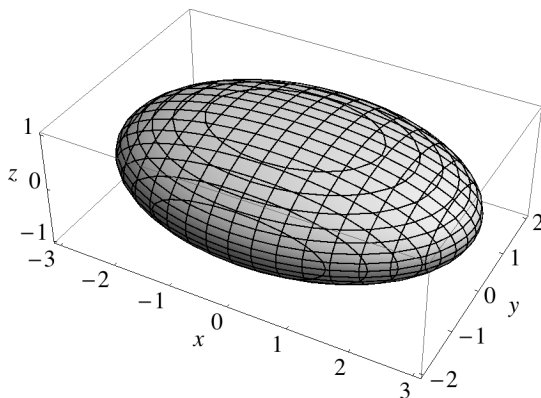
Then the integral is

$$\iiint_E (x+1) \, dA = \iiint_{E^\#} (2u+1) \cdot |12| \, dA = 12 \iiint_{E^\#} (2u+1) \, dA.$$

To perform the integration, use spherical coordinates:

$$\begin{aligned}
 12 \iiint_{E^\#} (2u + 1) dA &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} (2\rho \cos \theta \sin \phi + 1) \cdot \rho^2 \sin \phi d\theta d\phi d\rho \\
 &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} (2\rho^3 \cos \theta \sin^2 \phi + \rho^2 \sin \phi) d\theta d\phi d\rho \\
 &= 12 \int_0^1 \int_0^\pi [2\rho^3 \sin \theta \sin^2 \phi + \rho^2 \theta \sin \phi]_0^{2\pi} d\phi d\rho \\
 &= 12 \int_0^1 \int_0^\pi 2\pi \rho^2 \sin \phi d\phi d\rho \\
 &= 24\pi \int_0^1 [-\rho^2 \cos \phi]_0^\pi d\rho \\
 &= 24\pi \int_0^1 2\rho^2 d\rho \\
 &= 24\pi \left[\frac{2}{3} \rho^3 \right]_0^1 = \boxed{16\pi}.
 \end{aligned}$$

26. The solid E is



Solving the given change of variables for x , y , and z gives $x = 3u$, $y = 2v$, and $z = w$. Then $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ becomes $\frac{(3u)^2}{9} + \frac{(2v)^2}{4} + w^2 = 1$, which is just $u^2 + v^2 + w^2 = 1$, so that $E^\#$ is the region bounded by the sphere of radius 1 centered at the origin. The Jacobian of the transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6.$$

Then the integral is

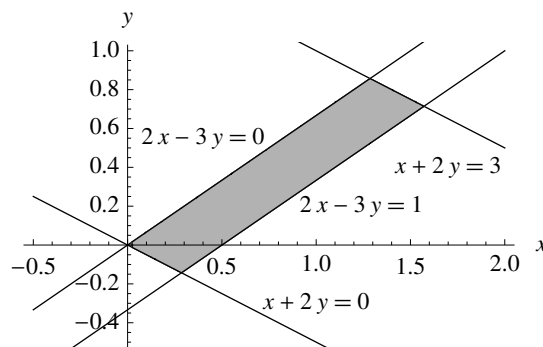
$$\iiint_E (x + 2y) dA = \iiint_{E^\#} (3u + 4v) \cdot |12| dA = 12 \iiint_{E^\#} (3u + 4v) dA.$$

To perform the integration, use spherical coordinates:

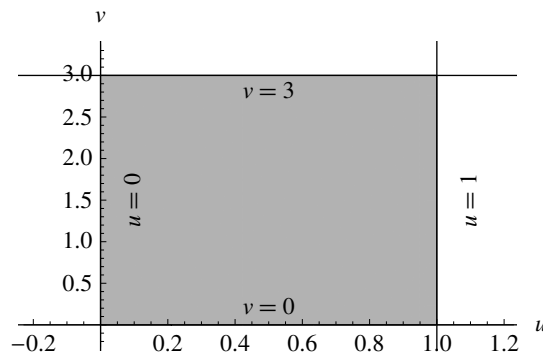
$$\begin{aligned}
 12 \iiint_{E^\#} (3u + 4v) dA &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} (3\rho \cos \theta \sin \phi + 4\rho \sin \theta \sin \phi) \cdot \rho^2 \sin \phi d\theta d\phi d\rho \\
 &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} (3\rho^3 \cos \theta \sin^2 \phi + 4\rho^3 \sin \theta \sin^2 \phi) d\theta d\phi d\rho \\
 &= 12 \int_0^1 \int_0^\pi [3\rho^3 \sin \theta \sin^2 \phi - 4\rho^3 \cos \theta \sin^2 \phi]_0^{2\pi} d\phi d\rho \\
 &= \boxed{0}.
 \end{aligned}$$

Applications and Extensions

27. (a) The region R is



- (b) Use the transformation $u = 2x - 3y$, $v = x + 2y$. Then the boundaries of $R^\#$ are $u = 0$ and $u = 1$, and $v = 0$ and $v = 3$, so this is a rectangular region:



- (c) Solve the transformation equations for x and y , giving $x = \frac{2}{7}u + \frac{3}{7}v$, $y = -\frac{1}{7}u + \frac{2}{7}v$. Then the integrand becomes

$$x^2 y = \left(\frac{2}{7}u + \frac{3}{7}v \right)^2 \left(-\frac{1}{7}u + \frac{2}{7}v \right) = \frac{1}{343} (18v^3 + 15uv^2 - 4u^2v - 4u^3).$$

The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} = \frac{1}{7}.$$

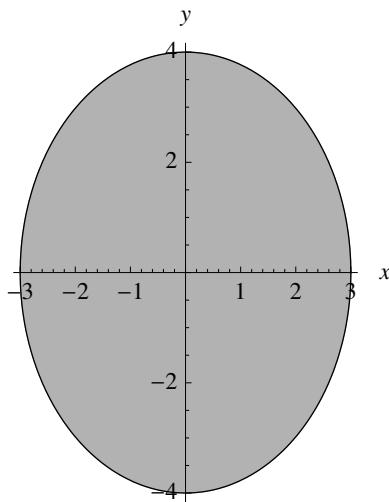
Therefore the integral is

$$\iint_R x^2 y dA = \boxed{\iint_{R^\#} \left(\frac{2}{7}u + \frac{3}{7}v \right)^2 \left(-\frac{1}{7}u + \frac{2}{7}v \right) \frac{1}{7} dv du}.$$

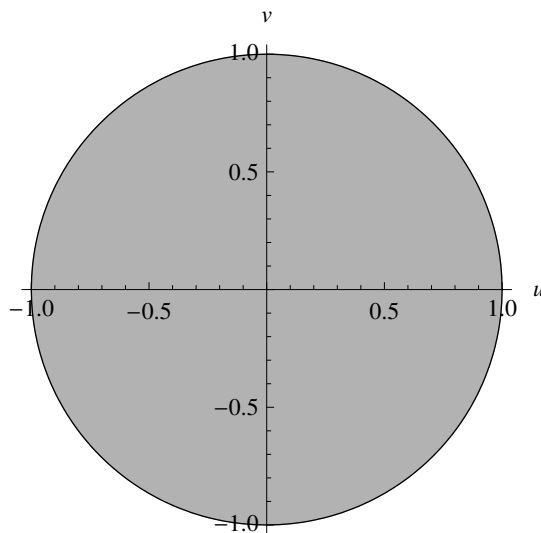
(d) Evaluating the integral, we get

$$\begin{aligned}
 \iint_R x^2 y \, dA &= \iint_{R^\#} \left(\frac{2}{7}u + \frac{3}{7}v \right)^2 \left(-\frac{1}{7}u + \frac{2}{7}v \right) \frac{1}{7} \, dv \, du \\
 &= \frac{1}{2401} \int_0^1 \int_0^3 (18v^3 + 15uv^2 - 4u^2v - 4u^3) \, dv \, du \\
 &= \frac{1}{2401} \int_0^1 \left[\frac{9}{2}v^4 + 5uv^3 - 2u^2v^2 - 4u^3v \right]_0^3 \, du \\
 &= \frac{1}{2401} \int_0^1 \left(\frac{729}{2} + 135u - 18u^2 - 12u^3 \right) \, du \\
 &= \frac{1}{2401} \left[\frac{729}{2}u + \frac{135}{2}u^2 - 6u^3 - 3u^4 \right]_0^1 \\
 &= \boxed{\frac{423}{2401}}.
 \end{aligned}$$

28. (a) The region R is



(b) Use the transformation $u = \frac{x}{3}$, $v = \frac{y}{4}$, so that $x = 3u$ and $y = 4v$. The equation in the uv -coordinate system is then $\frac{(3u)^2}{9} + \frac{(4v)^2}{16} = u^2 + v^2 = 1$, so it is the region bounded by the circle of radius 1 centered at the origin:



(c) The integrand becomes $x = 3u$, and the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = 12.$$

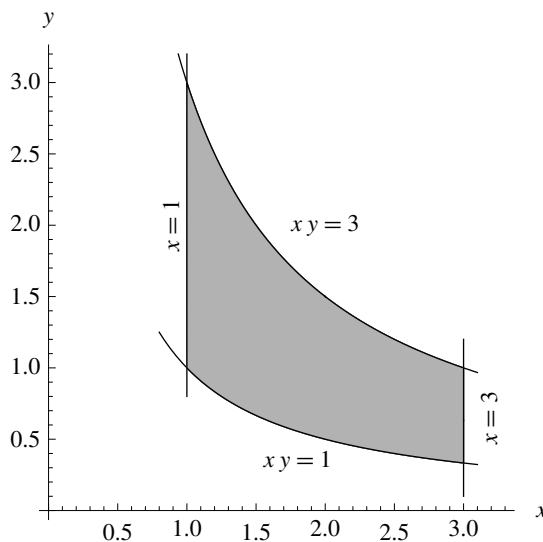
Therefore the integral is

$$\iint_R x \, dA = \iint_{R^\#} 3u \cdot |12| \, dA = \boxed{36 \iint_{R^\#} u \, dA}.$$

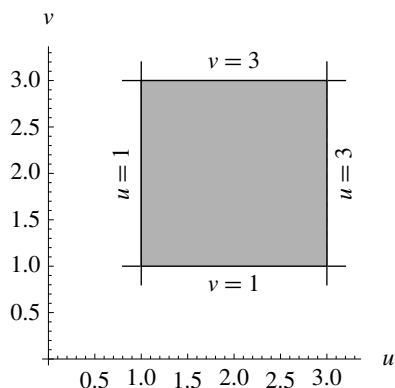
(d) Evaluating the integral using polar coordinates, we get

$$36 \iint_{R^\#} u \, dA = 36 \int_0^1 \int_0^{2\pi} r \cos \theta \cdot r \, d\theta \, dr = 36 \int_0^1 [r^2 \sin \theta]_0^{2\pi} \, dr = \boxed{0}.$$

29. (a) The region R is



(b) Use the transformation $u = x$, $v = xy$, so that $x = u$ and $y = \frac{v}{x} = \frac{v}{u}$. Then $xy = 1$ and $xy = 3$ becomes $v = 1$ and $v = 3$, while $x = 1$ and $x = 3$ become $u = 1$ and $u = 3$. So $R^\#$ is a square region:



- (c) The integrand becomes $x \cos(xy) = u \cos v$, and the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}.$$

Therefore the integral is

$$\iint_R x \cos(xy) dA = \iint_{R^\#} u \cos v \cdot \frac{1}{u} dA = \iint_{R^\#} \cos v dA = \boxed{\int_1^3 \int_1^3 \cos v dv du}.$$

- (d) Evaluating the integral we get

$$\int_1^3 \int_1^3 \cos v du dv = 2 \int_1^3 \cos v dv = 2 [\sin v]_1^3 = \boxed{2(\sin 3 - \sin 1)}.$$

30. (a) The region R is the same as the region in Problem 29.
 (b) Use the same transformation as in Problem 29, giving the same region $R^\#$.
 (c) The integrand becomes $ye^{xy} = \frac{v}{u}e^v$, and the Jacobian of the transformation is, as in Problem 29, $\frac{1}{u}$. Therefore the integral is

$$\iint_R ye^{xy} dA = \iint_{R^\#} \frac{v}{u}e^v \cdot \frac{1}{u} dA = \boxed{\iint_{R^\#} \frac{v}{u^2}e^v dA}.$$

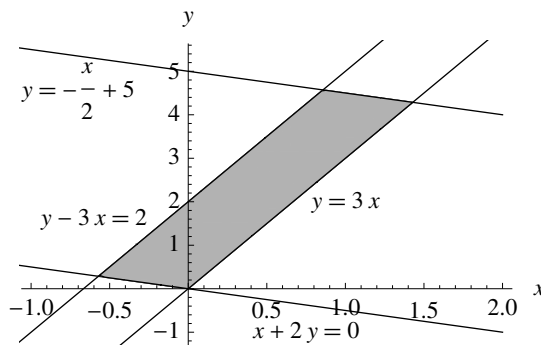
- (d) Evaluating the integral we get

$$\begin{aligned} \iint_{R^\#} \frac{v}{u^2}e^v dA &= \int_1^3 \int_1^3 \frac{v}{u^2}e^v du dv \\ &= \int_1^3 \left[-\frac{v}{u}e^v \right]_1^3 dv \\ &= \int_1^3 \left(ve^v - \frac{1}{3}ve^v \right) dv \\ &= \frac{2}{3} \int_1^3 ve^v dv. \end{aligned}$$

Now use Formula 115 in the Table of Integrals:

$$\begin{aligned} &= \frac{2}{3} [(v-1)e^v]_1^3 \\ &= \frac{2}{3} \cdot 2e^3 = \boxed{\frac{4}{3}e^3}. \end{aligned}$$

31. The region R is



These four equations may be rewritten $y - 3x = 0$, $y - 3x = 2$, $2y + x = 0$, and $2y + x = 10$. This suggests the transformations $u = y - 3x$ and $v = 2y + x$. Solving these for x and y gives $x = -\frac{2}{7}u + \frac{1}{7}v$ and $y = \frac{1}{7}u + \frac{3}{7}v$. With these transformation, the region $R^\#$ is the rectangular region bounded by $u = 0$, $u = 2$, $v = 0$, and $v = 10$. The Jacobian of this transformation is

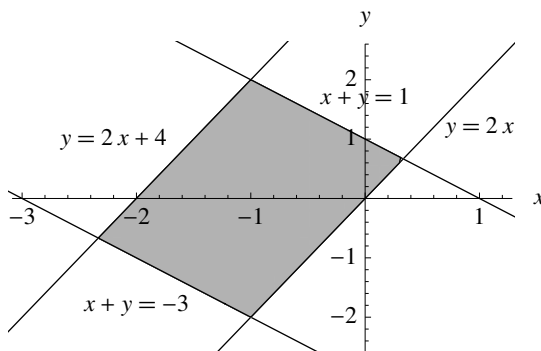
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{vmatrix} = -\frac{1}{7}.$$

The area of the region is therefore

$$\iint_R 1 \, dA = \iint_{R^\#} \left| -\frac{1}{7} \right| dA = \frac{1}{7} \iint_{R^\#} 1 \, dA = \frac{1}{7} \cdot 20 = \boxed{\frac{20}{7} \text{ square units}}$$

since the area of the rectangle is 20 square units.

32. The region R is



The given equations suggest the transformations $u = x + y$ and $v = y - 2x$. With these transformation, the region $R^\#$ is the rectangular region bounded by $u = -3$, $u = 1$, $v = 0$, and $v = 4$. Solving for x and y gives $x = \frac{1}{3}u - \frac{1}{3}v$ and $y = \frac{2}{3}u + \frac{1}{3}v$. Therefore the Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

The area of the region is therefore

$$\iint_R 1 \, dA = \iint_{R^\#} \left| \frac{1}{3} \right| dA = \frac{1}{3} \iint_{R^\#} 1 \, dA = \frac{1}{3} \cdot 16 = \boxed{\frac{16}{3} \text{ square units}}$$

since the area of the rectangle is 16 square units.

33. The given equations suggest the transformations

$$u = x + 2y, \quad v = y - z, \quad w = z.$$

With these transformations, the solid $E^\#$ is the rectangular solid bounded by the planes $u = 0$, $u = 3$, $v = 0$, $v = 2$, $w = 0$, and $w = 6$, so that it has a volume of $3 \cdot 2 \cdot 6 = 36$ cubic units. Solving the transformation equations for x , y , and z gives $x = u - 2v - 2w$, $y = v + w$, and $z = w$. Therefore the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

So the volume of E is

$$\iiint_E 1 \, dV = \iiint_{E^\#} 1 \, dV = 1 \cdot 36 = \boxed{36 \text{ cubic units}}.$$

34. The given equations suggest the transformations

$$u = x - y, \quad v = z - 2x, \quad w = z.$$

With these transformations, the solid $E^\#$ is the rectangular solid bounded by the planes $u = 0$, $u = 3$, $v = 0$, $v = 4$, $w = 1$, and $w = 5$, so that it has a volume of $3 \cdot 4 \cdot 4 = 48$ cubic units. Solving the transformation equations for x , y , and z gives $x = \frac{1}{2}w - \frac{1}{2}v$, $y = -u + \frac{1}{2}w - \frac{1}{2}v$, and $z = w$. Therefore the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{2}.$$

So the volume of E is

$$\iiint_E 1 \, dV = \iiint_{E^\#} \left| -\frac{1}{2} \right| dV = \frac{1}{2} \cdot 48 = \boxed{24 \text{ cubic units}}.$$

35. With the given change of variables, the plane
- $z = 1$
- becomes the plane
- $w = 1$
- , and the paraboloid becomes

$$w = 9 - (3u \cos v)^2 - 9(u \sin v)^2 = 9 - 9u^2 \cos^2 v - 9u^2 \sin^2 v = 9 - 9u^2.$$

The trace of the surface of the solid in the xy -plane is the ellipse $x^2 + 9y^2 = 9$, so that $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$; $z = w$ ranges from 1 to $9 - 9u^2$. The Jacobian of the transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3 \cos v & -3u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3u.$$

Therefore the volume of the solid is

$$\begin{aligned} \iint_E 1 \, dA &= \int_0^1 \int_0^{2\pi} \int_1^{9-9u^2} 3u \, dw \, dv \, du \\ &= \int_0^1 \int_0^{2\pi} [3uw]_1^{9-9u^2} \, dv \, du \\ &= \int_0^1 \int_0^{2\pi} (3u(9-9u^2) - 3u) \, dv \, du \\ &= \int_0^1 \int_0^{2\pi} (-27u^3 + 24u) \, dv \, du \\ &= 2\pi \int_0^1 (-27u^3 + 24u) \, du \\ &= 2\pi \left[-\frac{27}{4}u^4 + 12u^2 \right]_0^1 = \boxed{\frac{21}{2}\pi \text{ cubic units}}. \end{aligned}$$

36. Solving the transformation for x , y , and z gives $x = 2u$, $y = 3v$, $z = w$, and the ellipsoid becomes

$$\frac{(2u)^2}{4} + \frac{(3v)^2}{9} + w^2 = u^2 + v^2 + w^2 = 1.$$

Since $w = z$, we are considering only the portion of that region above the uv -plane. So the transformed solid $E^\#$ is the upper hemispherical half of the solid bounded by sphere of radius 1, which has volume $\frac{2}{3}\pi$ cubic units. The Jacobian of the transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6.$$

Therefore the volume of E is

$$\iiint_E 1 \, dV = \iiint_{E^\#} 6 \, dV = 6 \cdot \frac{2}{3}\pi = \boxed{4\pi \text{ cubic units}}.$$

37. The transformation from rectangular to spherical coordinates is given by

$$x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi,$$

so that the Jacobian of the transformation is

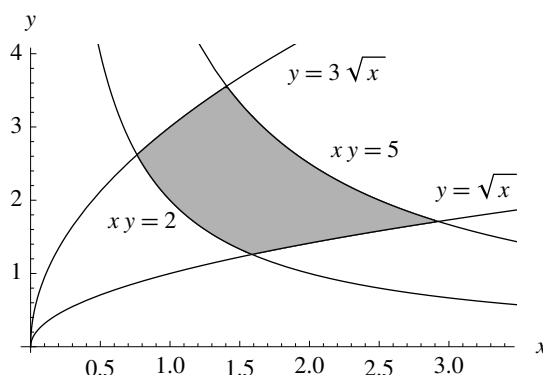
$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos \theta \sin \phi & -\rho \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\ \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi \begin{vmatrix} -\rho \sin \theta \sin \phi & \rho \cos \theta \cos \phi \\ \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \end{vmatrix} - \rho \sin \phi \begin{vmatrix} \cos \theta \sin \phi & -\rho \sin \theta \sin \phi \\ \sin \theta \sin \phi & \rho \cos \theta \sin \phi \end{vmatrix} \\ &= \cos \phi (-\rho^2 \sin^2 \theta \sin \phi \cos \phi - \rho^2 \cos^2 \theta \sin \phi \cos \phi) - \rho \sin \phi (\rho \cos^2 \theta \sin^2 \phi + \rho \sin^2 \theta \sin^2 \phi) \\ &= \cos \phi (-\rho^2 \sin \phi \cos \phi) - \rho \sin \phi (\rho \sin^2 \phi) \\ &= -\rho^2 (\sin \phi \cos^2 \phi + \sin^3 \phi) \\ &= -\rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ &= -\rho^2 \sin \phi. \end{aligned}$$

Note that $\rho^2 \geq 0$, and since $0 \leq \phi \leq \pi$, we also have $\sin \phi \geq 0$, so that $|\rho^2 \sin \phi| = \rho^2 \sin \phi$. Therefore

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \iiint_{E^\#} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \cdot |\rho^2 \sin \phi| \, dV \\ &= \iiint_{E^\#} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi. \end{aligned}$$

Challenge Problems

38. (a) The region R is



(b) Use the transformation $u = xy$ and $v = \frac{y}{\sqrt{x}}$. To solve for x and y , note that we have

$$y = \frac{u}{x} = v\sqrt{x},$$

so that $x^{3/2} = \frac{u}{v}$ and therefore $x = u^{2/3}v^{-2/3}$. Then $u = xy$ gives $y = u^{1/3}v^{2/3}$. The transformed region $R^\#$ is the rectangular region bounded by $u = 2$ and $u = 5$, and by $v = \frac{y}{\sqrt{x}} = 1$ and $v = 3$.

(c) The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{-2/3} & -\frac{2}{3}u^{2/3}v^{-5/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix} = \frac{4}{9v} + \frac{2}{9v} = \frac{2}{3v}.$$

Therefore the area of R is

$$\begin{aligned} \iint_R 1 \, dV &= \iint_{R^\#} \frac{2}{3v} \, dv \, du \\ &= \int_2^5 \int_1^3 \frac{2}{3v} \, dv \, du \\ &= \int_2^5 \left[\frac{2}{3} \ln v \right]_1^3 \, du \\ &= \int_2^5 \frac{2}{3} \ln 3 \, du = \boxed{2 \ln 3 \text{ square units}}. \end{aligned}$$

Review Exercises

1. In this iterated integral, first we integrate partially with respect to x :

$$\begin{aligned}\int_0^1 \int_1^e \frac{y}{x} dx dy &= \int_0^1 y \cdot [\ln x]_1^e dy = \int_0^1 y \cdot (\ln e - \ln 1) dy \\ &= \int_0^1 y \cdot (1 - 0) dy = \int_0^1 y dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}.\end{aligned}$$

2. In this iterated integral, first we integrate partially with respect to y :

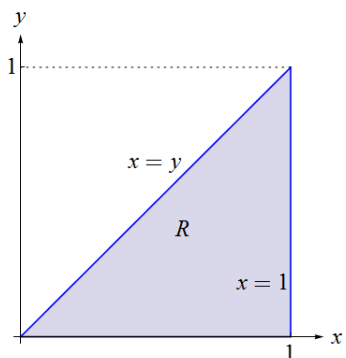
$$\begin{aligned}\int_1^3 \int_3^6 (x^2 + y^2) dy dx &= \int_1^3 \left[x^2 y + \frac{y^3}{3} \right]_3^6 dx = \int_1^3 [(6x^2 + 72) - (3x^2 + 9)] dx \\ &= \int_1^3 (3x^2 + 63) dx = [x^3 + 63x]_1^3 = (27 + 189) - (1 + 63) = \boxed{152}.\end{aligned}$$

3. We begin by graphing the region R associated with the integral $\int_0^1 \int_y^1 ye^{-x^3} dx dy$: it is the region between the graphs of $x = y$ and $x = 1$, where $0 \leq y \leq 1$; see the figure. Observe that R is also an x -simple region: y varies from $y = 0$ to $y = x$, where $0 \leq x \leq 1$. Then according to Fubini's Theorem,

$$\begin{aligned}\int_0^1 \int_y^1 ye^{-x^3} dx dy &= \int_0^1 \int_0^x ye^{-x^3} dy dx = \int_0^1 e^{-x^3} \left[\frac{y^2}{2} \right]_0^x dx \\ &= \int_0^1 e^{-x^3} \left(\frac{x^2}{2} - 0 \right) dx = \frac{1}{2} \int_0^1 e^{-x^3} \cdot x^2 dx.\end{aligned}$$

To evaluate this integral, we make a substitution $u = -x^3$; then $du = -3x^2 dx$, so $x^2 dx = \frac{du}{-3}$. We also change the limits of integration: if $x = 0$, then $u = -0^3 = 0$; if $x = 1$, then $u = -1^3 = -1$. So

$$\begin{aligned}\frac{1}{2} \int_0^1 e^{-x^3} \cdot x^2 dx &= \frac{1}{2} \int_0^{-1} e^u \cdot \left(\frac{du}{-3} \right) = -\frac{1}{6} \int_0^{-1} e^u du \\ &= \frac{1}{6} \int_{-1}^0 e^u du = \frac{1}{6} [e^u]_{-1}^0 = \frac{1}{6} (e^0 - e^{-1}) = \frac{1}{6} \left(1 - \frac{1}{e} \right) = \boxed{\frac{e-1}{6e}}.\end{aligned}$$



4. In this iterated integral, first we integrate partially with respect to y :

$$\begin{aligned}
 \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{x^2 + y^2 + 1} dy dx &= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2) + y^2} dy dx \\
 &= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx \\
 &= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \right) - \tan^{-1} \left(\frac{0}{\sqrt{1+x^2}} \right) \right] dx \\
 &= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0) dx \\
 &= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4} - 0 \right) dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx.
 \end{aligned}$$

To evaluate this integral, we make a substitution $x = \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; then $dx = \sec^2 \theta d\theta$. We also change the limits of integration: if $x = 0$, then $0 = \tan \theta$, so $\theta = 0$; if $x = 1$, then $1 = \tan \theta$, so $\theta = \frac{\pi}{4}$. The radical in the integrand simplifies as $\sqrt{1+x^2} = \sqrt{1+\tan^2 \theta} = \sqrt{\sec^2 \theta} = \sec \theta$, where we used the fact that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and so $\sec \theta > 0$. So the integral transforms into

$$\begin{aligned}
 \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx &= \frac{\pi}{4} \int_0^{\pi/4} \frac{1}{\sec \theta} \cdot \sec^2 \theta d\theta = \frac{\pi}{4} \int_0^{\pi/4} \sec \theta d\theta = \frac{\pi}{4} [\ln |\sec \theta + \tan \theta|]_0^{\pi/4} \\
 &= \frac{\pi}{4} (\ln |\sec \frac{\pi}{4} + \tan \frac{\pi}{4}| - \ln |\sec 0 + \tan 0|) = \frac{\pi}{4} (\ln |\sqrt{2} + 1| - \ln |1 + 0|) \\
 &= \frac{\pi}{4} (\ln (\sqrt{2} + 1) - \ln 1) = \boxed{\frac{\pi \ln (\sqrt{2} + 1)}{4}}.
 \end{aligned}$$

5. In this iterated integral, first we integrate partially with respect to y :

$$\begin{aligned}
 \int_0^{\pi/4} \int_0^{\tan x} \sec x dy dx &= \int_0^{\pi/4} \sec x \cdot [y]_0^{\tan x} dx = \int_0^{\pi/4} \sec x \cdot (\tan x - 0) dx \\
 &= \int_0^{\pi/4} \sec x \tan x dx = [\sec x]_0^{\pi/4} = \sec \left(\frac{\pi}{4} \right) - \sec 0 = \boxed{\sqrt{2} - 1}.
 \end{aligned}$$

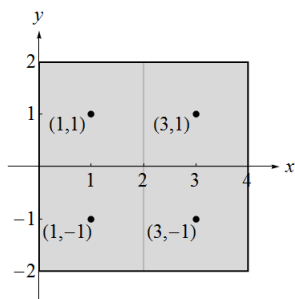
6. In this iterated integral, first we integrate partially with respect to y :

$$\begin{aligned}
 \int_0^1 \int_{-x}^x e^{x+y} dy dx &= \int_0^1 \int_{-x}^x e^x e^y dy dx = \int_0^1 e^x [e^y]_{-x}^x dx = \int_0^1 e^x (e^x - e^{-x}) dx \\
 &= \int_0^1 (e^{2x} - 1) dx = \left[\frac{1}{2} e^{2x} - x \right]_0^1 = \left(\frac{1}{2} e^2 - 1 \right) - \left(\frac{1}{2} e^0 - 0 \right) \\
 &= \frac{1}{2} e^2 - 1 - \frac{1}{2} = \frac{1}{2} e^2 - \frac{3}{2} = \boxed{\frac{e^2 - 3}{2}}.
 \end{aligned}$$

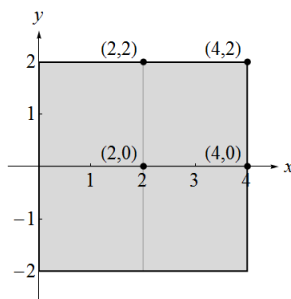
7. (a) Let $f(x, y) = x + 2y$, and let R be the closed square region defined by $0 \leq x \leq 4$, $-2 \leq y \leq 2$. Partition R into four congruent subsquares with sides $\Delta x_i = 2$, $i = 1, 2$, and $\Delta y_j = 2$, $j = 1, 2$, each with area $\Delta A_k = 4$. Choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the center of the k th subsquare: $(u_1, v_1) = (1, -1)$, $(u_2, v_2) = (3, -1)$, $(u_3, v_3) = (1, 1)$, and $(u_4, v_4) = (3, 1)$. See figure (a) below.

Then the corresponding Riemann sum is:

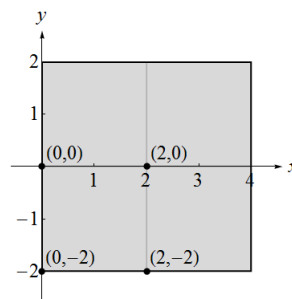
$$\begin{aligned}
 \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(1, -1) \cdot 4 + f(3, -1) \cdot 4 + f(1, 1) \cdot 4 + f(3, 1) \cdot 4 \\
 &= [f(1, -1) + f(3, -1) + f(1, 1) + f(3, 1)] \cdot 4 \\
 &= (-1 + 1 + 3 + 5) \cdot 4 = \boxed{32}.
 \end{aligned}$$



(a)



(b)



(c)

- (b) Since $f(x, y) = x + 2y$ is increasing with respect to both x and y , the greatest value of the function on each subsquare is attained at its upper-right corner and the smallest value of the function on each subsquare is attained at its lower-left corner.

To find the largest possible Riemann sum (for these subsquares), choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the upper-right corner of the k th subsquare: $(u_1, v_1) = (2, 0)$, $(u_2, v_2) = (4, 0)$, $(u_3, v_3) = (2, 2)$, and $(u_4, v_4) = (4, 2)$. See figure (b) above. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(2, 0) \cdot 4 + f(4, 0) \cdot 4 + f(2, 2) \cdot 4 + f(4, 2) \cdot 4 \\
 &= [f(2, 0) + f(4, 0) + f(2, 2) + f(4, 2)] \cdot 4 \\
 &= (2 + 4 + 6 + 8) \cdot 4 = \boxed{80}.
 \end{aligned}$$

To find the smallest possible Riemann sum (for these subsquares), choose (u_k, v_k) , $k = 1, 2, 3, 4$, as the lower-left corner of the k th subsquare: $(u_1, v_1) = (0, -2)$, $(u_2, v_2) = (2, -2)$, $(u_3, v_3) = (0, 0)$, and $(u_4, v_4) = (2, 0)$. See figure (c) above. Then the corresponding Riemann sum is:

$$\begin{aligned}
 \sum_{k=1}^4 f(u_k, v_k) \Delta A_k &= f(0, -2) \cdot 4 + f(2, -2) \cdot 4 + f(0, 0) \cdot 4 + f(2, 0) \cdot 4 \\
 &= [f(0, -2) + f(2, -2) + f(0, 0) + f(2, 0)] \cdot 4 \\
 &= (-4 - 2 + 0 + 2) \cdot 4 = \boxed{-16}.
 \end{aligned}$$

The average of these two values is $\frac{80 + (-16)}{2} = \frac{64}{2} = \boxed{32}$.

- (c) The actual value of the double integral $\iint_R f(x, y) dA$ can be found by applying Fubini's Theorem for rectangular regions:

$$\begin{aligned}
 \iint_R f(x, y) dA &= \int_0^4 \int_{-2}^2 (x + 2y) dy dx = \int_0^4 [xy + y^2]_{-2}^2 dx \\
 &= \int_0^4 [(2x + 4) - (-2x + 4)] dx = \int_0^4 4x dx = [2x^2]_0^4 = 32 - 0 = \boxed{32}.
 \end{aligned}$$

8. Using Fubini's Theorem, the given double integral $\iint_R x e^y dA$, where R is the closed rectangular region defined by $0 \leq x \leq 2$, $0 \leq y \leq 1$, equals either of the iterated integrals:

$$\int_0^2 \int_0^1 x e^y dy dx \quad \text{or} \quad \int_0^1 \int_0^2 x e^y dx dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x :

$$\begin{aligned} \iint_R x e^y dA &= \int_0^1 \int_0^2 x e^y dx dy = \int_0^1 e^y \left[\frac{x^2}{2} \right]_0^2 dy \\ &= \int_0^1 e^y (2 - 0) dy = 2 \int_0^1 e^y dy = 2 [e^y]_0^1 = 2(e^1 - e^0) = \boxed{2(e - 1)}. \end{aligned}$$

9. Using Fubini's Theorem, the given double integral $\iint_R \sin x \cos y dA$, where R is the closed rectangular region defined by $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq \frac{\pi}{3}$, equals either of the iterated integrals:

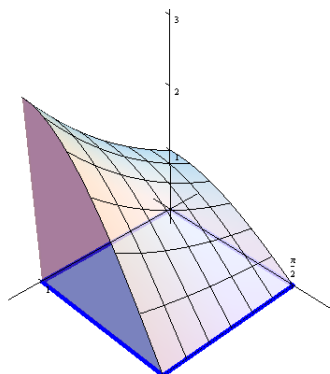
$$\int_0^{\pi/2} \int_0^{\pi/3} \sin x \cos y dy dx \quad \text{or} \quad \int_0^{\pi/3} \int_0^{\pi/2} \sin x \cos y dx dy.$$

Using the latter iterated integral, we begin by integrating partially with respect to x :

$$\begin{aligned} \iint_R \sin x \cos y dA &= \int_0^{\pi/3} \int_0^{\pi/2} \sin x \cos y dx dy = \int_0^{\pi/3} \cos y \cdot [-\cos x]_0^{\pi/2} dy \\ &= \int_0^{\pi/3} \cos y \cdot \left[-\cos\left(\frac{\pi}{2}\right) + \cos 0 \right] dy = \int_0^{\pi/3} \cos y \cdot (-0 + 1) dy \\ &= \int_0^{\pi/3} \cos y dy = [\sin y]_0^{\pi/3} = \sin\left(\frac{\pi}{3}\right) - \sin 0 = \frac{\sqrt{3}}{2} - 0 = \boxed{\frac{\sqrt{3}}{2}}. \end{aligned}$$

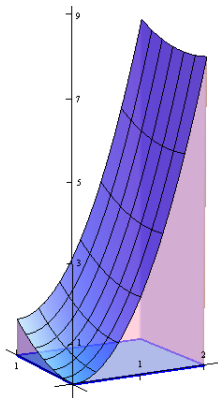
10. We begin by graphing $z = f(x, y) = e^x \cos y$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \frac{\pi}{2}$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R e^x \cos y dA$. Using Fubini's Theorem, we have

$$\begin{aligned} V &= \int_0^1 \int_0^{\pi/2} e^x \cos y dy dx = \int_0^1 e^x [\sin y]_0^{\pi/2} dx = \int_0^1 e^x \left[\sin\left(\frac{\pi}{2}\right) - \sin 0 \right] dx \\ &= \int_0^1 e^x (1 - 0) dx = \int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = \boxed{e - 1 \text{ cubic units}}. \end{aligned}$$



11. We begin by graphing $z = f(x, y) = 2x^2 + y^2$ and the rectangular region R as shown in the figure below. Since $z = f(x, y)$ is continuous and $z \geq 0$ over the rectangular region $0 \leq x \leq 2$, $0 \leq y \leq 1$, the volume V under the surface $z = f(x, y)$ is given by $V = \iint_R f(x, y) dA = \iint_R (2x^2 + y^2) dA$. Using Fubini's Theorem, we have

$$\begin{aligned} V &= \int_0^1 \int_0^2 (2x^2 + y^2) dx dy = \int_0^1 \left[\frac{2}{3}x^3 + xy^2 \right]_0^2 dy \\ &= \int_0^1 \left(\frac{16}{3} + 2y^2 \right) dy = \left[\frac{16}{3}y + \frac{2}{3}y^3 \right]_0^1 = \left(\frac{16}{3} + \frac{2}{3} \right) - 0 = \boxed{6 \text{ cubic units}}. \end{aligned}$$



12. (a) Using properties of double integrals,

$$\begin{aligned} \iint_R 2[f(x, y) + 3g(x, y)] dA &= \iint_R [2f(x, y) + 6g(x, y)] dA = \iint_R 2f(x, y) dA + \iint_R 6g(x, y) dA \\ &= 2 \iint_R f(x, y) dA + 6 \iint_R g(x, y) dA = 2 \cdot (-2) + 6 \cdot 3 = \boxed{14}. \end{aligned}$$

- (b) Using properties of double integrals,

$$\begin{aligned} \iint_R [g(x, y) - 2f(x, y)] dA &= \iint_R g(x, y) dA - \iint_R 2f(x, y) dA \\ &= \iint_R g(x, y) dA - 2 \iint_R f(x, y) dA = 3 - 2 \cdot (-2) = \boxed{7}. \end{aligned}$$

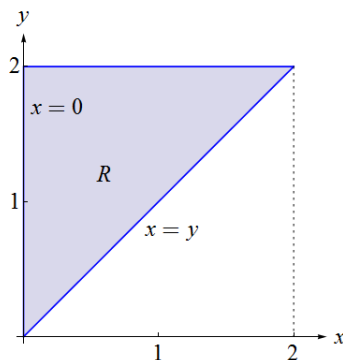
13. We begin by graphing the region R enclosed by the triangle with vertices $(0, 0)$, $(0, 2)$, and $(2, 2)$; see the figure below. Observe that R is a closed bounded region that is y -simple: it is bounded by the graphs of $x = 0$ and $x = y$, where $0 \leq y \leq 2$. So we can apply Fubini's Theorem for y -simple regions:

$$\begin{aligned} \iint_R x \sin(y^3) dA &= \int_0^2 \int_0^y x \sin(y^3) dx dy = \int_0^2 \sin(y^3) \cdot \left[\frac{x^2}{2} \right]_0^y dy \\ &= \int_0^2 \sin(y^3) \cdot \left(\frac{y^2}{2} - 0 \right) dy = \frac{1}{2} \int_0^2 \sin(y^3) \cdot y^2 dy. \end{aligned}$$

To complete evaluating this integral, we make a substitution $u = y^3$; then $du = 3y^2 dy$, so $y^2 dy = \frac{du}{3}$.

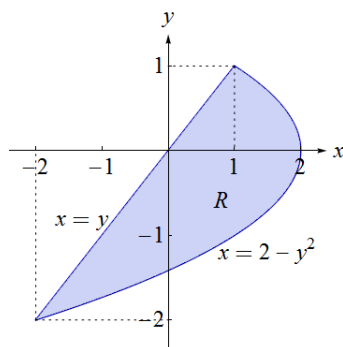
We also change the limits of integration: if $y = 0$, then $u = 0^3 = 0$; if $y = 2$, then $u = 2^3 = 8$. So

$$\begin{aligned} \frac{1}{2} \int_0^2 \sin(y^3) \cdot y^2 dy &= \frac{1}{2} \int_0^8 \sin u \cdot \left(\frac{du}{3}\right) = \frac{1}{6} \int_0^8 \sin u du \\ &= \frac{1}{6} [-\cos u]_0^8 = \frac{1}{6} (-\cos 8 + \cos 0) = \boxed{\frac{1 - \cos 8}{6}}. \end{aligned}$$



14. We begin by graphing the region R enclosed by $y = x$ and $y^2 = 2 - x$; see the figure below. Observe that R is a closed bounded region, that is y -simple: it is bounded by the graphs of $x = y$ and $x = 2 - y^2$, where $-2 \leq y \leq 1$. So we can apply Fubini's Theorem for y -simple regions:

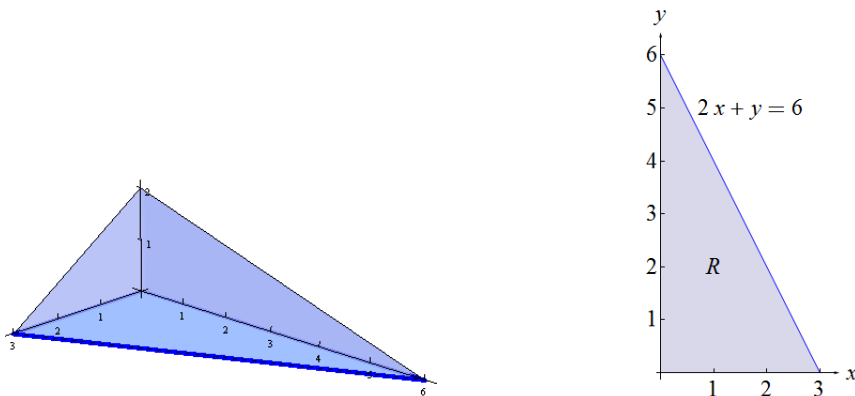
$$\begin{aligned} \iint_R (x + y) dA &= \int_{-2}^1 \int_y^{2-y^2} (x + y) dx dy = \int_{-2}^1 \left[\frac{x^2}{2} + xy \right]_y^{2-y^2} dy \\ &= \int_{-2}^1 \left\{ \left[\frac{(2-y^2)^2}{2} + (2-y^2)y \right] - \left[\frac{y^2}{2} + y^2 \right] \right\} dy \\ &= \int_{-2}^1 \left(2 - 2y^2 + \frac{y^4}{2} + 2y - y^3 - \frac{3y^2}{2} \right) dy \\ &= \int_{-2}^1 \left(\frac{y^4}{2} - y^3 - \frac{7y^2}{2} + 2y + 2 \right) dy \\ &= \left[\frac{y^5}{10} - \frac{y^4}{4} - \frac{7y^3}{6} + y^2 + 2y \right]_{-2}^1 \\ &= \left(\frac{1}{10} - \frac{1}{4} - \frac{7}{6} + 1 + 2 \right) - \left(-\frac{16}{5} - 4 + \frac{28}{3} + 4 - 4 \right) = \boxed{-\frac{9}{20}}. \end{aligned}$$



15. We begin by graphing the solid S enclosed by the plane $2x + y + 3z = 6$ and the coordinate planes. This solid lies in the first octant under the graph of $z = f(x, y) = 2 - \frac{2x}{3} - \frac{y}{3}$ and above the region R

in the xy -plane bounded by the coordinate axes and the line $2x + y = 6$. Observe that R is an x -simple region: y varies from $y = 0$ to $y = 6 - 2x$, where $0 \leq x \leq 3$; see the figures below. So the volume V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^3 \int_0^{6-2x} \left(2 - \frac{2x}{3} - \frac{y}{3} \right) \, dy \, dx = \int_0^3 \left[2y - \frac{2xy}{3} - \frac{y^2}{6} \right]_0^{6-2x} \, dx \\
 &= \int_0^3 \left\{ \left[2(6-2x) - \frac{2x(6-2x)}{3} - \frac{(6-2x)^2}{6} \right] - 0 \right\} \, dx \\
 &= \int_0^3 \left[(12-4x) - \left(4x - \frac{4x^2}{3} \right) - \left(6-4x + \frac{2x^2}{3} \right) \right] \, dx \\
 &= \int_0^3 \left(\frac{2x^2}{3} - 4x + 6 \right) \, dx = \left[\frac{2x^3}{9} - 2x^2 + 6x \right]_0^3 = (6 - 18 + 18) - 0 = \boxed{6 \text{ cubic units}}.
 \end{aligned}$$



16. The ellipsoid $4x^2 + y^2 + \frac{z^2}{4} = 1$, shown in figure (a) below, is symmetric about all coordinate planes. So we can find its volume if we find the volume of its part lying in the first octant and multiply it by 8. The solid S that is the part of the ellipsoid $4x^2 + y^2 + \frac{z^2}{4} = 1$ lying in the first octant is shown in figure (b) below. It is enclosed by the surface $z = f(x, y) = \sqrt{4 - 16x^2 - 4y^2} = 2\sqrt{1 - 4x^2 - y^2}$ and the xy -plane (that is $z = 0$), and it lies above the region R in the xy -plane, that is the quarter of the ellipse $4x^2 + y^2 \leq 1$ in the first quadrant, shown in figure (c) below. So the volume V of the entire ellipsoid is 8 times the volume of the solid S :

$$V = 8 \iint_R f(x, y) \, dA = 8 \iint_R 2\sqrt{1 - 4x^2 - y^2} \, dA = 16 \iint_R \sqrt{1 - 4x^2 - y^2} \, dA.$$

To evaluate this integral, we can make the following change of variables:

$$x = \frac{1}{2}r \cos \theta, \quad y = r \sin \theta.$$

The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{1}{2} \cos \theta, \quad \frac{\partial x}{\partial \theta} = -\frac{1}{2}r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta.$$

Then the Jacobian of x, y with respect to r, θ is

$$\begin{aligned}
 \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} \cos \theta & -\frac{1}{2}r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\
 &= \frac{1}{2} \cos \theta \cdot r \cos \theta - \left(-\frac{1}{2}r \sin \theta \right) \cdot \sin \theta = \frac{1}{2}r \cos^2 \theta + \frac{1}{2}r \sin^2 \theta = \frac{1}{2}r.
 \end{aligned}$$

In the new coordinates r, θ , the equation of the boundary of the quarter-ellipse region R takes the form

$$4x^2 + y^2 = 4 \cdot \left(\frac{1}{2}r \cos \theta\right)^2 + (r \sin \theta)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 = 1, \quad \text{so} \quad r = 1.$$

The integrand can be rewritten (using the computation above) as

$$\sqrt{1 - 4x^2 - y^2} = \sqrt{1 - (4x^2 + y^2)} = \sqrt{1 - r^2}.$$

So in the new coordinates the integral for the volume is

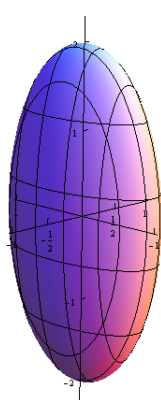
$$V = 16 \iint_R \sqrt{1 - 4x^2 - y^2} dA = 16 \int_0^{\pi/2} \int_0^1 \sqrt{1 - r^2} \cdot \frac{1}{2} r dr d\theta = 8 \int_0^{\pi/2} \int_0^1 \sqrt{1 - r^2} \cdot r dr d\theta.$$

To evaluate the “inside” integral $\int_0^1 \sqrt{1 - r^2} \cdot r dr$, we make a substitution $u = 1 - r^2$; then $du = -2r dr$, so $r dr = \frac{du}{-2}$. We also change the limits of integration: if $r = 0$, then $u = 1 - 0^2 = 1$; if $r = 1$, then $u = 1 - 1^2 = 0$. So

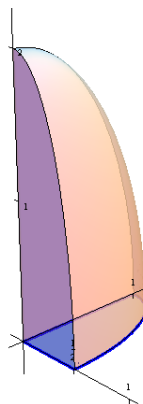
$$\begin{aligned} \int_0^1 \sqrt{1 - r^2} \cdot r dr &= \int_1^0 \sqrt{u} \left(\frac{du}{-2}\right) = -\frac{1}{2} \int_1^0 \sqrt{u} du \\ &= \frac{1}{2} \int_0^1 u^{1/2} du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_0^1 = \frac{1}{3} \left[u^{3/2} \right]_0^1 = \frac{1}{3} (1 - 0) = \frac{1}{3}. \end{aligned}$$

Going back to the computation of the volume,

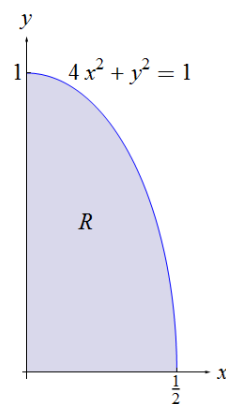
$$V = 8 \int_0^{\pi/2} \int_0^1 \sqrt{1 - r^2} \cdot r dr d\theta = 8 \int_0^{\pi/2} \frac{1}{3} d\theta = \frac{8}{3} [\theta]_0^{\pi/2} = \frac{8}{3} \left(\frac{\pi}{2} - 0\right) = \boxed{\frac{4\pi}{3} \text{ cubic units}}.$$



(a)



(b)

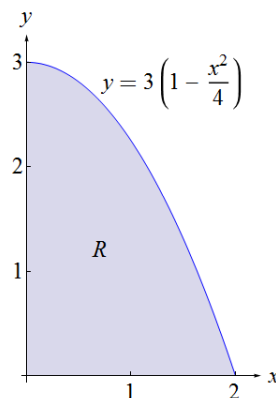
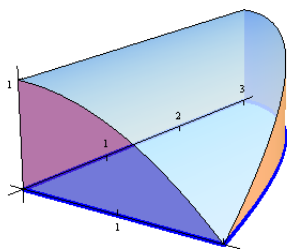


(c)

17. We begin by graphing the solid S enclosed in the first octant by the coordinate planes and the graphs of $4y = 3(4 - x^2)$ and $4z = 4 - x^2$. This solid lies under the surface $4z = 4 - x^2$, or equivalently $z = f(x, y) = 1 - \frac{x^2}{4}$, over the region R in the xy -plane bounded by the coordinate axes and the graph of $4y = 3(4 - x^2)$, or equivalently $y = 3 - \frac{3x^2}{4}$, in the first quadrant; see the figures below. Observe that R is an x -simple region: y varies from $y = 0$ to $y = 3 \left(1 - \frac{x^2}{4}\right)$, where $0 \leq x \leq 2$. So the volume

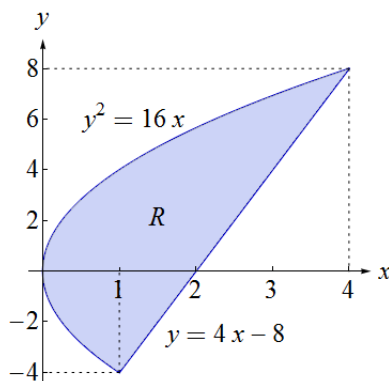
V of the solid S is

$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^2 \int_0^{3(1-x^2/4)} \left(1 - \frac{x^2}{4}\right) dy \, dx = \int_0^2 \left(1 - \frac{x^2}{4}\right) [y]_0^{3(1-x^2/4)} dx \\
 &= \int_0^2 \left(1 - \frac{x^2}{4}\right) \cdot \left[3\left(1 - \frac{x^2}{4}\right) - 0\right] dx = 3 \int_0^2 \left(1 - \frac{x^2}{4}\right)^2 dx \\
 &= 3 \int_0^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{16}\right) dx = 3 \left[x - \frac{x^3}{6} + \frac{x^5}{80}\right]_0^2 \\
 &= 3 \left[\left(2 - \frac{8}{6} + \frac{32}{80}\right) - 0\right] = 3 \cdot \frac{16}{15} = \boxed{\frac{16}{5} \text{ cubic units}}.
 \end{aligned}$$



18. We begin by graphing the region R enclosed by the parabola $y^2 = 16x$ and the line $y = 4x - 8$; see the figure below. Observe that R is a closed bounded region, that is y -simple: x varies from $x = \frac{y^2}{16}$ to $x = 2 + \frac{y}{4}$, where $-4 \leq y \leq 8$. Then the area A of the region R is

$$\begin{aligned}
 A &= \iint_R dA = \int_{-4}^8 \int_{y^2/16}^{2+y/4} dx \, dy = \int_{-4}^8 [x]_{y^2/16}^{2+y/4} dy = \int_{-4}^8 \left(2 + \frac{y}{4} - \frac{y^2}{16}\right) dy \\
 &= \left[2y + \frac{y^2}{8} - \frac{y^3}{48}\right]_{-4}^8 = \left[2 \cdot 8 + \frac{8^2}{8} - \frac{8^3}{48}\right] - \left[2 \cdot (-4) + \frac{(-4)^2}{8} - \frac{(-4)^3}{48}\right] \\
 &= \left(16 + 8 - \frac{32}{3}\right) - \left(-8 + 2 + \frac{4}{3}\right) = \boxed{18 \text{ square units}}.
 \end{aligned}$$

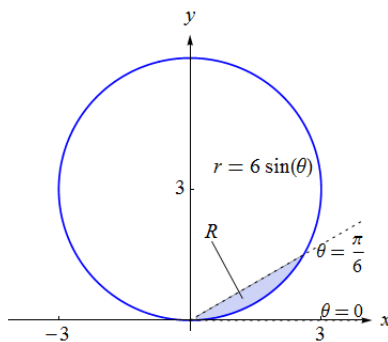


19. The region R enclosed by the rays $\theta = 0$ and $\theta = \frac{\pi}{6}$ and the circle $r = 6 \sin \theta$ is shown in the figure below. To find the given double integral, we rewrite it as an iterated integral and then evaluate.

$$\begin{aligned} \iint_R \sin \theta \, dA &= \int_0^{\pi/6} \int_0^{6 \sin \theta} \sin \theta \cdot r \, dr \, d\theta = \int_0^{\pi/6} \sin \theta \cdot \left[\frac{r^2}{2} \right]_0^{6 \sin \theta} d\theta \\ &= \int_0^{\pi/6} \sin \theta \cdot (18 \sin^2 \theta - 0) \, d\theta = 18 \int_0^{\pi/6} (1 - \cos^2 \theta) \sin \theta \, d\theta. \end{aligned}$$

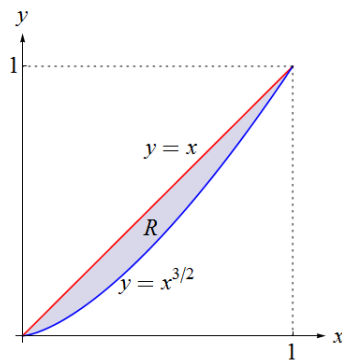
To evaluate this integral, we make a substitution $u = \cos \theta$; then $du = -\sin \theta \, d\theta$, so $\sin \theta \, d\theta = -du$. We also change the limits of integration: if $\theta = 0$, then $u = \cos 0 = 1$; if $\theta = \frac{\pi}{6}$, then $u = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. So

$$\begin{aligned} 18 \int_0^{\pi/6} (1 - \cos^2 \theta) \sin \theta \, d\theta &= 18 \int_1^{\sqrt{3}/2} (1 - u^2) (-du) = -18 \int_1^{\sqrt{3}/2} (1 - u^2) \, du \\ &= 18 \int_{\sqrt{3}/2}^1 (1 - u^2) \, du = 18 \left[u - \frac{u^3}{3} \right]_{\sqrt{3}/2}^1 = [18u - 6u^3]_{\sqrt{3}/2}^1 \\ &= [18 \cdot 1 - 6 \cdot 1^3] - \left[18 \cdot \frac{\sqrt{3}}{2} - 6 \cdot \left(\frac{\sqrt{3}}{2} \right)^3 \right] \\ &= 12 - \left(9\sqrt{3} - 6 \cdot \frac{3\sqrt{3}}{8} \right) = \boxed{12 - \frac{27\sqrt{3}}{4}}. \end{aligned}$$



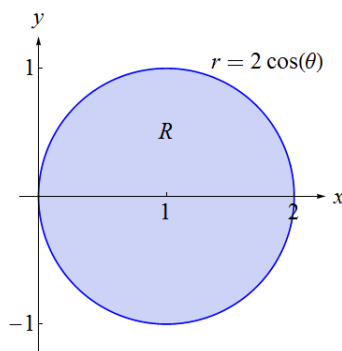
20. We begin by graphing the region R enclosed in the first quadrant by the graphs $y^2 = x^3$ and $y = x$; see the figure below. Observe that R is a closed bounded region, that is x -simple: y varies from $y = x^{3/2}$ to $y = x$, where $0 \leq x \leq 1$. Then the area A of the region R is

$$\begin{aligned} A &= \iint_R dA = \int_0^1 \int_{x^{3/2}}^x dy \, dx = \int_0^1 [y]_{x^{3/2}}^x \, dx = \int_0^1 (x - x^{3/2}) \, dx \\ &= \left[\frac{x^2}{2} - \frac{x^{5/2}}{5/2} \right]_0^1 = \left[\frac{x^2}{2} - \frac{2x^{5/2}}{5} \right]_0^1 = \left(\frac{1}{2} - \frac{2}{5} \right) - 0 = \boxed{\frac{1}{10} \text{ square units}}. \end{aligned}$$



21. We begin by graphing the region R lying inside the circle $r = 2 \cos \theta$; see the figure. The region R can be defined in polar coordinates as $0 \leq r \leq 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then the area of R can be found as

$$\begin{aligned}
 A &= \iint_R dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} \right]_0^{2 \cos \theta} d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{(2 \cos \theta)^2}{2} - 0 \right] d\theta \\
 &= \int_{-\pi/2}^{\pi/2} 2 \cos^2 \theta \, d\theta = \int_{-\pi/2}^{\pi/2} 2 \cdot \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\pi/2}^{\pi/2} \\
 &= \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} \right) \right] = \left[\left(\frac{\pi}{2} + 0 \right) - \left(-\frac{\pi}{2} + 0 \right) \right] = \boxed{\pi \text{ square units}}.
 \end{aligned}$$

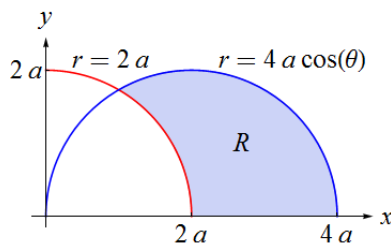


22. We begin by graphing the region R lying in the first quadrant outside the circle $r = 2a$ but inside the circle $r = 4a \cos \theta$; see the figure below. In the first quadrant, the graphs intersect at the point where

$$4a \cos \theta = 2a, \quad \text{so} \quad \cos \theta = \frac{1}{2}, \quad \text{so} \quad \theta = \frac{\pi}{3}.$$

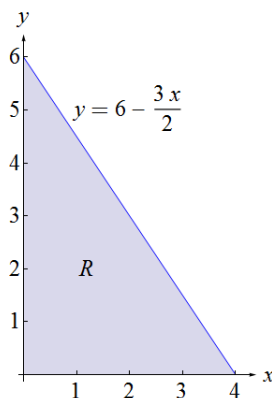
So the region R can be defined in polar coordinates as $2a \leq r \leq 4a \cos \theta$, $0 \leq \theta \leq \frac{\pi}{3}$. Then the area A of the region R can be found as

$$\begin{aligned}
 A &= \iint_R dA = \int_0^{\pi/3} \int_{2a}^{4a \cos \theta} r \, dr \, d\theta = \int_0^{\pi/3} \left[\frac{r^2}{2} \right]_{2a}^{4a \cos \theta} d\theta = \int_0^{\pi/3} (8a^2 \cos^2 \theta - 2a^2) \, d\theta \\
 &= 2a^2 \int_0^{\pi/3} (4 \cos^2 \theta - 1) \, d\theta = 2a^2 \int_0^{\pi/3} \left(4 \cdot \frac{1}{2} (1 + \cos 2\theta) - 1 \right) d\theta \\
 &= 2a^2 \int_0^{\pi/3} (1 + 2 \cos 2\theta) \, d\theta = 2a^2 [\theta + \sin 2\theta]_0^{\pi/3} = 2a^2 \left[\left(\frac{\pi}{3} + \sin \frac{2\pi}{3} \right) - (0 + 0) \right] \\
 &= 2a^2 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \boxed{a^2 \left(\frac{2\pi}{3} + \sqrt{3} \right) \text{ square units}}.
 \end{aligned}$$



23. Place the lamina on the cartesian coordinate plane so that its vertex at the right angle is at the origin and its base and height are on the positive directions of the x - and y -axes respectively; see the figure below. Then the region R occupied by the lamina can be described as bounded between $y = 0$ and $y = 6 - \frac{3x}{2}$, $0 \leq x \leq 4$. The mass density of the lamina is proportional to the square of the distance from the origin (where the vertex at the right angle has been placed); that is, $\rho(x, y) = k(x^2 + y^2)$, where k is a constant coefficient of proportionality. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R k(x^2 + y^2) dA = k \int_0^4 \int_0^{6-3x/2} (x^2 + y^2) dy dx \\
 &= k \int_0^4 \left[x^2 y + \frac{y^3}{3} \right]_0^{6-3x/2} dx = k \int_0^4 \left\{ x^2 \left(6 - \frac{3x}{2} \right) + \frac{\left(6 - \frac{3x}{2} \right)^3}{3} \right\} dx \\
 &= k \int_0^4 \left(6x^2 - \frac{3x^3}{2} + 72 - 54x + \frac{27x^2}{2} - \frac{9x^3}{8} \right) dx = k \int_0^4 \left(72 - 54x + \frac{39x^2}{2} - \frac{21x^3}{8} \right) dx \\
 &= k \left[72x - 27x^2 + \frac{13x^3}{2} - \frac{21x^4}{32} \right]_0^4 = k [(288 - 432 + 416 - 168) - 0] = \boxed{104k}.
 \end{aligned}$$



24. The region R enclosed by $3x^2 = y$ and $y = 3x$ is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 3x^2$ and $y = 3x$, $0 \leq x \leq 1$. The lamina is homogeneous; that is, its mass density is $\rho(x, y) = \rho$, a constant. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned}
 M &= \iint_R \rho(x, y) dA = \iint_R \rho dA = \rho \int_0^1 \int_{3x^2}^{3x} dy dx = \rho \int_0^1 [y]_{3x^2}^{3x} dx \\
 &= \rho \int_0^1 (3x - 3x^2) dx = \rho \left[\frac{3x^2}{2} - x^3 \right]_0^1 = \rho \left[\left(\frac{3}{2} - 1 \right) - 0 \right] = \frac{1}{2}\rho.
 \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x\rho dA = \rho \int_0^1 \int_{3x^2}^{3x} x dy dx = \rho \int_0^1 x[y]_{3x^2}^{3x} dx \\ &= \rho \int_0^1 x(3x - 3x^2) dx = \rho \int_0^1 (3x^2 - 3x^3) dx = \rho \left[x^3 - \frac{3x^4}{4} \right]_0^1 = \rho \left[\left(1 - \frac{3}{4}\right) - 0 \right] = \frac{1}{4}\rho, \end{aligned}$$

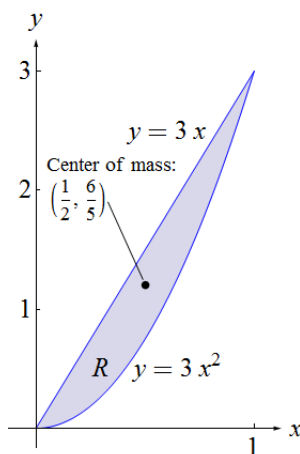
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(1/4)\rho}{(1/2)\rho} = \frac{1}{2}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y\rho dA = \rho \int_0^1 \int_{3x^2}^{3x} y dy dx = \rho \int_0^1 \left[\frac{y^2}{2} \right]_{3x^2}^{3x} dx \\ &= \rho \int_0^1 \left(\frac{9x^2}{2} - \frac{9x^4}{2} \right) dx = \frac{9}{2}\rho \int_0^1 (x^2 - x^4) dx \\ &= \frac{9}{2}\rho \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{9}{2}\rho \left[\left(\frac{1}{3} - \frac{1}{5} \right) - 0 \right] = \frac{9}{2}\rho \cdot \frac{2}{15} = \frac{3}{5}\rho, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{(3/5)\rho}{(1/2)\rho} = \frac{6}{5}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \boxed{\left(\frac{1}{2}, \frac{6}{5}\right)}$.



25. The region R enclosed in the first quadrant by $4x = y^2$, $y = 0$, and $x = 4$ is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = 0$ and $y = 2\sqrt{x}$, $0 \leq x \leq 4$. The lamina is homogeneous; that is, its mass density is $\rho(x, y) = \rho$, a constant. Then the mass M of the lamina can be found as a double integral:

$$\begin{aligned} M &= \iint_R \rho(x, y) dA = \iint_R \rho dA = \rho \int_0^4 \int_0^{2\sqrt{x}} dy dx = \rho \int_0^4 [y]_0^{2\sqrt{x}} dx \\ &= \rho \int_0^4 (2\sqrt{x} - 0) dx = 2\rho \int_0^4 x^{1/2} dx = 2\rho \left[\frac{x^{3/2}}{3/2} \right]_0^4 = \frac{4}{3}\rho \left[(\sqrt{x})^3 \right]_0^4 = \frac{4}{3}\rho(8 - 0) = \frac{32}{3}\rho. \end{aligned}$$

To find the center of mass of the lamina, we need to find the moments of mass about the x - and y -axes. The moment of mass about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\rho(x, y) dA = \iint_R x\rho dA = \rho \int_0^4 \int_0^{2\sqrt{x}} x dy dx = \rho \int_0^4 x[y]_0^{2\sqrt{x}} dx \\ &= \rho \int_0^4 x(2\sqrt{x} - 0) dx = 2\rho \int_0^4 x^{3/2} dx = 2\rho \left[\frac{x^{5/2}}{5/2} \right]_0^4 = \frac{4}{5}\rho \left[(\sqrt{x})^5 \right]_0^4 = \frac{4}{5}\rho(32 - 0) = \frac{128}{5}\rho, \end{aligned}$$

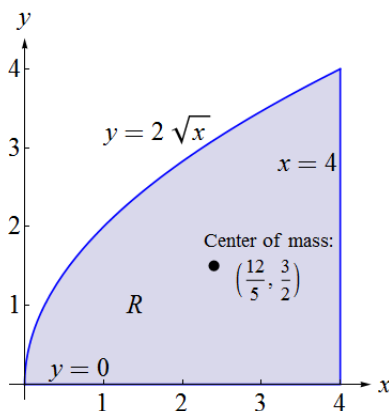
$$\text{so } \bar{x} = \frac{M_y}{M} = \frac{(128/5)\rho}{(32/3)\rho} = \frac{12}{5}.$$

Similarly, the moment of mass about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\rho(x, y) dA = \iint_R y\rho dA = \rho \int_0^4 \int_0^{2\sqrt{x}} y dy dx = \rho \int_0^4 \left[\frac{y^2}{2} \right]_0^{2\sqrt{x}} dx \\ &= \rho \int_0^4 (2x - 0) dx = \rho \int_0^4 2x dx = \rho [x^2]_0^4 = \rho(16 - 0) = 16\rho, \end{aligned}$$

$$\text{so } \bar{y} = \frac{M_x}{M} = \frac{16\rho}{(32/3)\rho} = \frac{3}{2}.$$

The center of mass of this lamina is at the point $(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{2} \right)$.



26. The ellipse $2x^2 + 9y^2 = 18$ representing this lamina is shown in the figure below. Notice that R is an x -simple region: it is bounded between $y = -\sqrt{2\left(1 - \frac{x^2}{9}\right)}$ and $y = \sqrt{2\left(1 - \frac{x^2}{9}\right)}$, $-3 \leq x \leq 3$. Assuming that the lamina is homogeneous of constant mass density ρ , its moment of inertia about the x -axis can be found as a double integral:

$$\begin{aligned} I_x &= \iint_R y^2 \rho(x, y) dA = \iint_R y^2 \rho dA = \rho \int_{-3}^3 \int_{-\sqrt{2(1-x^2/9)}}^{\sqrt{2(1-x^2/9)}} y^2 dy dx = \rho \int_{-3}^3 \left[\frac{y^3}{3} \right]_{-\sqrt{2(1-x^2/9)}}^{\sqrt{2(1-x^2/9)}} dx \\ &= \rho \int_{-3}^3 \frac{1}{3} \left\{ \left[\sqrt{2\left(1 - \frac{x^2}{9}\right)} \right]^3 - \left[-\sqrt{2\left(1 - \frac{x^2}{9}\right)} \right]^3 \right\} dx = \rho \int_{-3}^3 \frac{1}{3} \cdot 2 \left[\sqrt{2\left(1 - \frac{x^2}{9}\right)} \right]^3 dx \\ &= \frac{2}{3}\rho \int_{-3}^3 2\sqrt{2} \left(\sqrt{1 - \frac{x^2}{9}} \right)^3 dx = \frac{4\sqrt{2}}{3}\rho \int_{-3}^3 \left(\sqrt{1 - \frac{x^2}{9}} \right)^3 dx. \end{aligned}$$

To evaluate this integral, we make a substitution $x = 3 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dx = 3 \cos \theta d\theta$. We also change the limits of integration: if $x = -3$, then $-3 = 3 \sin \theta$, so $\sin \theta = -1$ and $\theta = -\frac{\pi}{2}$; if $x = 3$, then $3 = 3 \sin \theta$, so $\sin \theta = 1$ and $\theta = \frac{\pi}{2}$. The radical in the integrand simplifies as

$$\sqrt{1 - \frac{x^2}{9}} = \sqrt{1 - \frac{9 \sin^2 \theta}{9}} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta,$$

where we used the fact that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and so $\cos \theta \geq 0$. So the integral transforms into

$$\begin{aligned} I_x &= \frac{4\sqrt{2}}{3} \rho \int_{-3}^3 \left(\sqrt{1 - \frac{x^2}{9}} \right)^3 dx = \frac{4\sqrt{2}}{3} \rho \int_{-\pi/2}^{\pi/2} (\cos \theta)^3 \cdot 3 \cos \theta d\theta = 4\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta \\ &= 4\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} (\cos^2 \theta)^2 d\theta = 4\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}(1 + \cos 2\theta) \right]^2 d\theta \\ &= 4\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \frac{1}{4}(1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta = \sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \left[1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right] d\theta \\ &= \sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta = \sqrt{2} \rho \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_{-\pi/2}^{\pi/2} \\ &= \sqrt{2} \rho \left[\left(\frac{3}{2} \cdot \frac{\pi}{2} + \sin \pi + \frac{1}{8} \sin 2\pi \right) - \left(\frac{3}{2} \cdot \left(-\frac{\pi}{2} \right) + \sin(-\pi) + \frac{1}{8} \sin(-2\pi) \right) \right] \\ &= \sqrt{2} \rho \left[\left(\frac{3\pi}{4} + 0 + 0 \right) - \left(-\frac{3\pi}{4} + 0 + 0 \right) \right] = \sqrt{2} \rho \cdot \frac{3\pi}{2} = \boxed{\frac{3\sqrt{2}\pi\rho}{2}}. \end{aligned}$$

The moment of inertia of this lamina about the y -axis can be found as a double integral:

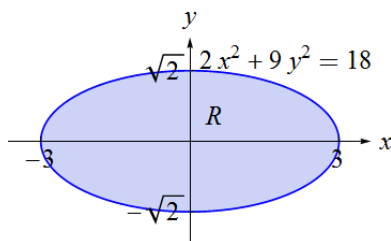
$$\begin{aligned} I_y &= \iint_R x^2 \rho(x, y) dA = \iint_R x^2 \rho dA = \rho \int_{-3}^3 \int_{-\sqrt{2(1-x^2/9)}}^{\sqrt{2(1-x^2/9)}} x^2 dy dx = \rho \int_{-3}^3 x^2 [y]_{-\sqrt{2(1-x^2/9)}}^{\sqrt{2(1-x^2/9)}} dx \\ &= \rho \int_{-3}^3 x^2 \left[\sqrt{2 \left(1 - \frac{x^2}{9} \right)} + \sqrt{2 \left(1 - \frac{x^2}{9} \right)} \right] dx = 2\sqrt{2} \rho \int_{-3}^3 x^2 \sqrt{1 - \frac{x^2}{9}} dx. \end{aligned}$$

Applying the same substitution $x = 3 \sin \theta$ as above (in the computation of I_x), we transform this integral into

$$\begin{aligned} I_y &= 2\sqrt{2} \rho \int_{-3}^3 x^2 \sqrt{1 - \frac{x^2}{9}} dx = 2\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} (3 \sin \theta)^2 \cos \theta \cdot 3 \cos \theta d\theta \\ &= 2\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} 27 \sin^2 \theta \cos^2 \theta d\theta = 54\sqrt{2} \rho \int_{-\pi/2}^{\pi/2} \frac{1}{4} \cdot 4 \sin^2 \theta \cos^2 \theta d\theta = \frac{27\sqrt{2}}{2} \rho \int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta \\ &= \frac{27\sqrt{2}}{2} \rho \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{27\sqrt{2}}{4} \rho \int_{-\pi/2}^{\pi/2} (1 - \cos 4\theta) d\theta = \frac{27\sqrt{2}}{4} \rho \left[\theta - \frac{1}{4} \sin 4\theta \right]_{-\pi/2}^{\pi/2} \\ &= \frac{27\sqrt{2}}{4} \rho \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin 2\pi \right) - \left(\left(-\frac{\pi}{2} \right) - \frac{1}{4} \sin(-2\pi) \right) \right] \\ &= \frac{27\sqrt{2}}{4} \rho \left[\left(\frac{\pi}{2} - 0 \right) - \left(-\frac{\pi}{2} - 0 \right) \right] = \frac{27\sqrt{2}}{4} \rho \cdot \pi = \boxed{\frac{27\sqrt{2}\pi\rho}{4}}. \end{aligned}$$

Finally, the moment of inertia of this lamina about the origin is

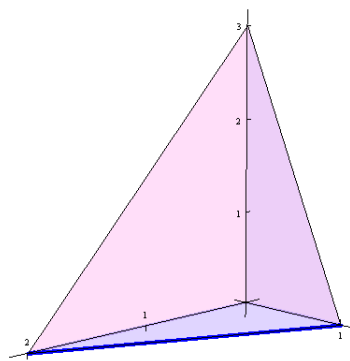
$$I_O = I_x + I_y = \frac{3\sqrt{2}\pi\rho}{2} + \frac{27\sqrt{2}\pi\rho}{4} = \boxed{\frac{33\sqrt{2}\pi\rho}{4}}.$$



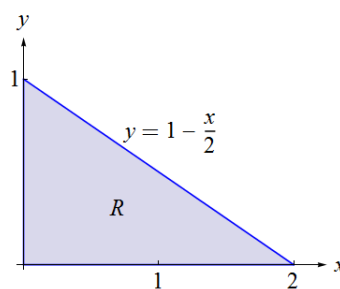
27. The equation of the plane $\frac{x}{2} + y + \frac{z}{3} = 1$ can be rewritten as $z = f(x, y) = 3 - \frac{3x}{2} - 3y$. The first-octant portion of this plane is shown in figure (a) below. It lies over the triangular region R in the xy -plane bounded by the line $\frac{x}{2} + y = 1$, or equivalently $y = 1 - \frac{x}{2}$, and the coordinate axes in the first quadrant; see figure (b) below. In the xy -plane, R is given by $0 \leq y \leq 1 - \frac{x}{2}$, $0 \leq x \leq 2$.

The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = -\frac{3}{2}$ and $f_y(x, y) = -3$. Since both f_x and f_y are continuous on R , the surface area S of the part of the plane that lies in the first octant, that is above the region R , is

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA = \iint_R \sqrt{\left(-\frac{3}{2}\right)^2 + (-3)^2 + 1} dA = \iint_R \sqrt{\frac{49}{4}} dA \\ &= \frac{7}{2} \iint_R dA = \frac{7}{2} \int_0^2 \int_0^{1-x/2} dy dx = \frac{7}{2} \int_0^2 [y]_0^{1-x/2} dx = \frac{7}{2} \int_0^2 \left[\left(1 - \frac{x}{2}\right) - 0 \right] dx \\ &= \frac{7}{2} \int_0^2 \left(1 - \frac{x}{2}\right) dx = \frac{7}{2} \left[x - \frac{x^2}{4} \right]_0^2 = \frac{7}{2} [(2 - 1) - 0] = \boxed{\frac{7}{2} \text{ square units}}. \end{aligned}$$



(a)



(b)

28. The part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$ is shown in figure (a) below. If we substitute $z = 1$ into the equation of the paraboloid, we obtain $x^2 + y^2 = 1$. So the projection of this part of the paraboloid onto the xy -plane is the circular region R shown in figure (b) below; in polar coordinates R is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

The partial derivatives of $z = f(x, y) = x^2 + y^2$ are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$. Since both f_x and

f_y are continuous on R , the surface area S we seek is

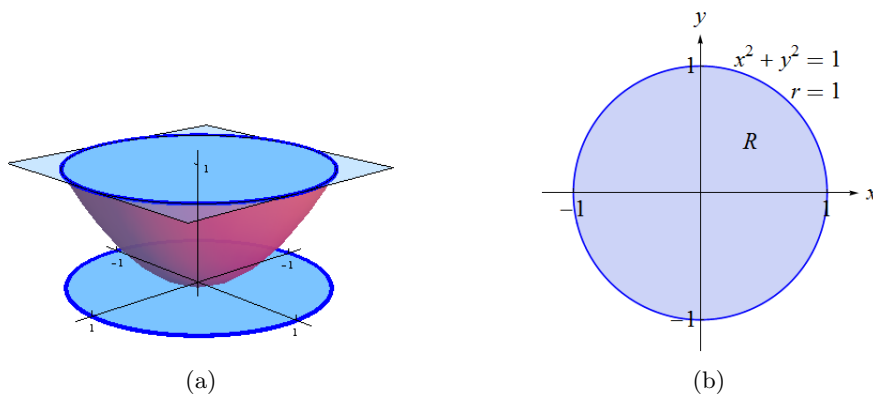
$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1} \, dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA = \iint_R \sqrt{4(x^2 + y^2) + 1} \, dA = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta. \end{aligned}$$

To evaluate the “inside” integral $\int_0^1 \sqrt{4r^2 + 1} \cdot r \, dr$, we make a substitution $u = 4r^2 + 1$; then $du = 8r \, dr$, so $r \, dr = \frac{1}{8} du$. We also change the limits of integration: if $r = 0$, then $u = 4 \cdot 0^2 + 1 = 1$; if $r = 1$, then $u = 4 \cdot 1^2 + 1 = 5$. So

$$\int_0^1 \sqrt{4r^2 + 1} \cdot r \, dr = \int_1^5 \sqrt{u} \cdot \frac{1}{8} \, du = \frac{1}{8} \int_1^5 u^{1/2} \, du = \frac{1}{8} \left[\frac{u^{3/2}}{3/2} \right]_1^5 = \frac{1}{12} \left[(\sqrt{u})^3 \right]_1^5 = \frac{5\sqrt{5} - 1}{12}.$$

Going back to the calculation of S ,

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \cdot r \, dr \, d\theta = \int_0^{2\pi} \frac{5\sqrt{5} - 1}{12} \, d\theta \\ &= \frac{5\sqrt{5} - 1}{12} \cdot [\theta]_0^{2\pi} = \frac{5\sqrt{5} - 1}{12} \cdot (2\pi - 0) = \boxed{\frac{(5\sqrt{5} - 1)\pi}{6} \text{ square units}}. \end{aligned}$$

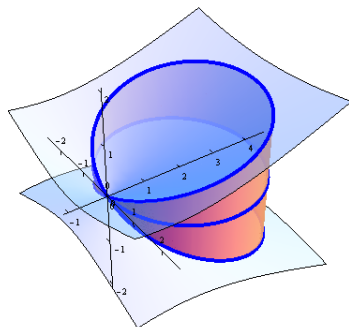


29. The part of the cone $x^2 + y^2 = 3z^2$ that lies inside the cylinder $x^2 + y^2 = 4y$ is shown in figure (a) below. Notice that it is symmetric about the xy -plane, so to find its surface area we can find the surface area of the part lying above the xy -plane and multiply it by 2. The upper nappe of this cone can be thought of as the graph of the function $z = f(x, y) = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2}$. It lies over the region R in the xy -plane that is the trace of the cylinder in this plane: R is defined by $x^2 + y^2 \leq 4y$ in Cartesian coordinates, and it is a disk of radius 2 centered at the point $(0, 2)$; see figure (b) below.

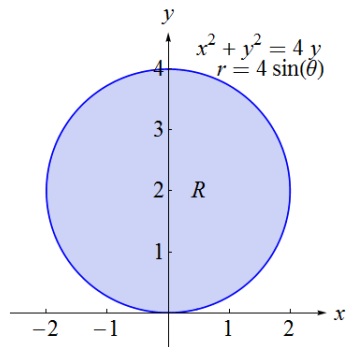
The partial derivatives of $z = f(x, y)$ are $f_x(x, y) = \frac{x}{\sqrt{3}\sqrt{x^2 + y^2}}$ and $f_y(x, y) = \frac{y}{\sqrt{3}\sqrt{x^2 + y^2}}$. Then

the surface area S we seek is

$$\begin{aligned}
 S &= 2 \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA \\
 &= 2 \iint_R \sqrt{\left(\frac{x}{\sqrt{3}\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{3}\sqrt{x^2 + y^2}}\right)^2 + 1} dA \\
 &= 2 \iint_R \sqrt{\frac{x^2}{3(x^2 + y^2)} + \frac{y^2}{3(x^2 + y^2)} + 1} dA = 2 \iint_R \sqrt{\frac{x^2 + y^2}{3(x^2 + y^2)} + 1} dA \\
 &= 2 \iint_R \sqrt{\frac{1}{3} + 1} dA = 2 \iint_R \sqrt{\frac{4}{3}} dA = \frac{4}{\sqrt{3}} \iint_R dA \\
 &= \frac{4\sqrt{3}}{3} \cdot (\text{Area of the disk } R) = \frac{4\sqrt{3}}{3} \cdot (\pi \cdot 2^2) = \boxed{\frac{16\sqrt{3}\pi}{3} \text{ square units}}.
 \end{aligned}$$



(a)



(b)

30. Since the function $f(x, y, z) = e^z \cos x$ is continuous on the solid E defined by $0 \leq x \leq \frac{\pi}{2}$, $0 \leq y \leq 2$, $0 \leq z \leq 1$, we can use Fubini's Theorem for triple integrals:

$$\begin{aligned}
 \iiint_E e^z \cos x dV &= \int_0^{\pi/2} \int_0^2 \int_0^1 e^z \cos x dz dy dx = \int_0^{\pi/2} \int_0^2 \cos x \cdot [e^z]_0^1 dy dx \\
 &= \int_0^{\pi/2} \int_0^2 \cos x \cdot (e^1 - e^0) dy dx = (e - 1) \int_0^{\pi/2} \int_0^2 \cos x dy dx \\
 &= (e - 1) \int_0^{\pi/2} \cos x \cdot [y]_0^2 dx = (e - 1) \int_0^{\pi/2} \cos x \cdot (2 - 0) dx \\
 &= 2(e - 1) \int_0^{\pi/2} \cos x dx = 2(e - 1) \cdot [\sin x]_0^{\pi/2} \\
 &= 2(e - 1) \cdot \left(\sin \frac{\pi}{2} - \sin 0\right) = 2(e - 1) \cdot (1 - 0) = \boxed{2(e - 1)}.
 \end{aligned}$$

31. We evaluate the given iterated integral as follows:

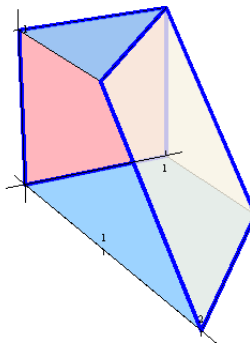
$$\begin{aligned}
 \int_0^1 \int_0^{2-3x} \int_0^{2y} x^2 dz dy dx &= \int_0^1 \int_0^{2-3x} x^2 [z]_0^{2y} dy dx = \int_0^1 \int_0^{2-3x} x^2 (2y - 0) dy dx \\
 &= \int_0^1 \int_0^{2-3x} x^2 \cdot 2y dy dx = \int_0^1 x^2 [y^2]_0^{2-3x} dx \\
 &= \int_0^1 x^2 [(2-3x)^2 - 0] dx = \int_0^1 x^2 (4 - 12x + 9x^2) dx \\
 &= \int_0^1 (4x^2 - 12x^3 + 9x^4) dx = \left[\frac{4x^3}{3} - 3x^4 + \frac{9x^5}{5} \right]_0^1 \\
 &= \left(\frac{4}{3} - 3 + \frac{9}{5} \right) - 0 = \boxed{\frac{2}{15}}.
 \end{aligned}$$

32. Since the function $f(x, y, z) = ye^x$ is continuous on the solid E defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $y^2 \leq z \leq y$, we can use Fubini's Theorem for triple integrals:

$$\begin{aligned}
 \iiint_E ye^x dV &= \int_0^1 \int_0^1 \int_{y^2}^y ye^x dz dy dx = \int_0^1 \int_0^1 ye^x \cdot [z]_{y^2}^y dy dx = \int_0^1 \int_0^1 ye^x \cdot (y - y^2) dy dx \\
 &= \int_0^1 \int_0^1 e^x (y^2 - y^3) dy dx = \int_0^1 e^x \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 dx = \int_0^1 e^x \left[\left(\frac{1}{3} - \frac{1}{4} \right) - 0 \right] dx \\
 &= \frac{1}{12} \int_0^1 e^x dx = \frac{1}{12} [e^x]_0^1 = \frac{1}{12} (e^1 - e^0) = \boxed{\frac{e-1}{12}}.
 \end{aligned}$$

33. We begin by graphing the solid E enclosed by $x = 0$ and $x = 2 - y - z$ whose projection onto the yz -plane is the region enclosed by the rectangle $0 \leq y \leq 1$, $0 \leq z \leq 1$; see the figure below. Since the function $f(x, y, z) = xyz$ is continuous on E , we can use Fubini's Theorem for triple integrals:

$$\begin{aligned}
 \iiint_E xyz dV &= \int_0^1 \int_0^1 \int_0^{2-y-z} xyz dx dy dz = \int_0^1 \int_0^1 yz \cdot \left[\frac{x^2}{2} \right]_0^{2-y-z} dy dz \\
 &= \frac{1}{2} \int_0^1 \int_0^1 yz \cdot [(2-y-z)^2 - 0] dy dz \\
 &= \frac{1}{2} \int_0^1 \int_0^1 yz (4 + y^2 + z^2 - 4y - 4z + 2yz) dy dz \\
 &= \frac{1}{2} \int_0^1 \int_0^1 (4yz + y^3z + yz^3 - 4y^2z - 4yz^2 + 2y^2z^2) dy dz \\
 &= \frac{1}{2} \int_0^1 \left[2y^2z + \frac{1}{4}y^4z + \frac{1}{2}y^2z^3 - \frac{4}{3}y^3z - 2y^2z^2 + \frac{2}{3}y^3z^2 \right]_0^1 dz \\
 &= \frac{1}{2} \int_0^1 \left[\left(2z + \frac{1}{4}z + \frac{1}{2}z^3 - \frac{4}{3}z - 2z^2 + \frac{2}{3}z^2 \right) - 0 \right] dz \\
 &= \frac{1}{2} \int_0^1 \left(\frac{11}{12}z - \frac{4}{3}z^2 + \frac{1}{2}z^3 \right) dz = \frac{1}{2} \left[\frac{11}{24}z^2 - \frac{4}{9}z^3 + \frac{1}{8}z^4 \right]_0^1 \\
 &= \frac{1}{2} \left[\left(\frac{11}{24} - \frac{4}{9} + \frac{1}{8} \right) - 0 \right] = \frac{1}{2} \cdot \frac{5}{36} = \boxed{\frac{5}{72}}.
 \end{aligned}$$



34. First, we need to find the mass of the given solid E . In spherical coordinates, this hemispherical shell is described by $a \leq r \leq b$, $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$. Since it is homogeneous, its mass density is $\rho(x, y, z) = \delta$, a constant. Then the mass M of the solid E is

$$\begin{aligned}
 M &= \iiint_E \rho(x, y, z) dV = \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \delta \cdot \rho^2 \sin \phi d\rho d\phi d\theta = \delta \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^2 \sin \phi d\rho d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_a^b d\phi d\theta = \delta \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cdot \left(\frac{b^3}{3} - \frac{a^3}{3} \right) d\phi d\theta \\
 &= \frac{\delta(b^3 - a^3)}{3} \int_0^{2\pi} \int_0^{\pi/2} \sin \phi d\phi d\theta = \frac{\delta(b^3 - a^3)}{3} \int_0^{2\pi} [-\cos \phi]_0^{\pi/2} d\theta \\
 &= \frac{\delta(b^3 - a^3)}{3} \int_0^{2\pi} \left(-\cos \frac{\pi}{2} + \cos 0 \right) d\theta = \frac{\delta(b^3 - a^3)}{3} \int_0^{2\pi} (-0 + 1) d\theta = \frac{\delta(b^3 - a^3)}{3} \int_0^{2\pi} d\theta \\
 &= \frac{\delta(b^3 - a^3)}{3} [\theta]_0^{2\pi} = \frac{\delta(b^3 - a^3)}{3} (2\pi - 0) = \frac{2\pi\delta(b^3 - a^3)}{3}.
 \end{aligned}$$

The homogeneous solid E is symmetric about both yz - and xz -planes, so $\bar{x} = \bar{y} = 0$. The moment about the xy -plane is

$$\begin{aligned}
 M_{xy} &= \iiint_E z\rho(x, y, z) dV = \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho \cos \phi \cdot \delta \cdot \rho^2 \sin \phi d\rho d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^3 \sin \phi \cos \phi d\rho d\phi d\theta = \delta \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos \phi \cdot \left[\frac{\rho^4}{4} \right]_a^b d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos \phi \cdot \left(\frac{b^4}{4} - \frac{a^4}{4} \right) d\phi d\theta = \frac{\delta(b^4 - a^4)}{4} \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos \phi d\phi d\theta \\
 &= \frac{\delta(b^4 - a^4)}{4} \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{2} \sin 2\phi d\phi d\theta = \frac{\delta(b^4 - a^4)}{4} \int_0^{2\pi} \frac{1}{4} [-\cos 2\phi]_0^{\pi/2} d\theta \\
 &= \frac{\delta(b^4 - a^4)}{16} \int_0^{2\pi} (-\cos \pi + \cos 0) d\theta = \frac{\delta(b^4 - a^4)}{16} \int_0^{2\pi} (1 + 1) d\theta = \frac{\delta(b^4 - a^4)}{8} \int_0^{2\pi} d\theta \\
 &= \frac{\delta(b^4 - a^4)}{8} [\theta]_0^{2\pi} = \frac{\delta(b^4 - a^4)}{8} (2\pi - 0) = \frac{\pi\delta(b^4 - a^4)}{4},
 \end{aligned}$$

so $\bar{z} = \frac{M_{xy}}{M} = \frac{\pi\delta(b^4 - a^4)/4}{2\pi\delta(b^3 - a^3)/3} = \frac{3(b^4 - a^4)}{8(b^3 - a^3)}$, and the center of mass is $\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3(b^4 - a^4)}{8(b^3 - a^3)} \right)}$.

Due to the symmetry of the hemispherical shell E , its moments of inertia about the x - and y -axes are the same. Using the fact that the distance from a point (x, y, z) to the x -axis is $r = \sqrt{y^2 + z^2}$, and so $r^2 = y^2 + z^2 = (\rho \sin \phi \cos \theta)^2 + (\rho \cos \phi)^2 = \rho^2(\sin^2 \phi \cos^2 \theta + \cos^2 \phi)$, we find that the moment of

inertia about the x -axis is

$$\begin{aligned}
 I_x &= \iiint_E r^2 \rho(x, y, z) dV = \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^2 (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \cdot \delta \cdot \rho^2 \sin \phi d\rho d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} \int_a^b \rho^4 (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi d\rho d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi \cdot \left[\frac{\rho^5}{5} \right]_a^b d\phi d\theta \\
 &= \delta \int_0^{2\pi} \int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi \cdot \left(\frac{b^5}{5} - \frac{a^5}{5} \right) d\phi d\theta \\
 &= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi d\phi d\theta.
 \end{aligned}$$

Now, to evaluate the “inside” integral $\int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi d\phi$, which must be integrated partially with respect to ϕ , we make a substitution $u = \cos \phi$; then $du = -\sin \phi d\phi$, so $\sin \phi d\phi = -du$. Using a trigonometric identity, in the integrand we can rewrite $\sin^2 \phi = 1 - \cos^2 \phi = 1 - u^2$. We also change the limits of integration: if $\phi = 0$, then $u = \cos 0 = 1$; if $\phi = \frac{\pi}{2}$, then $u = \cos \frac{\pi}{2} = 0$. So

$$\begin{aligned}
 \int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi d\phi &= \int_1^0 [(1 - u^2) \cos^2 \theta + u^2] (-du) \\
 &= - \int_1^0 [\cos^2 \theta + u^2(1 - \cos^2 \theta)] du \\
 &= \int_0^1 (\cos^2 \theta + u^2 \sin^2 \theta) du = \left[u \cos^2 \theta + \frac{u^3}{3} \sin^2 \theta \right]_0^1 \\
 &= \left(\cos^2 \theta + \frac{1}{3} \sin^2 \theta \right) - (0 + 0) = \cos^2 \theta + \frac{1}{3} \sin^2 \theta.
 \end{aligned}$$

Going back to the computation of I_x ,

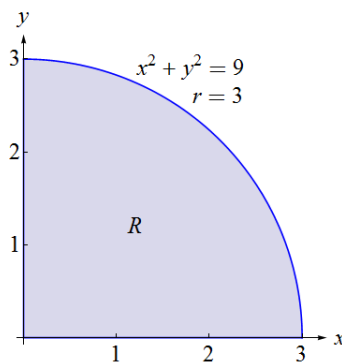
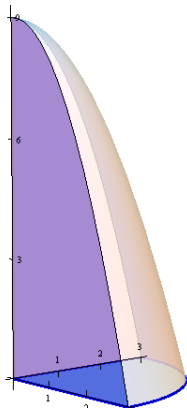
$$\begin{aligned}
 I_x &= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi/2} (\sin^2 \phi \cos^2 \theta + \cos^2 \phi) \sin \phi d\phi d\theta \\
 &= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left(\cos^2 \theta + \frac{1}{3} \sin^2 \theta \right) d\theta \\
 &= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left[\frac{1}{2}(1 + \cos 2\theta) + \frac{1}{3} \cdot \frac{1}{2}(1 - \cos 2\theta) \right] d\theta \\
 &= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \left(\frac{2}{3} + \frac{1}{3} \cos 2\theta \right) d\theta = \frac{\delta(b^5 - a^5)}{5} \left[\frac{2}{3}\theta + \frac{1}{6} \sin 2\theta \right]_0^{2\pi} \\
 &= \frac{\delta(b^5 - a^5)}{5} \left[\left(\frac{2}{3} \cdot 2\pi + \frac{1}{6} \sin 4\pi \right) - \left(\frac{2}{3} \cdot 0 + \frac{1}{6} \sin 0 \right) \right] = \frac{\delta(b^5 - a^5)}{5} \cdot \frac{4\pi}{3} = \frac{4\pi\delta(b^5 - a^5)}{15}.
 \end{aligned}$$

$$\text{So } I_x = I_y = \boxed{\frac{4\pi\delta(b^5 - a^5)}{15}}.$$

35. The part of the paraboloid $x^2 + y^2 + z = 9$ lying over the first quadrant and its projection onto the xy -plane are shown in the figures below. Since the equation of the paraboloid involves $x^2 + y^2$, it is convenient to switch to cylindrical coordinates: the given solid is bounded between $z = 0$ and

$z = 9 - (x^2 + y^2) = 9 - r^2$, where $0 \leq r \leq 3$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the volume is

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^{\pi/2} \int_0^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 r[z]_0^{9-r^2} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^3 r[(9-r^2) - 0] \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 (9r - r^3) \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 d\theta \\
 &= \int_0^{\pi/2} \left[\left(\frac{81}{2} - \frac{81}{4} \right) - (0 - 0) \right] d\theta = \int_0^{\pi/2} \frac{81}{4} d\theta = \frac{81}{4} [\theta]_0^{\pi/2} = \frac{81}{4} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{81\pi}{8} \text{ cubic units}}.
 \end{aligned}$$



36. The solid E enclosed in the first quadrant by $y = 0$, $z = 0$, $x + y = 2$, $x + 2y = 6$, and $y^2 + z^2 = 4$ is shown in figure (a) below; it is bounded between $z = 0$ and $z = \sqrt{4 - y^2}$. Its projection onto the xy -plane is the region R shown in figure (b) below; in the xy -plane it is bounded by the lines $x = 2 - y$ and $x = 6 - 2y$, where $0 \leq y \leq 2$. Then the volume V of the solid E is

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^2 \int_{2-y}^{6-2y} \int_0^{\sqrt{4-y^2}} dz \, dx \, dy = \int_0^2 \int_{2-y}^{6-2y} [z]_0^{\sqrt{4-y^2}} dx \, dy \\
 &= \int_0^2 \int_{2-y}^{6-2y} (\sqrt{4-y^2} - 0) dx \, dy = \int_0^2 \int_{2-y}^{6-2y} \sqrt{4-y^2} dx \, dy = \int_0^2 \sqrt{4-y^2} [x]_{2-y}^{6-2y} dy \\
 &= \int_0^2 \sqrt{4-y^2} [(6-2y) - (2-y)] dy = \int_0^2 \sqrt{4-y^2} (4-y) dy \\
 &= 4 \int_0^2 \sqrt{4-y^2} dy - \int_0^2 \sqrt{4-y^2} y dy.
 \end{aligned}$$

To evaluate the first integral, we make a substitution $y = 2 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; then $dy = 2 \cos \theta d\theta$. Using a trigonometric identity, we can rewrite the integrand as $\sqrt{4-y^2} = \sqrt{4-4\sin^2 \theta} = \sqrt{4(1-\sin^2 \theta)} = \sqrt{4\cos^2 \theta} = 2 \cos \theta$, where we used the fact that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and so $\cos \theta \geq 0$. We also change the limits of integration: if $y = 0$, then $2 \sin \theta = 0$ or $\sin \theta = 0$ and so $\theta = 0$; if $y = 2$, then $2 \sin \theta = 2$ or $\sin \theta = 1$ and so $\theta = \frac{\pi}{2}$. So the first integral becomes

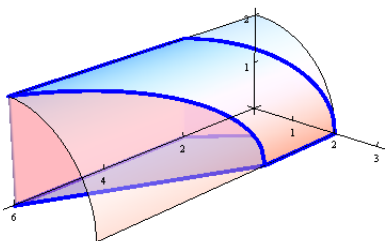
$$\begin{aligned}
 4 \int_0^2 \sqrt{4-y^2} dy &= 4 \int_0^{\pi/2} 2 \cos \theta \cdot 2 \cos \theta d\theta = 16 \int_0^{\pi/2} \cos^2 \theta d\theta \\
 &= 16 \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) d\theta = 8 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= 8 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 8 \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right] = 8 \cdot \frac{\pi}{2} = 4\pi.
 \end{aligned}$$

To evaluate the second integral, we make a substitution $u = 4 - y^2$; then $du = -2y dy$, so $y dy = \frac{du}{-2}$. We also change the limits of integration: if $y = 0$, then $u = 4 - 0^2 = 4$; if $y = 2$, then $u = 4 - 2^2 = 0$. So the second integral becomes

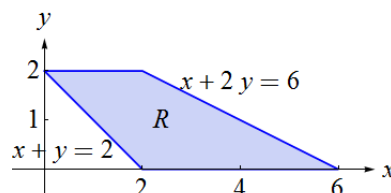
$$\begin{aligned} \int_0^2 \sqrt{4-y^2} y dy &= \int_4^0 \sqrt{u} \left(\frac{du}{-2} \right) = -\frac{1}{2} \int_4^0 \sqrt{u} du = \frac{1}{2} \int_0^4 u^{1/2} du \\ &= \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_0^4 = \frac{1}{3} \left[(\sqrt{u})^3 \right]_0^4 = \frac{1}{3} \left[(\sqrt{4})^3 - (\sqrt{0})^3 \right] = \frac{1}{3}(8-0) = \frac{8}{3}. \end{aligned}$$

So the final answer is

$$V = 4 \int_0^2 \sqrt{4-y^2} dy - \int_0^2 \sqrt{4-y^2} y dy = \boxed{4\pi - \frac{8}{3} \text{ cubic units}}.$$



(a)



(b)

37. (a) Since the z -coordinate in rectangular coordinates is 4, it is the z -coordinate in cylindrical coordinates as well. The value of r is $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$. In the xy -plane, the point $(3, 0, 0)$ lies on the positive half of the x -axis, so $\theta = 0$. So the cylindrical coordinates of the cartesian point $(3, 0, 4)$ are $(r, \theta, z) = \boxed{(3, 0, 4)}$.
- (b) Since θ in cylindrical coordinates is 0, that is the value of θ in spherical coordinates as well. The value of ρ is $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{25} = 5$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{4}{5}$, so $\phi = \cos^{-1} \left(\frac{4}{5} \right)$. So the spherical coordinates of the cartesian point $(3, 0, 4)$ are $(\rho, \theta, \phi) = \boxed{\left(5, 0, \cos^{-1} \left(\frac{4}{5} \right) \right)}$.
38. (a) Since the z -coordinate in rectangular coordinates is 3, it is the z -coordinate in cylindrical coordinates as well. The value of r is $r = \sqrt{x^2 + y^2} = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$. In the xy -plane, the point $(-2\sqrt{2}, 2\sqrt{2}, 0)$ lies in the second quadrant; and since $\tan \theta = \frac{y}{x} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1$, then $\theta = \frac{3\pi}{4}$. So the cylindrical coordinates of the cartesian point $(-2\sqrt{2}, 2\sqrt{2}, 3)$ are $(r, \theta, z) = \boxed{\left(4, \frac{3\pi}{4}, 3 \right)}$.
- (b) Since θ in cylindrical coordinates is $\frac{3\pi}{4}$, that is the value of θ in spherical coordinates as well. The value of ρ is $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + 3^2} = \sqrt{25} = 5$. Finally, $\cos \phi =$

$\frac{z}{\rho} = \frac{3}{5}$, so $\phi = \cos^{-1}\left(\frac{3}{5}\right)$. So the spherical coordinates of the cartesian point $(-2\sqrt{2}, 2\sqrt{2}, 3)$ are $(\rho, \theta, \phi) = \left(5, \frac{3\pi}{4}, \cos^{-1}\left(\frac{3}{5}\right)\right)$.

39. (a) Since the z -coordinate in rectangular coordinates is -2 , it is the z -coordinate in cylindrical coordinates as well. The value of r is $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. In the xy -plane, the point $(-1, 1, 0)$ lies in the second quadrant; and since $\tan \theta = \frac{y}{x} = \frac{1}{-1} = -1$, then $\theta = \frac{3\pi}{4}$. So the cylindrical coordinates of the cartesian point $(-1, 1, -2)$ are $(r, \theta, z) = \left(\sqrt{2}, \frac{3\pi}{4}, -2\right)$.

- (b) Since θ in cylindrical coordinates is $\frac{3\pi}{4}$, that is the value of θ in spherical coordinates as well. The value of ρ is $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{-2}{\sqrt{6}}$, so $\phi = \cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)$. So the spherical coordinates of the cartesian point $(-1, 1, -2)$ are $(\rho, \theta, \phi) = \left(\sqrt{6}, \frac{3\pi}{4}, \cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)\right)$.

40. (a) Since the z -coordinate in rectangular coordinates is 4 , it is the z -coordinate in cylindrical coordinates as well. The value of r is $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. In the xy -plane, the point $(1, \sqrt{3}, 0)$ lies in the first quadrant; and since $\tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$, then $\theta = \frac{\pi}{3}$. So the cylindrical coordinates of the cartesian point $(1, \sqrt{3}, 4)$ are $(r, \theta, z) = \left(2, \frac{\pi}{3}, 4\right)$.

- (b) Since θ in cylindrical coordinates is $\frac{\pi}{3}$, that is the value of θ in spherical coordinates as well. The value of ρ is $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (\sqrt{3})^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. Finally, $\cos \phi = \frac{z}{\rho} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$, so $\phi = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$. So the spherical coordinates of the cartesian point $(1, \sqrt{3}, 4)$ are $(\rho, \theta, \phi) = \left(2\sqrt{5}, \frac{\pi}{3}, \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$.

41. In spherical coordinates, the solid E between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$ is described by $1 \leq \rho \leq 2$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$. Then the volume V of the solid E is

$$\begin{aligned} V &= \iiint_E dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \sin \phi \cdot \left[\frac{\rho^3}{3}\right]_1^2 d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^\pi \sin \phi \cdot \left(\frac{8}{3} - \frac{1}{3}\right) d\phi \, d\theta = \frac{7}{3} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta = \frac{7}{3} \int_0^{2\pi} [-\cos \phi]_0^\pi d\theta \\ &= \frac{7}{3} \int_0^{2\pi} (-\cos \pi + \cos 0) d\theta = \frac{7}{3} \int_0^{2\pi} (1 + 1) d\theta = \frac{14}{3} \int_0^{2\pi} d\theta \\ &= \frac{14}{3} [\theta]_0^{2\pi} = \frac{14}{3} (2\pi - 0) = \boxed{\frac{28\pi}{3} \text{ cubic units}}. \end{aligned}$$

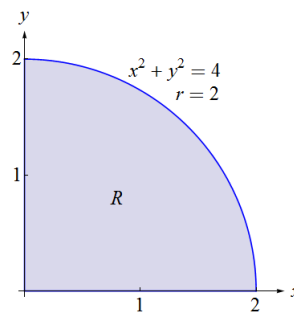
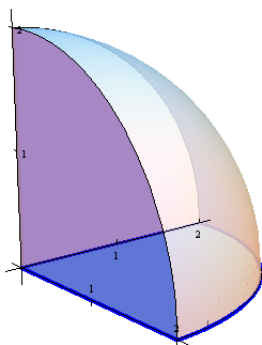
42. In spherical coordinates, the solid E cut from the sphere $\rho = 4$ by the cone $\phi = \frac{\pi}{4}$ is described by

$0 \leq \rho \leq 4$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{4}$. Then the volume V of the solid E is

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_0^4 d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cdot \left(\frac{64}{3} - 0 \right) d\phi \, d\theta = \frac{64}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \, d\phi \, d\theta = \frac{64}{3} \int_0^{2\pi} [-\cos \phi]_0^{\pi/4} d\theta \\
 &= \frac{64}{3} \int_0^{2\pi} \left(-\cos \frac{\pi}{4} + \cos 0 \right) d\theta = \frac{64}{3} \int_0^{2\pi} \left(-\frac{\sqrt{2}}{2} + 1 \right) d\theta = \frac{64}{3} \cdot \frac{2 - \sqrt{2}}{2} \int_0^{2\pi} d\theta \\
 &= \frac{32(2 - \sqrt{2})}{3} \cdot [\theta]_0^{2\pi} = \frac{32(2 - \sqrt{2})}{3} \cdot (2\pi - 0) = \boxed{\frac{64(2 - \sqrt{2})\pi}{3} \text{ cubic units}}.
 \end{aligned}$$

43. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $0 \leq z \leq \sqrt{4 - x^2 - y^2}$, $0 \leq y \leq \sqrt{4 - x^2}$, $0 \leq x \leq 2$; see the figures below. The limits of integration on z are $z = 0$, which is the xy -plane, and $z = \sqrt{4 - x^2 - y^2}$, or equivalently $x^2 + y^2 + z^2 = 4$, $z \geq 0$. So the solid E of integration is the portion of the ball of radius 2 lying in the first octant. In cylindrical coordinates, E is given by $0 \leq z \leq \sqrt{4 - r^2}$, $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the given integral can be converted to cylindrical coordinates as follows:

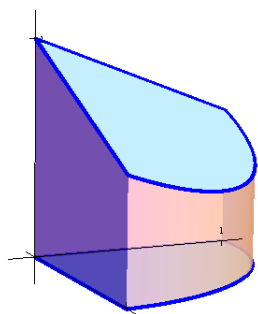
$$\begin{aligned}
 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{z}{\sqrt{x^2+y^2}} \, dz \, dy \, dx &= \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} \frac{z}{r} \cdot r \, dz \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} z \, dz \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^2 \left[\frac{z^2}{2} \right]_0^{\sqrt{4-r^2}} dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^2 \left(\frac{4-r^2}{2} - 0 \right) dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^2 \left(2 - \frac{r^2}{2} \right) dr \, d\theta = \int_0^{\pi/2} \left[2r - \frac{r^3}{6} \right]_0^2 d\theta \\
 &= \int_0^{\pi/2} \left[\left(4 - \frac{8}{6} \right) - 0 \right] d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta \\
 &= \frac{8}{3} [\theta]_0^{\pi/2} = \frac{8}{3} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{4\pi}{3}}.
 \end{aligned}$$



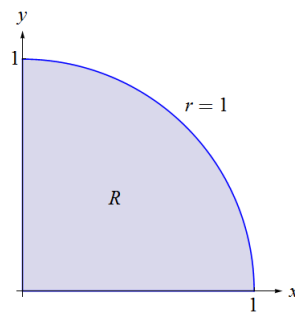
44. The solid E that is the part of the cylinder $r = 1$ lying in the first octant and below the plane $3x + 2y + 6z = 6$ is shown in figure (a) below. It is bounded between the surfaces $z = 0$ and

$z = 1 - \frac{x}{2} - \frac{y}{3} = 1 - \frac{r \cos \theta}{2} - \frac{r \sin \theta}{3}$. Its projection R onto the xy -plane is a quarter of a disk, as shown in figure (b) below; in cylindrical coordinates it is given by $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the volume V of the solid E is

$$\begin{aligned}
 V &= \iiint_E dV = \int_0^{\pi/2} \int_0^1 \int_0^{1-r \cos \theta/2 - r \sin \theta/3} r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r[z]_0^{1-r \cos \theta/2 - r \sin \theta/3} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^1 r \left[\left(1 - \frac{r \cos \theta}{2} - \frac{r \sin \theta}{3} \right) - 0 \right] \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 \left(r - \frac{r^2 \cos \theta}{2} - \frac{r^2 \sin \theta}{3} \right) \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3 \cos \theta}{6} - \frac{r^3 \sin \theta}{9} \right]_0^1 \, d\theta = \int_0^{\pi/2} \left[\left(\frac{1}{2} - \frac{\cos \theta}{6} - \frac{\sin \theta}{9} \right) - (0 - 0 - 0) \right] \, d\theta \\
 &= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos \theta}{6} - \frac{\sin \theta}{9} \right) \, d\theta = \left[\frac{\theta}{2} - \frac{\sin \theta}{6} + \frac{\cos \theta}{9} \right]_0^{\pi/2} \\
 &= \left(\frac{\pi}{4} - \frac{\sin \frac{\pi}{2}}{6} + \frac{\cos \frac{\pi}{2}}{9} \right) - \left(0 - \frac{\sin 0}{6} + \frac{\cos 0}{9} \right) = \left(\frac{\pi}{4} - \frac{1}{6} + 0 \right) - \left(0 - 0 + \frac{1}{9} \right) \\
 &= \boxed{\frac{\pi}{4} - \frac{5}{18} \text{ cubic units}}.
 \end{aligned}$$



(a)



(b)

45. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $0 \leq z \leq \sqrt{a^2 - x^2 - y^2}$, $0 \leq y \leq \sqrt{a^2 - x^2}$, $0 \leq x \leq a$; see the figures below. The limits of integration on z are $z = 0$, which is the xy -plane, and $z = \sqrt{a^2 - x^2 - y^2}$, or equivalently $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. So the solid E of integration is the portion of the ball of radius a lying in the first octant. In cylindrical coordinates, E is given by $0 \leq z \leq \sqrt{a^2 - r^2}$, $0 \leq r \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the given integral can be converted to cylindrical coordinates as follows:

$$\begin{aligned}
 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} dz \, dy \, dx &= \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2 - r^2}} r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^a r[z]_0^{\sqrt{a^2 - r^2}} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^a r \left(\sqrt{a^2 - r^2} - 0 \right) \, dr \, d\theta = \int_0^{\pi/2} \int_0^a r \sqrt{a^2 - r^2} \, dr \, d\theta.
 \end{aligned}$$

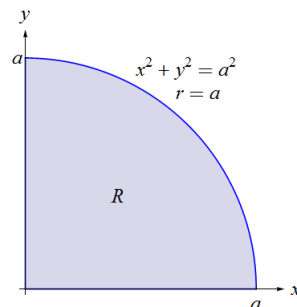
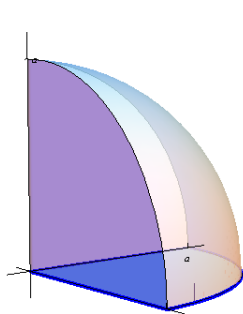
To evaluate the “inside” integral $\int_0^a r \sqrt{a^2 - r^2} \, dr$, we make a substitution $u = a^2 - r^2$; then $du = -2r \, dr$, so $r \, dr = \frac{du}{-2}$. We also change the limits of integration: if $r = 0$, then $u = a^2 - 0^2 = a^2$; if $r = a$, then

$u = a^2 - r^2 = 0$. So the second integral becomes

$$\begin{aligned} \int_0^a r \sqrt{a^2 - r^2} dr &= \int_{a^2}^0 \sqrt{u} \left(\frac{du}{-2} \right) = -\frac{1}{2} \int_{a^2}^0 \sqrt{u} du = \frac{1}{2} \int_0^{a^2} u^{1/2} du \\ &= \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_0^{a^2} = \frac{1}{3} \left[(\sqrt{u})^3 \right]_0^{a^2} = \frac{1}{3} \left[(\sqrt{a^2})^3 - (\sqrt{0})^3 \right] = \frac{1}{3} (a^3 - 0) = \frac{a^3}{3}. \end{aligned}$$

Going back to the double integral, we obtain

$$\int_0^{\pi/2} \int_0^a r \sqrt{a^2 - r^2} dr d\theta = \int_0^{\pi/2} \frac{a^3}{3} d\theta = \frac{a^3}{3} [\theta]_0^{\pi/2} = \frac{a^3}{3} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi a^3}{6}}.$$



46. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $0 \leq z \leq \sqrt{a^2 - x^2 - y^2}$, $0 \leq y \leq \sqrt{a^2 - x^2}$, $0 \leq x \leq a$; see the figures in the solution to Problem 45 above. The limits of integration on z are $z = 0$, which is the xy -plane, and $z = \sqrt{a^2 - x^2 - y^2}$, or equivalently $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. So the solid E of integration is the portion of the ball of radius a lying in the first octant. In spherical coordinates, E is given by $0 \leq \rho \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$. Then the given integral can be converted to spherical coordinates as follows:

$$\begin{aligned} \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} dz dy dx &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_0^a d\phi d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \cdot \left(\frac{a^3}{3} - 0 \right) d\phi d\theta \\ &= \frac{a^3}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi d\phi d\theta = \frac{a^3}{3} \int_0^{\pi/2} [-\cos \phi]_0^{\pi/2} d\theta \\ &= \frac{a^3}{3} \int_0^{\pi/2} \left(-\cos \frac{\pi}{2} + \cos 0 \right) d\theta = \frac{a^3}{3} \int_0^{\pi/2} (-0 + 1) d\theta \\ &= \frac{a^3}{3} \int_0^{\pi/2} d\theta = \frac{a^3}{3} [\theta]_0^{\pi/2} = \frac{a^3}{3} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi a^3}{6}}. \end{aligned}$$

47. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $0 \leq z \leq \sqrt{16 - x^2 - y^2}$, $-\sqrt{4 - y^2} \leq x \leq \sqrt{4 - y^2}$, $-2 \leq y \leq 2$; see the figures below. The limits of integration on z are $z = 0$, which is the xy -plane, and $z = \sqrt{16 - x^2 - y^2}$, or equivalently $x^2 + y^2 + z^2 = 16$, $z \geq 0$, which is the upper hemisphere of radius 4. The region R in the xy -plane is the disk of radius 2 centered at the origin. So the solid E of integration is the portion of the ball of radius 4 lying within the cylinder $x^2 + y^2 = 4$ and above the xy -plane. In cylindrical coordinates, E

is given by $0 \leq z \leq \sqrt{16-r^2}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Then the given integral can be converted to cylindrical coordinates as follows:

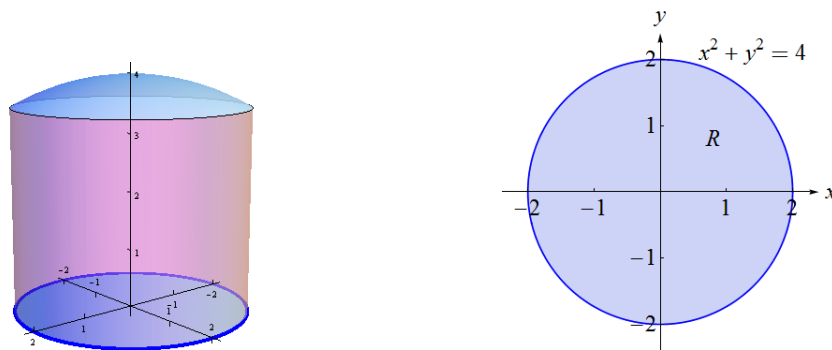
$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{\sqrt{16-x^2-y^2}} dz \, dx \, dy &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r[z]_0^{\sqrt{16-r^2}} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 r(\sqrt{16-r^2} - 0) \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r\sqrt{16-r^2} \, dr \, d\theta. \end{aligned}$$

To evaluate the “inside” integral $\int_0^2 r\sqrt{16-r^2} \, dr$, we make a substitution $u = 16-r^2$; then $du = -2r \, dr$, so $r \, dr = \frac{du}{-2}$. We also change the limits of integration: if $r = 0$, then $u = 16 - 0^2 = 16$; if $r = 2$, then $u = 16 - 2^2 = 12$. So the second integral becomes

$$\begin{aligned} \int_0^2 r\sqrt{16-r^2} \, dr &= \int_{16}^{12} \sqrt{u} \left(\frac{du}{-2} \right) = -\frac{1}{2} \int_{16}^{12} \sqrt{u} \, du = \frac{1}{2} \int_{12}^{16} u^{1/2} \, du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_{12}^{16} \\ &= \frac{1}{3} \left[(\sqrt{u})^3 \right]_{12}^{16} = \frac{1}{3} \left[(\sqrt{16})^3 - (\sqrt{12})^3 \right] = \frac{1}{3} (64 - 24\sqrt{3}) = \frac{64}{3} - 8\sqrt{3}. \end{aligned}$$

Going back to the double integral, we obtain

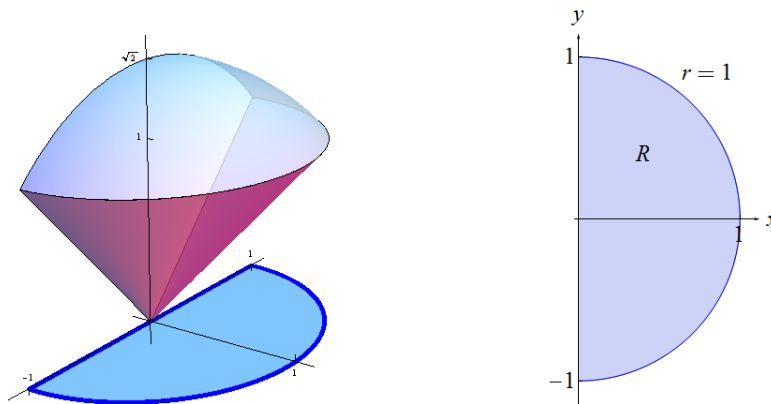
$$\begin{aligned} \int_0^{2\pi} \int_0^2 r\sqrt{16-r^2} \, dr \, d\theta &= \int_0^{2\pi} \left(\frac{64}{3} - 8\sqrt{3} \right) d\theta = \left(\frac{64}{3} - 8\sqrt{3} \right) [\theta]_0^{2\pi} \\ &= \left(\frac{64}{3} - 8\sqrt{3} \right) (2\pi - 0) = \boxed{\pi \left(\frac{128}{3} - 16\sqrt{3} \right)}. \end{aligned}$$



48. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $\sqrt{x^2 + y^2} \leq z \leq \sqrt{2 - x^2 - y^2}$, $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$; see the figures below. The limits of integration on z are from $z = \sqrt{x^2 + y^2}$, which is the cone $\phi = \frac{\pi}{4}$, to $z = \sqrt{2 - x^2 - y^2}$, which is a sphere of radius $\sqrt{2}$. In spherical coordinates, E is given by $0 \leq \rho \leq \sqrt{2}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{4}$.

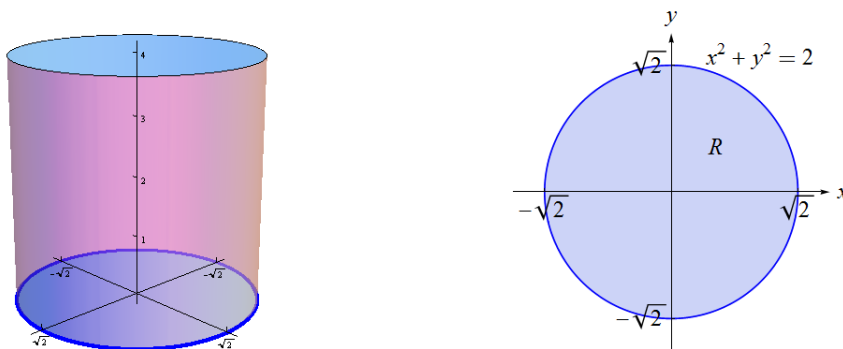
Then the given integral can be converted to spherical coordinates as follows:

$$\begin{aligned}
 \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz \, dy \, dx &= \int_{-\pi/2}^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \int_0^{\pi/4} \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_0^{\sqrt{2}} d\phi \, d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \int_0^{\pi/4} \sin \phi \cdot \left(\frac{2\sqrt{2}}{3} - 0 \right) d\phi \, d\theta \\
 &= \frac{2\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} [-\cos \phi]_0^{\pi/4} d\theta \\
 &= \frac{2\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(-\cos \frac{\pi}{4} + \cos 0 \right) d\theta \\
 &= \frac{2\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(-\frac{\sqrt{2}}{2} + 1 \right) d\theta = \frac{2\sqrt{2}}{3} \cdot \frac{2-\sqrt{2}}{2} \cdot [\theta]_{-\pi/2}^{\pi/2} \\
 &= \frac{2\sqrt{2}-2}{3} \cdot \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \boxed{\frac{(2\sqrt{2}-2)\pi}{3}}.
 \end{aligned}$$



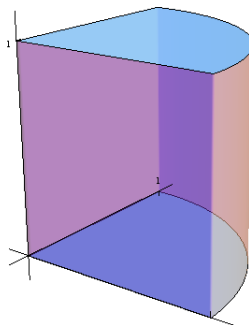
49. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $0 \leq z \leq 4$, $-\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2}$, $-\sqrt{2} \leq x \leq \sqrt{2}$; see the figures below. The region R in the xy -plane is the disk of radius $\sqrt{2}$ centered at the origin. So the solid E of integration is the portion of a vertical cylinder of radius $\sqrt{2}$ contained between the planes $z = 0$ and $z = 4$. In cylindrical coordinates, E is given by $0 \leq z \leq 4$, $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$. Then the given integral can be converted to cylindrical coordinates as follows:

$$\begin{aligned}
 \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^4 (x^2 + y^2)z \, dz \, dy \, dx &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^4 r^2 z \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^4 r^3 z \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 \left[\frac{z^2}{2} \right]_0^4 dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 (8 - 0) \, dr \, d\theta \\
 &= 8 \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 \, dr \, d\theta = 8 \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} d\theta \\
 &= 8 \int_0^{2\pi} (1 - 0) \, d\theta = 8 \int_0^{2\pi} d\theta = 8[\theta]_0^{2\pi} = 8(2\pi - 0) = \boxed{16\pi}.
 \end{aligned}$$



50. The solid E of integration can be described by the inequalities $0 \leq y \leq \sqrt{1-x^2}$, $0 \leq x \leq 1$, $0 \leq z \leq 1$; see the figure below. It is part of a vertical cylinder of radius 1 contained between the planes $z = 0$ and $z = 1$ in the first octant. In cylindrical coordinates, E is given by $0 \leq z \leq 1$, $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Then the given integral can be converted to cylindrical coordinates as follows:

$$\begin{aligned}
 \int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} dy \, dx \, dz &= \int_0^{\pi/2} \int_0^1 \int_0^1 r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r[z]_0^1 \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^1 r(1-0) \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 r \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^1 d\theta \\
 &= \int_0^{\pi/2} \left(\frac{1}{2} - 0 \right) d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} [\theta]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi}{4}}.
 \end{aligned}$$



51. The solid E of integration can be described by the inequalities $\frac{h}{a}\sqrt{x^2+y^2} \leq z \leq \sqrt{a^2-x^2-y^2}$, $0 \leq y \leq \sqrt{\frac{a^4}{a^2+h^2}-x^2}$, $0 \leq x \leq \frac{a^2}{\sqrt{a^2+h^2}}$; see the figure below. The limits of integration on z are from $z = \frac{h}{a}\sqrt{x^2+y^2}$, which is the upper nappe of the cone $\phi = \tan^{-1}\left(\frac{a}{h}\right)$, to $z = \sqrt{a^2-x^2-y^2}$, which is the upper half of the sphere of radius a ; and since $x \geq 0$ and $y \geq 0$, the entire solid lies in the first octant. The trace of the line of intersection of the cone and the sphere can be found by setting

their equations equal to each other:

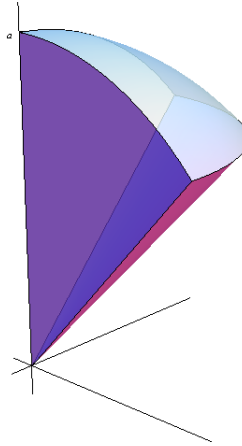
$$\begin{aligned}\sqrt{a^2 - x^2 - y^2} &= \frac{h}{a} \sqrt{x^2 + y^2} \\ a^2 - x^2 - y^2 &= \frac{h^2}{a^2} (x^2 + y^2) \\ a^2 &= \frac{h^2}{a^2} (x^2 + y^2) + x^2 + y^2 \\ a^2 &= \left(\frac{h^2}{a^2} + 1 \right) (x^2 + y^2) \\ x^2 + y^2 &= \frac{a^2}{\frac{h^2}{a^2} + 1} \\ x^2 + y^2 &= \frac{a^4}{h^2 + a^2}\end{aligned}$$

which is a circle of radius $\frac{a^2}{\sqrt{h^2 + a^2}}$ (or rather the first-quadrant quarter of this circle, since $x \geq 0$ and $y \geq 0$), in agreement with the given limits of integration. If we denote $\alpha = \tan^{-1} \left(\frac{a}{h} \right)$ for brevity, then in spherical coordinates E is given by $0 \leq \rho \leq a$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \alpha$. Then the given integral can be converted to spherical coordinates as follows:

$$\begin{aligned}\int_0^{\frac{a^2}{\sqrt{a^2+h^2}}} \int_0^{\sqrt{\frac{a^4}{a^2+h^2}-x^2}} \int_{\frac{h}{a}\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} dz \, dy \, dx &= \int_0^{\pi/2} \int_0^\alpha \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^\alpha \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_0^a d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^\alpha \sin \phi \cdot \left(\frac{a^3}{3} - 0 \right) d\phi \, d\theta \\ &= \frac{a^3}{3} \int_0^{\pi/2} [-\cos \phi]_0^\alpha d\theta \\ &= \frac{a^3}{3} \int_0^{\pi/2} (-\cos \alpha + \cos 0) d\theta \\ &= \frac{a^3}{3} (1 - \cos \alpha) \int_0^{\pi/2} d\theta = \frac{a^3}{3} (1 - \cos \alpha) [\theta]_0^{\pi/2} \\ &= \frac{a^3}{3} (1 - \cos \alpha) \left(\frac{\pi}{2} - 0 \right) = \frac{\pi a^3}{6} (1 - \cos \alpha).\end{aligned}$$

To obtain the final answer, recall that we denoted $\alpha = \tan^{-1} \left(\frac{a}{h} \right)$; notice that α lies in the first quadrant because $\tan \alpha = \frac{a}{h} > 0$. Then $\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \frac{a^2}{h^2} = \frac{h^2 + a^2}{h^2}$, so $\sec \alpha = \sqrt{\frac{h^2 + a^2}{h^2}} = \frac{\sqrt{h^2 + a^2}}{h}$, and $\cos \alpha = \frac{1}{\sec \alpha} = \frac{h}{\sqrt{h^2 + a^2}}$. Substituting this into the last expression, we obtain

$$\frac{\pi a^3}{6} (1 - \cos \alpha) = \boxed{\frac{\pi a^3}{6} \left(1 - \frac{h}{\sqrt{h^2 + a^2}} \right)}.$$

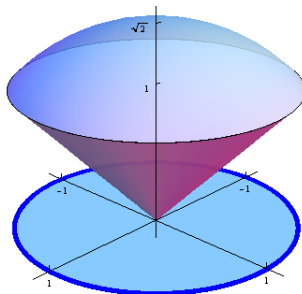


52. The region R associated with the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$ is enclosed between $y = 0$ and $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$. This is the first-quadrant quarter of the disk of radius 1 centered at the origin. In polar coordinates R is given by $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$. Also, in the integrand we can simplify $(x^2 + y^2)^{3/2} = (r^2)^{3/2} = r^3$. Then

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx &= \int_0^{\pi/2} \int_0^1 r^3 \cdot r dr d\theta = \int_0^{\pi/2} \int_0^1 r^4 dr d\theta = \int_0^{\pi/2} \left[\frac{r^5}{5} \right]_0^1 d\theta \\ &= \int_0^{\pi/2} \left(\frac{1}{5} - 0 \right) d\theta = \frac{1}{5} [\theta]_0^{\pi/2} = \frac{1}{5} \left(\frac{\pi}{2} - 0 \right) = \boxed{\frac{\pi}{10}}. \end{aligned}$$

53. The solid E of integration and its projection R onto the xy -plane can be described by the inequalities $\sqrt{x^2 + y^2} \leq z \leq \sqrt{2-x^2-y^2}$, $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$, $-1 \leq y \leq 1$; see the figure below. The limits of integration on z are from $z = \sqrt{x^2 + y^2}$, which is the cone $\phi = \frac{\pi}{4}$, to $z = \sqrt{2-x^2-y^2}$, which is a sphere of radius $\sqrt{2}$. In spherical coordinates, E is given by $0 \leq \rho \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{4}$. Then the given integral can be converted to spherical coordinates as follows:

$$\begin{aligned} \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz dx dy &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cdot \left[\frac{\rho^3}{3} \right]_0^{\sqrt{2}} d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cdot \left(\frac{2\sqrt{2}}{3} - 0 \right) d\phi d\theta \\ &= \frac{2\sqrt{2}}{3} \int_0^{2\pi} [-\cos \phi]_0^{\pi/4} d\theta \\ &= \frac{2\sqrt{2}}{3} \int_0^{2\pi} \left(-\cos \frac{\pi}{4} + \cos 0 \right) d\theta \\ &= \frac{2\sqrt{2}}{3} \int_0^{2\pi} \left(-\frac{\sqrt{2}}{2} + 1 \right) d\theta = \frac{2\sqrt{2}}{3} \cdot \frac{2-\sqrt{2}}{2} \cdot [\theta]_0^{2\pi} \\ &= \frac{2\sqrt{2}-2}{3} \cdot (2\pi-0) = \boxed{\frac{4(\sqrt{2}-1)\pi}{3}}. \end{aligned}$$

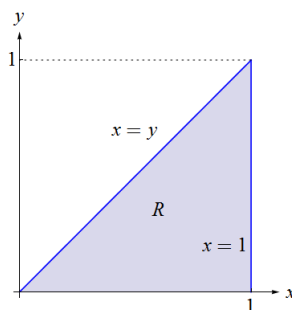


54. The limits of integration of the iterated integral $\int_0^1 \int_y^1 (x^2 + 1)^{2/3} dx dy$ describe the region R bounded between $x = y$ and $x = 1$, where $0 \leq y \leq 1$. This is a triangular region, as shown in the figure below. Observe that R is also an x -simple region: it is bounded between $y = 0$ and $y = x$, where $0 \leq x \leq 1$. So according to Fubini's Theorem,

$$\begin{aligned} \int_0^1 \int_y^1 (x^2 + 1)^{2/3} dx dy &= \iint_R (x^2 + 1)^{2/3} dA = \int_0^1 \int_0^x (x^2 + 1)^{2/3} dy dx = \int_0^1 (x^2 + 1)^{2/3} [y]_0^x dx \\ &= \int_0^1 (x^2 + 1)^{2/3} (x - 0) dx = \int_0^1 (x^2 + 1)^{2/3} x dx. \end{aligned}$$

In this integral we make a substitution $u = x^2 + 1$; then $du = 2x dx$, so $x dx = \frac{du}{2}$. We also change the limits of integration: if $x = 0$, then $u = 0^2 + 1 = 1$; if $x = 1$, then $u = 1^2 + 1 = 2$. So

$$\begin{aligned} \int_0^1 (x^2 + 1)^{2/3} x dx &= \int_1^2 u^{2/3} \frac{du}{2} = \frac{1}{2} \int_1^2 u^{2/3} du = \frac{1}{2} \left[\frac{u^{5/3}}{5/3} \right]_1^2 \\ &= \frac{3}{10} \left[u^{5/3} \right]_1^2 = \frac{3}{10} (2^{5/3} - 1^{5/3}) = \boxed{\frac{3(2^{5/3} - 1)}{10}}. \end{aligned}$$



55. The partial derivatives of the given change of variable functions $x = e^{u+v}$ and $y = e^{v-u}$ are

$$\frac{\partial x}{\partial u} = e^{u+v}, \quad \frac{\partial x}{\partial v} = e^{u+v}, \quad \frac{\partial y}{\partial u} = -e^{v-u}, \quad \frac{\partial y}{\partial v} = e^{v-u}.$$

Then the Jacobian of x, y with respect to u, v is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^{u+v} & e^{u+v} \\ -e^{v-u} & e^{v-u} \end{vmatrix} = e^{u+v} \cdot e^{v-u} - e^{u+v} \cdot (-e^{v-u}) = \boxed{2e^{2v}}.$$

56. The partial derivatives of the given change of variable functions $x = u + 3v$ and $y = 2u - v$ are

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 3, \quad \frac{\partial y}{\partial u} = 2, \quad \frac{\partial y}{\partial v} = -1.$$

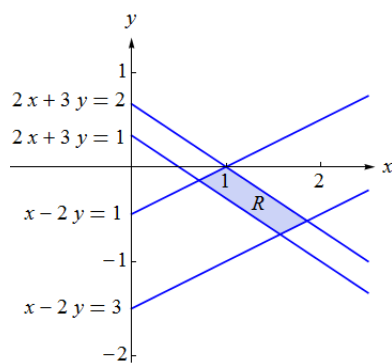
Then the Jacobian of x, y with respect to u, v is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 1 \cdot (-1) - 3 \cdot 2 = \boxed{-7}.$$

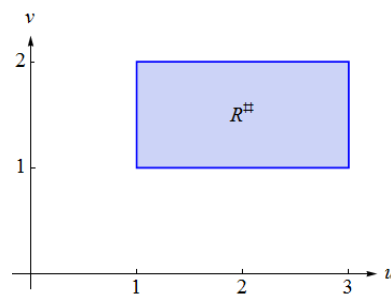
57. For the given change of variable functions $x = u^2$, $y = 3v$, and $z = u + w$, the Jacobian of x, y, z with respect to u, v, w is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2u \cdot 3 \cdot 1 = \boxed{6u}.$$

58. The region R enclosed by $x - 2y = 1$, $x - 2y = 3$, $2x + 3y = 1$, and $2x + 3y = 2$ is shown in figure (a) below. With the indicated change of variables $u = x - 2y$ and $v = 2x + 3y$, the equations $x - 2y = 1$ and $x - 2y = 3$ become $u = 1$ and $u = 3$, while $2x + 3y = 1$ and $2x + 3y = 2$ become $v = 1$ and $v = 2$. So the region $R^\#$ is the rectangle $1 \leq u \leq 3$, $1 \leq v \leq 2$, shown in figure (b) below.



(a)



(b)

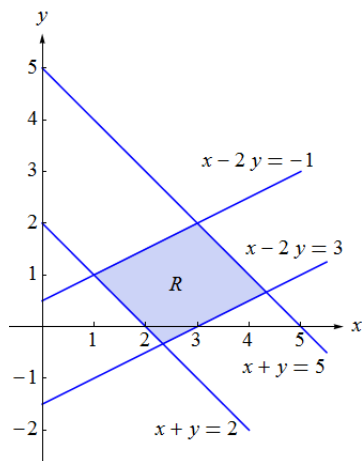
Solving the change of variable equations for x and y gives $x = \frac{3u + 2v}{7}$ and $y = \frac{v - 2u}{7}$. Then the Jacobian of x, y with respect to u, v is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{vmatrix} = \frac{3}{7} \cdot \frac{1}{7} - \frac{2}{7} \cdot \left(-\frac{2}{7}\right) = \frac{1}{7},$$

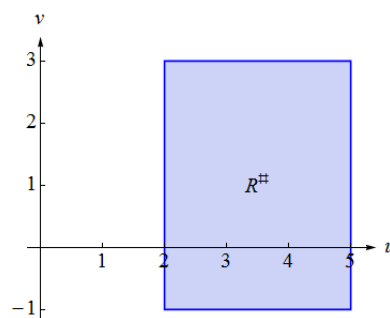
and $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \frac{1}{7} \right| = \frac{1}{7}$. Then in terms of the new variables u, v , the integral becomes

$$\begin{aligned} \iint_R y^2 dx dy &= \iint_{R^\#} \left(\frac{v - 2u}{7} \right)^2 \cdot \frac{1}{7} du dv = \frac{1}{343} \int_1^2 \int_1^3 (v^2 - 4uv + 4u^2) du dv \\ &= \frac{1}{343} \int_1^2 \left[uv^2 - 2u^2v + \frac{4u^3}{3} \right]_1^3 dv = \frac{1}{343} \int_1^2 \left[(3v^2 - 18v + 36) - \left(v^2 - 2v + \frac{4}{3} \right) \right] dv \\ &= \frac{1}{343} \int_1^2 \left(2v^2 - 16v + \frac{104}{3} \right) dv = \frac{1}{343} \left[\frac{2v^3}{3} - 8v^2 + \frac{104v}{3} \right]_1^2 \\ &= \frac{1}{343} \left[\left(\frac{16}{3} - 32 + \frac{208}{3} \right) - \left(\frac{2}{3} - 8 + \frac{104}{3} \right) \right] = \frac{1}{343} \cdot \frac{46}{3} = \boxed{\frac{46}{1029}}. \end{aligned}$$

59. The region R enclosed by $x + y = 2$, $x + y = 5$, $x - 2y = -1$, and $x - 2y = 3$ is shown in figure (a) below. With the indicated change of variables $u = x + y$ and $v = x - 2y$, the equations $x + y = 2$ and $x + y = 5$ become $u = 2$ and $u = 5$, while $x - 2y = -1$ and $x - 2y = 3$ become $v = -1$ and $v = 3$. So $R^\#$ is the rectangular region $2 \leq u \leq 5$, $-1 \leq v \leq 3$, shown in figure (b) below.



(a)



(b)

Solving the change of variable equations for x and y gives $x = \frac{2u + v}{3}$ and $y = \frac{u - v}{3}$. Then the Jacobian of x, y with respect to u, v is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \frac{2}{3} \cdot \left(-\frac{1}{3}\right) - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{3},$$

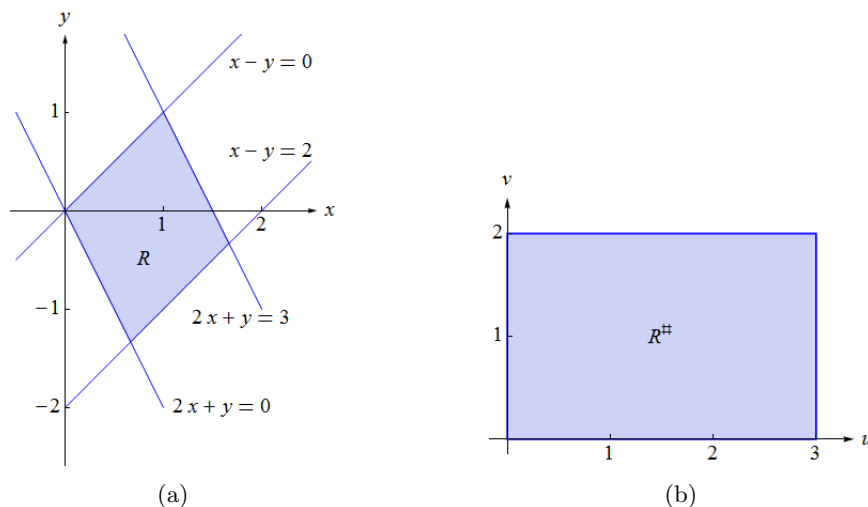
and $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| -\frac{1}{3} \right| = \frac{1}{3}$. Then in terms of the new variables u, v , the integral becomes

$$\begin{aligned} \iint_R (x + y)^3 dx dy &= \iint_{R^\#} u^3 \cdot \frac{1}{3} du dv = \frac{1}{3} \int_{-1}^3 \int_2^5 u^3 du dv = \frac{1}{3} \int_{-1}^3 \left[\frac{u^4}{4} \right]_2^5 dv \\ &= \frac{1}{3} \int_{-1}^3 \left(\frac{625}{4} - 4 \right) dv = \frac{1}{3} \cdot \frac{609}{4} \int_{-1}^3 dv = \frac{203}{4} [v]_{-1}^3 = \frac{203}{4} (3 + 1) = \boxed{203}. \end{aligned}$$

60. The region R enclosed by $2x + y = 0$, $2x + y = 3$, $x - y = 0$, and $x - y = 2$ is shown in figure (a) below. We will use the change of variables $u = 2x + y$ and $v = x - y$. The equations $2x + y = 0$ and $2x + y = 3$ become $u = 0$ and $u = 3$, while $x - y = 0$ and $x - y = 2$ become $v = 0$ and $v = 2$. So $R^\#$ is the rectangular region $0 \leq u \leq 3$, $0 \leq v \leq 2$, shown in figure (b) below.

Solving the change of variable equations for x and y gives $x = \frac{u + v}{3}$ and $y = \frac{u - 2v}{3}$. Then the Jacobian of x, y with respect to u, v is

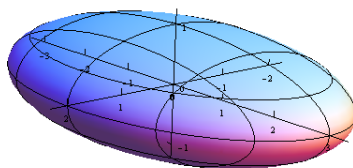
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{3} \cdot \left(-\frac{2}{3}\right) - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{3},$$



and $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{3} \right| = \frac{1}{3}$. Then in terms of the new variables u, v , the integral becomes

$$\begin{aligned}
 \iint_R xy \, dx \, dy &= \iint_{R^\#} \frac{u+v}{3} \cdot \frac{u-2v}{3} \cdot \frac{1}{3} \, du \, dv = \frac{1}{27} \int_0^2 \int_0^3 (u^2 - uv - 2v^2) \, du \, dv \\
 &= \frac{1}{27} \int_0^2 \left[\frac{u^3}{3} - \frac{u^2 v}{2} - 2uv^2 \right]_0^3 \, dv = \frac{1}{27} \int_0^2 \left[\left(9 - \frac{9v}{2} - 6v^2 \right) - 0 \right] \, dv \\
 &= \frac{1}{27} \int_0^2 \left(9 - \frac{9v}{2} - 6v^2 \right) \, dv = \frac{1}{27} \left[9v - \frac{9v^2}{4} - 2v^3 \right]_0^2 \\
 &= \frac{1}{27} [(18 - 9 - 16) - 0] = \frac{1}{27} \cdot (-7) = \boxed{-\frac{7}{27}}.
 \end{aligned}$$

61. The solid E enclosed by the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ is shown in the figure below. Solving the given change of variables $u = \frac{x}{2}$, $v = \frac{y}{3}$, $w = z$ for x, y , and z gives $x = 2u$, $y = 3v$, and $z = w$. Then the equation of the ellipsoid becomes $\frac{(2u)^2}{4} + \frac{(3v)^2}{9} + w^2 = 1$ or just $u^2 + v^2 + w^2 = 1$, so that $E^\#$ is the ball of radius 1 centered at the origin.



The Jacobian of the given transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot 1 = 6,$$

and $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = |6| = 6$. Then in terms of the new variables u, v, w , the integral becomes

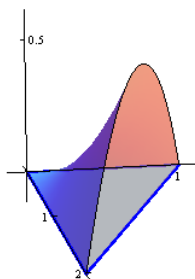
$$\iiint_E xz^2 dx dy dz = \iiint_{E^\#} 2u \cdot w^2 \cdot 6 dv du dw = 12 \iiint_{E^\#} uw^2 dv du dw.$$

To perform the integration, use spherical coordinates in the uvw -space:

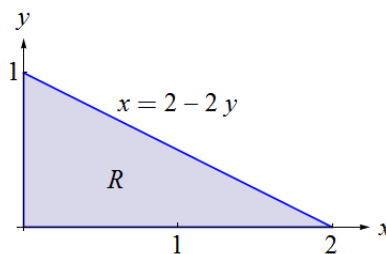
$$\begin{aligned} 12 \iiint_{E^\#} uw^2 dv du dw &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} \rho \cos \theta \sin \phi \cdot (\rho \cos \phi)^2 \cdot \rho^2 \sin \phi d\theta d\phi d\rho \\ &= 12 \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^5 \cos \theta \sin^2 \phi \cos^2 \phi d\theta d\phi d\rho \\ &= 12 \int_0^1 \int_0^\pi \rho^5 \sin^2 \phi \cos^2 \phi \cdot [\sin \theta]_0^{2\pi} d\phi d\rho \\ &= 12 \int_0^1 \int_0^\pi \rho^5 \sin^2 \phi \cos^2 \phi \cdot (\sin 2\pi - \sin 0) d\phi d\rho \\ &= 12 \int_0^1 \int_0^\pi \rho^5 \sin^2 \phi \cos^2 \phi \cdot (0 - 0) d\phi d\rho \\ &= 12 \int_0^1 \int_0^\pi 0 d\phi d\rho = \boxed{0}. \end{aligned}$$

62. Let E be the solid enclosed by the surface $z = xy$, the xy -plane, and the vertical cylinder whose base in the xy -plane is the triangular region R with vertices $(0, 0, 0)$, $(2, 0, 0)$, and $(0, 1, 0)$; the solid E and the region R are shown in the figures below. Using the given mass density $\rho(x, y, z) = x + y$, we find the mass of the solid E as follows:

$$\begin{aligned} M &= \iiint_E \rho(x, y, z) dV = \int_0^1 \int_0^{2-2y} \int_0^{xy} (x + y) dz dx dy = \int_0^1 \int_0^{2-2y} (x + y) [z]_0^{xy} dx dy \\ &= \int_0^1 \int_0^{2-2y} (x + y)(xy - 0) dx dy = \int_0^1 \int_0^{2-2y} (x^2 y + xy^2) dx dy = \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^{2-2y} dy \\ &= \int_0^1 \left[\left(\frac{(2-2y)^3 y}{3} + \frac{(2-2y)^2 y^2}{2} \right) - 0 \right] dy = \int_0^1 \left(\frac{8y}{3} - 6y^2 + 4y^3 - \frac{2y^4}{3} \right) dy \\ &= \left[\frac{4y^2}{3} - 2y^3 + y^4 - \frac{2y^5}{15} \right]_0^1 = \left(\frac{4}{3} - 2 + 1 - \frac{2}{15} \right) - 0 = \boxed{\frac{1}{5}}. \end{aligned}$$



(a)



(b)

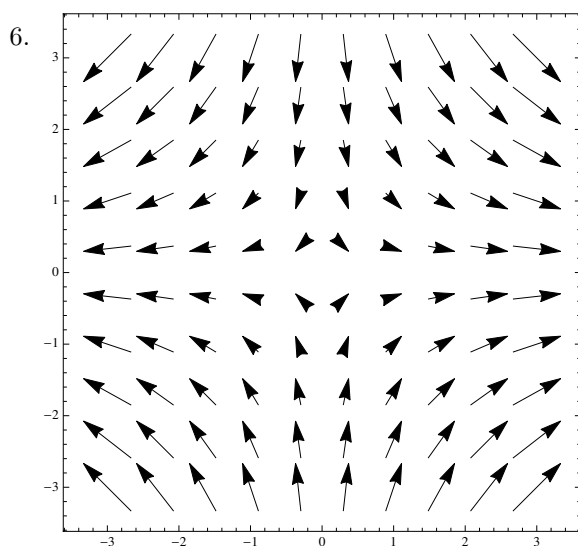
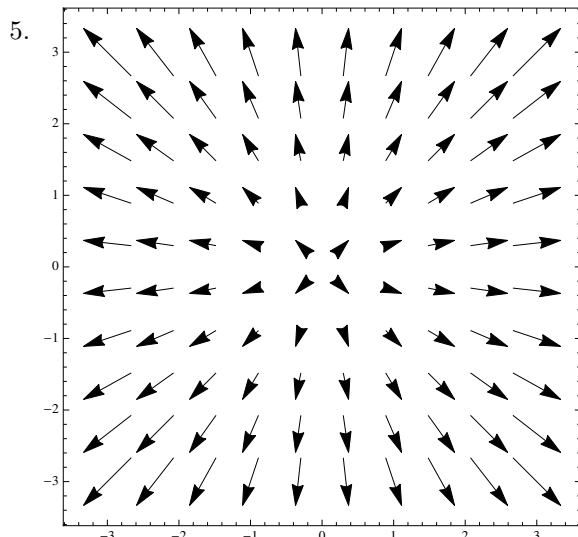
Chapter 15: Vector Calculus

15.1: Vector Fields

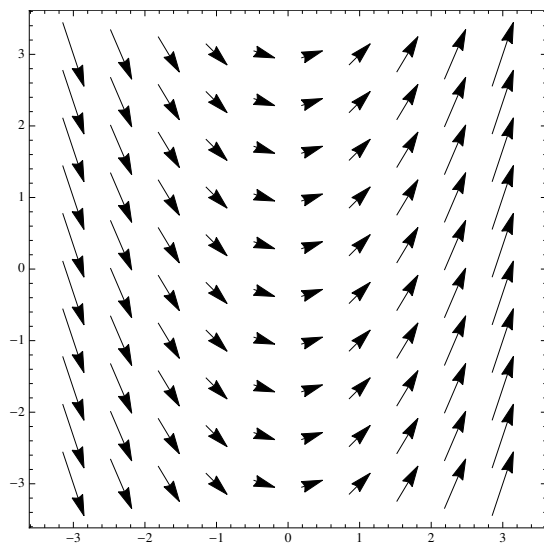
Concepts and Vocabulary

1. A *vector field* is a function that associates a vector to a point in the plane or to a point in space. See the introduction to objective 1.
2. False. The right-hand side as written is a number, not a vector. A vector field could be defined as $F(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$.
3. The range of $\mathbf{F}$ is a set of *vectors* in the plane. See objective 1.
4. True. See Example 6.

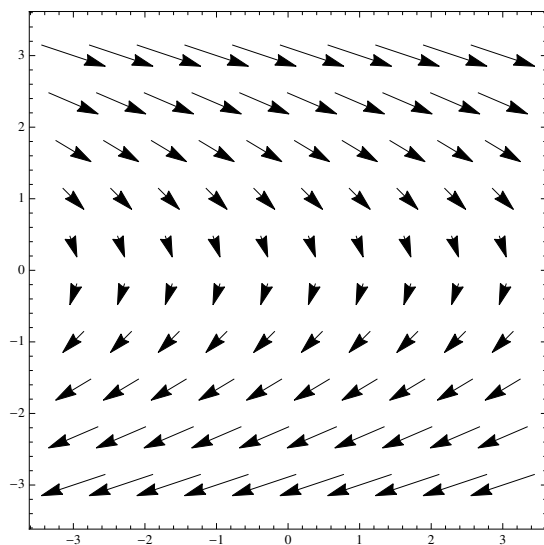
Skill Building



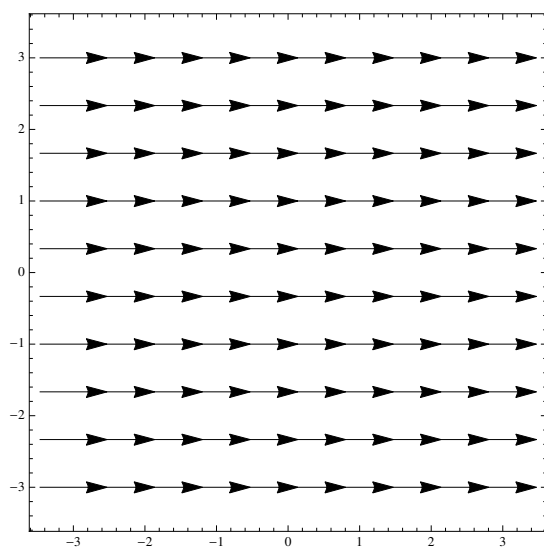
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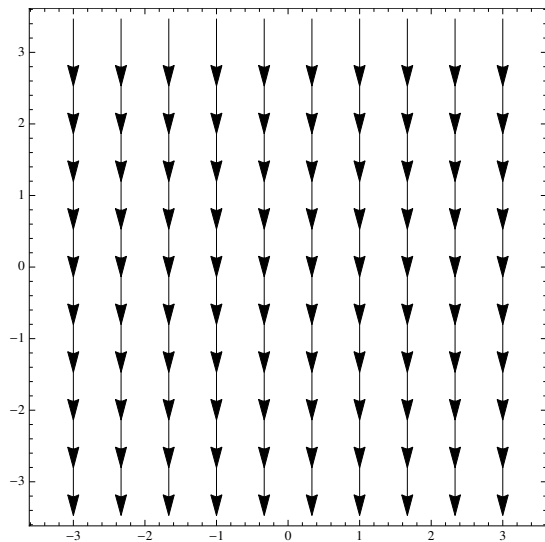
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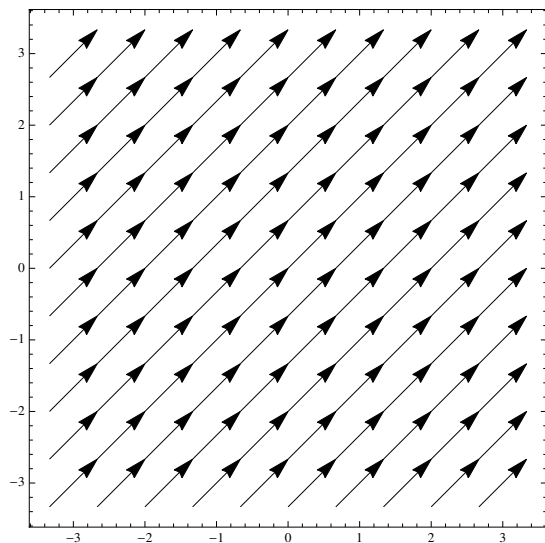
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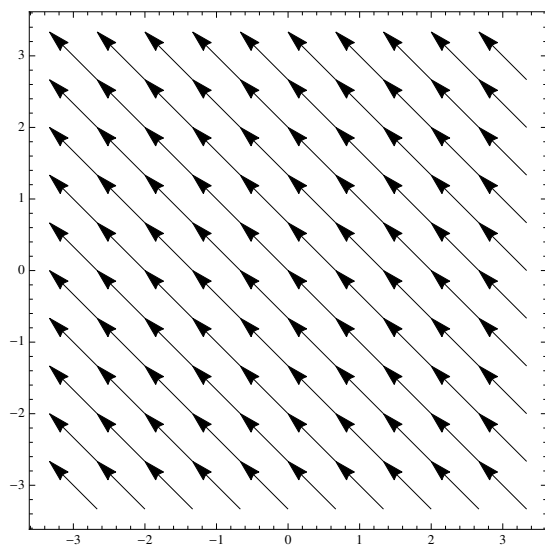
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11.

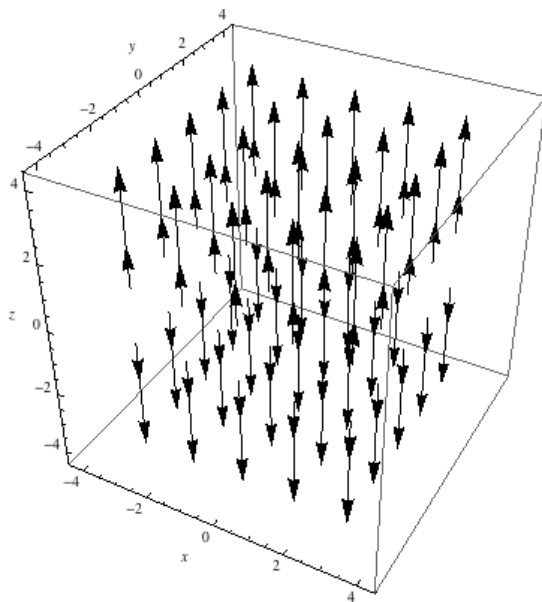


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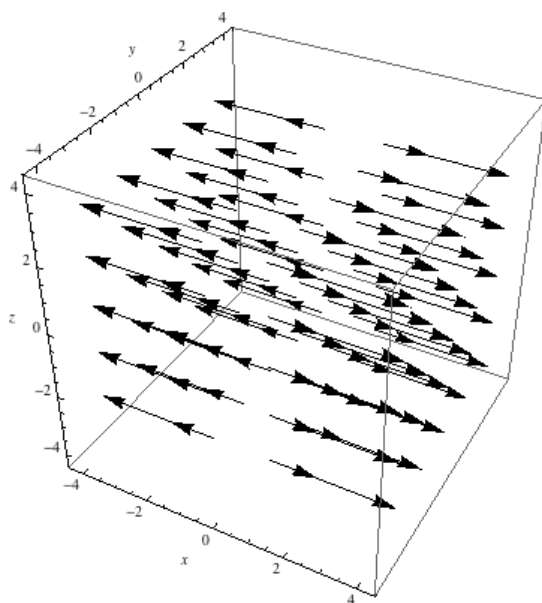


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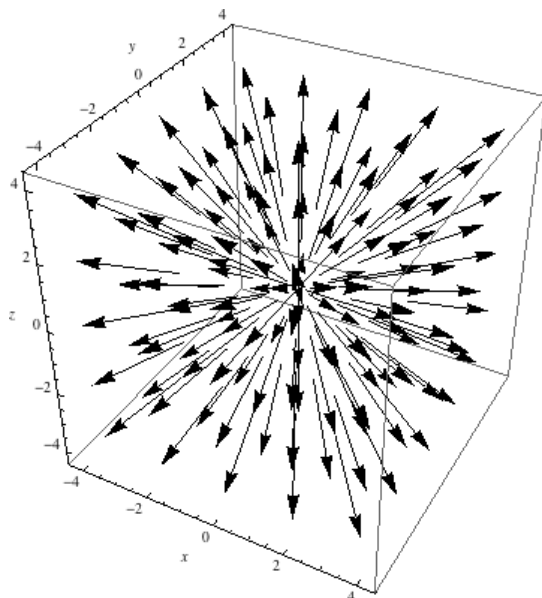
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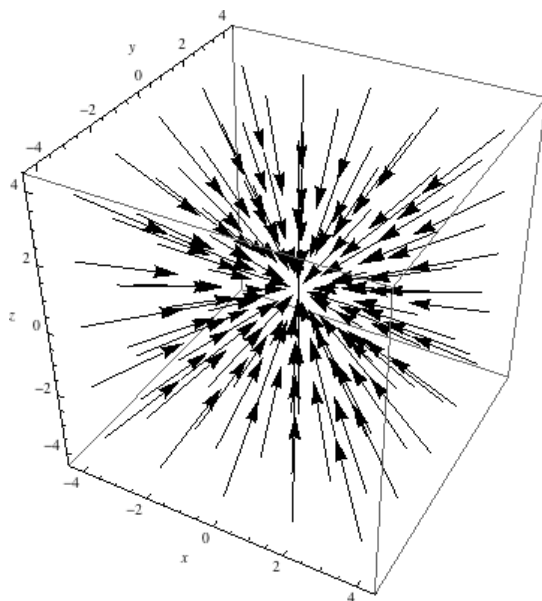


15.



The vector associated with any point except the origin is a unit vector pointing away from the origin. The vector field is not defined at the origin.

16.



The vector associated with any point except the origin is a unit vector pointing towards the origin. The vector field is not defined at the origin.

Applications and Extensions

17. Since $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$, and

$$f_x(x, y) = \frac{\partial}{\partial x}(x \sin y + \cos y) = \sin y, \quad f_y(x, y) = \frac{\partial}{\partial y}(x \sin y + \cos y) = x \cos y - \sin y,$$

the gradient is

$$\nabla f(x, y) = \boxed{\sin y \mathbf{i} + (x \cos y - \sin y) \mathbf{j}}.$$

18. Since $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$, and

$$f_x(x, y) = \frac{\partial}{\partial x}(xe^y) = e^y, \quad f_y(x, y) = \frac{\partial}{\partial y}(xe^y) = xe^y,$$

the gradient is

$$\nabla f(x, y) = \boxed{e^y \mathbf{i} + xe^y \mathbf{j}}.$$

19. Since $\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$, and

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial}{\partial x}(x^2y + xy + y^2z) = 2xy + y, \\ f_y(x, y, z) &= \frac{\partial}{\partial y}(x^2y + xy + y^2z) = x^2 + x + 2yz, \\ f_z(x, y, z) &= \frac{\partial}{\partial z}(x^2y + xy + y^2z) = y^2, \end{aligned}$$

the gradient is

$$\nabla f(x, y) = \boxed{(2xy + y)\mathbf{i} + (x^2 + x + 2yz)\mathbf{j} + y^2\mathbf{k}}.$$

20. Since $\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$, and

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial}{\partial x}(x^2y + xyz^2) = 2xy + yz^2, \\ f_y(x, y, z) &= \frac{\partial}{\partial y}(x^2y + xyz^2) = x^2 + xz^2, \\ f_z(x, y, z) &= \frac{\partial}{\partial z}(x^2y + xyz^2) = 2xyz, \end{aligned}$$

the gradient is

$$\nabla f(x, y) = \boxed{(2xy + yz^2)\mathbf{i} + (x^2 + xz^2)\mathbf{j} + 2xyz\mathbf{k}}.$$

21. (a) From plane geometry, we know that a tangent to a circle is perpendicular to the radius at that point. At a point (x, y) in the plane, the radius vector of the circle centered at the origin and passing through (x, y) is $x\mathbf{i} + y\mathbf{j}$, and the vector field there is $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Then

$$(x\mathbf{i} + y\mathbf{j}) \cdot (-y\mathbf{i} + x\mathbf{j}) = -xy + xy = 0,$$

so that the vector field is perpendicular to the radial vectors of the circle, so they are tangent to the unit circle.

- (b) The radius of the circle in part (a) is $\sqrt{x^2 + y^2}$, and the magnitude of the vector field is $\|\mathbf{F}\| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2}$. The two are equal.

22. Place the origin at the center of the large object. Then a point r meters away has coordinates (x, y, z) , where $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$u(x, y, z) = -\frac{Gm}{r} = -\frac{Gm}{\sqrt{x^2 + y^2 + z^2}} = -Gm(x^2 + y^2 + z^2)^{-1/2}.$$

Taking partial derivatives gives

$$\begin{aligned} u_x(x, y, z) &= -\frac{1}{2} \left(-Gm(x^2 + y^2 + z^2)^{-3/2} \right) \cdot 2x = -xGm(x^2 + y^2 + z^2)^{-3/2} = -\frac{Gm}{r^3}x \\ u_y(x, y, z) &= -\frac{1}{2} \left(-Gm(x^2 + y^2 + z^2)^{-3/2} \right) \cdot 2y = -yGm(x^2 + y^2 + z^2)^{-3/2} = -\frac{Gm}{r^3}y \\ u_z(x, y, z) &= -\frac{1}{2} \left(-Gm(x^2 + y^2 + z^2)^{-3/2} \right) \cdot 2z = -zGm(x^2 + y^2 + z^2)^{-3/2} = -\frac{Gm}{r^3}z. \end{aligned}$$

Then

$$\begin{aligned}
 \nabla u &= u_x(x, y, z)\mathbf{i} + u_y(x, y, z)\mathbf{j} + u_z(x, y, z)\mathbf{k} \\
 &= -\frac{Gm}{r^3}x\mathbf{i} - \frac{Gm}{r^3}y\mathbf{j} - \frac{Gm}{r^3}z\mathbf{k} \\
 &= -\frac{Gm}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
 &= -\frac{Gm}{r^3}\mathbf{r} = \mathbf{g}.
 \end{aligned}$$

15.2: Line Integrals

Concepts and Vocabulary

1. True. This is the content of the Value of a Line Integral Theorem.
2. $\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. See the Value of a Line Integral Theorem.
3. False. See Example 2. This fact is not proved in the text, but it is asserted.
4. $M = \int_C \rho(x, y) ds$. See Equation (2) and Example 3.
5. False. Reversing the direction of the curve negates the value of the line integral, so that $\int_{-C} (P dx + Q dy) = -\int_C (P dx + Q dy)$.
6. True. See Objective 4.
7. $\int_C (P dx + Q dy) = \int_{C_1} (P dx + Q dy) + \int_{C_1} (P dx + Q dy) + \cdots + \int_{C_n} (P dx + Q dy)$. See the first definition in Objective 4.
8. False. The factor of $x(t)$ in the right-hand side should be $x'(t)$:

$$\int_C P(x, y, z) dx = \int_a^b P(x(t), y(t), z(t))x'(t) dt.$$

Skill Building

9. With the given parametrization,

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = -\sin t,$$

so that

$$\begin{aligned}
 \int_C x^2 y ds &= \int_0^{\pi/2} \cos^2 t \cos t \sqrt{(-\sin t)^2 + (-\sin t)^2} dt = \int_0^{\pi/2} \sin t \cos^3 t \cdot \sqrt{2} dt \\
 &= \sqrt{2} \left[-\frac{1}{4} \cos^4 t \right]_0^{\pi/2} = \boxed{\frac{\sqrt{2}}{4}}.
 \end{aligned}$$

10. With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3,$$

so that

$$\int_C (y - x^2) ds = \int_0^1 (3t - t^2) \cdot \sqrt{1^2 + 3^2} dt = \sqrt{10} \left[\frac{3}{2}t^2 - \frac{1}{3}t^3 \right]_0^1 = \boxed{\frac{7\sqrt{10}}{6}}.$$

11. With the given parametrization,

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4,$$

so that

$$\int_C xy \, ds = \int_0^2 t^2 \cdot 4t \sqrt{(2t)^2 + 4^2} \, dt = \int_0^2 4t^3 \sqrt{4t^2 + 16} \, dt.$$

Now make the substitution $u = 4t^2 + 16$, so that $du = 8t \, dt$. Then

$$4t^3 \, dt = \frac{1}{2} t^2 \cdot 8t \, dt = \frac{1}{2} \cdot \frac{u - 16}{4} \, du = \frac{u - 16}{8} \, du.$$

Further $t = 0$ corresponds to $u = 16$ and $t = 2$ to $u = 32$, so we get

$$\begin{aligned} \int_{16}^{32} \frac{u - 16}{8} \cdot u^{1/2} \, du &= \frac{1}{8} \int_{16}^{32} (u^{3/2} - 16u^{1/2}) \, du \\ &= \frac{1}{8} \left[\frac{2}{5} u^{5/2} - \frac{32}{3} u^{3/2} \right]_{16}^{32} \\ &= \frac{1}{8} \left(\left(\frac{2}{5} \cdot 4096\sqrt{2} - \frac{32}{3} \cdot 128\sqrt{2} \right) - \left(\frac{2}{5} \cdot 1024 - \frac{32}{3} \cdot 64 \right) \right) \\ &= \boxed{\frac{512}{15}(1 + \sqrt{2})}. \end{aligned}$$

12. With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 6t,$$

so that

$$\int_C x \, ds = \int_0^1 t \sqrt{1^2 + (6t)^2} \, dt = \int_0^1 t \sqrt{36t^2 + 1} \, dt.$$

Now make the substitution $u = 36t^2 + 1$, so that $du = 72t \, dt$. Then $t = 0$ corresponds to $u = 1$, and $t = 1$ to $u = 37$, so we get

$$\int_1^{37} \frac{1}{72} u^{1/2} \, du = \frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_1^{37} = \boxed{\frac{1}{108}(37\sqrt{37} - 1)}.$$

13. With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1,$$

so that

$$\int_C \frac{y}{2x^2 - y^2} \, ds = \int_1^5 \frac{t}{2t^2 - t^2} \cdot \sqrt{1^2 + 1^2} \, dt = \sqrt{2} \int_1^5 \frac{1}{t} \, dt = \sqrt{2} [\ln t]_1^5 = \boxed{\sqrt{2} \ln 5}.$$

14. With the given parametrization,

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 4,$$

so that

$$\int_C \frac{x^2}{x^3 + y^3} \, ds = \int_1^2 \frac{(3t)^2}{(3t)^3 + (4t)^3} \cdot \sqrt{3^2 + 4^2} \, dt = \frac{45}{91} \int_1^2 \frac{1}{t} \, dt = \frac{45}{91} [\ln t]_1^2 = \boxed{\frac{45}{91} \ln 2}.$$

15. Parametrize the curve by $x = 2t$, $y = \frac{1}{(2t)^3} = 4t^3$ for $0 \leq t \leq 1$ (we use this parametrization instead of the more usual $x = t$ to avoid fractions). Then

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 12t^2,$$

so that

$$\int_C 2y \, ds = \int_0^1 8t^3 \sqrt{2^2 + (12t^2)^2} \, dt = \int_0^1 8t^3 \sqrt{144t^4 + 4} \, dt.$$

Now use the substitution $u = 144t^4 + 4$; then $du = 576t^3$, and $t = 0$ corresponds to $u = 4$ while $t = 1$ corresponds to $u = 148$, and we get

$$\frac{8}{576} \int_4^{148} u^{1/2} \, du = \frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_4^{148} = \frac{1}{108} (148^{3/2} - 4^{3/2}) = \boxed{\frac{2}{27} (37\sqrt{37} - 1)}.$$

16. Parametrize the curve by $y = t$, $x = 5t^3$ for $0 \leq t \leq 1$. Then

$$\frac{dx}{dt} = 15t^2, \quad \frac{dy}{dt} = 1,$$

so that

$$\int_C x \, ds = \int_0^1 5t^3 \sqrt{(15t^2)^2 + 1^2} \, dt = \int_0^1 5t^3 \sqrt{225t^4 + 1} \, dt.$$

Now use the substitution $u = 225t^4 + 1$; then $du = 900t^3$, and $t = 0$ corresponds to $u = 1$ while $t = 1$ corresponds to $u = 226$, and we get

$$\frac{5}{900} \int_1^{226} u^{1/2} \, du = \frac{1}{180} \left[\frac{2}{3} u^{3/2} \right]_1^{226} = \boxed{\frac{1}{270} (226\sqrt{226} - 1)}.$$

17. With the parametrization we get from $\mathbf{r}$, which is $x = \sin t$, $y = \cos t$, we have

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t,$$

so that

$$\begin{aligned} \int_C (xy + 1) \, ds &= \int_0^\pi (\sin t \cos t + 1) \sqrt{(\cos t)^2 + (-\sin t)^2} \, dt = \int_0^\pi (\sin t \cos t + 1) \, dt \\ &= \left[\frac{1}{2} \sin^2 t + t \right]_0^\pi = \boxed{\pi}. \end{aligned}$$

18. With the parametrization we get from $\mathbf{r}$, which is $x = \cos t$, $y = \sin t$, we have

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t,$$

so that

$$\begin{aligned} \int_C xy^2 \, ds &= \int_0^{\pi/2} \cos t \sin^2 t \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt \\ &= \int_0^{\pi/2} \cos t \sin^2 t \sqrt{\sin^2 t + \cos^2 t} \, dt = \int_0^{\pi/2} \cos t \sin^2 t \, dt = \left[\frac{1}{3} \sin^3 t \right]_0^{\pi/2} = \boxed{\frac{1}{3}}. \end{aligned}$$

19. (a) With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1,$$

so that

$$\int_C (x + y^2) \, ds = \int_0^1 (t + t^2) \sqrt{1^2 + 1^2} \, dt = \sqrt{2} \left[\frac{1}{2} t^2 + \frac{1}{3} t^3 \right]_0^1 = \boxed{\frac{5}{6} \sqrt{2}}.$$

(b) With the given parametrization,

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = \cos t,$$

so that

$$\begin{aligned} \int_C (x + y^2) ds &= \int_0^{\pi/2} (\sin t + \sin^2 t) \sqrt{\cos^2 t + \cos^2 t} dt \\ &= \sqrt{2} \int_0^{\pi/2} (\sin t \cos t + \sin^2 t \cos t) dt \\ &= \sqrt{2} \left[\frac{1}{2} \sin^2 t + \frac{1}{3} \sin^3 t \right]_0^{\pi/2} \\ &= \boxed{\frac{5}{6} \sqrt{2}}. \end{aligned}$$

20. (a) With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3,$$

so that

$$\int_C (x + y) ds = \int_0^1 4t \sqrt{1^2 + 3^2} dt = 4\sqrt{10} \int_0^1 t dt = 4\sqrt{10} \left[\frac{1}{2} t^2 \right]_0^1 = \boxed{2\sqrt{10}}.$$

(b) With the given parametrization,

$$\frac{dx}{dt} = 2 \cos(2t), \quad \frac{dy}{dt} = 6 \cos(2t),$$

so that

$$\int_C (x + y) ds = \int_0^{\pi/2} 4 \sin(2t) \cdot \sqrt{(2 \cos(2t))^2 + (6 \cos(2t))^2} dt = 8\sqrt{10} \int_0^{\pi/2} \sin(2t) \cdot \sqrt{\cos^2(2t)} dt.$$

We must now be careful: for $0 \leq t \leq \frac{\pi}{4}$, $\cos(2t) \geq 0$, but for $\frac{\pi}{4} \leq t \leq \frac{\pi}{2}$, $\cos(2t) \leq 0$. So when we take the square root of $\cos^2(2t)$, we must split this into two integrals:

$$\begin{aligned} 8\sqrt{10} \int_0^{\pi/2} \sin(2t) \cdot \sqrt{\cos^2(2t)} dt &= 8\sqrt{10} \left(\int_0^{\pi/4} \sin(2t) \cos(2t) dt - \int_{\pi/4}^{\pi/2} \sin(2t) \cos(2t) dt \right) \\ &= 8\sqrt{10} \left(\left[\frac{1}{4} \sin^2(2t) \right]_0^{\pi/4} - \left[\frac{1}{4} \sin^2(2t) \right]_{\pi/4}^{\pi/2} \right) \\ &= 8\sqrt{10} \left(\frac{1}{4} - \left(-\frac{1}{4} \right) \right) = \boxed{4\sqrt{10}}. \end{aligned}$$

21. (a) With the given parametrization,

$$\frac{dx}{dt} = -1, \quad \frac{dy}{dt} = 1,$$

so that

$$\int_C (x + e^y) ds = \int_0^1 (1 - t + e^t) \sqrt{(-1)^2 + 1^2} dt = \sqrt{2} \left[t - \frac{1}{2} t^2 + e^t \right]_0^1 = \boxed{\sqrt{2} e - \frac{1}{2} \sqrt{2}}.$$

(b) With the given parametrization,

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \sin t,$$

so that

$$\begin{aligned} \int_C (x + e^y) ds &= \int_0^{\pi/2} (\cos t + e^{1-\cos t}) \sqrt{(-\sin t)^2 + (\sin t)^2} dt \\ &= \sqrt{2} \int_0^{\pi/2} (\sin t \cos t + \sin t \cdot e^{1-\cos t}) dt \\ &= \sqrt{2} \left[\frac{1}{2} \sin^2 t + e^{1-\cos t} \right]_0^{\pi/2} \\ &= \sqrt{2} \left(\left(\frac{1}{2} + e \right) - (0 + 1) \right) \\ &= \boxed{\sqrt{2}e - \frac{1}{2}\sqrt{2}}. \end{aligned}$$

22. (a) With the given parametrization,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 1,$$

so that

$$\int_C x e^{x^2} ds = \int_0^1 (t+1) e^{(t+1)^2} \sqrt{1^2 + 1^2} dt = \sqrt{2} \int_0^1 (t+1) e^{(t+1)^2} dt.$$

Now use the substitution $u = (t+1)^2$, so that $du = 2(t+1) dt$; then $t = 0$ corresponds to $u = 1$ while $t = 1$ corresponds to $u = 4$, and we get

$$\sqrt{2} \int_0^1 (t+1) e^{(t+1)^2} dt = \sqrt{2} \int_1^4 \frac{1}{2} e^u du = \frac{\sqrt{2}}{2} [e^u]_1^4 = \boxed{\frac{\sqrt{2}}{2}(e^4 - e)}.$$

(b) With the given parametrization,

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = \cos t,$$

so that

$$\int_C x e^{x^2} ds = \int_0^{\pi/2} (1 + \sin t) e^{(1+\sin t)^2} \sqrt{\cos^2 t + \cos^2 t} dt = \sqrt{2} \int_0^{\pi/2} (1 + \sin t) e^{(1+\sin t)^2} \sqrt{\cos^2 t} dt.$$

To evaluate this integral, use the substitution $u = (1 + \sin t)^2$, so that $du = 2(1 + \sin t) \cos t$; then $t = 0$ corresponds to $u = 1$ and $t = \frac{\pi}{2}$ to $u = 4$. So the integral becomes

$$\sqrt{2} \int_1^4 \frac{1}{2} e^u du = \frac{\sqrt{2}}{2} [e^u]_1^4 = \boxed{\frac{\sqrt{2}}{2}(e^4 - e)}.$$

23. Since $g(x) = y = 3 - x$, we have $g'(x) = -1$, so that

$$\begin{aligned} \int_C f(x, y) dx &= \int_0^3 (2x + (3-x)^2) dx = \left[x^2 - \frac{1}{3}(3-x)^3 \right]_0^3 = \boxed{18} \\ \int_C f(x, y) dy &= \int_0^3 (2x + (3-x)^2) g'(x) dx = - \int_0^3 (2x + (3-x)^2) dx = - \left[x^2 - \frac{1}{3}(3-x)^3 \right]_0^3 = \boxed{-18}. \end{aligned}$$

24. $2x - y = 1$ is the same as $y = 2x - 1$, so with $g(x) = 2x - 1$ we have $g'(x) = 2$. Then

$$\begin{aligned}\int_C f(x, y) dx &= \int_0^1 f(x, y) dx = \int_0^1 x e^{2x-1} dx \\ \int_C f(x, y) dy &= \int_0^1 f(x, y) g'(x) dx = 2 \int_0^1 x e^{2x-1} dx.\end{aligned}$$

To perform this integration, use integration by parts with $u = x$, $dv = e^{2x-1} dx$, so that $du = dx$ and $v = \frac{1}{2}e^{2x-1}$. Then

$$\begin{aligned}\int_0^1 x e^{2x-1} dx &= \left[\frac{1}{2} x e^{2x-1} \right]_0^1 - \frac{1}{2} \int_0^1 e^{2x-1} dx \\ &= \left(\frac{1}{2} e - 0 \right) - \frac{1}{2} \left[\frac{1}{2} e^{2x-1} \right]_0^1 \\ &= \frac{1}{2} e - \frac{1}{4} (e - e^{-1}) \\ &= \frac{1}{4} \left(e + \frac{1}{e} \right)\end{aligned}$$

Then we get

$$\begin{aligned}\int_C f(x, y) dx &= \int_0^1 x e^{2x-1} dx = \boxed{\frac{1}{4} \left(e + \frac{1}{e} \right)} \\ \int_C f(x, y) dy &= 2 \int_0^1 x e^{2x-1} dx = 2 \left(\frac{1}{4} \left(e + \frac{1}{e} \right) \right) = \boxed{\frac{1}{2} \left(e + \frac{1}{e} \right)}.\end{aligned}$$

25. Solve for y to get $g(x) = y = \frac{1}{2} \ln x$, which is valid since the range of x we are considering is from 1 to e^2 , where $\ln x$ is defined. Then $g'(x) = \frac{1}{2x}$, and

$$\begin{aligned}\int_C f(x, y) dx &= \int_1^{e^2} \left(x e^{\frac{1}{2} \ln x} \right) dx = \int_1^{e^2} x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_1^{e^2} = \boxed{\frac{2}{5} (e^5 - 1)} \\ \int_C f(x, y) dy &= \int_1^{e^2} f(x, y) g'(x) dx = \int_1^{e^2} \left(x e^{\frac{1}{2} \ln x} \right) \cdot \frac{1}{2x} dx = \frac{1}{2} \int_1^{e^2} x^{1/2} dx \\ &= \frac{1}{2} \left[\frac{2}{3} x^{3/2} \right]_1^{e^2} = \boxed{\frac{1}{3} (e^3 - 1)}.\end{aligned}$$

26. Parametrize the curve by $y(t) = t$, $x(t) = 2t^2$ for $0 \leq t \leq \sqrt{\pi}$. Then $y'(t) = 1$ and $x'(t) = 4t$, so that

$$\begin{aligned}\int_C f(x, y) dx &= \int_0^{\sqrt{\pi}} f(x(t), y(t)) x'(t) dt = \int_0^{\sqrt{\pi}} (2t^2)^2 t \cdot 4t dt = 16 \int_0^{\sqrt{\pi}} t^6 dt = 16 \left[\frac{1}{7} t^7 \right]_0^{\sqrt{\pi}} \\ &= \boxed{\frac{16}{7} \pi^3 \sqrt{\pi}} \\ \int_C f(x, y) dy &= \int_0^{\sqrt{\pi}} f(x(t), y(t)) y'(t) dt = \int_0^{\sqrt{\pi}} (2t^2)^2 t \cdot 1 dt = 4 \int_0^{\sqrt{\pi}} t^5 dt = 4 \left[\frac{1}{6} t^6 \right]_0^{\sqrt{\pi}} = \boxed{\frac{2}{3} \pi^3}.\end{aligned}$$

27. (a) Using the given parametrization, $x'(t) = -\sin t$ and $y'(t) = \cos t$, so that

$$\begin{aligned}\int_C (x^2 y dx + xy dy) &= \int_C x^2 y dx + \int_C xy dy = \int_0^{\pi/2} x(t)^2 y(t) x'(t) dt + \int_0^{\pi/2} x(t) y(t) y'(t) dt \\ &= \int_0^{\pi/2} \cos^2 t \cdot \sin t \cdot (-\sin t) dt + \int_0^{\pi/2} \cos t \sin t \cdot \cos t dt \\ &= -\int_0^{\pi/2} \sin^2 t \cos^2 t dt + \int_0^{\pi/2} \cos^2 t \sin t dt.\end{aligned}$$

For the first integral, use the formulas for $\sin(2t)$ and $\cos(4t)$:

$$\sin^2 t \cos^2 t = (\sin t \cos t)^2 = \left(\frac{1}{2} \sin(2t)\right)^2 = \frac{1}{4} \sin^2(2t) = \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2} \cos(4t)\right) = \frac{1}{8} - \frac{1}{8} \cos(4t).$$

So the integral becomes

$$\begin{aligned}\int_C (x^2 y dx + xy dy) &= -\int_0^{\pi/2} \left(\frac{1}{8} - \frac{1}{8} \cos(4t)\right) dt + \int_0^{\pi/2} \cos^2 t \sin t dt \\ &= -\left[\frac{1}{8}t - \frac{1}{32} \sin(4t)\right]_0^{\pi/2} + \left[-\frac{1}{3} \cos^3 t\right]_0^{\pi/2} \\ &= -\left(\left(\frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{32} \sin(2\pi)\right) - (0 - 0)\right) + \left(0 + \frac{1}{3} \cos 0\right) \\ &= \boxed{\frac{1}{3} - \frac{\pi}{16}}.\end{aligned}$$

- (b) Set $y(t) = t$ and $x(t) = \sqrt{1-t^2}$; then $0 \leq t \leq 1$. So $y'(t) = 1$ and $x'(t) = -\frac{t}{\sqrt{1-t^2}}$, so that

$$\begin{aligned}\int_C (x^2 y dx + xy dy) &= \int_C x^2 y dx + \int_C xy dy = \int_0^1 x(t)^2 y(t) x'(t) dt + \int_0^1 x(t) y(t) y'(t) dt \\ &= \int_0^1 \left(\sqrt{1-t^2}\right)^2 \cdot t \cdot \frac{-t}{\sqrt{1-t^2}} dt + \int_0^1 \sqrt{1-t^2} \cdot t \cdot 1 dt \\ &= -\int_0^1 t^2 \sqrt{1-t^2} dt + \int_0^1 t \sqrt{1-t^2} dt.\end{aligned}$$

For the first integral, use the trigonometric substitution $t = \sin \theta$, so that $dt = \cos \theta d\theta$, and $\sqrt{1-t^2} = \cos \theta$. We are justified in using $\cos \theta$ rather than $-\cos \theta$ as the range of integration is from $t = 0$ to $t = 1$, which corresponds to $\theta = 0$ to $\theta = \frac{\pi}{2}$, and $\cos \theta$ is nonnegative there. For the second integral, use the substitution $u = 1 - t^2$, so that $du = -2t dt$; then $t = 0$ corresponds to $u = 1$, and $t = 1$ to $u = 0$. So we get

$$\begin{aligned}\int_C (x^2 y dx + xy dy) &= -\int_0^1 t^2 \sqrt{1-t^2} dt + \int_0^1 t \sqrt{1-t^2} dt \\ &= -\int_0^{\pi/2} \sin^2 \theta \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta - \frac{1}{2} \int_1^0 \sqrt{u} du \\ &= -\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta + \frac{1}{2} \int_0^1 \sqrt{u} du.\end{aligned}$$

We evaluated the first integral in part (a); use that value to continue:

$$\int_C (x^2 y dx + xy dy) = -\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta + \frac{1}{2} \int_0^1 \sqrt{u} du = -\frac{\pi}{16} + \frac{1}{2} \left[\frac{2}{3} u^{3/2}\right]_0^1 = \boxed{\frac{1}{3} - \frac{\pi}{16}}.$$

28. (a) Using the given parametrization, $x'(t) = -3 \sin t$ and $y'(t) = 3 \cos t$, so that

$$\begin{aligned}\int_C (xy \, dx + xy^2 \, dy) &= \int_C xy \, dx + \int_C xy^2 \, dy = \int_0^\pi x(t)y(t)x'(t) \, dt + \int_0^\pi x(t)y(t)^2 y'(t) \, dt \\ &= \int_0^\pi 3 \cos t \cdot 3 \sin t \cdot (-3 \sin t) \, dt + \int_0^\pi 3 \cos t \cdot (3 \sin t)^2 \cdot 3 \cos t \, dt \\ &= -27 \int_0^\pi \sin^2 t \cos t \, dt + 81 \int_0^\pi \sin^2 t \cos^2 t \, dt.\end{aligned}$$

For the second integral, use the formulas for $\sin(2t)$ and $\cos(4t)$:

$$\sin^2 t \cos^2 t = (\sin t \cos t)^2 = \left(\frac{1}{2} \sin(2t)\right)^2 = \frac{1}{4} \sin^2(2t) = \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2} \cos(4t)\right) = \frac{1}{8} - \frac{1}{8} \cos(4t).$$

So the integral becomes

$$\begin{aligned}\int_C (xy \, dx + xy^2 \, dy) &= -27 \int_0^\pi \sin^2 t \cos t \, dt + 81 \int_0^\pi \sin^2 t \cos^2 t \, dt \\ &= -27 \left[\frac{1}{3} \sin^3 t \right]_0^\pi + 81 \int_0^\pi \left(\frac{1}{8} - \frac{1}{8} \cos(4t) \right) dt \\ &= -27(0 - 0) + 81 \left[\frac{1}{8} t - \frac{1}{32} \sin(4t) \right]_0^\pi \\ &= 81 \left(\frac{1}{8} \pi - 0 \right) = \boxed{\frac{81}{8} \pi}.\end{aligned}$$

- (b) Set $g(x) = y = \sqrt{9 - x^2}$; then $g'(x) = -\frac{x}{\sqrt{9 - x^2}}$, so we get

$$\begin{aligned}\int_C (xy \, dx + xy^2 \, dy) &= \int_C xy \, dx + \int_C xy^2 \, dy \\ &= \int_{-3}^3 x \sqrt{9 - x^2} \, dx + \int_{-3}^3 x (\sqrt{9 - x^2})^2 \left(-\frac{x}{\sqrt{9 - x^2}} \right) dx \\ &= \int_{-3}^3 x \sqrt{9 - x^2} \, dx - \int_{-3}^3 x^2 \sqrt{9 - x^2} \, dx\end{aligned}$$

For the first integral, use the substitution $u = 9 - x^2$, so that $du = -2x \, dx$; then $x = -3$ corresponds to $u = 0$ as does $x = 3$. For the second integral, use the trigonometric substitution $x = 3 \cos \theta$; then $du = -3 \sin \theta \, d\theta$ and $\sqrt{9 - x^2} = \sqrt{9 - 9 \cos^2 \theta} = 3 \sin \theta$. We are justified in using this square root since $x = -3$ corresponds to $\theta = \pi$ and $x = 3$ to $\theta = 0$; $\sin \theta$ is nonnegative in this range. So the integral becomes

$$\begin{aligned}\int_C (xy \, dx + xy^2 \, dy) &= \int_{-3}^3 x \sqrt{9 - x^2} \, dx - \int_{-3}^3 x^2 \sqrt{9 - x^2} \, dx \\ &= -\frac{1}{2} \int_0^0 u^{1/2} \, du - \int_\pi^0 (3 \cos \theta)^2 \cdot 3 \sin \theta \cdot (-3 \sin \theta) \, d\theta \\ &= 0 + 81 \int_\pi^0 \sin^2 \theta \cos^2 \theta \, d\theta.\end{aligned}$$

We evaluated this integral in part (a); using that value, we get

$$\int_C (xy \, dx + xy^2 \, dy) = \boxed{-\frac{81}{8} \pi}.$$

Note that this value is the negative of the value in part (a). If we had integrated in the opposite direction, from $x = 3$ to $x = -3$, we would get the same value; this direction corresponds to x ranging from 0 to π in the first parametrization.

29. (a) Set $g(x) = y = 3x$; then $g'(x) = 3$, so we get

$$\begin{aligned}
 \int_C (y^2 dx + (xy - x^2) dy) &= \int_C y^2 dx + \int_C (xy - x^2)g'(x) dx \\
 &= \int_0^1 (3x)^2 dx + \int_0^1 (x \cdot 3x - x^2) \cdot 3 dx \\
 &= 9 \int_0^1 x^2 dx + 6 \int_0^1 x^2 dx \\
 &= 15 \int_0^1 x^2 dx \\
 &= 15 \left[\frac{1}{3} x^3 \right]_0^1 = \boxed{5}.
 \end{aligned}$$

(b) Solve for y to get $g(x) = y = 3\sqrt{x}$. Then $g'(x) = \frac{3}{2\sqrt{x}}$, so we get

$$\begin{aligned}
 \int_C (y^2 dx + (xy - x^2) dy) &= \int_C y^2 dx + \int_C (xy - x^2)g'(x) dx \\
 &= \int_0^1 9x dx + \int_0^1 (3x\sqrt{x} - x^2) \frac{3}{2\sqrt{x}} dx \\
 &= \int_0^1 9x dx + \int_0^1 \left(\frac{9}{2}x - \frac{3}{2}x^{3/2} \right) dx \\
 &= \left[\frac{9}{2}x^2 + \frac{9}{4}x^2 - \frac{3}{5}x^{5/2} \right]_0^1 \\
 &= \boxed{\frac{123}{20} = 6.15}.
 \end{aligned}$$

30. (a) Set $g(x) = y = x^2$; then $g'(x) = 2x$, so we get

$$\begin{aligned}
 \int_C ((x^2 + y^2) dx + 3x^2 y dy) &= \int_C (x^2 + y^2) dx + \int_C 3x^2 y g'(x) dx \\
 &= \int_{-2}^2 (x^2 + x^4) dx + \int_{-2}^2 3x^2 \cdot x^2 \cdot 2x dx \\
 &= \int_{-2}^2 (x^2 + x^4) dx + \int_{-2}^2 6x^5 dx \\
 &= \left[\frac{1}{3}x^3 + \frac{1}{5}x^5 \right]_{-2}^2 + [x^6]_{-2}^2 \\
 &= \left(\frac{1}{3}2^3 + \frac{1}{5}2^5 \right) - \left(\frac{1}{3}(-2)^3 + \frac{1}{5}(-2)^5 \right) + 0 \\
 &= \boxed{\frac{272}{15}}.
 \end{aligned}$$

(b) Set $g(x) = y = 4$; then $g'(x) = 0$, so we get

$$\begin{aligned}
 \int_C ((x^2 + y^2) dx + 3x^2 y dy) &= \int_C (x^2 + y^2) dx + \int_C 3x^2 y g'(x) dx \\
 &= \int_{-2}^2 (x^2 + 16) dx + \int_{-2}^2 3x^2 \cdot 4 \cdot 0 dx \\
 &= \int_{-2}^2 (x^2 + 16) dx \\
 &= \left[\frac{1}{3}x^3 + 16x \right]_{-2}^2 \\
 &= \boxed{\frac{208}{3}}.
 \end{aligned}$$

31. Let $g(x) = y = x^2$; then $g'(x) = 2x$, so that

$$\begin{aligned}
 \int_C ((x + 2y) dx + (2x + y) dy) &= \int_C (x + 2y) dx + \int_C (2x + y) g'(x) dx \\
 &= \int_0^1 (x + 2x^2) dx + \int_0^1 (2x + x^2) \cdot 2x dx \\
 &= \int_0^1 (x + 2x^2) dx + \int_0^1 (4x^2 + 2x^3) dx \\
 &= \left[\frac{1}{2}x^2 + \frac{2}{3}x^3 \right]_0^1 + \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 \right]_0^1 \\
 &= \boxed{3}.
 \end{aligned}$$

32. Let $g(x) = y = x^3$; then $g'(x) = 3x^2$, so that

$$\begin{aligned}
 \int_C ((x + 2y) dx + (2x + y) dy) &= \int_C (x + 2y) dx + \int_C (2x + y) g'(x) dx \\
 &= \int_0^1 (x + 2x^3) dx + \int_0^1 (2x + x^3) \cdot 3x^2 dx \\
 &= \int_0^1 (x + 2x^3) dx + \int_0^1 (6x^3 + 3x^5) dx \\
 &= \left[\frac{1}{2}x^2 + \frac{1}{2}x^4 \right]_0^1 + \left[\frac{3}{2}x^4 + \frac{1}{2}x^6 \right]_0^1 \\
 &= \boxed{3}.
 \end{aligned}$$

33. We have $x'(t) = -\sin t$ and $y'(t) = \cos t$, so that

$$\begin{aligned}
 \int_C ((x+2y) dx + (2x+y) dy) &= \int_C (x+2y) dx + \int_C (2x+y) dy \\
 &= \int_0^{\pi/2} (x(t) + 2y(t))x'(t) dt + \int_0^{\pi/2} (2x(t) + y(t))y'(t) dt \\
 &= \int_0^{\pi/2} (\cos t + 2\sin t)(-\sin t) dt + \int_0^{\pi/2} (2\cos t + \sin t)\cos t dt \\
 &= -\int_0^{\pi/2} (\sin t \cos t + 2\sin^2 t) dt + \int_0^{\pi/2} (2\cos^2 t + \sin t \cos t) dt \\
 &= \int_0^{\pi/2} (2\cos^2 t - 2\sin^2 t) dt \\
 &= \int_0^{\pi/2} 2\cos(2t) dt \\
 &= [\sin(2t)]_0^{\pi/2} = \boxed{0}.
 \end{aligned}$$

34. We have $x'(t) = -1$ and $y'(t) = 1$, so that

$$\begin{aligned}
 \int_C ((x+2y) dx + (2x+y) dy) &= \int_C (x+2y) dx + \int_C (2x+y) dy \\
 &= \int_0^1 (x(t) + 2y(t))x'(t) dt + \int_0^1 (2x(t) + y(t))y'(t) dt \\
 &= \int_0^1 (1-t+2t)(-1) dt + \int_0^1 (2(1-t) + t) \cdot 1 dt \\
 &= \int_0^1 (-t-1) dt + \int_0^1 (-t+2) dt \\
 &= \int_0^1 (1-2t) dt \\
 &= [t-t^2]_0^1 = \boxed{0}.
 \end{aligned}$$

35. The first curve may be parametrized by $x = t$, $y = 2 - t^2$, for $0 \leq t \leq 1$. Then $x'(t) = 1$, $y'(t) = -2t$. The second curve may be parametrized by $x = t$, $y = t$, for t from 1 to 0. Then $x'(t) = y'(t) = 1$. So the integral is

$$\int_C x ds = \int_0^1 t\sqrt{1^2 + (-2t)^2} dt + \int_1^0 t\sqrt{1^2 + 1^2} dt = \int_0^1 t\sqrt{4t^2 + 1} dt + \int_1^0 t\sqrt{2} dt.$$

For the first integral, use the substitution $u = 4t^2 + 1$, so that $du = 8t dt$; then $t = 0$ corresponds to $u = 1$ and $t = 1$ to $u = 5$, and we get

$$\begin{aligned}
 \int_C x ds &= \int_0^1 t\sqrt{4t^2 + 1} dt + \int_1^0 t\sqrt{2} dt \\
 &= \frac{1}{8} \int_1^5 u^{1/2} du + \left[\frac{\sqrt{2}}{2} t^2 \right]_1^0 \\
 &= \frac{1}{8} \left[\frac{2}{3} u^{3/2} \right]_1^5 - \frac{\sqrt{2}}{2} \\
 &= \boxed{\frac{5}{12}\sqrt{5} - \frac{1}{12} - \frac{1}{2}\sqrt{2}}.
 \end{aligned}$$

36. The first curve may be parametrized by $x = t$, $y = t^2$, for $0 \leq t \leq 1$. Then $x'(t) = 1$, $y'(t) = 2t$. The second curve may be parametrized by $x = t$, $y = 2 - t$, for $1 \leq t \leq 2$. Then $x'(t) = 1$ and $y'(t) = -1$. So the integral is

$$\int_C x \, ds = \int_0^1 t \sqrt{1^2 + (2t)^2} \, dt + \int_1^2 t \sqrt{1^2 + (-1)^2} \, dt = \int_0^1 t \sqrt{4t^2 + 1} \, dt + \int_1^2 t \sqrt{2} \, dt.$$

The first integral can be evaluated as in Problem 35 to get $\frac{1}{12}(5\sqrt{5} - 1)$, so we get

$$\int_C x \, ds = \frac{1}{12}(5\sqrt{5} - 1) + \left[\frac{\sqrt{2}}{2} t^2 \right]_1^2 = \boxed{\frac{1}{12}(5\sqrt{5} - 1) + \frac{3}{2}\sqrt{2}}.$$

37. The three curves, starting from $(1, 0)$, can be parametrized by:

$$\begin{aligned} x = t, \, y = t - 1, \, 1 \leq t \leq 2, \quad x'(t) = 1, \, y'(t) = 1 \\ x = t, \, y = 3 - t, \, 2 \leq t \leq 3, \quad x'(t) = 1, \, y'(t) = -1 \\ x = t, \, y = 0, \, t \text{ from } 3 \text{ to } 1, \quad x'(t) = 1, \, y'(t) = 0. \end{aligned}$$

So the integral is

$$\begin{aligned} \int_C x \, ds &= \int_1^2 t \sqrt{1^2 + 1^2} \, dt + \int_2^3 t \sqrt{1^2 + (-1)^2} \, dt + \int_3^1 t \sqrt{1^2 + 0^2} \, dt \\ &= \int_1^2 t \sqrt{2} \, dt + \int_2^3 t \sqrt{2} \, dt + \int_3^1 t \, dt \\ &= \sqrt{2} \int_1^2 t \, dt + \int_3^1 t \, dt \\ &= \sqrt{2} \left[\frac{1}{2} t^2 \right]_1^2 + \left[\frac{1}{2} t^2 \right]_3^1 \\ &= \sqrt{2} \left(\frac{9}{2} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{9}{2} \right) \\ &= \boxed{4\sqrt{2} - 4}. \end{aligned}$$

38. The three curves, starting from $(0, 0)$, can be parametrized by:

$$\begin{aligned} y = t, \, x = t^2, \, 0 \leq t \leq 1, \quad y'(t) = 1, \, x'(t) = 2t \\ x = t, \, y = 2 - t, \, 1 \leq t \leq 3, \quad x'(t) = 1, \, y'(t) = -1 \\ x = t, \, y = -\frac{1}{3}t, \, t \text{ from } 3 \text{ to } 0, \quad x'(t) = 1, \, y'(t) = -\frac{1}{3}. \end{aligned}$$

So the integral is

$$\begin{aligned} \int_C x \, ds &= \int_0^1 t^2 \sqrt{(2t)^2 + 1^2} \, dt + \int_1^3 t \sqrt{1^2 + (-1)^2} \, dt + \int_3^0 t \sqrt{1^2 + \left(-\frac{1}{3}\right)^2} \, dt \\ &= \int_0^1 t^2 \sqrt{4t^2 + 1} \, dt + \int_1^3 t \sqrt{2} \, dt + \int_3^0 t \frac{\sqrt{10}}{3} \, dt. \end{aligned}$$

For the first integral, we use the trigonometric substitution $t = \frac{1}{2} \tan \theta$; then $dt = \frac{1}{2} \sec^2 \theta$, and $\sqrt{4t^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta$. We are justified in using the positive square root, not the negative one, since $t = 0$ corresponds to $\theta = 0$ and $t = 1$ to $\theta = \tan^{-1} 2$, and $\sec \theta$ is positive on that range. So

we get

$$\begin{aligned}
 \int_C x \, ds &= \int_0^1 t^2 \sqrt{4t^2 + 1} \, dt + \int_1^3 t\sqrt{2} \, dt + \int_3^0 t \frac{\sqrt{10}}{3} \, dt \\
 &= \int_0^{\tan^{-1} 2} \left(\frac{1}{2} \tan \theta \right)^2 \cdot \sec \theta \cdot \frac{1}{2} \sec^2 \theta \, d\theta + \sqrt{2} \left[\frac{1}{2} t^2 \right]_1^3 + \frac{\sqrt{10}}{3} \left[\frac{1}{2} t^2 \right]_3^0 \\
 &= \frac{1}{8} \int_0^{\tan^{-1} 2} \tan^2 \theta \sec^3 \theta \, d\theta + 4\sqrt{2} - \frac{3}{2}\sqrt{10}.
 \end{aligned}$$

To evaluate the remaining integral, we use the substitution $\tan^2 \theta = \sec^2 \theta - 1$ and then the reduction formula from Example 8 in Section 7.1:

$$\begin{aligned}
 \int_C x \, ds &= \frac{1}{8} \int_0^{\tan^{-1} 2} \tan^2 \theta \sec^3 \theta \, d\theta + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{1}{8} \int_0^{\tan^{-1} 2} (\sec^2 \theta - 1) \sec^3 \theta \, d\theta + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{1}{8} \left(\int_0^{\tan^{-1} 2} \sec^5 \theta \, d\theta - \int_0^{\tan^{-1} 2} \sec^3 \theta \, d\theta \right) + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{1}{8} \left(\left[\frac{\sec^3 \theta \tan \theta}{4} \right]_0^{\tan^{-1} 2} + \frac{3}{4} \int_0^{\tan^{-1} 2} \sec^3 \theta \, d\theta - \int_0^{\tan^{-1} 2} \sec^3 \theta \, d\theta \right) + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{1}{8} \left(\frac{5\sqrt{5} \cdot 2}{4} - \frac{1}{4} \int_0^{\tan^{-1} 2} \sec^3 \theta \, d\theta \right) + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{1}{8} \left(\frac{5\sqrt{5}}{2} - \frac{1}{4} \left(\left[\frac{\sec \theta \tan \theta}{2} \right]_0^{\tan^{-1} 2} + \frac{1}{2} \int_0^{\tan^{-1} 2} \sec \theta \, d\theta \right) \right) + 4\sqrt{2} - \frac{3}{2}\sqrt{10} \\
 &= \frac{5\sqrt{5}}{16} + 4\sqrt{2} - \frac{3}{2}\sqrt{10} - \frac{1}{32} \cdot \frac{\sqrt{5} \cdot 2}{2} - \frac{1}{64} [\ln |\sec \theta + \tan \theta|]_0^{\tan^{-1} 2} \\
 &= \frac{9\sqrt{5}}{32} + 4\sqrt{2} - \frac{3}{2}\sqrt{10} - \frac{1}{64} (\ln(2 + \sqrt{5}) - \ln 1) \\
 &= \boxed{\frac{9\sqrt{5}}{32} + 4\sqrt{2} - \frac{3}{2}\sqrt{10} - \frac{1}{64} \ln(2 + \sqrt{5})}.
 \end{aligned}$$

39. Parametrize the segments as follows:

$$\begin{aligned}
 C_1 : (0, 0) \text{ to } (1, 0) : \quad x(t) &= t, \quad y(t) = 0, \quad 0 \leq t \leq 1 \\
 C_2 : (1, 0) \text{ to } (1, 1) : \quad x(t) &= 1, \quad y(t) = t, \quad 0 \leq t \leq 1 \\
 C_3 : (1, 1) \text{ to } (0, 1) : \quad x(t) &= 1 - t, \quad y(t) = 1, \quad 0 \leq t \leq 1.
 \end{aligned}$$

Then

$$\begin{aligned}
& \int_C (x \cos y \, dx - y \sin x \, dy) \\
&= \int_{C_1} (x \cos y \, dx - y \sin x \, dy) + \int_{C_2} (x \cos y \, dx - y \sin x \, dy) + \int_{C_3} (x \cos y \, dx - y \sin x \, dy) \\
&= \int_0^1 t \cos 0 \cdot 1 \, dt - \int_0^1 0 \cdot \sin t \cdot 0 \, dt + \int_0^1 1 \cos t \cdot 0 \, dt - \int_0^1 t \sin 1 \cdot 1 \, dt \\
&\quad + \int_0^1 (1-t) \cos 1 \cdot (-1) \, dt - \int_0^1 1 \cdot \sin(1-t) \cdot 0 \, dt \\
&= \int_0^1 t \, dt - 0 + 0 - \int_0^1 t \sin 1 \, dt - \int_0^1 (1-t) \cos 1 \, dt - 0 \\
&= \int_0^1 t \, dt - \int_0^1 t \sin 1 \, dt - \int_0^1 (1-t) \cos 1 \, dt \\
&= \left[\frac{1}{2} t^2 \right]_0^1 - \left[\frac{1}{2} t^2 \sin 1 \right]_0^1 - \left[-\frac{1}{2} (1-t)^2 \cos 1 \right]_0^1 \\
&= \boxed{\frac{1}{2} (1 - \sin 1 - \cos 1)}.
\end{aligned}$$

40. Parametrize the segments as follows:

$$\begin{aligned}
C_1 : (0, 0) \text{ to } (0, 1) : & \quad x(t) = 0, \, y(t) = t, \, 0 \leq t \leq 1 \\
C_2 : (0, 1) \text{ to } (1, 1) : & \quad x(t) = t, \, y(t) = 1, \, 0 \leq t \leq 1 \\
C_3 : (1, 1) \text{ to } (1, 0) : & \quad x(t) = 1, \, y(t) = 1 - t, \, 0 \leq t \leq 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_C (x \sin y \, dx - y \cos x \, dy) \\
&= \int_{C_1} (x \sin y \, dx - y \cos x \, dy) + \int_{C_2} (x \sin y \, dx - y \cos x \, dy) + \int_{C_3} (x \sin y \, dx - y \cos x \, dy) \\
&= \int_0^1 0 \sin t \cdot 0 \, dt - \int_0^1 t \cos 0 \cdot 1 \, dt + \int_0^1 t \sin 1 \cdot 1 \, dt - \int_0^1 1 \cos t \cdot 0 \, dt \\
&\quad + \int_0^1 1 \sin(1-t) \cdot 0 \, dt - \int_0^1 (1-t) \cos 1 \cdot (-1) \, dt \\
&= 0 - \int_0^1 t \, dt + \int_0^1 t \sin 1 \, dt - 0 + 0 + \int_0^1 (1-t) \cos 1 \, dt \\
&= -\int_0^1 t \, dt + \int_0^1 t \sin 1 \, dt + \int_0^1 (1-t) \cos 1 \, dt \\
&= \left[-\frac{1}{2} t^2 \right]_0^1 + \left[\frac{1}{2} t^2 \sin 1 \right]_0^1 + \left[-\frac{1}{2} (1-t)^2 \cos 1 \right]_0^1 \\
&= \boxed{-\frac{1}{2} + \frac{1}{2} (\sin 1 + \cos 1)}.
\end{aligned}$$

41. Parametrize the segments as follows:

$$\begin{aligned}
C_1 : (0, 0, 0) \text{ to } (1, 0, 0) : & \quad x(t) = t, \, y(t) = 0, \, z(t) = 0, \, 0 \leq t \leq 1 \\
C_2 : (1, 0, 0) \text{ to } (1, 1, 0) : & \quad x(t) = 1, \, y(t) = t, \, z(t) = 0, \, 0 \leq t \leq 1 \\
C_3 : (1, 1, 0) \text{ to } (1, 1, 1) : & \quad x(t) = 1, \, y(t) = 1, \, z(t) = t, \, 0 \leq t \leq 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_C (yz \, dx + xz \, dy + xy \, dz) \\
&= \int_{C_1} (yz \, dx + xz \, dy + xy \, dz) + \int_{C_2} (yz \, dx + xz \, dy + xy \, dz) + \int_{C_3} (yz \, dx + xz \, dy + xy \, dz) \\
&= \int_0^1 0 \cdot 0 \cdot 1 \, dt + \int_0^1 t \cdot 0 \cdot 0 \, dt + \int_0^1 t \cdot 0 \cdot 0 \, dt \\
&\quad + \int_0^1 t \cdot 0 \cdot 0 \, dt + \int_0^1 1 \cdot 0 \cdot 1 \, dt + \int_0^1 1 \cdot t \cdot 0 \, dt \\
&\quad + \int_0^1 1 \cdot t \cdot 0 \, dt + \int_0^1 1 \cdot t \cdot 0 \, dt + \int_0^1 1 \cdot 1 \cdot 1 \, dt \\
&= \int_0^1 1 \, dt \\
&= \boxed{1}.
\end{aligned}$$

42. Parametrize the segments as follows:

$$\begin{aligned}
C_1 : (0, 0, 0) \text{ to } (1, 0, 0) : \quad & x(t) = t, \, y(t) = 0, \, z(t) = 0, \, 0 \leq t \leq 1 \\
C_2 : (1, 0, 0) \text{ to } (1, 2, 0) : \quad & x(t) = 1, \, y(t) = t, \, z(t) = 0, \, 0 \leq t \leq 2 \\
C_3 : (1, 2, 0) \text{ to } (1, 2, 1) : \quad & x(t) = 1, \, y(t) = 2, \, z(t) = t, \, 0 \leq t \leq 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_C (yz \, dx + (y + zx) \, dy + xz \, dz) \\
&= \int_{C_1} (yz \, dx + (y + zx) \, dy + xz \, dz) + \int_{C_2} (yz \, dx + (y + zx) \, dy + xz \, dz) \\
&\quad + \int_{C_3} (yz \, dx + (y + zx) \, dy + xz \, dz) \\
&= \int_0^1 (0 \cdot 0 \cdot 1 \, dt) + \int_0^1 (0 + 0 \cdot t) \cdot 0 \, dt + \int_0^1 t \cdot 0 \cdot 0 \, dt \\
&\quad + \int_0^2 t \cdot 0 \cdot 0 \, dt + \int_0^2 (t + 0 \cdot 1) \cdot 1 \, dt + \int_0^2 1 \cdot 0 \cdot 0 \, dt \\
&\quad + \int_0^1 2t \cdot 0 \, dt + \int_0^1 (2 + t \cdot 1) \cdot 0 \, dt + \int_0^1 1 \cdot t \cdot 1 \, dt \\
&= \int_0^2 t \, dt + \int_0^1 t \, dt \\
&= \left[\frac{1}{2} t^2 \right]_0^2 + \left[\frac{1}{2} t^2 \right]_0^1 = \boxed{\frac{5}{2}}.
\end{aligned}$$

43. Parametrize the line segment joining $(2, 0)$ and $(0, 4)$ by $x(t) = 2 - 2t$, $y(t) = 4t$ for $0 \leq t \leq 1$. Then $x'(t) = -2$ and $y'(t) = 4$, and we get

$$\begin{aligned}
 \int_C (y \, dx + (x - 16y) \, dy) &= \int_0^1 y(t)x'(t) \, dt + \int_0^1 (x(t) - 16y(t))y'(t) \, dt \\
 &= - \int_0^1 8t \, dt + \int_0^1 (2 - 2t - 16 \cdot 4t) \cdot 4 \, dt \\
 &= - \int_0^1 8t \, dt + 4 \int_0^1 (2 - 66t) \, dt \\
 &= - [4t^2]_0^1 + 4 [2t - 33t^2]_0^1 \\
 &= \boxed{-128}.
 \end{aligned}$$

44. With $g(x) = y = 4 - x^2$, we get $g'(x) = -2x$, so the integral is

$$\begin{aligned}
 \int_C (y \, dx + (x - 16y) \, dy) &= \int_2^0 (4 - x^2) \, dx + \int_2^0 (x - 16(4 - x^2)) \cdot (-2x) \, dx \\
 &= \int_2^0 (4 - x^2) \, dx + \int_2^0 (-32x^3 - 2x^2 + 128x) \, dx \\
 &= \left[4x - \frac{1}{3}x^3 \right]_2^0 + \left[-8x^4 - \frac{2}{3}x^3 + 64x^2 \right]_2^0 \\
 &= \boxed{-128}.
 \end{aligned}$$

45. Parametrize the line segments as follows:

$$\begin{aligned}
 C_1 : (2, 0) \text{ to } (2, 2) : \quad x(t) &= 2, \quad y(t) = 2t, \quad 0 \leq t \leq 1 \\
 C_2 : (2, 2) \text{ to } (0, 4) : \quad x(t) &= 2 - 2t, \quad y(t) = 2 + 2t, \quad 0 \leq t \leq 1.
 \end{aligned}$$

Then

$$\begin{aligned}
 \int_C (y \, dx + (x - 16y) \, dy) &= \int_{C_1} (y \, dx + (x - 16y) \, dy) + \int_{C_2} (y \, dx + (x - 16y) \, dy) \\
 &= \int_0^1 2t \cdot 0 \, dt + \int_0^1 (2 - 32t) \cdot 2 \, dt + \int_0^1 (2 + 2t) \cdot (-2) \, dt \\
 &\quad + \int_0^1 (2 - 2t - 16(2 + 2t)) \cdot 2 \, dt \\
 &= \int_0^1 (4 - 64t) \, dt - \int_0^1 (4 + 4t) \, dt + \int_0^1 (-60 - 68t) \, dt \\
 &= [4t - 32t^2]_0^1 - [4t + 2t^2]_0^1 + [-60t - 34t^2]_0^1 \\
 &= \boxed{-128}.
 \end{aligned}$$

46. Parametrize the line segment by $x(t) = 1 - t$, $y(t) = 3 + t$, $0 \leq t \leq 1$, so that $x'(t) = -1$ and $y'(t) = 1$. Then the integral is

$$\begin{aligned}
 \int_C (y \, dx + (x - 16y) \, dy) &= \int_{C_1} (y \, dx + (x - 16y) \, dy) + \int_{C_2} (y \, dx + (x - 16y) \, dy) \\
 &= \int_2^1 (4 - x^2) \, dx + \int_2^1 (x - 16(4 - x^2)) \cdot (-2x) \, dx + \int_0^1 (3 + t) \cdot (-1) \, dt \\
 &\quad + \int_0^1 ((1 - t) - 16(3 + t)) \cdot 1 \, dt \\
 &= \int_2^1 (4 - x^2) \, dx - \int_2^1 (32x^3 + 2x^2 - 128x) \, dx \\
 &\quad - \int_0^1 (3 + t) \, dt - \int_0^1 (47 + 17t) \, dt \\
 &= \left[4x - \frac{1}{3}x^3 \right]_2^1 - \left[8x^4 + \frac{2}{3}x^3 - 64x^2 \right]_2^1 - \left[3t + \frac{1}{2}t^2 \right]_0^1 - \left[47t + \frac{17}{2}t^2 \right]_0^1 \\
 &= -\frac{5}{3} - \frac{202}{3} - \frac{7}{2} - \frac{111}{2} = \boxed{-128}.
 \end{aligned}$$

47. With the parametrization for $\mathbf{r}$, we have $\mathbf{F} = (t + 2t^2)\mathbf{i} + (2t + t^2)\mathbf{j}$. Also $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j}) \, dt = (\mathbf{i} + 2t\mathbf{j}) \, dt$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((t + 2t^2)\mathbf{i} + (2t + t^2)\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) \, dt = \int_0^1 (t + 6t^2 + 2t^3) \, dt = \left[\frac{1}{2}t^2 + 2t^3 + \frac{1}{2}t^4 \right]_0^1 = \boxed{3}.$$

48. With the parametrization for $\mathbf{r}$, we have

$$\mathbf{F} = \cos^2 t \, \mathbf{i} + \cos t \sin t \, \mathbf{j}, \quad d\mathbf{r} = (-\sin t \, \mathbf{i} + \cos t \, \mathbf{j}) \, dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (\cos^2 t \, \mathbf{i} + \cos t \sin t \, \mathbf{j}) \cdot (-\sin t \, \mathbf{i} + \cos t \, \mathbf{j}) \, dt = \int_0^\pi (-\sin t \cos^2 t + \sin t \cos^2 t) \, dt = \boxed{0}.$$

49. With the parametrization for $\mathbf{r}$, we have

$$\mathbf{F} = e^t \cdot e^{-t} \mathbf{i} + e^{2t} t^2 \mathbf{j} + e^t e^{-t} t^2 \mathbf{k} = \mathbf{i} + t^2 e^{2t} \mathbf{j} + t^2 \mathbf{k}, \quad d\mathbf{r} = (e^t \mathbf{i} - e^{-t} \mathbf{j} + 2t \mathbf{k}) \, dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\mathbf{i} + t^2 e^{2t} \mathbf{j} + t^2 \mathbf{k}) \cdot (e^t \mathbf{i} - e^{-t} \mathbf{j} + 2t \mathbf{k}) \, dt = \int_0^1 (e^t - t^2 e^t + 2t^3) \, dt.$$

From Example 7.1, we have $\int t^2 e^t \, dt = e^t(t^2 - 2t + 2) + C$, so the integral above becomes

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \left[e^t - e^t(t^2 - 2t + 2) + \frac{1}{2}t^4 \right]_0^1 \\
 &= \left(e - e(1 - 2 + 2) + \frac{1}{2} \right) - (1 - 1(0 - 0 + 2) + 0) = \boxed{\frac{3}{2}}.
 \end{aligned}$$

50. With the parametrization for $\mathbf{r}$, we have

$$\mathbf{F} = e^t \mathbf{i} + e^{4t} e^t \mathbf{j} + e^{2t} t \mathbf{k} = e^t \mathbf{i} + e^{5t} \mathbf{j} + t e^{2t} \mathbf{k}, \quad d\mathbf{r} = (2e^{2t} \mathbf{i} + e^t \mathbf{j} + \mathbf{k}) \, dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (e^t \mathbf{i} + e^{5t} \mathbf{j} + te^{2t} \mathbf{k}) \cdot (2e^{2t} \mathbf{i} + e^t \mathbf{j} + \mathbf{k}) dt = \int_0^1 (2e^{3t} + e^{6t} + te^{2t}) dt.$$

To integrate te^{2t} , use integration by parts, with $u = t$ and $dv = e^{2t} dt$, so that $du = dt$ and $v = \frac{1}{2}e^{2t}$. Then

$$\int te^{2t} dt = \frac{1}{2}te^{2t} - \frac{1}{2} \int e^{2t} dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C,$$

so that the integral above becomes

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (2e^{3t} + e^{6t} + te^{2t}) dt \\ &= \left[\frac{2}{3}e^{3t} + \frac{1}{6}e^{6t} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} \right]_0^1 \\ &= \boxed{\frac{2}{3}e^3 + \frac{1}{6}e^6 + \frac{1}{4}e^2 - \frac{7}{12}}. \end{aligned}$$

51. Parametrize $\mathbf{r}$ as $t\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = t\mathbf{i}$ for $0 \leq t \leq 1$. Then

$$\mathbf{F} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}, \quad d\mathbf{r} = \mathbf{i} dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{0} \cdot \mathbf{i} dt = \boxed{0}.$$

52. Parametrize $\mathbf{r}$ as $\mathbf{i} + 2t\mathbf{j} + 0\mathbf{k} = \mathbf{i} + 2t\mathbf{j}$ for $0 \leq t \leq 1$. Then

$$\mathbf{F} = 2t\mathbf{i} - 2t\mathbf{j} + 0\mathbf{k}, \quad d\mathbf{r} = 2\mathbf{j} dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t\mathbf{i} - 2t\mathbf{j}) \cdot (2\mathbf{j}) dt = \int_0^1 (-4t) dt = [-2t^2]_0^1 = \boxed{-2}.$$

53. Parametrize $\mathbf{r}$ as $\mathbf{i} + 2\mathbf{j} + 3t\mathbf{k}$ for $0 \leq t \leq 1$. Then

$$\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + 3t\mathbf{k}, \quad d\mathbf{r} = 3\mathbf{k} dt.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2\mathbf{i} - 2\mathbf{j} + 3t\mathbf{k}) \cdot (3\mathbf{k}) dt = \int_0^1 9t dt = \left[\frac{9}{2}t^2 \right]_0^1 = \boxed{\frac{9}{2}}.$$

54. Parametrize $\mathbf{r}$ as $t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$ for $0 \leq t \leq 1$. Then

$$\mathbf{F} = t \cdot 2t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k} = 2t^2\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}, \quad d\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} dt.$$

Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (2t^2\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) dt = \int_0^1 (2t^2 - 4t + 9t) dt \\ &= \int_0^1 (2t^2 + 5t) dt = \left[\frac{2}{3}t^3 + \frac{5}{2}t^2 \right]_0^1 = \boxed{\frac{19}{6}}. \end{aligned}$$

55. Along C , the differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1,$$

so that

$$\begin{aligned} A = \int_C f(x, y) ds &= \int_0^{2\pi} (1 - \sin^2 t) dt = \int_0^{2\pi} \left(1 - \left(\frac{1}{2} - \frac{1}{2} \cos(2t)\right)\right) dt \\ &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2t)\right) dt = \left[\frac{1}{2}t + \frac{1}{4} \sin(2t)\right]_0^{2\pi} = \boxed{\pi}. \end{aligned}$$

56. Along C , the differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{\cos^2 t + \sin^2 t} dt = dt,$$

so that

$$\begin{aligned} A = \int_C f(x, y) ds &= \int_0^{2\pi} (1 - \cos^2 t) dt = \int_0^{2\pi} \left(1 - \left(\frac{1}{2} + \frac{1}{2} \cos(2t)\right)\right) dt \\ &= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2t)\right) dt = \left[\frac{1}{2}t - \frac{1}{4} \sin(2t)\right]_0^{2\pi} = \boxed{\pi}. \end{aligned}$$

57. Along C , the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt = \sqrt{1 + \left(\frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)\right)^2} dt = \sqrt{1 + \frac{x^2}{1 - x^2}} dt = \frac{1}{\sqrt{1 - x^2}} dt,$$

so that

$$A = \int_C f(x, y) ds = \int_0^1 2x \sqrt{1 - x^2} \cdot \frac{1}{\sqrt{1 - x^2}} dx = \int_0^1 2x dx = [x^2]_0^1 = \boxed{1}.$$

58. Along C , the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt = \sqrt{1 + \left(\frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)\right)^2} dt = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} dt,$$

so that

$$\begin{aligned} A = \int_C f(x, y) ds &= \int_0^1 \left(x + \sqrt{1 - x^2}\right) \cdot \frac{1}{\sqrt{1 - x^2}} dx = \int_0^1 \left(\frac{x}{\sqrt{1 - x^2}} + 1\right) dx \\ &= \int_0^1 \frac{x}{\sqrt{1 - x^2}} dx + \int_0^1 1 dx. \end{aligned}$$

For the first integral, make the substitution $u = 1 - x^2$, so that $du = -2x dx$; then $x = 0$ corresponds to $u = 1$, and $x = 1$ to $u = 0$, and we get

$$A = \int_0^1 \frac{x}{\sqrt{1 - x^2}} dx + \int_0^1 1 dx = -\frac{1}{2} \int_1^0 u^{-1/2} du + 1 = -\frac{1}{2} [2u^{1/2}]_1^0 + 1 = \boxed{2}.$$

Applications and Extensions

59. Using the given parametrization for C , we have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt = \sqrt{a^2 + b^2} dt.$$

Then

$$\int_C \frac{ds}{x^2 + y^2 + z^2} = \int_0^1 \frac{\sqrt{a^2 + b^2}}{a^2 \cos^2 t + a^2 \sin^2 t + b^2 t^2} dt = \sqrt{a^2 + b^2} \int_0^1 \frac{1}{a^2 + b^2 t^2} dt.$$

Now make the substitution $u = \frac{b}{a}t$, so that $du = \frac{b}{a} dt$. Then $t = 0$ corresponds to $u = 0$, and $t = 1$ to $u = \frac{b}{a}$, so we get

$$\begin{aligned} \int_C \frac{ds}{x^2 + y^2 + z^2} &= \sqrt{a^2 + b^2} \int_0^1 \frac{1}{a^2 + b^2 t^2} dt \\ &= \frac{a\sqrt{a^2 + b^2}}{b} \int_0^{b/a} \frac{1}{a^2 + a^2 u^2} du \\ &= \frac{\sqrt{a^2 + b^2}}{ab} \int_0^{b/a} \frac{1}{1 + u^2} du \\ &= \frac{\sqrt{a^2 + b^2}}{ab} [\tan^{-1} u]_0^{b/a} \\ &= \boxed{\frac{\sqrt{a^2 + b^2}}{ab} \tan^{-1} \frac{b}{a}}. \end{aligned}$$

60. Using the given parametrization for C , we have $x'(t) = a \cos t$ and $y'(t) = -b \sin t$. Then

$$\begin{aligned} \int_C (y^n dx + x^n dy) &= \int_C y^n dx + \int_C x^n dy \\ &= \int_0^1 y(t)^n x'(t) dt + \int_0^1 x(t)^n y'(t) dt \\ &= \int_0^{2\pi} b^n \cos^n t \cdot a \cos t dt - \int_0^{2\pi} a^n \sin^n t \cdot b \sin t dt \\ &= ab^n \int_0^{2\pi} \cos^{n+1} t dt - ba^n \int_0^{2\pi} \sin^{n+1} t dt. \end{aligned}$$

We will use numbers 90 and 91 in the table of integrals in the Endnotes to evaluate these integrals.

We have

$$\begin{aligned} \int_0^{2\pi} \cos^{n+1} t dt &= \left[\frac{1}{n+1} \cos^n t \sin t \right]_0^{2\pi} + \frac{n}{n+1} \int_0^{2\pi} \cos^{n-1} t dt \\ &= 0 + \frac{n}{n+1} \int_0^{2\pi} \cos^{n-1} t dt \\ &= \frac{n}{n+1} \left(\left[\frac{1}{n-1} \cos^{n-2} t \sin t \right]_0^{2\pi} + \frac{n-2}{n-1} \int_0^{2\pi} \cos^{n-3} t dt \right) \\ &= \frac{n(n-2)}{(n+1)(n-1)} \int_0^{2\pi} \cos^{n-3} t dt. \end{aligned}$$

Continuing this process, we will end up with either

$$\begin{aligned} \int_0^{2\pi} \cos^{n+1} t dt &= \frac{n(n-2) \cdots 2}{(n+1)(n-1) \cdots 3} \int_0^{2\pi} \cos t dt = \frac{n(n-2) \cdots 2}{(n+1)(n-1) \cdots 3} [\sin t]_0^{2\pi} = 0 \\ \int_0^{2\pi} \cos^{n+1} t dt &= \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2} \int_0^{2\pi} 1 dt = 2\pi \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2}, \end{aligned}$$

where the first equation applies if n is even and the second if n is odd.

Similarly, we have

$$\begin{aligned}
 \int_0^{2\pi} \sin^{n+1} t \, dt &= \left[-\frac{1}{n+1} \sin^n t \cos t \right]_0^{2\pi} + \frac{n}{n+1} \int_0^{2\pi} \sin^{n-1} t \, dt \\
 &= 0 + \frac{n}{n+1} \int_0^{2\pi} \cos^{n-1} t \, dt \\
 &= \frac{n}{n+1} \left(\left[-\frac{1}{n-1} \sin^{n-2} t \cos t \right]_0^{2\pi} + \frac{n-2}{n-1} \int_0^{2\pi} \sin^{n-3} t \, dt \right) \\
 &= \frac{n(n-2)}{(n+1)(n-1)} \int_0^{2\pi} \sin^{n-3} t \, dt.
 \end{aligned}$$

Continuing this process, we will end up with either

$$\begin{aligned}
 \int_0^{2\pi} \sin^{n+1} t \, dt &= \frac{n(n-2) \cdots 2}{(n+1)(n-1) \cdots 3} \int_0^{2\pi} \sin t \, dt = -\frac{n(n-2) \cdots 2}{(n+1)(n-1) \cdots 3} [\cos t]_0^{2\pi} = 0 \\
 \int_0^{2\pi} \sin^{n+1} t \, dt &= \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2} \int_0^{2\pi} 1 \, dt = 2\pi \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2},
 \end{aligned}$$

where the first equation applies if n is even and the second if n is odd.

Using these computations in the original integral

$$\int_C (y^n \, dx + x^n \, dy) = ab^n \int_0^{2\pi} \cos^{n+1} t \, dt - ba^n \int_0^{2\pi} \sin^{n+1} t \, dt$$

shows that the integral is 0 if n is even, while if n is odd, we get

$$\begin{aligned}
 2\pi \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2} (ab^n - ba^n) &= 2\pi ab \frac{n(n-2) \cdots 1}{(n+1)(n-1) \cdots 2} (b^{n-1} - a^{n-1}) \\
 &= 2\pi ab \left(\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{n-2}{n-1} \cdot \frac{n}{n+1} \right) (b^{n-1} - a^{n-1}).
 \end{aligned}$$

61. Using the given parametrization, we have

$$\begin{aligned}
 ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} \, dt \\
 &= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \, dt = a\sqrt{2 - 2\cos t}.
 \end{aligned}$$

The first arch of the cycloid extends from $t = 0$ (since $y(0) = 0$) to $t = 2\pi$, which is the next value of t for which $y(t) = 0$. Then we get

$$\begin{aligned}
 \int_C y^2 \, ds &= \int_0^{2\pi} a^2(1 - \cos t)^2 \cdot a\sqrt{2 - 2\cos t} \, dt \\
 &= a^3\sqrt{2} \int_0^{2\pi} (1 - \cos t)^2 \sqrt{1 - \cos t} \, dt \\
 &= a^3\sqrt{2} \int_0^{2\pi} (1 - \cos t)^{5/2} \, dt.
 \end{aligned}$$

To integrate this, note that since

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \text{ or } 1 - \cos(2x) = 2\sin^2 x;$$

setting $x = \frac{t}{2}$ gives $1 - \cos t = 2 \sin^2 \frac{t}{2}$. Substituting for $1 - \cos t$ then gives

$$\int_C y^2 ds = a^3 \sqrt{2} \int_0^{2\pi} \left(2 \sin^2 \frac{t}{2} \right)^{5/2} dt = 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} dt.$$

Now use the fact that $\sin^2 \frac{t}{2} = 1 - \cos^2 \frac{t}{2}$:

$$\begin{aligned} \int_C y^2 ds &= 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} dt \\ &= 8a^3 \int_0^{2\pi} \sin \frac{t}{2} \left(1 - \cos^2 \frac{t}{2} \right)^2 dt \\ &= 8a^3 \int_0^{2\pi} \sin \frac{t}{2} \left(1 - 2 \cos^2 \frac{t}{2} + \cos^4 \frac{t}{2} \right) dt \\ &= 8a^3 \left[-2 \cos \frac{t}{2} + \frac{4}{3} \cos^3 \frac{t}{2} - \frac{2}{5} \cos^5 \frac{t}{2} \right]_0^{2\pi} \\ &= 8a^3 \left(\left(2 - \frac{4}{3} + \frac{2}{5} \right) - \left(-2 + \frac{4}{3} - \frac{2}{5} \right) \right) \\ &= \boxed{\frac{256}{15} a^3}. \end{aligned}$$

62. Using the given parametrization, we have

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt = \sqrt{a^2(-\sin t + \sin t + t \cos t)^2 + a^2(\cos t - \cos t + t \sin t)^2} dt \\ &= a \sqrt{t^2(\cos^2 t + \sin^2 t)} = at \end{aligned}$$

since $0 \leq t \leq 2\pi$. Also,

$$\begin{aligned} \sqrt{x^2 + y^2} &= \sqrt{a^2(\cos t + t \sin t)^2 + a^2(\sin t - t \cos t)^2} \\ &= a \sqrt{\cos^2 t + 2t \sin t \cos t + t^2 \sin^2 t + \sin^2 t - 2t \sin t \cos t + t^2 \cos^2 t} \\ &= a \sqrt{1 + t^2}. \end{aligned}$$

Therefore

$$\int_C \sqrt{x^2 + y^2} ds = \int_0^{2\pi} a \sqrt{1 + t^2} \cdot at dt = a^2 \int_0^{2\pi} t \sqrt{1 + t^2} dt.$$

Now use the substitution $u = 1 + t^2$, so that $du = 2t dt$; then $t = 0$ corresponds to $u = 1$ and $t = 2\pi$ to $u = 1 + 4\pi^2$, so we get

$$\int_C \sqrt{x^2 + y^2} ds = \frac{1}{2} a^2 \int_1^{1+4\pi^2} u^{1/2} du = \frac{1}{2} a^2 \left[\frac{2}{3} u^{3/2} \right]_1^{1+4\pi^2} = \boxed{\frac{1}{3} a^2 \left((1 + 4\pi^2)^{3/2} - 1 \right)}.$$

63. Using the given parametrization for C , we have

$$\begin{aligned} \int_C (z dx + x dy + y dz) &= \int_C z dx + \int_C x dy + \int_C y dz \\ &= \int_0^{2\pi} t \cdot (-a \sin t) dt + \int_0^{2\pi} a \cos t \cdot (a \cos t) dt + \int_0^{2\pi} a \sin t \cdot 1 dt \\ &= -a \int_0^{2\pi} t \sin t dt + a^2 \int_0^{2\pi} \cos^2 t dt + a \int_0^{2\pi} \sin t dt. \end{aligned}$$

For the first integral, use integration by parts with $u = t$, $dv = \sin t \, dt$, so that $du = dt$ and $v = -\cos t$. Then

$$\begin{aligned}\int_C (z \, dx + x \, dy + y \, dz) &= -a \left([-t \cos t]_0^{2\pi} + \int_0^{2\pi} \cos t \, dt \right) + a^2 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2t) \right) dt + a \int_0^{2\pi} \sin t \, dt \\ &= -a \left(-2\pi + [\sin t]_0^{2\pi} \right) + a^2 \left[\frac{1}{2}t + \frac{1}{4} \sin(2t) \right]_0^{2\pi} + a [-\cos t]_0^{2\pi} \\ &= \boxed{2\pi a + \pi a^2}.\end{aligned}$$

64. Using the given parametrization, we have

$$\begin{aligned}\int_C (xz \, dx + (y+z) \, dy + x \, dz) &= \int_C xz \, dx + \int_C (y+z) \, dy + \int_C x \, dz \\ &= \int_0^1 e^t \cdot e^{2t} \cdot e^t \, dt + \int_0^1 (e^{-t} + e^{2t}) \cdot (-e^{-t}) \, dt \\ &\quad + \int_0^1 e^t \cdot 2e^{2t} \, dt \\ &= \int_0^1 e^{4t} \, dt - \int_0^1 (e^{-2t} + e^t) \, dt + 2 \int_0^1 e^{3t} \, dt \\ &= \left[\frac{1}{4} e^{4t} \right]_0^1 - \left[-\frac{1}{2} e^{-2t} + e^t \right]_0^1 + 2 \left[\frac{1}{3} e^{3t} \right]_0^1 \\ &= \boxed{\frac{1}{4} e^4 + \frac{2}{3} e^3 - e + \frac{1}{2e^2} - \frac{5}{12}}.\end{aligned}$$

65. The mass is $M = \int_C \rho(x, y) \, ds$. Along the curve $y = x^2$, the element ds is

$$ds = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \sqrt{1 + (2x)^2} dx = \sqrt{4x^2 + 1} dx.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y) \, ds = \int_1^2 4x \sqrt{4x^2 + 1} \, dx.$$

Use the substitution $u = 4x^2 + 1$, so that $du = 8x \, dx$; then $x = 1$ corresponds to $u = 5$ while $x = 2$ corresponds to $u = 17$, and

$$M = \frac{1}{2} \int_5^{17} u^{1/2} \, du = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_5^{17} = \boxed{\frac{1}{3} (17\sqrt{17} - 5\sqrt{5})}.$$

66. The mass is $M = \int_C \rho(x, y) \, ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} = \sqrt{9 \sin^2 t + 9 \cos^2 t} = 3.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y) \, ds = \int_0^{\pi/2} (3 \sin t + 3 \cos t + 1) \cdot 3 \, dt = 3 [-3 \cos t + 3 \sin t + t]_0^{\pi/2} = \boxed{\frac{3\pi}{2} + 18}.$$

67. The mass is $M = \int_C \rho(x, y) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(4t)^2 + 1^2} = \sqrt{16t^2 + 1}.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y) ds = \int_{-2}^3 3 \cdot 2t^2 \sqrt{16t^2 + 1} dt = 6 \int_{-2}^3 t^2 \sqrt{16t^2 + 1} dt.$$

To evaluate the integral, we use the trigonometric substitution $t = \frac{1}{4} \tan \theta$; then $dt = \frac{1}{4} \sec^2 \theta$, and $\sqrt{16t^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta$. We are justified in using the positive square root, not the negative one, since $t = -2$ corresponds to $\theta = \tan^{-1}(-8)$ and $t = 3$ to $\theta = \tan^{-1} 12$, and $\sec \theta$ is positive on that range. So we get

$$\begin{aligned} M &= 6 \int_{-2}^3 t^2 \sqrt{16t^2 + 1} dt \\ &= 6 \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \left(\frac{1}{4} \tan \theta\right)^2 \cdot \sec \theta \cdot \frac{1}{4} \sec^2 \theta d\theta \\ &= \frac{3}{32} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \tan^2 \theta \sec^3 \theta d\theta. \end{aligned}$$

Now use the substitution $\tan^2 \theta = \sec^2 \theta - 1$ and then the reduction formula from Example 8 in Section 7.1:

$$\begin{aligned} \int_C x ds &= \frac{3}{32} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \tan^2 \theta \sec^3 \theta d\theta \\ &= \frac{3}{32} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} (\sec^2 \theta - 1) \sec^3 \theta d\theta \\ &= \frac{3}{32} \left(\int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec^5 \theta d\theta - \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec^3 \theta d\theta \right) \\ &= \frac{3}{32} \left(\left[\frac{\sec^3 \theta \tan \theta}{4} \right]_{\tan^{-1}(-8)}^{\tan^{-1} 12} + \frac{3}{4} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec^3 \theta d\theta - \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec^3 \theta d\theta \right) \\ &= \frac{3}{32} \left(\frac{145\sqrt{145} \cdot 12}{4} - \frac{65\sqrt{65} \cdot (-8)}{4} - \frac{1}{4} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec^3 \theta d\theta \right) \\ &= \frac{3}{32} \left(435\sqrt{145} + 130\sqrt{65} - \frac{1}{4} \left(\left[\frac{\sec \theta \tan \theta}{2} \right]_{\tan^{-1}(-8)}^{\tan^{-1} 12} + \frac{1}{2} \int_{\tan^{-1}(-8)}^{\tan^{-1} 12} \sec \theta d\theta \right) \right) \\ &= \frac{1305\sqrt{145} + 390\sqrt{65}}{32} - \frac{3\sqrt{145} \cdot 12}{256} + \frac{3\sqrt{65} \cdot (-8)}{256} - \frac{3}{256} [\ln |\sec \theta + \tan \theta|]_{\tan^{-1}(-8)}^{\tan^{-1} 12} \\ &= \frac{1305\sqrt{145} + 390\sqrt{65}}{32} - \frac{9\sqrt{145} + 6\sqrt{65}}{64} - \frac{3}{256} [\ln |\sec \theta + \tan \theta|]_{\tan^{-1}(-8)}^{\tan^{-1} 12} \\ &= \frac{2601\sqrt{145} + 774\sqrt{65}}{64} - \frac{3}{256} (\ln(\sqrt{145} + 12) - \ln(\sqrt{65} - 8)) \\ &= \boxed{\frac{2601}{64} \sqrt{145} + \frac{387}{32} \sqrt{65} - \frac{3}{256} (\ln(\sqrt{145} + 12) - \ln(\sqrt{65} - 8))}. \end{aligned}$$

68. The mass is $M = \int_C \rho(x, y) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + (2t)^2} dt = \sqrt{4t^2 + 1} dt.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y) ds = \int_0^3 t(t^2 - 1 + 1) \sqrt{4t^2 + 1} dt = \int_0^3 t^3 \sqrt{4t^2 + 1} dt.$$

To evaluate the integral, we use the trigonometric substitution $t = \frac{1}{2} \tan \theta$; then $dt = \frac{1}{2} \sec^2 \theta$, and $\sqrt{4t^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta$. We are justified in using the positive square root, not the negative one, since $t = 0$ corresponds to $\theta = 0$ and $t = 3$ to $\theta = \tan^{-1} 6$, and $\sec \theta$ is positive on that range. So we get

$$M = \int_0^{\tan^{-1} 6} \left(\frac{1}{2} \tan \theta\right)^3 \cdot \sec \theta \cdot \frac{1}{2} \sec^2 \theta d\theta = \frac{1}{16} \int_0^{\tan^{-1} 6} \tan^3 \theta \sec^3 \theta d\theta.$$

Now replace a factor of $\tan^2 \theta$ by $\sec^2 \theta - 1$, giving

$$M = \frac{1}{16} \int_0^{\tan^{-1} 6} \tan \theta (\sec^2 \theta - 1) \sec^3 \theta d\theta = \frac{1}{16} \int_0^{\tan^{-1} 6} (\sec^4 \theta - \sec^2 \theta) \tan \theta \sec \theta d\theta.$$

Finally, use the substitution $u = \sec \theta$, so that $du = \tan \theta \sec \theta$. Then $\theta = 0$ corresponds to $u = \sec 0 = 1$, and $\theta = \tan^{-1} 6$ corresponds to $u = \sec(\tan^{-1} 6) = \sqrt{37}$, and we get

$$M = \frac{1}{16} \int_1^{\sqrt{37}} (u^4 - u^2) du = \frac{1}{16} \left[\frac{1}{5} u^5 - \frac{1}{3} u^3 \right]_1^{\sqrt{37}} = \boxed{\frac{1 + 1961\sqrt{37}}{120}}.$$

69. The mass is $M = \int_C \rho(x, y) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}.$$

Two turns of the spring corresponds to the range $0 \leq \theta \leq 4\pi$. Therefore we get for the integral

$$M = \int_C \rho(x, y, z) ds = \int_0^{4\pi} \rho \sqrt{5} dt = \boxed{4\pi \rho \sqrt{5}}.$$

70. The mass is $M = \int_C \rho(x, y) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{9 \sin^2 t + 9 \cos^2 t + 1} = \sqrt{10}.$$

Two turns of the spring corresponds to the range $0 \leq \theta \leq 4\pi$. Therefore we get for the integral

$$M = \int_C \rho(x, y, z) ds = \int_0^{4\pi} (t + 2) \sqrt{10} dt = \sqrt{10} \left[\frac{1}{2} t^2 + 2t \right]_0^{4\pi} = \boxed{\sqrt{10} (8\pi^2 + 8\pi)}.$$

71. By analogy with the formula for the mass of a wire in the plane, the formula is

$$\boxed{M = \int_C \rho(x, y, z) ds}.$$

72. The mass is $M = \int_C \rho(x, y, z) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t + 25} = \sqrt{29}.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y, z) ds = \int_0^{2\pi} k \sqrt{29} dt = \boxed{2\pi k \sqrt{29}}.$$

73. The mass is $M = \int_C \rho(x, y, z) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{25} = 5.$$

Therefore we get for the integral

$$M = \int_C \rho(x, y, z) ds = \int_0^{2\pi} (2 + 4t) \cdot 5 dt = 5 [2t + 2t^2]_0^{2\pi} = 5(4\pi + 8\pi^2) = \boxed{20(\pi + 2\pi^2)}.$$

74. The mass is $M = \int_C \rho(x, y, z) ds$. Along the given parametrized curve, the element ds is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{4\sin^2 t + 4\cos^2 t + 25} = \sqrt{29}.$$

The mass density is

$$k(x^2 + y^2 + z^2) = k(4\cos^2 t + 4\sin^2 t + 25t^2) = k(25t^2 + 4).$$

Therefore we get for the integral

$$\begin{aligned} M &= \int_C \rho(x, y, z) ds \\ &= k \int_0^{2\pi} (25t^2 + 4) \cdot \sqrt{29} dt \\ &= k\sqrt{29} \int_0^{2\pi} (25t^2 + 4) dt \\ &= k\sqrt{29} \left[\frac{25}{3}t^3 + 4t \right]_0^{2\pi} \\ &= \boxed{k\sqrt{29} \left(\frac{200}{3}\pi^3 + 8\pi \right)}. \end{aligned}$$

75. Parametrize the circle by $x = R \cos t$, $y = R \sin t$, $0 \leq t \leq 2\pi$. Then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R.$$

Then we get for the lateral surface area

$$A = \int_C f(x, y) dx = \int_0^{2\pi} hR dt = \boxed{2\pi Rh}.$$

76. We have two parametrizations:

$$\begin{aligned} C_1 : x &= \cos \theta, \quad y = \sin \theta, \quad x' = -\sin \theta, \quad y' = \cos \theta, \quad 0 \leq \theta \leq 4\pi \\ C_2 : x &= \cos(t^2), \quad y = \sin(t^2), \quad x' = -2t \sin(t^2), \quad y' = 2t \cos(t^2), \quad 0 \leq t \leq 2\sqrt{\pi}. \end{aligned}$$

Then

$$\int_{C_1} (P dx + Q dy) = \int_0^{4\pi} (\cos^3 \theta \sin \theta \cdot (-\sin \theta) + \sin^2 \theta \cos \theta \cdot (\cos \theta)) d\theta.$$

Now make the substitution $t = \sqrt{\theta}$, so that $\theta = t^2$ and $d\theta = 2t dt$. Then $\theta = 0$ corresponds to $t = 0$ and $\theta = 4\pi$ to $t = 2\sqrt{\pi}$, and the integral becomes

$$\int_{C_1} (P dx + Q dy) = \int_0^{2\sqrt{\pi}} (\cos^3(t^2) \sin(t^2) \cdot (-\sin(t^2)) + \sin^2(t^2) \cos(t^2) \cos(t^2)) 2t dt.$$

Now look at the other integral:

$$\begin{aligned}\int_{C_2} (P dx + Q dy) &= \int_0^{2\sqrt{\pi}} (\cos^3(t^2) \sin(t^2) \cdot (-2t \sin(t^2)) + \sin^2(t^2) \cos(t^2) \cdot (2t \cos(t^2))) dt \\ &= \int_0^{2\sqrt{\pi}} (\cos^3(t^2) \sin(t^2) \cdot (-\sin(t^2)) + \sin^2(t^2) \cos(t^2) \cos(t^2)) 2t dt.\end{aligned}$$

This is the same as the first integral after the substitution. The reason that these two are the same is that the two curves trace exactly the same path, but at different speeds, as t and θ vary over their ranges.

Challenge Problems

77. With the given parametrization, we have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{1^2 + (3\sqrt{2}t)^2} dt = \sqrt{18t^2 + 1} dt.$$

Then the integral is

$$\int_C (x + y) ds = \int_0^1 \left(t + \frac{3}{\sqrt{2}}t^2\right) \sqrt{18t^2 + 1} dt = \int_0^1 t \sqrt{18t^2 + 1} dt + \frac{3}{\sqrt{2}} \int_0^1 t^2 \sqrt{18t^2 + 1} dt.$$

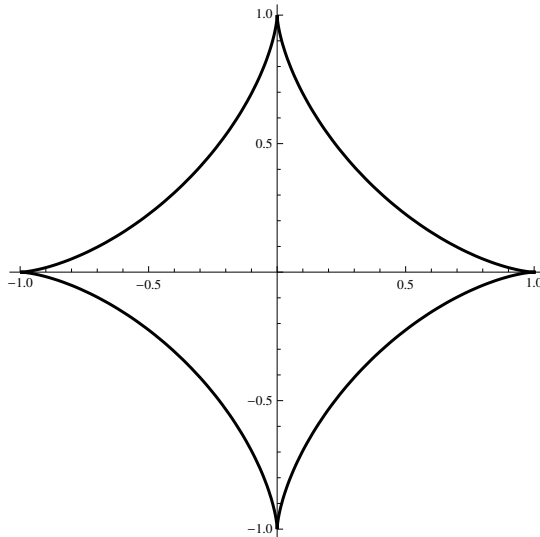
For the first integral, use the substitution $u = 18t^2 + 1$, so that $du = 36t dt$; then $t = 0$ corresponds to $u = 1$ while $t = 1$ corresponds to $u = 19$. For the second integral, use the trigonometric substitution $t = \frac{1}{\sqrt{18}} \tan \theta$; then $18t^2 + 1 = \sec^2 \theta$ and $dt = \frac{1}{\sqrt{18}} \sec^2 \theta d\theta$. Here $t = 0$ corresponds to $\theta = 0$ and $t = 1$ to $\theta = \tan^{-1} \sqrt{18}$. So we get

$$\begin{aligned}\int_C (x + y) ds &= \frac{1}{36} \int_1^{19} u^{1/2} du + \frac{3}{\sqrt{2}} \int_0^{\tan^{-1} \sqrt{18}} \left(\frac{1}{\sqrt{18}} \tan \theta\right)^2 \cdot \sec \theta \cdot \frac{1}{\sqrt{18}} \sec^2 \theta d\theta \\ &= \frac{1}{36} \left[\frac{2}{3} u^{3/2}\right]_1^{19} + \frac{1}{36} \int_0^{\tan^{-1} \sqrt{18}} \tan^2 \theta \sec^3 \theta d\theta\end{aligned}$$

Now use the substitution $\tan^2 \theta = \sec^2 \theta - 1$ and then the reduction formula from Example 8 in Section 7.1:

$$\begin{aligned}
 \int_C (x + y) ds &= \frac{1}{36} \left[\frac{2}{3} u^{3/2} \right]_1^{19} + \frac{1}{36} \int_0^{\tan^{-1} \sqrt{18}} \tan^2 \theta \sec^3 \theta d\theta \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{1}{36} \int_0^{\tan^{-1} \sqrt{18}} (\sec^2 \theta - 1) \sec^3 \theta d\theta \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{1}{36} \left(\int_0^{\tan^{-1} \sqrt{18}} \sec^5 \theta d\theta - \int_0^{\tan^{-1} \sqrt{18}} \sec^3 \theta d\theta \right) \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{1}{36} \left(\left[\frac{\sec^3 \theta \tan \theta}{4} \right]_0^{\tan^{-1} \sqrt{18}} \right. \\
 &\quad \left. + \frac{3}{4} \int_0^{\tan^{-1} \sqrt{18}} \sec^3 \theta d\theta - \int_0^{\tan^{-1} \sqrt{18}} \sec^3 \theta d\theta \right) \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{1}{36} \left(\frac{19\sqrt{19}\sqrt{18}}{4} - \frac{1}{4} \int_0^{\tan^{-1} \sqrt{18}} \sec^3 \theta d\theta \right) \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{1}{36} \left(\frac{19\sqrt{19}\sqrt{18}}{4} \right. \\
 &\quad \left. - \frac{1}{4} \left(\left[\frac{\sec \theta \tan \theta}{2} \right]_0^{\tan^{-1} \sqrt{18}} + \frac{1}{2} \int_0^{\tan^{-1} \sqrt{18}} \sec \theta d\theta \right) \right) \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{19\sqrt{19}\sqrt{18}}{144} - \frac{\sqrt{19}\sqrt{18}}{288} - \frac{1}{288} [\ln |\sec \theta + \tan \theta|]_0^{\tan^{-1} \sqrt{18}} \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{19\sqrt{19}\sqrt{18}}{144} - \frac{\sqrt{19}\sqrt{18}}{288} - \frac{1}{288} \ln(\sqrt{19} + \sqrt{18}) \\
 &= \frac{1}{54} (19\sqrt{19} - 1) + \frac{57\sqrt{38}}{144} - \frac{3\sqrt{38}}{288} - \frac{1}{288} \ln(\sqrt{19} + \sqrt{18}) \\
 &= \boxed{\frac{1}{54} (19\sqrt{19} - 1) + \frac{37\sqrt{38}}{96} - \frac{1}{288} \ln(\sqrt{19} + 3\sqrt{2})}.
 \end{aligned}$$

78. The hypocycloid (with $a = 1$) for $0 \leq t \leq 2\pi$ is



Since the wire is homogeneous, we assume its mass density is 1, by changing units if necessary.

(a) With the given parametrization, we have

$$\begin{aligned}
 ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \sqrt{(3a \cos^2 t (-\sin t))^2 + (3a \sin^2 t \cos t)^2} dt \\
 &= \sqrt{9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t} dt \\
 &= 3a \sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} dt \\
 &= 3a \sin t \cos t.
 \end{aligned}$$

Then the mass of one arm of the hypocycloid is

$$M = \int_0^{\pi/2} 1 ds = 3a \int_0^{\pi/2} \sin t \cos t dt = \frac{3a}{2} \int_0^{\pi/2} \sin(2t) dt = \frac{3a}{2} \left[-\frac{1}{2} \cos(2t) \right]_0^{\pi/2} = \frac{3a}{2}.$$

Then

$$\begin{aligned}
 M_y &= \int_C x ds \\
 &= \int_0^{\pi/2} a \cos^3 t \cdot 3a \sin t \cos t dt \\
 &= 3a^2 \int_0^{\pi/2} \cos^4 t \sin t dt \\
 &= 3a^2 \left[-\frac{1}{5} \cos^5 t \right]_0^{\pi/2} = \frac{3}{5} a^2,
 \end{aligned}$$

so that

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{3}{5}a^2}{\frac{3}{2}a} = \frac{2}{5}a.$$

By symmetry, $\bar{y} = \frac{2}{5}a$ as well, so that the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{2}{5}a, \frac{2}{5}a \right).$$

(b) By symmetry, the center of mass lies at the origin.

79. (a) In the definition of the line integral of f along C , Δs_i is the distance from $(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$, so it is

$$\begin{aligned}
 \Delta s_i &= \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2} \\
 &= \sqrt{\left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right)^2 + \left(\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\right)^2} (t_i - t_{i-1}).
 \end{aligned}$$

Now, $x(t)$ is continuous, so that by the Mean Value Theorem, there is some $t_i^* \in [t_{i-1}, t_i]$ such that

$$\frac{dx}{dt}(t_i^*) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}},$$

and similarly there is some $u_i^* \in [t_{i-1}, t_i]$ such that

$$\frac{dy}{dt}(u_i^*) = \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}.$$

So the equation for Δs_i becomes

$$\Delta s_i = \sqrt{\left(\frac{dx}{dt}(t_i^*)\right)^2 + \left(\frac{dy}{dt}(u_i^*)\right)^2} (t_i - t_{i-1}) = \sqrt{\left(\frac{dx}{dt}(t_i^*)\right)^2 + \left(\frac{dy}{dt}(u_i^*)\right)^2} \Delta t.$$

Substituting into the definition of the line integral gives

$$\begin{aligned} \int_C f(x, y) ds &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*) \sqrt{\left(\frac{dx}{dt}(t_i^*)\right)^2 + \left(\frac{dy}{dt}(u_i^*)\right)^2} \Delta t \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \end{aligned}$$

- (b) We know that $\int_C f(x, y) ds = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s$. A parametrization of C gives $x(t)$ and $y(t)$ for $a \leq t \leq b$, and then

$$s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2} du$$

is the length of arc traced at each value t , so that $\Delta s = s_{i+1} - s_i = s(t_{i+1}) - s(t_i)$. By the Mean Value Theorem, there is some $t_i^* \in [t_i, t_{i+1}]$ such that

$$s(t_{i+1}) - s(t_i) = s'(t_i^*) \Delta t = \sqrt{x'(t_i^*)^2 + y'(t_i^*)^2} \Delta t.$$

Therefore

$$\begin{aligned} \int_C f(x, y) ds &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x(t_i^*), y(t_i^*)) \sqrt{x'(t_i^*)^2 + y'(t_i^*)^2} \Delta t \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \end{aligned}$$

80. If C is expressed as $(x(s), y(s))$ where the parameter s represents arc length, then the arc length from $s = 0$ to $s = a$ is

$$\int_0^a 1 ds = \int_0^a \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} ds.$$

Therefore $\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1$, so the integral becomes

$$\int_C f(x, y) dx = \int_a^b f(x(s), y(s)) ds.$$

81. (a) If $P = P(x, y)$ and $Q = Q(x, y)$ and C is parametrized using the arc length parameter s , then

$$\begin{aligned} P(x, y) dx + Q(x, y) dy &= P(x, y) x'(s) ds + Q(x, y) y'(s) ds \\ &= (P(x(s), y(s)) x'(s) + Q(x(s), y(s)) y'(s)) ds. \end{aligned}$$

Then $f(s) = P(x(s), y(s)) x'(s) + Q(x(s), y(s)) y'(s)$.

(b) Suppose $s = s(x, y)$. Then

$$ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy,$$

so that

$$\int_C f ds = \int_C f \left(\frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy \right) = \int_C \left(f \frac{\partial s}{\partial x} \right) dx + \left(f \frac{\partial s}{\partial y} \right) dy.$$

Applying this to the example given, we have $s = \sqrt{x^2 + y^2}$, since C parametrizes the line $y = x$ for $0 \leq x \leq 1$. Then

$$\frac{\partial s}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial s}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}},$$

so that by the above,

$$\begin{aligned} \int_C (x + y^2) ds &= \int_C \left(\left((x + y^2) \cdot \frac{x}{\sqrt{x^2 + y^2}} \right) dx + \left((x + y^2) \cdot \frac{y}{\sqrt{x^2 + y^2}} \right) dy \right) \\ &= \int_C \left[\frac{x^2 + xy^2}{\sqrt{x^2 + y^2}} dx + \frac{xy + y^3}{\sqrt{x^2 + y^2}} dy \right]. \end{aligned}$$

15.3: Fundamental Theorem of Line Integrals

Concepts and Vocabulary

1. False. In fact f is a potential function for $\mathbf{F}$, not the other way around.
2. (d) is correct. See the discussion at the end of Objective 1.
3. True. This is part of the content of the Fundamental Theorem of Line Integrals.
4. (c) is correct. See the definition at the start of this section.
5. A piecewise-smooth curve C is *closed* if its initial point (x_0, y_0) and its terminal point (x_1, y_1) are the same.
6. $\int \mathbf{F} \cdot d\mathbf{r} = 0$. See Value of a Line Integral over a Closed Curve.
7. False. What is required is that *any* two points can be connected by such a curve. This is defined in Objective 2.
8. True. See the discussion of the converse in the text.
9. False. C is closed if $\mathbf{r}(a) = \mathbf{r}(b)$.
10. True. A simple curve is one that does not intersect itself; this is true of a parabola.
11. False. See the definition in Objective 4.
12. If and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; see Conservative Vector Field Test in the text.

Skill Building

13. The given vector field, $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j}$ is conservative. To see this, note that both P and Q as well as their derivatives are continuous on all of the plane, and that

$$\frac{\partial P}{\partial y} = -e^x \sin y = \frac{\partial Q}{\partial x}.$$

So once we find a potential function, we can evaluate the three line integrals given. To find a potential function, start by integrating $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int e^x \cos y dx = e^x \cos y + h(y).$$

Then we want the partial derivative with respect to y of this function to equal $Q(x, y)$:

$$\frac{\partial}{\partial y} (e^x \cos y + h(y)) = -e^x \sin y + h'(y) = -e^x \sin y,$$

so that $h'(y) = 0$ and h is a constant. So $f(x, y) = e^x \cos y + K$ is a potential function for $\mathbf{F}$. Therefore:

(a) This is a piecewise smooth parametrization with endpoints $(0, \frac{\pi}{3})$ and $(1, \pi)$, so that

$$\int_C (e^x \cos y dx - e^x \sin y dy) = f(1, \pi) - f\left(0, \frac{\pi}{3}\right) = \boxed{-e - \frac{1}{2}}.$$

(b) This is a smooth parametrization with endpoints $(0, \frac{\pi}{3})$ and $(1, \pi)$, so that again

$$\int_C (e^x \cos y dx - e^x \sin y dy) = f(1, \pi) - f\left(0, \frac{\pi}{3}\right) = \boxed{-e - \frac{1}{2}}.$$

(c) This is another piecewise smooth parametrization with endpoints $(0, \frac{\pi}{3})$ and $(1, \pi)$, so that

$$\int_C (e^x \cos y dx - e^x \sin y dy) = f(1, \pi) - f\left(0, \frac{\pi}{3}\right) = \boxed{-e - \frac{1}{2}}.$$

14. The given vector field, $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = e^y \cos x \mathbf{i} + e^y \sin x \mathbf{j}$ is conservative. To see this, note that both P and Q as well as their derivatives are continuous on all of the plane, and that

$$\frac{\partial P}{\partial y} = e^y \cos x = \frac{\partial Q}{\partial x}.$$

So once we find a potential function, we can evaluate the three line integrals given. To find a potential function, start by integrating $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int e^y \cos x dx = e^y \sin x + h(y).$$

Then we want the partial derivative with respect to y of this function to equal $Q(x, y)$:

$$\frac{\partial}{\partial y} (e^y \sin x + h(y)) = e^y \sin x + h'(y) = e^y \sin x,$$

so that $h'(y) = 0$ and h is a constant. So $f(x, y) = e^y \sin x + K$ is a potential function for $\mathbf{F}$. Therefore:

(a) This is a smooth parametrization with endpoints $(0, 0)$ and $(\pi, 1)$, so that

$$\int_C (e^y \cos x dx + e^y \sin x dy) = f(\pi, 1) - f(0, 0) = 0 - 0 = \boxed{0}.$$

(b) This is a piecewise smooth parametrization with endpoints $(0, 0)$ and $(\pi, 1)$, so that again

$$\int_C (e^y \cos x dx + e^y \sin x dy) = f(\pi, 1) - f(0, 0) = 0 - 0 = \boxed{0}.$$

(c) This is another piecewise smooth parametrization with endpoints $(0, 0)$ and $(\pi, 1)$, so that

$$\int_C (e^y \cos x \, dx + e^y \sin x \, dy) = f(\pi, 1) - f(0, 0) = 0 - 0 = \boxed{0}.$$

15. By the Fundamental Theorem of Line Integrals,

$$\int_C [(2x - y) \, dx + (2y - x) \, dy] = f(5, 4) - f(3, -1) = (5^2 - 5 \cdot 4 + 4^2) - (3^2 - 3 \cdot (-1) + (-1)^2) = \boxed{8}.$$

16. By the Fundamental Theorem of Line Integrals,

$$\int_C [(y^2 + 2x) \, dx + 2xy \, dy] = f(-3, 8) - f(2, 2) = (-3 \cdot 8^2 + (-3)^2) - (2 \cdot 2^2 + 2^2) = \boxed{-195}.$$

17. By the Fundamental Theorem of Line Integrals,

$$\int_C (3x^2y \, dx + x^3 \, dy) = f(3, 4) - f(0, -1) = 3^3 \cdot 4 - 0^3 \cdot (-1) = \boxed{108}.$$

18. By the Fundamental Theorem of Line Integrals,

$$\int_C (2xe^{x^2+y^2} \, dx + 2ye^{x^2+y^2} \, dy) = f(1, 1) - f(0, 0) = e^{1^2+1^2} - e^{0^2+0^2} = \boxed{e^2 - 1}.$$

19. By the Fundamental Theorem of Line Integrals,

$$\begin{aligned} \int_C \left(\frac{x}{x^2 + y^2} \, dx + \frac{y}{x^2 + y^2} \, dy \right) &= f(5, 12) - f(3, 4) = \ln \sqrt{5^2 + 12^2} - \ln \sqrt{3^2 + 4^2} \\ &= \boxed{\ln 13 - \ln 5 = \ln \frac{13}{5}}. \end{aligned}$$

20. By the Fundamental Theorem of Line Integrals,

$$\int_C \left(\frac{x}{\sqrt{x^2 + y^2}} \, dx + \frac{y}{\sqrt{x^2 + y^2}} \, dy \right) = f(3, 4) - f(0, -4) = \sqrt{3^2 + 4^2} - \sqrt{0^2 + (-4)^2} = 5 - 4 = \boxed{1}.$$

21. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(1, 1)$ to $(2, 4)$. Choose the path consisting of

$$\begin{aligned} C_1 : (1, 1) \text{ to } (2, 1), \quad x &= 1 + t, \, y = 1, \, 0 \leq t \leq 1 \\ C_2 : (2, 1) \text{ to } (2, 4), \quad x &= 2, \, y = 1 + 3t, \, 0 \leq t \leq 1. \end{aligned}$$

Then

$$\begin{aligned} \int_C (x^3 \, dx + y^3 \, dy) &= \int_{C_1} (x^3 \, dx + y^3 \, dy) + \int_{C_2} (x^3 \, dx + y^3 \, dy) \\ &= \int_0^1 ((1+t)^3 \cdot 1 + 1^3 \cdot 0) \, dt + \int_0^1 (2^3 \cdot 0 + (1+3t)^3 \cdot 3) \, dt \\ &= \int_0^1 (1+t)^3 \, dt + 3 \int_0^1 (1+3t)^3 \, dt \\ &= \left[\frac{1}{4}(1+t)^4 \right]_0^1 + 3 \left[\frac{1}{12}(1+3t)^4 \right]_0^1 \\ &= \left(4 - \frac{1}{4} \right) + 3 \left(\frac{64}{3} - \frac{1}{12} \right) \\ &= \boxed{\frac{135}{2}}. \end{aligned}$$

22. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(-1, 2)$ to $(2, 3)$. Choose the path consisting of

$$C_1 : (-1, 2) \text{ to } (2, 2), \quad x = -1 + t, \quad y = 2, \quad 0 \leq t \leq 3$$

$$C_2 : (2, 2) \text{ to } (2, 3), \quad x = 2, \quad y = 2 + t, \quad 0 \leq t \leq 1.$$

Then

$$\begin{aligned} \int_C (2x^2 dx - 4y dy) &= \int_{C_1} (2x^2 dx - 4y dy) + \int_{C_2} (2x^2 dx - 4y dy) \\ &= \int_0^3 (2(-1+t)^2 \cdot 1 - 4 \cdot 2 \cdot 0) dt + \int_0^1 (2 \cdot 2^2 \cdot 0 - 4(2+t) \cdot 1) dt \\ &= \int_0^3 2(t-1)^2 dt - \int_0^1 (8+4t) dt \\ &= \left[\frac{2}{3}(t-1)^3 \right]_0^3 - [8t + 2t^2]_0^1 \\ &= \left(\frac{16}{3} + \frac{2}{3} \right) - (10 - 0) = \boxed{-4}. \end{aligned}$$

23. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(0, 0)$ to $(2, 0)$. Choose the path parametrized as $x = 2t$, $y = 0$ for $0 \leq t \leq 1$. Then

$$\begin{aligned} \int_C [(x^2 + y^2) dx + (2xy + \sin y) dy] &= \int_0^1 [((2t)^2 + 0^2) \cdot 2 + (2 \cdot 2t \cdot 0 + \sin 0) \cdot 0] dt \\ &= \int_0^1 8t^2 dt \\ &= \left[\frac{8}{3}t^3 \right]_0^1 = \boxed{\frac{8}{3}}. \end{aligned}$$

24. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(1, \frac{\pi}{2})$ to $(2, \pi)$. Choose the path consisting of

$$C_1 : \left(1, \frac{\pi}{2}\right) \text{ to } \left(2, \frac{\pi}{2}\right), \quad x = 1 + t, \quad y = \frac{\pi}{2}, \quad 0 \leq t \leq 1$$

$$C_2 : \left(2, \frac{\pi}{2}\right) \text{ to } (2, \pi), \quad x = 2, \quad y = \frac{\pi}{2}(t+1), \quad 0 \leq t \leq 1.$$

Then

$$\begin{aligned} \int_C [(x - \cos y) dx + x \sin y dy] &= \int_{C_1} [(x - \cos y) dx + x \sin y dy] + \int_{C_2} [(x - \cos y) dx + x \sin y dy] \\ &= \int_0^1 \left[\left((t+1) - \cos \frac{\pi}{2} \right) \cdot 1 + (t+1) \sin \frac{\pi}{2} \cdot 0 \right] dt \\ &\quad + \int_0^1 \left[\left(2 - \cos \left(\frac{\pi}{2}(t+1) \right) \right) \cdot 0 + 2 \sin \left(\frac{\pi}{2}(t+1) \right) \cdot \frac{\pi}{2} \right] dt \\ &= \int_0^1 (t+1) dt + \int_0^1 \pi \sin \left(\frac{\pi}{2}(t+1) \right) dt \\ &= \left[\frac{1}{2}(t+1)^2 \right]_0^1 + \pi \left[-\frac{2}{\pi} \cos \left(\frac{\pi}{2}(t+1) \right) \right]_0^1 \\ &= \left(2 - \frac{1}{2} \right) + \pi \left(-\frac{2}{\pi} \cdot (-1) + \frac{2}{\pi} \cdot 0 \right) \\ &= \boxed{\frac{7}{2}}. \end{aligned}$$

25. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(0, 0)$ to $(\ln 2, 2)$. Choose the path consisting of

$$\begin{aligned} C_1 : (0, 0) \text{ to } (\ln 2, 0), \quad x = t \ln 2, \quad y = 0, \quad 0 \leq t \leq 1 \\ C_2 : (\ln 2, 0) \text{ to } (\ln 2, 2), \quad x = \ln 2, \quad y = 2t, \quad 0 \leq t \leq 1. \end{aligned}$$

Then

$$\begin{aligned} \int_C (y^2 e^x dx + 2ye^x dy) &= \int_{C_1} (y^2 e^x dx + 2ye^x dy) + \int_{C_2} (y^2 e^x dx + 2ye^x dy) \\ &= \int_0^1 (0^2 \cdot e^{t \ln 2} \cdot \ln 2 + 2 \cdot 0 \cdot e^{t \ln 2} \cdot 0) dt \\ &\quad + \int_0^1 ((2t)^2 e^{\ln 2} \cdot 0 + 2 \cdot 2t \cdot e^{\ln 2} \cdot 2) dt \\ &= \int_0^1 16t dt \\ &= [8t^2]_0^1 = \boxed{8}. \end{aligned}$$

26. Since $\mathbf{F}$ is conservative, and $P(x, y)$, $Q(x, y)$, and their derivatives are continuous everywhere, we may choose any piecewise smooth path from $(0, 0)$ to $(2, 1)$. Choose the path consisting of

$$\begin{aligned} C_1 : (0, 0) \text{ to } (2, 0), \quad x = 2t, \quad y = 0, \quad 0 \leq t \leq 1 \\ C_2 : (2, 0) \text{ to } (2, 1), \quad x = 2, \quad y = t, \quad 0 \leq t \leq 1. \end{aligned}$$

Then

$$\begin{aligned} \int_C \left[xe^y dx + \frac{1}{2}(x^2 - 4)e^y dy \right] &= \int_{C_1} \left[xe^y dx + \frac{1}{2}(x^2 - 4)e^y dy \right] + \int_{C_2} \left[xe^y dx + \frac{1}{2}(x^2 - 4)e^y dy \right] \\ &= \int_0^1 \left[2t \cdot e^0 \cdot 2 + \frac{1}{2}((2t)^2 - 4)e^0 \cdot 0 \right] dt \\ &\quad + \int_0^1 \left[2 \cdot e^t \cdot 0 + \frac{1}{2}(2^2 - 4) \cdot e^t \cdot 1 \right] dt \\ &= \int_0^1 4t dt \\ &= [2t^2]_0^1 = \boxed{2}. \end{aligned}$$

27. Since $\mathbf{F}$ is conservative and C is a closed curve, the line integral is zero by Value of a Line Integral over a Closed Curve.
28. Since $\mathbf{F}$ is conservative and C is a closed curve, the line integral is zero by Value of a Line Integral over a Closed Curve.
29. We have $P(x, y) = 5x - 2y + 1$ and $Q(x, y) = 4 - 2x$. Start by integrating P with respect to x :

$$\int P(x, y) dx = \int (5x - 2y + 1) dx = \frac{5}{2}x^2 - 2xy + x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 - 2x + 0 + h'(y) = 4 - 2x,$$

so that $h'(y) = 4$. Integrating with respect to y gives $h(y) = 4y + K$ where K is a constant. So

$$\boxed{f(x, y) = \frac{5}{2}x^2 - 2xy + x + 4y + K}.$$

30. We have $P(x, y) = 4xy$ and $Q(x, y) = 2x^2 + y$. Start by integrating P with respect to x :

$$\int P(x, y) dx = \int 4xy dx = 2x^2y + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$2x^2 + h'(y) = 2x^2 + y,$$

so that $h'(y) = y$. Integrating with respect to y gives $h(y) = \frac{1}{2}y^2 + K$ where K is a constant. So

$$\boxed{f(x, y) = 2x^2y + \frac{1}{2}y^2 + K}.$$

31. We have $P(x, y) = \ln y + 2x$ and $Q(x, y) = \frac{x}{y}$. Start by integrating P with respect to x :

$$\int P(x, y) dx = \int (\ln y + 2x) dx = x \ln y + x^2 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$\frac{x}{y} + 0 + h'(y) = \frac{x}{y},$$

so that $h'(y) = 0$. Integrating with respect to y gives $h(y) = K$ where K is a constant. So

$$\boxed{f(x, y) = x \ln y + x^2 + K}.$$

32. We have $P(x, y) = ye^x$ and $Q(x, y) = e^x$. Start by integrating P with respect to x :

$$\int P(x, y) dx = \int ye^x dx = ye^x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$e^x + h'(y) = e^x,$$

so that $h'(y) = 0$. Integrating with respect to y gives $h(y) = K$ where K is a constant. So

$$\boxed{f(x, y) = ye^x + K}.$$

33. We have $P(x, y) = x^2$ and $Q(x, y) = y^2$. Since P , Q , and their derivatives are continuous everywhere, and $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$, the Conservative Vector Field Test tells us that $\mathbf{F}$ is conservative.
34. We have $P(x, y) = xy$ and $Q(x, y) = xy$. Since P , Q , and their derivatives are continuous everywhere, and $\frac{\partial P}{\partial y} = x$ while $\frac{\partial Q}{\partial x} = y$, the Conservative Vector Field Test tells us that $\mathbf{F}$ is not conservative.
35. We have $P(x, y) = xe^y$ and $Q(x, y) = \frac{1}{2}x^2e^y$. Since P , Q , and their derivatives are continuous everywhere, and $\frac{\partial P}{\partial y} = xe^y = \frac{\partial Q}{\partial x}$, the Conservative Vector Field Test tells us that $\mathbf{F}$ is conservative.
36. We have $P(x, y) = x^2 + y^2$ and $Q(x, y) = 2xy - \sin y$. Since P , Q , and their derivatives are continuous everywhere, and $\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$, the Conservative Vector Field Test tells us that $\mathbf{F}$ is conservative.
37. (a) With $P(x, y) = x$ and $Q(x, y) = y$, note that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int x dx = \frac{1}{2}x^2 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + h'(y) = y,$$

so that $h'(y) = y$. Integrating with respect to y gives $h(y) = \frac{1}{2}y^2 + K$ where K is a constant. So

$$\boxed{f(x, y) = \frac{1}{2}(x^2 + y^2) + K}.$$

- (c) By the Fundamental Theorem of Line Integrals, we have

$$\int_C (P dx + Q dy) = f(2, 5) - f(1, 3) = \left(\frac{1}{2}(2^2 + 5^2) + K \right) - \left(\frac{1}{2}(1^2 + 3^2) + K \right) = \boxed{\frac{19}{2}}.$$

38. (a) With $P(x, y) = 2xy$ and $Q(x, y) = x^2 + 1$, note that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int 2xy dx = x^2y + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$x^2 + h'(y) = x^2 + 1,$$

so that $h'(y) = 1$. Integrating with respect to y gives $h(y) = y + K$ where K is a constant. So

$$\boxed{f(x, y) = x^2y + y + K}.$$

- (c) By the Fundamental Theorem of Line Integrals, we have

$$\int_C (P dx + Q dy) = f(-2, 3) - f(1, -4) = ((-2)^2 \cdot 3 + 3 + K) - (1^2 \cdot (-4) - 4 + K) = \boxed{23}.$$

39. (a) With $P(x, y) = x^2 + 3y$ and $Q(x, y) = 3x$, note that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (x^2 + 3y) dx = \frac{1}{3}x^3 + 3xy + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + 3x + h'(y) = 3x,$$

so that $h'(y) = 0$. Integrating with respect to y gives $h(y) = K$ where K is a constant. So

$$\boxed{f(x, y) = \frac{1}{3}x^3 + 3xy + K}.$$

(c) By the Fundamental Theorem of Line Integrals, we have

$$\begin{aligned}\int_C (P dx + Q dy) &= f(-3, 5) - f(1, 2) \\ &= \left(\frac{1}{3}(-3)^3 + 3 \cdot (-3) \cdot 5 + K \right) - \left(\frac{1}{3} \cdot 1^3 + 3 \cdot 1 \cdot 2 + K \right) = \boxed{-\frac{181}{3}}.\end{aligned}$$

40. (a) With $P(x, y) = 2x + y + 1$ and $Q(x, y) = x + 3y + 2$, note that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (2x + y + 1) dx = x^2 + xy + x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + x + 0 + h'(y) = x + 3y + 2,$$

so that $h'(y) = 3y + 2$. Integrate $h'(y)$ with respect to y :

$$\int h'(y) dy = \int (3y + 2) dy = \frac{3}{2}y^2 + 2y + K.$$

Therefore

$$f(x, y) = x^2 + xy + x + \frac{3}{2}y^2 + 2y + K = \boxed{x^2 + xy + \frac{3}{2}y^2 + x + 2y + K}.$$

(c) By the Fundamental Theorem of Line Integrals, we have

$$\begin{aligned}\int_C (P dx + Q dy) &= f(1, 2) - f(0, 0) \\ &= \left(1^2 + 1 \cdot 2 + \frac{3}{2} \cdot 2^2 + 1 + 2 \cdot 2 + K \right) - \left(0^2 + 0 \cdot 0 + \frac{3}{2} \cdot 0^2 + 0 + 2 \cdot 0 + K \right) \\ &= \boxed{14}.\end{aligned}$$

41. (a) With $P(x, y) = 4x^3 + 20xy^3 - 3y^4$ and $Q(x, y) = 30x^2y^2 - 12xy^3 + 5y^4$, note that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 60xy^2 - 12y^3 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (4x^3 + 20xy^3 - 3y^4) dx = x^4 + 10x^2y^3 - 3xy^4 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + 30x^2y^2 - 12xy^3 + h'(y) = 30x^2y^2 - 12xy^3 + 5y^4,$$

so that $h'(y) = 5y^4$. Integrate $h'(y)$ with respect to y :

$$\int h'(y) dy = \int 5y^4 dy = y^5 + K.$$

Therefore

$$\boxed{f(x, y) = x^4 + 10x^2y^3 - 3xy^4 + y^5 + K}.$$

(c) By the Fundamental Theorem of Line Integrals, we have

$$\int_C (P dx + Q dy) = f(1, 1) - f(0, 0) = (1 + 10 - 3 + 1 + K) - (0 + 0 - 0 + 0 + K) = \boxed{9}.$$

42. (a) With $P(x, y) = 2yx^{-1}$ and $Q(x, y) = \ln(x^2)$, note that P , Q , and their derivatives are continuous for $x > 0$, and that $\frac{\partial P}{\partial y} = \frac{2}{x} = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path as long as the path stays to the right of the y -axis.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int \frac{2y}{x} dx = 2y \ln x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$2 \ln x + h'(y) = \ln(x^2) = 2 \ln x,$$

so that $h'(y) = 0$. Then $h(y) = K$, where K is a constant. Therefore

$$\boxed{f(x, y) = 2y \ln x + K}.$$

(c) By the Fundamental Theorem of Line Integrals, we have

$$\int_C (P dx + Q dy) = f(5, 5) - f(1, 1) = 10 \ln 5 - 2 \ln 1 = \boxed{10 \ln 5}.$$

43. (a) With $P(x, y) = 3x^2y^2$ and $Q(x, y) = 2x^3y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 6x^2y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int 3x^2y^2 dx = x^3y^2 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$2x^3y + h'(y) = 2x^3y,$$

so that $h'(y) = 0$. Then $h(y) = K$, where K is a constant. Therefore

$$\boxed{f(x, y) = x^3y^2 + K}.$$

44. (a) With $P(x, y) = 2x + y$ and $Q(x, y) = x - 2y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (2x + y) dx = x^2 + xy + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + x + h'(y) = x - 2y,$$

so that $h'(y) = -2y$. Integrating $h'(y)$ with respect to y gives

$$h(y) = \int h'(y) dy = \int (-2y) dy = -y^2 + K.$$

Therefore

$$\boxed{f(x, y) = x^2 + xy - y^2 + K}.$$

45. (a) With $P(x, y) = x + 3y$ and $Q(x, y) = 3x$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (x + 3y) dx = \frac{1}{2}x^2 + 3xy + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + 3x + h'(y) = 3x,$$

so that $h'(y) = 0$. Therefore $h(y) = K$, where K is a constant, and then

$$\boxed{f(x, y) = \frac{1}{2}x^2 + 3xy + K}.$$

46. (a) With $P(x, y) = 2x + y$ and $Q(x, y) = 2y + x$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (2x + y) dx = x^2 + xy + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + x + h'(y) = 2y + x,$$

so that $h'(y) = 2y$. Integrating $h'(y)$ with respect to y gives

$$h(y) = \int h'(y) dy = \int 2y dy = y^2 + K.$$

Therefore

$$\boxed{f(x, y) = x^2 + xy + y^2 + K}.$$

47. (a) With $P(x, y) = 2xy - y^2$ and $Q(x, y) = x^2 - 2xy$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 2x - 2y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (2xy - y^2) dx = x^2y - xy^2 + h(y)$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$x^2 - 2xy + h'(y) = x^2 - 2xy,$$

so that $h'(y) = 0$. Therefore $h(y) = K$, where K is a constant, and then

$$\boxed{f(x, y) = x^2y - xy^2 + K}.$$

48. (a) With $P(x, y) = y^2$ and $Q(x, y) = 2yx - e^y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int y^2 dx = xy^2 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$2xy + h'(y) = 2yx - e^y,$$

so that $h'(y) = -e^y$. Integrating $h'(y)$ with respect to y gives

$$h(y) = \int h'(y) dy = \int (-e^y) dy = -e^y + K.$$

Therefore

$$\boxed{f(x, y) = xy^2 - e^y + K}.$$

49. (a) With $P(x, y) = x^2 - x + y^2$ and $Q(x, y) = -(ye^y - 2xy) = 2xy - ye^y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (x^2 - x + y^2) dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + xy^2 + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 - 0 + 2xy + h'(y) = 2xy - ye^y,$$

so that $h'(y) = -ye^y$. To integrate $h'(y)$, use integration by parts, with $u = y$ and $dv = -e^y dy$; then $du = dy$ and $v = -e^y$, so that

$$h(y) = \int (-ye^y) dy = -ye^y + \int e^y dy = e^y(1 - y) + K.$$

Therefore

$$\boxed{f(x, y) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + xy^2 + e^y(1 - y) + K}.$$

50. (a) With $P(x, y) = 3x^2y + xy^2 + e^x$ and $Q(x, y) = x^3 + x^2y + \sin y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = 3x^2 + 2xy = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (3x^2y + xy^2 + e^x) dx = x^3y + \frac{1}{2}x^2y^2 + e^x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$x^3 + x^2y + 0 + h'(y) = x^3 + x^2y + \sin y,$$

so that $h'(y) = \sin y$. Integrating $h'(y)$ gives $h(y) = -\cos y + K$. Therefore

$$f(x, y) = x^3y + \frac{1}{2}x^2y^2 + e^x - \cos y + K.$$

51. (a) With $P(x, y) = y \cos x - 2 \sin y$ and $Q(x, y) = -(2x \cos y - \sin x) = \sin x - 2x \cos y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = \cos x - 2 \cos y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (y \cos x - 2 \sin y) dx = y \sin x - 2x \sin y + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$\sin x - 2x \cos y + h'(y) = \sin x - 2x \cos y,$$

so that $h'(y) = 0$. It follows that $h(y) = K$, where K is a constant. Therefore

$$f(x, y) = y \sin x - 2x \sin y + K.$$

52. (a) With $P(x, y) = e^x \sin y + 2y \sin x$ and $Q(x, y) = e^x \cos y - 2 \cos x$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = e^x \cos y + 2 \sin x = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

- (b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (e^x \sin y + 2y \sin x) dx = e^x \sin y - 2y \cos x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$e^x \cos y - 2 \cos x + h'(y) = e^x \cos y - 2 \cos x,$$

so that $h'(y) = 0$. It follows that $h(y) = K$, where K is a constant. Therefore

$$f(x, y) = e^x \cos y - 2y \cos x + K.$$

53. (a) With $P(x, y) = 2x + y \cos x$ and $Q(x, y) = \sin x$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = \cos x = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (2x + y \cos x) dx = x^2 + y \sin x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$0 + \sin x + h'(y) = \sin x,$$

so that $h'(y) = 0$. It follows that $h(y) = K$, where K is a constant. Therefore

$$\boxed{f(x, y) = x^2 + y \sin x + K}.$$

54. (a) With $P(x, y) = \cos y - \cos x$ and $Q(x, y) = e^y - x \sin y$, note that P , Q , and their derivatives are continuous on the entire plane, and that $\frac{\partial P}{\partial y} = -\sin y = \frac{\partial Q}{\partial x}$. So the Conservative Vector Field Test tells us that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative, and then the Fundamental Theorem of Line Integrals ensures that $\int_C (P dx + Q dy)$ is independent of path on the entire plane.

(b) First integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int (\cos y - \cos x) dx = x \cos y - \sin x + h(y).$$

Differentiating this expression with respect to y and setting it equal to $Q(x, y)$ gives

$$-x \sin y - 0 + h'(y) = e^y - x \sin y,$$

so that $h'(y) = e^y$. It follows that $h(y) = e^y + K$, where K is a constant. Therefore

$$\boxed{f(x, y) = x \cos y - \sin x + e^y + K}.$$

Applications and Extensions

55. With $P(x, y) = -\frac{y}{x^2+y^2}$ and $Q(x, y) = \frac{x}{x^2+y^2}$, we have

$$\begin{aligned} \frac{\partial P}{\partial y} &= -\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}. \end{aligned}$$

Since these two are equal, it follows that $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative in any simply connected region where P , Q , $\frac{\partial P}{\partial y}$, and $\frac{\partial Q}{\partial x}$ are continuous. These functions fail to be continuous only where their denominators vanish; that happens only at $x = y = 0$, which is the origin. Since $a > 1$, all four points given are in the fourth quadrant, so that the origin does not lie in R . Since a rectangle is simply connected, and the four relevant functions are all continuous there, it follows that $\mathbf{F}$ is conservative in R . So by the Fundamental Theorem of Line Integrals, if C_1 and C_2 are two paths with the same endpoints lying completely in R , then $\int_{C_1} (P(x, y) dx + Q(x, y) dy) = \int_{C_2} (P(x, y) dx + Q(x, y) dy)$.

To find a potential function for $\mathbf{F}$, first integrate $P(x, y)$ with respect to x . Use the substitution $u = \frac{x}{y}$, so that $du = \frac{1}{y} dx$. Then

$$\int \left(-\frac{y}{x^2 + y^2} \right) dx = \int \left(-\frac{y}{(uy)^2 + y^2} \right) \cdot y du = - \int \frac{1}{u^2 + 1} du = -\tan^{-1} u + h(y) = -\tan^{-1} \frac{x}{y} + h(y).$$

Now differentiate this expression with respect to y and set it equal to $Q(x, y)$:

$$\begin{aligned}\frac{\partial}{\partial y} \left(-\tan^{-1} \frac{x}{y} + h(y) \right) &= -\frac{1}{\left(\frac{x}{y}\right)^2 + 1} \cdot \left(-\frac{x}{y^2} \right) + h'(y) \\ &= \frac{x}{y^2 \left(\left(\frac{x}{y}\right)^2 + 1 \right)} + h'(y) \\ &= \frac{x}{x^2 + y^2} + h'(y) \\ &= \frac{x}{x^2 + y^2}.\end{aligned}$$

It follows that $h'(y) = 0$, so that $h(y) = K$ is a constant. Therefore

$$\boxed{f(x, y) = -\tan^{-1} \frac{x}{y} + K}$$

is a potential function for $\mathbf{F}$.

56. Let $\mathbf{c} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Then

$$\begin{aligned}\nabla(\mathbf{c} \cdot \mathbf{r}) &= \nabla(x_0x + y_0y + z_0z) \\ &= \frac{\partial}{\partial x}(x_0x + y_0y + z_0z)\mathbf{i} + \frac{\partial}{\partial y}(x_0x + y_0y + z_0z)\mathbf{j} + \frac{\partial}{\partial z}(x_0x + y_0y + z_0z)\mathbf{k} \\ &= x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} = \mathbf{c}.\end{aligned}$$

So if we set $f(x, y, z) = x_0x + y_0y + z_0z$, we have $\nabla f = \mathbf{c}$. Since the components of $\mathbf{c}$ are all continuous everywhere in space, as are its derivatives, it follows that $\mathbf{c}$ is conservative, so that if C is any closed piecewise smooth curve in space, $\int_C \mathbf{c} \cdot d\mathbf{r} = 0$ by the Value of a Line Integral over a Closed Curve.

57. (a) Since $\mathbf{F}(x, y)$ is directed towards the origin, a direction vector is $-x\mathbf{i} - y\mathbf{j}$. Normalizing gives a unit direction vector for $\mathbf{F}$ equal to

$$-\frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}.$$

We are looking for a vector in the same direction, but with magnitude equal to $\frac{k}{x^2 + y^2}$; multiplying this by the above unit vector gives

$$\mathbf{F}(x, y) = \frac{k}{x^2 + y^2} \left(-\frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} \right) = -\frac{kx}{(x^2 + y^2)^{3/2}}\mathbf{i} - \frac{ky}{(x^2 + y^2)^{3/2}}\mathbf{j}.$$

Note that $\mathbf{F}(x, y)$ is defined everywhere except at the origin. Let R be any simply connected open region not containing the origin. Then

$$P(x, y) = -\frac{kx}{(x^2 + y^2)^{3/2}}, \quad Q(x, y) = -\frac{ky}{(x^2 + y^2)^{3/2}}$$

are continuous on R , and

$$\begin{aligned}\frac{\partial P}{\partial y} &= -k \frac{(x^2 + y^2)^{3/2} \cdot 0 - x \cdot \frac{3}{2}(x^2 + y^2)^{1/2} \cdot 2y}{(x^2 + y^2)^3} = \frac{3kxy(x^2 + y^2)^{1/2}}{(x^2 + y^2)^3} = \frac{3kxy}{(x^2 + y^2)^{5/2}} \\ \frac{\partial Q}{\partial x} &= -k \frac{(x^2 + y^2)^{3/2} \cdot 0 - y \cdot \frac{3}{2}(x^2 + y^2)^{1/2} \cdot 2x}{(x^2 + y^2)^3} = \frac{3kxy(x^2 + y^2)^{1/2}}{(x^2 + y^2)^3} = \frac{3kxy}{(x^2 + y^2)^{5/2}}.\end{aligned}$$

Therefore both partials are continuous on R , and they are equal. It follows that $\mathbf{F}$ is conservative on R .

(b) To find a potential function for $\mathbf{F}$, first integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = - \int \frac{kx}{(x^2 + y^2)^{3/2}} dx = -k \int x(x^2 + y^2)^{-3/2} dx.$$

Use the substitution $u = x^2 + y^2$, so that $du = 2x dx$. Then we get

$$-k \int x(x^2 + y^2)^{-3/2} dx = -\frac{1}{2}k \int u^{-3/2} du = -\frac{1}{2}k \cdot (-2) \cdot u^{-1/2} + h(y) = k(x^2 + y^2)^{-1/2} + h(y).$$

Now differentiate this expression with respect to y and set it equal to $Q(x, y)$:

$$\begin{aligned} \frac{\partial}{\partial y} \left(k(x^2 + y^2)^{-1/2} + h(y) \right) &= -\frac{1}{2}k(x^2 + y^2)^{-3/2} \cdot 2y + h'(y) \\ &= -\frac{ky}{(x^2 + y^2)^{3/2}} + h'(y) \\ &= Q(x, y) = -\frac{ky}{(x^2 + y^2)^{3/2}}. \end{aligned}$$

Therefore $h'(y) = 0$, so that $h(y) = K$ is a constant. It follows that a potential function for $\mathbf{F}$ is

$$f(x, y) = k(x^2 + y^2)^{-1/2} + K = \boxed{\frac{k}{\sqrt{x^2 + y^2}} + K}.$$

58. Regard f as a function on the plane whose value at any point (x, y) is $f(x)$; similarly, regard g as a function on the plane whose value at any point (x, y) is $g(y)$. Then since f and g are differentiable, they are continuous everywhere on the plane. Further,

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x},$$

so that the derivatives are continuous everywhere, and are equal. Therefore $f(x)\mathbf{i} + g(y)\mathbf{j}$ is a conservative vector field, so that by the Value of a Line Integral over a Closed Curve, $\int_C [f(x) dx + g(y) dy] = 0$ if C is any circle in the plane.

Challenge Problem

59. Consider the function fg . We have by the product rule

$$\frac{\partial(fg)}{\partial x} = f(x, y) \frac{\partial g}{\partial x} + g(x, y) \frac{\partial f}{\partial x}, \quad \frac{\partial(fg)}{\partial y} = f(x, y) \frac{\partial g}{\partial y} + g(x, y) \frac{\partial f}{\partial y},$$

so that

$$\begin{aligned} \nabla(fg) &= \frac{\partial(fg)}{\partial x} \mathbf{i} + \frac{\partial(fg)}{\partial y} \mathbf{j} \\ &= \left(f(x, y) \frac{\partial g}{\partial x} + g(x, y) \frac{\partial f}{\partial x} \right) \mathbf{i} + \left(f(x, y) \frac{\partial g}{\partial y} + g(x, y) \frac{\partial f}{\partial y} \right) \mathbf{j} \\ &= \left(f(x, y) \frac{\partial g}{\partial x} \mathbf{i} + f(x, y) \frac{\partial g}{\partial y} \mathbf{j} \right) + \left(g(x, y) \frac{\partial f}{\partial x} \mathbf{i} + g(x, y) \frac{\partial f}{\partial y} \mathbf{j} \right) \\ &= f(x, y) \left(\frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} \right) + g(x, y) \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right) \\ &= f \nabla g + g \nabla f. \end{aligned}$$

Therefore $f \nabla g + g \nabla f$ is a conservative vector field on R , since it is equal to $\nabla(fg)$. So if C is any piecewise smooth curve in R from A to B , the Fundamental Theorem of Line Integrals gives

$$\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = (fg)(B) - (fg)(A) = f(B)g(B) - f(A)g(A),$$

so that

$$\int_C f \nabla g \cdot d\mathbf{r} = f(B)g(B) - f(A)g(A) - \int_C g \nabla f \cdot d\mathbf{r}.$$

15.4: An Application of Line Integrals: Work

Concepts and Vocabulary

1. Work is the *energy* transferred to or from an object by a force acting on the object. See the opening sentences of this section.
2. True. See the definition of Work.
3. False. The work done is the change in potential energy from B to A . See the discussion just prior to the Law of Conservation of Energy.
4. (d) is correct. See Work on a Closed Path.
5. Kinetic energy K is the energy of *motion*. Potential energy U is the energy of *position*.
6. False. The Law of Conservation of Energy says that the *sum* of the potential and kinetic energies of an object is a constant (not necessarily zero).

Skill Building

7. Along the given curve $\mathbf{r}$ we have $x(t) = t$ and $y(t) = t^2$, so that $\mathbf{F} = t^2\mathbf{i} + t\mathbf{j}$. Then the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2\mathbf{i} + t\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) dt = \int_0^1 3t^2 dt = [t^3]_0^1 = \boxed{1}.$$

8. Along the given curve $\mathbf{r}$ we have $x(t) = t$ and $y(t) = t^2$, so that $\mathbf{F} = t \cdot t^2\mathbf{i} + (t^2)^2\mathbf{j} = t^3\mathbf{i} + t^4\mathbf{j}$. Then the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^3\mathbf{i} + t^4\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) dt = \int_0^1 (t^3 + 2t^5) dt = \left[\frac{1}{4}t^4 + \frac{1}{3}t^6 \right]_0^1 = \boxed{\frac{7}{12}}.$$

9. Along the given curve $\mathbf{r}$ we have $x(t) = 3 \cos t$ and $y(t) = 2 \sin t$, so that

$$\mathbf{F} = (3 \cos t - 4 \sin t)\mathbf{i} + (3 \cos t)(2 \sin t)\mathbf{j} = (3 \cos t - 4 \sin t)\mathbf{i} + 6 \sin t \cos t\mathbf{j}.$$

Then the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{\pi/2} ((3 \cos t - 4 \sin t)\mathbf{i} + 6 \sin t \cos t\mathbf{j}) \cdot (-3 \sin t\mathbf{i} + 2 \cos t\mathbf{j}) dt \\ &= \int_0^{\pi/2} (-9 \sin t \cos t + 12 \sin^2 t + 12 \sin t \cos^2 t) dt \\ &= \int_0^{\pi/2} (-9 \sin t \cos t + 6(1 - \cos(2t)) + 12 \sin t \cos^2 t) dt \\ &= \left[-\frac{9}{2} \sin^2 t + 6 \left(t - \frac{1}{2} \sin(2t) \right) - 4 \cos^3 t \right]_0^{\pi/2} \\ &= \left(-\frac{9}{2} + 6 \left(\frac{\pi}{2} - 0 \right) - 0 \right) - (0 + 6 \cdot 0 - 4 \cdot 1) \\ &= \boxed{3\pi - \frac{1}{2}}. \end{aligned}$$

10. With $P(x, y) = x$ and $Q(x, y) = y$, we see that P , Q , and their derivatives are continuous everywhere, and that $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$. Therefore this is a conservative field, and since the path is closed, the integral along the path is $\boxed{0}$.

11. With $P(x, y) = x^2y$ and $Q(x, y) = x^2 - y^2$, on the curve $y = 2x^2$ we have $dy = 4x dx$. Then

$$\begin{aligned} \int_{(0,0)}^{(1,2)} (P(x, y) dx + Q(x, y) dy) &= \int_0^1 (x^2 \cdot 2x^2 + (x^2 - (2x^2)^2) \cdot 4x) dx \\ &= \int_0^1 (-16x^5 + 2x^4 + 4x^3) dx \\ &= \left[-\frac{8}{3}x^6 + \frac{2}{5}x^5 + x^4 \right]_0^1 \\ &= \boxed{-\frac{19}{15}}. \end{aligned}$$

12. With $P(x, y) = y \sin x$ and $Q(x, y) = -x \cos y$, on the curve $y = x$ we have $dy = dx$. Then

$$\int_{(0,0)}^{(1,1)} (P(x, y) dx + Q(x, y) dy) = \int_0^1 (x \sin x - x \cos x) dx = \int_0^1 x(\sin x - \cos x) dx.$$

Use integration by parts, with $u = x$ and $dv = (\sin x - \cos x) dx$; then $du = dx$ and $v = -(\cos x + \sin x)$, so that

$$\begin{aligned} \int_0^1 x(\sin x - \cos x) dx &= [-x(\cos x + \sin x)]_0^1 + \int_0^1 (\cos x + \sin x) dx \\ &= -1(\cos 1 + \sin 1) + [\sin x - \cos x]_0^1 \\ &= (-\cos 1 - \sin 1) + (\sin 1 - \cos 1 + 1) = \boxed{1 - 2 \cos 1}. \end{aligned}$$

13. Parametrize the upper half of the circle as $x(t) = \cos t$, $y(t) = \sin t$ for $0 \leq t \leq \pi$. Then $dx = -\sin t dt$ and $dy = \cos t dt$. With $P(x, y) = y - x^2$ and $Q(x, y) = x$, we get for the work

$$\begin{aligned} \int_C (P(x, y) dx + Q(x, y) dy) &= \int_0^\pi ((\sin t - \cos^2 t) \cdot (-\sin t) + (\cos t) \cdot \cos t) dt \\ &= \int_0^\pi (-\sin^2 t + \cos^2 t \sin t + \cos^2 t) dt \\ &= \int_0^\pi (\cos(2t) + \cos^2 t \sin t) dt \\ &= \left[\frac{1}{2} \sin(2t) - \frac{1}{3} \cos^3 t \right]_0^\pi \\ &= \boxed{\frac{2}{3}}. \end{aligned}$$

14. For the first portion of the path, parametrize by $x = 1$, $y = t$ for $0 \leq t \leq 1$. Then the work is

$$\int_C (P(x, y) dx + Q(x, y) dy) = \int_0^1 ((t-1) \cdot 0 + 1 \cdot 1) dt = \int_0^1 1 dt = 1.$$

Parametrize the second portion of the path by $x = 1 - 2t$, $y = 1$ for $0 \leq t \leq 1$. Then the work is

$$\begin{aligned} \int_C (P(x, y) dx + Q(x, y) dy) &= \int_0^1 ((1 - (1 - 2t)^2) \cdot (-2) + (1 - 2t) \cdot 0) dt \\ &= \int_0^1 (8t^2 - 8t) dt = \left[\frac{8}{3}t^3 - 4t^2 \right]_0^1 = -\frac{4}{3}. \end{aligned}$$

Parametrize the third portion of the path by $x = -1$, $y = 1 - t$ for $0 \leq t \leq 1$. Then the work is

$$\int_C (P(x, y) dx + Q(x, y) dy) = \int_0^1 ((1 - t - (-1)^2) \cdot 0 - 1 \cdot (-1)) dt = \int_0^1 1 dt = 1.$$

The total work is the sum of the work along each piece of the path, so it is $1 - \frac{4}{3} + 1 = \boxed{\frac{2}{3}}$.

15. Along the given curve $\mathbf{r}$ we have $x(t) = a \cos t$ and $y(t) = b \sin t$, so that

$$\mathbf{F} = (b \sin t)^3 \mathbf{i} + (a \cos t)^3 \mathbf{j} = b^3 \sin^3 t \mathbf{i} + a^3 \cos^3 t \mathbf{j}.$$

Then the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} (b^3 \sin^3 t \mathbf{i} + a^3 \cos^3 t \mathbf{j}) \cdot (-a \sin t \mathbf{i} + b \cos t \mathbf{j}) dt \\ &= \int_0^{2\pi} (-ab^3 \sin^4 t + ba^3 \cos^4 t) dt. \end{aligned}$$

Now use numbers 90 and 91 in the Endnotes to evaluate these integrals.

$$\begin{aligned} \int_0^{2\pi} \sin^4 t dt &= \left[-\frac{1}{4} \sin^3 t \cos t \right]_0^{2\pi} + \frac{3}{4} \int_0^{2\pi} \sin^2 t dt \\ &= \frac{3}{4} \int_0^{2\pi} \sin^2 t dt \\ &= \frac{3}{4} \int_0^{2\pi} \frac{1}{2} (1 - \cos(2t)) dt \\ &= \frac{3}{4} \left[\frac{1}{2} t - \sin(2t) \right]_0^{2\pi} \\ &= \frac{3}{4} \pi, \end{aligned}$$

and

$$\begin{aligned} \int_0^{2\pi} \cos^4 t dt &= \left[\frac{1}{4} \cos^3 t \sin t \right]_0^{2\pi} + \frac{3}{4} \int_0^{2\pi} \cos^2 t dt \\ &= \frac{3}{4} \int_0^{2\pi} \cos^2 t dt \\ &= \frac{3}{4} \int_0^{2\pi} \frac{1}{2} (1 + \cos(2t)) dt \\ &= \frac{3}{4} \left[\frac{1}{2} t + \sin(2t) \right]_0^{2\pi} \\ &= \frac{3}{4} \pi, \end{aligned}$$

So the work performed is

$$\int_0^{2\pi} (-ab^3 \sin^4 t + ba^3 \cos^4 t) dt = -ab^3 \cdot \frac{3}{4} \pi + ba^3 \cdot \frac{3}{4} \pi = \boxed{\frac{3}{4} ab \pi (a^2 - b^2)}.$$

16. Parametrize the upper half of the ellipse by $x(t) = a \cos t$ and $y(t) = b \sin t$ for $0 \leq t \leq \pi$. Then

$$\mathbf{F} = -(b \sin t)^2 \mathbf{i} + (a \cos t)^2 \mathbf{j} = -b^2 \sin^2 t \mathbf{i} + a^2 \cos^2 t \mathbf{j}.$$

Then the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} (-b^2 \sin^2 t \mathbf{i} + a^2 \cos^2 t \mathbf{j}) \cdot (-a \sin t \mathbf{i} + b \cos t \mathbf{j}) dt \\ &= \int_0^{2\pi} (ab^2 \sin^3 t + ba^2 \cos^3 t) dt. \end{aligned}$$

Now use numbers 90 and 91 in the Endnotes to evaluate these integrals.

$$\begin{aligned} \int_0^\pi \sin^3 t dt &= \left[-\frac{1}{3} \sin^3 t \cos t \right]_0^\pi + \frac{2}{3} \int_0^\pi \sin t dt \\ &= \frac{2}{3} \int_0^\pi \sin t dt \\ &= \frac{2}{3} [-\cos t]_0^\pi = \frac{4}{3} \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi \cos^3 t dt &= \left[\frac{1}{3} \cos^2 t \sin t \right]_0^\pi + \frac{2}{3} \int_0^\pi \cos t dt \\ &= \frac{2}{3} \int_0^\pi \cos t dt \\ &= \frac{2}{3} [\sin t]_0^\pi = 0. \end{aligned}$$

So the work performed is

$$\int_0^{2\pi} (ab^2 \sin^3 t + ba^2 \cos^3 t) dt = ab^2 \cdot \frac{4}{3} + ba^2 \cdot 0 = \boxed{\frac{4}{3} ab^2}.$$

17. Since $P(x, y) = 3x$ and $Q(x, y) = 2y$ are continuous everywhere in the plane, their derivatives are as well, and

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x},$$

it follows that $\mathbf{F}$ is conservative. Since the path pictured is a closed path, the work done is $\boxed{0}$.

18. With $P(x, y) = 2xy$ and $Q(x, y) = xy^2$, we have

$$\frac{\partial P}{\partial y} = 2x \neq y^2 = \frac{\partial Q}{\partial x},$$

so that $\mathbf{F}$ is not conservative. Therefore we must compute the integrals. Parametrize the three portions of the path as follows:

$$\begin{aligned} C_1 &: (0, 0) \text{ to } (1, 1), \quad \mathbf{r}_1(t) = t \mathbf{i} + t \mathbf{j}, \quad 0 \leq t \leq 1 \\ C_2 &: (1, 1) \text{ to } (-1, 1), \quad \mathbf{r}_2(t) = (1 - 2t) \mathbf{i} + \mathbf{j}, \quad 0 \leq t \leq 1 \\ C_3 &: (-1, 1) \text{ to } (0, 0), \quad \mathbf{r}_3(t) = (-1 + t) \mathbf{i} + (1 - t) \mathbf{j}, \quad 0 \leq t \leq 1. \end{aligned}$$

Then the work is

$$\begin{aligned}
 W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3 \\
 &= \int_0^1 (2t \cdot t \mathbf{i} + t \cdot t^2 \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) \\
 &\quad + \int_0^1 (2 \cdot (1 - 2t) \cdot 1 \mathbf{i} + (1 - 2t) \cdot 1^2 \mathbf{j}) \cdot (-2\mathbf{i}) \\
 &\quad + \int_0^1 (2(-1 + t)(1 - t)\mathbf{i} + (-1 + t)(1 - t)^2 \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) \\
 &= \int_0^1 (2t^2 + t^3) dt + \int_0^1 (-4(1 - 2t)) dt + \int_0^1 (2(-1 + t)(1 - t) - (-1 + t)(1 - t)^2) dt \\
 &= \int_0^1 (2t^2 + t^3) dt + \int_0^1 (8t - 4) dt + \int_0^1 (-t^3 + t^2 + t - 1) dt \\
 &= \left[\frac{2}{3}t^3 + \frac{1}{4}t^4 \right]_0^1 + [4t^2 - 4t]_0^1 + \left[-\frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 - t \right]_0^1 \\
 &= \frac{2}{3} + \frac{1}{4} + 4 - 4 - \frac{1}{4} + \frac{1}{3} + \frac{1}{2} - 1 \\
 &= \boxed{\frac{1}{2}}.
 \end{aligned}$$

19. Since $P(x, y) = e^x y$ and $Q(x, y) = e^x$ are continuous everywhere in the plane, their derivatives are as well, and

$$\frac{\partial P}{\partial y} = e^x = \frac{\partial Q}{\partial x},$$

it follows that $\mathbf{F}$ is conservative. To find a potential function, integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int e^x y dx = e^x y + h(y).$$

Now differentiate this expression with respect to y and set it equal to $Q(x, y)$:

$$e^x + h'(y) = e^x,$$

so that $h'(y) = 0$ and therefore $h(y) = K$ is a constant. So

$$f(x, y) = e^x y + K$$

is a potential function. So the integral, which is the work performed, is

$$f(1, 0) - f(-1, -1) = (e^1 \cdot 0 + K) - (e^{-1} \cdot (-1) + K) = \boxed{\frac{1}{e}}.$$

20. Since $P(x, y) = e^x$ and $Q(x, y) = 5y$ are continuous everywhere in the plane, their derivatives are as well, and

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x},$$

it follows that $\mathbf{F}$ is conservative. To find a potential function, integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int e^x dx = e^x + h(y).$$

Now differentiate this expression with respect to y and set it equal to $Q(x, y)$:

$$0 + h'(y) = 5y,$$

so that $h'(y) = 5y$ and therefore $h(y) = \frac{5}{2}y^2 + K$. So

$$f(x, y) = e^x + \frac{5}{2}y^2 + K$$

is a potential function. So the integral, which is the work performed, is

$$f(1, 0) - f(-1, 2) = \left(e^1 + \frac{5}{2} \cdot 0^2 + K \right) - \left(e^{-1} + \frac{5}{2} \cdot 2^2 + K \right) = \boxed{e - \frac{1}{e} - 10}.$$

Applications and Extensions

21. To see that $\mathbf{F}$ is not conservative, note that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1 \neq \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-x) = -1.$$

Since these partials are unequal, $\mathbf{F}$ is not conservative.

Now consider the path C given by $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$ for $0 \leq t \leq 1$ from $(0, 0)$ to $(1, 1)$. The integral along this path is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} - t\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) dt = \int_0^1 0 dt = \boxed{0}.$$

For the other path, let C_1 be given by $\mathbf{r}_1(t) = t\mathbf{i}$ for $0 \leq t \leq 1$ from $(0, 0)$ to $(1, 0)$, and then C_2 be given by $\mathbf{r}_2(t) = \mathbf{i} + t\mathbf{j}$ for $0 \leq t \leq 1$ from $(1, 0)$ to $(1, 1)$. The integral along this path is

$$\int_{C_1 C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 (-t\mathbf{j}) \cdot \mathbf{i} dt + \int_0^1 (t\mathbf{i} - \mathbf{j}) \cdot \mathbf{j} dt = \int_0^1 0 dt + \int_0^1 (-1) dt = \boxed{-1}.$$

22. With the given parametrization, we have

$$\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} = t^2\mathbf{i} + t^2\mathbf{j},$$

so the work performed is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 (t^2\mathbf{i} + t^2\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) dt = \int_0^3 (t^2 + 2t^3) dt = \left[\frac{1}{3}t^3 + \frac{1}{2}t^4 \right]_0^3 = \boxed{\frac{99}{2}}.$$

23. Since $P(x, y) = \cos x$ and $Q(x, y) = \sin y$ are continuous everywhere, as are their derivatives, and also

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x},$$

the field is conservative. To find a potential function, integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int \cos x dx = \sin x + h(y).$$

Now differentiate this expression with respect to y and set equal to $Q(x, y)$:

$$0 + h'(y) = \sin y.$$

Therefore $h'(y) = \sin y$, so that $h(y) = -\cos y + K$. It follows that a potential function is (choosing $K = 0$)

$$f(x, y) = \sin x - \cos y.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi, \sqrt{\pi}) - f(0, 0) = (\sin \pi - \cos \sqrt{\pi}) - (0 - 1) = \boxed{1 - \cos \sqrt{\pi}}.$$

24. $\mathbf{F}$ has magnitude 3 in the direction of $\mathbf{i} + \mathbf{j}$, so

$$\mathbf{F} = \frac{3}{\sqrt{2}}\mathbf{i} + \frac{3}{\sqrt{2}}\mathbf{j}.$$

Since $P(x, y) = \frac{3}{\sqrt{2}}$ and $Q(x, y) = \frac{3}{\sqrt{2}}$ are continuous everywhere, as are their derivatives, and also

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x},$$

the field is conservative. To find a potential function, integrate $P(x, y)$ with respect to x :

$$\int P(x, y) dx = \int \frac{3}{\sqrt{2}} dx = \frac{3}{\sqrt{2}}x + h(y).$$

Now differentiate this expression with respect to y and set equal to $Q(x, y)$:

$$0 + h'(y) = \frac{3}{\sqrt{2}}.$$

Therefore $h'(y) = \frac{3}{\sqrt{2}}$, so that $h(y) = \frac{3}{\sqrt{2}}y + K$. It follows that a potential function is (choosing $K = 0$)

$$f(x, y) = \frac{3}{\sqrt{2}}x + \frac{3}{\sqrt{2}}y.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2a, 0) - f(0, 0) = \frac{3}{\sqrt{2}}(2a + 0) - \frac{3}{\sqrt{2}}(0 + 0) = \boxed{3a\sqrt{2}}.$$

25. The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 64\sqrt{3}\mathbf{i} + (64 - 32t)\mathbf{j},$$

so that the speed of the object is $s(t) = \|\mathbf{v}(t)\|$.

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{(64\sqrt{3})^2 + (64 - 32t)^2} = \sqrt{1024t^2 - 4096t + 16384} = 32\sqrt{t^2 - 4t + 16}.$$

Then $\mathbf{F}$ has direction $-\mathbf{v}(t) = -64\sqrt{3}\mathbf{i} + (32t - 64)\mathbf{j}$ and magnitude $ks(t) = k\|\mathbf{v}(t)\|$ for some constant of proportionality $k > 0$. So

$$\mathbf{F} = ks(t) \frac{-\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{k\|\mathbf{v}(t)\|}{\|\mathbf{v}(t)\|} (-\mathbf{v}(t)) = -k\mathbf{v}(t).$$

for some constant $k > 0$. Then the work done by the force is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^4 (-k\mathbf{v}(t)) \cdot \mathbf{v}(t) dt \\ &= -k \int_0^4 \|\mathbf{v}(t)\|^2 dt \\ &= -k \int_0^4 s(t)^2 dt \\ &= -1024k \int_0^4 (t^2 - 4t + 16) dt \\ &= -1024k \left[\frac{1}{3}t^3 - 2t^2 + 16t \right]_0^4 \\ &= \boxed{-\frac{163840}{3}k}. \end{aligned}$$

26. We have $P(x, y) = y^2 + 1$ and $Q(x, y) = x + y$. Further, on the given curve, using the product rule gives $dy = (a(1 - x) - ax) dx = (a - 2ax) dx$. Then the work done is

$$\begin{aligned}
 W &= \int_C (P(x, y) dx + Q(x, y) dy) \\
 &= \int_0^1 ((ax(1 - x))^2 + 1) \cdot 1 + (x + ax(1 - x)) \cdot (a - 2ax) dx \\
 &= \int_0^1 (a^2x^2 - 2a^2x^3 + a^2x^4 + 1 + ((a + 1)x - ax^2)(a - 2ax)) dx \\
 &= \int_0^1 (a^2x^2 - 2a^2x^3 + a^2x^4 + 1 + a(a + 1)x - 2a(a + 1)x^2 - a^2x^2 + 2a^2x^3) dx \\
 &= \left[\frac{1}{3}a^2x^3 - \frac{1}{2}a^2x^4 + \frac{1}{5}a^2x^5 + x + \frac{1}{2}a(a + 1)x^2 - \frac{2}{3}a(a + 1)x^3 - \frac{1}{3}a^2x^3 + \frac{1}{2}a^2x^4 \right]_0^1 \\
 &= \frac{1}{3}a^2 - \frac{1}{2}a^2 + \frac{1}{5}a^2 + 1 + \frac{1}{2}a(a + 1) - \frac{2}{3}a(a + 1) - \frac{1}{3}a^2 + \frac{1}{2}a^2 \\
 &= \frac{1}{30}a^2 - \frac{1}{6}a + 1.
 \end{aligned}$$

We want to find the value of a that minimizes the work, so taking the derivative of this quadratic with respect to a gives

$$\frac{1}{15}a - \frac{1}{6} = 0,$$

so that $a = \boxed{\frac{5}{2}}$.

27. Exercise 57 in Section 15.3 shows that the vector field

$$-\mathbf{F} = -\frac{x}{(x^2 + y^2)^{3/2}} - \frac{y}{(x^2 + y^2)^{3/2}}$$

is conservative, with potential function $\frac{1}{\sqrt{x^2 + y^2}} + K$. It follows that $\mathbf{F}$ is also conservative, since

$$\frac{\partial P}{\partial y} = -\frac{\partial(-P)}{\partial y} = -\frac{\partial(-Q)}{\partial x} = \frac{\partial Q}{\partial x},$$

and that its potential function is

$$f(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} + K.$$

Therefore the work done between the two given points is

$$W = f(-1, 2) - f(1, 0) = -\frac{1}{\sqrt{(-1)^2 + 2^2}} - \left(-\frac{1}{\sqrt{1^2 + 0^2}} \right) = \boxed{1 - \frac{\sqrt{5}}{5}}.$$

28. Along the given helix, the vector field is

$$\mathbf{F} = x \mathbf{i} + y \mathbf{j} + \mathbf{k} = \sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}.$$

Then the work done is

$$\begin{aligned}
 W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_0^\pi (\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}) \cdot (\cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}) dt \\
 &= \int_0^\pi (\sin t \cos t - \sin t \cos t + 1) dt \\
 &= [t]_0^\pi = \boxed{\pi}.
 \end{aligned}$$

29. Along the given helix, the vector field is

$$\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k} = (2 \cos t)^2 \mathbf{i} + (2 \sin t)^2 \mathbf{j} + (2t)^2 \mathbf{k} = 4 \cos^2 t \mathbf{i} + 4 \sin^2 t \mathbf{j} + 4t^2 \mathbf{k}.$$

Then the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^\pi (4 \cos^2 t \mathbf{i} + 4 \sin^2 t \mathbf{j} + 4t^2 \mathbf{k}) \cdot (-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 2 \mathbf{k}) dt \\ &= \int_0^\pi (-8 \cos^2 t \sin t + 8 \sin^2 t \cos t + 8t^2) dt \\ &= 8 \int_0^\pi (\sin^2 t \cos t - \cos^2 t \sin t + t^2) dt \\ &= 8 \left[\frac{1}{3} \sin^3 t + \frac{1}{3} \cos^3 t + \frac{1}{3} t^3 \right]_0^\pi \\ &= 8 \left(\left(0 - \frac{1}{3} + \frac{\pi^3}{3} \right) - \left(0 + \frac{1}{3} + 0 \right) \right) \\ &= 8 \left(\frac{\pi^3}{3} - \frac{2}{3} \right) = \boxed{\frac{8}{3}(\pi^3 - 2)}. \end{aligned}$$

30. For any piecewise smooth curve $C : \mathbf{r} = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$, the work expended by gravity along that path is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C mg \mathbf{k} \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) = \int_C mg dz = \int_{z_1}^{z_2} mg dz = mg(z_2 - z_1).$$

31. (a) The direction of the path at any given point is $\mathbf{r}'$, so saying that $\mathbf{F}$ is orthogonal to the path at each point is the same as saying that $\mathbf{F} \cdot \mathbf{r}' = 0$. Therefore the work done is

$$W = \int_C \mathbf{F} \cdot \mathbf{r}' = \int_C 0 dt = 0.$$

(b) Answers will vary.

32. Suppose that C_1 and C_2 are two paths in S from A to B . Consider the piecewise smooth path C consisting of C_1 followed by $-C_2$ (that is, C_1 followed by C_2 in the opposite direction). Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Rearranging this equation gives

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Challenge Problem

33. (a) From Example 4 in Section 15.1, the gravitational field is defined by

$$\mathbf{F}(x, y, z) = -\frac{GmM}{x^2 + y^2 + z^2} \mathbf{u},$$

where an object of mass M is centered at the origin and an object of mass m is centered at (x, y, z) . Here G is the universal gravitational constant and $\mathbf{u}$ is a unit vector in the direction

of $\langle x, y, z \rangle$. Writing $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$, we can rewrite the above formula as

$$\mathbf{F}(x, y, z) = -\frac{GmM}{r^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{GmM}{r^3} \mathbf{r}.$$

Problem 15.1.22 shows that $\mathbf{F}$ is conservative, with potential function $U(r) = -\frac{GmM}{r}$. The equation for $\mathbf{F}$ shows that the gravitational force acts arbitrarily far away from the object with mass M , but with a magnitude approaching zero. Since the kinetic energy is given by

$$K = \frac{m}{2} \|\mathbf{r}'(t)\|^2 = \frac{m}{2} \|\mathbf{v}\|^2,$$

and escape velocity is the velocity at which the object just escapes the gravitational force, we see that escape velocity is the velocity for which the kinetic energy approaches but never reaches zero as the object gets further away. But as the object gets further away, we also have

$$\lim_{r \rightarrow \infty} U(r) = \lim_{r \rightarrow \infty} \left(-\frac{GmM}{r} \right) = 0.$$

So both the kinetic and potential energies approach zero as the object gets further and further away when launched at escape velocity.

Now let $K(t)$ and $U(t)$ be the kinetic and potential energies at any time t , with $t = 0$ corresponding to launch. By Conservation of Energy, the sum of the kinetic and potential energies at any pair of times is the same, so that $K(0) + U(0) = \lim_{t \rightarrow \infty} (K(t) + U(t)) = 0$. If the initial speed is $\|\mathbf{v}_0\| = s$, then $K(0) = \frac{1}{2}ms^2$ by the discussion of Work and Kinetic Energy in the text. Also, by the discussion of Work and Potential Energy in the text, the initial potential energy is $U(0) = -\frac{GmM}{R}$, since the initial distance of the object is R . So we get

$$K(0) + U(0) = \frac{1}{2}ms^2 - \frac{GmM}{R} = 0.$$

Divide through by m and rearrange to get

$$\frac{1}{2}s^2 = \frac{GM}{R}, \text{ or } s = \sqrt{\frac{2GM}{R}}$$

- (b) $\sqrt{\frac{2GM}{R}} \approx 11173 \text{ m/s}$ $\approx 11.173 \text{ km/s}$.
- (c) $\sqrt{\frac{2GM}{R}} \approx 5019 \text{ m/s}$ $\approx 5.019 \text{ km/s}$.
- (d) $\sqrt{\frac{2GM}{R}} \approx 617589 \text{ m/s}$ $\approx 617.589 \text{ km/s}$.

15.5: Green's Theorem

Concepts and Vocabulary

- Green's Theorem relates the value of a *line* integral along a simple closed piecewise smooth curve C to the value of a *double* integral over the simply connected closed region R enclosed by C .
- $\oint_C (P(x, y) dx + Q(x, y) dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$.
- False. The direction of traversal is counterclockwise.
- It equals the *area* of R . See the discussion in Objective 2.

Skill Building

5. Here $P(x, y) = y$ and $Q(x, y) = 0$, so that

$$\begin{aligned}\oint_C y \, dx &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R (-1) \, dx \, dy = \int_0^3 \int_0^4 (-1) \, dx \, dy \\ &= \int_0^3 [-x]_0^4 \, dy = \int_0^3 (-4) \, dy = [-4y]_0^3 = \boxed{-12}.\end{aligned}$$

6. Here $P(x, y) = 0$ and $Q(x, y) = x$, so that

$$\oint_C x \, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R 1 \, dx \, dy = \int_0^3 \int_0^4 1 \, dx \, dy = \int_0^3 [x]_0^4 \, dy = \int_0^3 4 \, dy = \boxed{12}.$$

7. Here $P(x, y) = xy$ and $Q(x, y) = x + y$, so that

$$\begin{aligned}\oint_C (xy \, dx + (x + y) \, dy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R (1 - x) \, dx \, dy = \int_0^3 \int_0^4 (1 - x) \, dx \, dy \\ &= \int_0^3 \left[x - \frac{1}{2}x^2 \right]_0^4 \, dy = \int_0^3 (-4) \, dy = \boxed{-12}.\end{aligned}$$

8. Here $P(x, y) = 3y$ and $Q(x, y) = -2x$, so that

$$\begin{aligned}\oint_C (3y \, dx - 2x \, dy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R (-2 - 3) \, dx \, dy \\ &= \int_0^3 \int_0^4 (-5) \, dx \, dy = \int_0^3 (-20) \, dy = \boxed{-60}.\end{aligned}$$

9. Here $P(x, y) = xy^2$ and $Q(x, y) = x^2y$, so that

$$\oint_C (xy^2 \, dx + x^2y \, dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R (2xy - 2xy) \, dx \, dy = \iint_R 0 \, dx \, dy = \boxed{0}.$$

10. Here $P(x, y) = y \sin(xy)$ and $Q(x, y) = x \sin(xy)$, so that

$$\begin{aligned}\oint_C (y \sin(xy) \, dx + x \sin(xy) \, dy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy \\ &= \iint_R ((\sin(xy) + xy \cos(xy)) - (\sin(xy) + xy \cos(xy))) \, dx \, dy = \iint_R 0 \, dx \, dy = \boxed{0}.\end{aligned}$$

11. Here $P(x, y) = -x^2y$ and $Q(x, y) = y^2x$, so that

$$\oint_C (-x^2y \, dx + y^2x \, dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \iint_R (y^2 + x^2) \, dx \, dy.$$

Convert to polar coordinates; the integral becomes

$$\iint_R (y^2 + x^2) \, dx \, dy = \int_0^1 \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr = 2\pi \int_0^1 r^3 \, dr = 2\pi \left[\frac{1}{4}r^4 \right]_0^1 = \boxed{\frac{\pi}{2}}.$$

12. Here $P(x, y) = y(x^2 + y^2) = x^2y + y^3$ and $Q(x, y) = -x(x^2 + y^2) = -x^3 - xy^2$, so that

$$\begin{aligned}\oint_C [y(x^2 + y^2) dx - x(x^2 + y^2) dy] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R ((-3x^2 - y^2) - (x^2 + 3y^2)) dx dy = -4 \iint_R (x^2 + y^2) dx dy.\end{aligned}$$

Using the integral from Problem 11, we get

$$-4 \iint_R (x^2 + y^2) dx dy = -4 \cdot \frac{\pi}{2} = \boxed{-2\pi}.$$

13. Here $P(x, y) = x^2 - y^3$ and $Q(x, y) = x^2 + y^2$, so that

$$\begin{aligned}\oint_C [(x^2 - y^3) dx + (x^2 + y^2) dy] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R (2x - (-3y^2)) dx dy = \iint_R (2x + 3y^2) dx dy.\end{aligned}$$

Convert to polar coordinates; then $x = r \cos \theta$, $y^2 = r^2 \sin^2 \theta$, and the integral becomes

$$\begin{aligned}\iint_R (2x + 3y^2) dx dy &= \int_0^1 \int_0^{2\pi} (2r \cos \theta + 3r^2 \sin^2 \theta) \cdot r d\theta dr \\ &= \int_0^1 \int_0^{2\pi} \left(2r^2 \cos \theta + \frac{3}{2}r^3 - \frac{3}{2}r^3 \cos(2\theta) \right) d\theta dr \\ &= \int_0^1 \left[2r^2 \sin \theta + \frac{3}{2}r^3 \theta - \frac{3}{4}r^3 \sin(2\theta) \right]_0^{2\pi} dr \\ &= \int_0^1 \frac{3}{2}r^3 \cdot 2\pi dr \\ &= \int_0^1 3\pi r^3 dr \\ &= \left[\frac{3\pi}{4} r^4 \right]_0^1 = \boxed{\frac{3\pi}{4}}.\end{aligned}$$

14. Here $P(x, y) = xy^3 + \sin x$ and $Q(x, y) = x^2y^2 + 4x$, so that

$$\oint_C [(xy^3 + \sin x) dx + (x^2y^2 + 4x) dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (2xy + 4 - 3xy^2) dx dy.$$

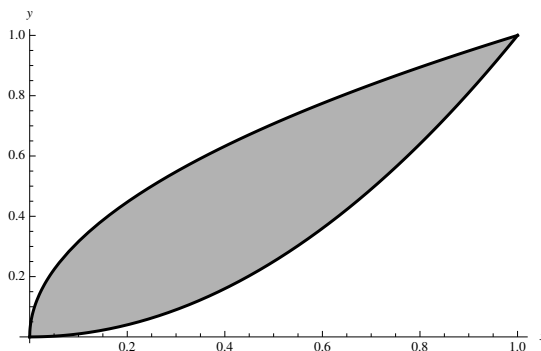
Convert to polar coordinates; then $x = r \cos \theta$, $y = r \sin \theta$, and the integral becomes

$$\begin{aligned}\iint_R (2xy + 4 - 3xy^2) dx dy &= \int_0^1 \int_0^{2\pi} (2(r \cos \theta)(r \sin \theta) + 4 - 3(r \cos \theta)(r \sin \theta)^2) r d\theta dr \\ &= \int_0^1 \int_0^{2\pi} (2r^3 \cos \theta \sin \theta + 4r - 3r^4 \cos \theta \sin^2 \theta) d\theta dr \\ &= \int_0^1 [r^3 \sin^2 \theta + 4r\theta - r^4 \sin^3 \theta]_0^{2\pi} dr \\ &= \int_0^1 8\pi r dr \\ &= [4\pi r^2]_0^1 = \boxed{4\pi}.\end{aligned}$$

15. With $P(x, y) = x^2 + y$ and $Q(x, y) = x - y^2$, we get

$$\oint_C [(x^2 + y) dx + (x - y^2) dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (1 - 1) dx dy = \boxed{0}.$$

16. The region is



Since R is both x -simple and y -simple, we can choose the order of integration; integrate first with respect to y . With $P(x, y) = 4x^2 - 8y^2$ and $Q(x, y) = y - 6xy$, we get

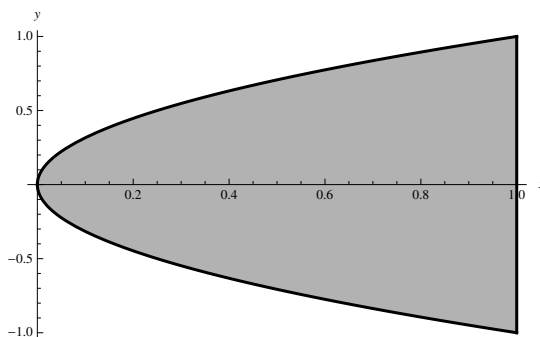
$$\begin{aligned} \oint_C [(4x^2 - 8y^2) dx + (y - 6xy) dy] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R (-6y - (-16y)) dx dy \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y dy dx \\ &= \int_0^1 [5y^2]_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 (5x - 5x^4) dx \\ &= \left[\frac{5}{2}x^2 - x^5 \right]_0^1 = \boxed{\frac{3}{2}}. \end{aligned}$$

17. Note that $x = y^2$ is the same as $y = \pm\sqrt{x}$, so the region bounded by $y = x^2$ and $x = y^2$ is the same as that bounded by $y = x^2$ and $y = \sqrt{x}$, which is shown in Problem 16. Since R is both x -simple and y -simple, we can choose the order of integration; integrate first with respect to y . With

$P(x, y) = x^2 - x^2y$ and $Q(x, y) = xy^2$, we get

$$\begin{aligned}
 \oint_C [(x^2 - x^2y) dx + xy^2 dy] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \iint_R (y^2 - (-x^2)) dx dy \\
 &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) dy dx \\
 &= \int_0^1 \left[x^2y + \frac{1}{3}y^3 \right]_{x^2}^{\sqrt{x}} dx \\
 &= \int_0^1 \left(\left(x^{5/2} + \frac{1}{3}x^{3/2} \right) - \left(x^4 + \frac{1}{3}x^6 \right) \right) dx \\
 &= \int_0^1 \left(x^{5/2} + \frac{1}{3}x^{3/2} - x^4 - \frac{1}{3}x^6 \right) dx \\
 &= \left[\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{5}x^5 - \frac{1}{21}x^7 \right]_0^1 \\
 &= \boxed{\frac{6}{35}}.
 \end{aligned}$$

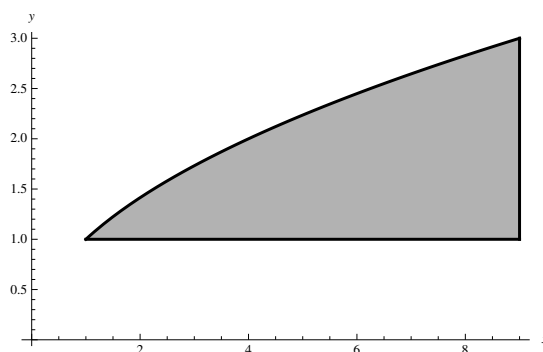
18. The region is



Since R is y -simple, we integrate first with respect to x . With $P(x, y) = x^2 - y^2$ and $Q(x, y) = xy$, we get

$$\begin{aligned}
 \oint_C [(x^2 - y^2) dx + xy dy] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \iint_R (y - (-2y)) dx dy \\
 &= \int_{-1}^1 \int_{y^2}^1 3y dx dy \\
 &= \int_{-1}^1 [3xy]_{y^2}^1 dy \\
 &= \int_{-1}^1 (3y - 3y^3) dy \\
 &= \left[\frac{3}{2}y^2 - \frac{3}{4}y^4 \right]_{-1}^1 = \boxed{0}.
 \end{aligned}$$

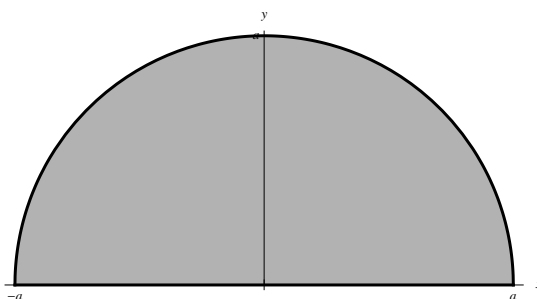
19. The region is



The curve is more easily described as a function of y , namely $x = y^2$, so regard this as a y -simple region (it is x -simple as well) and integrate first with respect to x . With $P(x, y) = \frac{1}{y}$ and $Q(x, y) = \frac{1}{x}$, we get

$$\begin{aligned}
 \oint_C \left[\frac{1}{y} dx + \frac{1}{x} dy \right] &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \iint_R \left(-\frac{1}{x^2} + \frac{1}{y^2} \right) dx dy \\
 &= \int_1^3 \int_{y^2}^9 \left(-\frac{1}{x^2} + \frac{1}{y^2} \right) dx dy \\
 &= \int_1^3 \left[\frac{1}{x} + \frac{x}{y^2} \right]_{y^2}^9 dy \\
 &= \int_1^3 \left(\frac{1}{9} + \frac{9}{y^2} - \frac{1}{y^2} - \frac{y^2}{y^2} \right) dy \\
 &= \int_1^3 \left(\frac{8}{y^2} - \frac{8}{9} \right) dy \\
 &= \left[-\frac{8}{y} - \frac{8}{9}y \right]_1^3 \\
 &= -\frac{8}{3} - \frac{8}{3} + 8 + \frac{8}{9} = \boxed{\frac{32}{9}}.
 \end{aligned}$$

20. The region is



With $P(x, y) = x^2y$ and $Q(x, y) = -y^2x$, we get

$$\oint_C [x^2y dx - y^2x dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (-y^2 - x^2) dx dy.$$

Convert to polar coordinates, giving

$$\begin{aligned}
 \iint_R (-y^2 - x^2) \, dx \, dy &= \int_0^a \int_0^\pi (-r^2) \cdot r \, d\theta \, dr \\
 &= \int_0^a [-r^3 \theta]_0^\pi \, dr \\
 &= \pi \int_0^a (-r^3) \, dr \\
 &= \pi \left[-\frac{1}{4} r^4 \right]_0^a = \boxed{-\frac{\pi}{4} a^4}.
 \end{aligned}$$

21. Since the region is the interior of a circle of radius 2 centered at the origin, and $P(x, y) = \frac{y}{x^2+y^2}$ and $Q(x, y) = -\frac{x}{x^2+y^2}$ are not defined at the origin, Green's Theorem does not apply, so we must compute the line integral. Parametrize the circle by $x(t) = 2 \cos t$, $y(t) = 2 \sin t$ for $0 \leq t \leq 2\pi$. Then

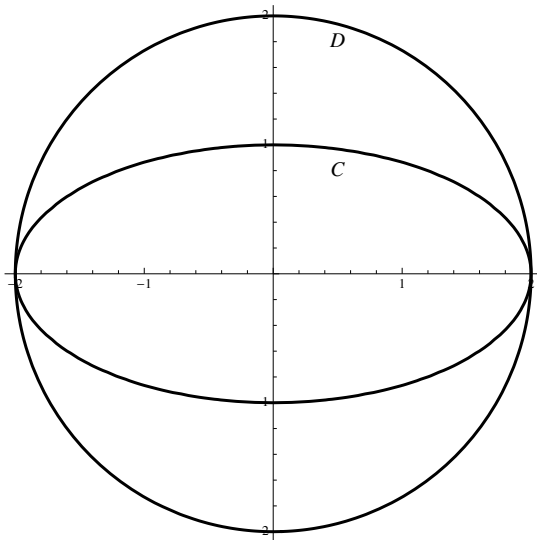
$$\begin{aligned}
 \oint_C (P(x, y) \, dx + Q(x, y) \, dy) &= \int_0^{2\pi} \left(\frac{2 \sin t}{(2 \sin t)^2 + (2 \cos t)^2} \cdot (-2 \sin t) - \frac{2 \cos t}{(2 \sin t)^2 + (2 \cos t)^2} \cdot (2 \cos t) \right) dt \\
 &= \int_0^{2\pi} \left(-\frac{4 \sin^2 t + 4 \cos^2 t}{4 \sin^2 t + 4 \cos^2 t} \right) dt \\
 &= \int_0^{2\pi} (-1) \, dt = \boxed{-2\pi}.
 \end{aligned}$$

22. The curve is a circle of radius 1 centered at $(2, -3)$, so it does not contain the origin (which is at a distance of $\sqrt{2^2 + 3^2} = \sqrt{13} > 1$ from $(2, -3)$). Therefore $P(x, y)$, $Q(x, y)$, and their derivatives are continuous on the interior of the circle. Since

$$\begin{aligned}
 \frac{\partial Q(x, y)}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{x}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
 \frac{\partial P(x, y)}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2},
 \end{aligned}$$

the vector field is conservative by the converse of the Conservative Vector Field Theorem. Since the curve C is closed, the line integral is $\boxed{\text{zero}}$.

23. Since the region is the interior of an ellipse centered at the origin, and $P(x, y) = \frac{y}{x^2+y^2}$ and $Q(x, y) = -\frac{x}{x^2+y^2}$ are not defined at the origin, Green's Theorem does not apply. However, P , Q , and their derivatives are continuous away from the origin, and this ellipse can be continuously deformed to the circle of radius 2 centered at the origin by intermediate curves that do not contain the origin; see the following figure:

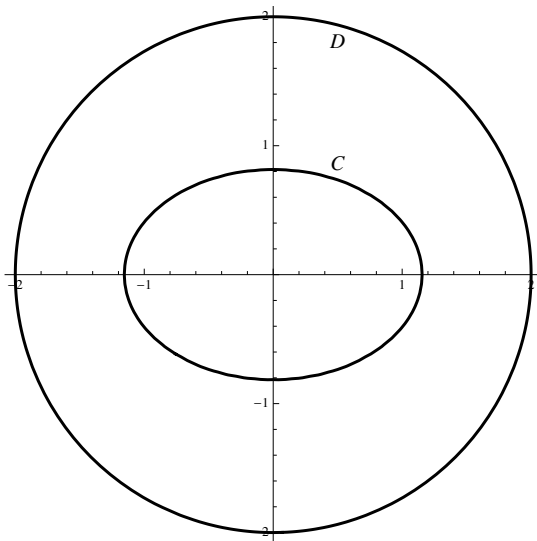


So by the theorem preceding Example 4, if D is the circle $x^2 + y^2 = 4$, then

$$\oint_C \frac{y \, dx - x \, dy}{x^2 + y^2} = \oint_D \frac{y \, dx - x \, dy}{x^2 + y^2}.$$

By Problem 21, the latter integral is $\boxed{-2\pi}$.

24. Since the region is the interior of an ellipse centered at the origin, and $P(x, y) = \frac{y}{x^2 + y^2}$ and $Q(x, y) = -\frac{x}{x^2 + y^2}$ are not defined at the origin, Green's Theorem does not apply. However, P , Q , and their derivatives are continuous away from the origin, and this ellipse can be continuously deformed to the circle of radius 2 centered at the origin by intermediate curves that do not contain the origin; see the following figure:

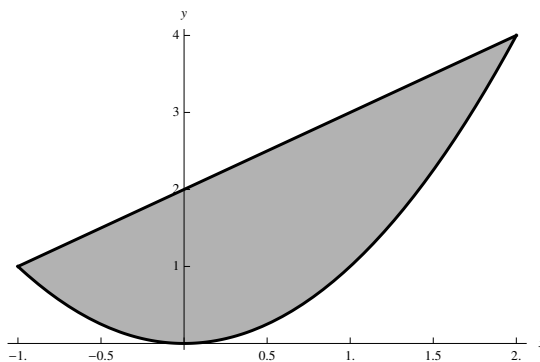


So by the theorem preceding Example 4, if D is the circle $x^2 + y^2 = 4$, then

$$\oint_C \frac{y \, dx - x \, dy}{x^2 + y^2} = \oint_D \frac{y \, dx - x \, dy}{x^2 + y^2}.$$

By Problem 21, the latter integral is $\boxed{-2\pi}$.

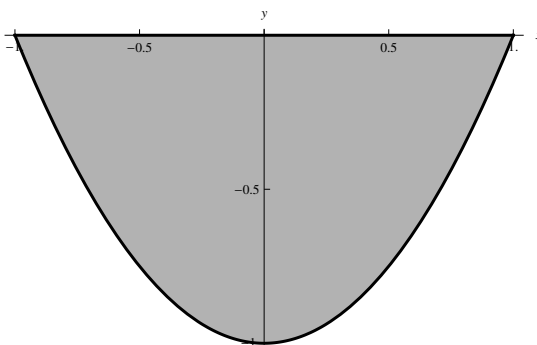
25. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_{-1}^2 \int_{x^2}^{x+2} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_{-1}^2 [2y]_{x^2}^{x+2} dx \\
 &= \frac{1}{2} \int_{-1}^2 (2x + 4 - 2x^2) \, dx \\
 &= \frac{1}{2} \left[x^2 + 4x - \frac{2}{3}x^3 \right]_{-1}^2 \\
 &= \frac{1}{2} \left(\left(4 + 8 - \frac{16}{3} \right) - \left(1 - 4 + \frac{2}{3} \right) \right) \\
 &= \boxed{\frac{9}{2}}.
 \end{aligned}$$

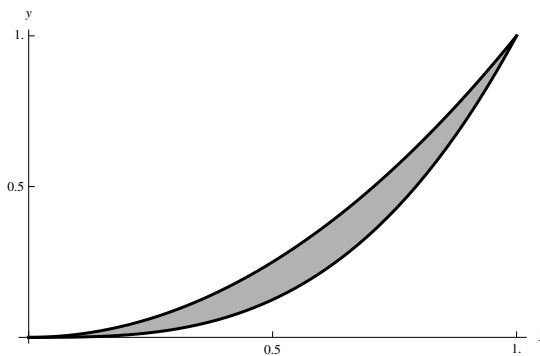
26. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_{-1}^1 \int_{x^2-1}^0 2 \, dy \, dx \\
 &= \frac{1}{2} \int_{-1}^1 [2y]_{x^2-1}^0 \, dx \\
 &= \frac{1}{2} \int_{-1}^1 (2 - 2x^2) \, dx \\
 &= \frac{1}{2} \left[2x - \frac{2}{3}x^3 \right]_{-1}^1 \\
 &= \boxed{\frac{4}{3}}.
 \end{aligned}$$

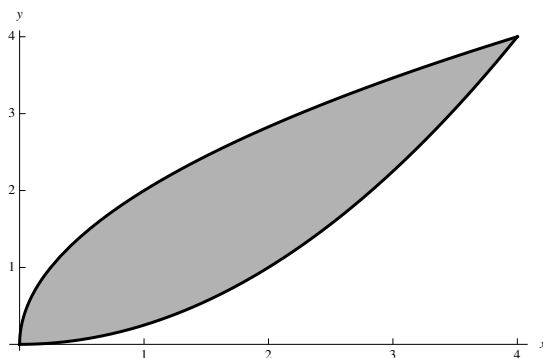
27. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_0^1 \int_{x^3}^{x^2} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_0^1 [2y]_{x^3}^{x^2} \, dx \\
 &= \frac{1}{2} \int_0^1 (2x^2 - 2x^3) \, dx \\
 &= \frac{1}{2} \left[\frac{2}{3}x^3 - \frac{1}{2}x^4 \right]_0^1 \\
 &= \boxed{\frac{1}{12}}.
 \end{aligned}$$

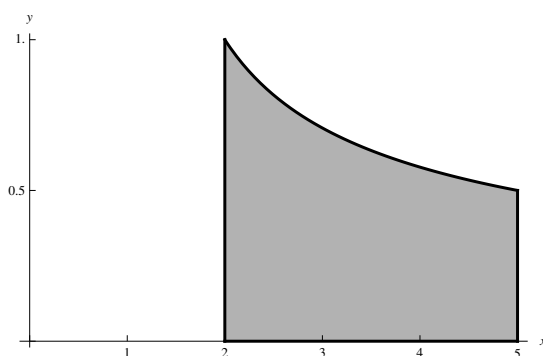
28. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_0^4 \int_{x^2/4}^{2\sqrt{x}} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_0^4 [2y]_{x^2/4}^{2\sqrt{x}} dx \\
 &= \frac{1}{2} \int_0^4 \left(4\sqrt{x} - \frac{x^2}{2} \right) dx \\
 &= \frac{1}{2} \left[\frac{8}{3} x^{3/2} - \frac{1}{6} x^3 \right]_0^4 \\
 &= \boxed{\frac{16}{3}}.
 \end{aligned}$$

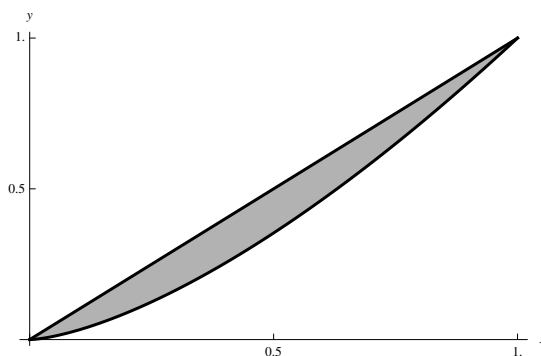
29. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_2^5 \int_0^{1/\sqrt{x-1}} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_2^5 [2y]_0^{1/\sqrt{x-1}} dx \\
 &= \frac{1}{2} \int_2^5 \frac{2}{\sqrt{x-1}} dx \\
 &= \frac{1}{2} \int_2^5 2(x-1)^{-1/2} dx \\
 &= \frac{1}{2} \left[4(x-1)^{1/2} \right]_2^5 \\
 &= \boxed{2}.
 \end{aligned}$$

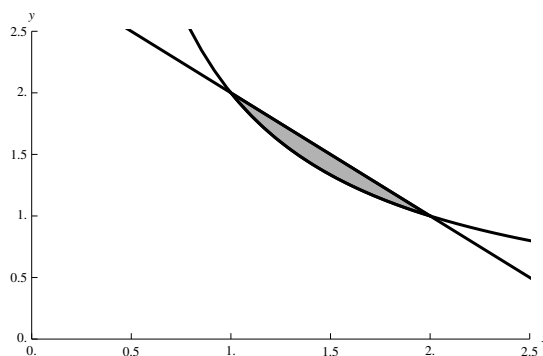
30. The region is



Treat this as an x -simple region. Then by Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_0^1 \int_{x^{3/2}}^x 2 \, dy \, dx \\
 &= \frac{1}{2} \int_0^1 [2y]_{x^{3/2}}^x dx \\
 &= \frac{1}{2} \int_0^1 (2x - 2x^{3/2}) dx \\
 &= \frac{1}{2} \left[x^2 - \frac{4}{5} x^{5/2} \right]_0^1 \\
 &= \boxed{\frac{1}{10}}.
 \end{aligned}$$

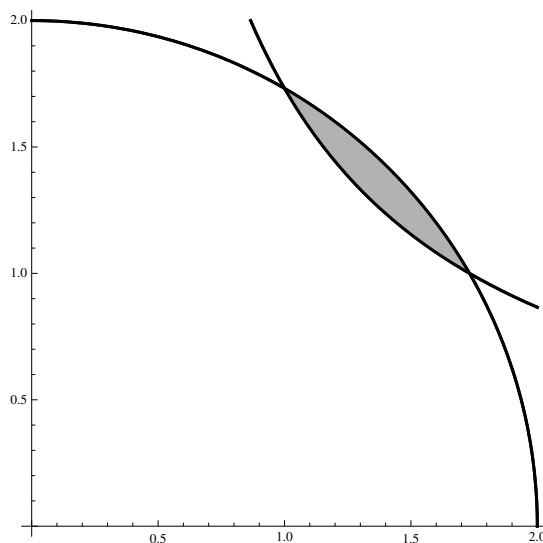
31. The region is



Treat this as an x -simple region; then $xy = 2$ is the curve $y = \frac{2}{x}$. By Green's Theorem for Area, the area of the region is

$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_1^2 \int_{2/x}^{3-x} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_1^2 [2y]_{2/x}^{3-x} dx \\
 &= \frac{1}{2} \int_1^2 \left(6 - 2x - \frac{4}{x} \right) dx \\
 &= \frac{1}{2} [6x - x^2 - 4 \ln x]_1^2 \\
 &= \frac{1}{2} ((12 - 4 - 4 \ln 2) - (6 - 1 - 4 \ln 1)) \\
 &= \boxed{\frac{3}{2} - 2 \ln 2}.
 \end{aligned}$$

32. The region is



To find the points of intersection of the two curves, solve each for y , giving $y = \frac{\sqrt{3}}{x}$ and $y = \sqrt{4-x^2}$; these are equal when $\frac{\sqrt{3}}{x} = \sqrt{4-x^2}$. Squaring gives $\frac{3}{x^2} = 4-x^2$, so that $x^4-4x^2+3 = (x^2-3)(x^2-1) = 0$. Since we are working in the first quadrant only, the points of intersection are at $x = 1$ and $x = \sqrt{3}$, which correspond to $y = \sqrt{3}$ and $y = 1$. So the intersection points are $(1, \sqrt{3})$ and $(\sqrt{3}, 1)$. By Green's Theorem for Area, the area of the region is

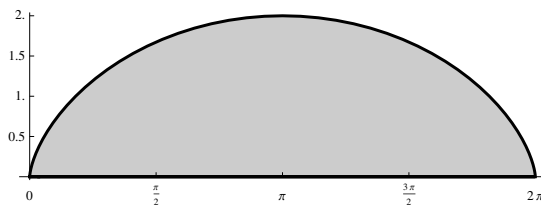
$$\begin{aligned}
 A &= \frac{1}{2} \oint_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dy \, dx \\
 &= \frac{1}{2} \int_1^{\sqrt{3}} \int_{\sqrt{3}/x}^{\sqrt{4-x^2}} 2 \, dy \, dx \\
 &= \frac{1}{2} \int_1^{\sqrt{3}} [2y]_{\sqrt{3}/x}^{\sqrt{4-x^2}} dx \\
 &= \frac{1}{2} \int_1^{\sqrt{3}} \left(2\sqrt{4-x^2} - \frac{2\sqrt{3}}{x} \right) dx \\
 &= \int_1^{\sqrt{3}} \sqrt{4-x^2} \, dx - \int_1^{\sqrt{3}} \frac{\sqrt{3}}{x} \, dx.
 \end{aligned}$$

To integrate the first term, use the trigonometric substitution $x = 2 \sin \theta$; then $4 - x^2 = 4 - 4 \sin^2 \theta = \cos^2 \theta$; since we are working in the first quadrant, we can take $\sqrt{4-x^2} = \cos \theta$. Further, $dx = 2 \cos \theta \, d\theta$. Also, $x = 1$ corresponds to $\theta = \frac{\pi}{6}$ and $x = \sqrt{3}$ to $\theta = \frac{\pi}{3}$. So the integral becomes

$$\begin{aligned}
 A &= \int_1^{\sqrt{3}} \sqrt{4-x^2} \, dx - \int_1^{\sqrt{3}} \frac{\sqrt{3}}{x} \, dx \\
 &= \int_{\pi/6}^{\pi/3} 2 \cos \theta \cdot 2 \cos \theta \, d\theta - \int_1^{\sqrt{3}} \frac{\sqrt{3}}{x} \, dx \\
 &= \int_{\pi/6}^{\pi/3} 4 \cos^2 \theta \, d\theta - \left[\sqrt{3} \ln x \right]_1^{\sqrt{3}} \\
 &= \int_{\pi/6}^{\pi/3} 4 \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta - \sqrt{3} \ln \sqrt{3} \\
 &= \int_{\pi/6}^{\pi/3} (2 + 2 \cos(2\theta)) \, d\theta - \frac{\sqrt{3}}{2} \ln 3 \\
 &= [2\theta + \sin(2\theta)]_{\pi/6}^{\pi/3} - \frac{\sqrt{3}}{2} \ln 3 \\
 &= \left(\frac{2\pi}{3} + \sin \frac{2\pi}{3} \right) - \left(\frac{\pi}{3} + \sin \frac{\pi}{3} \right) - \frac{\sqrt{3}}{2} \ln 3 \\
 &= \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2} \ln 3}.
 \end{aligned}$$

Applications and Extensions

33. We are given a parametrization of the cycloid. In order to use Green's formula for the area, we want a parametrization of the boundary of that area under the cycloid; a plot of the cycloid is



While the area is given by a counterclockwise parametrization, the parametrization we are given for the curve is clockwise (left to right). So we will parametrize the line along the bottom clockwise as well, by

$$C_1 : x(t) = 2\pi(1 - t), \quad y(t) = 0, \quad 0 \leq t \leq 1,$$

and negate the sum of the two line integrals to get the area. Note that

$$C_1 : x'(t) = -2\pi, \quad y'(t) = 0, \quad C_2 : x'(t) = 2\pi - 2\pi \cos(2\pi t), \quad y'(t) = 2\pi \sin(2\pi t).$$

Then the area is

$$\begin{aligned} A &= -\frac{1}{2} \left(\int_{C_1} (-y \, dx + x \, dy) + \int_{C_2} (-y \, dx + x \, dy) \right) \\ &= -\frac{1}{2} \int_0^1 (-0 \cdot (-2\pi) - 2\pi(1 - t) \cdot 0) \, dt \\ &\quad - \frac{1}{2} \int_0^1 [-(1 - \cos(2\pi t)) \cdot (2\pi - 2\pi \cos(2\pi t)) + (2\pi t - \sin(2\pi t)) \cdot (2\pi \sin(2\pi t))] \, dt \\ &= -\frac{1}{2} \int_0^1 [-2\pi + 2\pi \cos(2\pi t) + 2\pi \cos(2\pi t) - 2\pi \cos^2(2\pi t) + 4\pi^2 t \sin(2\pi t) - 2\pi \sin^2(2\pi t)] \, dt \\ &= -\frac{1}{2} \int_0^1 [-2\pi + 4\pi \cos(2\pi t) + 4\pi^2 t \sin(2\pi t) - 2\pi] \, dt \\ &= \int_0^1 [2\pi - 2\pi \cos(2\pi t) - 2\pi^2 t \sin(2\pi t)] \, dt. \end{aligned}$$

Integrate $t \sin(2\pi t)$ using integration by parts with $u = t$ and $dv = \sin(2\pi t) \, dt$, so that $du = dt$ and $v = -\frac{1}{2\pi} \cos(2\pi t)$. Then

$$\begin{aligned} A &= \int_0^1 [2\pi - 2\pi \cos(2\pi t) - 2\pi^2 t \sin(2\pi t)] \, dt \\ &= [2\pi t - \sin(2\pi t)]_0^1 - 2\pi^2 \left[\left[-\frac{1}{2\pi} t \cos(2\pi t) \right]_0^1 + \frac{1}{2\pi} \int_0^1 \cos(2\pi t) \, dt \right] \\ &= 2\pi - 2\pi^2 \left[-\frac{1}{2\pi} + \frac{1}{2\pi} \left[\frac{1}{2\pi} \sin(2\pi t) \right]_0^1 \right] \\ &= 2\pi - \pi \left[-1 + \left[\frac{1}{2\pi} \sin(2\pi t) \right]_0^1 \right] \\ &= \boxed{3\pi}. \end{aligned}$$

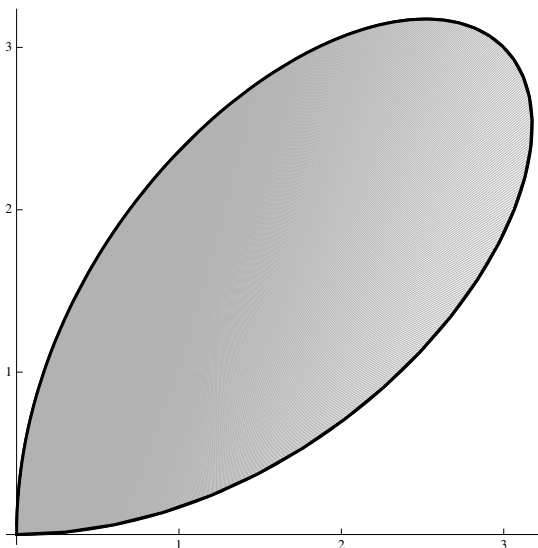
34. A diagram of the hypocycloid is given in the text. Using the given parametrization for $0 \leq t \leq 2\pi$, we have

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

Then the area enclosed by the hypocycloid is

$$\begin{aligned}
 A &= \frac{1}{2} \int_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \int_0^{2\pi} (-\sin^3 t \cdot (-3 \cos^2 t \sin t) + \cos^3 t \cdot 3 \sin^2 t \cos t) \, dt \\
 &= \frac{3}{2} \int_0^{2\pi} (\sin^4 t \cos^2 t + \sin^2 t \cos^4 t) \, dt \\
 &= \frac{3}{2} \int_0^{2\pi} ((\sin^2 t + \cos^2 t) \sin^2 t \cos^2 t) \, dt \\
 &= \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt \\
 &= \frac{3}{2} \int_0^{2\pi} \frac{1}{4} \sin^2(2t) \, dt \\
 &= \frac{3}{8} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(4t) \right) \, dt \\
 &= \frac{3}{16} \int_0^{2\pi} (1 - \cos(4t)) \, dt \\
 &= \frac{3}{16} \left[t - \frac{1}{4} \sin(4t) \right]_0^{2\pi} \\
 &= \boxed{\frac{3}{8}\pi}.
 \end{aligned}$$

35. The loop of the Folium of Descartes is



With the given parametrization,

$$x'(t) = \frac{6(1 - 2t^3)}{(1 + t^3)^2}, \quad y'(t) = \frac{6(2t - t^4)}{(1 + t^3)^2}.$$

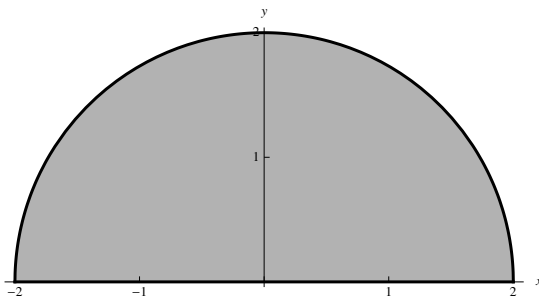
Note that $(x(t), y(t)) \rightarrow (0, 0)$ as $t \rightarrow \infty$, so the area is

$$\begin{aligned}
 A &= \frac{1}{2} \int_C (-y \, dx + x \, dy) \\
 &= \frac{1}{2} \int_0^\infty \left(-\frac{6t^2}{1+t^3} \cdot \frac{6(1-2t^3)}{(1+t^3)^2} + \frac{6t}{1+t^3} \cdot \frac{6(2t-t^4)}{(1+t^3)^2} \right) dt \\
 &= \frac{1}{2} \int_0^\infty \frac{-36t^2(1-2t^3) + 36t(2t-t^4)}{(1+t^3)^3} dt \\
 &= 36 \cdot \frac{1}{2} \int_0^\infty \frac{-t^2 + 2t^5 + 2t^2 - t^5}{(1+t^3)^3} dt \\
 &= 18 \int_0^\infty \frac{t^5 + t^2}{(1+t^3)^3} dt \\
 &= 18 \int_0^\infty \frac{t^2}{(1+t^3)^2} dt.
 \end{aligned}$$

Now make the substitution $u = 1 + t^3$, so that $du = 3t^2 \, dt$. $t = 0$ corresponds to $u = 1$, and the upper bound remains ∞ . Then we get

$$\begin{aligned}
 A &= 18 \int_0^\infty \frac{t^2}{(1+t^3)^2} dt \\
 &= 18 \cdot \frac{1}{3} \int_1^\infty u^{-2} du \\
 &= 6 \lim_{b \rightarrow \infty} \int_1^b u^{-2} du \\
 &= 6 \lim_{b \rightarrow \infty} [-u^{-1}]_1^b \\
 &= 6 \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) \\
 &= \boxed{6}.
 \end{aligned}$$

36. The region is



Since this is a semicircle of radius 2, its area is $A = \frac{1}{2} \cdot \pi \cdot 2^2 = 2\pi$. We wish to compute

$$\bar{x} = \frac{1}{4\pi} \oint_C x^2 \, dy, \quad \bar{y} = -\frac{1}{4\pi} \oint_C y^2 \, dx.$$

For the first integral, using Green's Theorem with $P(x, y) = 0$ and $Q(x, y) = x^2$, we get

$$\bar{x} = \frac{1}{4\pi} \oint_C x^2 \, dy = \frac{1}{4\pi} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \frac{1}{4\pi} \iint_R 2x \, dx \, dy.$$

Convert to polar coordinates, giving

$$\bar{x} = \frac{1}{4\pi} \iint_R 2x \, dx \, dy = \frac{1}{4\pi} \int_0^2 \int_0^\pi 2r \cos \theta \cdot r \, d\theta \, dr = \frac{1}{4\pi} \int_0^2 [2r^2 \sin \theta]_0^\pi \, dr = 0.$$

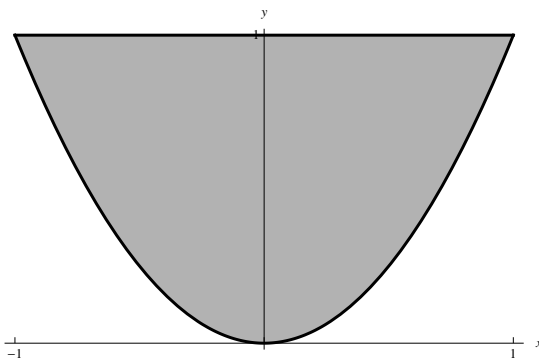
(Note that this is what one would expect from symmetry.) For the second integral, again use Green's Theorem, now with $P(x, y) = y^2$ and $Q(x, y) = 0$, to get

$$\begin{aligned} \bar{y} &= -\frac{1}{4\pi} \oint_C y^2 \, dx = -\frac{1}{4\pi} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \\ &= -\frac{1}{4\pi} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \\ &= -\frac{1}{4\pi} \iint_R (-2y) \, dx \, dy \\ &= \frac{1}{4\pi} \int_{-2}^2 \int_0^{\sqrt{4-x^2}} 2y \, dy \, dx \\ &= \frac{1}{4\pi} \int_{-2}^2 [y^2]_0^{\sqrt{4-x^2}} \, dx \\ &= \frac{1}{4\pi} \int_{-2}^2 (4 - x^2) \, dx \\ &= \frac{1}{4\pi} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{8}{3\pi}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \boxed{\left(0, \frac{8}{3\pi} \right)}.$$

37. The region is



To compute its area, we integrate

$$A = \int_{-1}^1 (1 - x^2) \, dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{4}{3}.$$

Then $\frac{1}{2A} = \frac{3}{8}$, and we wish to compute

$$\bar{x} = \frac{3}{8} \oint_C x^2 \, dy, \quad \bar{y} = -\frac{3}{8} \oint_C y^2 \, dx.$$

For the first integral, using Green's Theorem with $P(x, y) = 0$ and $Q(x, y) = x^2$, we get

$$\begin{aligned}
 \bar{x} &= \frac{3}{8} \oint_C x^2 dy = \frac{3}{8} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \frac{3}{8} \iint_R 2x dx dy \\
 &= \frac{3}{8} \int_{-1}^1 \int_{x^2}^1 2x dy dx \\
 &= \frac{3}{8} \int_{-1}^1 [2xy]_{x^2}^1 dx \\
 &= \frac{3}{8} \int_{-1}^1 (2x - 2x^3) dx \\
 &= \frac{3}{8} \left[x^2 - \frac{1}{2} x^4 \right]_{-1}^1 = 0.
 \end{aligned}$$

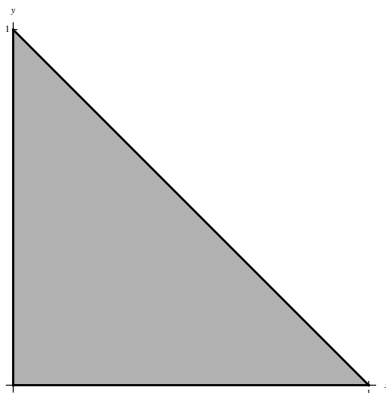
(Note that this is what one would expect from symmetry.) For the second integral, again use Green's Theorem, now with $P(x, y) = y^2$ and $Q(x, y) = 0$, to get

$$\begin{aligned}
 \bar{y} &= -\frac{3}{8} \oint_C y^2 dx = -\frac{3}{8} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= -\frac{3}{8} \iint_R (-2y) dx dy \\
 &= \frac{3}{8} \int_{-1}^1 \int_{x^2}^1 2y dy dx \\
 &= \frac{3}{8} \int_{-1}^1 [y^2]_{x^2}^1 dx \\
 &= \frac{3}{8} \int_{-1}^1 (1 - x^4) dx \\
 &= \frac{3}{8} \left[x - \frac{1}{5} x^5 \right]_{-1}^1 = \frac{3}{5}.
 \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \boxed{\left(0, \frac{3}{5} \right)}.$$

38. The region is



The figure is a triangle with base and height each equal to 1, so its area is $A = \frac{1}{2}$, so that $\frac{1}{2A} = 1$. Note that the region is defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$. We wish to compute

$$\bar{x} = \oint_C x^2 dy, \quad \bar{y} = - \oint_C y^2 dx.$$

For the first integral, using Green's Theorem with $P(x, y) = 0$ and $Q(x, y) = x^2$, we get

$$\begin{aligned} \bar{x} &= \oint_C x^2 dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R 2x dx dy \\ &= \int_0^1 \int_0^{1-x} 2x dy dx \\ &= \int_0^1 [2xy]_0^{1-x} dx \\ &= \int_0^1 2x(1-x) dx \\ &= \int_0^1 (2x - 2x^2) dx \\ &= \left[x^2 - \frac{2}{3}x^3 \right]_0^1 = \frac{1}{3}. \end{aligned}$$

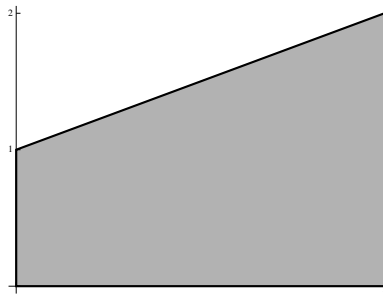
For the second integral, again use Green's Theorem, now with $P(x, y) = y^2$ and $Q(x, y) = 0$, to get

$$\begin{aligned} \bar{y} &= - \oint_C y^2 dx = - \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= - \iint_R (-2y) dx dy \\ &= \int_0^1 \int_0^{1-x} 2y dy dx \\ &= \int_0^1 [y^2]_0^{1-x} dx \\ &= \int_0^1 (1-x)^2 dx \\ &= \left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{3}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{1}{3} \right).$$

39. The region is



The figure is a trapezoid with bases 1 and 2 and height 1, so its area is $A = \frac{1}{2}(1+2) \cdot 1 = \frac{3}{2}$, and therefore $\frac{1}{2A} = \frac{1}{3}$. Note that the region is defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1+x$. We wish to compute

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy, \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx.$$

For the first integral, using Green's Theorem with $P(x, y) = 0$ and $Q(x, y) = x^2$, we get

$$\begin{aligned} \bar{x} &= \frac{1}{3} \oint_C x^2 dy = \frac{1}{3} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \frac{1}{3} \iint_R 2x dx dy \\ &= \frac{1}{3} \int_0^1 \int_0^{1+x} 2x dy dx \\ &= \frac{1}{3} \int_0^1 [2xy]_0^{1+x} dx \\ &= \frac{1}{3} \int_0^1 2x(1+x) dx \\ &= \frac{1}{3} \int_0^1 (2x + 2x^2) dx \\ &= \frac{1}{3} \left[x^2 + \frac{2}{3}x^3 \right]_0^1 = \frac{5}{9}. \end{aligned}$$

For the second integral, again use Green's Theorem, now with $P(x, y) = y^2$ and $Q(x, y) = 0$, to get

$$\begin{aligned} \bar{y} &= -\frac{1}{3} \oint_C y^2 dx = -\frac{1}{3} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= -\frac{1}{3} \iint_R (-2y) dx dy \\ &= \frac{1}{3} \int_0^1 \int_0^{1+x} 2y dy dx \\ &= \frac{1}{3} \int_0^1 [y^2]_0^{1+x} dx \\ &= \frac{1}{3} \int_0^1 (1+x)^2 dx \\ &= \frac{1}{3} \left[\frac{1}{3}(1+x)^3 \right]_0^1 = \frac{7}{9}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{5}{9}, \frac{7}{9} \right).$$

40. A diagram of the hypocycloid is in the text. Using the given parametrization for $0 \leq t \leq \frac{\pi}{2}$, we have

$$x'(t) = -3a \cos^2 t \sin t, \quad y'(t) = 3a \sin^2 t \cos t.$$

Parametrize the boundary of the first-quadrant region in question as

$$C_1 : x(t) = at, \quad y(t) = 0, \quad 0 \leq t \leq 1 \quad x'(t) = a, \quad y'(t) = 0$$

$$C_2 : x(t) = a \cos^3 t, \quad y(t) = a \sin^3 t, \quad 0 \leq t \leq \frac{\pi}{2} \quad x'(t) = -3a \cos^2 t \sin t, \quad y'(t) = 3a \sin^2 t \cos t$$

$$C_3 : x(t) = 0, \quad y(t) = a(1-t), \quad 0 \leq t \leq 1 \quad x'(t) = 0, \quad y'(t) = -a.$$

Then the area enclosed by the hypocycloid in the first quadrant is

$$A = \frac{1}{2} \oint_C (-y dx + x dy) = \frac{1}{2} \int_{C_1} (-y dx + x dy) + \frac{1}{2} \int_{C_2} (-y dx + x dy) + \frac{1}{2} \int_{C_3} (-y dx + x dy).$$

The first and third integrals are each zero, since in the first integral $-y dx + x dy = -0 \cdot a + at \cdot 0 = 0$ and in the third, $-y dx + x dy = -a(1-t) \cdot 0 + 0 \cdot (-a) = 0$. So the area is determined by the second integral:

$$\begin{aligned} A &= \frac{1}{2} \int_{C_2} (-y dx + x dy) = \frac{1}{2} \int_0^{\pi/2} (-a \sin^3 t \cdot (-3a \cos^2 t \sin t) + a \cos^3 t \cdot 3a \sin^2 t \cos t) dt \\ &= \frac{3}{2} a^2 \int_0^{\pi/2} (\sin^4 t \cos^2 t + \sin^2 t \cos^4 t) dt \\ &= \frac{3}{2} a^2 \int_0^{\pi/2} ((\sin^2 t + \cos^2 t) \sin^2 t \cos^2 t) dt \\ &= \frac{3}{2} a^2 \int_0^{\pi/2} \sin^2 t \cos^2 t dt \\ &= \frac{3}{2} a^2 \int_0^{\pi/2} \frac{1}{4} \sin^2(2t) dt \\ &= \frac{3}{8} a^2 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2} \cos(4t) \right) dt \\ &= \frac{3}{16} a^2 \int_0^{\pi/2} (1 - \cos(4t)) dt \\ &= \frac{3}{16} a^2 \left[t - \frac{1}{4} \sin(4t) \right]_0^{\pi/2} = \frac{3}{32} \pi a^2. \end{aligned}$$

Then $\frac{1}{2A} = \frac{16}{3\pi a^2}$. We wish to compute

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy, \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx,$$

As before, C consists of C_1 , C_2 , and C_3 , and both integrals are zero along both C_1 and C_3 , since along C_1

$$x^2 dy = (at)^2 \cdot 0 = 0, \quad y^2 dx = 0^2 \cdot a = 0,$$

and along C_3

$$x^2 dy = 0^2 \cdot (-a), \quad y^2 dx = (a(1-t))^2 \cdot 0 = 0.$$

So the coordinates of the center of mass are determined by C_2 . For the first integral, we get

$$\begin{aligned} \bar{x} &= \frac{16}{3\pi a^2} \int_{C_2} x^2 dy = \frac{16}{3\pi a^2} \int_0^{\pi/2} (a \cos^3 t)^2 \cdot 3a \sin^2 t \cos t dt \\ &= \frac{16}{3\pi a^2} \cdot 3a^3 \int_0^{\pi/2} \sin^2 t \cos^7 t dt \\ &= \frac{16}{\pi} a \int_0^{\pi/2} \sin^2 t (1 - \sin^2 t)^3 \cos t dt \\ &= \frac{16}{\pi} a \int_0^{\pi/2} (\sin^2 t - 3\sin^4 t + 3\sin^6 t - \sin^8 t) \cos t dt \\ &= \frac{16}{\pi} a \left[\frac{1}{3} \sin^2 t - \frac{3}{5} \sin^5 t + \frac{3}{7} \sin^7 t - \frac{1}{9} \sin^9 t \right]_0^{\pi/2} \\ &= \frac{256a}{315\pi}. \end{aligned}$$

For the second integral, again use Green's Theorem, now with $P(x, y) = y^2$ and $Q(x, y) = 0$, to get

$$\begin{aligned}
 \bar{y} &= -\frac{16}{3\pi a^2} \int_{C_2} y^2 dx = -\frac{16}{3\pi a^2} \int_0^{\pi/2} (a \sin^3 t)^2 \cdot (-3a \cos^2 t \sin t) dt \\
 &= \frac{16}{3\pi a^2} \cdot 3a^3 \int_0^{\pi/2} \cos^2 t \sin^7 t dt \\
 &= \frac{16}{\pi} a \int_0^{\pi/2} \cos^2 t (1 - \cos^2 t)^3 \sin t dt \\
 &= \frac{16}{\pi} a \int_0^{\pi/2} (\cos^2 t - 3\cos^4 t + 3\cos^6 t - \cos^8 t) \sin t dt \\
 &= \frac{16}{\pi} a \left[-\frac{1}{3} \cos^3 t + \frac{3}{5} \cos^5 t - \frac{3}{7} \cos^7 t + \frac{1}{9} \cos^9 t \right]_0^{\pi/2} \\
 &= \frac{256a}{315\pi}.
 \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{256a}{315\pi}, \frac{256a}{315\pi} \right).$$

41. Since the two curves form a closed path C , both components of $\mathbf{F}$ as well as their derivatives are continuous everywhere, and

$$\frac{\partial Q}{\partial x} = 0 = \frac{\partial P}{\partial y},$$

Green's Theorem tells us that

$$\oint_C \mathbf{F} \cdot \mathbf{r} = \oint_C (P dx + Q dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R 0 dx dy = \boxed{0}.$$

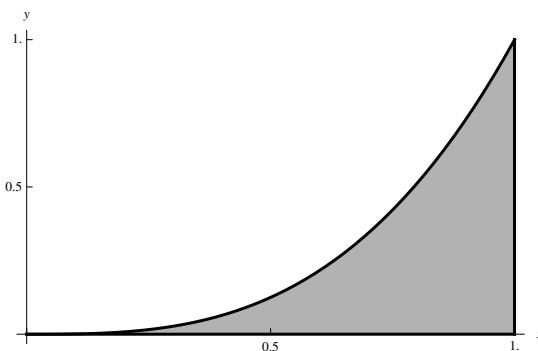
Therefore no work is done.

42. This path is a circle C of radius 3 centered at the origin. We have $P(x, y) = e^x + 2y$ and $Q(x, y) = e^y - x$, so the work done is

$$W = \oint_C P(x, y) dx + Q(x, y) dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (-1 - 2) dx dy = -3 \iint_R 1 dx dy.$$

But this integral is -3 times the area of the circle, which is $\pi \cdot 3^2 = 9\pi$, so the work done is $\boxed{-27\pi}$.

43. The region is



We have

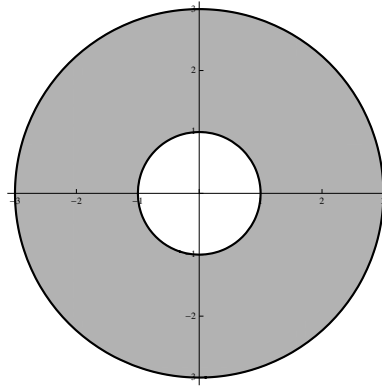
$$P(x, y) = \ln(1 + x^2) - x^2 \cos y, \quad Q(x, y) = \frac{x^3}{3} \cos y - e^{\cos y},$$

$$\frac{\partial P}{\partial y} = x^2 \sin y, \quad \frac{\partial Q}{\partial x} = x^2 \cos y.$$

Then by Green's Theorem,

$$\begin{aligned} \oint_C \left\{ [\ln(1 + x^2) - x^2 \cos y] dx + \left(\frac{x^3}{3} \cos y - e^{\cos y} \right) dy \right\} &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_R x^2 (\cos y - \sin y) dx dy \\ &= \int_0^1 \int_0^{x^3} x^2 (\cos y - \sin y) dy dx \\ &= \int_0^1 [x^2 (\sin y + \cos y)]_0^{x^3} dx \\ &= \int_0^1 (x^2 \sin x^3 + x^2 \cos x^3 - x^2) dx \\ &= \left[-\frac{1}{3} \cos x^3 + \frac{1}{3} \sin x^3 - \frac{1}{3} x^3 \right]_0^1 \\ &= \boxed{\frac{1}{3} (\sin 1 - \cos 1)}. \end{aligned}$$

44. The region R is the area between circles centered at the origin of radii 1 and 3:



Let C_3 be the circle of radius 3, and R_3 be the region inside that circle; let C_1 be the circle of radius 1, and R_1 be the region inside that circle. Denote by C the properly oriented boundary of the region in question. We have $P(x, y) = y^3$ and $Q(x, y) = -x^3$, so that $\frac{\partial P}{\partial y} = 3y^2$ and $\frac{\partial Q}{\partial x} = -3x^2$. Then by Green's Theorem for Multiply Connected Regions

$$\begin{aligned} \oint_C [y^3 dx - x^3 dy] &= \oint_{C_3} [y^3 dx - x^3 dy] - \oint_{C_1} [y^3 dx - x^3 dy] \\ &= \iint_{R_3} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy - \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_{R_3} (-3x^2 - 3y^2) dx dy - \iint_{R_1} (-3x^2 - 3y^2) dx dy \\ &= -3 \left(\iint_{R_3} (x^2 + y^2) dx dy - \iint_{R_1} (x^2 + y^2) dx dy \right). \end{aligned}$$

To compute these integrals, convert to polar coordinates:

$$\begin{aligned}
 & -3 \left(\iint_{R_3} (x^2 + y^2) \, dx \, dy - \iint_{R_1} (x^2 + y^2) \, dx \, dy \right) \\
 &= -3 \left(\int_0^3 \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr - \int_0^1 \int_0^{2\pi} r^2 \cdot r \, d\theta \, dr \right) \\
 &= -3 \left(\int_0^3 [r^3 \theta]_0^{2\pi} \, dr - \int_0^1 [r^3 \theta]_0^{2\pi} \, dr \right) \\
 &= -6\pi \left(\int_0^3 r^3 \, dr - \int_0^1 r^3 \, dr \right) \\
 &= -6\pi \left(\left[\frac{1}{4} r^4 \right]_0^3 - \left[\frac{1}{4} r^4 \right]_0^1 \right) \\
 &= \boxed{-120\pi}.
 \end{aligned}$$

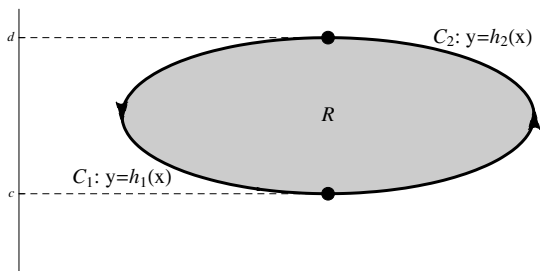
45. Answers will vary.

46. Answers will vary.

47. Answers will vary.

48. Answers will vary.

49. Since R is y -simple, suppose that the boundary of R consists of two smooth curves, $C_1 : x = h_1(y)$ and $C_2 : x = h_2(y)$, where $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$. The situation is:



Then

$$\iint_R \frac{\partial Q}{\partial x} \, dx \, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x} \, dx \right] dy = \int_c^d [Q(x, y)]_{h_1(y)}^{h_2(y)} \, dy = \int_c^d [Q(h_2(y), y) - Q(h_1(y), y)] \, dy.$$

Convert this to two separate integrals by reversing the integration bounds:

$$\int_c^d [Q(h_2(y), y) - Q(h_1(y), y)] \, dy = \int_c^d Q(h_2(y), y) \, dy + \int_d^c Q(h_1(y), y) \, dy.$$

By definition of the line integral, this is equal to

$$\int_{C_2} Q(x, y) \, dy + \int_{C_1} Q(x, y) \, dy = \oint_C Q(x, y) \, dy,$$

as desired.

50. Using Green's Theorem for Area, we know that

$$A = \frac{1}{2} \oint_C (-y \, dx + x \, dy).$$

Here $P(x, y) = -y$ and $Q(x, y) = x$, so applying Green's Theorem gives

$$A = \frac{1}{2} \oint_C (-y dx + x dy) = \frac{1}{2} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \frac{1}{2} \iint_R 2 dx dy = \iint_R 1 dx dy.$$

Similarly, we get

$$\begin{aligned} \oint_C x dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (1 - 0) dx dy = \iint_R 1 dx dy \\ \oint_C (-y) dx &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (0 - (-1)) dx dy = \iint_R 1 dx dy. \end{aligned}$$

So each of these integrals is equal to the area integral and we are done.

51. In order to apply Green's Theorem to convert the line integral to a double integral, P and Q must be continuous on the interior of the curve, and their first derivatives must be as well. Here,

$$P(x, y) = \frac{y}{(x^2 + y^2)^{3/2}} = y(x^2 + y^2)^{-3/2}, \quad Q(x, y) = -\frac{x}{(x^2 + y^2)^{3/2}} = -x(x^2 + y^2)^{-3/2},$$

so that P and Q are continuous everywhere except at the origin. Their derivatives are

$$\begin{aligned} \frac{\partial P}{\partial x} &= -\frac{3}{2}y(x^2 + y^2)^{-5/2} \cdot 2x = -3xy(x^2 + y^2)^{-5/2} \\ \frac{\partial P}{\partial y} &= (x^2 + y^2)^{-3/2} + y \cdot \left(-\frac{3}{2}(x^2 + y^2)^{-5/2} \cdot 2y \right) \\ &= (x^2 + y^2)^{-3/2} - 3y^2(x^2 + y^2)^{-5/2} \\ &= ((x^2 + y^2) - 3y^2)(x^2 + y^2)^{-5/2} = (x^2 - 2y^2)(x^2 + y^2)^{-5/2} \\ \frac{\partial Q}{\partial x} &= -(x^2 + y^2)^{-3/2} - x \cdot \left(-\frac{3}{2}(x^2 + y^2)^{-5/2} \cdot 2x \right) \\ &= -(x^2 + y^2)^{-3/2} + 3x^2(x^2 + y^2)^{-5/2} \\ &= (-(x^2 + y^2) + 3x^2)(x^2 + y^2)^{-5/2} = (2x^2 - y^2)(x^2 + y^2)^{-5/2} \\ \frac{\partial Q}{\partial y} &= -\left(-\frac{3}{2}x(x^2 + y^2)^{-5/2} \cdot 2y \right) = 3xy(x^2 + y^2)^{-5/2}. \end{aligned}$$

All of these are continuous everywhere except at the origin. Therefore Green's Theorem can be applied as long as neither the curve C nor its interior contains the origin.

Challenge Problems

52. Assume that the mass density of the lamina is 1 (otherwise, we would need to replace A by M , the mass, in the formula). With $P(x, y) = 0$ and $Q(x, y) = x^2$ we have, using Green's Theorem,

$$\frac{1}{2A} \oint_C x^2 dy = \frac{1}{2A} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \frac{1}{2A} \iint_R 2x dx dy = \frac{\iint_R x dx dy}{A}.$$

By the Mass of a Lamina Theorem in Section 14.4, this is the x -coordinate of the center of mass, $\bar{x}$. Next, with $P(x, y) = y^2$ and $Q(x, y) = 0$, we again have, using Green's Theorem,

$$-\frac{1}{2A} \oint_C y^2 dx = -\frac{1}{2A} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = -\frac{1}{2A} \iint_R (-2y) dx dy = \frac{\iint_R y dx dy}{A}.$$

Using the same theorem, we see that this is the y -coordinate of the center of mass, $\bar{y}$.

53. With $P(x, y) = -y^3$ and $Q(x, y) = x^3$, we have, using Green's Theorem,

$$\begin{aligned} \frac{1}{3} \oint_C (x^3 dy - y^3 dx) &= \frac{1}{3} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \frac{1}{3} \iint_R (3x^2 - (-3y^2)) dx dy \\ &= \iint_R (x^2 + y^2) dx dy. \end{aligned}$$

Using the formula for moments of inertia in Objective 2 of Section 4.4, we see that this is the moment of inertia I_O about the origin.

54. $\iint_R dx dy$ is the area of R . From Green's Formula for Area, we know that this integral is equal to

$$\iint_R dx dy = \frac{1}{2} \oint_C (x dy - y dx).$$

Now parametrize x and y as given in the exercise: $x = x(s, t)$ and $y = y(s, t)$. Then

$$dx = x_s ds + x_t dt, \quad dy = y_s ds + y_t dt,$$

so we get

$$\begin{aligned} \frac{1}{2} \oint_C (x dy - y dx) &= \frac{1}{2} \oint_{C'} (x(s, t) (y_s ds + y_t dt) - y(s, t) (x_s ds + x_t dt)) \\ &= \frac{1}{2} \oint_{C'} ((x(s, t)y_s - y(s, t)x_s) ds + (x(s, t)y_t - y(s, t)x_t) dt). \end{aligned} \quad (1)$$

Let

$$P(s, t) = x(s, t)y_s - y(s, t)x_s, \quad Q(s, t) = x(s, t)y_t - y(s, t)x_t.$$

Then

$$\begin{aligned} \frac{\partial P}{\partial t} &= x_t y_s + x(s, t)y_{st} - y_t x_s - y(s, t)x_{st} \\ \frac{\partial Q}{\partial s} &= x_s y_t + x(s, t)y_{ts} - y_s x_t - y(s, t)x_{ts}, \end{aligned}$$

so that (since the mixed partial derivatives are equal)

$$\frac{\partial Q}{\partial s} - \frac{\partial P}{\partial t} = x_s y_t - y_s x_t - x_t y_s + y_t x_s = 2(x_s y_t - x_t y_s).$$

Then applying Green's Theorem to (1), we get

$$\begin{aligned} \frac{1}{2} \oint_{C'} ((x(s, t)y_s - y(s, t)x_s) ds + (x(s, t)y_t - y(s, t)x_t) dt) &= \frac{1}{2} \iint_{R'} \left(\frac{\partial Q}{\partial s} - \frac{\partial P}{\partial t} \right) ds dt \\ &= \iint_{R'} (x_s y_t - x_t y_s) ds dt. \end{aligned}$$

Note that care should be taken that both regions are traversed in a counterclockwise direction.

55. By the Fundamental Theorem of Calculus, we have

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left(- \int \phi(x, y) dy \right) = -\phi(x, y), \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left(\int \phi(x, y) dx \right) = \phi(x, y),$$

so that using Green's Theorem

$$\iint_R \phi(x, y) dx dy = \frac{1}{2} \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \frac{1}{2} \oint_C (P dx + Q dy).$$

Now, using the Chain Rule,

$$dx = x_s ds + x_t dt, \quad dy = y_s ds + y_t dt,$$

so in the st -plane, the above integral is equal to

$$\frac{1}{2} \oint_{C'} (P(x_s ds + x_t dt) + Q(y_s ds + y_t dt)) = \frac{1}{2} \oint_{C'} ((Px_s + Qy_s) ds + (Px_t + Qy_t) dt).$$

Apply Green's Theorem again, in the st -plane, to get

$$\frac{1}{2} \iint_{R'} \left(\frac{\partial}{\partial s}(Px_t + Qy_t) - \frac{\partial}{\partial t}(Px_s + Qy_s) \right) ds dt. \quad (1)$$

Now expand each of these derivatives using the product rule together with the Chain Rule:

$$\begin{aligned} \frac{\partial}{\partial s}(Px_t + Qy_t) &= \frac{\partial P}{\partial s}x_t + Px_{ts} + \frac{\partial Q}{\partial s}y_t + Qy_{ts} \\ &= \left(\frac{\partial P}{\partial x}x_s + \frac{\partial P}{\partial y}y_s \right)x_t + Px_{ts} + \left(\frac{\partial Q}{\partial x}x_s + \frac{\partial Q}{\partial y}y_s \right)y_t + Qy_{ts} \\ &= \frac{\partial P}{\partial x}x_sx_t + \frac{\partial P}{\partial y}x_t y_s + Px_{ts} + \frac{\partial Q}{\partial x}x_s y_t + \frac{\partial Q}{\partial y}y_s y_t + Qy_{ts} \\ \frac{\partial}{\partial t}(Px_s + Qy_s) &= \frac{\partial P}{\partial t}x_s + Px_{st} + \frac{\partial Q}{\partial t}y_s + Qy_{st} \\ &= \left(\frac{\partial P}{\partial x}x_t + \frac{\partial P}{\partial y}y_t \right)x_s + Px_{st} + \left(\frac{\partial Q}{\partial x}x_t + \frac{\partial Q}{\partial y}y_t \right)y_s + Qy_{st} \\ &= \frac{\partial P}{\partial x}x_sx_t + \frac{\partial P}{\partial y}x_s y_t + Px_{st} + \frac{\partial Q}{\partial x}x_t y_s + \frac{\partial Q}{\partial y}y_s y_t + Qy_{st}. \end{aligned}$$

Subtract the second equation from the first. Note that mixed partials are equal, so we get

$$\begin{aligned} \frac{\partial}{\partial s}(Px_t + Qy_t) - \frac{\partial}{\partial t}(Px_s + Qy_s) &= \frac{\partial P}{\partial y}x_t y_s - \frac{\partial P}{\partial y}x_s y_t + \frac{\partial Q}{\partial x}x_s y_t - \frac{\partial Q}{\partial x}x_t y_s \\ &= \frac{\partial Q}{\partial x}(x_s y_t - x_t y_s) - \frac{\partial P}{\partial y}(x_s y_t - x_t y_s) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)(x_s y_t - x_t y_s) \\ &= 2\phi(x(s, t), y(s, t))(x_s y_t - x_t y_s). \end{aligned}$$

Substitute into (1) to get

$$\begin{aligned} \frac{1}{2} \iint_{R'} \left(\frac{\partial}{\partial s}(Px_t + Qy_t) - \frac{\partial}{\partial t}(Px_s + Qy_s) \right) ds dt &= \frac{1}{2} \iint_{R'} 2\phi(x(s, t), y(s, t))(x_s y_t - x_t y_s) ds dt \\ &= \iint_{R'} \phi(x(s, t), y(s, t))(x_s y_t - x_t y_s) ds dt, \end{aligned}$$

as required.

15.6: Parametric Surfaces

Concepts and Vocabulary

1. (b), *tangent plane*, is correct. See the discussion following the definition of a smooth parametrization.
2. (c), *parametric surface*, is correct. See the definition at the start of the section.

3. (c), *sphere*, is correct. These are the parametric equations in spherical coordinates for a sphere of radius 5. See Example 4.
4. Since

$$\mathbf{r}(u, v) = (1 - u + 2v)\mathbf{i} + 4u\mathbf{j} + (2u + 3v - 7)\mathbf{k} = \langle 1, 0, -7 \rangle + \langle -1, 4, 2 \rangle u + \langle 2, 0, 3 \rangle v,$$

this is a plane through the point $(1, 0, -7)$ containing the two direction vectors $\langle -1, 4, 2 \rangle$ and $\langle 2, 0, 3 \rangle$.
 (a), *plane*, is correct.

Skill Building

5. (a) When $u = 0$, we get

$$\mathbf{r}(0, v) = (0 - 5v)\mathbf{i} + 2 \cdot 0\mathbf{j} + (-0 + v + 1)\mathbf{k} = \boxed{-5v\mathbf{i} + (v + 1)\mathbf{k}}.$$

This is a line through $(0, 0, 1)$ with direction vector $\langle -5, 0, 1 \rangle$. When $v = 0$, we get

$$\mathbf{r}(u, 0) = (u - 5 \cdot 0)\mathbf{i} + 2u\mathbf{j} + (-u + 0 + 1)\mathbf{k} = \boxed{u\mathbf{i} + 2u\mathbf{j} + (1 - u)\mathbf{k}}.$$

This is a line through $(0, 0, 1)$ with direction vector $\langle 1, 2, -1 \rangle$.

- (b) We start with the given system of equations:

$$\begin{aligned} x &= u - 5v \\ y &= 2u \\ z &= -u + v + 1. \end{aligned}$$

Add five times the third equation to the first to eliminate v , giving the system

$$\begin{aligned} x + 5z &= -4u + 5 \\ y &= 2u. \end{aligned}$$

Now add twice the second equation to the first, giving $\boxed{x + 2y + 5z = 5}$.

6. (a) When $u = 0$, we get

$$\mathbf{r}(0, v) = 0\mathbf{i} + v\mathbf{j} + (9 - 0^2 - v^2)\mathbf{k} = \boxed{v\mathbf{j} + (9 - v^2)\mathbf{k}}.$$

This is the parabola $z = 9 - y^2$ in the yz -plane. When $v = 0$, we get

$$\mathbf{r}(u, 0) = u\mathbf{i} + 0\mathbf{j} + (9 - u^2 - 0^2)\mathbf{k} = \boxed{u\mathbf{i} + (9 - u^2)\mathbf{k}}.$$

This is the parabola $z = 9 - x^2$ in the xz -plane.

- (b) We start with the given system of equations:

$$\begin{aligned} x &= u \\ y &= v \\ z &= 9 - u^2 - v^2. \end{aligned}$$

Square each of the first two equations and add them to the third, giving $z + x^2 + y^2 = 9$, or $\boxed{z = 9 - x^2 - y^2}$. This is a paraboloid.

7. (a) When $u = 0$, we get

$$\mathbf{r}(0, v) = 0 \cos v \mathbf{i} + 0 \sin v \mathbf{j} + 0 \mathbf{k} = \boxed{\mathbf{0}}.$$

This is a single point, the origin. When $v = 0$, we get

$$\mathbf{r}(u, 0) = u \cos 0 \mathbf{i} + u \sin 0 \mathbf{j} + u \mathbf{k} = \boxed{u \mathbf{i} + u \mathbf{k}}.$$

This is the line through the origin with direction vector $\langle 1, 0, 1 \rangle$.

- (b) We start with the given system of equations:

$$x = u \cos v$$

$$y = u \sin v$$

$$z = u.$$

Square all three equations and subtract the first two from the third, giving

$$z^2 - x^2 - y^2 = u^2 - u^2 \cos^2 v - u^2 \sin^2 v = u^2 - u^2 = 0.$$

This is the portion of the cone $\boxed{z^2 = x^2 + y^2}$ lying between the planes $z = 0$ and $z = 2$ (since $0 \leq u \leq 2$) and having positive y -coordinates (since $0 \leq v \leq \pi$).

8. (a) When $u = 0$, we get

$$\mathbf{r}(0, v) = \cos 0 \sin v \mathbf{i} + \sin 0 \sin v \mathbf{j} + \cos v \mathbf{k} = \boxed{\sin v \mathbf{i} + \cos v \mathbf{k}}.$$

When $v = 0$, we get

$$\mathbf{r}(u, 0) = \cos u \sin 0 \mathbf{i} + \sin u \sin 0 \mathbf{j} + \cos 0 \mathbf{k} = \boxed{\mathbf{k}}.$$

- (b) We start with the given system of equations:

$$x = \cos u \sin v$$

$$y = \sin u \sin v$$

$$z = \cos v.$$

Square all three equations and add them together, giving

$$x^2 + y^2 + z^2 = \cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v = \sin^2 v + \cos^2 v = 1.$$

This is the hemisphere of radius 1 centered at the origin lying above the xy -plane (since $0 \leq v \leq \frac{\pi}{2}$), having equation $\boxed{x^2 + y^2 + z^2 = 1}$.

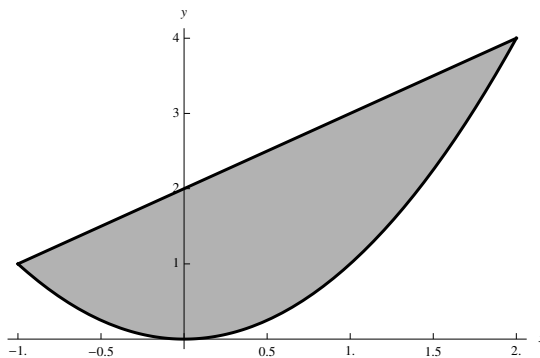
9. Let $x = u$ and $y = v$; then $z = 4 - u - 2v$. Since we want the portion in the first octant, we restrict $x = u \geq 0$ and $y = v \geq 0$. We also want $z \geq 0$, so that $4 - u - 2v \geq 0$ and therefore $0 \leq u \leq 4$ and $0 \leq v \leq 2 - \frac{1}{2}u$. So the parametrization is

$$\boxed{\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (4 - u - 2v) \mathbf{k}, \quad 0 \leq u \leq 4, \quad 0 \leq v \leq 2 - \frac{1}{2}u.}$$

10. Let $x = u$ and $y = v$; then $z = e^{-u^2+v^2}$. Since we want the portion of this surface inside the cylinder $x^2 + y^2 = 4$, we want $x^2 + y^2 = u^2 + v^2 < 4$. So the parametrization is

$$\boxed{\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + e^{-u^2+v^2} \mathbf{k}, \quad u^2 + v^2 < 4.}$$

11. Let $x = u$ and $y = v$; then $z = \sin(u^2v)$. The curves $y = x + 2$ and $y = x^2$ are



Since we want the portion of this surface lying over the shaded region, we want $-1 \leq x \leq 2$, so that $-1 \leq u \leq 2$, and $u^2 \leq y = v \leq u + 2$. So the parametrization is

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + \sin(u^2v) \mathbf{k}, \quad -1 \leq u \leq 2, \quad u^2 \leq v \leq u + 2.$$

12. Let $x = u$ and $z = v$; then $y = 5 - e^{uv}$. Since we want the portion of this surface inside the cylinder $x^2 + z^2 = 1$, we want $0 \leq x^2 + z^2 = u^2 + v^2 < 1$. So the parametrization is

$$\mathbf{r}(u, v) = u \mathbf{i} + (5 - e^{uv}) \mathbf{j} + v \mathbf{k}, \quad u^2 + v^2 < 1.$$

13. (a) At $(7, 5, 4)$, the values of the parameters are determined by solving the equations

$$\begin{aligned} 3u + 2v &= 7 \\ 5u^3 &= 5 \\ v^2 &= 4. \end{aligned}$$

The second equation gives $u = 1$; substituting this into the first equation gives $v = 2$, which satisfies the third equation, so this point is indeed on the surface. Now,

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial}{\partial u}(3u + 2v) \mathbf{i} + \frac{\partial}{\partial u}(5u^3) \mathbf{j} + \frac{\partial}{\partial u}(v^2) \mathbf{k} = 3 \mathbf{i} + 15u^2 \mathbf{j} \\ \mathbf{r}_v &= \frac{\partial}{\partial v}(3u + 2v) \mathbf{i} + \frac{\partial}{\partial v}(5u^3) \mathbf{j} + \frac{\partial}{\partial v}(v^2) \mathbf{k} = 2 \mathbf{i} + 2v \mathbf{k}. \end{aligned}$$

Then when $u = 1$ and $v = 2$ we have

$$\mathbf{r}_u(1, 2) = 3 \mathbf{i} + 15 \mathbf{j}, \quad \mathbf{r}_v(1, 2) = 2 \mathbf{i} + 4 \mathbf{k}.$$

Therefore a normal to the tangent plane at $(7, 5, 4)$ is given by

$$\langle 3, 15, 0 \rangle \times \langle 2, 0, 4 \rangle = \langle 60, -12, -30 \rangle,$$

so that the equation of the tangent plane is $60(x - 7) - 12(y - 5) - 30(z - 4) = 0$. Simplifying by dividing through by 6 and expanding the terms gives $10x - 2y - 5z = 40$.

- (b) The normal line to the tangent plane at the given point is the line whose direction vector is $\langle 60, -12, -30 \rangle$ through the point $(7, 5, 4)$. Divide the direction vector through by 6 to simplify, giving $\langle 10, -2, -5 \rangle$. A parametrization of the normal line is then

$$\mathbf{r}(t) = (7 + 10t) \mathbf{i} + (5 - 2t) \mathbf{j} + (4 - 5t) \mathbf{k}.$$

14. (a) At $(11, 1, -5)$, the values of the parameters are determined by solving the equations

$$\begin{aligned} 3u - v &= 11 \\ 2 - u - v &= 1 \\ 1 + 3v &= -5. \end{aligned}$$

The third equation gives $v = -2$; substituting that into the first equation gives $u = 3$. Those two values satisfy the remaining equation, so this point is indeed on the surface. Now,

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial}{\partial u}(3u - v)\mathbf{i} + \frac{\partial}{\partial u}(2 - u - v)\mathbf{j} + \frac{\partial}{\partial u}(1 + 3v)\mathbf{k} = 3\mathbf{i} - \mathbf{j} \\ \mathbf{r}_v &= \frac{\partial}{\partial v}(3u - v)\mathbf{i} + \frac{\partial}{\partial v}(2 - u - v)\mathbf{j} + \frac{\partial}{\partial v}(1 + 3v)\mathbf{k} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}. \end{aligned}$$

Then when $u = 3$ and $v = -2$ we have

$$\mathbf{r}_u(3, -2) = 3\mathbf{i} - \mathbf{j}, \quad \mathbf{r}_v(3, -2) = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Therefore a normal to the tangent plane at $(11, 1, -5)$ is given by

$$\langle 3, -1, 0 \rangle \times \langle -1, -1, 3 \rangle = \langle -3, -9, -4 \rangle.$$

We can instead negate this vector, giving a normal of $\langle 3, 9, 4 \rangle$. Then the equation of the tangent plane is $3(x - 11) + 9(y - 1) + 4(z + 5) = 0$. Simplifying gives $\boxed{3x + 9y + 4z = 22}$. (Note that the given parametric surface is a plane, and that this equation is the equation of the plane in rectangular coordinates.)

- (b) The normal line to the tangent plane at the given point is the line with direction vector $\langle 3, 9, 4 \rangle$ through the point $(11, 1, -5)$; a parametrization of this line is

$$\boxed{\mathbf{r}(t) = (11 + 3t)\mathbf{i} + (1 + 9t)\mathbf{j} + (-5 + 4t)\mathbf{k}}.$$

15. (a) At $\left(5, \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$, the values of the parameters are determined by solving the equations

$$\begin{aligned} u &= 5 \\ u \cos v &= \frac{5\sqrt{2}}{2} \\ u \sin v &= \frac{5\sqrt{2}}{2}. \end{aligned}$$

The first equation gives $u = 5$; then the other two equations give $v = \frac{\pi}{4}$. Now,

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial}{\partial u}(u)\mathbf{i} + \frac{\partial}{\partial u}(u \cos v)\mathbf{j} + \frac{\partial}{\partial u}(u \sin v)\mathbf{k} = \mathbf{i} + \cos v \mathbf{j} + \sin v \mathbf{k} \\ \mathbf{r}_v &= \frac{\partial}{\partial v}(u)\mathbf{i} + \frac{\partial}{\partial v}(u \cos v)\mathbf{j} + \frac{\partial}{\partial v}(u \sin v)\mathbf{k} = -u \sin v \mathbf{j} + u \cos v \mathbf{k}. \end{aligned}$$

Then when $u = 5$ and $v = \frac{\pi}{4}$ we have

$$\mathbf{r}_u\left(5, \frac{\pi}{4}\right) = \mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}, \quad \mathbf{r}_v\left(5, \frac{\pi}{4}\right) = -\frac{5\sqrt{2}}{2}\mathbf{j} + \frac{5\sqrt{2}}{2}\mathbf{k}$$

Therefore a normal to the tangent plane at $\left(5, \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$ is given by

$$\left\langle 1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \times \left\langle 0, -\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \right\rangle = \left\langle 5, -\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2} \right\rangle.$$

Therefore the equation of the tangent plane is

$$5(x-5) - \frac{5\sqrt{2}}{2} \left(y - \frac{5\sqrt{2}}{2} \right) - \frac{5\sqrt{2}}{2} \left(z - \frac{5\sqrt{2}}{2} \right) = 0, \text{ or}$$

$$5x - \frac{5\sqrt{2}}{2}y - \frac{5\sqrt{2}}{2}z = 25 - \frac{25}{2} - \frac{25}{2} = 0.$$

Multiply through by 2 to clear fractions and divide through by $5\sqrt{2}$ to simplify, giving

$$\boxed{\sqrt{2}x - y - z = 0}.$$

- (b) We determine a normal vector to the tangent plane at the given point from the equation above; it is $\langle \sqrt{2}, -1, -1 \rangle$. The line with this direction vector through the point $\left(5, \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2} \right)$ may be parametrized as

$$\boxed{\mathbf{r}(t) = (5 + \sqrt{2}t)\mathbf{i} + \left(\frac{5\sqrt{2}}{2} - t \right)\mathbf{j} + \left(\frac{5\sqrt{2}}{2} - t \right)\mathbf{k}}.$$

16. (a) At $\left(1, 0, \frac{\sqrt{3}}{2} \right)$, the values of the parameters are determined by solving the equations

$$\begin{aligned} 3 \cos u \sin v + 1 &= 1 \\ 2 \sin u \sin v - 1 &= 0 \\ \cos v &= \frac{\sqrt{3}}{2}. \end{aligned}$$

From $2 \sin u \sin v = 1$, we conclude that $\sin v \neq 0$. The first equation simplifies to $3 \cos u \sin v = 0$, so that since $\sin v \neq 0$, we must have $\cos u = 0$, so that $\sin u = \pm 1$. If $\sin u = 1$, then from the second equation $\sin v = \frac{1}{2}$; together with $\cos v = \frac{\sqrt{3}}{2}$ we get $v = \frac{\pi}{6}$ and $u = \frac{\pi}{2}$. If $\sin u = -1$, then the second equation gives $\sin v = -\frac{1}{2}$, so that $v = -\frac{\pi}{6}$ and $u = -\frac{\pi}{2}$. So the point actually has two possible values for u and v in the range $-\pi \leq u, v \leq \pi$; we use the first one, so that $u = \frac{\pi}{2}$ and $v = \frac{\pi}{6}$. Now,

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial}{\partial u}(3 \cos u \sin v + 1)\mathbf{i} + \frac{\partial}{\partial u}(2 \sin u \sin v - 1)\mathbf{j} + \frac{\partial}{\partial u}(\cos v)\mathbf{k} = -3 \sin u \sin v \mathbf{i} + 2 \cos u \sin v \mathbf{j} \\ \mathbf{r}_v &= \frac{\partial}{\partial v}(3 \cos u \sin v + 1)\mathbf{i} + \frac{\partial}{\partial v}(2 \sin u \sin v - 1)\mathbf{j} + \frac{\partial}{\partial v}(\cos v)\mathbf{k} \\ &= 3 \cos u \cos v \mathbf{i} + 2 \sin u \cos v \mathbf{j} - \sin v \mathbf{k}. \end{aligned}$$

Then when $u = \frac{\pi}{2}$ and $v = \frac{\pi}{6}$ we have

$$\mathbf{r}_u \left(\frac{\pi}{2}, \frac{\pi}{6} \right) = -\frac{3}{2} \mathbf{i} \quad \mathbf{r}_v \left(\frac{\pi}{2}, \frac{\pi}{6} \right) = \sqrt{3} \mathbf{j} - \frac{1}{2} \mathbf{k}.$$

Therefore a normal to the tangent plane at $\left(1, 0, \frac{\sqrt{3}}{2} \right)$ is given by

$$\left\langle -\frac{3}{2}, 0, 0 \right\rangle \times \left\langle 0, \sqrt{3}, -\frac{1}{2} \right\rangle = \left\langle 0, -\frac{3}{4}, -\frac{3\sqrt{3}}{2} \right\rangle,$$

and so the equation of the tangent plane is

$$\begin{aligned} -\frac{3}{4}(y-0) - \frac{3\sqrt{3}}{2} \left(z - \frac{\sqrt{3}}{2} \right) &= 0, \text{ or} \\ 3y + 6\sqrt{3}z &= 9. \end{aligned}$$

Divide through by 3 to simplify, giving

$$\boxed{y + 2\sqrt{3}z = 3}.$$

- (b) The normal line to the tangent plane at the given point is the line whose direction vector is $\left\langle 0, -\frac{3}{4}, -\frac{3\sqrt{3}}{2} \right\rangle$ through the point $\left(1, 0, \frac{\sqrt{3}}{2}\right)$; a parametrization of this line is

$$\boxed{\mathbf{r}(t) = \mathbf{i} - \frac{3}{4}t\mathbf{j} + \left(\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}t\right)\mathbf{k}}.$$

17. Start by finding the partial derivatives of $\mathbf{r}$:

$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + 3u^2 \mathbf{j} + \sin v \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{k}.$$

Then

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 3u^2 & \sin v \\ -u \sin v & 0 & u \cos v \end{vmatrix} \\ &= 3u^3 \cos v \mathbf{i} - (u \cos^2 v + u \sin^2 v) \mathbf{j} + 3u^3 \sin v \mathbf{k} \\ &= 3u^3 \cos v \mathbf{i} - u \mathbf{j} + 3u^3 \sin v \mathbf{k}. \end{aligned}$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{(3u^3 \cos v)^2 + u^2 + (3u^3 \sin v)^2} = \sqrt{9u^6 + u^2} = u\sqrt{9u^4 + 1}.$$

Then the surface area is

$$\int_0^1 \int_{-\pi}^{\pi} u\sqrt{9u^4 + 1} dv du = \int_0^1 \left[uv\sqrt{9u^4 + 1} \right]_{-\pi}^{\pi} du = 2\pi \int_0^1 u\sqrt{9u^4 + 1} du.$$

Now make the substitution $t = 3u^2$, so that $dt = 6u du$, or $du = \frac{1}{6u} dt = \frac{\sqrt{3}}{6\sqrt{t}} dt$. Then $u = 0$ corresponds to $t = 0$ and $u = 1$ to $t = 3$, and we get

$$2\pi \int_0^1 u\sqrt{9u^4 + 1} du = 2\pi \int_0^3 \frac{\sqrt{t}}{\sqrt{3}} \cdot \sqrt{t^2 + 1} \cdot \frac{\sqrt{3}}{6\sqrt{t}} dt = \frac{\pi}{3} \int_0^3 \sqrt{t^2 + 1} dt.$$

From Formula 49 in the tables in the Endnotes, this integral is

$$\begin{aligned} \frac{\pi}{3} \left[\frac{t}{2} \sqrt{t^2 + 1} + \frac{1}{2} \ln |t + \sqrt{t^2 + 1}| \right]_0^3 &= \frac{\pi}{3} \left(\frac{3}{2} \sqrt{10} + \frac{1}{2} \ln(3 + \sqrt{10}) \right) \\ &= \boxed{\pi \left(\frac{\sqrt{10}}{2} + \frac{1}{6} \ln(3 + \sqrt{10}) \right)}. \end{aligned}$$

18. Start by finding the partial derivatives of $\mathbf{r}$:

$$\mathbf{r}_u(u, v) = 6u \mathbf{i} + 2u \mathbf{j} - 2u \mathbf{k}$$

$$\mathbf{r}_v(u, v) = \mathbf{i} + \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6u & 2u & -2u \\ 1 & 0 & 1 \end{vmatrix} = 2u \mathbf{i} - 8u \mathbf{j} - 2u \mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4u^2 + 64u^2 + 4u^2} = u\sqrt{72} = 6u\sqrt{2}.$$

Then the surface area is

$$\int_0^2 \int_{-1}^1 6u\sqrt{2} \, du \, dv = 6\sqrt{2} \int_0^2 [uv]_{-1}^1 \, du = 6\sqrt{2} \int_0^2 2u \, du = 6\sqrt{2} [u^2]_0^2 = \boxed{24\sqrt{2}}.$$

19. Solve the equation of the plane for z , giving $z = -\frac{1}{2}x + \frac{1}{4}y + \frac{3}{4}$. Parametrize the plane by

$$x = u, \quad y = v, \quad z = -\frac{1}{2}u + \frac{1}{4}v + \frac{3}{4},$$

so that $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (-\frac{1}{2}u + \frac{1}{4}v + \frac{3}{4})\mathbf{k}$. The bounds on u and v are given by the fact that $(x-2)^2 + y^2$ must lie inside the given cylinder, so we get $(u-2)^2 + v^2 \leq 4$. Let $0 \leq u \leq 4$; then $-\sqrt{4-(u-2)^2} \leq v \leq \sqrt{4-(u-2)^2}$. Next find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \mathbf{i} - \frac{1}{2}\mathbf{k} \\ \mathbf{r}_v(u, v) &= \mathbf{j} + \frac{1}{4}\mathbf{k}.\end{aligned}$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{4} \end{vmatrix} = \frac{1}{2}\mathbf{i} - \frac{1}{4}\mathbf{j} + \mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + 1} = \frac{\sqrt{21}}{4}.$$

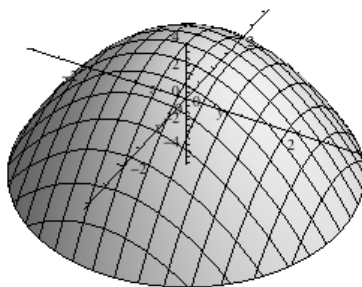
Then the surface area is

$$\begin{aligned}\int_0^4 \int_{-\sqrt{4-(u-2)^2}}^{\sqrt{4-(u-2)^2}} \frac{\sqrt{21}}{4} \, dv \, du &= \frac{\sqrt{21}}{4} \int_0^4 [v]_{-\sqrt{4-(u-2)^2}}^{\sqrt{4-(u-2)^2}} \, du \\ &= \frac{\sqrt{21}}{2} \int_0^4 \sqrt{4-(u-2)^2} \, du \\ &= \frac{\sqrt{21}}{2} \int_0^4 \sqrt{4u-u^2} \, du.\end{aligned}$$

By Formula 71 in the tables in the Endnotes, this is

$$\frac{\sqrt{21}}{2} \left[\frac{u-2}{2} \sqrt{4u-u^2} + 2 \cos^{-1} \left(\frac{2-x}{2} \right) \right]_0^4 = \frac{\sqrt{21}}{2} \cdot (2\pi) = \boxed{\pi\sqrt{21}}.$$

20. The paraboloid is



The paraboloid intersects $z = 0$ (the xy -plane) when $0 = 4 - x^2 - y^2$, so along the circle $x^2 + y^2 = 4$. Parametrize the paraboloid by

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2,$$

so that $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}$. The bounds on u and v are given by the fact that $x^2 + y^2$ must lie inside the circle of intersection, so we get $u^2 + v^2 \leq 4$. Let $-2 \leq u \leq 2$; then $-\sqrt{4 - u^2} \leq v \leq \sqrt{4 - u^2}$. Next find the partial derivatives of $\mathbf{r}$:

$$\mathbf{r}_u(u, v) = \mathbf{i} - 2u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = \mathbf{j} - 2v\mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4u^2 + 4v^2 + 1} = \sqrt{4(u^2 + v^2) + 1}$$

Then the surface area is

$$\iint_R \sqrt{4(u^2 + v^2) + 1} \, dA,$$

where R is a circle of radius 2 centered at the origin. Convert to polar coordinates, giving

$$\iint_R \sqrt{4(u^2 + v^2) + 1} \, dA = \int_0^2 \int_0^{2\pi} \sqrt{4r^2 + 1} \cdot r \, d\theta \, dr = 2\pi \int_0^2 r \sqrt{4r^2 + 1} \, dr.$$

Now make the substitution $t = 4r^2 + 1$, so that $dt = 8r \, dr$; then $r = 0$ corresponds to $t = 1$ and $r = 2$ to $t = 17$, and we get

$$2\pi \int_0^2 r \sqrt{4r^2 + 1} \, dr = \frac{\pi}{4} \int_1^{17} \sqrt{t} \, dt = \frac{\pi}{4} \left[\frac{2}{3} t^{3/2} \right]_1^{17} = \boxed{\frac{\pi}{6} (17\sqrt{17} - 1)}.$$

21. The given sphere intersects the plane $z = 2$ then $x^2 + y^2 + 4 = 16$, so when $x^2 + y^2 = 12$. Parametrize the sphere by

$$x = u, \quad y = v, \quad z = \sqrt{16 - u^2 - v^2},$$

so that $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{16 - u^2 - v^2}\mathbf{k}$. The bounds on u and v are given by the fact that $x^2 + y^2$ must lie inside the circle of intersection, so we get $u^2 + v^2 \leq 12$. Let $-\sqrt{12} \leq u \leq \sqrt{12}$; then $-\sqrt{12 - u^2} \leq v \leq \sqrt{12 - u^2}$. Next find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \mathbf{i} - \frac{u}{\sqrt{16 - u^2 - v^2}}\mathbf{k} \\ \mathbf{r}_v(u, v) &= \mathbf{j} - \frac{v}{\sqrt{16 - u^2 - v^2}}\mathbf{k}.\end{aligned}$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{u}{\sqrt{16 - u^2 - v^2}} \\ 0 & 1 & -\frac{v}{\sqrt{16 - u^2 - v^2}} \end{vmatrix} = \frac{u}{\sqrt{16 - u^2 - v^2}}\mathbf{i} + \frac{v}{\sqrt{16 - u^2 - v^2}}\mathbf{j} + \mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\left(\frac{u}{\sqrt{16 - u^2 - v^2}}\right)^2 + \left(\frac{v}{\sqrt{16 - u^2 - v^2}}\right)^2 + 1} = \sqrt{\frac{16}{16 - u^2 - v^2}} = \frac{4}{\sqrt{16 - u^2 - v^2}}.$$

Then the surface area is

$$\iint_R \frac{4}{\sqrt{16 - u^2 - v^2}} dA,$$

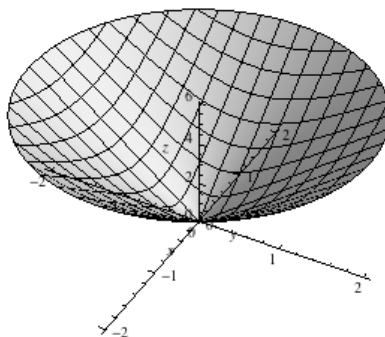
where R is a circle of radius $\sqrt{12}$ centered at the origin. Convert to polar coordinates, giving

$$\iint_R \frac{4}{\sqrt{16 - u^2 - v^2}} dA = \int_0^{\sqrt{12}} \int_0^{2\pi} \frac{4r}{\sqrt{16 - r^2}} d\theta dr = 2\pi \int_0^{\sqrt{12}} \frac{4r}{\sqrt{16 - r^2}} dr.$$

Now make the substitution $t = 16 - r^2$, so that $dt = -2r dr$; then $r = 0$ corresponds to $t = 16$ and $r = \sqrt{12}$ to $t = 4$, and we get

$$2\pi \int_0^{\sqrt{12}} \frac{4r}{\sqrt{16 - r^2}} dr = -4\pi \int_{16}^4 \frac{1}{\sqrt{t}} dt = -4\pi \left[2t^{1/2} \right]_{16}^4 = \boxed{16\pi}.$$

22. The cone is



The cone intersects $z = 3$ when $3 = 3\sqrt{x^2 + y^2}$, so when $x^2 + y^2 = 1$, and intersects $z = 6$ when $6 = 3\sqrt{x^2 + y^2}$, so when $x^2 + y^2 = 4$. Parametrize the cone by

$$x = u, \quad y = v, \quad z = 3\sqrt{u^2 + v^2},$$

so that $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 3\sqrt{u^2 + v^2}\mathbf{k}$. The bounds on u and v are given by the fact that $x^2 + y^2$ must lie between the two circles of intersection, so we get $1 \leq u^2 + v^2 \leq 4$. Next find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \mathbf{i} + \frac{3u}{\sqrt{u^2 + v^2}}\mathbf{k} \\ \mathbf{r}_v(u, v) &= \mathbf{j} + \frac{3v}{\sqrt{u^2 + v^2}}\mathbf{k}.\end{aligned}$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{3u}{\sqrt{u^2 + v^2}} \\ 0 & 1 & \frac{3v}{\sqrt{u^2 + v^2}} \end{vmatrix} = -\frac{3u}{\sqrt{u^2 + v^2}}\mathbf{i} - \frac{3v}{\sqrt{u^2 + v^2}}\mathbf{j} + \mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\left(\frac{3u}{\sqrt{u^2 + v^2}}\right)^2 + \left(\frac{3v}{\sqrt{u^2 + v^2}}\right)^2 + 1} = \sqrt{10}.$$

Then the surface area is

$$\iint_R \sqrt{10} dA,$$

where R is the area between the circle of radius 1 centered at the origin and the circle of radius 2 centered at the origin. Convert to polar coordinates, giving

$$\iint_R \sqrt{10} dA = \int_1^2 \int_0^{2\pi} \sqrt{10} \cdot r d\theta dr = 2\pi\sqrt{10} \int_1^2 r dr = 2\pi\sqrt{10} \left[\frac{1}{2}r^2\right]_1^2 = \boxed{3\pi\sqrt{10}}.$$

23. This surface extends from $u = 0$ to $u = 4$, so from $z = 0$ to $z = -4$. The coordinate curve for $u = a$ is parametrized as $a \cos v \mathbf{i} + a \sin v \mathbf{j} - a \mathbf{k}$, so each cross-section parallel to the xy -plane is a circle. This fits surfaces **(D)** and **(E)**. The rectangular equation for this curve is determined by eliminating u and v from the equations $x = u \cos v$, $y = u \sin v$, $z = -u$. Squaring the first two equations gives $x^2 + y^2 = u^2$, so the rectangular equation is $z = -\sqrt{x^2 + y^2}$. This is a downward pointing cone, so **(E)** is correct.
24. Since the x -coordinate is parametrized by u^3 , it extends from $x = 0$ to $x = 64$. At $u = a$, the coordinate curve is parametrized as $a^3 \mathbf{i} + a \sin v \mathbf{j} + a \cos v \mathbf{k}$, which is a circle of radius a , so each cross-section parallel to the yz -plane is a circle. This fits surface **(C)**.
25. This surface extends from $u = z = 0$ to $u = z = 4$. The coordinate curve for $u = a$ is parametrized as $a \cos v \mathbf{i} + a \sin v \mathbf{j} + a \mathbf{k}$, so each cross-section parallel to the xy -plane is a circle. This fits surface **(B)**.
26. For each coordinate $x = u$, $0 \leq u \leq 4$, the coordinate curve is a circle of radius 2, since the y and z -coordinates are $(4 \sin v, 4 \cos v)$. So this surface is a cylinder, which is **(A)**.
27. Find the rectangular equation for this surface by eliminating u and v . Square $x = u \cos v$ and $y = u \sin v$ to get $x^2 + y^2 = u^2$; then

$$z = -u^3 = -(x^2 + y^2)^{3/2}.$$

This matches only surface **(D)**.

28. The coordinate curve at any value of u is given by $\mathbf{r}(u, v)$; since the y -coordinate is the cube of the x -coordinate, the curve has equation $y = x^3$ at $z = u$. This matches **(F)**.

Applications and Extensions

29. Using cylindrical coordinates, the cylinder $x^2 + y^2 = 16$ is parametrized as $r = \sqrt{x^2 + y^2} = 4$ and $0 \leq \theta \leq 2\pi$; the restriction that we have $0 \leq z \leq 3$ provides the third constraint. So the parametrization is

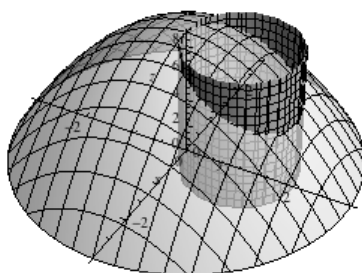
$$x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + z \mathbf{k} = \boxed{4 \cos \theta \mathbf{i} + 4 \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3}.$$

30. Using spherical coordinates, the sphere is parametrized as

$$\mathbf{r}(\theta, \phi) = 5 \cos \theta \sin \phi \mathbf{i} + 5 \sin \theta \sin \phi \mathbf{j} + 5 \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

See Example 4.

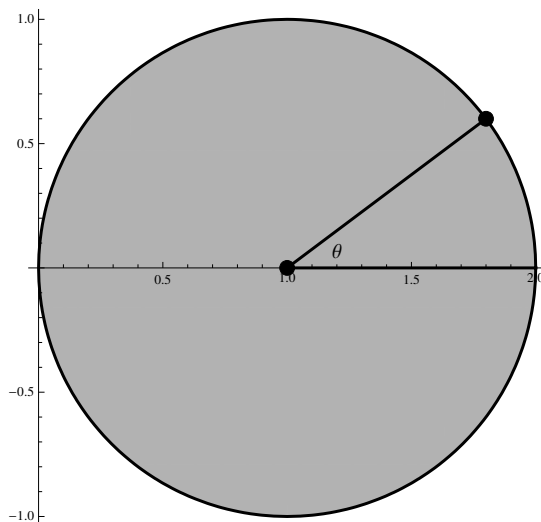
31. The two surfaces are



- (a) Using rectangular coordinates, since the surface lies above a disk in the xy -plane of radius 1 centered at $(1,0)$, we let $x = u$ for $0 \leq u \leq 2$ and $y = v$ for $-\sqrt{1-u^2} \leq v \leq \sqrt{1-u^2}$. These are a parametrization of the disk. Then the surface lying above that circle is given by $z = 9 - x^2 - y^2 = 9 - u^2 - v^2$. Therefore a parametrization of this portion of the paraboloid is

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (9 - u^2 - v^2) \mathbf{k}, \quad 0 \leq u \leq 2, \quad -\sqrt{1-u^2} \leq v \leq \sqrt{1-u^2}.$$

- (b) The disk above which the surface lies is



The point shown, which is on a line at an angle of θ with the x -axis from the center, has coordinates $(\cos \theta, \sin \theta)$ with respect to the center, so its coordinates in the xy -plane are $(1 + \cos \theta, \sin \theta)$. The line from $(1, 0)$ to this point is the line $(1, 0) + r(\cos \theta, \sin \theta) = (1 + r \cos \theta, r \sin \theta)$ for $0 \leq r \leq 1$. As θ ranges from 0 to 2π , these lines sweep out the entire disk. If $x = 1 + r \cos \theta$ and $y = r \sin \theta$, then $z = 9 - x^2 - y^2 = 8 - r^2 - 2r \cos \theta$, so we can parametrize the surface as

$$\mathbf{r}(r, \theta) = (1 + r \cos \theta) \mathbf{i} + r \sin \theta \mathbf{j} + (8 - r^2 - 2r \cos \theta) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

32. Converting the given equation to spherical coordinates gives $r^2 = 3r + r \cos \phi$, or $r(r - (3 + \cos \phi)) = 0$. Therefore $r = 3 + \cos \phi$. Then use the spherical coordinate transformations

$$x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

and substitute $3 + \cos \phi$ for r to get

$$\mathbf{r}(\theta, \phi) = (3 + \cos \phi) \cos \theta \sin \phi \mathbf{i} + (3 + \cos \phi) \sin \theta \sin \phi \mathbf{j} + (3 + \cos \phi) \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

33. (a) Considering only first octant points, we want $0 \leq x = u \leq 2$ and $0 \leq y = v \leq \sqrt{4 - x^2}$; then $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - u^2 - v^2}$. So the parametrization is

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + \sqrt{4 - u^2 - v^2} \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq \sqrt{4 - u^2}.$$

- (b) Using cylindrical coordinates, the surface lies over the quarter-circle of radius 2 in the first quadrant, which is parametrized by $x = r \cos \theta$, $y = r \sin \theta$, $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$; then $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. So the parametrization is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \sqrt{4 - r^2} \mathbf{k}, \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

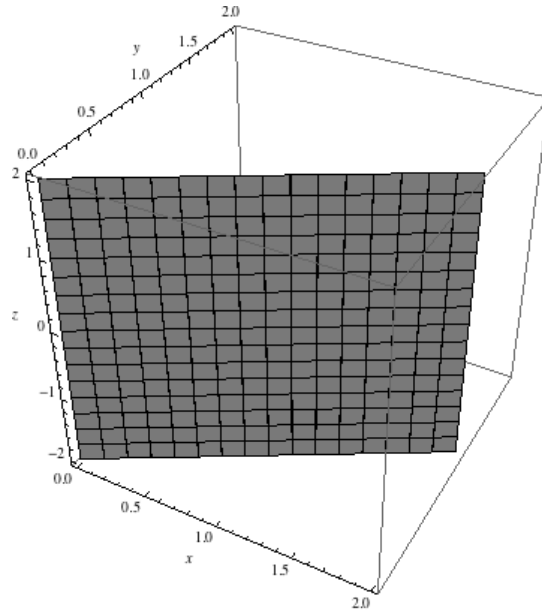
- (c) The full sphere is parametrized using spherical coordinates by

$$\mathbf{r}(\theta, \phi) = 2 \cos \theta \sin \phi \mathbf{i} + 2 \sin \theta \sin \phi \mathbf{j} + 2 \cos \phi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Referring to the definition of spherical coordinates, we see that the first octant corresponds to $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$, so the desired parametrization is

$$\mathbf{r}(\theta, \phi) = 2 \cos \theta \sin \phi \mathbf{i} + 2 \sin \theta \sin \phi \mathbf{j} + 2 \cos \phi, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

34. The surface is



- (a) Since z is a free variable, let $z = v$. Let $x = u$; then $y = \sqrt{3}u$; since $x \geq 0$ and $y \geq 0$ we want $u \geq 0$, but v is unrestricted. So the parametrization is

$$\mathbf{r}(u, v) = u\mathbf{i} + \sqrt{3}u\mathbf{j} + v\mathbf{k}, \quad u \geq 0.$$

- (b) Using cylindrical coordinates, z is again unrestricted, so let $z = v$. Since $\frac{y}{x} = \tan \theta = \frac{1}{\sqrt{3}}$, we have $\theta = \frac{\pi}{6}$. Finally, r is unrestricted since the graph is unbounded with increasing r . Therefore the desired parametrization is

$$\mathbf{r}(r, z) = r \cos \frac{\pi}{6} \mathbf{i} + r \sin \frac{\pi}{6} \mathbf{j} + z \mathbf{k} = \frac{\sqrt{3}}{2}r \mathbf{i} + \frac{1}{2}r \mathbf{j} + z \mathbf{k}.$$

- (c) As in the cylindrical case, $\frac{y}{x} = \tan \theta = \frac{1}{\sqrt{3}}$, so that $\theta = \frac{\pi}{6}$. Since z is unrestricted, we take $0 \leq \phi \leq \pi$. Finally, r is unrestricted since the graph is unbounded with increasing r . Therefore the desired parametrization is

$$\mathbf{r}(r, \phi) = r \cos \frac{\pi}{6} \sin \phi \mathbf{i} + r \sin \frac{\pi}{6} \sin \phi \mathbf{j} + r \cos \phi \mathbf{k} = \frac{\sqrt{3}}{2}r \sin \phi \mathbf{i} + \frac{1}{2}r \sin \phi \mathbf{j} + r \cos \phi \mathbf{k}.$$

35. (a) The values of u and v that give the point $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right)$ are found by solving the three equations

$$\begin{aligned} \cos u(3 + \cos v) &= \frac{3\sqrt{2}}{2} \\ \sin u(3 + \cos v) &= \frac{3\sqrt{2}}{2} \\ \sin v &= 1. \end{aligned}$$

So $v = \frac{\pi}{2}$, and then $\cos v = 0$, so the first two equations become $3 \cos u = \frac{3\sqrt{2}}{2}$ and $3 \sin u = \frac{3\sqrt{2}}{2}$, so that $u = \frac{\pi}{4}$. Next, find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned} \mathbf{r}_u(u, v) &= -\sin u(3 + \cos v)\mathbf{i} + \cos u(3 + \cos v)\mathbf{j} \\ \mathbf{r}_v(u, v) &= -\cos u \sin v \mathbf{i} - \sin u \sin v \mathbf{j} + \cos v \mathbf{k}. \end{aligned}$$

When $u = \frac{\pi}{4}$ and $v = \frac{\pi}{2}$, these are

$$\begin{aligned}\mathbf{r}_u\left(\frac{\pi}{4}, \frac{\pi}{2}\right) &= -\sin\frac{\pi}{4}\left(3 + \cos\frac{\pi}{2}\right)\mathbf{i} + \cos\frac{\pi}{4}\left(3 + \cos\frac{\pi}{2}\right)\mathbf{j} = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} \\ \mathbf{r}_v\left(\frac{\pi}{4}, \frac{\pi}{2}\right) &= -\cos\frac{\pi}{4}\sin\frac{\pi}{2}\mathbf{i} - \sin\frac{\pi}{4}\sin\frac{\pi}{2}\mathbf{j} + \cos\frac{\pi}{2}\mathbf{k} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}.\end{aligned}$$

Therefore a normal vector to the surface at the given point is

$$\left\langle -\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 0 \right\rangle \times \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right\rangle = \langle 0, 0, 3 \rangle.$$

So the equation of the tangent plane at $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right)$ is $3(z - 1) = 0$, or $\boxed{z = 1}$.

(b) The normal line has direction vector $\langle 0, 0, 3 \rangle$, so it has the parametrization

$$\boxed{\mathbf{r}(t) = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} + (1 + 3t)\mathbf{k}.$$

36. Start by finding the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= -\sin u(2 + \cos v)\mathbf{i} + \cos u(2 + \cos v)\mathbf{j} \\ \mathbf{r}_v(u, v) &= -\cos u \sin v \mathbf{i} - \sin u \sin v \mathbf{j} + \cos v \mathbf{k}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u(2 + \cos v) & \cos u(2 + \cos v) & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix} \\ &= \cos u \cos v(2 + \cos v)\mathbf{i} + \sin u \cos v(2 + \cos v)\mathbf{j} + \sin v(2 + \cos v)\mathbf{k}.\end{aligned}$$

Next,

$$\begin{aligned}\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{\cos^2 u \cos^2 v(2 + \cos v)^2 + \sin^2 u \cos^2 v(2 + \cos v)^2 + \sin^2 v(2 + \cos v)^2} \\ &= (2 + \cos v)\sqrt{(\cos^2 u + \sin^2 u) \cos^2 v + \sin^2 v} = 2 + \cos v.\end{aligned}$$

Then the surface area is

$$\int_0^{2\pi} \int_0^{2\pi} (2 + \cos v) dv du = \int_0^{2\pi} [2v + \sin v]_0^{2\pi} du = \int_0^{2\pi} 4\pi du = \boxed{8\pi^2}.$$

37. Start by finding the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \cos(2v)\mathbf{i} + \sin(2v)\mathbf{j} \\ \mathbf{r}_v(u, v) &= -2u \sin(2v)\mathbf{i} + 2u \cos(2v)\mathbf{j} + \mathbf{k}.\end{aligned}$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(2v) & \sin(2v) & 0 \\ -2u \sin(2v) & 2u \cos(2v) & 1 \end{vmatrix} = \sin(2v)\mathbf{i} - \cos(2v)\mathbf{j} + 2u\mathbf{k}.$$

Next,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\sin^2(2v) + \cos^2(2v) + 4u^2} = \sqrt{4u^2 + 1}.$$

Then the surface area is

$$\int_0^1 \int_0^{2\pi} \sqrt{4u^2 + 1} \, dv \, du = 2\pi \int_0^1 \sqrt{4u^2 + 1} \, du = 4\pi \int_0^1 \sqrt{u^2 + \frac{1}{4}} \, du.$$

Evaluate this integral using Formula 49 in the tables in the Endnotes, giving

$$\begin{aligned} 4\pi \int_0^1 \sqrt{u^2 + \frac{1}{4}} \, du &= 4\pi \left[\frac{u}{2} \sqrt{u^2 + \frac{1}{4}} + \frac{1}{8} \ln \left(u + \sqrt{u^2 + \frac{1}{4}} \right) \right]_0^1 \\ &= 4\pi \left(\frac{1}{2} \sqrt{\frac{5}{4}} + \frac{1}{8} \ln \left(1 + \sqrt{\frac{5}{4}} \right) - 0 - \frac{1}{8} \ln \left(0 + \sqrt{\frac{1}{4}} \right) \right) \\ &= 4\pi \left(\frac{\sqrt{5}}{4} + \frac{1}{8} \ln \left(1 + \frac{\sqrt{5}}{2} \right) - \frac{1}{8} \ln \frac{1}{2} \right) \\ &= \pi\sqrt{5} + \frac{\pi}{2} \ln \left(\frac{2 + \sqrt{5}}{2} \cdot 2 \right) = \boxed{\pi\sqrt{5} + \frac{\pi}{2} \ln(2 + \sqrt{5})}. \end{aligned}$$

38. Start by finding the partial derivatives of $\mathbf{r}$:

$$\begin{aligned} \mathbf{r}_u(u, v) &= \left(-2 \sin u - \frac{1}{2}v \sin \frac{u}{2} \right) \mathbf{i} + \left(2 \cos u - \frac{1}{2}v \sin \frac{u}{2} \right) \mathbf{j} - \frac{1}{2}v \cos \frac{u}{2} \mathbf{k} \\ \mathbf{r}_v(u, v) &= \cos \frac{u}{2} \mathbf{i} + \cos \frac{u}{2} \mathbf{j} - \sin \frac{u}{2} \mathbf{k}. \end{aligned}$$

Then (using a CAS to simplify)

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin u - \frac{1}{2}v \sin \frac{u}{2} & 2 \cos u - \frac{1}{2}v \sin \frac{u}{2} & -\frac{1}{2}v \cos \frac{u}{2} \\ \cos \frac{u}{2} & \cos \frac{u}{2} & -\sin \frac{u}{2} \end{vmatrix} \\ &= \left(\frac{v}{2} + \sin \frac{u}{2} - \sin \frac{3u}{2} \right) \mathbf{i} + \left(-\frac{v}{2} - \cos \frac{u}{2} + \cos \frac{3u}{2} \right) \mathbf{j} + \left(-2 \cos \frac{u}{2} (\cos u + \sin u) \right) \mathbf{k}. \end{aligned}$$

Next, again using technology,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 4 + \frac{v^2}{2} + v \cos \frac{u}{2} - v \cos \frac{3u}{2} + v \sin \frac{u}{2} + \sin u - v \sin \frac{3u}{2} + 2 \sin(2u) + \sin(3u).$$

Then the surface area is (using technology)

$$\int_0^{2\pi} \int_{-0.5}^{0.5} \|\mathbf{r}_u \times \mathbf{r}_v\| \, dv \, du \approx 12.236.$$

39. (a) Let $y' = \frac{y}{3}$ and $z' = \frac{z}{2}$; then the equation of the ellipsoid becomes

$$x^2 + (y')^2 + (z')^2 = 1.$$

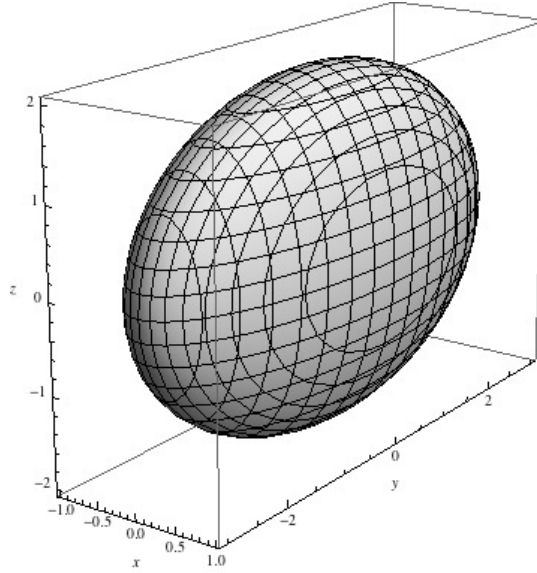
This is a unit sphere in (x, y', z') coordinates. Using the standard parametrization for spherical coordinates, we have

$$x = \cos \theta \sin \phi, \quad y' = \sin \theta \sin \phi, \quad z' = \cos \phi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Now convert back to the original coordinate system by substituting $\frac{y}{3}$ for y' and multiplying that equation through by 3, and substituting $\frac{z}{2}$ for z' and multiplying that equation through by 2. This gives the parametrization

$$\mathbf{r}(\theta, \phi) = \boxed{\cos \theta \sin \phi \mathbf{i} + 3 \sin \theta \sin \phi \mathbf{j} + 2 \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi}.$$

(b) The ellipsoid is



(c) Differentiating the parametrization from part (a) gives

$$\begin{aligned}\mathbf{r}_\theta(\theta, \phi) &= -\sin \theta \sin \phi \mathbf{i} + 3 \cos \theta \sin \phi \mathbf{j} \\ \mathbf{r}_\phi(\theta, \phi) &= \cos \theta \cos \phi \mathbf{i} + 3 \sin \theta \cos \phi \mathbf{j} - 2 \sin \phi \mathbf{k},\end{aligned}$$

so that

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \sin \phi & 3 \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & 3 \sin \theta \cos \phi & -2 \sin \phi \end{vmatrix} \\ &= -6 \cos \theta \sin^2 \phi \mathbf{i} - 2 \sin \theta \sin^2 \phi \mathbf{j} - 3 \sin \phi \cos \phi \mathbf{k}.\end{aligned}$$

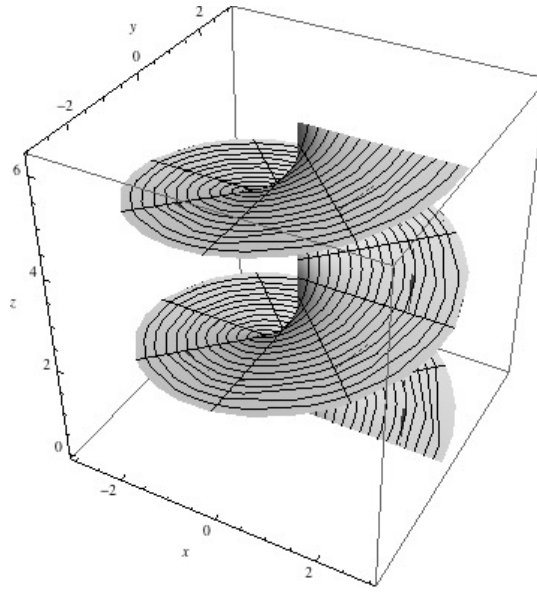
It follows that

$$\|\mathbf{r}_\theta \times \mathbf{r}_\phi\| = \sqrt{36 \cos^2 \theta \sin^4 \phi + 4 \sin^2 \theta \sin^4 \phi + 9 \sin^2 \phi \cos^2 \phi},$$

so that the surface area is

$$S = \int_0^{2\pi} \int_0^\pi \sqrt{36 \cos^2 \theta \sin^4 \phi + 4 \sin^2 \theta \sin^4 \phi + 9 \sin^2 \phi \cos^2 \phi} d\phi d\theta.$$

40. (a) The helicoid is



(b) A parametrization is given in the problem statement for part (a); that is the one we shall use. We have

$$\mathbf{r}(u, v) = u \cos(2v)\mathbf{i} + u \sin(2v)\mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

(c) Find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \cos(2v)\mathbf{i} + \sin(2v)\mathbf{j} \\ \mathbf{r}_v(u, v) &= -2u \sin(2v)\mathbf{i} + 2u \cos(2v)\mathbf{j} + \mathbf{k}.\end{aligned}$$

When $u = \frac{1}{3}$ and $v = \pi$, these are

$$\begin{aligned}\mathbf{r}_u\left(\frac{1}{3}, \pi\right) &= \cos(2\pi)\mathbf{i} + \sin(2\pi)\mathbf{j} = \mathbf{i} \\ \mathbf{r}_v\left(\frac{1}{3}, \pi\right) &= -\frac{2}{3}\sin(2\pi)\mathbf{i} + \frac{2}{3}\cos(2\pi)\mathbf{j} + \mathbf{k} = \frac{2}{3}\mathbf{j} + \mathbf{k}.\end{aligned}$$

Therefore a normal vector to the surface at the given point is

$$\langle 1, 0, 0 \rangle \times \left\langle 0, \frac{2}{3}, 1 \right\rangle = \left\langle 0, -1, \frac{2}{3} \right\rangle.$$

When $(u, v) = (\frac{1}{3}, \pi)$, the corresponding point on the helicoid is

$$\left(\frac{1}{3}\cos(2\pi), \frac{1}{3}\sin(2\pi), \pi\right) = \left(\frac{1}{3}, 0, \pi\right).$$

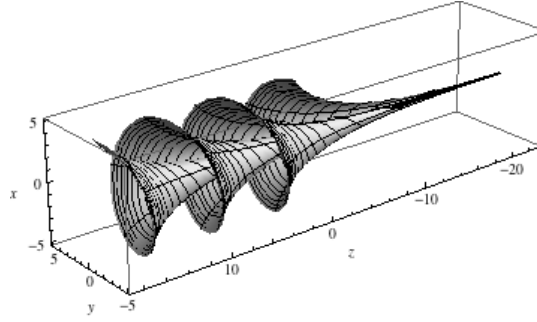
The equation of the tangent plane at that point is

$$0\left(x - \frac{1}{3}\right) - 1(y - 0) + \frac{2}{3}(z - \pi) = 0, \text{ or } -y + \frac{2}{3}z = \frac{2}{3}\pi, \text{ or } 3y - 2z = -2\pi.$$

(d) The normal line has direction vector $\langle 0, -1, \frac{2}{3} \rangle$ and passes through $(\frac{1}{3}, 0, \pi)$, so its parametric equations are

$$\boxed{\frac{1}{3}\mathbf{i} - t\mathbf{j} + \left(\pi + \frac{2}{3}t\right)\mathbf{k}}.$$

41. (a) Dini's surface is



(b) With the given parametrization, we have

$$\mathbf{r}(u, v) = 6 \cos u \sin v \mathbf{i} + 6 \sin u \sin v \mathbf{j} + \left(6 \left[\cos v + \ln \left(\tan \frac{v}{2} \right) \right] + u \right) \mathbf{k}.$$

Find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned} \mathbf{r}_u(u, v) &= -6 \sin u \sin v \mathbf{i} + 6 \cos u \sin v \mathbf{j} + \mathbf{k} \\ \mathbf{r}_v(u, v) &= 6 \cos u \cos v \mathbf{i} + 6 \sin u \cos v \mathbf{j} + 6 \left(-\sin v + \frac{1}{2} \frac{\sec^2 \frac{v}{2}}{\tan \frac{v}{2}} \right) \mathbf{k} \\ &= 6 \cos u \cos v \mathbf{i} + 6 \sin u \cos v \mathbf{j} + 6 \left(-\sin v + \frac{1}{2} \csc \frac{v}{2} \sec \frac{v}{2} \right) \mathbf{k} \\ &= 6 \cos u \cos v \mathbf{i} + 6 \sin u \cos v \mathbf{j} + 6 \left(-\sin v + \frac{1}{2 \sin \frac{v}{2} \cos \frac{v}{2}} \right) \mathbf{k} \\ &= 6 \cos u \cos v \mathbf{i} + 6 \sin u \cos v \mathbf{j} + 6 \left(-\sin v + \frac{1}{\sin v} \right) \mathbf{k} \\ &= 6 \cos u \cos v \mathbf{i} + 6 \sin u \cos v \mathbf{j} + 6(-\sin v + \csc v) \mathbf{k}. \end{aligned}$$

When $u = \frac{\pi}{3}$ and $v = \frac{\pi}{4}$, these are

$$\begin{aligned} \mathbf{r}_u \left(\frac{\pi}{3}, \frac{\pi}{4} \right) &= -6 \sin \frac{\pi}{3} \sin \frac{\pi}{4} \mathbf{i} + 6 \cos \frac{\pi}{3} \sin \frac{\pi}{4} \mathbf{j} + \mathbf{k} \\ &= -\frac{3\sqrt{6}}{2} \mathbf{i} + \frac{3\sqrt{2}}{2} \mathbf{j} + \mathbf{k} \\ \mathbf{r}_v \left(\frac{\pi}{3}, \frac{\pi}{4} \right) &= 6 \cos \frac{\pi}{3} \cos \frac{\pi}{4} \mathbf{i} + 6 \sin \frac{\pi}{3} \cos \frac{\pi}{4} \mathbf{j} + 6 \left(-\sin \frac{\pi}{4} + \csc \frac{\pi}{4} \right) \mathbf{k} \\ &= \frac{3\sqrt{2}}{2} \mathbf{i} + \frac{3\sqrt{6}}{2} \mathbf{j} + 6 \left(-\frac{\sqrt{2}}{2} + \sqrt{2} \right) \mathbf{k} \\ &= \frac{3\sqrt{2}}{2} \mathbf{i} + \frac{3\sqrt{6}}{2} \mathbf{j} + 3\sqrt{2} \mathbf{k}. \end{aligned}$$

Therefore a normal vector to the surface at the given point is

$$\left\langle -\frac{3\sqrt{6}}{2}, \frac{3\sqrt{2}}{2}, 1 \right\rangle \times \left\langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{6}}{2}, 3\sqrt{2} \right\rangle = \left\langle 9 - \frac{3\sqrt{6}}{2}, 9\sqrt{3} + \frac{3\sqrt{2}}{2}, -18 \right\rangle.$$

When $(u, v) = \left(\frac{\pi}{3}, \frac{\pi}{4} \right)$, the corresponding point on the helicoid is

$$\begin{aligned} &\left(6 \cos \frac{\pi}{3} \sin \frac{\pi}{4}, 6 \sin \frac{\pi}{3} \sin \frac{\pi}{4}, 6 \left[\cos \frac{\pi}{4} + \ln \left(\tan \frac{\pi}{8} \right) \right] + \frac{\pi}{3} \right) \\ &= \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{6}}{2}, \frac{\pi}{3} + 6 \left[\frac{\sqrt{2}}{2} + \ln \left(\tan \frac{\pi}{8} \right) \right] \right). \end{aligned}$$

The equation of the tangent plane at that point is then

$$\left(9 - \frac{3\sqrt{6}}{2}\right)\left(x - \frac{3\sqrt{2}}{2}\right) + \left(9\sqrt{3} + \frac{3\sqrt{2}}{2}\right)\left(y - \frac{3\sqrt{6}}{2}\right) - 18\left(z - \left(\frac{\pi}{3} + 6\left[\frac{\sqrt{2}}{2} + \ln\left(\tan\frac{\pi}{8}\right)\right]\right)\right) = 0.$$

Using technology, that equation “simplifies” to

$$(6 - \sqrt{6})x + (\sqrt{2} + 6\sqrt{3})y - 12z = -4\pi - 72\ln(\tan(\sqrt{2} - 1)).$$

(c) The normal line has direction vector $\left\langle 9 - \frac{3\sqrt{6}}{2}, 9\sqrt{3} + \frac{3\sqrt{2}}{2}, -18 \right\rangle$ and passes through

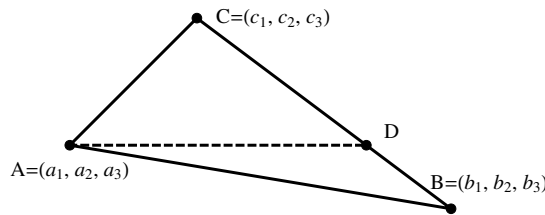
$$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{6}}{2}, \frac{\pi}{3} + 6\left[\frac{\sqrt{2}}{2} + \ln(\tan(\sqrt{2} - 1))\right]\right),$$

so it has the parametrization

$$\begin{aligned} \mathbf{r}(t) = & \left(\frac{3\sqrt{2}}{2} + \left(9 - \frac{3\sqrt{6}}{2}\right)t\right) \mathbf{i} + \left(\frac{3\sqrt{6}}{2} + \left(9\sqrt{3} + \frac{3\sqrt{2}}{2}\right)t\right) \mathbf{j} \\ & + \left(\frac{\pi}{3} + 6\left[\frac{\sqrt{2}}{2} + \ln(\tan(\sqrt{2} - 1))\right] - 18t\right) \mathbf{k}. \end{aligned}$$

Challenge Problems

42. An annotated diagram of the triangle is



The line segment from B to C can be parametrized by

$$\mathbf{r}_1(t) = \langle b_1, b_2, b_3 \rangle + t \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle, \quad 0 \leq t \leq 1.$$

Then if D is the point $\mathbf{r}(t_d)$, the line segment from A to D can be parametrized as

$$\begin{aligned} \mathbf{r}_2(u) &= \langle a_1, a_2, a_3 \rangle + u(\mathbf{r}_1(t_d) - \langle a_1, a_2, a_3 \rangle) \\ &= \langle a_1, a_2, a_3 \rangle + u(\langle b_1, b_2, b_3 \rangle + t_d \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle - \langle a_1, a_2, a_3 \rangle) \\ &= (1 - u) \langle a_1, a_2, a_3 \rangle + u \langle b_1, b_2, b_3 \rangle + ut_d \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle, \quad 0 \leq u \leq 1. \end{aligned}$$

Letting t range over all its values $0 \leq t \leq 1$, these line segments sweep out the triangle, so a parametrization is

$$\mathbf{r}(t, u) = (1 - u) \langle a_1, a_2, a_3 \rangle + u \langle b_1, b_2, b_3 \rangle + ut \langle c_1 - b_1, c_2 - b_2, c_3 - b_3 \rangle, \quad 0 \leq t \leq 1, \quad 0 \leq u \leq 1.$$

43. (a) Consider a circle of radius a and center $(b, 0, 0)$ lying in the xz -plane. Rotating that circle about the z -axis generates the required torus. Points on the circle are $(b + a \cos u, 0, a \sin u)$ for $0 \leq u \leq 2\pi$, so in cylindrical coordinates, the points on the circle are of the form

$$r = \sqrt{(b + a \cos u)^2 + 0^2} = b + a \cos u, \quad z = a \sin u, \quad \theta = 0, \quad 0 \leq u \leq 2\pi.$$

When the circle is rotated about the y -axis, the r and z coordinates of a given point on the circle remain constant, so a parametrization of the torus in cylindrical coordinates is

$$r = b + a \cos u, \quad z = a \sin u, \quad \theta = v, \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi.$$

In rectangular coordinates, this gives

$$\begin{aligned} x &= r \cos \theta = (b + a \cos u) \cos v \\ y &= r \sin \theta = (b + a \cos u) \sin v \\ z &= a \sin u, \end{aligned}$$

so that a parametrization of the torus in rectangular coordinates is

$$\mathbf{r}(u, v) = (b + a \cos u) \cos v \mathbf{i} + (b + a \cos u) \sin v \mathbf{j} + a \sin u \mathbf{k}.$$

- (b) To eliminate v , square and add the first two equations, giving

$$x^2 + y^2 = (b + a \cos u)^2 \cos^2 v + (b + a \cos u)^2 \sin^2 v = (b + a \cos u)^2.$$

To eliminate u , we need to get $a \cos u$ by itself so we can square this equation and the third and add them. So rewrite the above equation as

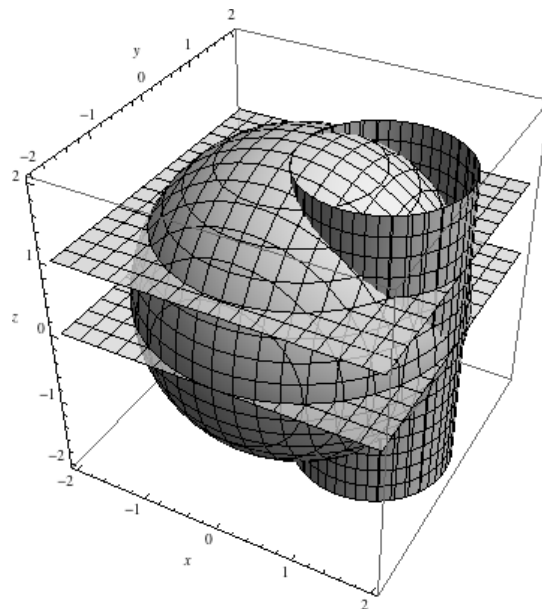
$$a \cos u = \sqrt{x^2 + y^2} - b;$$

then squaring this equation and adding it to the square of $z = a \sin u$ gives

$$a^2 \cos^2 u + a^2 \sin^2 u = z^2 + \left(\sqrt{x^2 + y^2} - b \right)^2, \quad \text{or}$$

$$z^2 + \left(\sqrt{x^2 + y^2} - b \right)^2 = a^2.$$

44. The surfaces involved are



The surface whose area we wish to find is the portion of the cylinder visible between the planes $z = 0$ and $z = 1$ on the right hand side of the diagram. To simplify things, note that this region is symmetric about the xz -plane, so we will compute the area for $y \geq 0$ and double it. Since the rectangular equation of the sphere is $x^2 + y^2 + z^2 = 4$, converting to cylindrical coordinates gives $r^2 + z^2 = 4$, or $z = \sqrt{4 - r^2}$. The cylinder lies above the circle $(x - 1)^2 + y^2 = 1$, which can be rewritten as $x^2 + y^2 - 2x = 0$. In cylindrical coordinates, this becomes $r^2 - 2r \cos \theta = 0$, or $r = 2 \cos \theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We want the portion of the cylinder lying outside (above) the sphere, so we require that $\sqrt{4 - r^2} \leq z \leq 1$. Substituting $2 \cos \theta$ for r gives the constraint $\sqrt{4 - 4 \cos^2 \theta} \leq z \leq 1$, or $|2 \sin \theta| \leq z \leq 1$. Since $\theta \geq 0$ when $y \geq 0$, we can write this as $2 \sin \theta \leq z \leq 1$, which implies that $0 \leq \theta \leq \frac{\pi}{6}$. We now have the bounds for the integration; next we must find the integrand. To do this, we find a rectangular parametrization of the circle $(x - 1)^2 + y^2 = 1$. From the above, we have $r^2 = x^2 + y^2 = 4 \cos^2 \theta$, so that

$$(x - 1)^2 + y^2 - 1 = x^2 + y^2 - 2x = 4 \cos^2 \theta - 2x = 0.$$

It follows that $x = 2 \cos^2 \theta = 1 + \cos(2\theta)$, so that

$$(x - 1)^2 + y^2 = \cos^2(2\theta) + y^2 = 1,$$

and therefore $y = \sin(2\theta)$. So a parametrization of the cylinder in rectangular coordinates is

$$\mathbf{r}(\theta, z) = \langle 1 + \cos(2\theta), \sin(2\theta), z \rangle.$$

Then

$$\mathbf{r}_\theta = \langle -2 \sin(2\theta), 2 \cos(2\theta), 0 \rangle, \quad \mathbf{r}_z = \langle 0, 0, 1 \rangle,$$

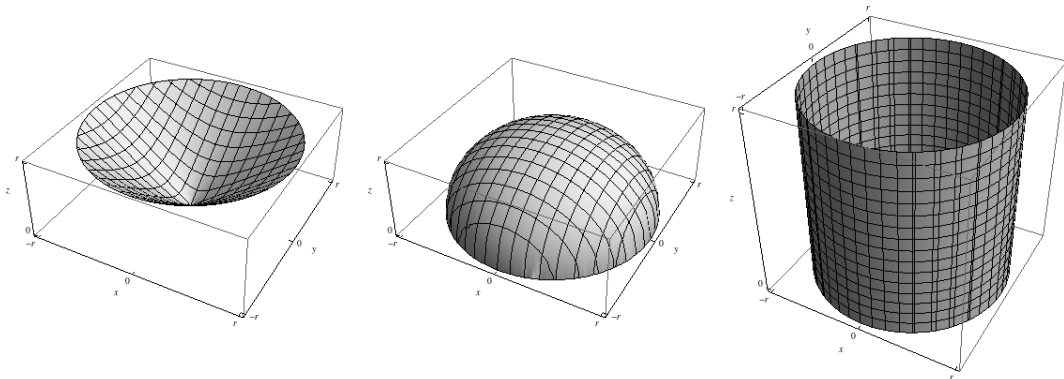
so that

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle -2 \sin(2\theta), 2 \cos(2\theta), 0 \rangle \times \langle 0, 0, 1 \rangle = \langle 2 \cos(2\theta), 2 \sin(2\theta), 0 \rangle.$$

Then $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = \sqrt{4 \cos^2(2\theta) + 4 \sin^2(2\theta)} = 2$. Therefore the area of the region is (converting to cylindrical coordinates)

$$\begin{aligned} 2 \int_0^{\pi/6} \int_{2 \sin \theta}^1 2r \, dz \, d\theta &= 2 \int_0^{\pi/6} \int_{2 \sin \theta}^1 4 \cos \theta \, dz \, d\theta \\ &= 2 \int_0^{\pi/6} [4z \cos \theta]_{2 \sin \theta}^1 \, d\theta \\ &= 2 \int_0^{\pi/6} (4 \cos \theta - 8 \sin \theta \cos \theta) \, d\theta \\ &= 2 [4 \sin \theta - 4 \sin^2 \theta]_0^{\pi/6} \\ &= 2(2 - 1) = 2. \end{aligned}$$

45. The three surfaces are



The cone has rectangular equation $z = \sqrt{x^2 + y^2}$, since a cross-section at height $z = r$ gives $\sqrt{x^2 + y^2} = r$, so that $x^2 + y^2 = r^2$ and the cross-section is a circle of radius r . Using rectangular coordinates, we have

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}, \quad -r \leq u \leq r, \quad -\sqrt{1 - u^2} \leq v \leq \sqrt{1 - u^2}.$$

Then

$$\mathbf{r}_u = \mathbf{i} + u(u^2 + v^2)^{-1/2}\mathbf{k}, \quad \mathbf{r}_v = \mathbf{j} + v(u^2 + v^2)^{-1/2}\mathbf{k},$$

so that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 0, u(u^2 + v^2)^{-1/2} \rangle \times \langle 0, 1, v(u^2 + v^2)^{-1/2} \rangle = \langle -u(u^2 + v^2)^{-1/2}, -v(u^2 + v^2)^{-1/2}, 1 \rangle,$$

and then

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\frac{u^2}{u^2 + v^2} + \frac{v^2}{u^2 + v^2} + 1} = \sqrt{\frac{2u^2 + 2v^2}{u^2 + v^2}} = \sqrt{2}.$$

Integrate using polar coordinates (ρ, θ) ; then $0 \leq \rho \leq r$ and $0 \leq \theta \leq 2\pi$, and we get for the surface area

$$\iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA = \int_0^r \int_0^{2\pi} \rho\sqrt{2} \, d\theta \, dR = \int_0^r 2\pi\rho\sqrt{2} \, dR = \left[\pi\sqrt{2}\frac{\rho^2}{2} \right]_0^r = \pi r^2\sqrt{2}.$$

Parametrize the sphere using spherical coordinates, so that

$$\mathbf{r}(\theta, \phi) = r \cos \theta \sin \phi \mathbf{i} + r \sin \theta \sin \phi \mathbf{j} + r \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\theta = -r \sin \theta \sin \phi \mathbf{i} + r \cos \theta \sin \phi \mathbf{j}, \quad \mathbf{r}_\phi = r \cos \theta \cos \phi \mathbf{i} + r \sin \theta \cos \phi \mathbf{j} - r \sin \phi \mathbf{k},$$

so that

$$\begin{aligned} \mathbf{r}_\theta \times \mathbf{r}_\phi &= \langle -r \sin \theta \sin \phi, r \cos \theta \sin \phi, 0 \rangle \times \langle r \cos \theta \cos \phi, r \sin \theta \cos \phi, -r \sin \phi \rangle \\ &= \langle -r^2 \cos \theta \cos^2 \phi, -r^2 \sin \theta \sin^2 \phi, -r^2 \sin \phi \cos \phi \rangle, \end{aligned}$$

and then

$$\|\mathbf{r}_\theta \times \mathbf{r}_\phi\| = r^2 \sin \phi.$$

Then the surface area is

$$\iint_R \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| \, dA = \int_0^{2\pi} \int_0^{\pi/2} r^2 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} [-r^2 \cos \phi]_0^{\pi/2} \, d\theta = \int_0^{2\pi} r^2 \, d\theta = 2\pi r^2.$$

Finally, parametrize the cylinder using cylindrical coordinates:

$$\mathbf{r}(\theta, z) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq r,$$

so that

$$\begin{aligned} \mathbf{r}_\theta \times \mathbf{r}_z &= \langle -r \sin \theta \sin \phi, r \cos \theta \cos \phi, 0 \rangle \times \langle 0, 0, 1 \rangle \\ &= \langle r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} \rangle, \end{aligned}$$

and then

$$\|\mathbf{r}_\theta \times \mathbf{r}_z\| = r.$$

Then the surface area is

$$\iint_R \|\mathbf{r}_\theta \times \mathbf{r}_z\| \, dA = \int_0^{2\pi} \int_0^r r \, dz \, d\theta = \int_0^{2\pi} [rz]_0^r \, d\theta = \int_0^{2\pi} r^2 \, d\theta = 2\pi r^2.$$

The ratios of the three surface areas are $\pi r^2\sqrt{2} : 2\pi r^2 : 2\pi r^2$, or $\sqrt{2} : 2 : 2$, as desired.

46. (a) The identity $\cosh^2 u - \sinh^2 u = 1$ is the key identity for this parametrization. We want $x^2 + y^2 - z^2 = c$, so we want to try to get $x^2 + y^2$ to equal $c \cosh^2 u$ and z^2 to equal $c \sinh^2 u$. The most natural way to do this is to let $z = \sqrt{c} \sinh u$ and then to introduce a new parameter v , with $x = \sqrt{c} \cosh u \cos v$ and $y = \sqrt{c} \cosh u \sin v$. Note that in the diagram in part (b), for a fixed value of z (that is, a fixed value of u), the cross-sections are circles, which is consistent with the parametrization of x and y above. With this parametrization, we have

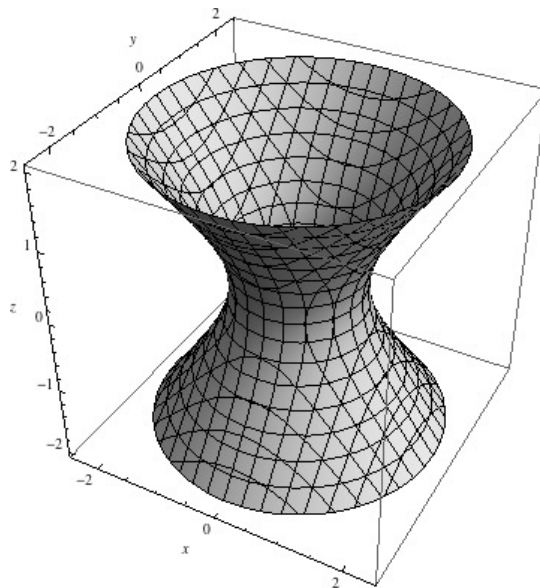
$$\begin{aligned} x^2 + y^2 - z^2 &= (\sqrt{c} \cosh u \cos v)^2 + (\sqrt{c} \cosh u \sin v)^2 - (\sqrt{c} \sinh u)^2 \\ &= c (\cosh^2 u (\cos^2 v + \sin^2 v) - \sinh^2 u) \\ &= c (\cosh^2 u - \sinh^2 u) = c, \end{aligned}$$

as desired. So the parametrization is

$$\mathbf{r}(u, v) = \sqrt{c} \cosh u \cos v \mathbf{i} + \sqrt{c} \cosh u \sin v \mathbf{j} + \sqrt{c} \sinh u \mathbf{k}.$$

The bounds of the parameters are $-\infty < u < \infty$ and $0 \leq v \leq 2\pi$.

- (b) The one-sheeted hyperboloid from part (a) with $c = 1$ is



47. (a) First change variables, setting $x = ax'$, $y = by'$, and $z = cz'$. The hyperboloid then has the equation $(x')^2 - (y')^2 - (z')^2 = 1$ in the (x', y', z') coordinate system. The identity $\cosh^2 u - \sinh^2 u = 1$ is the key identity for this parametrization, as it was in Problem 46. We want $(x')^2 - (y')^2 - (z')^2 = 1$, so we want to try to get $(x')^2$ to equal $\cosh^2 u$ and $(y')^2 + (z')^2$ to equal $\sinh^2 u$. The most natural way to do this is to let $x' = \cosh u$ and then to introduce a new parameter v , with $y' = \sinh u \cos v$ and $z' = \sinh u \sin v$. With this parametrization, we have

$$\begin{aligned} (x')^2 - (y')^2 - (z')^2 &= \cosh^2 u - \sinh^2 u \cos^2 v - \sinh^2 u \sin^2 v \\ &= \cosh^2 u - \sinh^2 u (\cos^2 v + \sin^2 v) \\ &= \cosh^2 u - \sinh^2 u = 1, \end{aligned}$$

as desired. Substituting back in $\frac{x}{a}$ for x' , $\frac{y}{b}$ for y' , and $\frac{z}{c}$ for z' , we get

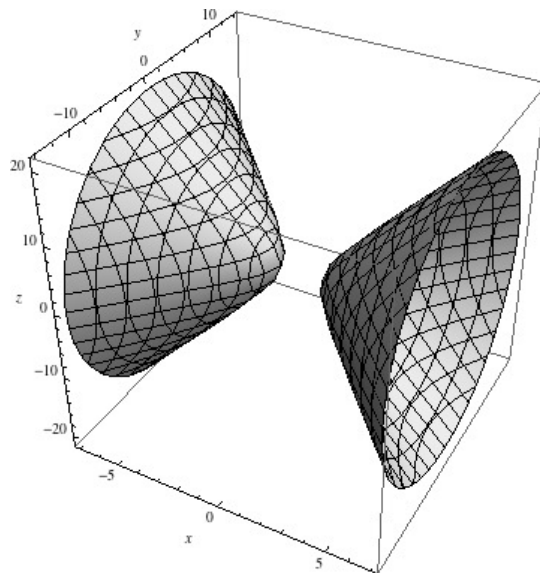
$$\begin{aligned} \frac{x}{a} &= \cosh u, & \frac{y}{b} &= \sinh u \cos v, & \frac{z}{c} &= \sinh u \sin v, \text{ or} \\ x &= a \cosh u, & y &= b \sinh u \cos v, & z &= c \sinh u \sin v. \end{aligned}$$

So the parametrization is

$$\mathbf{r}(u, v) = \boxed{a \cosh u \mathbf{i} + b \sinh u \cos v \mathbf{j} + c \sinh u \sin v \mathbf{k}}.$$

The bounds of the parameters are $-\infty < u < \infty$ and $0 \leq v \leq 2\pi$.

(b) The two-sheeted hyperboloid from part (a) with $a = 1$, $b = 2$, and $c = 3$ is



15.7: Surface and Flux Integrals

Concepts and Vocabulary

1. (c), *orientable*, is correct. See Objective 2.
2. (b), *flux*, is correct. See Objective 3.

Skill Building

3. Parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (2u + 3v + 5) \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2 \mathbf{i} - 3 \mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{14}$, so the surface integral is

$$\begin{aligned} \iint_S (x^2 + y^2) dS &= \int_0^1 \int_0^1 (u^2 + v^2) \cdot \sqrt{14} du dv = \sqrt{14} \int_0^1 \left[\frac{1}{3} u^3 + uv^2 \right]_0^1 dv \\ &= \sqrt{14} \int_0^1 \left(\frac{1}{3} + v^2 \right) dv = \sqrt{14} \left[\frac{1}{3} v + \frac{1}{3} v^3 \right]_0^1 = \boxed{\frac{2}{3} \sqrt{14}}. \end{aligned}$$

4. Parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (3u + 4v + 8) \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 0 & 1 & 4 \end{vmatrix} = -3\mathbf{i} - 4\mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{26}$, so the surface integral is

$$\begin{aligned} \iint_S x^2 y^2 dS &= \int_0^1 \int_0^1 u^2 v^2 \sqrt{26} du dv = \sqrt{26} \int_0^1 \left[\frac{1}{3} u^3 v^2 \right]_0^1 dv = \sqrt{26} \int_0^1 \left(\frac{1}{3} v^2 \right) dv \\ &= \sqrt{26} \left[\frac{1}{9} v^3 \right]_0^1 = \boxed{\frac{1}{9} \sqrt{26}}. \end{aligned}$$

5. Parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u + 5v^2)\mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 10v \end{vmatrix} = -1\mathbf{i} - 10v\mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{100v^2 + 2}$, so the surface integral is

$$\iint_S y dS = \int_0^1 \int_0^2 v \sqrt{100v^2 + 2} du dv = \int_0^1 \left[uv \sqrt{100v^2 + 2} \right]_0^2 dv = \int_0^1 2v \sqrt{100v^2 + 2} dv.$$

Use the substitution $t = 100v^2 + 2$, so that $dt = 200v dv$:

$$\int_0^1 2v \sqrt{100v^2 + 2} dv = \frac{1}{100} \int_2^{102} \sqrt{t} dt = \frac{1}{100} \left[\frac{2}{3} t^{3/2} \right]_2^{102} = \boxed{\frac{17}{25} \sqrt{102} - \frac{1}{75} \sqrt{2}}.$$

6. Parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (2u^2 + v)\mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4u \\ 0 & 1 & 1 \end{vmatrix} = -4u\mathbf{i} - 1\mathbf{j} + \mathbf{k}.$$

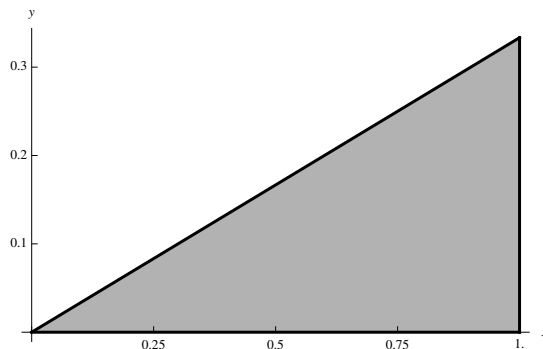
Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{16u^2 + 2}$, so the surface integral is

$$\iint_S 4x dS = \int_0^1 \int_0^2 4u \sqrt{16u^2 + 2} dv du = \int_0^1 \left[4uv \sqrt{16u^2 + 2} \right]_0^2 du = \int_0^1 8u \sqrt{u^2 + 2} du.$$

Use the substitution $t = 16u^2 + 2$, so that $dt = 32u du$:

$$\int_0^1 8u \sqrt{16u^2 + 2} du = \frac{1}{4} \int_2^{18} \sqrt{t} dt = \frac{1}{4} \left[\frac{2}{3} t^{3/2} \right]_2^{18} = \frac{1}{6} (18\sqrt{18} - 2\sqrt{2}) = \boxed{\frac{26}{3} \sqrt{2}}.$$

7. The region of interest in the xy -plane is



Parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (u - 3v) \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{vmatrix} = -1 \mathbf{i} + 3 \mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{11}$, so the surface integral is

$$\begin{aligned} \iint_S (x + y + z) dS &= \int_0^1 \int_0^{u/3} (u + v + u - 3v) \sqrt{11} dv du \\ &= \sqrt{11} \int_0^1 \int_0^{u/3} (2u - 2v) dv du \\ &= \sqrt{11} \int_0^1 [2uv - v^2]_0^{u/3} du \\ &= \sqrt{11} \int_0^1 \left(\frac{2}{3}u^2 - \frac{1}{9}u^2 \right) du \\ &= \sqrt{11} \int_0^1 \frac{5}{9}u^2 du \\ &= \sqrt{11} \left[\frac{5}{27}u^3 \right]_0^1 = \boxed{\frac{5}{27}\sqrt{11}}. \end{aligned}$$

8. The region in the xy -plane is a square of side 1 whose lower left corner is at the origin. Solve for z , giving $z = 2 - \frac{1}{2}x - \frac{1}{2}y$; then parametrize the surface by $x = u$ and $y = v$, so that

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + \left(2 - \frac{1}{2}u - \frac{1}{2}v \right) \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{6}}{2}$, so the surface integral is

$$\begin{aligned} \iint_S x dS &= \int_0^1 \int_0^1 u \cdot \frac{\sqrt{6}}{2} dv du \\ &= \frac{\sqrt{6}}{2} \int_0^1 [uv]_0^1 du \\ &= \frac{\sqrt{6}}{2} \int_0^1 u du \\ &= \frac{\sqrt{6}}{2} \left[\frac{1}{2}u^2 \right]_0^1 \\ &= \boxed{\frac{\sqrt{6}}{4}}. \end{aligned}$$

9. The region in the xy -plane is the disk of radius 1 inside the given cylinder. This suggests cylindrical coordinates, so parametrize the surface by $x = r \cos \theta$ and $y = r \sin \theta$, so that $z = 2 - r \cos \theta - r \sin \theta$. Therefore

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (2 - r \cos \theta - r \sin \theta) \mathbf{k}.$$

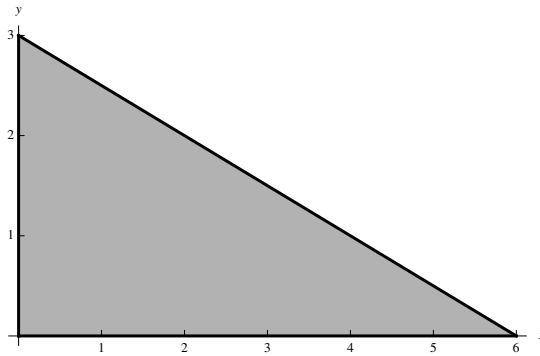
Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\cos \theta - \sin \theta \\ -r \sin \theta & r \cos \theta & r \sin \theta - r \cos \theta \end{vmatrix} = r \mathbf{i} + r \mathbf{j} + r \mathbf{k}.$$

Therefore $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = r\sqrt{3}$. The region of integration is over the unit circle, so that $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then the surface integral is

$$\begin{aligned} \iint_S y \, dS &= \int_0^1 \int_0^{2\pi} r \sin \theta \cdot r\sqrt{3} \, d\theta \, dr \\ &= \sqrt{3} \int_0^1 \int_0^{2\pi} r^2 \sin \theta \, d\theta \, dr \\ &= \sqrt{3} \int_0^1 [-r^2 \cos \theta]_0^{2\pi} \, dr \\ &= \sqrt{3} \int_0^1 0 \, dr = \boxed{0}. \end{aligned}$$

10. The plane $x + 2y + 3z = 6$ intersects the xy -plane when $z = 0$, so along the line $x + 2y = 6$. The region in the xy -plane is therefore the region bounded by this line and the two coordinate axes:



Parametrize the surface by $x = u$, $y = v$, $z = \frac{1}{3}(6 - u - 2v) = 2 - \frac{1}{3}u - \frac{2}{3}v$, so that

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + \left(2 - \frac{1}{3}u - \frac{2}{3}v\right) \mathbf{k}.$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{vmatrix} = \frac{1}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} + \mathbf{k}.$$

Therefore $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{\frac{1}{9} + \frac{4}{9} + 1} = \frac{\sqrt{14}}{3}$. The region is determined by $0 \leq y \leq 3$ and $0 \leq x \leq 6 - 2y$,

so the surface integral is

$$\begin{aligned}
 \iint_S yz \, dS &= \int_0^3 \int_0^{6-2v} v \left(2 - \frac{1}{3}u - \frac{2}{3}v \right) \cdot \frac{\sqrt{14}}{3} \, du \, dv \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \int_0^{6-2v} \left(2v - \frac{1}{3}uv - \frac{2}{3}v^2 \right) \, du \, dv \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \left[2uv - \frac{1}{6}u^2v - \frac{2}{3}uv^2 \right]_0^{6-2v} \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \left(2(6-2v)v - \frac{1}{6}(6-2v)^2v - \frac{2}{3}(6-2v)v^2 \right) \, dv \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \left(\frac{2}{3}v^3 - 4v^2 + 6v \right) \, dv \\
 &= \frac{\sqrt{14}}{3} \left[\frac{1}{6}v^4 - \frac{4}{3}v^3 + 3v^2 \right]_0^3 \\
 &= \frac{\sqrt{14}}{3} \left(\frac{27}{2} - 36 + 27 \right) = \boxed{\frac{3}{2}\sqrt{14}}.
 \end{aligned}$$

11. The surface is a cylinder of radius 1 and height 1. Parametrize the surface using cylindrical coordinates: $x = \cos \theta$, $y = \sin \theta$, $z = z$. Therefore

$$\mathbf{r}(\theta, z) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + z \mathbf{k}.$$

Then

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}.$$

Therefore $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$. The region of integration is $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$, so that the surface integral is

$$\begin{aligned}
 \iint_S x^2 z \, dS &= \int_0^1 \int_0^{2\pi} \cos^2 \theta \cdot z \cdot 1 \, d\theta \, dz \\
 &= \int_0^1 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) z \, d\theta \, dz \\
 &= \int_0^1 \left[\left(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right) z \right]_0^{2\pi} \, dz \\
 &= \int_0^1 \pi z \, dz \\
 &= \left[\frac{1}{2} \pi z^2 \right]_0^1 = \boxed{\frac{\pi}{2}}.
 \end{aligned}$$

12. The surface is a cylinder of radius 3 and height 2. Parametrize the surface using cylindrical coordinates: $x = 3 \cos \theta$, $y = 3 \sin \theta$, $z = z$. Therefore

$$\mathbf{r}(\theta, z) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + z \mathbf{k}.$$

Then

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \sin \theta & 3 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j}.$$

Therefore $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = \sqrt{9 \cos^2 \theta + 9 \sin^2 \theta} = 3$. The region of integration is $0 \leq \theta \leq 2\pi$ and $1 \leq z \leq 3$, so that the surface integral is

$$\begin{aligned} \iint_S (x+y)z \, dS &= \int_0^1 \int_0^{2\pi} (3 \cos \theta + 3 \sin \theta)z \, d\theta \, dz \\ &= 3 \int_0^1 [(\sin \theta - \cos \theta)z]_0^{2\pi} \, dz \\ &= 3 \int_0^1 0 \, dz = \boxed{0}. \end{aligned}$$

13. Since the surface is defined as $z = f(x, y) = \sqrt{16 - x^2 - y^2}$, the outer unit normal is the normal with a positive $\mathbf{k}$ component, which (see Objective 2) will be

$$\mathbf{n} = \frac{-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Here we have

$$\begin{aligned} f_x(x, y) &= \frac{1}{2} (16 - x^2 - y^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{16 - x^2 - y^2}} \\ f_y(x, y) &= \frac{1}{2} (16 - x^2 - y^2)^{-1/2} \cdot (-2y) = -\frac{y}{\sqrt{16 - x^2 - y^2}}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{\frac{x^2}{16 - x^2 - y^2} + \frac{y^2}{16 - x^2 - y^2} + 1}} \left(\frac{x}{\sqrt{16 - x^2 - y^2}} \mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \frac{\sqrt{16 - x^2 - y^2}}{4} \left(\frac{x}{\sqrt{16 - x^2 - y^2}} \mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \boxed{\frac{x}{4} \mathbf{i} + \frac{y}{4} \mathbf{j} + \frac{\sqrt{16 - x^2 - y^2}}{4} \mathbf{k}}. \end{aligned}$$

14. Since the surface is defined as $z = f(x, y) = \sqrt{1 - x^2 - y^2}$, the outer unit normal is the normal with a positive $\mathbf{k}$ component, which (see Objective 2) will be

$$\mathbf{n} = \frac{-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Here we have

$$\begin{aligned} f_x(x, y) &= \frac{1}{2} (1 - x^2 - y^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2 - y^2}} \\ f_y(x, y) &= \frac{1}{2} (1 - x^2 - y^2)^{-1/2} \cdot (-2y) = -\frac{y}{\sqrt{1 - x^2 - y^2}}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{\frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2} + 1}} \left(\frac{x}{\sqrt{1 - x^2 - y^2}} \mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \sqrt{1 - x^2 - y^2} \left(\frac{x}{\sqrt{1 - x^2 - y^2}} \mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \boxed{x \mathbf{i} + y \mathbf{j} + \sqrt{1 - x^2 - y^2} \mathbf{k}}. \end{aligned}$$

15. Since the surface is defined as $z = f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$, the outer unit normal is the normal with a positive $\mathbf{k}$ component, which (see Objective 2) will be

$$\mathbf{n} = \frac{-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Here we have

$$\begin{aligned} f_x(x, y) &= \frac{1}{2} (36 - 9x^2 - 4y^2)^{-1/2} \cdot (-18x) = -\frac{9x}{\sqrt{36 - 9x^2 - 4y^2}} \\ f_y(x, y) &= \frac{1}{2} (36 - 9x^2 - 4y^2)^{-1/2} \cdot (-8y) = -\frac{4y}{\sqrt{36 - 9x^2 - 4y^2}}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{\frac{81x^2}{36-9x^2-4y^2} + \frac{16y^2}{36-9x^2-4y^2} + 1}} \left(\frac{9x}{\sqrt{36-9x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{36-9x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \frac{\sqrt{36-9x^2-4y^2}}{\sqrt{36+72x^2+12y^2}} \left(\frac{9x}{\sqrt{36-9x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{36-9x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \frac{\sqrt{36-9x^2-4y^2}}{2\sqrt{9+18x^2+3y^2}} \left(\frac{9x}{\sqrt{36-9x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{36-9x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \boxed{\frac{9x}{2\sqrt{9+18x^2+3y^2}} \mathbf{i} + \frac{4y}{2\sqrt{9+18x^2+3y^2}} \mathbf{j} + \frac{\sqrt{36-9x^2-4y^2}}{2\sqrt{9+18x^2+3y^2}} \mathbf{k}}. \end{aligned}$$

16. Since the surface is defined as $z = f(x, y) = \sqrt{4 - x^2 - 4y^2}$, the outer unit normal is the normal with a positive $\mathbf{k}$ component, which (see Objective 2) will be

$$\mathbf{n} = \frac{-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Here we have

$$\begin{aligned} f_x(x, y) &= \frac{1}{2} (4 - x^2 - 4y^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{4 - x^2 - 4y^2}} \\ f_y(x, y) &= \frac{1}{2} (4 - x^2 - 4y^2)^{-1/2} \cdot (-8y) = -\frac{4y}{\sqrt{4 - x^2 - 4y^2}}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{\frac{x^2}{4-x^2-4y^2} + \frac{16y^2}{4-x^2-4y^2} + 1}} \left(\frac{x}{\sqrt{4-x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{4-x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \frac{\sqrt{4-x^2-4y^2}}{\sqrt{4+12y^2}} \left(\frac{x}{\sqrt{4-x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{4-x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \frac{\sqrt{4-x^2-4y^2}}{2\sqrt{1+3y^2}} \left(\frac{x}{\sqrt{4-x^2-4y^2}} \mathbf{i} + \frac{4y}{\sqrt{4-x^2-4y^2}} \mathbf{j} + \mathbf{k} \right) \\ &= \boxed{\frac{x}{2\sqrt{1+3y^2}} \mathbf{i} + \frac{4y}{2\sqrt{1+3y^2}} \mathbf{j} + \frac{\sqrt{4-x^2-4y^2}}{2\sqrt{1+3y^2}} \mathbf{k}}. \end{aligned}$$

17. A normal vector is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 1 & 0 & 2v \end{vmatrix} = 4uv \mathbf{i} - 2v \mathbf{j} - 2u \mathbf{k}.$$

Then

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{(4uv)^2 + (-2v)^2 + (-2u)^2} = 2\sqrt{u^2 + 4uv + v^2},$$

so that the two unit normals are

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{4uv}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{i} + \frac{-2v}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{j} + \frac{-2u}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{k} \\ -\mathbf{n} &= -\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{-4uv}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{i} + \frac{2v}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{j} + \frac{2u}{2\sqrt{u^2 + 4uv + v^2}} \mathbf{k} \end{aligned}$$

18. A normal vector is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 0 & e^u \\ u & 1 & 0 \end{vmatrix} = -e^u \mathbf{i} + ue^u \mathbf{j} + v \mathbf{k}.$$

Then

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{(-e^u)^2 + (ue^u)^2 + v^2} = \sqrt{(u^2 + 1)e^{2u} + v^2},$$

so that the two unit normals are

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{(u^2 + 1)e^{2u} + v^2}} (-e^u \mathbf{i} + ue^u \mathbf{j} + v \mathbf{k}) \\ -\mathbf{n} &= -\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{(u^2 + 1)e^{2u} + v^2}} (e^u \mathbf{i} - ue^u \mathbf{j} - v \mathbf{k}). \end{aligned}$$

19. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = x \mathbf{i}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$x \mathbf{i}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$x \mathbf{i}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$\mathbf{0}$	0	0
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$1 \mathbf{i}$	1	ρ
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$x \mathbf{i}$	0	0
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$x \mathbf{i}$	0	0

The only integral that has a nonzero integrand is S_4 ; for that integral, we get

$$\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS = \rho \int_0^1 \int_0^1 dy dz = \rho.$$

So the total flux across the surface is $\boxed{\rho}$.

20. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = y \mathbf{i}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$y \mathbf{i}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$\mathbf{0}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$y \mathbf{i}$	$-y$	$-\frac{1}{2}\rho$
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$y \mathbf{i}$	y	$\frac{1}{2}\rho$
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$1 \mathbf{i}$	0	0
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$y \mathbf{i}$	0	0

Only S_3 and S_4 have nonzero integrands, and

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 (-y) dy dz = \rho \int_0^1 \left[-\frac{1}{2}y^2 \right]_0^1 dz = -\frac{1}{2}\rho \int_0^1 1 dz = -\frac{1}{2}\rho \\ \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 y dy dz = \rho \int_0^1 \left[\frac{1}{2}y^2 \right]_0^1 dz = \frac{1}{2}\rho \int_0^1 1 dz = \frac{1}{2}\rho.\end{aligned}$$

So the total flux across the surface is $\boxed{0}$.

21. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = z \mathbf{i}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$\mathbf{0}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$z \mathbf{i}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$z \mathbf{i}$	$-z$	$-\frac{1}{2}\rho$
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$z \mathbf{i}$	z	$\frac{1}{2}\rho$
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$z \mathbf{i}$	0	0
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$\mathbf{i}$	0	0

Only S_3 and S_4 have nonzero integrands, and

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 (-z) dz dy = \rho \int_0^1 \left[-\frac{1}{2}z^2 \right]_0^1 dy = -\frac{1}{2}\rho \int_0^1 1 dy = -\frac{1}{2}\rho \\ \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 z dz dy = \rho \int_0^1 \left[\frac{1}{2}z^2 \right]_0^1 dy = \frac{1}{2}\rho \int_0^1 1 dy = \frac{1}{2}\rho.\end{aligned}$$

So the total flux across the surface is $\boxed{0}$.

22. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$x \mathbf{i} + y \mathbf{j}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$x \mathbf{i}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$y \mathbf{j}$	0	0
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$\mathbf{i} + y \mathbf{j}$	1	ρ
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$x \mathbf{i} + \mathbf{j}$	1	ρ
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$x \mathbf{i} + y \mathbf{j}$	0	0

Only S_4 and S_5 have nonzero integrands, so

$$\begin{aligned}\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dy = \rho \int_0^1 [z]_0^1 dy = \rho \int_0^1 1 dy = \rho \\ \iint_{S_5} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dx = \rho \int_0^1 [z]_0^1 dx = \rho \int_0^1 1 dx = \rho.\end{aligned}$$

So the total flux across the surface is $\boxed{2\rho}$.

23. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$x \mathbf{i} + y \mathbf{j}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$x \mathbf{i} + z \mathbf{k}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$y \mathbf{j} + z \mathbf{k}$	0	0
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$\mathbf{i} + y \mathbf{j} + z \mathbf{k}$	1	ρ
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$x \mathbf{i} + \mathbf{j} + z \mathbf{k}$	1	ρ
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$x \mathbf{i} + y \mathbf{j} + \mathbf{k}$	1	ρ

The values for S_4 , S_5 , and S_6 are determined as follows:

$$\begin{aligned}\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dy = \rho \int_0^1 [z]_0^1 dy = \rho \int_0^1 1 dy = \rho \\ \iint_{S_5} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dx = \rho \int_0^1 [z]_0^1 dx = \rho \int_0^1 1 dx = \rho \\ \iint_{S_6} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dy dx = \rho \int_0^1 [y]_0^1 dx = \rho \int_0^1 1 dx = \rho.\end{aligned}$$

So the total flux across the surface is $\boxed{3\rho}$.

24. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = z^2 \mathbf{i}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$\mathbf{0}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$z^2 \mathbf{i}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$z^2 \mathbf{i}$	$-z^2$	$-\frac{1}{3}\rho$
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$z^2 \mathbf{i}$	z^2	$\frac{1}{3}\rho$
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$z^2 \mathbf{i}$	0	0
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$\mathbf{i}$	0	0

The values for S_3 and S_4 are determined as follows:

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 (-z^2) dz dy = \rho \int_0^1 \left[-\frac{1}{3}z^3\right]_0^1 dy = \rho \int_0^1 \left(-\frac{1}{3}\right) dy = -\frac{1}{3}\rho \\ \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 z^2 dz dx = \rho \int_0^1 \left[\frac{1}{3}z^3\right]_0^1 dx = \rho \int_0^1 \frac{1}{3} dx = \frac{1}{3}\rho.\end{aligned}$$

So the total flux across the surface is $\boxed{0}$.

25. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$x^2 \mathbf{i} + y^2 \mathbf{j}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$x^2 \mathbf{i} + z^2 \mathbf{k}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$y^2 \mathbf{j} + z^2 \mathbf{k}$	0	0
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$\mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$	1	ρ
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$x^2 \mathbf{i} + \mathbf{j} + z^2 \mathbf{k}$	1	ρ
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$x^2 \mathbf{i} + y^2 \mathbf{j} + \mathbf{k}$	1	ρ

The values for S_4 , S_5 , and S_6 are determined as follows:

$$\begin{aligned}\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dy = \rho \int_0^1 [z]_0^1 dy = \rho \int_0^1 1 dy = \rho \\ \iint_{S_5} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dx = \rho \int_0^1 [z]_0^1 dx = \rho \int_0^1 1 dx = \rho \\ \iint_{S_6} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dy dx = \rho \int_0^1 [y]_0^1 dx = \rho \int_0^1 1 dx = \rho.\end{aligned}$$

So the total flux across the surface is $\boxed{3\rho}$.

26. Using the notation in Example 8 and Figure 66, we construct a similar table, with $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j}$:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$	$\rho \iint_S \mathbf{F} \cdot \mathbf{n} dS$
$S_1 = ABCO$ $z = 0$	$-\mathbf{k}$	$x^2 \mathbf{i} + y^2 \mathbf{j}$	0	0
$S_2 = OAEG$ $y = 0$	$-\mathbf{j}$	$x^2 \mathbf{i}$	0	0
$S_3 = OCDG$ $x = 0$	$-\mathbf{i}$	$y^2 \mathbf{j}$	0	0
$S_4 = ABFE$ $x = 1$	$\mathbf{i}$	$\mathbf{i} + y^2 \mathbf{j}$	1	ρ
$S_5 = BCDF$ $y = 1$	$\mathbf{j}$	$x^2 \mathbf{i} + \mathbf{j}$	1	ρ
$S_6 = DFEG$ $z = 1$	$\mathbf{k}$	$x^2 \mathbf{i} + y^2 \mathbf{j}$	0	0

The values for S_4 and S_5 are determined as follows:

$$\begin{aligned} \iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dy = \rho \int_0^1 [z]_0^1 dy = \rho \int_0^1 1 dy = \rho \\ \iint_{S_5} \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^1 \int_0^1 1 dz dx = \rho \int_0^1 [z]_0^1 dx = \rho \int_0^1 1 dx = \rho. \end{aligned}$$

So the total flux across the surface is $\boxed{2\rho}$.

Applications and Extensions

27. The surface is the upper hemisphere of the unit sphere. Parametrizing using spherical coordinates, we get

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = -\cos \theta \sin^2 \phi \mathbf{i} - \sin \theta \sin^2 \phi \mathbf{j} - \cos \phi \sin \phi \mathbf{k}.$$

We want the outward normal, so we use instead

$$-(\mathbf{r}_\theta \times \mathbf{r}_\phi) = \cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}.$$

Then with $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}$ the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^{\pi/2} \mathbf{F} \cdot \frac{\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} \cdot \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}) \cdot (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\cos^2 \theta \sin^3 \phi + \sin^2 \theta \sin^3 \phi + \sin \phi \cos^2 \phi) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\sin^3 \phi + \sin \phi \cos^2 \phi) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\sin \phi (\sin^2 \phi + \cos^2 \phi)) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} [-\cos \phi]_0^{\pi/2} d\theta = \int_0^{2\pi} 1 d\theta = \boxed{2\pi}. \end{aligned}$$

28. The surface is the upper hemisphere of the unit sphere. Parametrizing using spherical coordinates, we get

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = -\cos \theta \sin^2 \phi \mathbf{i} - \sin \theta \sin^2 \phi \mathbf{j} - \cos \phi \sin \phi \mathbf{k}.$$

We want the outward normal, so we use instead

$$-(\mathbf{r}_\theta \times \mathbf{r}_\phi) = \cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}.$$

Then with $\mathbf{F} = -y \mathbf{i} + x \mathbf{j} + z \mathbf{k} = -\sin \theta \sin \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}$ the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^{\pi/2} \mathbf{F} \cdot \frac{\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} \cdot \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (-\sin \theta \sin \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}) \cdot (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \cos \phi \sin \phi \mathbf{k}) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (-\sin \theta \cos \theta \sin^3 \phi + \sin \theta \cos \theta \sin^3 \phi + \sin \phi \cos^2 \phi) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \cos^2 \phi d\phi d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} d\theta = \boxed{\frac{2}{3}\pi}. \end{aligned}$$

29. The region in the xy -plane over which this surface lies is determined by setting $z = 0$ in the equation, giving $2x + 2y = 8$, or $y = 4 - x$; this line, together with the x and y -axes, bounds the region:



Therefore the region of integration will be $0 \leq x \leq 4$, $0 \leq y \leq 4 - x$. Parametrize the surface by $x = u$, $y = v$, $z = 8 - 2u - 2v$. Then

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (8 - 2u - 2v) \mathbf{k},$$

so that

$$\mathbf{r}_u = \mathbf{i} - 2\mathbf{k}, \quad \mathbf{r}_v = \mathbf{j} - 2\mathbf{k}.$$

Therefore

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Then $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2^2 + 2^2 + 1^2} = 3$; since the $\mathbf{k}$ -component of the cross product is positive, this is the normal we want. So the unit normal is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}).$$

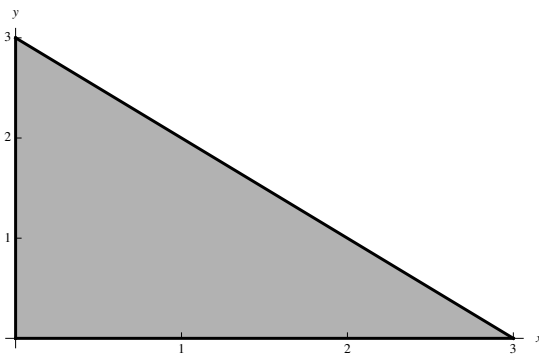
The vector field is

$$\mathbf{F} = (x+y)\mathbf{i} + (2x-z)\mathbf{j} + y\mathbf{k} = (u+v)\mathbf{i} + (2u - (8-2u-2v))\mathbf{j} + v\mathbf{k} = (u+v)\mathbf{i} + (4u+2v-8)\mathbf{j} + v\mathbf{k}.$$

It follows that the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^4 \int_0^{4-u} ((u+v)\mathbf{i} + (4u+2v-8)\mathbf{j} + v\mathbf{k}) \cdot \left(\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\right) \cdot 3 \, dv \, du \\ &= \int_0^4 \int_0^{4-u} ((u+v)\mathbf{i} + (4u+2v-8)\mathbf{j} + v\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \, dv \, du \\ &= \int_0^4 \int_0^{4-u} (2(u+v) + 2(4u+2v-8) + v) \, dv \, du \\ &= \int_0^4 \int_0^{4-u} (10u + 7v - 16) \, dv \, du \\ &= \int_0^4 \left[10uv + \frac{7}{2}v^2 - 16v \right]_0^{4-u} \, du \\ &= \int_0^4 \left(10u(4-u) + \frac{7}{2}(4-u)^2 - 16(4-u) \right) \, du \\ &= \int_0^4 \left(-\frac{13}{2}u^2 + 28u - 8 \right) \, du \\ &= \left[-\frac{13}{6}u^3 + 14u^2 - 8u \right]_0^4 = \boxed{\frac{160}{3}}. \end{aligned}$$

30. The region in the xy -plane over which this surface lies is determined by setting $z = 0$ in the equation, giving $2x + 2y = 6$, or $y = 3 - x$; this line, together with the x and y -axes, bounds the region:



Therefore the region of integration will be $0 \leq x \leq 3$, $0 \leq y \leq 3 - x$. Parametrize the surface by $x = u$, $y = v$, $z = 6 - 2u - 2v$. Then

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - 2u - 2v)\mathbf{k},$$

so that

$$\mathbf{r}_u = \mathbf{i} - 2\mathbf{k}, \quad \mathbf{r}_v = \mathbf{j} - 2\mathbf{k}.$$

Therefore

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Then $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2^2 + 2^2 + 1^2} = 3$; since the $\mathbf{k}$ -component of the cross product is positive, this is the normal we want. So the unit normal is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}).$$

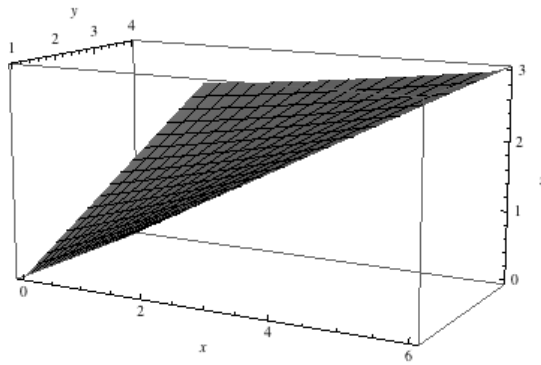
The vector field is

$$\mathbf{F} = 2x\mathbf{i} - x^2\mathbf{j} + (z - 2x + 2y)\mathbf{k} = 2u\mathbf{i} - u^2\mathbf{j} + (6 - 2u - 2v - 2u + 2v)\mathbf{k} = 2u\mathbf{i} - u^2\mathbf{j} + (6 - 4u)\mathbf{k}.$$

It follows that the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^3 \int_0^{3-u} (2u\mathbf{i} - u^2\mathbf{j} + (6 - 4u)\mathbf{k}) \cdot \left(\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\right) \cdot 3 \, dv \, du \\ &= \int_0^3 \int_0^{3-u} (2u\mathbf{i} - u^2\mathbf{j} + (6 - 4u)\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \, dv \, du \\ &= \int_0^3 \int_0^{3-u} (4u - 2u^2 + 6 - 4u) \, dv \, du \\ &= \int_0^3 \int_0^{3-u} (6 - 2u^2) \, dv \, du \\ &= \int_0^3 (6 - 2u^2)(3 - u) \, du \\ &= \int_0^3 (2u^3 - 6u^2 - 6u + 18) \, du \\ &= \left[\frac{1}{2}u^4 - 2u^3 - 3u^2 + 18u \right]_0^3 = \boxed{\frac{27}{2}}. \end{aligned}$$

31. The surface is



Differentiating gives

$$\mathbf{r}_u = v\mathbf{i} + 2u\mathbf{j}, \quad \mathbf{r}_v = u\mathbf{i} + \mathbf{k}.$$

Therefore

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 2u & 0 \\ u & 0 & 1 \end{vmatrix} = 2u\mathbf{i} - v\mathbf{j} - 2u^2\mathbf{k}.$$

We can choose either orientation; we choose the one given by this normal vector (if we choose the other orientation, we will get the negative of the answer below). Then $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4u^2 + v^2 + 4u^4}$. So the unit normal is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{4u^2 + v^2 + 4u^4}}(2u\mathbf{i} - v\mathbf{j} - 2u^2\mathbf{k}).$$

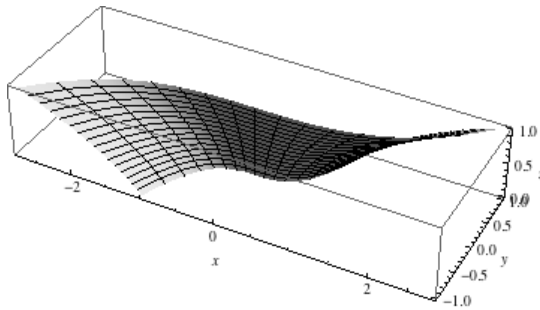
The vector field is

$$\mathbf{F} = -z\mathbf{j} + \mathbf{k} = -v\mathbf{j} + \mathbf{k}.$$

It follows that the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_1^2 \int_0^3 (-v\mathbf{j} + \mathbf{k}) \cdot \frac{1}{\sqrt{4u^2 + v^2 + 4u^4}} (2u\mathbf{i} - v\mathbf{j} - 2u^2\mathbf{k}) \cdot \sqrt{4u^2 + v^2 + 4u^4} dv du \\ &= \int_1^2 \int_0^3 (-v\mathbf{j} + \mathbf{k}) \cdot (2u\mathbf{i} - v\mathbf{j} - 2u^2\mathbf{k}) dv du \\ &= \int_1^2 \int_0^3 (v^2 - 2u^2) dv du \\ &= \int_1^2 \left[\frac{1}{3}v^3 - 2u^2v \right]_0^3 du \\ &= \int_1^2 (9 - 6u^2) du \\ &= [9u - 2u^3]_1^2 = \boxed{-5}. \end{aligned}$$

32. The surface is



Differentiating gives

$$\mathbf{r}_u = ve^u\mathbf{i} + \mathbf{k}, \quad \mathbf{r}_v = e^u\mathbf{i} + 3v^2\mathbf{j}.$$

Therefore

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ve^u & 0 & 1 \\ e^u & 3v^2 & 0 \end{vmatrix} = -3v^2\mathbf{i} + e^u\mathbf{j} + 3e^uv^3\mathbf{k}.$$

We can choose either orientation; we choose the one given by this normal vector (if we choose the other orientation, we will get the negative of the answer below). Then $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9v^4 + e^{2u}(1 + 9v^6)}$. So the unit normal is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{1}{\sqrt{9v^4 + e^{2u}(1 + 9v^6)}} (-3v^2\mathbf{i} + e^u\mathbf{j} + 3e^uv^3\mathbf{k}).$$

The vector field is

$$\mathbf{F} = xy\mathbf{i} + 2z\mathbf{j} = v^4e^u\mathbf{i} + 2u\mathbf{j}.$$

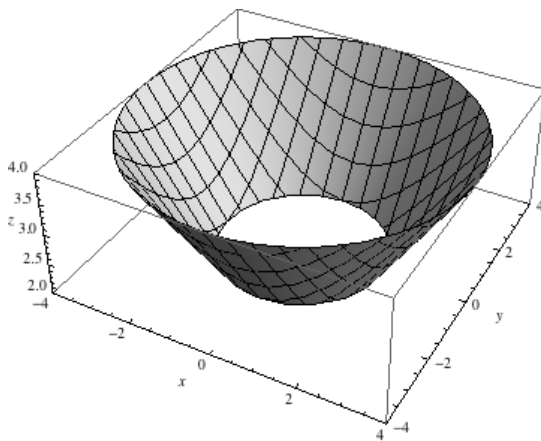
It follows that the flux is

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^1 \int_{-1}^1 (v^4 e^u \mathbf{i} + 2u \mathbf{j}) \cdot \frac{1}{\sqrt{9v^4 + e^{2u}(1 + 9v^6)}} (-3v^2 \mathbf{i} + e^u \mathbf{j} + 3e^u v^3 \mathbf{k}) \\
 &\quad \cdot \sqrt{9v^4 + e^{2u}(1 + 9v^6)} \, dv \, du \\
 &= \int_0^1 \int_{-1}^1 (v^4 e^u \mathbf{i} + 2u \mathbf{j}) \cdot (-3v^2 \mathbf{i} + e^u \mathbf{j} + 3e^u v^3 \mathbf{k}) \, dv \, du \\
 &= \int_0^1 \int_{-1}^1 (-3v^6 e^u + 2ue^u) \, dv \, du \\
 &= \int_0^1 \left[-\frac{3}{7} v^7 e^u + 2uv e^u \right]_{-1}^1 \, du \\
 &= \int_0^1 \left(4ue^u - \frac{6}{7} e^u \right) \, du.
 \end{aligned}$$

To integrate ue^u , use integration by parts with $r = u$ and $ds = e^u du$; then $dr = du$ and $s = e^u$, so we get

$$\begin{aligned}
 &= 4 \int_0^1 ue^u \, du - \frac{6}{7} \int_0^1 e^u \, du \\
 &= 4 \left([ue^u]_0^1 - \int_0^1 e^u \, du \right) - \frac{6}{7} \int_0^1 e^u \, du \\
 &= 4e - \frac{34}{7} \int_0^1 e^u \, du \\
 &= 4e - \frac{34}{7} [e^u]_0^1 = \boxed{-\frac{6}{7}e + \frac{34}{7}}.
 \end{aligned}$$

33. The surface is



Parametrize using cylindrical coordinates; the equation of the cone is $z = r$. Then the range $2 \leq z \leq 4$ corresponds to $2 \leq r \leq 4$, so the parametrization is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k}.$$

Differentiating gives

$$\mathbf{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \mathbf{k}, \quad \mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}.$$

Therefore

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} + r \mathbf{k}.$$

Then $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = r\sqrt{2}$. The mass density of the lamina is $\rho = 8 - z = 8 - r$, so the mass of the lamina is

$$\begin{aligned} M &= \iint_S \rho \, dS = \iint_R \rho \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, dr \, d\theta \\ &= \iint_R (8 - r) \cdot r\sqrt{2} \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_2^4 (8r - r^2) \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[4r^2 - \frac{1}{3}r^3 \right]_2^4 \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \frac{88}{3} \, d\theta = \boxed{\frac{176}{3} \pi \sqrt{2}}. \end{aligned}$$

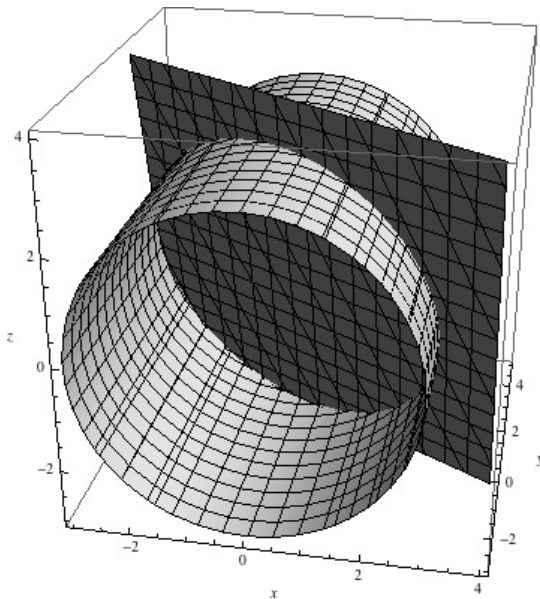
34. The surface is the same as in Problem 33, so the parametrization is the same:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k},$$

as is $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = r\sqrt{2}$. See Problem 33 for details. The mass density of the lamina is $\rho = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$, so the mass of the lamina is

$$\begin{aligned} M &= \iint_S \rho \, dS = \iint_R \rho \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, dr \, d\theta \\ &= \iint_R r \cdot r\sqrt{2} \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_2^4 r^2 \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{1}{3}r^3 \right]_2^4 \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \frac{56}{3} \, d\theta = \boxed{\frac{112}{3} \pi \sqrt{2}}. \end{aligned}$$

35. The surface is



Parametrize using a variant of cylindrical coordinates:

$$x = r \cos \theta, \quad y = y, \quad z = r \sin \theta.$$

Then since $y = \frac{1}{3}(4 - 2x - z)$, the equation of the plane is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + \frac{1}{3}(4 - 2r \cos \theta - r \sin \theta) \mathbf{j} + r \sin \theta \mathbf{k}.$$

Then

$$\mathbf{r}_r = \cos \theta \mathbf{i} + \frac{1}{3}(-2 \cos \theta - \sin \theta) \mathbf{j} + \sin \theta \mathbf{k}, \quad \mathbf{r}_\theta = -r \sin \theta \mathbf{i} + \frac{1}{3}(2r \sin \theta - r \cos \theta) \mathbf{j} + r \cos \theta \mathbf{k},$$

so that

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \frac{1}{3}(-2 \cos \theta - \sin \theta) & \sin \theta \\ -r \sin \theta & \frac{1}{3}(2r \sin \theta - r \cos \theta) & r \cos \theta \end{vmatrix} = -\frac{2r}{3} \mathbf{i} - r \mathbf{j} - \frac{r}{3} \mathbf{k}.$$

Then

$$\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \sqrt{\frac{4r^2}{9} + r^2 + \frac{r^2}{9}} = r \frac{\sqrt{14}}{3}.$$

Since we want the portion of the plane lying inside the cylinder $x^2 + z^2 = r^2 = 9$, the region describing that portion of the plane is $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$. Also, the mass density of the lamina is $\rho = \sqrt{x^2 + z^2} = \sqrt{r^2} = r$.

Then the mass of the lamina is

$$\begin{aligned}
 M &= \iint_S \rho \, dS = \iint_R \rho \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, dr \, d\theta \\
 &= \iint_R r \cdot r \frac{\sqrt{14}}{3} \, dr \, d\theta \\
 &= \frac{\sqrt{14}}{3} \int_0^{2\pi} \int_0^3 r^2 \, dr \, d\theta \\
 &= \frac{\sqrt{14}}{3} \int_0^{2\pi} \left[\frac{1}{3} r^3 \right]_0^3 \, d\theta \\
 &= \frac{\sqrt{14}}{3} \int_0^{2\pi} 9 \, d\theta = 6\pi\sqrt{14}.
 \end{aligned}$$

The moments are

$$\begin{aligned}
 M_{xy} &= \iint_S \rho z \, dS = \iint_R \rho r \sin \theta \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, dr \\
 &= \iint_R r \cdot r \sin \theta \cdot r \frac{\sqrt{14}}{3} \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \int_0^{2\pi} r^3 \sin \theta \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{3} \int_0^3 [-r^3 \cos \theta]_0^{2\pi} \, dr = 0 \\
 M_{yz} &= \iint_S \rho x \, dS = \iint_R \rho r \cos \theta \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, dr \\
 &= \iint_R r \cdot r \cos \theta \cdot r \frac{\sqrt{14}}{3} \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{3} \int_0^3 \int_0^{2\pi} r^3 \cos \theta \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{3} \int_0^3 [r^3 \sin \theta]_0^{2\pi} \, dr = 0 \\
 M_{xz} &= \iint_S \rho y \, dS = \iint_R \rho \left(\frac{1}{3}(4 - 2r \cos \theta - r \sin \theta) \right) \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, dr \\
 &= \frac{1}{3} \iint_R r(4 - 2r \cos \theta - r \sin \theta) \cdot r \frac{\sqrt{14}}{3} \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{9} \int_0^3 \int_0^{2\pi} (4r^2 - 2r^3 \cos \theta - r^3 \sin \theta) \, d\theta \, dr \\
 &= \frac{\sqrt{14}}{9} \int_0^3 [4r^2\theta - 2r^3 \sin \theta + r^3 \cos \theta]_0^{2\pi} \, dr \\
 &= \frac{\sqrt{14}}{9} \int_0^3 8\pi r^2 \, dr \\
 &= \frac{\sqrt{14}}{9} \left[\frac{8}{3} \pi r^3 \right]_0^3 = 8\pi\sqrt{14}.
 \end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) = \boxed{\left(0, \frac{4}{3}, 0 \right)}.$$

36. Parametrize the unit sphere using spherical coordinates, so that

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}.$$

Then

$$\mathbf{r}_\theta = -\sin \theta \sin \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j}, \quad \mathbf{r}_\phi = \cos \theta \cos \phi \mathbf{i} + \sin \theta \cos \phi \mathbf{j} - \sin \phi \mathbf{k},$$

so that

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = -\cos \theta \sin^2 \phi \mathbf{i} - \sin \theta \sin^2 \phi \mathbf{j} - \cos \phi \sin \phi \mathbf{k}.$$

Then

$$\begin{aligned} \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{\cos^2 \theta \sin^4 \phi + \sin^2 \theta \sin^4 \phi + \cos^2 \phi \sin^2 \phi} \\ &= \sqrt{(\cos^2 \theta + \sin^2 \theta) \sin^4 \phi + \cos^2 \phi \sin^2 \phi} \\ &= \sqrt{(\sin^2 \phi + \cos^2 \phi) \sin^2 \phi} = \sin \phi \end{aligned}$$

since $0 \leq \phi \leq \pi$ and $\sin \phi \geq 0$ there. The mass density of the lamina is $\rho = z + 1 = 1 + \cos \phi$.

Therefore the mass of the lamina is

$$\begin{aligned} M &= \iint_S \rho \, dS = \iint_R \rho \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\ &= \iint_R (1 + \cos \phi) \sin \phi \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} (\sin \phi + \cos \phi \sin \phi) \, d\theta \, d\phi \\ &= 2\pi \int_0^\pi (\sin \phi + \cos \phi \sin \phi) \, d\phi \\ &= 2\pi \left[-\cos \phi + \frac{1}{2} \sin^2 \phi \right]_0^\pi = 4\pi. \end{aligned}$$

The moments are

$$\begin{aligned} M_{xy} &= \iint_S \rho z \, dS = \iint_R \rho \cos \phi \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} (1 + \cos \phi) \cos \phi \sin \phi \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} (\cos \phi \sin \phi + \cos^2 \phi \sin \phi) \, d\theta \, d\phi \\ &= 2\pi \int_0^\pi (\cos \phi \sin \phi + \cos^2 \phi \sin \phi) \, d\phi \\ &= 2\pi \left[\frac{1}{2} \sin^2 \phi - \frac{1}{3} \cos^3 \phi \right]_0^\pi = \frac{4}{3} \pi \end{aligned}$$

$$\begin{aligned}
M_{yz} &= \iint_S \rho x \, dS = \iint_R \rho \cos \theta \sin \phi \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\
&= \int_0^\pi \int_0^{2\pi} (1 + \cos \phi) \cos \theta \sin \phi \sin \phi \, d\theta \, d\phi \\
&= \int_0^\pi \int_0^{2\pi} \cos \theta (\sin^2 \phi + \cos \phi \sin^2 \phi) \, d\theta \, d\phi \\
&= \int_0^\pi [\sin \theta (\sin^2 \phi + \cos \phi \sin^2 \phi)]_0^{2\pi} \, d\phi \\
&= \int_0^\pi 0 \, d\phi = 0
\end{aligned}$$

$$\begin{aligned}
M_{xz} &= \iint_S \rho y \, dS = \iint_R \rho \sin \theta \sin \phi \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\
&= \int_0^\pi \int_0^{2\pi} (1 + \cos \phi) \sin \theta \sin \phi \sin \phi \, d\theta \, d\phi \\
&= \int_0^\pi \int_0^{2\pi} \sin \theta (\sin^2 \phi + \cos \phi \sin^2 \phi) \, d\theta \, d\phi \\
&= \int_0^\pi [-\cos \theta (\sin^2 \phi + \cos \phi \sin^2 \phi)]_0^{2\pi} \, d\phi \\
&= \int_0^\pi 0 \, d\phi = 0.
\end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) = \boxed{\left(0, 0, \frac{1}{3} \right)}.$$

37. Parametrize the cylinder using cylindrical coordinates; then

$$x = 3 \cos \theta, \quad y = 3 \sin \theta, \quad z = z, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 5.$$

Then

$$\mathbf{r}(\theta, z) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + z \mathbf{k},$$

so that

$$\mathbf{r}_\theta = -3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j}, \quad \mathbf{r}_z = \mathbf{k}.$$

Then

$$\begin{aligned}
\mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \sin \theta & 3 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} \\
\|\mathbf{r}_\theta \times \mathbf{r}_z\| &= \sqrt{9 \cos^2 \theta + 9 \sin^2 \theta} = 3.
\end{aligned}$$

Then the surface integral is

$$\begin{aligned}
 \iint_S x^2 z \, dS &= \iint_R (3 \cos \theta)^2 \cdot z \cdot 3 \, dz \, d\theta \\
 &= \int_0^{2\pi} \int_0^5 27z \cos^2 \theta \, dz \, d\theta \\
 &= \int_0^{2\pi} \left[\frac{27}{2} z^2 \cos^2 \theta \right]_0^5 d\theta \\
 &= \frac{675}{2} \int_0^{2\pi} \cos^2 \theta \, d\theta \\
 &= \frac{675}{2} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \\
 &= \frac{675}{2} \left[\frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} = \boxed{\frac{675}{2} \pi}.
 \end{aligned}$$

38. With the given parametrization, we have

$$\mathbf{r}_u = \cos v \mathbf{i} + 2u \mathbf{j} + \cos v \mathbf{k}, \quad \mathbf{r}_v = -u \sin v \mathbf{i} - u \sin v \mathbf{k}.$$

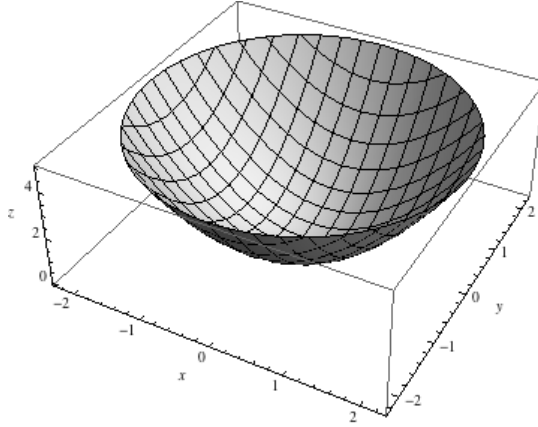
Then

$$\begin{aligned}
 \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 2u & \cos v \\ -u \sin v & 0 & -u \sin v \end{vmatrix} = -2u^2 \sin v \mathbf{i} + 2u^2 \sin v \mathbf{k} \\
 \|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{4u^4 \sin^2 v + 4u^4 \sin^2 v} = 2u^2 \sqrt{2} \sin v.
 \end{aligned}$$

Then the surface integral is

$$\begin{aligned}
 \iint_S (x + 2y) \, dS &= \iint_R (u \cos v + 2(u^2 + 1)) \cdot 2u^2 \sqrt{2} \sin v \, dv \, du \\
 &= \sqrt{2} \int_0^1 \int_0^{\pi/2} (2u^3 \cos v \sin v + 4u^2(u^2 + 1) \sin v) \, dv \, du \\
 &= \sqrt{2} \int_0^1 \left[2u^3 \cdot \frac{1}{2} \sin^2 v + 4u^2(u^2 + 1)(-\cos v) \right]_0^{\pi/2} du \\
 &= \sqrt{2} \int_0^1 (u^3 + 4u^2(u^2 + 1)) \, du \\
 &= \sqrt{2} \int_0^1 (4u^4 + u^3 + 4u^2) \, du \\
 &= \sqrt{2} \left[\frac{4}{5} u^5 + \frac{1}{4} u^4 + \frac{4}{3} u^3 \right]_0^1 \\
 &= \boxed{\frac{143}{60} \sqrt{2}}.
 \end{aligned}$$

39. The surface is



Parametrize the surface using cylindrical coordinates; then $z = r^2$ and we have

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r^2 \mathbf{k}.$$

Differentiating gives

$$\mathbf{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 2r \mathbf{k}, \quad \mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}.$$

Therefore

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}.$$

We use this as the outward (upward) normal, so that

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} = \frac{-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|}.$$

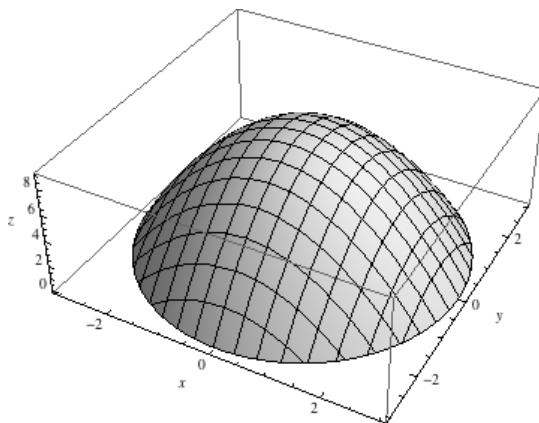
The vector field is

$$\mathbf{F} = -x \mathbf{i} - y \mathbf{j} - z \mathbf{k} = -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} - r^2 \mathbf{k}.$$

Let the mass density of the fluid be ρ . It follows that the flux is

$$\begin{aligned} \iint_S \rho \mathbf{F} \cdot \mathbf{n} dS &= \rho \int_0^{2\pi} \int_0^{\sqrt{5}} (-r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} - r^2 \mathbf{k}) \cdot \frac{-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| dr d\theta \\ &= \rho \int_0^{2\pi} \int_0^{\sqrt{5}} (-r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} - r^2 \mathbf{k}) \cdot (-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) dr d\theta \\ &= \rho \int_0^{2\pi} \int_0^{\sqrt{5}} (2r^3 \cos^2 \theta + 2r^3 \sin^2 \theta - r^3) dr d\theta \\ &= \rho \int_0^{2\pi} \int_0^{\sqrt{5}} r^3 dr d\theta \\ &= \rho \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^{\sqrt{5}} d\theta \\ &= \rho \int_0^{2\pi} \frac{25}{4} d\theta = \boxed{\frac{25}{2} \pi \rho}. \end{aligned}$$

40. The surface is



Parametrize the surface using cylindrical coordinates; then $z = 9 - r^2$ and we have

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (9 - r^2) \mathbf{k}.$$

Differentiating gives

$$\mathbf{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} - 2r \mathbf{k}, \quad \mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}.$$

Therefore

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = 2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}.$$

We use this as the outward (upward) normal, so that

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} = \frac{2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|}.$$

The vector field is

$$\mathbf{F} = x \mathbf{i} + y \mathbf{j} + 3 \mathbf{k} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 3 \mathbf{k}.$$

It follows that the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^3 (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 3 \mathbf{k}) \cdot \frac{2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 3 \mathbf{k}) \cdot (2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (2r^3 \cos^2 \theta + 2r^3 \sin^2 \theta + 3r) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (2r^3 + 3r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2} r^4 + \frac{3}{2} r^2 \right]_0^3 \, d\theta \\ &= \int_0^{2\pi} 54 \, d\theta = \boxed{108\pi}. \end{aligned}$$

41. From equation (3) in Objective 2, the upward pointing normal $\mathbf{n}$ to $z = f(x, y)$ is

$$\mathbf{n} = \frac{-f_x(x, y) \mathbf{i} - f_y(x, y) \mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Use the natural parametrization of the surface $x = x$, $y = y$, $z = f(x, y)$; then

$$\begin{aligned}\mathbf{r}(x, y) &= x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k} \\ \mathbf{r}_x &= \mathbf{i} + f_x(x, y)\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + f_y(x, y)\mathbf{k} \\ \mathbf{r}_x \times \mathbf{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k} \\ \|\mathbf{r}_x \times \mathbf{r}_y\| &= \sqrt{(-f_x(x, y))^2 + (-f_y(x, y))^2 + 1} = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}.\end{aligned}$$

Therefore the integral is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \frac{-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}} \\ &\quad \cdot \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dx \, dy \\ &= \iint_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (-f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}) \, dx \, dy \\ &= \iint_D (-Pf_x(x, y) - Qf_y(x, y) + R) \, dx \, dy \\ &= \iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R \right) \, dx \, dy.\end{aligned}$$

42. Parametrizing the cylinder using cylindrical coordinates gives

$$\mathbf{r}(\theta, z) = 3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j} + z\mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 2.$$

Then

$$\begin{aligned}\mathbf{r}_\theta &= -3\sin\theta\mathbf{i} + 3\cos\theta\mathbf{j}, \quad \mathbf{r}_z = \mathbf{k} \\ \mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j} \\ \|\mathbf{r}_\theta \times \mathbf{r}_z\| &= 3.\end{aligned}$$

It follows that the flux is

$$\begin{aligned}\iint_S \mathbf{E} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^3 (3\cos\theta\mathbf{i} + 2 \cdot 3\sin\theta\mathbf{j} + 3z\mathbf{k}) \cdot \frac{3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}}{3} \cdot 3 \, dz \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (3\cos\theta\mathbf{i} + 6\sin\theta\mathbf{j} + 3z\mathbf{k}) \cdot (3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}) \, dz \, d\theta \\ &= \int_0^{2\pi} \int_0^3 (9\cos^2\theta + 18\sin^2\theta) \, dz \, d\theta \\ &= 27 \int_0^{2\pi} (\cos^2\theta + 2\sin^2\theta) \, d\theta \\ &= 27 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) + 1 - \cos(2\theta) \right) \, d\theta \\ &= 27 \int_0^{2\pi} \left(\frac{3}{2} - \frac{1}{2}\cos(2\theta) \right) \, d\theta \\ &= 27 \left[\frac{3}{2}\theta - \frac{1}{4}\sin(2\theta) \right]_0^{2\pi} = \boxed{81\pi}.\end{aligned}$$

43. Given the parametrization, we have

$$\begin{aligned}\mathbf{r}_u &= \cos(2v)\mathbf{i} + \sin(2v)\mathbf{j}, & \mathbf{r}_v &= -2u\sin(2v)\mathbf{i} + 2u\cos(2v)\mathbf{j} + \mathbf{k} \\ \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(2v) & \sin(2v) & 0 \\ -2u\sin(2v) & 2u\cos(2v) & 1 \end{vmatrix} = \sin(2v)\mathbf{i} - \cos(2v)\mathbf{j} + 2u\mathbf{k} \\ \|\mathbf{r}_\theta \times \mathbf{r}_z\| &= \sqrt{4u^2 + 1}.\end{aligned}$$

Then the surface integral is

$$\begin{aligned}\iint_S (x^2 + z^2) dS &= \int_0^3 \int_0^{2\pi} (u^2 \cos^2(2v) + v^2) \cdot \sqrt{4u^2 + 1} dv du \\ &= \int_0^3 \int_0^{2\pi} \left(u^2 \left(\frac{1}{2} + \frac{1}{2} \cos(4v) \right) + v^2 \right) \cdot \sqrt{4u^2 + 1} dv du \\ &= \int_0^3 \left[\frac{1}{2} u^2 v + \frac{1}{8} u^2 \sin(4v) + \frac{1}{3} v^3 \right]_0^{2\pi} \sqrt{4u^2 + 1} du \\ &= \int_0^3 \left(\pi u^2 + \frac{8}{3} \pi^3 \right) \sqrt{4u^2 + 1} du \\ &= \pi \int_0^3 u^2 \sqrt{4u^2 + 1} du + \frac{8}{3} \pi^3 \int_0^3 \sqrt{4u^2 + 1} du.\end{aligned}$$

For both integrals use the substitution $t = 2u$, so that $dt = 2 du$; then $u = 0$ corresponds to $t = 0$ and $u = 3$ to $t = 6$. We get

$$\begin{aligned}&= \frac{\pi}{2} \int_0^6 \left(\frac{t}{2} \right)^2 \sqrt{t^2 + 1} dt + \frac{4}{3} \pi^3 \int_0^6 \sqrt{t^2 + 1} dt \\ &= \frac{\pi}{8} \int_0^6 t^2 \sqrt{t^2 + 1} dt + \frac{4}{3} \pi^3 \int_0^6 \sqrt{t^2 + 1} dt\end{aligned}$$

Now use Formulas 49 and 51 from the integral formulas in the Endsheets, giving

$$\begin{aligned}&= \frac{\pi}{8} \left[\frac{t}{8} (2t^2 + 1) \sqrt{t^2 + 1} - \frac{1}{8} \ln |t + \sqrt{t^2 + 1}| \right]_0^6 \\ &\quad + \frac{4}{3} \pi^3 \left[\frac{t}{2} \sqrt{t^2 + 1} + \frac{1}{2} \ln |t + \sqrt{t^2 + 1}| \right]_0^6 \\ &= \frac{\pi}{8} \left(\frac{3}{4} \cdot 73 \cdot \sqrt{37} - \frac{1}{8} \ln(6 + \sqrt{37}) \right) + \frac{4}{3} \pi^3 \left(3\sqrt{37} + \frac{1}{2} \ln(6 + \sqrt{37}) \right) \\ &= \boxed{\left(\frac{219}{32} \pi + 4\pi^3 \right) \sqrt{37} + \left(\frac{2}{3} \pi^3 - \frac{1}{64} \pi \right) \ln(6 + \sqrt{37})}.\end{aligned}$$

Challenge Problems

44. Since both the surface and the density function are symmetric with respect to the x and y coordinates, we have $\bar{x} = \bar{y} = 0$, so we need only compute $\bar{z}$. Parametrize the hemisphere using spherical coordinates:

$$x = a \cos \theta \sin \phi, \quad y = a \sin \theta \sin \phi, \quad z = a \cos \phi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2},$$

so that

$$\mathbf{r}(\theta, \phi) = a \cos \theta \sin \phi \mathbf{i} + a \sin \theta \sin \phi \mathbf{j} + a \cos \phi \mathbf{k}.$$

Then

$$\mathbf{r}_\theta = -a \sin \theta \sin \phi \mathbf{i} + a \cos \theta \sin \phi \mathbf{j}, \quad \mathbf{r}_\phi = a \cos \theta \cos \phi \mathbf{i} + a \sin \theta \cos \phi \mathbf{j} - a \sin \phi \mathbf{k},$$

so that

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin \theta \sin \phi & a \cos \theta \sin \phi & 0 \\ a \cos \theta \cos \phi & a \sin \theta \cos \phi & -a \sin \phi \end{vmatrix} = -a^2 \cos \theta \sin^2 \phi \mathbf{i} - a^2 \sin \theta \sin^2 \phi \mathbf{j} - a^2 \cos \phi \sin \phi \mathbf{k}.$$

Then

$$\begin{aligned} \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{a^4 \cos^2 \theta \sin^4 \phi + a^4 \sin^2 \theta \sin^4 \phi + a^4 \cos^2 \phi \sin^2 \phi} \\ &= a^2 \sqrt{(\cos^2 \theta + \sin^2 \theta) \sin^4 \phi + \cos^2 \phi \sin^2 \phi} \\ &= a^2 \sqrt{(\sin^2 \phi + \cos^2 \phi) \sin^2 \phi} = a^2 \sin \phi \end{aligned}$$

since $0 \leq \phi \leq \frac{\pi}{2}$ and $\sin \phi \geq 0$ there. The mass density of the lamina is the distance from the xy -plane, which is just the z -coordinate, so that $\rho = z = \cos \phi$.

Therefore the mass of the lamina is

$$\begin{aligned} M &= \iint_S \rho \, dS = \iint_R \rho \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\ &= \iint_R \cos \phi \cdot a^2 \sin \phi \, d\theta \, d\phi \\ &= a^2 \int_0^{\pi/2} \int_0^{2\pi} \cos \phi \sin \phi \, d\theta \, d\phi \\ &= 2\pi a^2 \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \\ &= 2\pi a^2 \left[\frac{1}{2} \sin^2 \phi \right]_0^{\pi/2} = \pi a^2. \end{aligned}$$

The xy moment is

$$\begin{aligned} M_{xy} &= \iint_S \rho z \, dS = \iint_R \rho a \cos \phi \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, d\phi \\ &= \int_0^{\pi/2} \int_0^{2\pi} \cos \phi \cdot a \cos \phi \cdot a^2 \sin \phi \, d\theta \, d\phi \\ &= a^3 \int_0^{\pi/2} \int_0^{2\pi} \cos^2 \phi \sin \phi \, d\theta \, d\phi \\ &= 2\pi a^3 \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \\ &= 2\pi a^3 \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} = \frac{2}{3} \pi a^3. \end{aligned}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) = \boxed{\left(0, 0, \frac{2a}{3} \right)}.$$

45. Suppose the level surface S is $f(x, y, z) = a$ for some constant a . Then

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

But from The Gradient is Normal to the Level Curve Theorem in Section 13.1, generalized to one additional dimension, we see that ∇f is normal to the level curve of $w = f(x, y, z)$ at each point, so

that ∇f is also a normal vector. Since

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = 1,$$

it follows that $\|\nabla f\| = 1$, so that ∇f is a unit normal. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S (\nabla f) \cdot (\nabla f) \, dS = \iint_S \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right) dS = \iint_S 1 \, dS,$$

which is just the area of S .

15.8: The Divergence Theorem

Concepts and Vocabulary

1. False. $\operatorname{div} \mathbf{F}$ is a function; see the definition at the start of the section.
2. $\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$; see the definition at the start of the section.

Skill Building

3. We have $\operatorname{div} \mathbf{F} = \operatorname{div}(x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{j}) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = \boxed{2x + 2y + 2z}$.
4. We have $\operatorname{div} \mathbf{F} = \operatorname{div}(x \mathbf{i} + xy \mathbf{j} + xyz \mathbf{j}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xyz) = \boxed{1 + x + xy}$.
5. We have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \operatorname{div}((x + \cos x) \mathbf{i} + (y + y \sin x) \mathbf{j} + 2z \mathbf{j}) \\ &= \frac{\partial}{\partial x}(x + \cos x) + \frac{\partial}{\partial y}(y + y \sin x) + \frac{\partial}{\partial z}(2z) = 1 - \sin x + 1 + \sin x + 2 = \boxed{4}. \end{aligned}$$

6. We have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \operatorname{div}(xye^z \mathbf{i} + x^2e^z \mathbf{j} + x^2ye^z \mathbf{j}) \\ &= \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(x^2e^z) + \frac{\partial}{\partial z}(x^2ye^z) = ye^z + 0 + x^2ye^z = \boxed{(1 + x^2)ye^z}. \end{aligned}$$

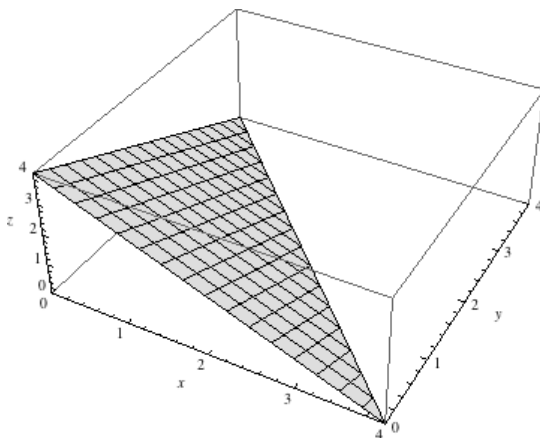
7. We have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \operatorname{div}(\sqrt{x^2 + y^2} \mathbf{i} + \sqrt{x^2 + y^2} \mathbf{j} + z \mathbf{j}) = \frac{\partial}{\partial x}(\sqrt{x^2 + y^2}) + \frac{\partial}{\partial y}(\sqrt{x^2 + y^2}) + \frac{\partial}{\partial z}(z) \\ &= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x + \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y + 1 = \boxed{\frac{x + y}{\sqrt{x^2 + y^2}} + 1}. \end{aligned}$$

8. We have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \operatorname{div}(xy^2 \mathbf{i} + x^2y \mathbf{j} + xyz \mathbf{j}) \\ &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(xyz) = y^2 + x^2 + xy = \boxed{x^2 + xy + y^2}. \end{aligned}$$

9. The surface is



Compute $\operatorname{div} \mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2xy + 2z) + \frac{\partial}{\partial y}(y^2 + 1) + \frac{\partial}{\partial z}(-(x + y)) = 2y + 2y = 4y.$$

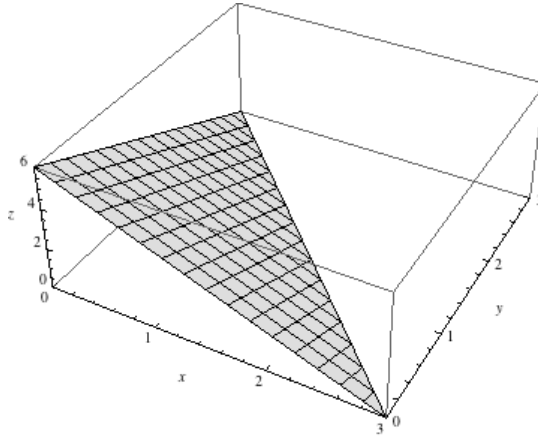
Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 4y \, dx \, dy \, dz$$

where E is the interior of the given surface. This solid is given by $0 \leq x \leq 4$, $0 \leq y \leq 4 - x$, $0 \leq z \leq 4 - x - y$, so that

$$\begin{aligned} \iiint_E 4y \, dx \, dy \, dz &= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} 4y \, dz \, dy \, dx \\ &= \int_0^4 \int_0^{4-x} [4yz]_0^{4-x-y} \, dy \, dx \\ &= \int_0^4 \int_0^{4-x} 4y(4-x-y) \, dy \, dx \\ &= \int_0^4 \int_0^{4-x} (16y - 4xy - 4y^2) \, dy \, dx \\ &= \int_0^4 \left[8y^2 - 2xy^2 - \frac{4}{3}y^3 \right]_0^{4-x} \, dx \\ &= \int_0^4 \left(8(4-x)^2 - 2x(4-x)^2 - \frac{4}{3}(4-x)^3 \right) \, dx \\ &= \int_0^4 \left(-\frac{2}{3}x^3 + 8x^2 - 32x + \frac{128}{3} \right) \, dx \\ &= \left[-\frac{1}{6}x^4 + \frac{8}{3}x^3 - 16x^2 + \frac{128}{3}x \right]_0^4 \\ &= \boxed{\frac{128}{3}}. \end{aligned}$$

10. The surface is



Compute $\operatorname{div} \mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2xy + z) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-(x + 4y)) = 2y + 2y = 4y.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 4y \, dx \, dy \, dz$$

where E is the interior of the given surface. This solid is given by $0 \leq x \leq 3$, $0 \leq y \leq 3 - x$, $0 \leq z \leq 6 - 2x - 2y$, so that

$$\begin{aligned} \iiint_E 4y \, dx \, dy \, dz &= \int_0^3 \int_0^{3-x} \int_0^{6-2x-2y} 4y \, dz \, dy \, dx \\ &= \int_0^3 \int_0^{3-x} [4yz]_0^{6-2x-2y} \, dy \, dx \\ &= \int_0^3 \int_0^{3-x} 4y(6 - 2x - 2y) \, dy \, dx \\ &= \int_0^3 \int_0^{3-x} (24y - 8xy - 8y^2) \, dy \, dx \\ &= \int_0^3 \left[12y^2 - 4xy^2 - \frac{8}{3}y^3 \right]_0^{3-x} \, dx \\ &= \int_0^3 \left(12(3-x)^2 - 4x(3-x)^2 - \frac{8}{3}(3-x)^3 \right) \, dx \\ &= \int_0^3 \left(-\frac{4}{3}x^3 + 12x^2 - 36x + 36 \right) \, dx \\ &= \left[-\frac{1}{3}x^4 + 4x^3 - 18x^2 + 36x \right]_0^3 \\ &= \boxed{27}. \end{aligned}$$

11. The given solid is the unit cube, and

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E (2x + 2y + 2z) dx dy dz$$

where E is the interior of the given surface. This solid is given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, so that

$$\begin{aligned} \iiint_E (2x + 2y + 2z) dx dy dz &= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx dy dz \\ &= \int_0^1 \int_0^1 [x^2 + 2xy + 2xz]_0^1 dy dz \\ &= \int_0^1 \int_0^1 (2y + 2z + 1) dy dz \\ &= \int_0^1 [y^2 + 2yz + y]_0^1 dz \\ &= \int_0^1 (2z + 2) dz \\ &= [z^2 + 2z]_0^1 = \boxed{3}. \end{aligned}$$

12. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x - y) + \frac{\partial}{\partial y}(y - z) + \frac{\partial}{\partial z}(x - y) = 2.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 2 dx dy dz$$

where E is the interior of the given surface. This is twice the volume of the solid; since the solid is a cube whose side length is 2, it has volume 8, so

$$\iiint_E 2 dx dy dz = \boxed{16}.$$

13. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(4z^2) = 2x + 8z + 2.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E (2x + 8z + 2) dx dy dz$$

where E is the interior of the given surface. Convert to cylindrical coordinates to compute the integral; then $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ for $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 2$. We get

$$\begin{aligned} \iiint_E (2x + 8z + 2) dx dy dz &= \int_0^2 \int_0^{2\pi} \int_0^2 (2r \cos \theta + 8z + 2) \cdot r dz d\theta dr \\ &= \int_0^2 \int_0^{2\pi} [2r^2 z \cos \theta + 4rz^2 + 2rz]_0^2 d\theta dr \\ &= \int_0^2 \int_0^{2\pi} (4r^2 \cos \theta + 20r) d\theta dr \\ &= \int_0^2 [4r^2 \sin \theta + 20r\theta]_0^{2\pi} dr \\ &= \int_0^2 40r\pi dr \\ &= [20r^2\pi]_0^2 = \boxed{80\pi}. \end{aligned}$$

14. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(3z^2) = 4y + 6z + 1.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E (4y + 6z + 1) \, dx \, dy \, dz$$

where E is the interior of the given surface. Convert to cylindrical coordinates to compute the integral; then $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ for $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 1$. We get

$$\begin{aligned} \iiint_E (4y + 6z + 1) \, dx \, dy \, dz &= \int_0^3 \int_0^{2\pi} \int_0^1 (4r \cos \theta + 6z + 1) \cdot r \, dz \, d\theta \, dr \\ &= \int_0^3 \int_0^{2\pi} [4r^2 z \cos \theta + 3rz^2 + rz]_0^1 \, d\theta \, dr \\ &= \int_0^3 \int_0^{2\pi} (4r^2 \cos \theta + 4r) \, d\theta \, dr \\ &= \int_0^3 [4r^2 \sin \theta + 4r\theta]_0^{2\pi} \, dr \\ &= \int_0^3 8r\pi \, dr \\ &= [4r^2\pi]_0^3 = \boxed{36\pi}. \end{aligned}$$

15. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3 \, dx \, dy \, dz$$

where E is the interior of the given sphere. The volume of a sphere of radius 1 is $\frac{4}{3}\pi$, and this integral is three times that volume, so

$$\iiint_E 3 \, dx \, dy \, dz = \boxed{4\pi}.$$

16. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z) = 2 + 2 + 2 = 6.$$

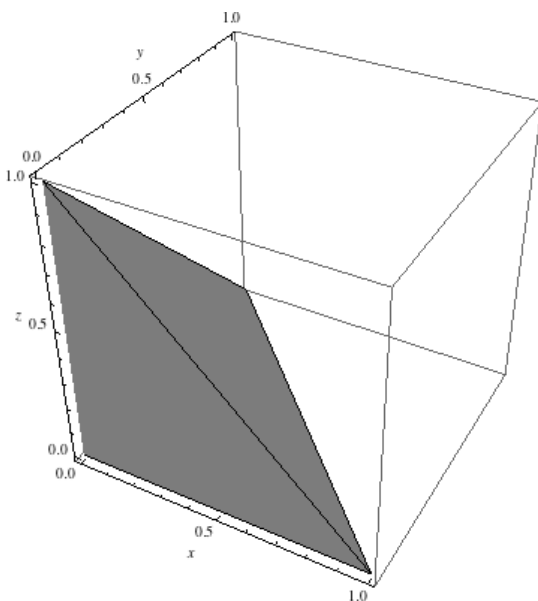
Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 6 \, dx \, dy \, dz$$

where E is the interior of the given sphere. The volume of a sphere of radius $\sqrt{2}$ is $\frac{4}{3}\pi \cdot 2\sqrt{2} = \frac{8}{3}\pi\sqrt{2}$, and this integral is six times that volume, so

$$\iiint_E 6 \, dx \, dy \, dz = 6 \cdot \frac{8}{3}\pi\sqrt{2} = \boxed{16\pi\sqrt{2}}.$$

17. The surface is



Compute $\operatorname{div} \mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x + \cos x) + \frac{\partial}{\partial y}(y + y \sin x) + \frac{\partial}{\partial z}(2z) = 1 - \sin x + 1 + \sin x + 2 = 4.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 4 \, dx \, dy \, dz$$

where E is the interior of the given surface. This solid is given by $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$, $0 \leq z \leq 1 - x - y$, so that

$$\begin{aligned} \iiint_E 4 \, dx \, dy \, dz &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 4 \, dz \, dy \, dx \\ &= \int_0^1 \int_0^{1-x} [4z]_0^{1-x-y} \, dy \, dx \\ &= \int_0^1 \int_0^{1-x} (4 - 4x - 4y) \, dy \, dx \\ &= \int_0^1 [4y - 4xy - 2y^2]_0^{1-x} \, dx \\ &= \int_0^1 (4(1-x) - 4x(1-x) - 2(1-x)^2) \, dx \\ &= \int_0^1 (2x^2 - 4x + 2) \, dx \\ &= \left[\frac{2}{3}x^3 - 2x^2 + 2x \right]_0^1 = \boxed{\frac{2}{3}}. \end{aligned}$$

Alternatively, if you recall that the volume of a pyramid is $\frac{1}{3}$ times the area of the base times the height, the result follows without integration since $4 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{2}{3}$.

18. A plot of the surface is given in Problem 17. Compute $\operatorname{div} \mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0.$$

Then by the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 0 dx dy dz = \boxed{0}.$$

Applications and Extensions

19. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3,$$

so that by the Divergence Theorem

$$\frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} dS = \frac{1}{3} \iiint_E \operatorname{div} \mathbf{F} dV = \frac{1}{3} \iiint_E 3 dx dy dz = \iiint_E dx dy dz,$$

which is an integral expressing the volume V of E .

20. Placing the solid in the first octant with one vertex at the origin, the sides of the parallelepiped are given by $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$, $z = c$. Then

$$\frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

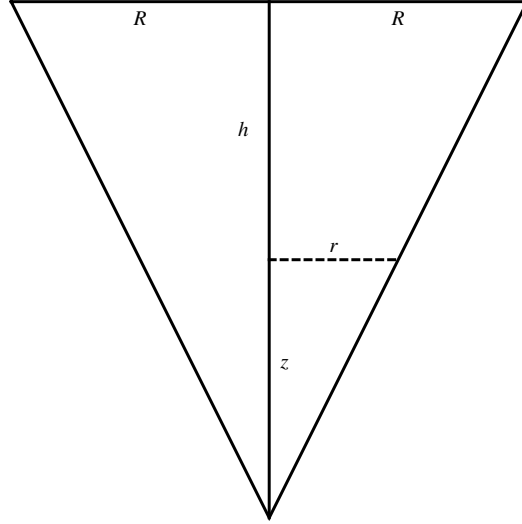
is composed of six separate integrals, one along each face:

Face	$\mathbf{n}$	$\mathbf{F}$	$\mathbf{F} \cdot \mathbf{n}$
$S_1 : x = 0$	$-\mathbf{i}$	$y\mathbf{j} + z\mathbf{k}$	0
$S_2 : y = 0$	$-\mathbf{j}$	$x\mathbf{i} + z\mathbf{k}$	0
$S_3 : z = 0$	$-\mathbf{k}$	$x\mathbf{i} + y\mathbf{j}$	0
$S_4 : x = a$	$\mathbf{i}$	$a\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	a
$S_5 : y = b$	$\mathbf{j}$	$x\mathbf{i} + b\mathbf{j} + z\mathbf{k}$	b
$S_6 : z = c$	$\mathbf{k}$	$x\mathbf{i} + y\mathbf{j} + c\mathbf{k}$	c

The integrals over S_1 , S_2 , and S_3 are clearly zero. For S_4 , the integral is over $0 \leq y \leq b$ and $0 \leq z \leq c$; for S_5 , it is over $0 \leq x \leq a$ and $0 \leq z \leq c$; and for S_6 , it is over $0 \leq x \leq a$ and $0 \leq y \leq b$. Therefore we get

$$\begin{aligned} V &= \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} dS \\ &= \frac{1}{3} \sum_{i=1}^6 \iint_{S_i} \mathbf{F} \cdot \mathbf{n} dS \\ &= \frac{1}{3} \left(\int_0^b \int_0^c a dz dy + \int_0^a \int_0^c b dz dx + \int_0^a \int_0^b c dy dx \right) \\ &= \frac{1}{3} \left(\int_0^b [az]_0^c dy + \int_0^a [bz]_0^c dx + \int_0^a [cy]_0^b dx \right) \\ &= \frac{1}{3} \left(\int_0^b ac dy + \int_0^a bc dx + \int_0^a bc dx \right) \\ &= \frac{1}{3} \left([acy]_0^b + [bcx]_0^a + [bcx]_0^a \right) \\ &= \frac{1}{3} (abc + abc + abc) = \boxed{abc}. \end{aligned}$$

21. Assume the cone is oriented as shown in the figure in the text. To determine the volume of the cone using the required formula, we must integrate both over the lateral surface of the cone and over its top. Do this first for the lateral surface; start by determining parametric equations for this surface using cylindrical coordinates. A cross-section of the cone in the yz -plane looks like this:



Using similar triangles, we see that $\frac{h}{R} = \frac{z}{r}$, so that $z = \frac{h}{R}r$. Then a cylindrical parametrization for the cone is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \frac{h}{R}r \mathbf{k}, \quad 0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi.$$

We get

$$\begin{aligned} \mathbf{r}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \frac{h}{R} \mathbf{k}, & \mathbf{r}_\theta &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & \frac{h}{R} \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -\frac{h}{R}r \cos \theta \mathbf{i} - \frac{h}{R}r \sin \theta \mathbf{j} + r \mathbf{k} \\ \|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{\frac{h^2}{R^2}r^2 \cos^2 \theta + \frac{h^2}{R^2}r^2 \sin^2 \theta + r^2} = r\sqrt{1 + \frac{h^2}{R^2}} = \frac{r}{R}\sqrt{R^2 + h^2}. \end{aligned}$$

With $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \frac{h}{R}r \mathbf{k}$, we get

$$\begin{aligned} \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \frac{1}{3} \int_0^R \int_0^{2\pi} \left(r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \frac{h}{R}r \mathbf{k} \right) \cdot \left(-\frac{h}{R}r \cos \theta \mathbf{i} - \frac{h}{R}r \sin \theta \mathbf{j} + r \mathbf{k} \right) \\ &\quad \cdot \frac{1}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| d\theta dr \\ &= \frac{1}{3} \int_0^R \int_0^{2\pi} \left(r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + \frac{h}{R}r \mathbf{k} \right) \cdot \left(-\frac{h}{R}r \cos \theta \mathbf{i} - \frac{h}{R}r \sin \theta \mathbf{j} + r \mathbf{k} \right) d\theta dr \\ &= \frac{1}{3} \int_0^R \int_0^{2\pi} \left(-\frac{h}{R}r^2 \cos^2 \theta - \frac{h}{R}r^2 \sin^2 \theta + \frac{h}{R}r^2 \right) d\theta dr \\ &= \frac{1}{3} \int_0^R \int_0^{2\pi} 0 d\theta dr = 0. \end{aligned}$$

So this piece of the surface integral is zero. For the top of the cone, note that it is a disk of radius R , parametrized by

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + h \mathbf{k}, \quad 0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi.$$

We get

$$\begin{aligned}\mathbf{r}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, & \mathbf{r}_\theta &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \mathbf{k} \\ \|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \|r \mathbf{k}\| = r.\end{aligned}$$

With $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + h \mathbf{k}$, we get

$$\begin{aligned}\frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \frac{1}{3} \int_0^R \int_0^{2\pi} (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + h \mathbf{k}) \cdot r \mathbf{k} \cdot \frac{1}{r} \cdot r \, d\theta \, dr \\ &= \frac{1}{3} \int_0^R \int_0^{2\pi} r h \, d\theta \, dr \\ &= \frac{2\pi}{3} \int_0^R r h \, dr \\ &= \frac{2\pi}{3} \left[\frac{1}{2} r^2 h \right]_0^R = \frac{1}{3} \pi R^2 h.\end{aligned}$$

Since the first integral was zero, the volume of the cone is just the second integral, or $\boxed{\frac{1}{3} \pi R^2 h}$.

22. Center the sphere at the origin, and parametrize it using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = R \cos \theta \sin \phi \mathbf{i} + R \sin \theta \sin \phi \mathbf{j} + R \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Then

$$\begin{aligned}\mathbf{r}_\theta &= -R \sin \theta \sin \phi \mathbf{i} + R \cos \theta \sin \phi \mathbf{j}, & \mathbf{r}_\phi &= R \cos \theta \cos \phi \mathbf{i} + R \sin \theta \cos \phi \mathbf{j} - R \sin \phi \mathbf{k} \\ \mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R \sin \theta \sin \phi & R \cos \theta \sin \phi & 0 \\ R \cos \theta \cos \phi & R \sin \theta \cos \phi & -R \sin \phi \end{vmatrix} \\ &= -R^2 \cos \theta \sin^2 \phi \mathbf{i} - R^2 \sin \theta \sin^2 \phi \mathbf{j} - R^2 \sin \phi \cos \phi \mathbf{k} \\ \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{R^4 \cos^2 \theta \sin^4 \phi + R^4 \sin^2 \theta \sin^4 \phi + R^4 \cos^4 \theta \sin^2 \phi \cos^2 \phi} \\ &= R^2 \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = R^2 \sin \phi.\end{aligned}$$

Then with

$$\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = R \cos \theta \sin \phi \mathbf{i} + R \sin \theta \sin \phi \mathbf{j} + R \cos \phi \mathbf{k},$$

we get

$$\begin{aligned}
\frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \frac{1}{3} \int_0^{2\pi} \int_0^\pi (R \cos \theta \sin \phi \mathbf{i} + R \sin \theta \sin \phi \mathbf{j} + R \cos \phi \mathbf{k}) \cdot \\
&\quad (-R^2 \cos \theta \sin^2 \phi \mathbf{i} - R^2 \sin \theta \sin^2 \phi \mathbf{j} - R^2 \sin \phi \cos \phi \mathbf{k}) \\
&\quad \cdot \frac{1}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} \cdot \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| \, d\phi \, d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^\pi (R \cos \theta \sin \phi \mathbf{i} + R \sin \theta \sin \phi \mathbf{j} + R \cos \phi \mathbf{k}) \cdot \\
&\quad (-R^2 \cos \theta \sin^2 \phi \mathbf{i} - R^2 \sin \theta \sin^2 \phi \mathbf{j} - R^2 \sin \phi \cos \phi \mathbf{k}) \, d\phi \, d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^\pi (-R^3 \cos^2 \theta \sin^3 \phi - R^3 \sin^2 \theta \sin^3 \phi - R^3 \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
&= -\frac{1}{3} R^3 \int_0^{2\pi} \int_0^\pi (\sin^3 \phi + \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
&= -\frac{1}{3} R^3 \int_0^{2\pi} \int_0^\pi \sin \phi (\sin^2 \phi + \cos^2 \phi) \, d\phi \, d\theta \\
&= -\frac{1}{3} R^3 \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta \\
&= -\frac{1}{3} R^3 \int_0^{2\pi} [-\cos \phi]_0^\pi \, d\theta \\
&= \frac{1}{3} R^3 \int_0^{2\pi} 2 \, d\theta = \boxed{\frac{4}{3} \pi R^3}.
\end{aligned}$$

23. Parametrize the surface of the sphere, which has radius 10, using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = 10 \cos \theta \sin \phi \mathbf{i} + 10 \sin \theta \sin \phi \mathbf{j} + 10 \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Then

$$\begin{aligned}
\mathbf{r}_\theta &= -10 \sin \theta \sin \phi \mathbf{i} + 10 \cos \theta \sin \phi \mathbf{j} & \mathbf{r}_\phi &= 10 \cos \theta \cos \phi \mathbf{i} + 10 \sin \theta \cos \phi \mathbf{j} - 10 \sin \phi \mathbf{k} \\
\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 \sin \theta \sin \phi & 10 \cos \theta \sin \phi & 0 \\ 10 \cos \theta \cos \phi & 10 \sin \theta \cos \phi & -10 \sin \phi \end{vmatrix} \\
&= -100 \cos \theta \sin^2 \phi \mathbf{i} - 100 \sin \theta \sin^2 \phi \mathbf{j} - 100 \sin \phi \cos \phi \mathbf{k} \\
\|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{10000 \cos^2 \theta \sin^4 \phi + 10000 \sin^2 \theta \sin^4 \phi + 10000 \cos^4 \theta \sin^2 \phi \cos^2 \phi} \\
&= 100 \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = 100 \sin \phi.
\end{aligned}$$

Then with

$$\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = 10 \cos \theta \sin \phi \mathbf{i} + 10 \sin \theta \sin \phi \mathbf{j} + 10 \cos \phi \mathbf{k},$$

we get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^\pi (10 \cos \theta \sin \phi \mathbf{i} + 10 \sin \theta \sin \phi \mathbf{j} + 10 \cos \phi \mathbf{k}) \cdot \\
&\quad (-100 \cos \theta \sin^2 \phi \mathbf{i} - 100 \sin \theta \sin^2 \phi \mathbf{j} - 100 \sin \phi \cos \phi \mathbf{k}) \\
&\quad \cdot \frac{1}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} \cdot \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^\pi (10 \cos \theta \sin \phi \mathbf{i} + 10 \sin \theta \sin \phi \mathbf{j} + 10 \cos \phi \mathbf{k}) \cdot \\
&\quad (-100 \cos \theta \sin^2 \phi \mathbf{i} - 100 \sin \theta \sin^2 \phi \mathbf{j} - 100 \sin \phi \cos \phi \mathbf{k}) \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^\pi (-1000 \cos^2 \theta \sin^3 \phi - 1000 \sin^2 \theta \sin^3 \phi - 1000 \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
&= -1000 \int_0^{2\pi} \int_0^\pi (\sin^3 \phi + \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
&= -1000 \int_0^{2\pi} \int_0^\pi \sin \phi (\sin^2 \phi + \cos^2 \phi) \, d\phi \, d\theta \\
&= -1000 \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta \\
&= -1000 \int_0^{2\pi} [-\cos \phi]_0^\pi \, d\theta \\
&= 1000 \int_0^{2\pi} 2 \, d\theta = 4000\pi.
\end{aligned}$$

For the second form, since

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

we parametrize again using spherical coordinates; the solid sphere is

$$\mathbf{r}(r, \theta, \phi) = r \cos \theta \sin \phi \mathbf{i} + r \sin \theta \sin \phi \mathbf{j} + r \cos \phi \mathbf{k}, \quad 0 \leq r \leq 10, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Then the solid integral is

$$\begin{aligned}
\iiint_E \operatorname{div} \mathbf{F} \, dV &= \int_0^{10} \int_0^{2\pi} \int_0^\pi 3r^2 \sin \phi \, d\phi \, d\theta \, dr \\
&= \int_0^{10} \int_0^{2\pi} [-3r^2 \cos \phi]_0^\pi \, d\theta \, dr \\
&= \int_0^{10} \int_0^{2\pi} 6r^2 \, d\theta \, dr \\
&= \int_0^{10} 12\pi r^2 \, dr \\
&= [4\pi r^3]_0^{10} = \boxed{4000\pi}.
\end{aligned}$$

The two results are the same.

24. The surface of the cylinder consists of three pieces: the top, S_1 , the lateral curved surface, S_2 , and the bottom, S_3 . We parametrize all three surfaces using cylindrical coordinates:

$$\begin{aligned}
S_1 : \quad \mathbf{r}(r, \theta) &= r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 2 \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \\
S_2 : \quad \mathbf{s}(\theta, z) &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 2 \\
S_3 : \quad \mathbf{t}(r, \theta) &= r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{r}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, & \mathbf{r}_\theta &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \mathbf{k} \\
\mathbf{s}_\theta &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, & \mathbf{s}_z &= \mathbf{k} \\
\mathbf{s}_\theta \times \mathbf{s}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \\
\mathbf{t}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, & \mathbf{t}_\theta &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\
\mathbf{t}_r \times \mathbf{t}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \mathbf{k}.
\end{aligned}$$

We need all of the normals we use to be outward-pointing. $r \mathbf{k}$ points upwards, which is outwards, along the top, and $\cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ points outwards along the lateral surface. However, $r \mathbf{k}$ points upwards, which is inwards, along the bottom, so we must use instead $-r \mathbf{k}$ along the bottom. Therefore

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} dS &= \iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} dS + \iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS \\
&= \int_0^1 \int_0^{2\pi} (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 2 \mathbf{k}) \cdot r \mathbf{k} \cdot \frac{1}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| d\theta dr \\
&\quad + \int_0^1 \int_0^{2\pi} (\cos \theta \mathbf{i} + \sin \theta \mathbf{j} + z \mathbf{k}) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \cdot \frac{1}{\|\mathbf{s}_\theta \times \mathbf{s}_z\|} \cdot \|\mathbf{s}_\theta \times \mathbf{s}_z\| dz d\theta \\
&\quad + \int_0^1 \int_0^{2\pi} (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}) \cdot (-r \mathbf{k}) \cdot \frac{1}{\|\mathbf{t}_r \times \mathbf{t}_\theta\|} \cdot \|\mathbf{t}_r \times \mathbf{t}_\theta\| d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 2r d\theta dr + \int_0^1 \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) dz d\theta + \int_0^1 \int_0^{2\pi} 0 d\theta dr \\
&= \int_0^1 4\pi r dr + \int_0^1 \int_0^{2\pi} 1 dz d\theta \\
&= [2\pi r^2]_0^1 + 4\pi = 6\pi.
\end{aligned}$$

For the second form, since

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

we parametrize again using cylindrical coordinates; the solid cylinder is

$$\mathbf{r}(r, \theta, \phi) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 2.$$

Then the solid integral is

$$\begin{aligned}
\iiint_E \operatorname{div} \mathbf{F} dV &= \int_0^1 \int_0^{2\pi} \int_0^2 3r dz d\theta dr \\
&= \int_0^1 \int_0^{2\pi} [3rz]_0^2 d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 6r d\theta dr \\
&= \int_0^1 12\pi r dr = [6\pi r^2]_0^1 = \boxed{6\pi}.
\end{aligned}$$

The two results are the same.

25. If $\mathbf{F}$ is constant, then $\operatorname{div} \mathbf{F} = 0$ since all three components of $\mathbf{F}$ are zero. Since E satisfies the conditions of the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 0 dV = 0.$$

26. Since

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(7z + 2x) = 3 + 4 + 7 = 14 \\ \operatorname{div} \mathbf{G} &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(9z + 6y) = 2 + 3 + 9 = 14, \end{aligned}$$

the two vector fields have the same divergence. Since E satisfies the assumptions of the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E \operatorname{div} \mathbf{G} dV = \iint_S \mathbf{G} \cdot \mathbf{n} dS.$$

27. (a) The velocity is a vector field $\mathbf{F}$; saying that the velocity is constant is the same as saying that the vector field is constant. Let the closed surface be S and the solid it bounds be E . Then by Problem 25 (as long as we assume that E satisfies the conditions of the Divergence Theorem) we know that the flux, which is $\iint_S \mathbf{F} \cdot \mathbf{n} dS$, is zero.

(b) Answers will vary.

- 28.

$$\mathbf{F} = \frac{1}{\rho}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}.$$

Then

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \\ &= \frac{(x^2 + y^2 + z^2)^{1/2} \cdot 1 - x \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x}{x^2 + y^2 + z^2} \\ &\quad + \frac{(x^2 + y^2 + z^2)^{1/2} \cdot 1 - y \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y}{x^2 + y^2 + z^2} \\ &\quad + \frac{(x^2 + y^2 + z^2)^{1/2} \cdot 1 - z \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2z}{x^2 + y^2 + z^2} \\ &= \frac{3(x^2 + y^2 + z^2)^{1/2} - (x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-1/2}}{x^2 + y^2 + z^2} \\ &= \frac{3(x^2 + y^2 + z^2)^{1/2} - (x^2 + y^2 + z^2)^{1/2}}{x^2 + y^2 + z^2} \\ &= \frac{2}{\sqrt{x^2 + y^2 + z^2}} = \frac{2}{\rho}. \end{aligned}$$

29. (a) Let $\mathbf{F}$ be as in Problem 28. Then

$$\iiint_E \frac{dV}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{2} \iiint_E \frac{2}{\rho} dV = \frac{1}{2} \iiint_E \operatorname{div} \mathbf{F} dV = \frac{1}{2} \iint_S \mathbf{F} \cdot \mathbf{n} dS,$$

where S is the surface of the unit sphere. Parametrizing the unit sphere using spherical coordinates gives

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Then

$$\begin{aligned}
 \mathbf{r}_\theta &= -\sin \theta \sin \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} & \mathbf{r}_\phi &= \cos \theta \cos \phi \mathbf{i} + \sin \theta \cos \phi \mathbf{j} - \sin \phi \mathbf{k} \\
 \mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = -\cos \theta \sin^2 \phi \mathbf{i} - \sin \theta \sin^2 \phi \mathbf{j} - \sin \phi \cos \phi \mathbf{k} \\
 \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{\cos^2 \theta \sin^4 \phi + \sin^2 \theta \sin^4 \phi + \cos^4 \theta \sin^2 \phi \cos^2 \phi} \\
 &= \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = \sin \phi.
 \end{aligned}$$

Note that $\mathbf{r}_\theta \times \mathbf{r}_\phi$ is the inward-pointing normal, so we will use its negative, which will be the outward-pointing normal. Then with

$$\mathbf{F} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k},$$

the surface integral is

$$\begin{aligned}
 \frac{1}{2} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi (\cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}) \\
 &\quad \cdot (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}) \cdot \frac{1}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} \\
 &\quad \cdot \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| \, d\phi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi (\cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}) \\
 &\quad \cdot (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}) \, d\phi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi (\cos^2 \theta \sin^3 \phi + \sin^2 \theta \sin^3 \phi + \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi (\sin^3 \phi + \sin \phi \cos^2 \phi) \, d\phi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} [-\cos \phi]_0^\pi \, d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} 2 \, d\theta = \boxed{2\pi}.
 \end{aligned}$$

(b) Evaluate the integral using spherical coordinates; then $\sqrt{x^2 + y^2 + z^2} = \sqrt{r^2} = r$, so we get

$$\begin{aligned}
 \iiint_E \frac{dV}{\sqrt{x^2 + y^2 + z^2}} &= \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{r^2 \sin \phi}{r} \, d\phi \, d\theta \, dr \\
 &= \int_0^1 \int_0^{2\pi} \int_0^\pi r \sin \phi \, d\phi \, d\theta \, dr \\
 &= \int_0^1 \int_0^{2\pi} [-r \cos \phi]_0^\pi \, d\theta \, dr \\
 &= \int_0^1 \int_0^{2\pi} 2r \, d\theta \, dr \\
 &= \int_0^1 4\pi r \, dr \\
 &= [2\pi r^2]_0^1 = 2\pi.
 \end{aligned}$$

30. We have

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k},$$

so that

$$\operatorname{div} \nabla f = \frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_y) + \frac{\partial}{\partial z}(f_z) = f_{xx} + f_{yy} + f_{zz} = 0$$

if f satisfies the Laplace equation. Now, ∇f is a vector field, and by the hypotheses, invoking the Divergence Theorem gives

$$\iint_S \nabla f \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \nabla f \, dV = \iiint_E 0 \, dV = 0.$$

Challenge Problems

31. Rearranging the equation gives

$$x^2 + y^2 + z^2 - 2qz = 0, \text{ or } x^2 + y^2 + (z - q)^2 = q^2,$$

which is a sphere centered at $(0, 0, q)$ of radius q . Now,

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z,$$

so that using the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E (2x + 2y + 2z) \, dV.$$

Evaluate this integral using spherical coordinates:

$$\begin{aligned} \iiint_E (2x + 2y + 2z) \, dV &= \int_0^\pi \int_0^{2\pi} \int_0^q (2x + 2y + 2z) \, dV \\ &= \int_0^\pi \int_0^{2\pi} \int_0^q (2r \cos \theta \sin \phi + 2r \sin \theta \sin \phi + 2q + 2r \cos \phi) \cdot r^2 \sin \phi \, dr \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} \int_0^q (2r^3 \cos \theta \sin^2 \phi + 2r^3 \sin \theta \sin^2 \phi + 2qr^2 \sin \phi + 2r^3 \cos \phi \sin \phi) \, dr \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} \left[\frac{1}{2} r^4 \cos \theta \sin^2 \phi + \frac{1}{2} r^4 \sin \theta \sin^2 \phi + \frac{2}{3} q r^3 \sin \phi + \frac{1}{2} r^4 \cos \phi \sin \phi \right]_0^q \, d\theta \, d\phi \\ &= \int_0^\pi \int_0^{2\pi} \left(\frac{q^4}{2} \sin^2 \phi (\cos \theta + \sin \theta) + \frac{2}{3} q^4 \sin \phi + \frac{1}{2} q^4 \cos \phi \sin \phi \right) \, d\theta \, d\phi \\ &= q^4 \int_0^\pi \left[\frac{1}{2} \sin^2 \phi (\sin \theta - \cos \theta) + \frac{2}{3} \theta \sin \phi + \frac{1}{2} \theta \cos \phi \sin \phi \right]_0^{2\pi} \, d\phi \\ &= q^4 \int_0^\pi \left(\frac{4\pi}{3} \sin \phi + \pi \cos \phi \sin \phi \right) \, d\phi \\ &= q^4 \left[-\frac{4\pi}{3} \cos \phi + \frac{\pi}{2} \sin^2 \phi \right]_0^\pi \\ &= \boxed{\frac{8\pi q^4}{3}}. \end{aligned}$$

32. The interior of the cube is defined by $0 \leq x \leq q$, $0 \leq y \leq q$, $0 \leq z \leq q$, and

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z.$$

So by the Divergence Theorem,

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iiint_E \operatorname{div} \mathbf{F} \, dV \\
 &= \int_0^q \int_0^q \int_0^q (2x + 2y + 2z) \, dx \, dy \, dz \\
 &= \int_0^q \int_0^q [x^2 + 2xy + 2xz]_0^q \, dy \, dz \\
 &= \int_0^q \int_0^q (2qy + 2qz + q^2) \, dy \, dz \\
 &= \int_0^q [qy^2 + 2qyz + q^2y]_0^q \, dz \\
 &= \int_0^q (q^3 + 2q^2z + q^3) \, dz \\
 &= [2q^3z + q^2z^2]_0^q \\
 &= 2q^4 + q^4 = \boxed{3q^4}.
 \end{aligned}$$

33. With

$$\mathbf{F} = f(\nabla g) = fg_x \mathbf{i} + fg_y \mathbf{j} + fg_z \mathbf{k},$$

we have

$$\begin{aligned}
 \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(fg_x) + \frac{\partial}{\partial y}(fg_y) + \frac{\partial}{\partial z}(fg_z) \\
 &= f_x g_x + f g_{xx} + f_y g_y + f g_{yy} + f_z g_z + f g_{zz} \\
 &= \langle f_x, f_y, f_z \rangle \cdot \langle g_x, g_y, g_z \rangle + f(g_{xx} + g_{yy} + g_{zz}) \\
 &= f(\nabla^2 g) + \nabla f \cdot \nabla g.
 \end{aligned}$$

Then since E satisfies the assumptions of the Divergence Theorem, we have

$$\iint_S f(\nabla g) \cdot \mathbf{n} \, dS = \iiint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E [f(\nabla^2 g) + \nabla f \cdot \nabla g] \, dV.$$

15.9: Stokes' Theorem

Concepts and Vocabulary

1. False. $\operatorname{curl} \mathbf{F}$ is a vector; see the introduction to this section.
2. We have

$$\begin{aligned}
 \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x^2/2} - yz & e^{-y^2/2} + xz + 2x & e^{-z^2/2} + 5 \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y}(e^{-z^2/2} + 5) - \frac{\partial}{\partial z}(e^{-y^2/2} + xz + 2x) \right) \mathbf{i} \\
 &\quad + \left(\frac{\partial}{\partial z}(e^{-x^2/2} - yz) - \frac{\partial}{\partial x}(e^{-z^2/2} + 5) \right) \mathbf{j} \\
 &\quad + \left(\frac{\partial}{\partial x}(e^{-y^2/2} + xz + 2x) - \frac{\partial}{\partial y}(e^{-x^2/2} - yz) \right) \mathbf{k} \\
 &= (0 - x) \mathbf{i} + (-y - 0) \mathbf{j} + ((z + 2) - (-z)) \mathbf{k} \\
 &= -x \mathbf{i} - y \mathbf{j} + (2z + 2) \mathbf{k}.
 \end{aligned}$$

3. (c), $\mathbf{0}$, is correct. See the first theorem in Objective 4.
4. False. See the discussion in Objective 5; the component of the curl in the direction of the normal to the surface measures this circulation.
5. By the theorem on Curl and Angular Velocity in Objective 5, $\text{curl } \mathbf{F} = 2\omega$.
6. False. See the Summary in Objective 5.

Skill Building

7. We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(0) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right) \mathbf{k} = \boxed{\mathbf{0}}.$$

8. We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(0) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) \right) \mathbf{k} = \boxed{\mathbf{0}}.$$

9. We have

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xz & z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(xz) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(z) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(xyz) \right) \mathbf{k} \\ &= \boxed{-x \mathbf{i} + xy \mathbf{j} + (z - xz) \mathbf{k}}. \end{aligned}$$

10. We have

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & -y & -2z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(-y) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(4x) - \frac{\partial}{\partial x}(-2z) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(4x) \right) \mathbf{k} = \boxed{\mathbf{0}}. \end{aligned}$$

11. We have

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz^2 & y^2 \sin z & xe^{2z} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(xe^{2z}) - \frac{\partial}{\partial z}(y^2 \sin z) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(3xyz^2) - \frac{\partial}{\partial x}(xe^{2z}) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(y^2 \sin z) - \frac{\partial}{\partial y}(3xyz^2) \right) \mathbf{k} \\ &= \boxed{-y^2 \cos z \mathbf{i} + (6xyz - e^{2z}) \mathbf{j} - 3xz^2 \mathbf{k}}. \end{aligned}$$

12. We have

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & z^2x & yz \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(z^2x) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(yz) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(z^2x) - \frac{\partial}{\partial y}(yz) \right) \mathbf{k} \\ &= \boxed{(z - 2zx) \mathbf{i} + y \mathbf{j} + (z^2 - z) \mathbf{k}}. \end{aligned}$$

13. We have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{1}{x^2+y^2+z^2} \end{vmatrix}.$$

Now,

$$\frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2+z^2} \right) = \frac{\partial}{\partial y} (x(x^2+y^2+z^2)^{-1}) = -x(x^2+y^2+z^2)^{-2} \cdot 2y = -\frac{2xy}{(x^2+y^2+z^2)^2}.$$

In a similar manner, we get

$$\begin{aligned} \frac{\partial}{\partial z} \left(\frac{x}{x^2+y^2+z^2} \right) &= -\frac{2xz}{(x^2+y^2+z^2)^2} \\ \frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2+z^2} \right) &= -\frac{2xy}{(x^2+y^2+z^2)^2} \\ \frac{\partial}{\partial z} \left(\frac{y}{x^2+y^2+z^2} \right) &= -\frac{2yz}{(x^2+y^2+z^2)^2}. \end{aligned}$$

For the $\mathbf{k}$ component, we have

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{1}{x^2+y^2+z^2} \right) &= -\frac{2x}{(x^2+y^2+z^2)^2} \\ \frac{\partial}{\partial y} \left(\frac{1}{x^2+y^2+z^2} \right) &= -\frac{2y}{(x^2+y^2+z^2)^2}. \end{aligned}$$

Putting these together gives

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \left(-\frac{2y}{(x^2+y^2+z^2)^2} + \frac{2yz}{(x^2+y^2+z^2)^2} \right) \mathbf{i} \\ &\quad + \left(\frac{2x}{(x^2+y^2+z^2)^2} - \frac{2xz}{(x^2+y^2+z^2)^2} \right) \mathbf{j} \\ &\quad + \left(-\frac{2xy}{(x^2+y^2+z^2)^2} + \frac{2xy}{(x^2+y^2+z^2)^2} \right) \mathbf{k} \\ &= \boxed{\frac{2y(z-1)}{(x^2+y^2+z^2)^2} \mathbf{i} + \frac{2x(1-z)}{(x^2+y^2+z^2)^2} \mathbf{j}}. \end{aligned}$$

14. We have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & x^2y & e^z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(e^z) - \frac{\partial}{\partial z}(x^2y) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(e^x) - \frac{\partial}{\partial x}(e^z) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(e^x) \right) \mathbf{k} \\ &= \boxed{2xy \mathbf{k}}. \end{aligned}$$

15. We have

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & \sin y & e^{xz} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(e^{xz}) - \frac{\partial}{\partial z}(\sin y) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(\cos x) - \frac{\partial}{\partial x}(e^{xz}) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(\sin y) - \frac{\partial}{\partial y}(\cos x) \right) \mathbf{k} \\ &= \boxed{-ze^{xz} \mathbf{j}}. \end{aligned}$$

16. We have

$$\begin{aligned}
 \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(xy) & \cos(xy^2) & x \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(\cos(xy^2)) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(\sin(xy)) - \frac{\partial}{\partial x}(x) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(\cos(xy^2)) - \frac{\partial}{\partial y}(\sin(xy)) \right) \mathbf{k} \\
 &= -\mathbf{j} + (-\sin(xy^2) \cdot y^2 - \cos(xy) \cdot x) \mathbf{k} \\
 &= \boxed{-\mathbf{j} - (y^2 \sin(xy^2) + x \cos(xy)) \mathbf{k}}.
 \end{aligned}$$

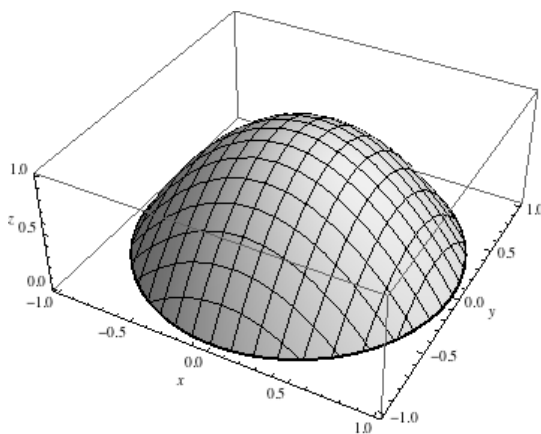
17. We have

$$\begin{aligned}
 \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & z+x \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y}(z+x) - \frac{\partial}{\partial z}(y+z) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(x+y) - \frac{\partial}{\partial x}(z+x) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial y}(x+y) \right) \mathbf{k} \\
 &= \boxed{-\mathbf{i} - \mathbf{j} - \mathbf{k}}.
 \end{aligned}$$

18. We have

$$\begin{aligned}
 \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & z+y+x \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y}(z+y+x) - \frac{\partial}{\partial z}(z+x) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(y+z) - \frac{\partial}{\partial x}(z+y+x) \right) \mathbf{j} \\
 &\quad + \left(\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z) \right) \mathbf{k} \\
 &= \boxed{\mathbf{0}}.
 \end{aligned}$$

19. The paraboloid together with its bounding curve is



The bounding curve C is the circle $x^2 + y^2 = 1$ in the xy -plane. Parametrize the curve by

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = (z-y) \mathbf{i} + (z+x) \mathbf{j} - (x+y) \mathbf{k} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} - (\sin \theta + \cos \theta) \mathbf{k},$$

so that the line integral is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j} - (\sin \theta + \cos \theta) \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = \int_0^{2\pi} 1 d\theta = 2\pi.\end{aligned}$$

For the other integral, parametrize the paraboloid using cylindrical coordinates:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (1 - r^2) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = 2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}.$$

This is upward-pointing, so after normalization, it is the normal we want. Computing $\text{curl } \mathbf{F}$ gives

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & z + x & -(x + y) \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

Then

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS &= \int_0^1 \int_0^{2\pi} (-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot \frac{1}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} (2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| d\theta dr \\ &= \int_0^1 \int_0^{2\pi} (-4r^2 \cos \theta + 4r^2 \sin \theta + 2r) d\theta dr \\ &= \int_0^1 [-4r^2 \sin \theta - 4r^2 \cos \theta + 2r\theta]_0^{2\pi} dr \\ &= \int_0^1 4\pi r dr \\ &= [2\pi r^2]_0^1 = \boxed{2\pi}.\end{aligned}$$

The two answers agree.

20. See Problem 19 for a plot of the surface and boundary. The bounding curve C is the circle $x^2 + y^2 = 1$ in the xy -plane. Parametrize the curve by

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} = \sin \theta \mathbf{i} + \cos \theta \mathbf{k}$$

so that the line integral is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin \theta \mathbf{i} + \cos \theta \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (-\sin^2 \theta) d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{2} + \frac{1}{2} \cos(2\theta)\right) d\theta \\ &= \left[-\frac{1}{2}\theta + \frac{1}{4} \sin(2\theta)\right]_0^{2\pi} = -\pi.\end{aligned}$$

For the other integral, parametrize the paraboloid using cylindrical coordinates:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (1 - r^2) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Then the normal vector is

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = 2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}.$$

This is upward-pointing, so after normalization, it is the normal we want. Computing $\text{curl } \mathbf{F}$ gives

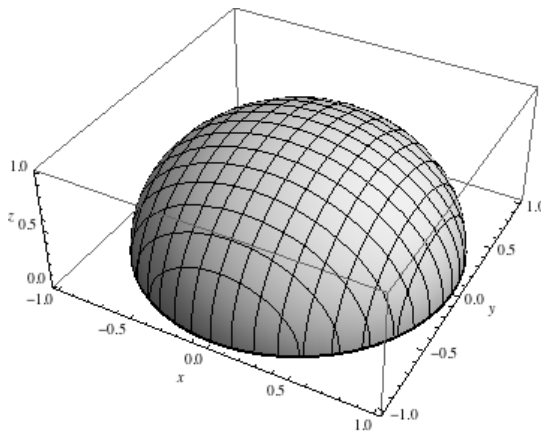
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -1 \mathbf{i} - 1 \mathbf{j} - 1 \mathbf{k}.$$

Then

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^1 \int_0^{2\pi} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot \frac{1}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} (2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} (-2r^2 \cos \theta - 2r^2 \sin \theta - r) \, d\theta \, dr \\ &= \int_0^1 [-2r^2 \sin \theta + 2r^2 \cos \theta - r\theta]_0^{2\pi} \\ &= - \int_0^1 2\pi r \, dr \\ &= - [\pi r^2]_0^1 = \boxed{-\pi}. \end{aligned}$$

The two answers agree.

21. The hemisphere together with its bounding curve is



The bounding curve C is the circle $x^2 + y^2 = 1$ in the xy -plane. Parametrize the curve by

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = y \mathbf{i} - x \mathbf{j} = \sin \theta \mathbf{i} - \cos \theta \mathbf{j}$$

so that the line integral is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin \theta \mathbf{i} - \cos \theta \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (-\sin^2 \theta - \cos^2 \theta) d\theta = -\int_0^{2\pi} 1 d\theta = -2\pi.\end{aligned}$$

For the other integral, parametrize the hemisphere using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \end{vmatrix} = \cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}.$$

This is upward-pointing in the given range of ϕ , so after normalization, it is the normal we want. Computing $\text{curl } \mathbf{F}$ gives

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\mathbf{k}.$$

Then

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^{\pi/2} -2\mathbf{k} \cdot \frac{1}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|} (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_\phi \times \mathbf{r}_\theta\| d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (-2 \sin \phi \cos \phi) d\phi d\theta \\ &= \int_0^{2\pi} [-\sin^2 \phi]_0^{\pi/2} d\theta \\ &= \int_0^{2\pi} (-1) d\theta = \boxed{-2\pi}.\end{aligned}$$

The two answers agree.

22. See Problem 21 for a plot of the surface and boundary. The bounding curve C is the circle $x^2 + y^2 = 1$ in the xy -plane. Parametrize the curve by

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} = \cos \theta \mathbf{j} + \sin \theta \mathbf{k}$$

so that the line integral is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos \theta \mathbf{j} + \sin \theta \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} \cos^2 \theta d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \\ &= \left[\frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} = \pi.\end{aligned}$$

For the other integral, parametrize the hemisphere using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \end{vmatrix} = \cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}.$$

This is upward-pointing in the given range of ϕ , so after normalization, it is the normal we want. Computing $\text{curl } \mathbf{F}$ gives

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Then

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^{\pi/2} (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \frac{1}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|} (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_\phi \times \mathbf{r}_\theta\| \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\cos \theta \sin^2 \phi + \sin \theta \sin^2 \phi + \sin \phi \cos \phi) \, d\phi \, d\theta \\ &= \int_0^{\pi/2} \int_0^{2\pi} (\cos \theta \sin^2 \phi + \sin \theta \sin^2 \phi + \sin \phi \cos \phi) \, d\theta \, d\phi \\ &= \int_0^{\pi/2} [\sin \theta \sin^2 \phi - \cos \theta \sin^2 \phi + \theta \sin \phi \cos \phi]_0^{2\pi} \, d\phi \\ &= \int_0^{\pi/2} 2\pi \sin \phi \cos \phi \, d\phi \\ &= [\pi \sin^2 \phi]_0^{\pi/2} = \boxed{\pi}. \end{aligned}$$

The two answers agree.

23. See Problem 19 for a plot of the surface and boundary. The bounding curve C is the circle $x^2 + y^2 = 1$ in the xy -plane. Parametrize the curve by

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = y^2 \mathbf{i} + x \mathbf{j} - xz \mathbf{k} = \sin^2 \theta \mathbf{i} + \cos \theta \mathbf{j}$$

so that the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin^2 \theta \mathbf{i} + \cos \theta \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta \\ &= \int_0^{2\pi} (-\sin^3 \theta + \cos^2 \theta) \, d\theta \\ &= \int_0^{2\pi} \left(-\sin \theta (1 - \cos^2 \theta) + \frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) \, d\theta \\ &= \left[\cos \theta - \frac{1}{3} \cos^3 \theta + \frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} \\ &= \pi. \end{aligned}$$

For the other integral, parametrize the paraboloid using cylindrical coordinates:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (1 - r^2) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Then the normal vector is

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = 2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}.$$

This is upward-pointing, so after normalization, it is the normal we want. Computing $\text{curl } \mathbf{F}$ gives

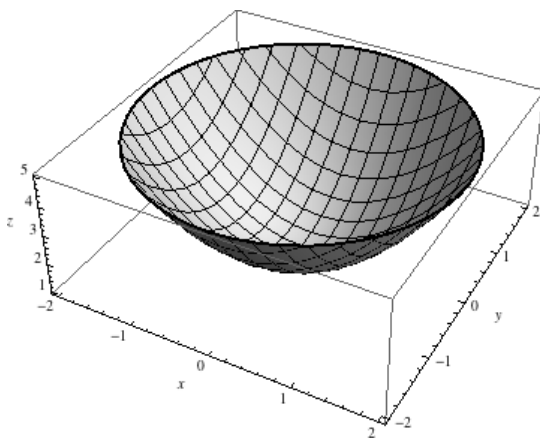
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & -xz \end{vmatrix} = z \mathbf{j} + (1 - 2y) \mathbf{k} = (1 - r^2) \mathbf{j} + (1 - 2r \sin \theta) \mathbf{k}.$$

Then

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^1 \int_0^{2\pi} ((1 - r^2) \mathbf{j} + (1 - 2r \sin \theta) \mathbf{k}) \cdot \frac{1}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} (2r^2 \cos \theta \mathbf{i} + 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} ((1 - r^2) 2r^2 \sin \theta + (1 - 2r \sin \theta) r) \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} 2r^2 \sin \theta - 2r^4 \sin \theta + r - 2r^2 \sin \theta \, d\theta \, dr \\ &= \int_0^1 [-2r^2 \cos \theta + 2r^4 \cos \theta + r\theta + 2r^2 \cos \theta]_0^{2\pi} \, dr \\ &= \int_0^1 2\pi r \, dr \\ &= [\pi r^2]_0^1 = \boxed{\pi}. \end{aligned}$$

The two answers agree.

24. The surface together with its bounding curve is



The bounding curve C is a circle; since $z = 5$, the circle is $5 = 1 + x^2 + y^2$, so it is a circle of radius 2. Parametrize the curve by

$$\mathbf{r}(\theta) = 2 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{j} + 5 \mathbf{k}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{F} = 4x \mathbf{i} - y \mathbf{j} + 2z \mathbf{k} = 8 \cos \theta \mathbf{i} - 2 \sin \theta \mathbf{j} + 10 \mathbf{k}$$

so that the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (8 \cos \theta \mathbf{i} - 2 \sin \theta \mathbf{j} + 10 \mathbf{k}) \cdot (-2 \sin \theta \mathbf{i} + 2 \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (-20 \sin \theta \cos \theta) d\theta = [-10 \sin^2 \theta]_0^{2\pi} = 0. \end{aligned}$$

For the other integral, computing $\text{curl } \mathbf{F}$ gives

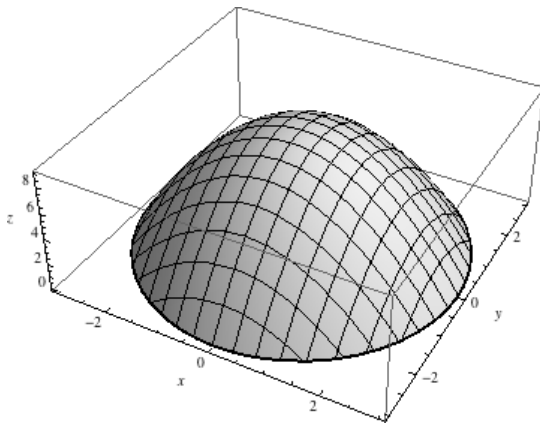
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & -y & 2z \end{vmatrix} = \mathbf{0}.$$

Therefore

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \iint_S 0 dS = \boxed{0}.$$

The two answers agree.

25. The surface together with its bounding curve is



The bounding curve C is the circle of radius 3 centered at the origin in the xy -plane; parametrize it in the counterclockwise direction by

$$\mathbf{r}(\theta) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

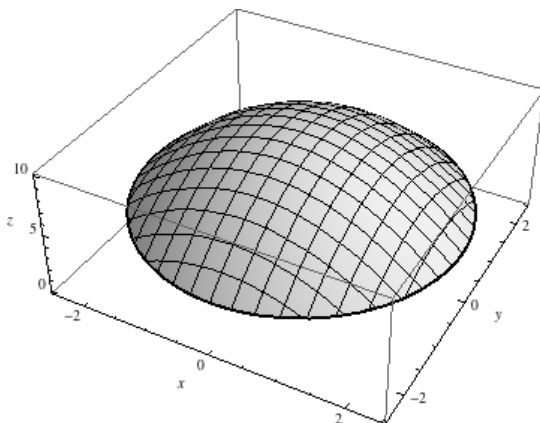
Then

$$\mathbf{F} = y \mathbf{i} + x \mathbf{j} + x^2 \mathbf{k} = 3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j} + 9 \cos^2 \theta \mathbf{k}$$

so that the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j} + 9 \cos^2 \theta \mathbf{k}) \cdot (-3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} 3 (\cos^2 \theta - \sin^2 \theta) d\theta \\ &= \int_0^{2\pi} 3 \cos(2\theta) d\theta \\ &= \left[\frac{3}{2} \sin(2\theta) \right]_0^{2\pi} = \boxed{0}. \end{aligned}$$

26. The surface together with its bounding curve is



When $z = 4$, we have $4 = 10 - x^2 - y^2$, so that $x^2 + y^2 = 6$. Therefore the boundary of the surface is the circle $x^2 + y^2 = 6$ centered on the z -axis at $z = 4$. Parametrize it in the counterclockwise direction by

$$\mathbf{r}(\theta) = \sqrt{6} \cos \theta \mathbf{i} + \sqrt{6} \sin \theta \mathbf{j} + 4 \mathbf{k}, \quad 0 \leq \theta \leq 2\pi.$$

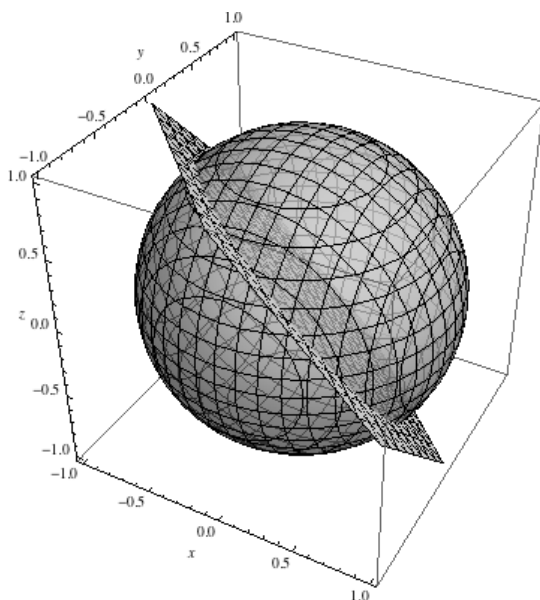
Then

$$\mathbf{F} = 4z \mathbf{i} + 3x \mathbf{j} + 3y \mathbf{k} = 16 \mathbf{i} + 3\sqrt{6} \cos \theta \mathbf{j} + 3\sqrt{6} \sin \theta \mathbf{k}$$

so that the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(16 \mathbf{i} + 3\sqrt{6} \cos \theta \mathbf{j} + 3\sqrt{6} \sin \theta \mathbf{k} \right) \cdot \left(-\sqrt{6} \sin \theta \mathbf{i} + \sqrt{6} \cos \theta \mathbf{j} \right) d\theta \\ &= \int_0^{2\pi} \left(-16\sqrt{6} \sin \theta + 18 \cos^2 \theta \right) d\theta \\ &= \int_0^{2\pi} \left(-16\sqrt{6} \sin \theta + 18 \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) \right) d\theta \\ &= \left[16\sqrt{6} \cos \theta + 9\theta + \frac{9}{2} \sin(2\theta) \right]_0^{2\pi} = \boxed{18\pi}. \end{aligned}$$

27. The two surfaces are



Using Stokes' Theorem, note that

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = \mathbf{0},$$

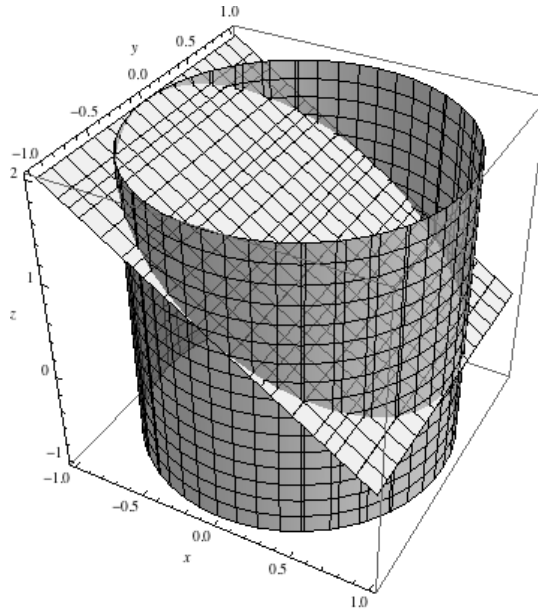
so that

$$\oint_C [(y+z) dx + (z+x) dy + (x+y) dz] = \iint_S \mathbf{0} \cdot \mathbf{n} dS = 0.$$

Calculating directly, note that on the curve of intersection, $x+y=-z$, $y+z=-x$, and $x+z=-y$. Further, $x^2+y^2+z^2=1$ implies that $2x dx + 2y dy + 2z dz = 0$, so that $x dx + y dy + z dz = 0$. Therefore

$$\oint_C [(y+z) dx + (z+x) dy + (x+y) dz] = \oint_C (-x dx - y dy - z dz) = \oint_C 0 = \boxed{0}.$$

28. The two surfaces are



Using Stokes' Theorem, we can parametrize the portion of the plane inside the cylinder using cylindrical coordinates by

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (1 - r \cos \theta) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\cos \theta \\ -r \sin \theta & r \cos \theta & r \sin \theta \end{vmatrix} = r \mathbf{i} + r \mathbf{k}.$$

Since this is upward-pointing, we use this normal. Further,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

Then

$$\begin{aligned}
 \oint_C [(y-z) dx + (z-x) dy + (x-y) dz] &= \iint_S \mathbf{F} \cdot \mathbf{n} dS \\
 &= \int_0^1 \int_0^{2\pi} (-2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \cdot (r\mathbf{i} + r\mathbf{k}) d\theta dr \\
 &= \int_0^1 \int_0^{2\pi} (-4r) d\theta dr \\
 &= \int_0^1 (-8\pi r) dr \\
 &= [-4\pi r^2]_0^1 = -4\pi.
 \end{aligned}$$

To compute the line integral directly, use the same parametrization as above, but with $r = 1$ to get the boundary; then

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + (1 - \cos \theta) \mathbf{k}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\begin{aligned}
 \oint_C [(y-z) dx + (z-x) dy + (x-y) dz] &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_0^{2\pi} ((\sin \theta + \cos \theta - 1)\mathbf{i} + (1 - 2\cos \theta)\mathbf{j} + (\cos \theta - \sin \theta)\mathbf{k}) \\
 &\quad \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j} + \sin \theta \mathbf{k}) d\theta \\
 &= \int_0^{2\pi} (-2\sin^2 \theta + \sin \theta + \cos \theta - 2\cos^2 \theta) d\theta \\
 &= \int_0^{2\pi} (-2 + \sin \theta + \cos \theta) d\theta \\
 &= [-2\theta - \cos \theta + \sin \theta]_0^{2\pi} = \boxed{-4\pi}.
 \end{aligned}$$

29. The given curve is a circle of radius 2 centered on the z -axis, in the plane $z = 2$. Using Stokes' Theorem, we consider the disk in that plane that it bounds; we can parametrize that disk by

$$\mathbf{r}(r, \theta) = 2r \cos \theta \mathbf{i} + 2r \sin \theta \mathbf{j} + 2\mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos \theta & 2\sin \theta & 0 \\ -2r\sin \theta & 2r\cos \theta & 0 \end{vmatrix} = 4r \mathbf{k}.$$

Since this is upward-pointing, we use this normal. Further,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x+y & x+y+z \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Then

$$\begin{aligned}
 \oint_C [x dx + (x+y) dy + (x+y+z) dz] &= \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS \\
 &= \int_0^1 \int_0^{2\pi} (\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot 4r \mathbf{k} d\theta dr \\
 &= \int_0^1 \int_0^{2\pi} 4r d\theta dr \\
 &= \int_0^1 [4r\theta]_0^{2\pi} dr \\
 &= \int_0^1 8\pi r dr \\
 &= [4\pi r^2]_0^1 = 4\pi.
 \end{aligned}$$

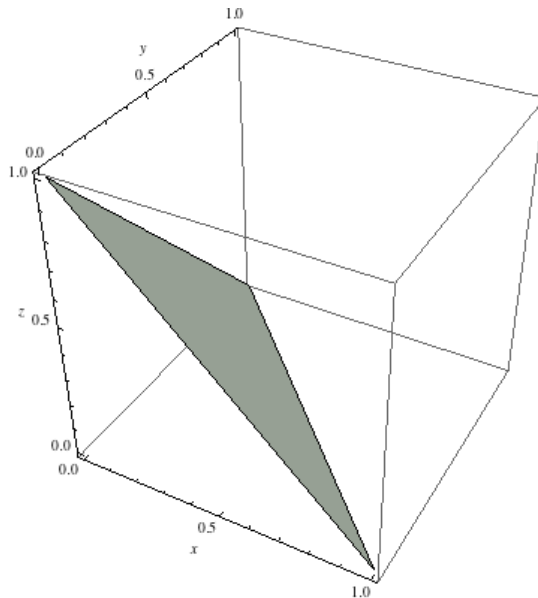
To compute the line integral directly, use the same parametrization as above, but with $r = 1$ to get the boundary; then as in the problem statement

$$\mathbf{r}(\theta) = 2 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{j} + 2 \mathbf{k}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\begin{aligned}
 \oint_C [x dx + (x+y) dy + (x+y+z) dz] &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
 &= \int_0^{2\pi} (2 \cos \theta \mathbf{i} + (2 \cos \theta + 2 \sin \theta) \mathbf{j} + (2 \cos \theta + 2 \sin \theta + 2) \mathbf{k}) \\
 &\quad \cdot (-2 \sin \theta \mathbf{i} + 2 \cos \theta \mathbf{j}) d\theta \\
 &= \int_0^{2\pi} 4 \cos^2 \theta d\theta \\
 &= \int_0^{2\pi} (2 + 2 \cos(2\theta)) d\theta \\
 &= [2\theta + \sin(2\theta)]_0^{2\pi} = \boxed{4\pi}.
 \end{aligned}$$

30. The triangle is



We parametrize the triangle by

$$\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + (1 - x - y) \mathbf{k}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

Then

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Since this is upward-pointing, we use this normal. Further,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z \mathbf{i} - 2x \mathbf{j} - 2y \mathbf{k}.$$

Then

$$\begin{aligned} \oint_C [y^2 dx + z^2 dy + x^2 dz] &= \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS \\ &= \int_0^1 \int_0^{1-x} (-2(1-x-y) \mathbf{i} - 2x \mathbf{j} - 2y \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) dy dx \\ &= \int_0^1 \int_0^{1-x} (-2) dy dx \\ &= \int_0^1 [-2y]_0^{1-x} dx \\ &= \int_0^1 2(x-1) dx \\ &= [(x-1)^2]_0^1 = -1. \end{aligned}$$

To compute the line integral directly, we parametrize the boundary using three separate curves:

$$\begin{aligned} C_1 : & \text{ from } (1, 0, 0) \text{ to } (0, 1, 0) & \mathbf{r}(t) &= (1-t) \mathbf{i} + t \mathbf{j}, \quad 0 \leq t \leq 1 \\ C_2 : & \text{ from } (0, 1, 0) \text{ to } (0, 0, 1) & \mathbf{r}(t) &= (1-t) \mathbf{j} + t \mathbf{k}, \quad 0 \leq t \leq 1 \\ C_3 : & \text{ from } (0, 0, 1) \text{ to } (1, 0, 0) & \mathbf{r}(t) &= t \mathbf{i} + (1-t) \mathbf{k}. \end{aligned}$$

Then

$$\begin{aligned} \oint_C (y^2 dx + z^2 dy + x^2 dz) &= \int_{C_1} (y^2 dx + z^2 dy + x^2 dz) + \int_{C_2} (y^2 dx + z^2 dy + x^2 dz) \\ &\quad + \int_{C_3} (y^2 dx + z^2 dy + x^2 dz) \\ &= \int_0^1 (t^2 \mathbf{i} + (1-t)^2 \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j}) dt + \int_0^1 ((1-t)^2 \mathbf{i} + t^2 \mathbf{j}) \cdot (-\mathbf{j} + \mathbf{k}) dt \\ &\quad + \int_0^1 ((1-t)^2 \mathbf{j} + t^2 \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k}) dt \\ &= \int_0^1 (-t^2) dt + \int_0^1 (-t^2) dt + \int_0^1 (-t^2) dt \\ &= -3 \int_0^1 t^2 dt = -3 \left[\frac{1}{3} t^3 \right]_0^1 = \boxed{-1}. \end{aligned}$$

31. We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \mathbf{0},$$

so that $\mathbf{F}$ is conservative, by the theorem in Objective 4.

32. We have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \mathbf{0},$$

so that $\mathbf{F}$ is conservative, by the theorem in Objective 4.

33. We have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k} \neq \mathbf{0},$$

so that $\mathbf{F}$ is not conservative, by the theorem in Objective 4.

34. We have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \mathbf{0},$$

so that $\mathbf{F}$ is conservative, by the theorem in Objective 4.

Applications and Extensions

35. $\mathbf{F}$ is conservative if and only if its curl is zero:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{y} & c\frac{xz}{y^2} & 0 \end{vmatrix} = (-x + 2cx)\mathbf{k}.$$

This is $\mathbf{0}$ if and only if $c = \frac{1}{2}$.

36. $\mathbf{F}$ is conservative if and only if its curl is zero:

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{y} & c\frac{xz}{y^2} & \frac{x}{y} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y} \left(\frac{x}{y} \right) - \frac{\partial}{\partial z} \left(c\frac{xz}{y^2} \right) \right) \mathbf{i} + \left(\frac{\partial}{\partial z} \left(\frac{z}{y} \right) - \frac{\partial}{\partial x} \left(\frac{x}{y} \right) \right) \mathbf{j} + \left(\frac{\partial}{\partial x} \left(c\frac{xz}{y^2} \right) - \frac{\partial}{\partial y} \left(\frac{z}{y} \right) \right) \mathbf{k} \\ &= \left(-\frac{x}{y^2} - c\frac{x}{y^2} \right) \mathbf{i} + \left(\frac{1}{y} - \frac{1}{y} \right) \mathbf{j} + \left(c\frac{z}{y^2} + \frac{z}{y^2} \right) \mathbf{k}. \end{aligned}$$

This is $\mathbf{0}$ if and only if $c = -1$.

37. Let $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ and $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$. Then

$$\mathbf{F} + \mathbf{G} = (f_1 + g_1)\mathbf{i} + (f_2 + g_2)\mathbf{j} + (f_3 + g_3)\mathbf{k},$$

so that

$$\begin{aligned} \operatorname{curl}(\mathbf{F} + \mathbf{G}) &= \left(\frac{\partial}{\partial y}(f_3 + g_3) - \frac{\partial}{\partial z}(f_2 + g_2) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(f_3 + g_3) - \frac{\partial}{\partial z}(f_1 + g_1) \right) \mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}(f_2 + g_2) - \frac{\partial}{\partial y}(f_1 + g_1) \right) \mathbf{k} \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \\ &\quad + \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial g_3}{\partial x} - \frac{\partial g_1}{\partial z} \right) \mathbf{j} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right) \mathbf{k} \\ &= \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}. \end{aligned}$$

38. Let $\mathbf{F} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$. Then

$$c\mathbf{F} = (cf_1)\mathbf{i} + (cf_2)\mathbf{j} + (cf_3)\mathbf{k},$$

so that

$$\begin{aligned} \operatorname{curl}(c\mathbf{F}) &= \left(\frac{\partial}{\partial y}(cf_3) - \frac{\partial}{\partial z}(cf_2) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(cf_3) - \frac{\partial}{\partial z}(cf_1) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(cf_2) - \frac{\partial}{\partial y}(cf_1) \right) \mathbf{k} \\ &= c \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} - c \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \mathbf{j} + c \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \\ &= c \operatorname{curl} \mathbf{F}. \end{aligned}$$

39. Parametrize the hemisphere using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = \cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

Then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \end{vmatrix} = \cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}.$$

This is upward-pointing in the given range of ϕ , so after normalization, it is the normal we want. Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^{\pi/2} (\cos \phi \mathbf{i} + \cos \theta \sin \phi \mathbf{j} + \sin \theta \sin \phi \mathbf{k}) \\ &\quad \cdot \frac{1}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|} (\cos \theta \sin^2 \phi \mathbf{i} + \sin \theta \sin^2 \phi \mathbf{j} + \sin \phi \cos \phi \mathbf{k}) \\ &\quad \cdot \|\mathbf{r}_\phi \times \mathbf{r}_\theta\| d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (\cos \theta \sin^2 \phi \cos \phi + \sin \theta \cos \theta \sin^3 \phi + \sin \theta \sin^2 \phi \cos \phi) d\phi d\theta \\ &= \int_0^{\pi/2} \int_0^{2\pi} (\cos \theta \sin^2 \phi \cos \phi + \sin \theta \cos \theta \sin^3 \phi + \sin \theta \sin^2 \phi \cos \phi) d\theta d\phi \\ &= \int_0^{\pi/2} \left[\sin \theta \sin^2 \phi \cos \phi + \frac{1}{2} \sin^2 \theta \sin^3 \phi - \cos \theta \sin^2 \phi \cos \phi \right]_0^{2\pi} d\phi \\ &= \boxed{0}. \end{aligned}$$

40. Parametrize the unit disk via

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \mathbf{k}.$$

Since this is an upward-pointing normal, we use this normal. Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^1 \int_0^{2\pi} (r \cos \theta \mathbf{j} + r \sin \theta \mathbf{k}) \cdot r \mathbf{k} d\theta dr \\ &= \int_0^1 \int_0^{2\pi} r^2 \sin \theta d\theta dr \\ &= \int_0^1 [-r^2 \cos \theta]_0^{2\pi} dr = \boxed{0}. \end{aligned}$$

Note that the answers to Problems 38 and 39 are the same. In fact, $\mathbf{F} = \operatorname{curl}(xz \mathbf{i} + xy \mathbf{j} + xz \mathbf{k})$, so the comment after Example 3 ensures that the two integrals will be equal.

41. Computing $\text{curl } \mathbf{F}$ gives

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix} = -2\mathbf{k}.$$

Since the curl is nonzero, the vector field is not conservative. To find a path closed path C on which the line integral is zero, note that by Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = \iint_S (-2\mathbf{k}) \cdot \mathbf{n} dS,$$

so that any surface whose normal has no $\mathbf{k}$ component will suffice. So for example we can choose the square with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$; it is parametrized as

$$\mathbf{r}(y, z) = y\mathbf{j} + z\mathbf{k}, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1,$$

so that its normal is

$$\mathbf{r}_y \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}.$$

Then $\text{curl } \mathbf{F} \cdot \mathbf{n} = 0$, so the path forming the boundary of this square is a path along which the line integral is zero.

42. Since

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k},$$

we have

$$\begin{aligned} \text{div curl } \mathbf{F} &= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\ &= R_{xy} - Q_{xz} - R_{yx} + P_{yz} + Q_{zx} - P_{zy} = 0 \end{aligned}$$

since mixed partials are equal (because the functions are twice differentiable with continuous derivatives).

43. (a) Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field with continuous and differentiable components. Suppose $Q_x = P_y$, $R_y = Q_z$, and $P_z = R_x$. Then

$$\text{curl } \mathbf{F} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} = \mathbf{0},$$

which implies that $\mathbf{F}$ is the gradient of some function. Therefore $\mathbf{F}$ is a gradient field.

(b) Answers will vary.

44. Here

$$P(x, y, z) = yz - y - z, \quad Q(x, y, z) = xz - x - z, \quad R(x, y, z) = xy - x - y,$$

so that

$$\begin{aligned} P_y &= z - 1, & Q_x &= z - 1, & R_x &= y - 1, \\ P_z &= y - 1, & Q_z &= x - 1, & R_y &= x - 1. \end{aligned}$$

Then $Q_x = P_y$, $R_y = Q_z$, and $R_x = P_z$, so by Problem 43, the given integral is independent of the path.

45. Here

$$P(x, y, z) = x + y, \quad Q(x, y, z) = x - z, \quad R(x, y, z) = z - y,$$

so that

$$\begin{aligned} P_y &= 1, & Q_x &= 1, & R_x &= 0, \\ P_z &= 0, & Q_z &= -1, & R_y &= -1. \end{aligned}$$

Then $Q_x = P_y$, $R_y = Q_z$, and $R_x = P_z$, so by Problem 43, the given integral is independent of the path. To find the value of the integral, parametrize the path as

$$\mathbf{r}(t) = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k}, \quad 0 \leq t \leq 1.$$

Then the work performed is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 ((at + bt)\mathbf{i} + (at - ct)\mathbf{j} + (ct - bt)\mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \, dt \\ &= \int_0^1 (a^2 + ab + ab - bc + c^2 - bc)t \, dt \\ &= (a^2 + 2ab - 2bc + c^2) \int_0^1 t \, dt \\ &= (a^2 + 2ab - 2bc + c^2) \left[\frac{1}{2}t^2 \right]_0^1 \\ &= \boxed{\frac{a^2 + 2ab - 2bc + c^2}{2}}. \end{aligned}$$

46. By Problem 42, if the given vector field is $\text{curl } \mathbf{F}$ for some vector field $\mathbf{F}$, then

$$\text{div curl } \mathbf{F} = \text{div}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 0.$$

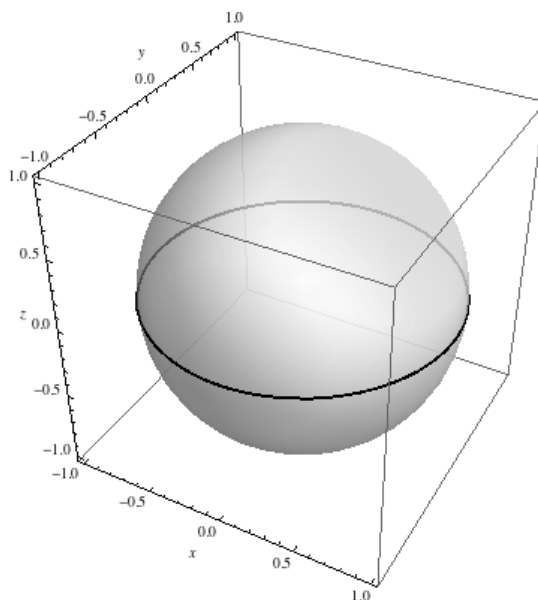
But in fact

$$\text{div}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0.$$

Therefore the given vector field is not the curl of any twice-differentiable vector field.

Challenge Problems

47. The simplest argument is to apply Stokes' Theorem directly to the surface of the sphere. Since the boundary of the sphere is the empty curve, the line integral around that curve is zero. Alternatively, consider the diagram



Denote the upper hemisphere by U and the lower by L . Then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_U \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS + \iint_L \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS.$$

Applying Stokes' Theorem to each piece, we see that each boundary curve is the unit circle C in the xy -plane. However, the direction of the parametrization for U is determined (see the discussion in Objective 1) by walking around the circle so that the surface is on your left, which is counter-clockwise; for L , keeping the surface on your left results in a clockwise parametrization. So the two parametrizations have derivatives that are negatives of each other, so that

$$\iint_U \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS + \iint_L \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_C \mathbf{F} \cdot (-d\mathbf{r}) = 0.$$

48. The conditions on E , S , and $\mathbf{F}$ ensure that the Divergence Theorem applies, so that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \operatorname{curl} \mathbf{F} \, dV.$$

But by Problem 42, $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$, so the integral is zero.

49. The conditions of the problem statement ensure that Stokes' Theorem applies. Then

$$\begin{aligned} \operatorname{curl}(f\nabla g) &= \operatorname{curl} \left(f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k} \right) \\ &= \left(\frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial z} \right) - \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial y} \right) \right) \mathbf{i} - \left(\frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial z} \right) - \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial x} \right) \right) \mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial y} \right) - \frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial x} \right) \right) \mathbf{k}. \end{aligned}$$

Computing the derivatives above gives:

$$\begin{aligned} \frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial z} \right) &= f_y g_z + f g_{yz}, & \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial y} \right) &= f_z g_y + f g_{zy}, \\ \frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial z} \right) &= f_x g_z + f g_{xz}, & \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial x} \right) &= f_z g_x + f g_{zx}, \\ \frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial y} \right) &= f_x g_y + f g_{xy}, & \frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial x} \right) &= f_y g_x + f g_{yx}. \end{aligned}$$

Therefore, since mixed partials are equal,

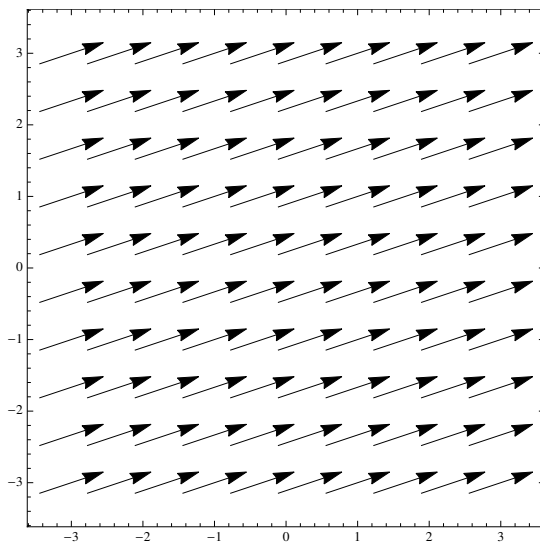
$$\begin{aligned} \operatorname{curl}(f\nabla g) &= (f_y g_z - f_z g_y) \mathbf{i} - (f_x g_z - f_z g_x) \mathbf{j} + (f_x g_y - f_y g_x) \mathbf{k} \\ &= (f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}) \times (g_x \mathbf{i} + g_y \mathbf{j} + g_z \mathbf{k}) \\ &= \nabla f \times \nabla g. \end{aligned}$$

Then by Stokes' Theorem,

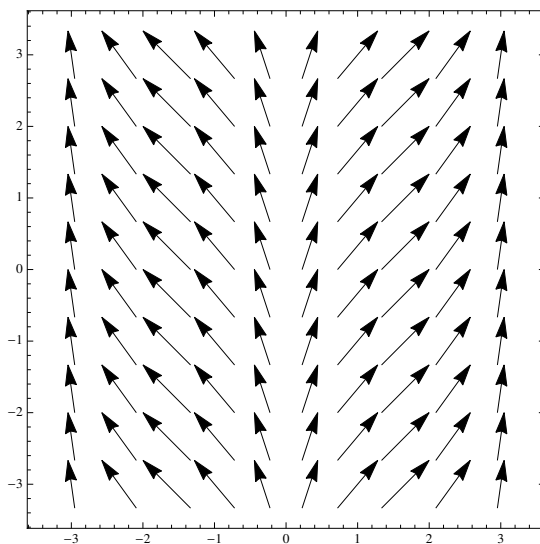
$$\int_C (f\nabla g) \cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g) \cdot \mathbf{n} \, dS = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, dS.$$

Chapter Review

1. The vector field is



2. The vector field is



3. Since $y = x^2$, we have $dy = 2x \, dx$. The range of integration is from $x = 1$ to $x = 2$. Therefore we get

$$\int_C \left(\frac{dx}{y} + \frac{dy}{x} \right) = \int_1^2 \left(\frac{dx}{x^2} + \frac{2x \, dx}{x} \right) = \int_1^2 \left(\frac{1}{x^2} + 2 \right) dx = \left[-\frac{1}{x} + 2x \right]_1^2 = \boxed{\frac{5}{2}}.$$

4. With the given parametrization, we have

$$dx = 2t \, dt, \quad dy = 3t^2 \, dt,$$

so that

$$\int_C [y \cos(xy) \, dx + x \cos(xy) \, dy] = \int_0^1 [t^3 \cos(t^2 \cdot t^3) \cdot 2t + t^2 \cos(t^2 \cdot t^3) \cdot 3t^2] \, dt = \int_0^1 5t^4 \cos(t^5) \, dt.$$

Now use the substitution $u = t^5$, so that $du = 5t^4 dt$; then $t = 0$ corresponds to $u = 0$ and $t = 1$ to $u = 1$, so the integral becomes

$$\int_0^1 5t^4 \cos(t^5) dt = \int_0^1 \cos u du = [\sin u]_0^1 = \boxed{\sin 1}.$$

5. (a) Since $P(x, y) = y \cos(xy)$ and $Q(x, y) = x \cos(xy)$ are continuous with continuous derivatives everywhere in the plane, to see if the vector field is conservative we must check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x \cos(xy)) = \cos(xy) - xy \sin(xy), \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y \cos(xy)) = \cos(xy) - xy \sin(xy).$$

Since the partials are equal, this vector field is conservative so the integral is independent of path.

- (b) The given path goes from $(0, 0)$ to $(1, 1)$, as does the path in Problem 4. Since the vector fields are the same, we conclude that the value of the integral along the right-angle path is the same as the integral in Problem 4, or $\boxed{\sin 1}$.

6. (a) First integrate $P(x, y) = y \cos(xy)$ with respect to x :

$$\int P(x, y) dx = \int y \cos(xy) dx = y \cdot \frac{1}{y} \sin(xy) + h(y) = \sin(xy) + h(y).$$

Then the y -derivative of this expression must be equal to $Q(x, y)$, so that

$$\frac{\partial}{\partial y}(\sin(xy) + h(y)) = x \cos(xy) + h'(y) = Q(x, y) = x \cos(xy),$$

so that $h'(y) = 0$. Therefore $h(y) = C$ for an arbitrary constant C , so that $\boxed{f(x, y) = \sin(xy) + C}$.

- (b) By the Fundamental Theorem of Line Integrals, the value of the integral along any path is the difference of the value of f at the two endpoints, so that in this case we get $f(1, 1) - f(0, 0) = \sin 1 - \sin 0 = \boxed{\sin 1}$.

7. With the given parametrization, we have

$$dx = dt, \quad dy = 2t dt, \quad dz = 3t^2 dt,$$

so that

$$\begin{aligned} \int_C [(yz - y - z) dx + (xz - x - z) dy + (xy - x - y) dz] \\ &= \int_0^1 [(t^2 \cdot t^3 - t^2 - t^3) dt + (t \cdot t^3 - t - t^3) \cdot 2t dt + (t \cdot t^2 - t - t^2) \cdot 3t^2 dt] \\ &= \int_0^1 (6t^5 - 5t^4 - 4t^3 - 3t^2) dt \\ &= [t^6 - t^5 - t^4 - t^3]_0^1 = \boxed{-2}. \end{aligned}$$

8. Parametrize the line segment by

$$\mathbf{r}(t) = t \mathbf{i} - 2t \mathbf{j} + 3t \mathbf{k}, \quad 0 \leq t \leq 1,$$

so that $\mathbf{r}_t = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{r} &= \int_0^1 (t \cdot (-2t) \mathbf{i} + t \cdot 3t \mathbf{j} + (-2t - t) \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) dt \\ &= \int_0^1 (-2t^2 - 6t^2 - 9t) dt \\ &= \int_0^1 (-8t^2 - 9t) dt \\ &= \left[-\frac{8}{3}t^3 - \frac{9}{2}t^2 \right]_0^1 = \boxed{-\frac{43}{6}}. \end{aligned}$$

9. (a) We have

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = 6x \mathbf{i} + \mathbf{j} = \mathbf{F}(x, y).$$

- (b) By the Fundamental Theorem of Line Integrals, we have

$$\int_C (6x \, dx + dy) = f(6, 3) - f(4, 1) = (3 \cdot 6^2 + 3 - 2) - (3 \cdot 4^2 + 1 - 2) = \boxed{62}.$$

10. We are given that $\nabla(e^x \sin y) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$, so the vector field we are integrating is conservative. Write $f(x, y) = e^x \sin y$. Then by the Fundamental Theorem of Line Integrals

$$\int_C (e^x \sin y \, dx + e^x \cos y \, dy) = f(1, 1) - f(0, 0) = e^1 \sin 1 - e^0 \sin 0 = \boxed{e \sin 1}.$$

11. The mass is given by the line integral of the density function along the wire; the density function is $\rho(t) = x + 1 = \cos t + 1$. Therefore the mass is

$$\begin{aligned} \int_C \rho(x, y) \, ds &= \int_0^{3\pi/4} (x + 1) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\ &= \int_0^{3\pi/4} (\cos t + 1) \sqrt{(-\sin t)^2 + \cos^2 t} \, dt \\ &= \int_0^{3\pi/4} (\cos t + 1) \, dt \\ &= [\sin t + t]_0^{3\pi/4} \\ &= \frac{3\pi}{4} + \sin \frac{3\pi}{4} = \boxed{\frac{3\pi}{4} + \frac{\sqrt{2}}{2}}. \end{aligned}$$

12. (a) Parametrize the line segment by $x = 1 + t$, $y = 1 + 3t$, $0 \leq t \leq 1$. Then $dx = dt$ and $dy = 3 \, dt$, so

$$\begin{aligned} \int_C (x^3 \, dx + y^3 \, dy) &= \int_0^1 ((1+t)^3 + (1+3t)^3 \cdot 3) \, dt \\ &= \int_0^1 ((1+t)^3 + 3(1+3t)^3) \, dt \\ &= \left[\frac{1}{4}(1+t)^4 + \frac{1}{4}(1+3t)^4 \right]_0^1 \\ &= (4 + 64) - \left(\frac{1}{4} + \frac{1}{4} \right) = \boxed{\frac{135}{2}}. \end{aligned}$$

- (b) Parametrize the three line segments as follows:

$$\begin{aligned} C_1, \text{ from } (1, 1) \text{ to } (1, 5): & \quad x = 1, \, y = 1 + 4t, \, 0 \leq t \leq 1, \quad dx = 0, \, dy = 4 \, dt \\ C_2, \text{ from } (1, 5) \text{ to } (2, 3): & \quad x = 1 + t, \, y = 5 - 2t, \, 0 \leq t \leq 1, \quad dx = dt, \, dy = -2 \, dt \\ C_3, \text{ from } (2, 3) \text{ to } (2, 4): & \quad x = 2, \, y = 3 + t, \, 0 \leq t \leq 1, \quad dx = 0, \, dy = dt. \end{aligned}$$

Then

$$\begin{aligned}
 \int_C (x^3 dx + y^3 dy) &= \int_{C_1} (x^3 dx + y^3 dy) + \int_{C_2} (x^3 dx + y^3 dy) + \int_{C_3} (x^3 dx + y^3 dy) \\
 &= \int_0^1 (1+4t)^3 \cdot 4 dt + \int_0^1 ((1+t)^3 + (5-2t)^3 \cdot (-2)) dt \\
 &\quad + \int_0^1 (3+t)^3 \cdot 1 dt \\
 &= \int_0^1 4(1+4t)^3 dt + \int_0^1 ((1+t)^3 - 2(5-2t)^3) dt + \int_0^1 (3+t)^3 dt \\
 &= \left[\frac{1}{4}(1+4t)^4 + \frac{1}{4}(1+t)^4 + \frac{1}{4}(5-2t)^4 + \frac{1}{4}(3+t)^4 \right]_0^1 \\
 &= \left(\frac{625}{4} + 4 + \frac{81}{4} + 64 \right) - \left(\frac{1}{4} + \frac{1}{4} + \frac{625}{4} + \frac{81}{4} \right) \\
 &= \boxed{\frac{135}{2}}.
 \end{aligned}$$

(c) Parametrize the parabolic piece by $x = t$, $y = t^2$, $1 \leq t \leq 2$; then $dx = dt$ and $dy = 2t dt$, so

$$\begin{aligned}
 \int_C (x^3 dx + y^3 dy) &= \int_1^2 (t^3 \cdot 1 + t^6 \cdot 2t) dt \\
 &= \int_1^2 (t^3 + 2t^7) dt \\
 &= \left[\frac{1}{4}t^4 + \frac{1}{4}t^8 \right]_1^2 \\
 &= (4 + 64) - \left(\frac{1}{4} + \frac{1}{4} \right) = \boxed{\frac{135}{2}}.
 \end{aligned}$$

13. Parametrize the lamina using cylindrical coordinates; then

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k}, \quad 1 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\begin{aligned}
 \mathbf{r}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \mathbf{k}, \quad \mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\
 \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} + r \mathbf{k} \\
 \|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = r\sqrt{2}.
 \end{aligned}$$

Then the mass of the lamina is the integral of the mass density function over the surface, so it is

$$\begin{aligned}
 M &= \iint_S \rho(x, y, z) dS = \int_1^4 \int_0^{2\pi} r^2 \cdot \|\mathbf{r}_r \times \mathbf{r}_\theta\| d\theta dr \\
 &= \int_1^4 \int_0^{2\pi} r^3 \sqrt{2} d\theta dr \\
 &= \int_1^4 2\pi \sqrt{2} r^3 dr \\
 &= \left[\frac{1}{2} \pi \sqrt{2} r^4 \right]_1^4 = \boxed{\frac{255}{2} \sqrt{2} \pi}.
 \end{aligned}$$

14. Parametrize the curve by $y = t$, $x = t^2$ for $0 \leq t \leq 1$; then $dy = dt$ and $dx = 2t dt$, so

$$\begin{aligned}\int_C (x^2 y^3 dx - xy^4 dy) &= \int_0^1 (t^4 t^3 \cdot 2t - t^2 t^4) dt \\ &= \int_0^1 (2t^8 - t^6) dt \\ &= \left[\frac{2}{9} t^9 - \frac{1}{7} t^7 \right]_0^1 = \boxed{\frac{5}{63}}.\end{aligned}$$

15. With the given parametrization, we have $dx = -\sin t dt$ and $dy = \cos t dt$, so that

$$\begin{aligned}\int_C x^2 y^2 ds &= \int_0^\pi \cos^2 t \sin^2 t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi \cos^2 t \sin^2 t \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^\pi \cos^2 t \sin^2 t dt \\ &= \int_0^\pi \frac{1}{4} \sin^2(2t) dt \\ &= \frac{1}{4} \int_0^\pi \left(\frac{1}{2} - \frac{1}{2} \cos(4t) \right) dt \\ &= \frac{1}{4} \left[\frac{1}{2} t - \frac{1}{8} \sin(4t) \right]_0^\pi \\ &= \boxed{\frac{\pi}{8}}.\end{aligned}$$

16. Since the given vector field is discontinuous at the origin, we must compute the line integral. We can parametrize the unit circle C as

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then $(x^2 + y^2)^{3/2} = 1$, so the work done is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin \theta \mathbf{i} - \cos \theta \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\ &= \int_0^{2\pi} (-\sin^2 \theta - \cos^2 \theta) d\theta \\ &= \int_0^{2\pi} (-1) d\theta = \boxed{-2\pi}.\end{aligned}$$

17. Parametrize the curve by $\mathbf{r}(t) = t \mathbf{i} + \sin t \mathbf{j}$ for $0 \leq t \leq 2\pi$. Then the work is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin^2 t \mathbf{i} + \sin t \mathbf{j}) \cdot (\mathbf{i} + \cos t \mathbf{j}) dt \\ &= \int_0^{2\pi} (\sin^2 t + \sin t \cos t) dt \\ &= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2t) + \sin t \cos t \right) dt \\ &= \left[\frac{1}{2} t - \frac{1}{4} \sin(2t) + \frac{1}{2} \sin^2 t \right]_0^{2\pi} = \boxed{\pi}.\end{aligned}$$

18. The given vector field is conservative. To see this, start by integrating $\frac{x}{x^2+y^2}$ with respect to x , using the substitution $x^2 + y^2 = u$:

$$\int \frac{x}{x^2 + y^2} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + y^2) + h(y).$$

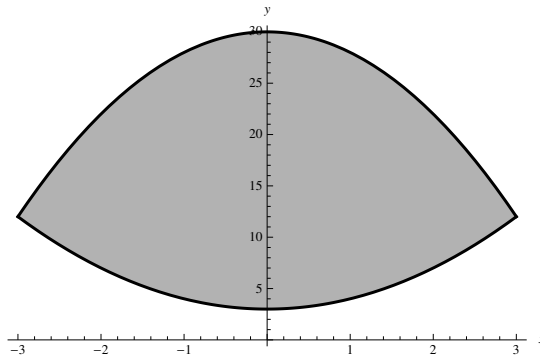
Differentiating this expression and setting it equal to $\frac{y}{x^2+y^2}$ gives

$$\frac{\partial}{\partial y} \left(\frac{1}{2} \ln(x^2 + y^2) + h(y) \right) = \frac{y}{x^2 + y^2} + h'(y) = \frac{y}{x^2 + y^2},$$

so that $h'(y) = 0$ and therefore $h(y) = C$ for an arbitrary constant C . So $\mathbf{F} = \nabla f$ where $f(x, y)$ is (taking $C = 0$) $f(x, y) = \frac{1}{2} \ln(x^2 + y^2)$. Then by the Fundamental Theorem of Line Integrals, the work done in moving the object along the given curve is

$$f(2\pi, 0) - f(-\pi, 0) = \frac{1}{2} \ln(4\pi^2) - \frac{1}{2} \ln(\pi^2) = \ln 2 + \ln \pi - \ln \pi = \boxed{\ln 2}.$$

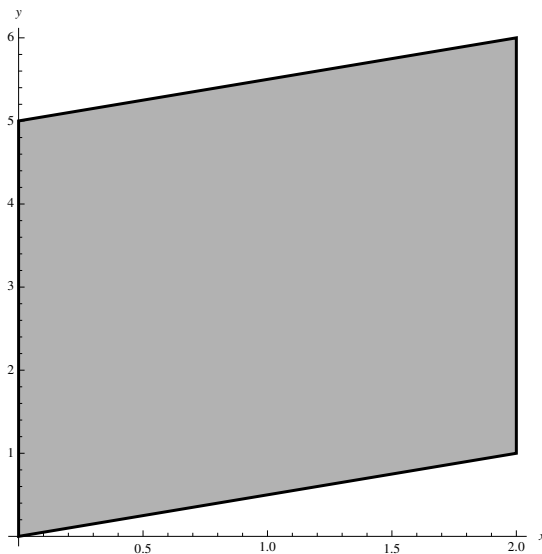
19. The region is



The lower curve is $t^2 + 3$, since that is an upward-opening parabola. The two curves intersect where $t^2 + 3 = 30 - 2t^2$, or $3t^2 = 27$. So they intersect when $t = \pm 3$ (which corresponds to the points $(\pm 3, 12)$). To determine the area, we compute the line integral around the boundary in a counterclockwise direction, so we walk along $t^2 + 3$ from $t = -3$ to $t = 3$, and then along $30 - 2t^2$ from $t = 3$ to $t = -3$. For $t^2 + 3$, we have $dx = dt$ and $dy = 2t dt$, while for $30 - 2t^2$ we have $dx = dt$ and $dy = -4t dt$. Using Green's formula for area, then, the area is

$$\begin{aligned} A &= \frac{1}{2} \int_{C_{\text{lower}}} (-y dx + x dy) + \frac{1}{2} \int_{C_{\text{higher}}} (-y dx + x dy) \\ &= \frac{1}{2} \int_{-3}^3 (-(t^2 + 3) + t \cdot 2t) dt + \frac{1}{2} \int_3^{-3} (-(30 - 2t^2) + t \cdot (-4t)) dt \\ &= \frac{1}{2} \int_{-3}^3 (t^2 - 3) dt + \frac{1}{2} \int_3^{-3} (-2t^2 - 30) dt \\ &= \frac{1}{2} \left[\frac{1}{3} t^3 - 3t \right]_{-3}^3 + \frac{1}{2} \left[-\frac{2}{3} t^3 - 30t \right]_3^{-3} \\ &= 0 + 108 = \boxed{108}. \end{aligned}$$

20. The parallelogram is



The lower edge of the parallelogram passes through $(0, 0)$ and $(2, 1)$, so it has the equation $y = \frac{1}{2}x$. The upper edge passes through $(0, 5)$ and $(2, 6)$, so its equation is $y = 5 + \frac{1}{2}x$. Since both $P(x, y) = \ln(1 + y)$ and $Q(x, y) = \frac{xy}{1+y}$ are continuous and differentiable for $y > -1$, Green's Theorem applies, so that

$$\begin{aligned}
 \int_C \left[\ln(1 + y) dx + \frac{xy}{1 + y} dy \right] &= \iint_R \left(\frac{\partial}{\partial x} \left(\frac{xy}{1 + y} \right) - \frac{\partial}{\partial y} (\ln(1 + y)) \right) dx dy \\
 &= \int_0^2 \int_{x/2}^{5+x/2} \left(\frac{y}{1 + y} - \frac{1}{1 + y} \right) dy dx \\
 &= \int_0^2 \int_{x/2}^{5+x/2} \left(\frac{1 + y}{1 + y} - \frac{1}{1 + y} - \frac{1}{1 + y} \right) dy dx \\
 &= \int_0^2 \int_{x/2}^{5+x/2} \left(1 - \frac{2}{1 + y} \right) dy dx \\
 &= \int_0^2 [y - 2 \ln(1 + y)]_{x/2}^{5+x/2} dx \\
 &= \int_0^2 \left(5 - 2 \ln \left(6 + \frac{x}{2} \right) + 2 \ln \left(1 + \frac{x}{2} \right) \right) dx \\
 &= \int_0^2 \left(5 - 2 \ln \frac{1}{2} - 2 \ln(12 + x) + 2 \ln \frac{1}{2} + 2 \ln(2 + x) \right) dx \\
 &= \int_0^2 (5 - 2 \ln(12 + x) + 2 \ln(2 + x)) dx.
 \end{aligned}$$

To integrate $\ln(12 + x)$ and $\ln(2 + x)$, use Formula 118 in the endnotes together with the substitution $u = 12 + x$ or $u = 2 + x$. For example,

$$\int \ln(12 + x) dx = \int \ln u du = u \ln u - u + C = (12 + x) \ln(12 + x) - (12 + x) + C.$$

Similarly, $\int \ln(2+x) dx = (2+x) \ln(2+x) - (2+x) + C$. So the integral above is

$$\begin{aligned}
 \int_0^2 (5 - 2 \ln(12+x) + 2 \ln(2+x)) dx \\
 &= [5x - 2((12+x) \ln(12+x) - (12+x)) + 2((2+x) \ln(2+x) - (2+x))]_0^2 \\
 &= [5x + 20 - 2(12+x) \ln(12+x) + 2(2+x) \ln(2+x)]_0^2 \\
 &= (10 + 20 - 28 \ln 14 + 8 \ln 4) - (0 + 20 - 24 \ln 12 + 4 \ln 2) \\
 &= 10 - 28 \ln 7 - 28 \ln 2 + 16 \ln 2 + 48 \ln 2 + 24 \ln 3 - 4 \ln 2 \\
 &= \boxed{10 - 28 \ln 7 + 32 \ln 2 + 24 \ln 3}.
 \end{aligned}$$

21. Parametrize the four edges of the square as follows:

$$\begin{aligned}
 C_1 : (0,0) \rightarrow (1,0) : \quad x &= t, \ y = 0, \quad 0 \leq t \leq 1, \quad dx = dt, \ dy = 0 \\
 C_2 : (1,0) \rightarrow (1,1) : \quad x &= 1, \ y = t, \quad 0 \leq t \leq 1, \quad dx = 0, \ dy = dt \\
 C_3 : (1,1) \rightarrow (0,1) : \quad x &= 1-t, \ y = 1, \quad 0 \leq t \leq 1, \quad dx = -dt, \ dy = 0 \\
 C_4 : (0,1) \rightarrow (0,0) : \quad x &= 0, \ y = 1-t, \quad 0 \leq t \leq 1, \quad dx = 0, \ dy = -dt.
 \end{aligned}$$

Then the line integral is the sum of the line integrals on each piece:

$$\begin{aligned}
 \oint_C (y^2 dx - x^2 dy) &= \int_{C_1} (y^2 dx - x^2 dy) + \int_{C_2} (y^2 dx - x^2 dy) \\
 &\quad + \int_{C_3} (y^2 dx - x^2 dy) + \int_{C_4} (y^2 dx - x^2 dy) \\
 &= \int_0^1 0 dt + \int_0^1 (-1) dt + \int_0^1 (-1) dt + \int_0^1 0 dt \\
 &= -\int_0^1 1 dt - \int_0^1 1 dt = \boxed{-2}.
 \end{aligned}$$

22. Since the component functions $P(x, y) = y^2$ and $Q(x, y) = -x^2$ are continuous and differentiable everywhere, Green's Theorem applies, and

$$\begin{aligned}
 \oint_C (y^2 dx - x^2 dy) &= \iint_R \left(\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(y^2) \right) dx dy \\
 &= \int_0^1 \int_0^1 (-2x - 2y) dx dy \\
 &= \int_0^1 [-x^2 - 2xy]_0^1 dy \\
 &= \int_0^1 (-1 - 2y) dy \\
 &= [-y - y^2]_0^1 = \boxed{-2}.
 \end{aligned}$$

23. With the given parametrization, $dx = -2 \sin t dt$ and $dy = 3 \cos t dt$, so the line integral is

$$\begin{aligned}
 \oint_C [(x-y) dx + (x+y) dy] &= \int_0^{2\pi} [(2 \cos t - 3 \sin t)(-2 \sin t) + (2 \cos t + 3 \sin t)(3 \cos t)] dt \\
 &= \int_0^{2\pi} (5 \sin t \cos t + 6 \sin^2 t + 6 \cos^2 t) dt \\
 &= \int_0^{2\pi} (5 \sin t \cos t + 6) dt \\
 &= \left[\frac{5}{2} \sin^2 t + 6t \right]_0^{2\pi} = \boxed{12\pi}.
 \end{aligned}$$

24. Since the component functions $P(x, y) = x - y$ and $Q(x, y) = x + y$ are continuous and differentiable everywhere, Green's Theorem applies, and

$$\oint_C ((x - y) dx + (x + y) dy) = \iint_R \left(\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x - y) \right) dx dy = \iint_R 2 dx dy.$$

The region R is the ellipse $9x^2 + 4y^2 = 36$, so this integral is

$$\iint_R 2 dx dy = \int_{-2}^2 \int_{-\sqrt{36-9x^2}/2}^{\sqrt{36-9x^2}/2} 2 dy dx.$$

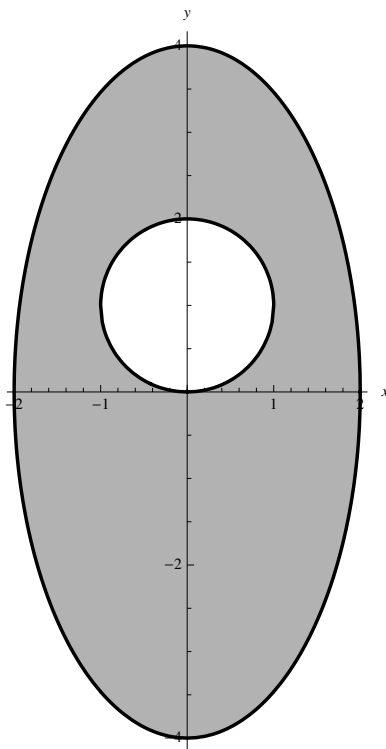
Since the ellipse is symmetric about the origin, we compute this area in the first quadrant and multiply by 4, so we get

$$4 \int_0^2 \int_0^{\sqrt{36-9x^2}/2} 2 dy dx = 4 \int_0^2 [2y]_0^{\sqrt{36-9x^2}/2} dx = 4 \int_0^2 \sqrt{36-9x^2} dx.$$

Use the trigonometric substitution $x = 2 \sin u$; then $dx = 2 \cos u du$. Since we are working in the first quadrant, $\sqrt{36-9x^2} = \sqrt{36-36 \sin^2 u} = 6 \cos u$. Then $x = 0$ corresponds to $u = 0$ and $x = 2$ to $u = \frac{\pi}{2}$. So we get

$$4 \int_0^2 \sqrt{36-9x^2} dx = 4 \int_0^{\pi/2} 12 \cos^2 u du = 4 \int_0^{\pi/2} (6 + 6 \cos(2u)) du = 4 [6u + 3 \sin(2u)]_0^{\pi/2} = \boxed{12\pi}.$$

25. The region is



Using Green's Theorem, the area of the region is one half the sum of the line integrals, properly oriented, of $-y dx + x dy$ along each of the boundary curves. Parametrize the outer curve, the ellipse C_1 , by

$$\mathbf{r}_1(\theta) = 2 \cos \theta \mathbf{i} + 4 \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi;$$

here we choose a counterclockwise orientation so as to keep the region on our left as we walk the curve. Similarly, we parametrize the inner circle C_2 in a clockwise parametrization so as to keep the region on our left:

$$\mathbf{r}_2(\theta) = \cos \theta \mathbf{i} + (1 - \sin \theta) \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then the area is

$$\begin{aligned} A &= \frac{1}{2} \oint_{C_1} (-y dx + x dy) + \frac{1}{2} \oint_{C_2} (-y dx + x dy) \\ &= \frac{1}{2} \int_0^{2\pi} (-4 \sin \theta \cdot (-2 \sin \theta) + 2 \cos \theta \cdot (4 \cos \theta)) d\theta \\ &\quad + \frac{1}{2} \int_0^{2\pi} ((\sin \theta - 1) \cdot (-\sin \theta) + \cos \theta \cdot (-\cos \theta)) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 8 d\theta + \frac{1}{2} \int_0^{2\pi} (-1 + \sin \theta) d\theta \\ &= 8\pi + \frac{1}{2} [-\theta - \cos \theta]_0^{2\pi} = \boxed{7\pi}. \end{aligned}$$

26. (a) When $u = 0$, the equation is $\mathbf{r}(0, v) = \mathbf{0}$, so the coordinate curve consists of only the origin. Otherwise, for $0 < u \leq 1$, the coordinate curve is an ellipse whose semimajor axis is $3u$ and whose semiminor axis is $2u$, lying in the plane $z = u^2$. When $v = 0$, the coordinate curve is $\mathbf{r}(u, 0) = 3u \mathbf{i} + u^2 \mathbf{k}$, which is a parabolic arc since $z = u^2 = \frac{1}{9}(3u \cos 0)^2 = \frac{1}{9}x^2$. When $v = \frac{\pi}{2}$, the coordinate curve is $\mathbf{r}(u, \frac{\pi}{2}) = 2u \mathbf{j} + u^2 \mathbf{k}$, which is again a parabolic arc. In general, the coordinate curves are all parabolic arcs.

- (b) We have $x = 3u \cos v$, $y = 2u \sin v$, $z = u^2$. Then

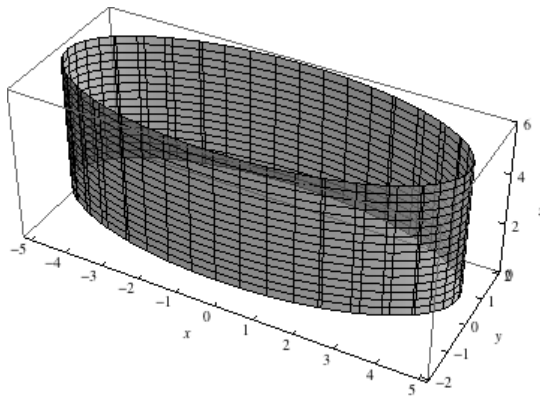
$$\left(\frac{1}{3}x\right)^2 + \left(\frac{1}{2}y\right)^2 = (u \cos v)^2 + (u \sin v)^2 = u^2 = z.$$

Expanding gives

$$z = \frac{x^2}{9} + \frac{y^2}{4},$$

which is a paraboloid.

27. The region is



The level curve for any value of z is the ellipse $4x^2 + 25y^2 = 100$, which is parametrized in polar coordinates by $x = 5 \cos \theta$, $y = 2 \sin \theta$ for $0 \leq \theta \leq 2\pi$. Then using cylindrical coordinates, $z = z$, and we get for the desired parametrization

$$\mathbf{r}(\theta, z) = 5 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 1 \leq z \leq 6.$$

28. (a) The values of u and v that give the point $(\sqrt{3}, 4, 1)$ are found by solving the three equations

$$\begin{aligned}u \sin v &= \sqrt{3}, \\u^2 &= 4, \\u \cos v &= 1.\end{aligned}$$

So $u = \pm 2$. If $u = 2$, then $v = \frac{\pi}{3}$; if $u = -2$, then $v = \frac{2\pi}{3}$. We consider only the values $u = 2$, $v = \frac{\pi}{3}$ (choosing the other pair would give the same answer). Next, find the partial derivatives of $\mathbf{r}$:

$$\begin{aligned}\mathbf{r}_u(u, v) &= \sin v \mathbf{i} + 2u \mathbf{j} + \cos v \mathbf{k} \\ \mathbf{r}_v(u, v) &= u \cos v \mathbf{i} - u \sin v \mathbf{k}.\end{aligned}$$

When $u = 2$ and $v = \frac{\pi}{3}$, these are

$$\begin{aligned}\mathbf{r}_u\left(2, \frac{\pi}{3}\right) &= \sin \frac{\pi}{3} \mathbf{i} + 4 \mathbf{j} + \cos \frac{\pi}{3} \mathbf{k} = \frac{\sqrt{3}}{2} \mathbf{i} + 4 \mathbf{j} + \frac{1}{2} \mathbf{k} \\ \mathbf{r}_v\left(2, \frac{\pi}{3}\right) &= 2 \cos \frac{\pi}{3} \mathbf{i} - 2 \sin \frac{\pi}{3} \mathbf{k} = \mathbf{i} - \sqrt{3} \mathbf{k}.\end{aligned}$$

Therefore a normal vector to the surface at the given point is

$$\left\langle \frac{\sqrt{3}}{2}, 4, \frac{1}{2} \right\rangle \times \langle 1, 0, -\sqrt{3} \rangle = \langle -4\sqrt{3}, 2, -4 \rangle.$$

So the equation of the tangent plane at $(\sqrt{3}, 4, 1)$ is $-4\sqrt{3}(x - \sqrt{3}) + 2(y - 4) - 4(z - 1) = 0$, or $\boxed{2\sqrt{3}x - y + 2z = 4}$.

- (b) The normal line has direction vector $\langle -4\sqrt{3}, 2, -4 \rangle$, so it can be parametrized as

$$\boxed{(\sqrt{3} - 4\sqrt{3}t) \mathbf{i} + (4 + 2t) \mathbf{j} + (1 - 4t) \mathbf{k}}.$$

29. With the given parametrization, we have

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin v & 2u & \cos v \\ u \cos v & 0 & -u \sin v \end{vmatrix} = -2u^2 \sin v \mathbf{i} + u \mathbf{j} - 2u^2 \cos v \mathbf{k}.$$

Then

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}.$$

Therefore the surface area is

$$\iint_S \|\mathbf{r}_u \times \mathbf{r}_v\| dS = \int_0^4 \int_0^{2\pi} u\sqrt{4u^2 + 1} dv du = 2\pi \int_0^4 u\sqrt{4u^2 + 1} du.$$

Now use the substitution $t = 4u^2 + 1$, so that $dt = 8u du$; then $u = 0$ corresponds to $t = 1$, and $u = 4$ to $t = 65$, so we get

$$2\pi \int_0^4 u\sqrt{4u^2 + 1} du = 2\pi \int_1^{65} \frac{1}{8} t^{1/2} dt = 2\pi \left[\frac{1}{12} t^{3/2} \right]_1^{65} = \boxed{\frac{65\sqrt{65}}{6} \pi - \frac{\pi}{6}}.$$

30. With the given parametrization, we have

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 \cos u \sin v & -3 \sin u \sin v \\ -\sin v & 3 \sin u \cos v & 3 \cos u \cos v \end{vmatrix} = 9 \cos v \sin v \mathbf{i} + 3 \sin u \sin^2 v \mathbf{j} + 3 \cos u \sin^2 v \mathbf{k}.$$

Then

$$\begin{aligned}\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{81 \cos^2 v \sin^2 v + 9 \sin^2 u \sin^4 v + 9 \cos^2 u \sin^4 v} \\ &= \sqrt{81 \cos^2 v \sin^2 v + 9 \sin^4 v} = 3 \sin v \sqrt{9 \cos^2 v + \sin^2 v} = 3 \sin v \sqrt{8 \cos^2 v + 1}.\end{aligned}$$

Therefore the integral is

$$\begin{aligned}\iint_S x \|\mathbf{r}_u \times \mathbf{r}_v\| dS &= \int_0^{\pi/2} \int_0^{2\pi} 3 \sin v \cos v \sqrt{8 \cos^2 v + 1} du dv \\ &= 2\pi \int_0^{\pi/2} 3 \sin v \cos v \sqrt{8 \cos^2 v + 1} dv.\end{aligned}$$

Now use the substitution $t = 8 \cos^2 v + 1$, so that $dt = -16 \sin v \cos v dv$; then $v = 0$ corresponds to $t = 9$ and $v = \frac{\pi}{2}$ to $t = 1$, so we get

$$2\pi \int_0^{\pi/2} 3 \sin v \cos v \sqrt{8 \cos^2 v + 1} dv = 2\pi \int_9^1 \left(-\frac{3}{16} t^{1/2} dt \right) = -\frac{3}{8} \pi \left[\frac{2}{3} t^{3/2} \right]_9^1 = \frac{1}{4} \pi (27 - 1) = \boxed{\frac{13}{2} \pi}.$$

31. Parametrize the sphere using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = 2 \cos \theta \sin \phi \mathbf{i} + 2 \sin \theta \sin \phi \mathbf{j} + 2 \cos \phi \mathbf{k};$$

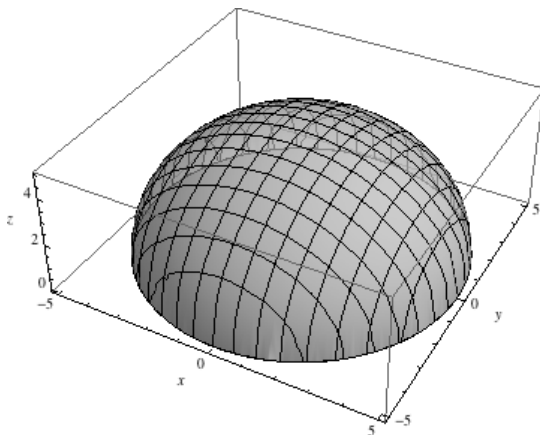
then we have

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin \theta \sin \phi & 2 \cos \theta \sin \phi & 0 \\ 2 \cos \theta \cos \phi & 2 \sin \theta \cos \phi & -2 \sin \phi \end{vmatrix} \\ &= -4 \cos \theta \sin^2 \phi \mathbf{i} - 4 \sin \theta \sin^2 \phi \mathbf{j} - 4 \sin \phi \cos \phi \mathbf{k} \\ \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| &= \sqrt{16 \cos^2 \theta \sin^4 \phi + 16 \sin^2 \theta \sin^4 \phi + 16 \sin^2 \phi \cos^2 \phi} \\ &= \sqrt{16 \sin^4 \phi + 16 \sin^2 \phi \cos^2 \phi} \\ &= 4 \sin \phi \sqrt{\sin^2 \phi + \cos^2 \phi} = 4 \sin \phi.\end{aligned}$$

Then the integral is

$$\iint_S z^2 dS = \int_0^{2\pi} \int_0^\pi 4 \cos^2 \phi \cdot 4 \sin \phi d\phi d\theta = 16 \int_0^{2\pi} \left[-\frac{1}{3} \cos^3 \phi \right]_0^\pi d\theta = 16 \int_0^{2\pi} \frac{2}{3} d\theta = \boxed{\frac{64\pi}{3}}.$$

32. The solid is



The top surface is parametrized using spherical coordinates:

$$\mathbf{r}(\theta, \phi) = 5 \cos \theta \sin \phi \mathbf{i} + 5 \sin \theta \sin \phi \mathbf{j} + 5 \cos \phi \mathbf{k};$$

then

$$\begin{aligned} \mathbf{r}_\theta &= -5 \sin \theta \sin \phi \mathbf{i} + 5 \cos \theta \sin \phi \mathbf{j}, & \mathbf{r}_\phi &= 5 \cos \theta \cos \phi \mathbf{i} + 5 \sin \theta \cos \phi \mathbf{j} - 5 \sin \phi \mathbf{k} \\ \mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 \sin \theta \sin \phi & 5 \cos \theta \sin \phi & 0 \\ 5 \cos \theta \cos \phi & 5 \sin \theta \cos \phi & -5 \sin \phi \end{vmatrix} = -25 \cos \theta \sin^2 \phi \mathbf{i} - 25 \sin \theta \sin^2 \phi \mathbf{j} - 25 \sin \phi \cos \phi \mathbf{k}. \end{aligned}$$

At the point $(1, 0, 0)$, corresponding to $\theta = 0$ and $\phi = \frac{\pi}{2}$, this gives $-25 \mathbf{i}$, which is inward-pointing. So the outward-pointing normals are the negative of the above, or

$$25 \cos \theta \sin^2 \phi \mathbf{i} + 25 \sin \theta \sin^2 \phi \mathbf{j} + 25 \sin \phi \cos \phi \mathbf{k}.$$

The magnitude of this vector is

$$\begin{aligned} \sqrt{25^2 \cos^2 \theta \sin^4 \phi + 25^2 \sin^2 \theta \sin^4 \phi + 25^2 \sin^2 \phi \cos^2 \phi} &= \sqrt{25^2 \sin^4 \phi + 25^2 \sin^2 \phi \cos^2 \phi} \\ &= 25 \sin \phi \sqrt{\sin^2 \phi + \cos^2 \phi} = 25 \sin \phi, \end{aligned}$$

so that the unit normal is

$$\cos \theta \sin \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k}.$$

Along the bottom edge we have the plane $z = 0$; clearly an outward unit normal to this plane is $-\mathbf{k}$.

33. Parametrize the surface using cylindrical coordinates:

$$\mathbf{r}(\theta, z) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad -1 \leq z \leq 1.$$

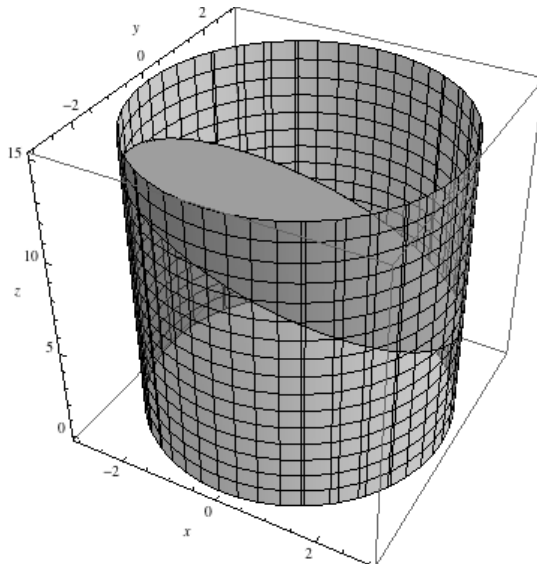
Then

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \sin \theta & 3 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j}.$$

Therefore $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = 3$, and the integral is

$$\iint_S x \, dS = \int_{-1}^1 \int_0^{2\pi} 3 \cdot 3 \cos \theta \, d\theta \, dz = \int_{-1}^1 [9 \sin \theta]_0^{2\pi} \, dz = \int_{-1}^1 0 \, dz = \boxed{0}.$$

34. The surface in question is the plane



For any values of x and y such that $x^2 + y^2 \leq 9$, the corresponding point on the plane is $z = 9 - x - y$. So we can parametrize the surface using cylindrical coordinates:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (9 - r \cos \theta - r \sin \theta) \mathbf{k}, \quad 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -\cos \theta - \sin \theta \\ -r \sin \theta & r \cos \theta & r \sin \theta - r \cos \theta \end{vmatrix} = r \mathbf{i} + r \mathbf{j} + r \mathbf{k},$$

so that $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = r\sqrt{3}$. Then the integral is

$$\begin{aligned} \iint_S z \, dS &= \int_0^3 \int_0^{2\pi} (9 - r \cos \theta - r \sin \theta) \cdot r\sqrt{3} \, d\theta \, dr \\ &= \sqrt{3} \int_0^3 \int_0^{2\pi} (9r - r^2 \cos \theta - r^2 \sin \theta) \, d\theta \, dr \\ &= \sqrt{3} \int_0^3 [9r\theta - r^2 \sin \theta + r^2 \cos \theta]_0^{2\pi} \, dr \\ &= \sqrt{3} \int_0^3 18\pi r \, dr \\ &= \sqrt{3} [9\pi r^2]_0^3 = \boxed{81\pi\sqrt{3}}. \end{aligned}$$

35. We can parametrize this plane by $y = t$, $z = u$, $x = t + u$ for $0 \leq t \leq \pi$ and $0 \leq u \leq \pi - t$. Then

$$\mathbf{r}(t, u) = (t + u) \mathbf{i} + t \mathbf{j} + u \mathbf{k},$$

so that

$$\mathbf{r}_t \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Then $\|\mathbf{r}_t \times \mathbf{r}_u\| = \sqrt{3}$, so the integral is

$$\begin{aligned} \iint_S \cos x \, dS &= \int_0^\pi \int_0^{\pi-t} \cos(t + u) \cdot \sqrt{3} \, du \, dt \\ &= \sqrt{3} \int_0^\pi [\sin(t + u)]_0^{\pi-t} \, dt = \sqrt{3} \int_0^\pi (\sin \pi - \sin t) \, dt. \end{aligned}$$

But $\sin \pi = 0$, so this integral becomes

$$\sqrt{3} \int_0^\pi (-\sin t) \, dt = \sqrt{3} [\cos t]_0^\pi = \boxed{-2\sqrt{3}}.$$

36. Parametrize this cylinder using cylindrical coordinates:

$$\mathbf{r}(\theta, z) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + z \mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1.$$

Differentiating gives

$$\mathbf{r}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad \mathbf{r}_z = \mathbf{k}.$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}.$$

This normal points outwards, so we use this one rather than its negative. The vector field is

$$\mathbf{F} = x^2 \mathbf{i} + y \mathbf{j} - z \mathbf{k} = \cos^2 \theta \mathbf{i} + \sin \theta \mathbf{j} - z \mathbf{k}.$$

It follows that the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^1 (\cos^2 \theta \mathbf{i} + \sin \theta \mathbf{j} - z \mathbf{k}) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \, dz \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (\cos^3 \theta + \sin^2 \theta) \, dz \, d\theta \\ &= \int_0^{2\pi} \left(\cos \theta (1 - \sin^2 \theta) + \frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta \\ &= \left[\sin \theta - \frac{1}{3} \sin^3 \theta + \frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} = \boxed{\pi}. \end{aligned}$$

37. (a) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(z \cos x) + \frac{\partial}{\partial y}(\sin y) + \frac{\partial}{\partial z}(e^x) = \boxed{\cos y - z \sin x}.$

(b) $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z \cos x & \sin y & e^x \end{vmatrix} = \boxed{(\cos x - e^x) \mathbf{j}}.$

(c) Parametrize the surface of the paraboloid using cylindrical coordinates:

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r^2 \mathbf{k}.$$

Then

$$\begin{aligned} \mathbf{r}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 2r \mathbf{k}, & \mathbf{r}_\theta &= -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k} \\ \|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{4r^4 + r^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^1 \int_0^{2\pi} (\cos x - e^x) \mathbf{j} \cdot \frac{1}{\sqrt{4r^4 + r^2}} (-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \\ &\quad \cdot \sqrt{4r^4 + r^2} \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} (-2r^2 (\cos(r \cos \theta) - e^{r \cos \theta}) \sin \theta) \, d\theta \, dr. \end{aligned}$$

Now make the substitution $t = \cos \theta$; then $dt = -\sin \theta \, d\theta$; $\theta = 0$ corresponds to $t = 1$, as does $\theta = 2\pi$, so we get

$$\int_0^1 \int_1^1 (2r^2 (\cos(rt) - e^{rt})) \, dt = 0$$

since the upper and lower limits of the inner integral are the same. For the other half of Stokes' Theorem, parametrize the unit circle clockwise, via

$$\mathbf{r}(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}, 0 \leq t \leq 2\pi,$$

since we want to keep the surface on our left as we walk the circle. Then

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos(\cos t) \mathbf{i} - \sin(\sin t) \mathbf{j} + e^{\cos t} \mathbf{k}) \cdot (-\sin t \mathbf{i} - \cos t \mathbf{j}) \, dt \\ &= \int_0^{2\pi} (-\cos(\cos t) \sin t + \sin(\sin t) \cos t) \, dt \\ &= [\sin(\cos t) - \cos(\sin t)]_0^{2\pi} = \boxed{0}. \end{aligned}$$

38. (a) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-3y) + \frac{\partial}{\partial z}(4z^2) = \boxed{2x + 8z - 3}.$

(b) $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -3y & 4z^2 \end{vmatrix} = \boxed{\mathbf{0}}.$

(c) Since $\operatorname{curl} \mathbf{F} = \mathbf{0}$, we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = 0.$$

For the other half of Stokes' Theorem, parametrize the unit circle as in Problem 37. Then

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (x^2 \mathbf{i} - 3y \mathbf{j} + 4z^2 \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta \\ &= \int_0^{2\pi} (\cos^2 \theta \mathbf{i} - 3 \sin^2 \theta \mathbf{j} + 4 \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta \\ &= \int_0^{2\pi} (-\sin \theta \cos^2 \theta - 3 \sin^3 \theta \cos \theta) \, d\theta \\ &= \left[\frac{1}{3} \cos^3 \theta - \sin^3 \theta \right]_0^{2\pi} = \boxed{0}. \end{aligned}$$

39. (a) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = \boxed{3}.$

(b) $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \boxed{\mathbf{0}}.$

(c) Since $\operatorname{curl} \mathbf{F} = \mathbf{0}$, we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS = 0.$$

For the other half of Stokes' Theorem, parametrize the unit circle as in Problem 37. Then

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta \\ &= \int_0^{2\pi} (\cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 1 \mathbf{k}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \, d\theta \\ &= \int_0^{2\pi} (-\sin \theta \cos \theta + \sin \theta \cos \theta) \, d\theta = \boxed{0}. \end{aligned}$$

40. (a) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-ye^z) + \frac{\partial}{\partial z}(ze^x) = \boxed{e^y - e^z + e^x}.$

(b) $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y & -ye^z & ze^x \end{vmatrix} = \boxed{ye^z \mathbf{i} - ze^x \mathbf{j} - xe^y \mathbf{k}}.$

(c) Parametrize the surface of the paraboloid as in Exercise 37. Then

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^1 \int_0^{2\pi} (ye^z \mathbf{i} - ze^x \mathbf{j} - xe^y \mathbf{k}) \cdot \frac{1}{\sqrt{4r^4 + r^2}} (-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \\ &\quad \cdot \sqrt{4r^4 + r^2} \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} (r \sin \theta e^{r^2} \mathbf{i} - r^2 e^{r \cos \theta} \mathbf{j} - r \cos \theta e^{r \sin \theta} \mathbf{k}) \\ &\quad \cdot (-2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}) \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} (-2r^3 \sin \theta \cos \theta + 2r^4 \sin \theta e^{r \cos \theta} - r^2 \cos \theta e^{r \sin \theta}) \, d\theta \, dr. \end{aligned}$$

For the second term, use the substitution $t = \cos \theta$; $dt = -\sin \theta d\theta$; $\theta = 0$ corresponds to $t = 1$, as does $\theta = 2\pi$. Since the upper and lower bounds of the t -integration will be the same, the integral of this term will be zero. Similarly, for the third term, use the substitution $t = \sin \theta$; $dt = \cos \theta d\theta$, and here $\theta = 0$ corresponds to $t = 0$ as does $\theta = 2\pi$. Since the upper and lower bounds of the t -integration will be the same, the integral of this term will be zero as well. We are left with

$$\int_0^1 \int_0^{2\pi} (-2r^3 \sin \theta \cos \theta) d\theta dr = \int_0^1 [-r^3 \sin^2 \theta]_0^{2\pi} = 0.$$

So the surface integral is zero. For the other half of Stokes' Theorem, parametrize the unit circle clockwise as in Problem 37. Then, via

$$\mathbf{r}(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}, 0 \leq t \leq 2\pi,$$

since we want to keep the surface on our left as we walk the circle. Then

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos t e^{-\sin t} \mathbf{i} + e \sin t \mathbf{j} + e^{\cos t} \mathbf{k}) \cdot (-\sin t \mathbf{i} - \cos t \mathbf{j}) dt \\ &= \int_0^{2\pi} (-\sin t \cos t e^{-\sin t} - e \sin t \cos t) dt \\ &= \int_0^{2\pi} (-\sin t \cos t e^{-\sin t}) dt - \int_0^{2\pi} e \sin t \cos t dt \\ &= \int_0^{2\pi} (-\sin t \cos t e^{-\sin t}) dt - \left[\frac{1}{2} e \sin^2 t \right]_0^{2\pi} \\ &= \int_0^{2\pi} (-\sin t \cos t e^{-\sin t}) dt. \end{aligned}$$

To evaluate the remaining integral, use integration by parts with $u = -\sin t$, $dv = \cos t e^{-\sin t} dt$. We can integrate dv with respect to t easily since $\frac{d}{dt}(-\sin t) = -\cos t$, to get $v = -e^{-\sin t}$. Also $du = -\cos t dt$, so the integral is

$$\int_0^{2\pi} (-\sin t \cos t e^{-\sin t}) dt = [\sin t e^{-\sin t}]_0^{2\pi} - \int_0^{2\pi} \cos t e^{-\sin t} dt = - \int_0^{2\pi} \cos t e^{-\sin t} dt.$$

As above, this is easily integrable since $\frac{d}{dt}(-\sin t) = -\cos t$, and we get finally

$$- \int_0^{2\pi} \cos t e^{-\sin t} dt = -[-e^{-\sin t}]_0^{2\pi} = \boxed{0}.$$

41. The given surface bounds the solid cylinder of radius 1 extending from $z = 0$ to $z = 1$, and the given vector field is differentiable everywhere, so the Divergence Theorem applies, and

$$\operatorname{div} \mathbf{F} = 2x + 1 - 1 = 2x.$$

To integrate, convert to cylindrical coordinates; then

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV = \int_0^1 \int_0^1 \int_0^{2\pi} 2r \cos \theta \cdot r d\theta dr dz = \int_0^1 \int_0^1 [2r^2 \sin \theta]_0^{2\pi} dr dz = \boxed{0}.$$

42. (a) If $\nabla f = (yz - y - z)\mathbf{i} + (xz - x - z)\mathbf{j} + (xy - x - y)\mathbf{k}$, then $\frac{\partial f}{\partial x} = yz - y - z$. Integrating both sides with respect to x gives

$$f(x, y, z) = xyz - xy - xz + h(y, z).$$

Now differentiate with respect to y and set the result equal to the y -component:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xyz - xy - xz + h(y, z)) = xz - x + h_y = xz - x - z.$$

Therefore $h_y = -z$, so that $h(y, z) = -yz + k(z)$, and then

$$f(x, y, z) = xyz - xy - xz - yz + k(z).$$

Now differentiate with respect to z and set the result equal to the z -component:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(xyz - xy - xz - yz + k(z)) = xy - x - y + k'(z) = xy - x - y.$$

Therefore $k'(z) = 0$, so that $k(z) = C$ for C an arbitrary constant. Choosing $C = 0$, we get

$$f(x, y, z) = \boxed{xyz - xy - xz - yz}.$$

- (b) Since the given vector field is a gradient, it is conservative. So by the Fundamental Theorem of Line Integrals, since C is a path from $(0, 0, 0)$ to $(1, 1, 1)$, we have

$$\int_C [(yz - y - z) dx + (xz - x - z) dy + (xy - x - y) dz] = f(1, 1, 1) - f(0, 0, 0) = \boxed{-2},$$

which agrees with the result from Problem 7.

43. Let $P(x, y, z) = xz$, $Q(x, y, z) = yz$, $R(x, y, z) = x^2$. Then from the second form of the Divergence Theorem in Section 15.8,

$$\iint_S (xz \cos \alpha + yz \cos \beta + x^2 \cos \gamma) dS = \iiint_E \left(\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(x^2) \right) dV = \iiint_E 2z dV,$$

where E is the interior of the upper hemisphere of the unit sphere. Convert to spherical coordinates to evaluate the integral:

$$\begin{aligned} \iiint_E 2z dV &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} 2\rho \cos \phi \cdot \rho^2 \sin \phi d\phi d\theta d\rho \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} 2\rho^3 \sin \phi \cos \phi d\phi d\theta d\rho \\ &= \int_0^1 \int_0^{2\pi} [\rho^3 \sin^2 \phi]_0^{\pi/2} d\theta d\rho \\ &= \int_0^1 \int_0^{2\pi} \rho^3 d\theta d\rho \\ &= \int_0^1 2\pi \rho^3 d\rho \\ &= \left[\frac{1}{2} \pi \rho^4 \right]_0^1 = \boxed{\frac{\pi}{2}}. \end{aligned}$$

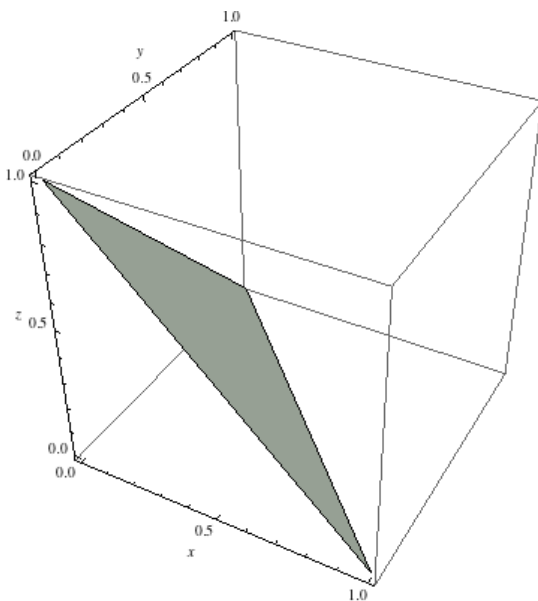
44. Write $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; then

$$\begin{aligned} \operatorname{div} \|\mathbf{r}\|^2 \mathbf{r} &= \operatorname{div} (x(x^2 + y^2 + z^2)\mathbf{i} + y(x^2 + y^2 + z^2)\mathbf{j} + z(x^2 + y^2 + z^2)\mathbf{k}) \\ &= (x^2 + y^2 + z^2) + x(2x) + (x^2 + y^2 + z^2) + y(2y) + (x^2 + y^2 + z^2) + z(2z) \\ &= 5(x^2 + y^2 + z^2). \end{aligned}$$

Perform the integration using spherical coordinates:

$$\begin{aligned}
 \iiint_E \operatorname{div} \mathbf{F} \, dV &= \int_0^1 \int_0^{2\pi} \int_0^\pi 5\rho^2 \cdot \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\
 &= \int_0^1 \int_0^{2\pi} [-5\rho^4 \cos \phi]_0^\pi \, d\theta \, d\rho \\
 &= \int_0^1 \int_0^{2\pi} 10\rho^4 \, d\theta \, d\rho \\
 &= \int_0^1 20\pi\rho^4 \, d\rho \\
 &= [4\pi\rho^5]_0^1 = \boxed{4\pi}.
 \end{aligned}$$

45. The curve is the boundary of the plane



Since all three components of the vector field are everywhere differentiable, Stokes' Theorem applies. We parametrize the triangle by

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (1 - x - y)\mathbf{k}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

Then

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Since this is upward-pointing, we use this normal, since it is consistent with a counterclockwise orientation of the boundary. Next,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & y - z & z - x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Then

$$\begin{aligned}
 \oint_C [(x-y) dx + (y-z) dy + (z-x) dz] &= \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS \\
 &= \int_0^1 \int_0^{1-x} (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) dy dx \\
 &= \int_0^1 \int_0^{1-x} 3 dy dx \\
 &= \int_0^1 [3y]_0^{1-x} dx \\
 &= \int_0^1 3(1-x) dx \\
 &= \left[-\frac{3}{2}(1-x)^2 \right]_0^1 = \boxed{\frac{3}{2}}.
 \end{aligned}$$

46. (a) Since the three component functions are differentiable everywhere, the Divergence Theorem applies, so that the flux is

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \operatorname{div} \mathbf{F} dV.$$

Here E is the interior of the unit sphere, and

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2 + y^2 + z^2).$$

So the flux is

$$\iiint_E 3(x^2 + y^2 + z^2) dV.$$

Convert to spherical coordinates, giving

$$\begin{aligned}
 \int_0^1 \int_0^{2\pi} \int_0^\pi 3\rho^2 \cdot \rho^2 \sin \phi d\phi d\theta d\rho &= \int_0^1 \int_0^{2\pi} [-3\rho^4 \cos \phi]_0^\pi d\theta d\rho \\
 &= \int_0^1 \int_0^{2\pi} 6\rho^4 d\theta d\rho \\
 &= \int_0^1 12\pi\rho^4 d\rho \\
 &= \left[\frac{12}{5}\pi\rho^5 \right]_0^1 = \boxed{\frac{12}{5}\pi}.
 \end{aligned}$$

- (b) To determine the circulation, we must parametrize the unit circle in the xy -plane:

$$\mathbf{r}(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Then the circulation is

$$\begin{aligned}
 \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos^3 \theta \mathbf{i} + \sin^3 \theta \mathbf{j}) \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta \\
 &= \int_0^{2\pi} (-\sin \theta \cos^3 \theta + \cos \theta \sin^3 \theta) d\theta \\
 &= \left[\frac{1}{4} \cos^4 \theta + \frac{1}{4} \sin^4 \theta \right]_0^{2\pi} = \boxed{0}.
 \end{aligned}$$

47. Computing $\text{curl } \mathbf{F}$ gives

$$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xy & xy \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xy) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(yz) \right) \mathbf{k} \\ &= (x - 0) \mathbf{i} + (y - y) \mathbf{j} + (y - z) \mathbf{k} = x \mathbf{i} + (y - z) \mathbf{k} \neq \mathbf{0}.\end{aligned}$$

Since $\text{curl } \mathbf{F} \neq \mathbf{0}$, $\mathbf{F}$ is not conservative (see Objective 4 in Section 15.9).

48. Computing $\text{curl } \mathbf{F}$ gives

$$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z & 2x^2yz & x^2y^2 - 2z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(x^2y^2 - 2z) - \frac{\partial}{\partial z}(2x^2yz) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(2xy^2z) - \frac{\partial}{\partial x}(x^2y^2 - 2z) \right) \mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(2xy^2z) \right) \mathbf{k} \\ &= (2x^2y - 2x^2y) \mathbf{i} + (2xy^2 - 2xy^2) \mathbf{j} + (4xyz - 4xyz) \mathbf{k} = \mathbf{0}.\end{aligned}$$

Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative (see Objective 4 in Section 15.9).

Differential Equations

16.1 Classification of Ordinary Differential Equations

Concepts and Vocabulary

1. The differential equation $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - y = 0$ is of order two and degree 1. It is second-order because the highest-order derivative is the second derivative, and it is degree 1 because the exponent of the second derivative is 1.
2. True. The differential equation $(x^2 + 2)\frac{dy}{dx} - \frac{1}{x}y = 0$ is linear. Since $P_0(x) = x^2 + 2$ and $P_1(x) = \frac{1}{x}$ are both functions of x alone and no term contains more than one derivative, the differential equation is linear.
3. True. $y = -4x^2 + 100$ is a solution of the differential equation $\frac{dy}{dx} = -8x$.
4. False. $y = 8e^{5x}$ is not a solution of the differential equation $\frac{dy}{dx} + 3 = 8e^{5x}$. The derivative of y is $\frac{dy}{dx} = 40e^{5x}$.

Skill Building

5. The differential equation $\frac{dy}{dx} + x^2y = xe^x$ is first-order because the highest-order derivative is the first derivative. It has degree 1 because the exponent of the first derivative is 1. Since $P_0(x) = 1$, $P_1(x) = x^2$, and $Q(x) = xe^x$ are all functions of x alone and no term contains more than one derivative, the differential equation is linear.
6. The differential equation $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = \sin x$ is third-order because the highest-order derivative is the third derivative. It has degree 1 because the exponent of the third derivative is 1. Since $P_0(x) = 1$, $P_1(x) = 4$, $P_2(x) = -5$, $P_3(x) = 3$, and $Q(x) = \sin x$ are all functions of x alone and no term contains more than one derivative, the differential equation is linear.
7. The differential equation $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} + 5y = 0$ is fourth-order because the highest-order derivative is the fourth derivative. It has degree 1 because the exponent of the fourth derivative is 1. Since $P_0(x) = 1$, $P_1(x) = 0$, $P_2(x) = 3$, $P_3(x) = 0$, and $P_4(x) = 5$ are all functions of x alone and no term contains more than one derivative, the differential equation is linear.
8. The differential equation $\frac{d^2y}{dx^2} + y \sin x = 0$ is second-order because the highest-order derivative is the second derivative. It has degree 1 because the exponent of the second derivative is 1. Since $P_0(x) = 1$, $P_1(x) = 0$, and $P_2(x) = \sin x$ are all functions of x alone and no term contains more than one derivative, the differential equation is linear.

9. The differential equation $\frac{d^2y}{dx^2} + x \sin y = 0$ is second-order because the highest-order derivative is the second derivative. It has degree 1 because the exponent of the second derivative is 1. Since $\sin y$ is not a polynomial expression of y , the differential equation is nonlinear.
10. The differential equation $\frac{d^6x}{dt^6} + \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + x = t$ is sixth-order because the highest-order derivative is the sixth derivative. It has degree 1 because the exponent of the first derivative is 1. Since $P_0(t) = P_2(t) = P_3(t) = P_6(t) = 1$, $P_1(t) = P_4(t) = P_5(t) = 0$, and $Q(t) = t$ are all functions of t alone and no term contains more than one derivative, the differential equation is linear.
11. The differential equation $\left(\frac{dr}{ds}\right)^2 = \left(\frac{d^2r}{ds^2}\right)^3 + 1$ is second-order because the highest-order derivative is the second derivative. It has degree 3 because the exponent of the first derivative is 3. Since the first (or second) derivative is not of degree 1, the differential equation is nonlinear.
12. The differential equation $x(y'')^3 + (y')^4 - y = 0$ is second-order because the highest-order derivative is the second derivative. It has degree 3 because the exponent of the second derivative is 3. Since the first (or second) derivative is not of degree 1, the differential equation is nonlinear.
13. The differential equation $\frac{d^2y}{ds^2} + 3s \frac{dy}{ds} - y = 0$ is second-order because the highest-order derivative is the second derivative. It has degree 1 because the exponent of the second derivative is 1. Since $P_0(s) = 1$, $P_1(s) = 3s$, and $P_2(s) = -1$ are all functions of s alone and no term contains more than one derivative, the differential equation is linear.
14. The differential equation $\frac{dy}{dx} = 1 - xy + y^2$ is first-order because the highest-order derivative is the first derivative. It has degree 1 because the exponent of the first derivative is 1. Since y^2 is not of degree 1, the differential equation is nonlinear.
15. For $y = e^x + 3e^{-x}$, we have $\frac{dy}{dx} = e^x - 3e^{-x}$ and $\frac{d^2y}{dx^2} = e^x + 3e^{-x}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} - y &= 0 \\ e^x + 3e^{-x} - (e^x + 3e^{-x}) &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

16. For $y = 5 \sin x + 2 \cos x$, we have $\frac{dy}{dx} = 5 \cos x - 2 \sin x$ and $\frac{d^2y}{dx^2} = -5 \sin x - 2 \cos x$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} + y &= 0 \\ -5 \sin x - 2 \cos x + 5 \sin x + 2 \cos x &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

17. For $y = \sin(2x)$, we have $\frac{dy}{dx} = 2 \cos(2x)$ and $\frac{d^2y}{dx^2} = -4 \sin(2x)$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} + 4y &= 0 \\ -4 \sin(2x) + 4 \sin(2x) &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

18. For $y = \cos(2x)$, we have $\frac{dy}{dx} = -2\sin(2x)$ and $\frac{d^2y}{dx^2} = -4\cos(2x)$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} + 4y &= 0 \\ -4\cos(2x) + 4\cos(2x) &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

19. For $y = \frac{3}{1-x^3}$, we have $\frac{dy}{dx} = \frac{9x^2}{(1-x^3)^2}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= x^2 y^2 \\ \frac{9x^2}{(1-x^3)^2} &= x^2 \left(\frac{3}{1-x^3} \right)^2 \\ \frac{9x^2}{(1-x^3)^2} &= x^2 \frac{9}{(1-x^3)^2}\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

20. For $y = 1 + Ce^{-x^2/2}$, C is a constant, we have $\frac{dy}{dx} = -Cxe^{-x^2/2}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} + xy &= x \\ -Cxe^{-x^2/2} + x(1 + Ce^{-x^2/2}) &= x \\ -Cxe^{-x^2/2} + x + Cxe^{-x^2/2} &= x \\ x &= x\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

21. For $y = e^{-2x}$, we have $y' = -2e^{-2x}$ and $y'' = 4e^{-2x}$. Substitute these into the differential equation:

$$\begin{aligned}y'' + 3y' + 2y &= 0 \\ 4e^{-2x} + 3(-2e^{-2x}) + 2(e^{-2x}) &= 0 \\ 4e^{-2x} - 6e^{-2x} + 2e^{-2x} &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

22. For $y = e^x$, we have $y' = e^x$ and $y'' = e^x$. Substitute these into the differential equation:

$$\begin{aligned}2y'' + y' - 3y &= 0 \\ 2e^x + e^x - 3e^x &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

23. Differentiate $y^2 + 2xy - x^2 = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned}2y \frac{dy}{dx} + 2y + 2x \frac{dy}{dx} - 2x &= 0 \\ 2(x+y) \frac{dy}{dx} &= 2(x-y) \\ \frac{dy}{dx} &= \frac{x-y}{x+y}\end{aligned}$$

The function $y = f(x)$ defined by the equation $y^2 + 2xy - x^2 = C$ satisfies the first-order differential equation and so is a solution.

24. Solve for C in the equation $x^2 - y^2 = Cx^2y^2$. Differentiate $y^{-2} - x^{-2} = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} -2y^{-3} \frac{dy}{dx} + 2x^{-3} &= 0 \\ -\frac{2}{y^3} \frac{dy}{dx} + \frac{2}{x^3} &= 0 \\ -\frac{2}{y^3} \frac{dy}{dx} &= -\frac{2}{x^3} \\ \frac{dy}{dx} &= \frac{y^3}{x^3} \end{aligned}$$

The function $y = f(x)$ defined by the equation $x^2 - y^2 = Cx^2y^2$ satisfies the first-order differential equation and so is a solution.

25. Solve for C in the equation $x^2 - y^2 = Cx$. Differentiate $\frac{x^2 - y^2}{x} = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{x \left(2x - 2y \frac{dy}{dx} \right) - (x^2 - y^2)}{x^2} &= 0 \\ x \left(2x - 2y \frac{dy}{dx} \right) - (x^2 - y^2) &= 0 \\ 2x^2 - 2xy \frac{dy}{dx} - x^2 + y^2 &= 0 \\ -2xy \frac{dy}{dx} &= -x^2 - y^2 \\ \frac{dy}{dx} &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

The function $y = f(x)$ defined by the equation $x^2 - y^2 = Cx$ satisfies the first-order differential equation and so is a solution.

26. Differentiate $x^2y + 12x^2 + 16y = 9$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} 2xy + x^2 \frac{dy}{dx} + 24x + 16 \frac{dy}{dx} &= 0 \\ (x^2 + 16) \frac{dy}{dx} &= -(2xy + 24x) \\ \frac{dy}{dx} &= -\frac{2xy + 24x}{x^2 + 16} \end{aligned}$$

The function $y = f(x)$ defined by the equation $x^2y + 12x^2 + 16y = 9$ satisfies the first-order differential equation and so is a solution.

27. Differentiate $2(x-1)e^x + y^2 = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} 2e^x + 2(x-1)e^x + 2y\frac{dy}{dx} &= 0 \\ 2e^x + 2xe^x - 2e^x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2xe^x \\ \frac{dy}{dx} &= -\frac{xe^x}{y} \end{aligned}$$

The function $y = f(x)$ defined by the equation $2(x-1)e^x + y^2 = C$ satisfies the first-order differential equation and so is a solution.

28. Differentiate $x^2 - y^2 + 2(e^{-x} - e^y) = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} 2x - 2y\frac{dy}{dx} - 2e^{-x} - 2e^y\frac{dy}{dx} &= 0 \\ -2(y + e^y)\frac{dy}{dx} &= -2(x - e^{-x}) \\ \frac{dy}{dx} &= \frac{x - e^{-x}}{y + e^y} \end{aligned}$$

The function $y = f(x)$ defined by the equation $x^2 - y^2 + 2(e^{-x} - e^y) = C$ satisfies the first-order differential equation and so is a solution.

29. Differentiate $\tan^{-1} y = x + \frac{x^2}{2} + C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{1}{1+y^2}\frac{dy}{dx} &= 1+x \\ \frac{dy}{dx} &= (1+x)(1+y^2) \\ \frac{dy}{dx} &= xy^2 + y^2 + x + 1 \end{aligned}$$

The function $y = f(x)$ defined by the equation $\tan^{-1} y = x + \frac{x^2}{2} + C$ satisfies the first-order differential equation and so is a solution.

30. Solve for C in the equation $1 + 2y^2\sqrt{1+x^2} = Cy^2$. Differentiate $y^{-2} + 2\sqrt{1+x^2} = C$ implicitly with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} -\frac{2}{y^3}\frac{dy}{dx} + \frac{2x}{\sqrt{1+x^2}} &= 0 \\ -\frac{2}{y^3}\frac{dy}{dx} &= -\frac{2x}{\sqrt{1+x^2}} \\ \frac{dy}{dx} &= \frac{xy^3}{\sqrt{1+x^2}} \end{aligned}$$

The function $y = f(x)$ defined by the equation $1 + 2y^2\sqrt{1+x^2} = Cy^2$ satisfies the first-order differential equation and so is a solution.

Applications and Extensions

31. For $y = 3x$, we have $\frac{dy}{dx} = 3$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} \\ 3 &= \frac{3x}{x}\end{aligned}$$

The function y satisfies the differential equation when $x \neq 0$ and so y is a solution. It further satisfies the initial condition: when $x = 1$, $y = 3 \cdot 1 = 3$.

32. For $y = 2e^{3x}$, we have $\frac{dy}{dx} = 6e^{3x}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= 3y \\ 6e^{3x} &= 3 \cdot 2e^{3x}\end{aligned}$$

The function y satisfies the differential equation and so y is a solution. It further satisfies the initial condition: when $x = 0$, $y = 2e^0 = 2 \cdot 1 = 2$.

33. For $y = \frac{1}{1-x}$, we have $\frac{dy}{dx} = \frac{1}{(1-x)^2}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= y^2 \\ \frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)^2\end{aligned}$$

The function y satisfies the differential equation and so y is a solution. It further satisfies the initial condition: when $x = 0$, $y = \frac{1}{1-0} = 1$.

34. For $y = \frac{x^2}{3} + \frac{3}{x}$, we have $\frac{dy}{dx} = \frac{2x}{3} - \frac{3}{x^2}$. Substitute these into the differential equation:

$$\begin{aligned}x \frac{dy}{dx} + y &= x^2 \\ x \left(\frac{2x}{3} - \frac{3}{x^2} \right) + \frac{x^2}{3} + \frac{3}{x} &= x^2 \\ \frac{2x^2}{3} - \frac{3}{x} + \frac{x^2}{3} + \frac{3}{x} &= x^2 \\ x^2 &= x^2\end{aligned}$$

The function y satisfies the differential equation and so y is a solution. It further satisfies the initial condition: when $x = 3$, $y = \frac{3^2}{3} + \frac{3}{3} = 3 + 1 = 4$.

35. For $y = C_1e^{ax} + C_2e^{-ax}$, C_1 and C_2 are constants, we have $\frac{dy}{dx} = aC_1e^{ax} - aC_2e^{-ax}$ and $\frac{d^2y}{dx^2} = a^2C_1e^{ax} + a^2C_2e^{-ax}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} - a^2y &= 0 \\ a^2C_1e^{ax} + a^2C_2e^{-ax} - a^2(C_1e^{ax} + C_2e^{-ax}) &= 0 \\ a^2C_1e^{ax} + a^2C_2e^{-ax} - a^2C_1e^{ax} - a^2C_2e^{-ax} &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

36. For $y = \sinh x$, we have $\frac{dy}{dx} = \cosh x$ and $\frac{d^2y}{dx^2} = \sinh x$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} &= y \\ \sinh x &= \sinh x\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

37. For $y = \frac{a^2kt}{1+akt}$, $a > 0$ and k are constants, we have

$$\frac{dy}{dt} = \frac{(1+akt)a^2k - a^2kt \cdot ak}{(1+akt)^2} = \frac{a^2k}{(1+akt)^2}$$

Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dt} &= k(a-y)^2 \\ \frac{a^2k}{(1+akt)^2} &= k\left(a - \frac{a^2kt}{1+akt}\right)^2 \\ \frac{a^2k}{(1+akt)^2} &= k\left(\frac{a+a^2kt}{1+akt} - \frac{a^2kt}{1+akt}\right)^2 \\ \frac{a^2k}{(1+akt)^2} &= k\left(\frac{a}{1+akt}\right)^2 \\ \frac{a^2k}{(1+akt)^2} &= k\frac{a^2}{(1+akt)^2} \\ \frac{a^2k}{(1+akt)^2} &= \frac{a^2k}{(1+akt)^2}\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

38. For $y = \ln(C - e^{-x})$, C is a constant, we have $\frac{dy}{dx} = \frac{e^{-x}}{C - e^{-x}}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= e^{-(x+y)} \\ \frac{e^{-x}}{C - e^{-x}} &= e^{-(x+\ln(C-e^{-x}))} \\ \frac{e^{-x}}{C - e^{-x}} &= e^{-x}e^{-\ln(C-e^{-x})} \\ \frac{e^{-x}}{C - e^{-x}} &= \frac{e^{-x}}{e^{\ln(C-e^{-x})}} \\ \frac{e^{-x}}{C - e^{-x}} &= \frac{e^{-x}}{C - e^{-x}}\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

39. For $y = e^{nx}$, n a constant, we have $y' = ne^{nx}$ and $y'' = n^2e^{nx}$. Substitute these into the differential equation:

$$\begin{aligned}y'' + y' - 6y &= 0 \\ n^2e^{nx} + ne^{nx} - 6e^{nx} &= 0 \\ e^{nx}(n^2 + n - 6) &= 0\end{aligned}$$

Since $e^{nx} \neq 0$, for the differential equation to be satisfied, it must be that $n^2 + n - 6 = (n+3)(n-2) = 0$. Solving for n , we have $n = -3$ or $n = 2$. For $y = e^{nx}$ to be a solution of $y'' + y' - 6y = 0$, $\boxed{n = -3 \text{ or } 2}$.

40. Answers will vary. The function $y = e^x + e^{-x}$ is a solution of the first-order differential equation $y' = e^x - e^{-x}$.
41. The Schrödinger equation $-\frac{h^2}{2m} \frac{d^2 Y(x)}{dx^2} + U(x)Y(x) = EY(x)$ is a second-order differential equation of degree 1. It is second-order because the highest-order derivative is the second derivative, and it is degree 1 because the exponent of the second derivative is 1.

16.2 Separation of Variables in First-Order Differential Equations

Concepts and Vocabulary

1. Express $\frac{dy}{dx} = xy$ in the differential form

$$\frac{1}{y} dy = x dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \frac{1}{y} dy &= \int x dx \\ \ln |y| &= 0.5x^2 + C_1 \\ e^{\ln |y|} &= e^{0.5x^2 + C_1} \\ |y| &= e^{0.5x^2} e^{C_1} \\ y &= \pm e^{0.5x^2} e^{C_1} \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{C_1}$, where C is a nonzero constant. Then $y = Ce^{0.5x^2}$ is a general solution.

Notice that the function $y = 0$ also satisfies the differential equation $\frac{dy}{dx} = xy$, so it too is a solution. This particular solution matches the general solution above when $C = 0$. So the general solution of the differential equation $\frac{dy}{dx} = xy$ is $y = Ce^{0.5x^2}$ where C is any constant.

2. True. If $f(tx, ty) = t^k f(x, y)$, for all $t > 0$, where k is a real number, then the function f is said to be homogeneous of degree k in x and y .
3. If $F(x, y, C) = 0$ and $G(x, y, K) = 0$ are one-parameter families of curves in which each member of one family intersects the members of the other family orthogonally, then the two families are said to be orthogonal trajectories of each other.
4. If $M(x, y) dx + N(x, y) dy = 0$ is a homogeneous differential equation, then it can be transformed into an equation whose variables are separable by using the substitution $y = xv(x)$ where $v = v(x)$ is a differentiable function of x .

Skill Building

5. Express $\frac{dy}{dx} = x \sec y = \frac{x}{\cos y}$ in the differential form

$$\cos y dy = x dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \cos y dy &= \int x dx \\ \sin y &= \frac{1}{2}x^2 + C \end{aligned}$$

Then the general solution is $\boxed{\sin y = \frac{1}{2}x^2 + C}$, where C is a constant.

6. Express $\cos y \frac{dy}{dx} = \frac{2}{x}$ in the differential form

$$\cos y \, dy = \frac{2}{x} \, dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \cos y \, dy &= \int \frac{2}{x} \, dx \\ \sin y &= 2 \ln |x| + C \end{aligned}$$

Then the general solution is $\boxed{\sin y = 2 \ln |x| + C}$, where C is a constant.

7. Express $\frac{dy}{dx} = e^{y-x} = e^y e^{-x}$ in the differential form

$$e^{-y} \, dy = e^{-x} \, dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int e^{-y} \, dy &= \int e^{-x} \, dx \\ -e^{-y} &= -e^{-x} + C_1 \\ e^{-y} &= e^{-x} - C_1 \\ \ln(e^{-y}) &= \ln(e^{-x} - C_1) \\ -y &= \ln(e^{-x} - C_1) \\ y &= -\ln(e^{-x} - C_1) \end{aligned}$$

Since C_1 is a constant, we can write $C = -C_1$, where C is a constant. Then the general solution can be written as $\boxed{y = -\ln(e^{-x} + C)}$, where C is a constant.

8. Express $x \frac{dy}{dx} + 2y = 5$ first as $x \frac{dy}{dx} = 5 - 2y$, then in the differential form

$$\frac{1}{5-2y} \, dy = \frac{1}{x} \, dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \frac{1}{5-2y} \, dy &= \int \frac{1}{x} \, dx \\ -\frac{1}{2} \ln |5-2y| &= \ln |x| + C_1 \\ \ln |5-2y| &= -2 \ln |x| - 2C_1 \\ e^{\ln |5-2y|} &= e^{-2 \ln |x| - 2C_1} \\ |5-2y| &= e^{\ln |x|^{-2}} e^{-2C_1} \\ 5-2y &= \pm |x|^{-2} e^{-2C_1} \\ -2y &= -5 \pm \frac{e^{-2C_1}}{x^2} \\ y &= \frac{5}{2} \pm \frac{e^{-2C_1}}{2x^2} \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm \frac{e^{-2C_1}}{2}$, where C is a constant. Then the general solution can be written as $\boxed{y = \frac{5}{2} + \frac{C}{x^2}}$, where C is a constant.

9. Express $\frac{dy}{dx} = \cos x \frac{dy}{dx} - y \sin x$ first as $(1 - \cos x) \frac{dy}{dx} = -y \sin x$, then in the differential form

$$\frac{1}{y} dy = -\frac{\sin x}{1 - \cos x} dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \frac{1}{y} dy &= \int \left(-\frac{\sin x}{1 - \cos x} \right) dx && \text{Use } u = 1 - \cos x. \\ \ln |y| &= -\ln |1 - \cos x| + C_1 \\ e^{\ln |y|} &= e^{-\ln |1 - \cos x| + C_1} \\ |y| &= e^{-\ln |1 - \cos x|} e^{C_1} \\ y &= \pm \frac{e^{C_1}}{e^{\ln |1 - \cos x|}} \\ y &= \pm \frac{e^{C_1}}{|1 - \cos x|} \\ y &= \pm \frac{e^{C_1}}{1 - \cos x} \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{C_1}$, where C is a constant. Then the general solution can be written as $y = \frac{C}{1 - \cos x}$, where $C \neq 0$ is a constant.

Notice that the function $y = 0$ also satisfies the differential equation $\frac{dy}{dx} = \cos x \frac{dy}{dx} - y \sin x$, so it too is a solution. This particular solution matches the general solution above when $C = 0$. So the general solution is $\boxed{y = \frac{C}{1 - \cos x}}$, where C is any constant.

10. Express $\frac{dy}{dx} = x + y \frac{dy}{dx}$ first as $(1 - y) \frac{dy}{dx} = x$, then in the differential form

$$(1 - y) dy = x dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int (1 - y) dy &= \int x dx \\ y - \frac{1}{2} y^2 &= \frac{1}{2} x^2 + C \end{aligned}$$

Then the general solution is $\boxed{y - \frac{1}{2} y^2 = \frac{1}{2} x^2 + C}$, where C is a constant.

11. Express $\frac{dy}{dx} + xy = x$ first as $\frac{dy}{dx} = x - xy = x(1 - y)$, then in the differential form

$$\frac{1}{1 - y} dy = x dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \frac{1}{1 - y} dy &= \int x dx \\ -\ln |1 - y| &= 0.5x^2 + C_1 \\ \ln |1 - y| &= -0.5x^2 - C_1 \\ e^{\ln |1 - y|} &= e^{-0.5x^2 - C_1} \\ |1 - y| &= e^{-0.5x^2} e^{-C_1} \\ 1 - y &= \pm e^{-0.5x^2} e^{-C_1} \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{-C_1}$, where C is a nonzero constant. Solve for y in the equation $1 - y = Ce^{-0.5x^2}$. Then the general solution is $y = 1 - Ce^{-0.5x^2}$, where $C \neq 0$ is a constant.

Notice that the function $y = 1$ also satisfies the differential equation $\frac{dy}{dx} + xy = x$, so it too is a solution. This particular solution matches the general solution above when $C = 0$. So the general solution is

$$\boxed{y = 1 - Ce^{-0.5x^2}}, \text{ where } C \text{ is any constant.}$$

12. Express $(3x + 1)dx + e^{x+y}dy = 0$ first as $(3x + 1)dx + e^x e^y dy = 0$, then as $(3x + 1)e^{-x}dx + e^y dy = 0$. Integrate to obtain the general solution.

$$\begin{aligned} \int (3x + 1)e^{-x}dx + \int e^y dy &= 0 && \text{Use integration by parts.} \\ -(3x + 4)e^{-x} + e^y + C &= 0 \end{aligned}$$

Then the general solution is $\boxed{-(3x + 4)e^{-x} + e^y + C = 0}$, where C is a constant.

13. Express $\ln x \frac{dx}{dy} = \frac{x}{y}$ in the differential form

$$\frac{\ln x}{x} dx = \frac{1}{y} dy$$

Integrate to obtain the general solution.

$$\begin{aligned} \int \frac{\ln x}{x} dx &= \int \frac{1}{y} dy \\ \frac{1}{2}(\ln x)^2 &= \ln |y| + C_1 \\ \frac{1}{2}(\ln x)^2 - C_1 &= \ln |y| \\ e^{[1/2(\ln x)^2 - C_1]} &= |y| \\ e^{1/2(\ln x)^2} e^{-C_1} &= |y| \\ \pm e^{1/2(\ln x)^2} e^{-C_1} &= y \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{-C_1}$, where C is a nonzero constant.

$$\begin{aligned} y &= Ce^{1/2(\ln x)^2} \\ &= C \left((e^{\ln x})^{1/2} \right)^{\ln x} \\ &= C \left((x)^{1/2} \right)^{\ln x} \\ &= C (\sqrt{x})^{\ln x} \end{aligned}$$

Then the general solution $\boxed{y = C (\sqrt{x})^{\ln x}}$ is where $C \neq 0$ is a constant.

14. Express $\frac{dy}{dx} = \frac{x+2}{2-y}$ in the differential form

$$(2 - y)dy = (x + 2)dx$$

Integrate to obtain the general solution.

$$\begin{aligned} \int (2 - y)dy &= \int (x + 2)dx \\ 2y - \frac{1}{2}y^2 &= \frac{1}{2}x^2 + 2x + C \end{aligned}$$

Then the general solution is $\boxed{2y - \frac{1}{2}y^2 = \frac{1}{2}x^2 + 2x + C}$, where C is a constant.

15. The general solution of the differential equation $\frac{dy}{dx} = x \sec y$ is $\sin y = \frac{1}{2}x^2 + C$, see the solution to Problem 5. To find the particular solution, substitute $x = 1$ and $y = \frac{\pi}{4}$ and solve for C .

$$\begin{aligned}\sin y &= \frac{1}{2}x^2 + C \\ \sin \frac{\pi}{4} &= \frac{1}{2}1^2 + C \\ \frac{\sqrt{2}}{2} &= \frac{1}{2} + C \\ \frac{\sqrt{2} - 1}{2} &= C\end{aligned}$$

Solve for y in the equation $\sin y = \frac{1}{2}x^2 + \frac{\sqrt{2} - 1}{2}$. The particular solution is

$$\boxed{y = \sin^{-1} \left(\frac{1}{2}x^2 + \frac{\sqrt{2} - 1}{2} \right)}$$

16. The general solution of the differential equation $\cos y \frac{dy}{dx} = \frac{2}{x}$ is $\sin y = 2 \ln |x| + C$, see the solution to Problem 6. To find the particular solution, substitute $x = -1$ and $y = \frac{\pi}{3}$ and solve for C .

$$\begin{aligned}\sin y &= 2 \ln |x| + C \\ \sin \frac{\pi}{3} &= 2 \ln |-1| + C \\ \frac{\sqrt{3}}{2} &= 2 \cdot 0 + C \\ \frac{\sqrt{3}}{2} &= C\end{aligned}$$

The particular solution is

$$\boxed{\sin y = 2 \ln |x| + \frac{\sqrt{3}}{2}}$$

17. The general solution of the differential equation $\frac{dy}{dx} = e^{y-x}$ is $y = -\ln(e^{-x} + C)$, see the solution to Problem 7. To find the particular solution, substitute $x = 0$ and $y = 0$ and solve for C .

$$\begin{aligned}y &= -\ln(e^{-x} + C) \\ 0 &= -\ln(e^0 + C) \\ 0 &= -\ln(1 + C) \\ 0 &= \ln(1 + C) \\ e^0 &= 1 + C \\ 1 &= 1 + C \\ 0 &= C\end{aligned}$$

Replace C with zero to get $y = -\ln(e^{-x} + 0) = -\ln(e^{-x}) = -(-x) = x$. The particular solution is

$$\boxed{y = x}$$

18. The general solution of the differential equation $x \frac{dy}{dx} + 2y = 5$ is $y = \frac{5}{2} + \frac{C}{x^2}$, see the solution to Problem 8. To find the particular solution, substitute $x = -1$ and $y = 1$ and solve for C .

$$\begin{aligned} y &= \frac{5}{2} + \frac{C}{x^2} \\ 1 &= \frac{5}{2} + \frac{C}{(-1)^2} \\ 1 &= \frac{5}{2} + C \\ -\frac{3}{2} &= C \end{aligned}$$

The particular solution is

$$y = \frac{5}{2} - \frac{3}{2x^2}$$

19. The general solution of the differential equation $\frac{dy}{dx} = \cos x \frac{dy}{dx} - y \sin x$ is $y = \frac{C}{1 - \cos x}$, see the solution to Problem 9. To find the particular solution, substitute $x = \pi$ and $y = \frac{1}{2}$ and solve for C .

$$\begin{aligned} y &= \frac{C}{1 - \cos x} \\ \frac{1}{2} &= \frac{C}{1 - \cos \pi} \\ \frac{1}{2} &= \frac{C}{2} \\ 1 &= C \end{aligned}$$

The particular solution is

$$y = \frac{1}{1 - \cos x}$$

on the interval $0 < x < 2\pi$.

20. The general solution of the differential equation $\frac{dy}{dx} = x + y \frac{dy}{dx}$ is $y - \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, see the solution to Problem 10. To find the particular solution, substitute $x = 2$ and $y = 1$ and solve for C .

$$\begin{aligned} y - \frac{1}{2}y^2 &= \frac{1}{2}x^2 + C \\ 1 - \frac{1}{2}1^2 &= \frac{1}{2}2^2 + C \\ \frac{1}{2} &= 2 + C \\ -\frac{3}{2} &= C \end{aligned}$$

The particular solution is

$$y - \frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{3}{2}$$

21. $f(x, y) = 2x^2 - 3xy - y^2$ is a homogeneous function of degree 2, since

$$\begin{aligned} f(tx, ty) &= 2(tx)^2 - 3(tx)(ty) - (ty)^2 = 2t^2x^2 - 3t^2xy - t^2y^2 \\ &= t^2(2x^2 - 3xy - y^2) = t^2f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

22. $f(x, y) = x^3 - xy^2 + y^3$ is a homogeneous function of degree 3, since

$$\begin{aligned} f(tx, ty) &= (tx)^3 - (tx)(ty)^2 + (ty)^3 \\ &= t^3(x^3 - xy^2 + y^3) = t^3 f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

23. $f(x, y) = x^3 - xy + y^3$ is not a homogeneous function, since

$$f(tx, ty) = (tx)^3 - (tx)(ty) + (ty)^3 = t^2(tx^3 - xy + ty^3) \neq t^k(x^3 - xy + y^3)$$

for all $t > 0$ and some k .

24. $f(x, y) = x^2 - xy^2 + y^2$ is not a homogeneous function, since

$$f(tx, ty) = (tx)^2 - tx(ty)^2 + (ty)^2 = t^2(x^2 - txy^2 + y^2) \neq t^k(x^2 - xy^2 + y^2)$$

for all $t > 0$ and some k .

25. $f(x, y) = 2x + \sqrt{x^2 + y^2}$ is a homogeneous function of degree 1, since

$$\begin{aligned} f(tx, ty) &= 2tx + \sqrt{(tx)^2 + (ty)^2} \\ &= 2tx + \sqrt{t^2(x^2 + y^2)} \\ &= 2tx + \sqrt{t^2} \sqrt{x^2 + y^2} \\ &= 2tx + t\sqrt{x^2 + y^2} \\ &= t(2x + \sqrt{x^2 + y^2}) = t f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

26. $f(x, y) = \sqrt{x + y}$ is a homogeneous function of degree 1/2, since

$$\begin{aligned} f(tx, ty) &= \sqrt{tx + ty} \\ &= \sqrt{t(x + y)} \\ &= \sqrt{t} \sqrt{x + y} \\ &= t^{1/2} \sqrt{x + y} = t^{1/2} f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

27. $f(x, y) = \tan\left(\frac{3x}{y}\right)$ is a homogeneous function of degree 0, since

$$\begin{aligned} f(tx, ty) &= \tan\left(\frac{3tx}{ty}\right) \\ &= t^0 \left(\tan\left(\frac{3x}{y}\right) \right) = t^0 f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

28. $f(x, y) = e^{x/y}$ is a homogeneous function of degree 0, since

$$\begin{aligned} f(tx, ty) &= e^{tx/ty} \\ &= t^0 e^{x/y} = t^0 f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

29. $f(x, y) = \ln \frac{x}{y}$ is a homogeneous function of degree 0, since

$$\begin{aligned} f(tx, ty) &= \ln \frac{tx}{ty} \\ &= t^0 \ln \frac{x}{y} = t^0 f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

30. $f(x, y) = x \ln x - x \ln y$ is a homogeneous function of degree 1, since

$$\begin{aligned} f(tx, ty) &= tx \ln tx - tx \ln ty \\ &= tx(\ln tx - \ln ty) \\ &= tx \ln \frac{tx}{ty} \\ &= tx \ln \frac{x}{y} \\ &= t(x \ln x - x \ln y) \\ &= t^1 f(x, y) \end{aligned}$$

for all $t > 0$.

31. Both $x - y$ and x are homogeneous of degree 1. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} (x - y) dx + x dy &= 0 \\ (x - xv) dx + x(x dv + v dx) &= 0 \\ x dx + x^2 dv &= 0 \end{aligned}$$

For $x \neq 0$, separate the variables by dividing by x^2 , then integrate.

$$\begin{aligned} \frac{1}{x} dx + dv &= 0 \\ \int \frac{1}{x} dx + \int dv &= 0 \\ \ln |x| + v &= C \\ \ln |x| + \frac{y}{x} &= C \\ x \ln |x| + y &= Cx \\ y &= Cx - x \ln |x| \end{aligned}$$

Then the general solution is $y = Cx - x \ln |x|$, where C is a constant.

32. Both $x + y$ and x are homogeneous of degree 1. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} (x + y) dx + x dy &= 0 \\ (x + xv) dx + x(x dv + v dx) &= 0 \\ (x + 2xv) dx + x^2 dv &= 0 \\ x(1 + 2v) dx + x^2 dv &= 0 \end{aligned}$$

For $x \neq 0$ and $v \neq -1/2$, we can separate the variables by dividing by x^2 and $1 + 2v$.

$$\begin{aligned} x(1 + 2v) dx + x^2 dv &= 0 \\ \frac{x(1 + 2v)}{x^2(1 + 2v)} dx + \frac{x^2}{x^2(1 + 2v)} dv &= 0 \\ \frac{1}{x} dx + \frac{1}{1 + 2v} dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{1}{1+2v} dv &= 0 \\
 \ln|x| + \frac{1}{2} \ln|1+2v| &= C_1 \\
 2 \ln|x| + \ln|1+2v| &= 2C_1 \\
 \ln|x^2| + \ln|1+2v| &= 2C_1 \\
 \ln|x^2(1+2v)| &= 2C_1 \\
 |x^2(1+2v)| &= e^{2C_1} \\
 x^2(1+2v) &= \pm e^{2C_1} \\
 x^2(1+2\frac{y}{x}) &= \pm e^{2C_1} \\
 x^2 + 2xy &= \pm e^{2C_1} \\
 2xy &= -x^2 \pm e^{2C_1} \\
 y &= -\frac{1}{2}x \pm \frac{e^{2C_1}}{2x}
 \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm \frac{e^{2C_1}}{2}$, where C is a nonzero constant. Then the general solution can be written as $y = -\frac{1}{2}x + \frac{C}{x}$, where $C \neq 0$ is a constant.

Notice that the function $y = -\frac{1}{2}x$ also satisfies the differential equation $(x+y) + x \frac{dy}{dx} = 0$, so it too is a solution. This particular solution matches the general solution above when $C = 0$. Then the general solution is $\boxed{y = -\frac{1}{2}x + \frac{C}{x}}$, where C is a constant.

33. Both $x^2 + y^2$ and $x^2 - xy$ are homogeneous of degree 2. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}
 (x^2 + y^2) dx + (x^2 - xy) dy &= 0 \\
 (x^2 + (xv)^2) dx + (x^2 - x(xv))(x dv + v dx) &= 0 \\
 (x^2 + x^2v) dx + (x^3 - x^3v) dv &= 0 \\
 x^2(1+v) dx + x^3(1-v) dv &= 0
 \end{aligned}$$

For $x \neq 0$ and $v \neq -1$, we can separate the variables by dividing by x^3 and $1+v$.

$$\begin{aligned}
 x^2(1+v) dx + x^3(1-v) dv &= 0 \\
 \frac{x^2(1+v)}{x^3(1+v)} dx + \frac{x^3(1-v)}{x^3(1+v)} dv &= 0 \\
 \frac{1}{x} dx + \frac{1-v}{1+v} dv &= 0
 \end{aligned}$$

Integrate.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{1-v}{1+v} dv &= 0 \\
 \int \frac{1}{x} dx + \int \left(-1 + \frac{2}{1+v} \right) dv &= 0 \\
 \ln |x| - v + 2 \ln |1+v| &= C \\
 \ln |x| - \frac{y}{x} + 2 \ln \left| 1 + \frac{y}{x} \right| &= C \\
 \ln |x| - \frac{y}{x} + 2 \ln \left| \frac{x+y}{x} \right| &= C \\
 \ln |x| - \frac{y}{x} + 2 \ln |x+y| - 2 \ln |x| &= C \\
 2 \ln |x+y| - \ln |x| - \frac{y}{x} &= C
 \end{aligned}$$

Then the general solution is $\boxed{\frac{y}{x} - 2 \ln |x+y| + \ln |x| + C = 0}$, where C is a constant.

34. Both xy and $x^2 + y^2$ are homogeneous of degree 2. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}
 xy dx + (x^2 + y^2) dy &= 0 \\
 x(xv) dx + (x^2 + (xv)^2)(x dv + v dx) &= 0 \\
 (2x^2v + x^2v^3) dx + (x^3 + x^3v^2) dv &= 0 \\
 x^2(2v + v^3) dx + x^3(1 + v^2) dv &= 0
 \end{aligned}$$

For $x \neq 0$ and $v \neq 0$, we can separate the variables by dividing by x^3 and $2v + v^3$.

$$\begin{aligned}
 x^2(2v + v^3) dx + x^3(1 + v^2) dv &= 0 \\
 \frac{x^2(2v + v^3)}{x^3(2v + v^3)} dx + \frac{x^3(1 + v^2)}{x^3(2v + v^3)} dv &= 0 \\
 \frac{1}{x} dx + \frac{1 + v^2}{2v + v^3} dv &= 0
 \end{aligned}$$

Integrate using a partial fraction decomposition.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{1 + v^2}{2v + v^3} dv &= 0 \\
 \int \frac{1}{x} dx + \frac{1}{2} \int \left(\frac{1}{v} + \frac{v}{2 + v^2} \right) dv &= 0 \\
 \ln |x| + \frac{1}{2} \ln |v| + \frac{1}{4} \ln |2 + v^2| &= C_1 \\
 4 \ln |x| + 2 \ln |v| + \ln |2 + v^2| &= 4C_1 \\
 4 \ln |x| + 2 \ln \left| \frac{y}{x} \right| + \ln \left| 2 + \frac{y^2}{x^2} \right| &= 4C_1 \\
 4 \ln |x| + 2 \ln \left| \frac{y}{x} \right| + \ln \left| \frac{2x^2 + y^2}{x^2} \right| &= 4C_1 \\
 4 \ln |x| + 2 \ln |y| - 2 \ln |x| + \ln |2x^2 + y^2| - \ln |x^2| &= 4C_1 \\
 4 \ln |x| + 2 \ln |y| - 2 \ln |x| + \ln |2x^2 + y^2| - 2 \ln |x| &= 4C_1 \\
 2 \ln |y| + \ln |2x^2 + y^2| &= 4C_1
 \end{aligned}$$

Since C_1 is a constant, we can write $C = 4C_1$, where C is a constant. Then the general solution is $\boxed{2 \ln |y| + \ln |2x^2 + y^2| = C}$, where C is a constant.

35. Write the differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ in the form $(y+x)dy = (y-x)dx$. Both $y+x$ and $y-x$ are homogeneous of degree 1. Let $y = xv$. Then $dy = xdv + vdx$. Substitute these into the differential equation:

$$\begin{aligned}(y+x)dy &= (y-x)dx \\(xv+x)(xdv+vdx) &= (xv-x)dx \\(xv^2+x)dx + (x^2v+x^2)dv &= 0 \\x(v^2+1)dx + x^2(v+1)dv &= 0\end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^2 and v^2+1 .

$$\begin{aligned}x(v^2+1)dx + x^2(v+1)dv &= 0 \\ \frac{x(v^2+1)}{x^2(v^2+1)}dx + \frac{x^2(v+1)}{x^2(v^2+1)}dv &= 0 \\ \frac{1}{x}dx + \frac{v+1}{v^2+1}dv &= 0\end{aligned}$$

Integrate.

$$\begin{aligned}\int \frac{1}{x}dx + \int \frac{v+1}{v^2+1}dv &= 0 \\ \int \frac{1}{x}dx + \int \frac{v}{v^2+1}dv + \int \frac{1}{v^2+1}dv &= 0 \\ \ln|x| + \frac{1}{2}\ln(v^2+1) + \tan^{-1}v &= C \\ \ln|x| + \frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \tan^{-1}\left(\frac{y}{x}\right) &= C \\ \ln|x| + \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) &= C \\ \ln|x| + \frac{1}{2}\ln(x^2+y^2) - \frac{1}{2}\ln x^2 + \tan^{-1}\left(\frac{y}{x}\right) &= C \\ \ln|x| + \frac{1}{2}\ln(x^2+y^2) - \ln\sqrt{x^2} + \tan^{-1}\left(\frac{y}{x}\right) &= C \\ \ln|x| + \frac{1}{2}\ln(x^2+y^2) - \ln|x| + \tan^{-1}\left(\frac{y}{x}\right) &= C \\ \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) &= C\end{aligned}$$

Then the general solution is $\boxed{\frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = C}$, where C is a constant.

36. Write the differential equation $\frac{dy}{dx} = \frac{x+2y}{2x+y}$ in the form $(2x+y)dy = (x+2y)dx$. Both $2x+y$ and $x+2y$ are homogeneous of degree 1. Let $y = xv$. Then $dy = xdv + vdx$. Substitute these into the differential equation:

$$\begin{aligned}(2x+y)dy &= (x+2y)dx \\(2x+xv)(xdv+vdx) &= (x+2xv)dx \\(xv^2-x)dx + (2x^2+x^2v)dv &= 0 \\x(v^2-1)dx + x^2(2+v)dv &= 0\end{aligned}$$

For $x \neq 0$ and $v \neq \pm 1$, we can separate the variables by dividing by x^2 and $v^2 - 1$.

$$\begin{aligned} x(v^2 - 1) dx + x^2(2 + v) dv &= 0 \\ \frac{x(v^2 - 1)}{x^2(v^2 - 1)} dx + \frac{x^2(2 + v)}{x^2(v^2 - 1)} dv &= 0 \\ \frac{1}{x} dx + \frac{2 + v}{v^2 - 1} dv &= 0 \end{aligned}$$

Integrate using a partial fraction decomposition.

$$\begin{aligned} \int \frac{1}{x} dx + \int \frac{2 + v}{v^2 - 1} dv &= 0 \\ \int \frac{1}{x} dx + \int \left(\frac{3}{2} \frac{1}{v - 1} - \frac{1}{2} \frac{1}{v + 1} \right) dv &= 0 \\ \ln |x| + \frac{3}{2} \ln |v - 1| - \frac{1}{2} \ln |v + 1| &= C_1 \\ 2 \ln |x| + 3 \ln |v - 1| - \ln |v + 1| &= 2C_1 \\ 2 \ln |x| + 3 \ln \left| \frac{y}{x} - 1 \right| - \ln \left| \frac{y}{x} + 1 \right| &= 2C_1 \\ 2 \ln |x| + 3 \ln \left| \frac{y - x}{x} \right| - \ln \left| \frac{x + y}{x} \right| &= 2C_1 \\ 2 \ln |x| + 3 \ln |y - x| - 3 \ln |x| - \ln |x + y| + \ln |x| &= 2C_1 \\ 3 \ln |y - x| - \ln |x + y| &= 2C_1 \end{aligned}$$

Since C_1 is a constant, we can write $C = 2C_1$, where C is a constant. Then the general solution is $\boxed{3 \ln |y - x| - \ln |x + y| = C}$, where C is a constant.

37. Write the differential equation $x \frac{dy}{dx} = x + y$ in the form $x dy = (x + y) dx$. Both x and $x + y$ are homogeneous of degree 1. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} x dy &= (x + y) dx \\ x(x dv + v dx) &= (x + xv) dx \\ -x dx + x^2 dv &= 0 \end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^2 .

$$\begin{aligned} -x dx + x^2 dv &= 0 \\ -\frac{1}{x} dx + dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned} -\int \frac{1}{x} dx + \int dv &= 0 \\ -\ln |x| + v &= C \\ -\ln |x| + \frac{y}{x} &= C \\ -x \ln |x| + y &= Cx \\ y &= x \ln |x| + Cx = x(C + \ln |x|) \end{aligned}$$

Then the general solution is $\boxed{y = x(C + \ln |x|)}$, where C is a constant.

38. Write the differential equation $x(x^2 - y^2) \frac{dy}{dx} - y(x^2 + y^2) = 0$ in the form $x(x^2 - y^2) dy - y(x^2 + y^2) dx = 0$. Both $x(x^2 - y^2)$ and $-y(x^2 + y^2)$ are homogeneous of degree 3. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} x(x^2 - y^2) dy - y(x^2 + y^2) dx &= 0 \\ x(x^2 - (xv)^2)(x dv + v dx) - xv(x^2 + (xv)^2) dx &= 0 \\ -2x^3v^3 dx + (x^4 - x^4v^2) dv &= 0 \\ -2x^3v^3 dx + x^4(1 - v^2) dv &= 0 \end{aligned}$$

For $x \neq 0$ and $v \neq 0$, we can separate the variables by dividing by $-x^4$ and v^3 .

$$\begin{aligned} -2x^3v^3 dx + x^4(1 - v^2) dv &= 0 \\ \frac{-2x^3v^3}{-x^4v^3} dx + \frac{x^4(1 - v^2)}{-x^4v^3} dv &= 0 \\ \frac{2}{x} dx + \frac{v^2 - 1}{v^3} dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned} \int \frac{2}{x} dx + \int \frac{v^2 - 1}{v^3} dv &= 0 \\ \int \frac{2}{x} dx + \int \left(\frac{1}{v} - v^{-3} \right) dv &= 0 \\ 2 \ln |x| + \ln |v| + \frac{1}{2} v^{-2} &= C \\ 2 \ln |x| + \ln \left| \frac{y}{x} \right| + \frac{1}{2} \left(\frac{y}{x} \right)^{-2} &= C \\ 2 \ln |x| + \ln |y| - \ln |x| + \frac{1}{2} \left(\frac{x}{y} \right)^2 &= C \\ \ln |x| + \ln |y| + \frac{x^2}{2y^2} &= C \end{aligned}$$

Then the general solution is $\boxed{\ln |x| + \ln |y| + \frac{x^2}{2y^2} = C}$, where C is a constant.

39. Write the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$ in the form $(x^2 + y^2) dy = 2xy dx$. Both $x^2 + y^2$ and $2xy$ are homogeneous of degree 2. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} (x^2 + y^2) dy &= 2xy dx \\ (x^2 + (xv)^2)(x dv + v dx) &= 2x(xv) dx \\ (x^2v^3 - x^2v) dx + (x^3 + x^3v^2) dv &= 0 \\ x^2(v^3 - v) dx + x^3(1 + v^2) dv &= 0 \end{aligned}$$

For $x \neq 0$ and $v \neq 0, \pm 1$, we can separate the variables by dividing by x^3 and $v^3 - v$.

$$\begin{aligned} x^2(v^3 - v) dx + x^3(1 + v^2) dv &= 0 \\ \frac{x^2(v^3 - v)}{x^3(v^3 - v)} dx + \frac{x^3(1 + v^2)}{x^3(v^3 - v)} dv &= 0 \\ \frac{1}{x} dx + \frac{1 + v^2}{v^3 - v} dv &= 0 \end{aligned}$$

Integrate using a partial fraction decomposition.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{1+v^2}{v^3-v} dv &= 0 \\
 \int \frac{1}{x} dx + \int \left(-\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v+1} \right) dv &= 0 \\
 \ln|x| - \ln|v| + \ln|v-1| + \ln|v+1| &= C_1 \\
 \ln|x| - \ln\left|\frac{y}{x}\right| + \ln\left|\frac{y}{x}-1\right| + \ln\left|\frac{y}{x}+1\right| &= C_1 \\
 \ln|x| - \ln\left|\frac{y}{x}\right| + \ln\left|\frac{y-x}{x}\right| + \ln\left|\frac{y+x}{x}\right| &= C_1 \\
 \ln|x| + \ln\left|\frac{x}{y}\right| + \ln\left|\frac{y-x}{x}\right| + \ln\left|\frac{y+x}{x}\right| &= C_1 \\
 \ln\left|x \cdot \frac{x}{y} \cdot \frac{y-x}{x} \cdot \frac{y+x}{x}\right| &= C_1 \\
 \ln\left|\frac{y^2-x^2}{y}\right| &= C_1 \\
 \left|\frac{y^2-x^2}{y}\right| &= e^{C_1} \\
 \frac{y^2-x^2}{y} &= \pm e^{C_1}
 \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{C_1}$, where C is a nonzero constant. Then the general solution can be written as $\frac{y^2-x^2}{y} = C$, where $C \neq 0$ is a constant.

Notice that the functions $y = x$ and $y = -x$ also satisfy the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$, so they are also solutions. These particular solutions arise in the general solution above when $C = 0$. Then

the general solution is $\boxed{\frac{y^2-x^2}{y} = C}$, where C is a constant.

40. Write the differential equation $\frac{dy}{dx} = \frac{x+2y}{2x-y}$ in the form $(2x-y)dy = (x+2y)dx$. Both $2x-y$ and $x+2y$ are homogeneous of degree 1. Let $y = xv$. Then $dy = xdv + vdx$. Substitute these into the differential equation:

$$\begin{aligned}
 (2x-y)dy &= (x+2y)dx \\
 (2x-xv)(xdv+vdx) &= (x+2xv)dx \\
 (-xv^2-x)dx + (2x^2-x^2v)dv &= 0 \\
 x(-v^2-1)dx + x^2(2-v)dv &= 0
 \end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^2 and $-v^2-1$.

$$\begin{aligned}
 x(-v^2-1)dx + x^2(2-v)dv &= 0 \\
 \frac{x(-v^2-1)}{x^2(-v^2-1)}dx + \frac{x^2(2-v)}{x^2(-v^2-1)}dv &= 0 \\
 \frac{1}{x}dx + \frac{v-2}{v^2+1}dv &= 0
 \end{aligned}$$

Integrate.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{v-2}{v^2+1} dv &= 0 \\
 \int \frac{1}{x} dx + \int \left(\frac{v}{v^2+1} - \frac{2}{v^2+1} \right) dv &= 0 \\
 \ln |x| + \frac{1}{2} \ln(v^2+1) - 2 \tan^{-1} v &= C \\
 \ln |x| + \frac{1}{2} \ln \left(\frac{y^2}{x^2} + 1 \right) - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C \\
 \ln |x| + \frac{1}{2} \ln \left(\frac{x^2+y^2}{x^2} \right) - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C \\
 \ln |x| + \frac{1}{2} \ln(x^2+y^2) - \frac{1}{2} \ln(x^2) - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C \\
 \ln |x| + \frac{1}{2} \ln(x^2+y^2) - \ln(\sqrt{x^2}) - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C \\
 \ln |x| + \frac{1}{2} \ln(x^2+y^2) - \ln |x| - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C \\
 \frac{1}{2} \ln(x^2+y^2) - 2 \tan^{-1} \left(\frac{y}{x} \right) &= C
 \end{aligned}$$

Then the general solution is $\boxed{\frac{1}{2} \ln(x^2+y^2) - 2 \tan^{-1} \left(\frac{y}{x} \right) = C}$, where C is a constant.

41. Write the differential equation $x^2 \frac{dy}{dx} = x^2 + xy - 4y^2$ in the form $x^2 dy = (x^2 + xy - 4y^2) dx$. Both x^2 and $x^2 + xy - 4y^2$ are homogeneous of degree 2. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}
 x^2 dy &= (x^2 + xy - 4y^2) dx \\
 x^2(x dv + v dx) &= (x^2 + x(xv) - 4(xv)^2) dx \\
 (4x^2v^2 - x^2) dx + x^3 dv &= 0 \\
 x^2(4v^2 - 1) dx + x^3 dv &= 0
 \end{aligned}$$

For $x \neq 0$ and $v \neq \pm 1/2$, we can separate the variables by dividing by x^3 and $4v^2 - 1$.

$$\begin{aligned}
 x^2(4v^2 - 1) dx + x^3 dv &= 0 \\
 \frac{x^2(4v^2 - 1)}{x^3(4v^2 - 1)} dx + \frac{x^3}{x^3(4v^2 - 1)} dv &= 0 \\
 \frac{1}{x} dx + \frac{1}{4v^2 - 1} dv &= 0
 \end{aligned}$$

Integrate using a partial fraction decomposition.

$$\begin{aligned}
 \int \frac{1}{x} dx + \int \frac{1}{4v^2 - 1} dv &= 0 \\
 \int \frac{1}{x} dx + \frac{1}{2} \int \left(\frac{1}{2v - 1} - \frac{1}{2v + 1} \right) dv &= 0 \\
 \ln |x| + \frac{1}{4} \ln |2v - 1| - \frac{1}{4} \ln |2v + 1| &= C_1 \\
 4 \ln |x| + \ln |2v - 1| - \ln |2v + 1| &= 4C_1 \\
 4 \ln |x| + \ln \left| \frac{2y}{x} - 1 \right| - \ln \left| \frac{2y}{x} + 1 \right| &= 4C_1 \\
 4 \ln |x| + \ln \left| \frac{2y - x}{x} \right| - \ln \left| \frac{2y + x}{x} \right| &= 4C_1 \\
 \ln |x^4| + \ln \left| \frac{2y - x}{x} \right| + \ln \left| \frac{x}{2y + x} \right| &= 4C_1 \\
 \ln \left| x^4 \cdot \frac{2y - x}{x} \cdot \frac{x}{2y + x} \right| &= 4C_1 \\
 \ln \left| \frac{x^4(2y - x)}{2y + x} \right| &= 4C_1 \\
 \left| \frac{x^4(2y - x)}{2y + x} \right| &= e^{4C_1} \\
 \frac{x^4(2y - x)}{2y + x} &= \pm e^{4C_1}
 \end{aligned}$$

Since C_1 is a constant, we can write $C = \pm e^{4C_1}$, where $C \neq 0$ is a constant. Then the general solution is $\frac{x^4(2y - x)}{2y + x} = C$, where $C \neq 0$ is a constant.

To find the particular solution, substitute $x = -1$ and $y = 1$ and solve for C .

$$\begin{aligned}
 \frac{x^4(2y - x)}{2y + x} &= C \\
 \frac{(-1)^4(2 - (-1))}{2 - 1} &= C \\
 3 &= C
 \end{aligned}$$

The particular solution is $\boxed{\frac{x^4(2y - x)}{2y + x} = 3}$.

42. Write the differential equation $x(x^2 - y^2) \frac{dy}{dx} - y(x^2 + y^2) = 0$ in the form $x(x^2 - y^2) dy - y(x^2 + y^2) dx = 0$. Both $x(x^2 - y^2)$ and $-y(x^2 + y^2)$ are homogeneous of degree 3. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}
 x(x^2 - y^2) dy - y(x^2 + y^2) dx &= 0 \\
 x(x^2 - (xv)^2)(x dv + v dx) - xv(x^2 + (xv)^2) dx &= 0 \\
 -2x^3v^3 dx + (x^4 - x^4v^2) dv &= 0 \\
 -2x^3v^3 dx + x^4(1 - v^2) dv &= 0
 \end{aligned}$$

For $x \neq 0$ and $v \neq 0$, we can separate the variables by dividing by $-x^4$ and v^3 .

$$\begin{aligned} -2x^3v^3 dx + x^4(1-v^2) dv &= 0 \\ \frac{-2x^3v^3}{-x^4v^3} dx + \frac{x^4(1-v^2)}{-x^4v^3} dv &= 0 \\ \frac{2}{x} dx + \frac{v^2-1}{v^3} dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned} \int \frac{2}{x} dx + \int \frac{v^2-1}{v^3} dv &= 0 \\ \int \frac{2}{x} dx + \int \left(\frac{1}{v} - v^{-3} \right) dv &= 0 \\ 2 \ln |x| + \ln |v| + \frac{1}{2} v^{-2} &= C \\ 2 \ln |x| + \ln \left| \frac{y}{x} \right| + \frac{1}{2} \left(\frac{y}{x} \right)^{-2} &= C \\ 2 \ln |x| + \ln |y| - \ln |x| + \frac{1}{2} \left(\frac{x}{y} \right)^2 &= C \\ \ln |x| + \ln |y| + \frac{1}{2} \left(\frac{x^2}{y^2} \right) &= C \\ \ln |x| + \ln |y| + \frac{x^2}{2y^2} &= C \end{aligned}$$

To find the particular solution, substitute $x = -1$ and $y = -2$ and solve for C .

$$\begin{aligned} \ln |x| + \ln |y| + \frac{x^2}{2y^2} &= C \\ \ln |-1| + \ln |-2| + \frac{(-1)^2}{2(-2)^2} &= C \\ \ln(2) + \frac{1}{8} &= C \end{aligned}$$

The particular solution is $\boxed{\ln |x| + \ln |y| + \frac{x^2}{2y^2} = \ln(2) + \frac{1}{8}}$.

43. Find y' by differentiating $xy = C$ with respect to x .

$$\begin{aligned} xy' + y &= 0 \\ xy' &= -y \\ y' &= -\frac{y}{x} \end{aligned}$$

Replace y' with $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = \frac{x}{y}$$

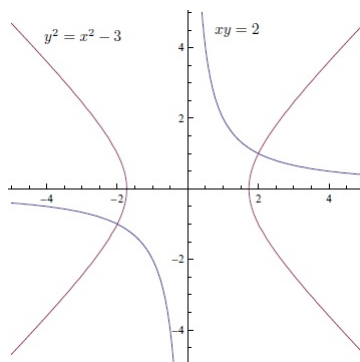
The differential equation $\frac{dy}{dx} = \frac{x}{y}$ is separable and can be written as

$$y dy = x dx$$

Integrate to find the general solution.

$$\begin{aligned}\int y \, dy &= \int x \, dx \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + K_1 \\ y^2 &= x^2 + 2K_1\end{aligned}$$

Since K_1 is a constant, we may write $K = 2K_1$, where K is a constant. The orthogonal trajectories for the family of curves $xy = C$ is the family $y^2 = x^2 + K$. To find the particular curves needed for the graph, substitute $x = 2$ and $y = 1$ into $xy = C$ and $y^2 = x^2 + K$ separately and solve for C and K . The graph of $xy = 2$ and $y^2 = x^2 - 3$ is shown.



44. Rewrite $y^2 = 2(C - x)$ in the form $x + \frac{1}{2}y^2 = C$. Find y' by differentiating $x + \frac{1}{2}y^2 = C$ with respect to x .

$$\begin{aligned}1 + yy' &= 0 \\ yy' &= -1 \\ y' &= -\frac{1}{y}\end{aligned}$$

Replace y' with $-\frac{1}{y}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = y$$

The differential equation $\frac{dy}{dx} = y$ is separable and can be written as

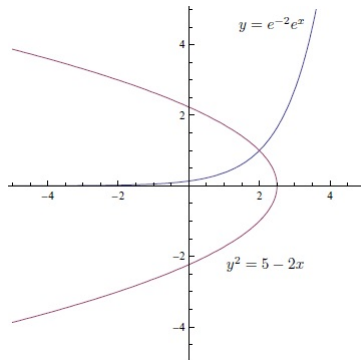
$$\frac{1}{y} dy = dx$$

Integrate to find the general solution.

$$\begin{aligned}\int \frac{1}{y} dy &= \int dx \\ \ln |y| &= x + K_1 \\ |y| &= e^{x+K_1} \\ |y| &= e^x e^{K_1} \\ y &= \pm e^x e^{K_1}\end{aligned}$$

Since K_1 is a constant, we can write $K = \pm e^{K_1}$, where K is a nonzero constant. Notice that $y = 0$ also satisfies the differential equation $y' = y$, so it too is a solution. This particular solution matches

the general solution when $K = 0$. The orthogonal trajectories for the family of curves $y^2 = 2(C - x)$ is the family $y = Ke^x$, where K is any constant. To find the particular curves needed for the graph, substitute $x = 2$ and $y = 1$ into $y^2 = 2(C - x)$ and $y = Ke^x$ separately and solve for C and K . The graph of $y^2 = 5 - 2x$ and $y = e^{-2}e^x$ is shown.



45. Rewrite $y = Cx^2$ in the form $\frac{y}{x^2} = C$. Find y' by differentiating $\frac{y}{x^2} = C$ with respect to x .

$$\begin{aligned}\frac{x^2 y' - y \cdot 2x}{x^4} &= 0 \\ x^2 y' - 2xy &= 0 \\ x^2 y' &= 2xy \\ y' &= \frac{2xy}{x^2} \\ y' &= \frac{2y}{x}\end{aligned}$$

Replace y' with $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = -\frac{x}{2y}$$

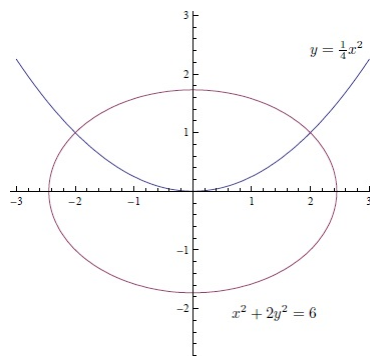
The differential equation $\frac{dy}{dx} = -\frac{x}{2y}$ is separable and can be written as

$$2y \, dy = -x \, dx$$

Integrate to find the general solution.

$$\begin{aligned}\int 2y \, dy &= -\int x \, dx \\ y^2 &= -\frac{1}{2}x^2 + K_1 \\ 2y^2 &= -x^2 + 2K_1 \\ x^2 + 2y^2 &= 2K_1\end{aligned}$$

Since K_1 is a constant, we can write $K = 2K_1$, where K is a constant. The orthogonal trajectories for the family of curves $y = Cx^2$ is the family $x^2 + 2y^2 = K$. To find the particular curves needed for the graph, substitute $x = 2$ and $y = 1$ into $y = Cx^2$ and $x^2 + 2y^2 = K$ separately and solve for C and K . The graph of $y = \frac{1}{4}x^2$ and $x^2 + 2y^2 = 6$ is shown.



46. Rewrite $x^2 = Cy^3$ in the form $\frac{x^2}{y^3} = C$. Find y' by differentiating $\frac{x^2}{y^3} = C$ with respect to x .

$$\begin{aligned}\frac{y^3 \cdot 2x - x^2 \cdot 3y^2 y'}{y^6} &= 0 \\ 2xy^3 - 3x^2 y^2 y' &= 0 \\ -3x^2 y^2 y' &= -2xy^3 \\ y' &= \frac{-2xy^3}{-3x^2 y^2} \\ y' &= \frac{2y}{3x}\end{aligned}$$

Replace y' with $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = -\frac{3x}{2y}$$

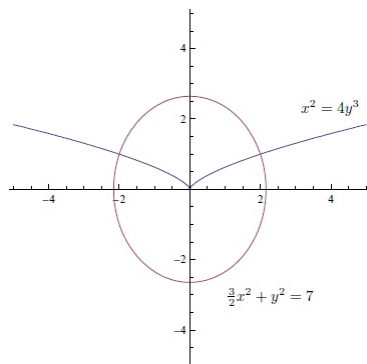
The differential equation $\frac{dy}{dx} = -\frac{3x}{2y}$ is separable and can be written as

$$2y \, dy = -3x \, dx$$

Integrate to find the general solution.

$$\begin{aligned}\int 2y \, dy &= -\int 3x \, dx \\ y^2 &= -\frac{3}{2}x^2 + K \\ \frac{3}{2}x^2 + y^2 &= K\end{aligned}$$

The orthogonal trajectories for the family of curves $x^2 = Cy^3$ is the family $\boxed{\frac{3}{2}x^2 + y^2 = K}$. To find the particular curves needed for the graph, substitute $x = 2$ and $y = 1$ into $x^2 = Cy^3$ and $\frac{3}{2}x^2 + y^2 = K$ separately and solve for C and K . The graph of $x^2 = 4y^3$ and $\frac{3}{2}x^2 + y^2 = 7$ is shown.



Applications and Extensions

47. Substitute $x = u + h$ and $y = v + k$ into the numerator $y - x - 3$ to eliminate the constant -3 : $(v + k) - (u + h) - 3 = v - u$ implies that $k - h - 3 = 0$. Substitute $x = u + h$ and $y = v + k$ into the denominator $y + x + 4$ to eliminate the constant 4 : $(v + k) + (u + h) + 4 = v + u$ implies that $k + h + 4 = 0$. Solve the system of equations

$$\begin{aligned} k - h &= 3 \\ k + h &= -4 \end{aligned}$$

to find $h = -\frac{7}{2}$ and $k = -\frac{1}{2}$.

Let $x = u - \frac{7}{2}$ and $y = v - \frac{1}{2}$. Then $\frac{dy}{dx} = \frac{dv}{du}$. Substitute these into the differential equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y - x - 3}{y + x + 4} \\ \frac{dv}{du} &= \frac{v - u}{v + u} \\ (v + u) dv &= (v - u) du \end{aligned}$$

Both $v + u$ and $v - u$ are homogeneous of degree 1. Let $v = uw$. Then $dv = u dw + w du$. Substitute these into the differential equation.

$$\begin{aligned} (v + u) dv &= (v - u) du \\ (uw + u)(u dw + w du) &= (uw - u) du \\ (uw^2 + u) du + (u^2 w + u^2) dw &= 0 \\ u(w^2 + 1) du + u^2(w + 1) dw &= 0 \end{aligned}$$

For $u \neq 0$, we can separate the variables by dividing by u^2 and $w^2 + 1$.

$$\begin{aligned} u(w^2 + 1) du + u^2(w + 1) dw &= 0 \\ \frac{u(w^2 + 1)}{u^2(w^2 + 1)} du + \frac{u^2(w + 1)}{u^2(w^2 + 1)} dw &= 0 \\ \frac{1}{u} du + \frac{w + 1}{w^2 + 1} dw &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned}
 \int \frac{1}{u} du + \int \frac{w+1}{w^2+1} dw &= 0 \\
 \int \frac{1}{u} du + \int \left(\frac{w}{w^2+1} + \frac{1}{w^2+1} \right) dw &= 0 \\
 \ln |u| + \frac{1}{2} \ln(w^2+1) + \tan^{-1} w &= C_1 \\
 \ln |u| + \frac{1}{2} \ln \left(\frac{v^2}{u^2} + 1 \right) + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \quad w = \frac{v}{u} \\
 \ln |u| + \frac{1}{2} \ln \left(\frac{u^2+v^2}{u^2} \right) + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \\
 \ln |u| + \frac{1}{2} \ln(u^2+v^2) - \frac{1}{2} \ln(u^2) + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \\
 \ln |u| + \frac{1}{2} \ln(u^2+v^2) - \ln(\sqrt{u^2}) + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \\
 \ln |u| + \frac{1}{2} \ln(u^2+v^2) - \ln |u| + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \\
 \frac{1}{2} \ln(u^2+v^2) + \tan^{-1} \left(\frac{v}{u} \right) &= C_1 \\
 \frac{1}{2} \ln \left(\left(x + \frac{7}{2} \right)^2 + \left(y + \frac{1}{2} \right)^2 \right) + \tan^{-1} \left(\frac{y + \frac{1}{2}}{x + \frac{7}{2}} \right) &= C_1 \quad u = x + \frac{7}{2}, v = y + \frac{1}{2} \\
 \frac{1}{2} \ln \left(\left(\frac{2x+7}{2} \right)^2 + \left(\frac{2y+1}{2} \right)^2 \right) + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) &= C_1 \\
 \frac{1}{2} \ln \left(\frac{(2x+7)^2}{4} + \frac{(2y+1)^2}{4} \right) + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) &= C_1 \\
 \frac{1}{2} \ln((2x+7)^2 + (2y+1)^2) - \frac{1}{2} \ln 4 + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) &= C_1 \\
 \frac{1}{2} \ln((2x+7)^2 + (2y+1)^2) - \ln \sqrt{4} + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) &= C_1 \\
 \frac{1}{2} \ln((2x+7)^2 + (2y+1)^2) + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) &= C_1 + \ln 2 = C
 \end{aligned}$$

Then the general solution is $\boxed{\frac{1}{2} \ln((2x+7)^2 + (2y+1)^2) + \tan^{-1} \left(\frac{2y+1}{2x+7} \right) = C}$, where C is a constant.

48. Substitute $x = u + h$ and $y = v + k$ into the numerator $x + 2y - 3$ to eliminate the constant -3 : $(u + h) + 2(v + k) - 3 = u + 2v$ implies that $h + 2k - 3 = 0$. Substitute $x = u + h$ and $y = v + k$ into the denominator $2x - y + 1$ to eliminate the constant 1 : $2(u + h) - (v + k) + 1 = 2u - v$ implies that $2h - k + 1 = 0$. Solve the system of equations

$$\begin{aligned}
 h + 2k &= 3 \\
 2h - k &= -1
 \end{aligned}$$

to find $h = \frac{1}{5}$ and $k = \frac{7}{5}$.

Let $x = u + \frac{1}{5}$ and $y = v + \frac{7}{5}$. Then $\frac{dy}{dx} = \frac{dv}{du}$. Substitute these into the differential equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x + 2y - 3}{2x - y + 1} \\ \frac{dv}{du} &= \frac{u + 2v}{2u - v} \\ (2u - v) dv &= (u + 2v) du\end{aligned}$$

Both $2u - v$ and $u + 2v$ are homogeneous of degree 1. Let $v = uw$. Then $dv = u dw + w du$. Substitute these into the differential equation.

$$\begin{aligned}(2u - v) dv &= (u + 2v) du \\ (2u - uw)(u dw + w du) &= (u + 2uw) du \\ (-uw^2 - u) du + (2u^2 - u^2w) dw &= 0 \\ -u(w^2 + 1) du + u^2(2 - w) dw &= 0\end{aligned}$$

For $u \neq 0$, we can separate the variables by dividing by u^2 and $w^2 + 1$.

$$\begin{aligned}-u(w^2 + 1) du + u^2(2 - w) dw &= 0 \\ \frac{-u(w^2 + 1)}{-u^2(w^2 + 1)} du + \frac{u^2(2 - w)}{-u^2(w^2 + 1)} dw &= 0 \\ \frac{1}{u} du + \frac{w - 2}{w^2 + 1} dw &= 0\end{aligned}$$

Integrate.

$$\begin{aligned}
 \int \frac{1}{u} du + \int \frac{w-2}{w^2+1} dw &= 0 \\
 \int \frac{1}{u} du + \int \left(\frac{w}{w^2+1} - \frac{2}{w^2+1} \right) dw &= 0 \\
 \ln |u| + \frac{1}{2} \ln(w^2+1) - 2 \tan^{-1} w &= 0 \\
 \ln |u| + \frac{1}{2} \ln \left(\frac{v^2}{u^2} + 1 \right) - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \quad w = \frac{v}{u} \\
 \ln |u| + \frac{1}{2} \ln \left(\frac{u^2 + v^2}{u^2} \right) - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \\
 \ln |u| + \frac{1}{2} \ln(u^2 + v^2) - \frac{1}{2} \ln(u^2) - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \\
 \ln |u| + \frac{1}{2} \ln(u^2 + v^2) - \ln(\sqrt{u^2}) - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \\
 \ln |u| + \frac{1}{2} \ln(u^2 + v^2) - \ln |u| - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \\
 \frac{1}{2} \ln(u^2 + v^2) - 2 \tan^{-1} \left(\frac{v}{u} \right) &= C \\
 \frac{1}{2} \ln \left(\left(x - \frac{1}{5} \right)^2 + \left(y - \frac{7}{5} \right)^2 \right) - 2 \tan^{-1} \left(\frac{y - \frac{7}{5}}{x - \frac{1}{5}} \right) &= C \quad u = x - \frac{1}{5}, \quad v = y - \frac{7}{5} \\
 \frac{1}{2} \ln \left(\left(\frac{5x-1}{5} \right)^2 + \left(\frac{5y-7}{2} \right)^2 \right) - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) &= C \\
 \frac{1}{2} \ln \left(\frac{(5x-1)^2}{25} + \frac{(5y-7)^2}{25} \right) - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) &= C \\
 \frac{1}{2} \ln((5x-1)^2 + (5y-7)^2) - \frac{1}{2} \ln 25 - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) &= C \\
 \frac{1}{2} \ln((5x-1)^2 + (5y-7)^2) - \ln \sqrt{25} - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) &= C \\
 \frac{1}{2} \ln((5x-1)^2 + (5y-7)^2) - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) &= C + \ln 5
 \end{aligned}$$

To find the particular solution, substitute $x = 3$ and $y = 4$. The particular solution is

$$\boxed{\frac{1}{2} \ln((5x-1)^2 + (5y-7)^2) - 2 \tan^{-1} \left(\frac{5y-7}{5x-1} \right) = \ln \sqrt{365} - 2 \tan^{-1} \left(\frac{13}{14} \right)}$$

49. Differentiate $y^2 = 4Cx$ with respect to x to get $2yy' = 4C$. Since $C = \frac{y^2}{4x}$ from the original equation, the differential equation becomes

$$\begin{aligned}
 2yy' &= 4C \\
 2yy' &= 4 \frac{y^2}{4x} \\
 y' &= \frac{y}{2x}
 \end{aligned}$$

Replace y' with its opposite reciprocal $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = -\frac{2x}{y}$$

The differential equation is separable and can be written as

$$y \, dy = -2x \, dx$$

The general solution is

$$\frac{1}{2}y^2 = -x^2 + K_1 \quad \text{or} \quad y^2 = -2x^2 + K$$

Note that $2x^2 + y^2 = K$ describes an ellipse when $K > 0$, describes a single point $(0, 0)$ when $K = 0$, and describes no points in the plane when $K < 0$. The orthogonal trajectories for the family of parabolas $y^2 = 4Cx$ is a family of ellipses $\boxed{2x^2 + y^2 = K}$, where $K > 0$.

50. We will find the orthogonal trajectories of all circles tangent to the x -axis at $(0, 0)$, then translate the family.

A line tangent to a circle is perpendicular to the radius at the point of contact. For a circle to be tangent to the x -axis, the radius that meets the x -axis must be vertical. For a circle to be tangent to the x -axis at the origin, the center is of the form $(0, b)$. The family of circles perpendicular to the x -axis at $(0, 0)$ is $x^2 + (y - C)^2 = C^2$, where $C \neq 0$, which can be rewritten as:

$$\begin{aligned} x^2 + (y - C)^2 &= C^2 \\ x^2 + y^2 - 2Cy + C^2 &= C^2 \\ x^2 + y^2 &= 2Cy \\ \frac{x^2 + y^2}{y} &= 2C \end{aligned}$$

Differentiate implicitly with respect to x to find y' .

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^2 + y^2}{y} \right) &= \frac{d}{dx} (2C) \\ \frac{y(2x + 2yy') - (x^2 + y^2)y'}{y^2} &= 0 \\ y(2x + 2yy') - (x^2 + y^2)y' &= 0 \\ 2xy + (y^2 - x^2)y' &= 0 \\ (y^2 - x^2)y' &= -2xy \\ y' &= -\frac{2xy}{y^2 - x^2} \end{aligned}$$

Replace y' with its opposite reciprocal $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = \frac{y^2 - x^2}{2xy}$$

Rewrite in differential form as $(x^2 - y^2) \, dy + 2xy \, dx = 0$. This is a homogenous differential equation. Let $y = xv$. Then $dy = x \, dv + v \, dx$. Substitute these into the differential equation:

$$\begin{aligned} (x^2 - y^2) \, dy + 2xy \, dx &= 0 \\ (x^2 - x^2v^2) \, dx + 2x^2v \, (x \, dv + v \, dx) &= 0 \\ (x^2 + x^2v^2) \, dx + 2x^3v \, dv &= 0 \\ x^2(1 + v^2) \, dx + 2x^3v \, dv &= 0 \end{aligned}$$

For $x \neq 0$, separate the variables by dividing by x^3 and $1 + v^2$, then integrate.

$$\begin{aligned}\frac{1}{x} dx + \frac{2v}{1+v^2} dv &= 0 \\ \int \frac{1}{x} dx + \int \frac{2v}{1+v^2} dv &= 0 \\ \ln|x| + \ln(1+v^2) &= K_1 \\ \ln|x| + \ln\left(1 + \frac{y^2}{x^2}\right) &= K_1 \\ \ln\left|\frac{x^2+y^2}{x}\right| &= K_1 \\ \left|\frac{x^2+y^2}{x}\right| &= e^{K_1} \\ \frac{x^2+y^2}{x} &= \pm e^{K_1}\end{aligned}$$

Since K_1 is a constant, we can write $K = \pm e^{K_1}$, where $K \neq 0$ is a constant. Then the family of orthogonal trajectories to the family of circles tangent to the x -axis at the origin is $x^2 + y^2 = Kx$, where K is a constant.

Translate this family to the right 3 units to find that the orthogonal trajectories of all circles tangent to the x -axis at $(3, 0)$ is the family of circles tangent to the line $x = 3$ at $(3, 0)$, $\boxed{(x-3)^2 + y^2 = K(x-3)}$, where K is a constant.

51. Consider the differential equation

$$\frac{du}{dt} = k(T - u)$$

where k is a positive constant.

- (a) The differential equation is separable, so separate the variables and integrate.

$$\begin{aligned}\frac{du}{dt} &= k(T - u) \\ \frac{1}{T - u} du &= k dt \\ \int \frac{1}{T - u} du &= \int k dt \\ -\ln|T - u| &= kt + C_1 \\ \ln|T - u| &= -kt - C_1 \\ |T - u| &= e^{-kt - C_1} \\ |T - u| &= e^{-kt} e^{-C_1} \\ T - u &= \pm e^{-kt} e^{-C_1}\end{aligned}$$

Since C_1 is a constant, we may write $C = \pm e^{-C_1}$, where $C \neq 0$ is a constant. It can be shown that $C = 0$ also provides a solution, so C is any constant. Then $T - u = Ce^{-kt}$ leads to the general solution of $\boxed{u = T - Ce^{-kt}}$, where C is a constant.

- (b) When $t = 0$, the initial temperature is $u = u_0$. Substitute these into the general solution to find C .

$$\begin{aligned}u &= T - Ce^{-kt} \\ u_0 &= T - Ce^0 \\ u_0 &= T - C \\ C &= T - u_0\end{aligned}$$

The particular solution is $\boxed{u = T - (T - u_0)e^{-kt}}$.

- (c) The initial temperature of the bar is $u_0 = 25$ and the medium temperature is the temperature of the boiling water $T = 100$. The equation that describes the temperature of the bar in degrees Celsius at time t in seconds is

$$u = 100 - 75e^{-kt}$$

When $t = 5$, the temperature is $u = 35$. Substitute these into the equation to find k .

$$\begin{aligned} u &= 100 - 75e^{-kt} \\ 35 &= 100 - 75e^{-5k} \\ -65 &= -75e^{-5k} \\ \frac{13}{15} &= e^{-5k} \\ \ln \frac{13}{15} &= -5k \\ -0.2 \ln \frac{13}{15} &= k \end{aligned}$$

Now the equation that describes the temperature of the bar at time t in seconds is

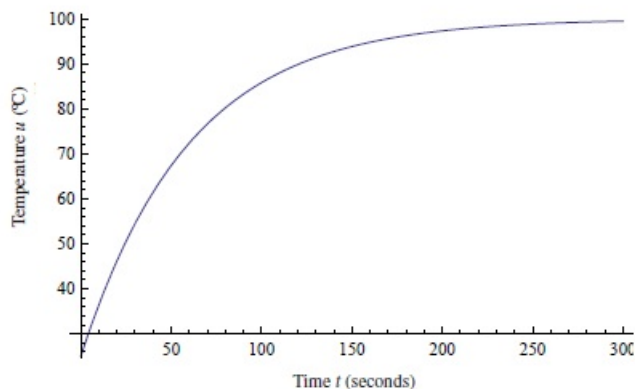
$$u = 100 - 75e^{0.2t \ln \frac{13}{15}}$$

After one minute, the time is $t = 60$ seconds, which is substituted into the equation to get

$$u = 100 - 75e^{(0.2)(60) \ln \frac{13}{15}} \approx 86.533$$

The temperature of the bar after 1 minute is approximately $\boxed{86.533^\circ\text{C}}$.

- (d) The graph of $u = 100 - 75e^{0.2t \ln \frac{13}{15}}$ is shown.



- (e) It will take approximately $\boxed{175.1 \text{ seconds}}$ for the temperature of the bar to be within 0.5°C of the temperature of the boiling water.

52. The differential equation from Problem 52 applies to this problem,

$$\frac{du}{dt} = k(T - u)$$

and the same general solution applies as well

$$u = T - (T - u_0)e^{-kt}$$

The initial temperature is $u_0 = 1230$ and the surrounding temperature is $T = 30$. Substitute these into the general solution to get

$$u = 30 + 1200e^{-kt}$$

where u is in degrees Celsius and t is in minutes. After 10 minutes, $t = 10$ and the temperature is $u = 1030$. Substitute these into the equation to find k .

$$\begin{aligned} u &= 30 + 1200e^{-kt} \\ 1030 &= 30 + 1200e^{-10k} \\ 1000 &= 1200e^{-10k} \\ \frac{5}{6} &= e^{-10k} \\ \ln \frac{5}{6} &= -10k \\ -0.1 \ln \frac{5}{6} &= k \end{aligned}$$

The function is now

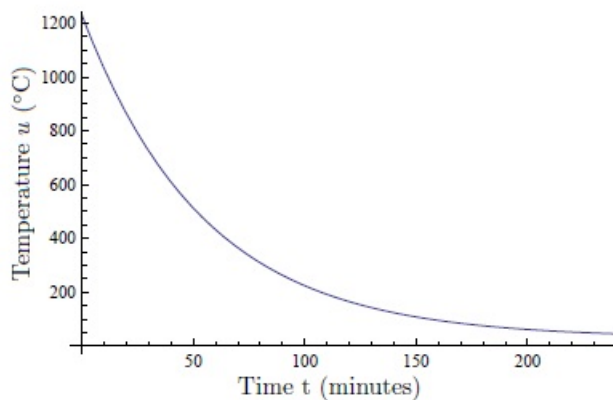
$$u = 30 + 1200e^{0.1t \ln \frac{5}{6}}$$

- (a) Find the time for the bar to cool to 80°C by substituting $u = 80$ into the equation.

$$\begin{aligned} u &= 30 + 1200e^{0.1t \ln \frac{5}{6}} \\ 80 &= 30 + 1200e^{0.1t \ln \frac{5}{6}} \\ 50 &= 1200e^{0.1t \ln \frac{5}{6}} \\ \frac{1}{24} &= e^{0.1t \ln \frac{5}{6}} \\ \ln \frac{1}{24} &= 0.1t \ln \frac{5}{6} \\ \frac{10 \ln \frac{1}{24}}{\ln \frac{5}{6}} &= t \\ 174.310 &\approx t \end{aligned}$$

The bar will cool to 80°C after 174.310 minutes.

- (b) The graph of the temperature of the steel bar over the interval $0 \leq t \leq 240$ is shown.



- (c) The steel bar cools to 50°C after approximately 225 minutes.
53. (a) Since there are no sales initially, use equation (4). Then since the upper limit of sales is \$20 million, the sales at time t are

$$y(t) = 20(1 - e^{-kt})$$

Since sales are \$3 million after 2 years, the boundary condition is $y(2) = 3$. Use the boundary

condition to find the constant k :

$$\begin{aligned} 3 &= 20(1 - e^{-2k}) \\ 0.15 &= 1 - e^{-2k} \\ e^{-2k} &= 0.85 \\ -2k &= \ln 0.85 \\ k &= -0.5 \ln 0.85 \end{aligned}$$

So the function

$$y(t) = 20 \left(1 - e^{(0.5 \ln 0.85)t} \right)$$

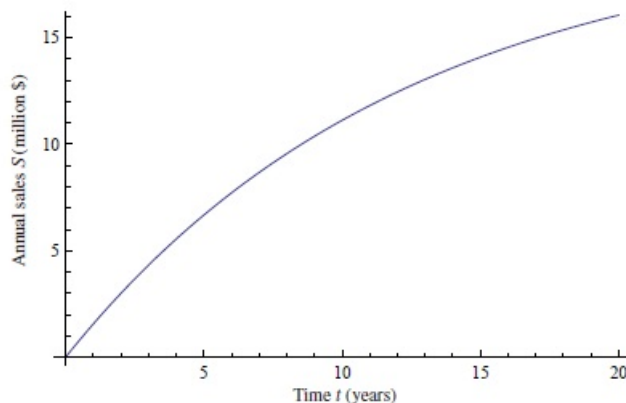
models the company's sales in year t .

- (b) According to the model, sales during the eighth year of operations will be $y(8) = 20 \left(1 - e^{(0.5)(8) \ln 0.85} \right) \approx$
9.560 million dollars.
- (c) To find out how long it will take for sales to reach \$12 million, solve the equation $y(t) = 12$ for t .
 Then

$$\begin{aligned} 12 &= 20(1 - e^{0.5t \ln 0.85}) \\ 0.6 &= 1 - e^{0.5t \ln 0.85} \\ e^{0.5t \ln 0.85} &= 0.4 \\ 0.5t \ln 0.85 &= \ln 0.4 \\ t &= \frac{2 \ln 0.4}{\ln 0.85} \approx 11.28 \text{ years} \end{aligned}$$

It will take approximately 11.28 years for annual sales to reach \$12 million.

- (d) The graph of the annual sales of the company for its first 20 years of operation is shown.



54. The bacteria experience inhibited growth, so the function

$$y(t) = M + (R - M)e^{-kt}$$

models the population at time t in hours. The maximum population is $M = 1,000,000$ and the initial population is $R = 1000$. Replacing M and R in the function gives

$$y(t) = 1,000,000 - 999,000e^{-kt}$$

Since the population is 1500 cells after 1 hour, the boundary condition $y(1) = 1500$, which we use to find the constant k :

$$\begin{aligned} 1500 &= 1,000,000 - 999,000e^{-k} \\ 999,000e^{-k} &= 998,500 \\ e^{-k} &= \frac{998,500}{999,000} \\ -k &= \ln \frac{998,500}{999,000} \\ k &= -\ln \frac{998,500}{999,000} \end{aligned}$$

Now the function is

$$y(t) = 1,000,000 - 999,000e^{t \ln \frac{998,500}{999,000}}$$

(a) To find the number of bacteria after 5 hours, find $y(5) = 1,000,000 - 999,000e^{5 \ln \frac{998,500}{999,000}} \approx 3497$. There will be about 3497 bacteria after 5 hours.

(b) To find when the culture contains 9950 bacteria, solve the equation $y(t) = 9950$ for t .

$$\begin{aligned} 9950 &= 1,000,000 - 999,000e^{t \ln \frac{998,500}{999,000}} \\ 999,000e^{t \ln \frac{998,500}{999,000}} &= 990,050 \\ e^{t \ln \frac{998,500}{999,000}} &= \approx 0.991041041 \\ t \ln \frac{998,500}{999,000} &\approx \ln 0.991041041 \approx -0.0089993317 \\ t &\approx 17.976 \end{aligned}$$

The culture will contain 9950 bacteria in approximately 17.976 hours.

55. The growth of the number of words per minute W is inhibited. Since a beginner cannot text at all ($W = 0$), use

$$W(t) = M(1 - e^{-kt})$$

The upper limit on W is 150 words per minute, so $M = 150$. Now the function becomes

$$W(t) = 150(1 - e^{-kt})$$

The condition that the beginner can text 30 words per minute after 10 hours of practice is $W(10) = 30$. Use this condition to find the constant k :

$$\begin{aligned} 30 &= 150(1 - e^{-10k}) \\ 0.2 &= 1 - e^{-10k} \\ e^{-10k} &= 0.8 \\ -10k &= \ln 0.8 \\ k &= -0.1 \ln 0.8 \end{aligned}$$

The function

$$W(t) = 150(1 - e^{0.1t \ln 0.8})$$

models the number of words per minute that the beginner can text after t hours of practice.

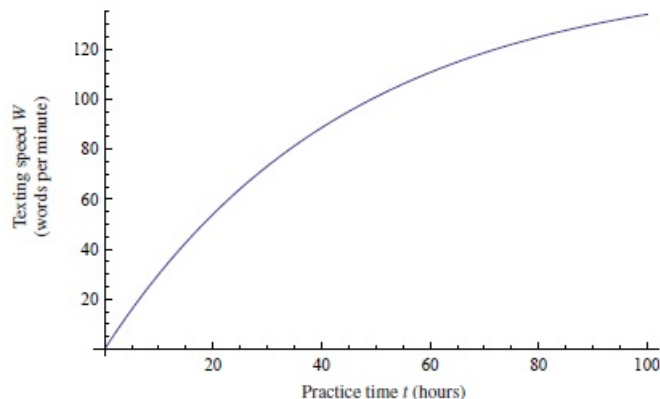
- (a) To find the number of words per minute he can text after 25 hours of practice, let $t = 25$, then $W(25) = 150(1 - e^{(0.1)(25) \ln 0.8}) \approx 64.135$. After 25 hours of practice, the beginner can text approximately 64.1 words per minute.

- (b) To find the number of hours of practice required, solve the equation $W(t) = 100$ for t .

$$\begin{aligned}
 100 &= 150(1 - e^{0.1t \ln 0.8}) \\
 \frac{2}{3} &= 1 - e^{0.1t \ln 0.8} \\
 e^{0.1t \ln 0.8} &= \frac{1}{3} \\
 0.1t \ln 0.8 &= \ln \frac{1}{3} \\
 t &= \frac{10 \ln \frac{1}{3}}{\ln 0.8} \approx 49.233
 \end{aligned}$$

It will take approximately 49.2 hours for a beginner to text 100 words per minute.

- (c) The graph of the number of words per minute W a beginner can text as a function of hours of practice is shown.



56. (a) Let $k > 0$ be the constant of proportionality referred to in the problem. Then the rate of change depends on the addition of the drug at a rate of r mg/s and the removal of the drug at a rate of ky mg/s. A differential equation that models the problem is $\frac{dy}{dt} = r - ky$.

- (b) The differential equation is separable and can be written and solved as

$$\begin{aligned}
 \frac{dy}{dt} &= r - ky \\
 \frac{dy}{r - ky} &= dt \\
 \int \frac{dy}{r - ky} &= \int dt \\
 -\frac{1}{k} \ln |r - ky| &= t + C_1 \\
 \ln |r - ky| &= -kt - kC_1 \\
 |r - ky| &= e^{-kt - kC_1} \\
 |r - ky| &= e^{-kt} e^{-kC_1} \\
 r - ky &= \pm e^{-kt} e^{-kC_1} \\
 -ky &= -r \pm e^{-kt} e^{-kC_1} \\
 y &= \frac{r}{k} \pm \frac{e^{-kt} e^{-kC_1}}{k}
 \end{aligned}$$

Since k and C_1 are constants, we let $C = \pm \frac{e^{-kC_1}}{k}$ and the general solution can be written as

$$\boxed{y = \frac{r}{k} + Ce^{-kt}}, \text{ where } k > 0 \text{ and } C \text{ are constants.}$$

- (c) Using the initial condition $y(0) = 0$, $0 = \frac{r}{k} + Ce^0$ so $C = -\frac{r}{k}$. The particular solution is

$$\boxed{y = \frac{r}{k} - \frac{r}{k}e^{-kt}}, \text{ where } k > 0 \text{ is a constant.}$$

57. The frictional force f of an object sliding over a surface is modeled by $f = -Av$, where A is a positive constant.

- (a) Since acceleration is the derivative of velocity, $a = \frac{dv}{dt}$, we can write Newton's Second Law $F = ma$ as

$$F = m \frac{dv}{dt}$$

When the force is due to friction, that is $F = f$, we have the differential equation

$$\boxed{m \frac{dv}{dt} = -Av}$$

where m is the mass of the object and A is a positive constant.

- (b) The differential equation $m \frac{dv}{dt} = -Av$ is separable. Separate the variables and integrate.

$$\begin{aligned} m \frac{dv}{dt} &= -Av \\ m \frac{1}{v} dv &= -A dt \\ m \int \frac{1}{v} dv &= -A \int dt \\ m \ln |v| &= -At + C_1 \\ \ln |v| &= -\frac{A}{m}t + \frac{C_1}{m} \\ |v| &= e^{(-\frac{A}{m}t + \frac{C_1}{m})} \\ |v| &= e^{-\frac{A}{m}t} e^{\frac{C_1}{m}} \\ v &= \pm e^{-\frac{A}{m}t} e^{\frac{C_1}{m}} \end{aligned}$$

Since C_1 and m are constants, we may write $C = \pm e^{\frac{C_1}{m}}$, where $C \neq 0$ is a constant. The speed function is now $v = Ce^{-\frac{A}{m}t}$. Given the initial velocity of v_0 , we have the boundary condition $v(0) = v_0$, and the speed of the object is

$$\boxed{v = v_0 e^{-\frac{A}{m}t}}$$

where A and m are constants.

- (c) The frictional force of a 25-kg sled with a speed of 10 m/s is 150N. The force is negative because the direction of the frictional force is opposite to the direction of the velocity, so $f = -150\text{N} = -150\text{kg} \frac{\text{m}}{\text{s}^2}$ and $v = 10 \frac{\text{m}}{\text{s}}$. Use $f = -Av$ to find A : $-150\text{kg} \frac{\text{m}}{\text{s}^2} = -A \left(10 \frac{\text{m}}{\text{s}}\right)$ so $\boxed{A = 15 \frac{\text{kg}}{\text{s}}}$.
- (d) To find the time required for the sled to slow down to 10% of its initial speed, we solve $v(t) = 0.1v_0$ for t , using $A = 15$ and $m = 25$:

$$\begin{aligned} 0.1v_0 &= v_0 e^{-\frac{15}{25}t} \\ 0.1 &= e^{-0.6t} \\ \ln 0.1 &= -0.6t \\ t &= -\frac{\ln 0.1}{0.6} \approx 3.838 \end{aligned}$$

The sled will down to 10% of its initial speed in approximately 3.838 seconds.

- (e) To slow down to 1.0 m/s, or 10% of the initial speed, takes 3.838 seconds. The distance traveled is the definite integral of the speed. Find the definite integral of $v = 10e^{-0.6t}$ over the interval $[0, 3.838]$.

$$\begin{aligned}\int_0^{3.838} v \, dt &= \int_0^{3.838} 10e^{-0.6t} \, dt \\ &= -\frac{10}{0.6} e^{-0.6t} \Big|_0^{3.838} \\ &\approx 15.0\end{aligned}$$

The sled will travel 15 meters while slowing down to 1.0 m/s.

58. The frictional force f of an object sliding over a surface is modeled by $f = -B\sqrt{v}$, where B is a positive constant.

- (a) Since acceleration is the derivative of velocity, $a = \frac{dv}{dt}$, we can write Newton's Second Law $F = ma$ as

$$F = m \frac{dv}{dt}$$

When the force is due to friction, that is $F = f$, we have the differential equation

$$\boxed{m \frac{dv}{dt} = -B\sqrt{v}}$$

where m is the mass of the object and B is a positive constant.

- (b) The differential equation $m \frac{dv}{dt} = -B\sqrt{v}$ is separable. Separate the variables and integrate.

$$\begin{aligned}m \frac{dv}{dt} &= -B\sqrt{v} \\ mv^{-1/2} dv &= -B \, dt \\ m \int v^{-1/2} dv &= -B \int dt \\ 2mv^{1/2} &= -Bt + C \\ v^{1/2} &= \frac{-Bt + C}{2m} \\ v &= \frac{(C - Bt)^2}{4m^2}\end{aligned}$$

The velocity function is $v = \frac{(C - Bt)^2}{4m^2}$. Solve for C using the initial velocity $v(0) = v_0 = \frac{C^2}{4m^2}$ so $C = 2m\sqrt{v_0}$. The speed of the object is

$$\boxed{v = \frac{(2m\sqrt{v_0} - Bt)^2}{4m^2}}$$

where B and m are constants and $0 \leq t \leq \frac{2m}{B}\sqrt{v_0}$.

- (c) The frictional force of a 25-kg sled with a speed of 10 m/s is 150N. The force is negative because the direction of the frictional force is opposite to the direction of the velocity, so $f = -150$ and $v = 10$. Use $f = -B\sqrt{v}$ to find B :

$$\begin{aligned}-150 &= -B\sqrt{10} \\ \frac{150}{\sqrt{10}} &= B \\ \frac{150\sqrt{10}}{10} &= B \\ 15\sqrt{10} &= B\end{aligned}$$

Therefore

$$\boxed{B = 15\sqrt{10}}$$

- (d) Using $B = 15\sqrt{10}$ and $m = 25$, the speed is $v = \frac{(50\sqrt{v_0} - 15t\sqrt{10})^2}{2500}$, where $0 \leq t \leq \frac{\sqrt{10v_0}}{3}$. To find the time required for the sled to slow down to 10% of its initial speed, we solve $v(t) = 0.1v_0$ for t :

$$\begin{aligned} 0.1v_0 &= \frac{(50\sqrt{v_0} - 15t\sqrt{10})^2}{2500} \\ 250v_0 &= (50\sqrt{v_0} - 15t\sqrt{10})^2 \\ \pm\sqrt{250}\sqrt{v_0} &= 50\sqrt{v_0} - 15t\sqrt{10} \\ 15t\sqrt{10} &= 50\sqrt{v_0} \pm \sqrt{250}\sqrt{v_0} \\ t &= \frac{50 \pm \sqrt{250}}{15\sqrt{10}}\sqrt{v_0} \end{aligned}$$

Only $t = \frac{50 - \sqrt{250}}{15\sqrt{10}}\sqrt{v_0} \leq \frac{\sqrt{10v_0}}{3}$, so the sled will down to 10% of its initial speed in approximately $\boxed{0.721\sqrt{v_0} \text{ seconds}}$.

- (e) To slow down to 1.0 m/s, or 10% of the initial speed, takes $0.721\sqrt{10} \approx 2.279$ seconds. The distance traveled is the definite integral of the speed. Find the definite integral of $v = \frac{(50\sqrt{10} - 15t)^2}{2500}$ over the interval $[0, 2.279]$.

$$\begin{aligned} \int_0^{2.279} v \, dt &= \int_0^{2.279} \frac{(50\sqrt{10} - 15t\sqrt{10})^2}{2500} \, dt \\ &= \frac{1}{3} \left(-\frac{1}{15\sqrt{10}} \right) \frac{(50\sqrt{10} - 15t)^3}{2500} \bigg|_0^{2.279} \\ &\approx 10.760 \end{aligned}$$

The sled will travel $\boxed{10.760 \text{ meters}}$ while slowing down to 1.0 m/s.

59. Consider a circuit with initial charge q_0 that satisfies the differential equation

$$\frac{dq}{dt} + \frac{q}{RC} = 0$$

- (a) The differential equation is separable and can be written in the differential form

$$\frac{1}{q} dq + \frac{1}{RC} dt = 0$$

Integrate and solve for q .

$$\begin{aligned} \int \frac{1}{q} dq + \frac{1}{RC} \int dt &= 0 \\ \ln |q| + \frac{1}{RC} t &= K_1 \\ \ln |q| &= K_1 - \frac{1}{RC} t \\ |q| &= e^{(K_1 - \frac{1}{RC} t)} \\ |q| &= e^{K_1} e^{-\frac{1}{RC} t} \\ q &= \pm e^{K_1} e^{-\frac{1}{RC} t} \end{aligned}$$

Since K_1 is a constant, we may write $K = \pm e^{K_1}$, where K is a constant. The charge is now $q = Ke^{-\frac{1}{RC}t}$. Using the initial charge is $q(0) = q_0$, we can solve for K : $q_0 = Ke^0$ so $K = q_0$. The charge of the capacitor is given by the function

$$q = q_0 e^{-\frac{1}{RC}t}$$

- (b) Using $R = 150$ and $C = 0.000005$ (a microfarad is one-millionth of a farad), the charge is given by the function

$$q = q_0 e^{-\frac{4000}{3}t}$$

For the capacitor to lose 99% of its charge means that the capacitor has only 1% of the initial charge left, so solve $q(t) = 0.01q_0$ for t .

$$\begin{aligned} 0.01q_0 &= q_0 e^{-\frac{4000}{3}t} \\ 0.01 &= e^{-\frac{4000}{3}t} \\ \ln 0.01 &= -\frac{4000}{3}t \\ t &= -\frac{3}{4000} \ln 0.01 \approx 0.00345 \end{aligned}$$

The discharging capacitor will lose 99% of its initial charge in approximately 0.00345 seconds.

60. Model the air resistance with

$$F_{\text{drag}} = -Av^2$$

where A is a positive constant.

- (a) Since acceleration is the derivative of velocity, $a = \frac{dv}{dt}$, we can write Newton's Second Law $F = ma$ as

$$F = m \frac{dv}{dt}$$

When the only force is drag, that is $F = F_{\text{drag}}$, we have the differential equation

$$m \frac{dv}{dt} = -Av^2$$

where A is a positive constant.

- (b) The differential equation $m \frac{dv}{dt} = -Av^2$ is separable. Separate the variables and integrate.

$$\begin{aligned} m \frac{dv}{dt} &= -Av^2 \\ mv^{-2} dv &= -A dt \\ m \int v^{-2} dv &= -A \int dt \\ -mv^{-1} &= -At + C \\ v^{-1} &= \frac{At - C}{m} \\ v &= \frac{m}{At - C} \end{aligned}$$

The velocity of the object is modeled by the function

$$v = \frac{m}{At - C}$$

where A is a positive constant and C is a constant.

(c) The limit of $v(t)$ as t approaches infinity is

$$\lim_{t \rightarrow \infty} \frac{m}{At - C} = 0$$

If the model for air resistance at high speeds is used for arbitrarily large values of t , then the speed of the projectile approaches zero.

Challenge Problems

61. Given the family of hyperbolas

$$\frac{x^2}{C^2} - \frac{y^2}{4 - C^2} = 1$$

where $0 < C < 2$, first clear the denominators and then find y' using implicit differentiation.

$$\begin{aligned} \frac{x^2}{C^2} - \frac{y^2}{4 - C^2} &= 1 \\ \frac{x^2}{C^2} \cdot C^2(4 - C^2) - \frac{y^2}{4 - C^2} \cdot C^2(4 - C^2) &= 1 \cdot C^2(4 - C^2) \\ (4 - C^2)x^2 - C^2y^2 &= C^2(4 - C^2) \\ (4 - C^2)(2x) - C^2(2yy') &= 0 \\ (4 - C^2)x - C^2yy' &= 0 \end{aligned}$$

It is necessary to find the derivative y' in terms of x and y alone, without using the constant C . To do this, first solve for $4 - C^2$ above.

$$\begin{aligned} (4 - C^2)x - C^2yy' &= 0 \\ (4 - C^2)x &= C^2yy' \\ 4 - C^2 &= \frac{C^2yy'}{x} \end{aligned}$$

Now replace $4 - C^2$ in the original equation, and solve for C^2 .

$$\begin{aligned} \frac{x^2}{C^2} - \frac{y^2}{4 - C^2} &= 1 \\ \frac{x^2}{C^2} - \frac{y^2}{\frac{C^2yy'}{x}} &= 1 \\ \frac{x^2}{C^2} - \frac{xy^2}{C^2yy'} &= 1 \\ \frac{x^2}{C^2} - \frac{xy}{C^2y'} &= 1 \\ x^2 - \frac{xy}{y'} &= C^2 \end{aligned}$$

Now replace C^2 in the differential equation found before.

$$\begin{aligned} (4 - C^2)x - C^2yy' &= 0 \\ \left(4 - \left(x^2 - \frac{xy}{y'}\right)\right)x - \left(x^2 - \frac{xy}{y'}\right)yy' &= 0 \\ 4x - x^3 + \frac{x^2y}{y'} - x^2yy' + xy^2 &= 0 \\ 4 - x^2 + \frac{xy}{y'} - xy y' + y^2 &= 0 \end{aligned}$$

We find the differential equation for the family of orthogonal curves by replacing y' in the differential equation with $-\frac{1}{y'}$.

$$\begin{aligned} 4 - x^2 + \frac{xy}{y'} - xy y' + y^2 &= 0 \\ 4 - x^2 + \frac{xy}{-\frac{1}{y'}} - xy \left(-\frac{1}{y'}\right) + y^2 &= 0 \\ 4 - x^2 - xy y' + \frac{xy}{y'} + y^2 &= 0 \end{aligned}$$

The differential equation for the family of orthogonal trajectories is the same as the differential equation for the original family, so the solutions are of the form

$$\frac{x^2}{C^2} - \frac{y^2}{4 - C^2} = 1$$

as well. The orthogonal trajectories consists of the family of ellipses

$$\boxed{\frac{x^2}{C^2} - \frac{y^2}{4 - C^2} = 1}$$

where $C > 2$. The hyperbolas and ellipses all have $(\pm 2, 0)$ as foci.

62. The family of parabolas with a common vertex at $(1, 2)$ and having a vertical axis is

$$y - 2 = C(x - 1)^2$$

Rewrite $y - 2 = C(x - 1)^2$ in the form $\frac{y - 2}{(x - 1)^2} = C$. Find y' by differentiating $\frac{y - 2}{(x - 1)^2} = C$ with respect to x .

$$\begin{aligned} \frac{(x - 1)^2 y' - (y - 2) \cdot 2(x - 1)}{(x - 1)^4} &= 0 \\ (x - 1)^2 y' - 2(x - 1)(y - 2) &= 0 \\ (x - 1)^2 y' &= 2(x - 1)(y - 2) \\ y' &= \frac{2(y - 2)}{(x - 1)} \end{aligned}$$

Replace y' with $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = -\frac{(x - 1)}{2(y - 2)}$$

The differential equation $\frac{dy}{dx} = -\frac{x - 1}{2(y - 2)}$ is separable and can be written as

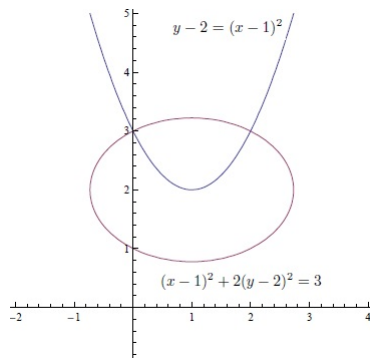
$$2(y - 2) dy = -(x - 1) dx$$

Integrate to find the general solution.

$$\begin{aligned} \int 2(y - 2) dy &= -\int (x - 1) dx \\ (y - 2)^2 &= -\frac{1}{2}(x - 1)^2 + K \end{aligned}$$

The orthogonal trajectories for the family of curves $y = Cx^2$ is the family $\boxed{\frac{1}{2}(x-1)^2 + (y-2)^2 = K}$.

To find the particular curves needed for the graph, substitute $x = 2$ and $y = 3$ into $y - 2 = C(x - 1)^2$ and $\frac{1}{2}(x - 1)^2 + (y - 2)^2 = K$ separately and solve for C and K . The graph of $y - 2 = (x - 1)^2$ and $(x - 1)^2 + 2(y - 2)^2 = 3$ is shown.



63. (a) Recall Newton's Second Law $F = ma$ and set it equal to the force due to gravity $F = \frac{k}{x^2}$:

$$ma = \frac{k}{x^2}$$

Let g be the magnitude of acceleration due to gravity at sea level, that is when $x = R$. Solve for k by substituting $a = -g$ and $x = R$ into the equation: $-mg = \frac{k}{R^2}$ so $k = -mgR^2$. The equation becomes $ma = -\frac{mgR^2}{x^2}$ or

$$a = -\frac{R^2g}{x^2}$$

Also consider that $a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$. The equation now becomes the differential equation

$$v \frac{dv}{dx} = -\frac{R^2g}{x^2}$$

The differential equation is separable, so separate the variables and integrate.

$$\begin{aligned} v dv &= -R^2g x^{-2} dx \\ \int v dv &= -R^2g \int x^{-2} dx \\ \frac{1}{2}v^2 &= \frac{R^2g}{x} + C \end{aligned}$$

where C is a constant. The initial velocity is v_0 when $x = R$. Solve for C by making the substitutions $v = v_0$ and $x = R$:

$$\begin{aligned} \frac{1}{2}v^2 &= \frac{R^2g}{x} + C \\ \frac{1}{2}v_0^2 &= \frac{R^2g}{R} + C \\ \frac{1}{2}v_0^2 &= Rg + C \\ \frac{1}{2}v_0^2 - Rg &= C \end{aligned}$$

Replacing C gives $\frac{1}{2}v^2 = \frac{1}{2}v_0^2 - Rg + \frac{R^2g}{x}$. The velocity of the projectile is given by

$$v^2 = v_0^2 - 2Rg + \frac{2R^2g}{x}$$

- (b) To keep the projectile going indefinitely, $f(x) = v_0^2 - 2Rg + \frac{2R^2g}{x} > 0$ for all $x > R$. Notice that f is a decreasing function with respect to x because

$$f'(x) = -\frac{2R^2g}{x^2} < 0$$

for $x > R$. To keep the projectile going indefinitely, it is enough then for the limit as x approaches infinity to be greater than or equal to zero. If

$$\lim_{x \rightarrow \infty} \left(v_0^2 - 2Rg + \frac{2R^2g}{x} \right) = v_0^2 - 2Rg \geq 0$$

then $v_0 \geq \sqrt{2Rg}$. The smallest value of v_0 is the escape velocity $\boxed{\sqrt{2Rg}}$.

64. The volume of water is a function of time, $V(t) = \pi r^2 h$. The radius of the cylinder remains constant $r = \frac{3}{2}$, but the height of the water is also a function of time, $h = h(t)$, so $V(t) = \frac{9\pi}{4}h(t)$. Differentiate to find

$$\frac{dV}{dt} = \frac{9\pi}{4} \frac{dh}{dt}$$

The circular hole has area $A = \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$, and we are given $k = 0.6$ and $g = 32$. The equation $\frac{dV}{dt} = -kA\sqrt{2gh}$ becomes

$$\frac{dV}{dt} = -(0.6)\frac{\pi}{4}\sqrt{2(32)}h^{1/2} = -\frac{6\pi}{5}h^{1/2}$$

- (a) Set these equations equal to each other to get

$$\frac{9\pi}{4} \frac{dh}{dt} = -\frac{6\pi}{5}h^{1/2}$$

This differential equation is separable, so separate the variables and integrate.

$$\begin{aligned} \frac{9\pi}{4} \frac{dh}{dt} &= -\frac{6\pi}{5}h^{1/2} \\ h^{-1/2} dh &= -\frac{8}{15} dt \\ \int h^{-1/2} dh &= -\frac{8}{15} \int dt \\ 2h^{1/2} &= -\frac{8}{15}t + C_1 \\ h^{1/2} &= -\frac{4}{15}t + \frac{C_1}{2} \end{aligned}$$

Since C_1 is a constant, we may write $C = \frac{C_1}{2}$ where C is a constant. The equation is

$$h^{1/2} = -\frac{4}{15}t + C$$

The initial height of the water is $h(0) = 6$. Use the boundary condition to find C : $6^{1/2} = \sqrt{6} = C$. Then the height of the water is

$$\boxed{h = \left(-\frac{4}{15}t + \sqrt{6}\right)^2}$$

at time t in seconds.

(b) To find the time required to empty the tank, solve the equation $h(t) = 0$ for t :

$$\begin{aligned} 0 &= \left(-\frac{4}{15}t + \sqrt{6}\right)^2 \\ 0 &= -\frac{4}{15}t + \sqrt{6} \\ \frac{4}{15}t &= \sqrt{6} \\ t &= \frac{15\sqrt{6}}{4} \approx 9.186 \end{aligned}$$

The tank will drain in approximately 9.186 seconds.

65. The volume of water is $V(t) = \pi r^2 h(t)$, where $V(t)$ is a function of t , r is the constant radius of the cylinder, and $h(t)$ is the height of the water which is also a function of t . The rate of flow is the derivative of the volume and is proportional to the square root of the height of the water:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = -k\sqrt{h}$$

where k is a positive constant and r is the constant radius. This differential equation is separable, so separate the variables and integrate.

$$\begin{aligned} \pi r^2 \frac{dh}{dt} &= -k h^{1/2} \\ \pi r^2 h^{-1/2} dh &= -k dt \\ \pi r^2 \int h^{-1/2} dh &= -k \int dt \\ 2\pi r^2 h^{1/2} &= -kt + C \end{aligned}$$

Denote the initial height h_0 . Use the initial height to find C : $2\pi r^2 \sqrt{h_0} = C$. So we can write

$$2\pi r^2 h^{1/2} = -kt + 2\pi r^2 \sqrt{h_0}$$

To find the time required to leak three-fourths of the water (when the volume is one-fourth of the original volume), solve the equation when $h = \frac{1}{4}h_0$ for t :

$$\begin{aligned} 2\pi r^2 \sqrt{\frac{1}{4}h_0} &= -kt + 2\pi r^2 \sqrt{h_0} \\ \pi r^2 \sqrt{h_0} &= -kt + 2\pi r^2 \sqrt{h_0} \\ kt &= \pi r^2 \sqrt{h_0} \\ t &= \frac{\pi r^2 \sqrt{h_0}}{k} \end{aligned}$$

The time required to leak three-fourths of the water is $\frac{\pi r^2 \sqrt{h_0}}{k}$. To find the time required to empty the tank completely, solve the equation when $h = 0$ for t :

$$\begin{aligned} 0 &= -kt + 2\pi r^2 \sqrt{h_0} \\ kt &= 2\pi r^2 \sqrt{h_0} \\ t &= \frac{2\pi r^2 \sqrt{h_0}}{k} \end{aligned}$$

The time required to empty the entire tank is $\frac{2\pi r^2 \sqrt{h_0}}{k}$, twice as long as the time required to leak three-fourths of the water. Therefore, the time required to leak out three-fourths of the water is equal to the time required to leak out the remaining one-fourth of the water.

66. Let r denote the radius of the tank, and let $y(t)$ denote the depth and $V(t)$ the volume of the water at time t in hours. Pressure is proportional to the water depth, so the rate of change of volume is proportional to the water depth:

$$\frac{dV}{dt} = k_1 y$$

Differentiating $V = \pi r^2 y$ with respect to t yields

$$\begin{aligned}\frac{dV}{dt} &= \pi r^2 \frac{dy}{dt} \\ k_1 y &= \pi r^2 \frac{dy}{dt} \\ \frac{k_1}{\pi r^2} y &= \frac{dy}{dt} \\ ky &= \frac{dy}{dt}\end{aligned}$$

where $k = \frac{k_1}{\pi r^2}$. This is a separable equation with a general solution of $y = y_0 e^{kt}$. The tank loses 2% of its water in 24 h, so the depth of the water is 98% of the original in 24 h, $y(24) = 0.98y_0$.

$$\begin{aligned}0.98y_0 &= y_0 e^{24k} \\ 0.98 &= e^{24k} \\ \ln(0.98) &= 24k \\ \frac{\ln(0.98)}{24} &= k\end{aligned}$$

Now solve for the t such that $y(t) = 0.5y_0$,

$$\begin{aligned}0.5y_0 &= y_0 e^{t \ln(0.98)/24} \\ 0.5 &= e^{t \ln(0.98)/24} \\ \ln(0.5) &= \frac{t \ln(0.98)}{24} \\ \frac{24 \ln(0.5)}{\ln(0.98)} &= t \\ 823.431 &\approx t\end{aligned}$$

The tank will be half empty in $\boxed{\frac{24 \ln(0.5)}{\ln(0.98)} \approx 823.431}$ hours.

16.3 Exact Differential Equations

Concepts and Vocabulary

1. True. Let $M = xy$ and $N = \frac{x^2}{2}$. Then

$$\frac{\partial M}{\partial y} = x \quad \text{and} \quad \frac{\partial N}{\partial x} = x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , then differential equation $xy \, dx + \frac{x^2}{2} \, dy = 0$ is exact.

2. False. Let $M = y \cos x$ and $N = -\sin x$. Then

$$\frac{\partial M}{\partial y} = \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = -\cos x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ for all x and y , then differential equation $y \cos x \, dx - \sin x \, dy = 0$ is not exact.

3. False. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $M(x, y) dx + N(x, y) dy = 0$ is an exact differential equation.
4. If the differential equation $M(x, y) dx + N(x, y) dy = 0$ is not exact, but $a(x, y)M(x, y) dx + a(x, y)N(x, y) dy = 0$ is an exact differential equation, then the expression $a(x, y)$ is called an integrating factor.

Skill Building

5. Consider the differential equation $(4x - 2y + 5) dx + (2y - 2x) dy = 0$.

(a) Let $M = 4x - 2y + 5$ and $N = 2y - 2x$. Then

$$\frac{\partial M}{\partial y} = -2 \quad \text{and} \quad \frac{\partial N}{\partial x} = -2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = 4x - 2y + 5$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (4x - 2y + 5) dx = 2x^2 - 2xy + 5x + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [2x^2 - 2xy + 5x + B(y)] = -2x + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ -2x + \frac{\partial B}{\partial y} &= 2y - 2x \\ \frac{\partial B}{\partial y} &= 2y \end{aligned}$$

Now we integrate $\frac{dB}{dy} = 2y$ with respect to y .

$$B(y) = \int 2y dy = y^2 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = 2x^2 - 2xy + 5x + y^2 + C$$

The general solution of the differential equation is

$$\boxed{2x^2 - 2xy + 5x + y^2 + C = 0}$$

where C is a constant.

6. Consider the differential equation $(3x^2 + 3xy^2) dx + (3x^2y - 3y^2 + 2y) dy = 0$.

(a) Let $M = 3x^2 + 3xy^2$ and $N = 3x^2y - 3y^2 + 2y$. Then

$$\frac{\partial M}{\partial y} = 6xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = 3x^2 + 3xy^2$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (3x^2 + 3xy^2) dx = x^3 + \frac{3}{2}x^2y^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[x^3 + \frac{3}{2}x^2y^2 + B(y) \right] = 3x^2y + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 3x^2y + \frac{\partial B}{\partial y} &= 3x^2y - 3y^2 + 2y \\ \frac{\partial B}{\partial y} &= -3y^2 + 2y \end{aligned}$$

Now we integrate $\frac{dB}{dy} = -3y^2 + 2y$ with respect to y .

$$B(y) = \int (-3y^2 + 2y) dy = -y^3 + y^2 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x^3 + \frac{3}{2}x^2y^2 - y^3 + y^2 + C$$

The general solution of the differential equation is

$$\boxed{x^3 + \frac{3}{2}x^2y^2 - y^3 + y^2 + C = 0}$$

where C is a constant.

7. Consider the differential equation $(a^2 - 2xy - y^2) dx - (x + y)^2 dy = 0$, where a is a constant.

- (a) Let $M = a^2 - 2xy - y^2$ and $N = -(x + y)^2$. Then

$$\frac{\partial M}{\partial y} = -2x - 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2x - 2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = a^2 - 2xy - y^2$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (a^2 - 2xy - y^2) dx = a^2x - x^2y - xy^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [a^2x - x^2y - xy^2 + B(y)] = -x^2 - 2xy + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ -x^2 - 2xy + \frac{\partial B}{\partial y} &= -(x+y)^2 \\ -x^2 - 2xy + \frac{\partial B}{\partial y} &= -x^2 - 2xy - y^2 \\ \frac{\partial B}{\partial y} &= -y^2\end{aligned}$$

Now we integrate $\frac{dB}{dy} = -y^2$ with respect to y .

$$B(y) = -\int y^2 dy = -\frac{1}{3}y^3 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = a^2x - x^2y - xy^2 - \frac{1}{3}y^3 + C$$

The general solution of the differential equation is

$$a^2x - x^2y - xy^2 - \frac{1}{3}y^3 + C = 0$$

where C is a constant.

8. Consider the differential equation $(2ax + by + g)dx + (2e^y + bx + h)dy = 0$, where a, b, g, h are constants.

(a) Let $M = 2ax + by + g$ and $N = 2e^y + bx + h$. Then

$$\frac{\partial M}{\partial y} = b \quad \text{and} \quad \frac{\partial N}{\partial x} = b$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = 2ax + by + g$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (2ax + by + g) dx = ax^2 + bxy + gx + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [ax^2 + bxy + gx + B(y)] = bx + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ bx + \frac{\partial B}{\partial y} &= 2e^y + bx + h \\ \frac{\partial B}{\partial y} &= 2e^y + h\end{aligned}$$

Now we integrate $\frac{dB}{dy} = 2e^y + h$ with respect to y .

$$B(y) = \int (2e^y + h) dy = 2e^y + hy + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = ax^2 + bxy + gx + 2e^y + hy + C$$

The general solution of the differential equation is

$$\boxed{ax^2 + bxy + gx + 2e^y + hy + C = 0}$$

where C is a constant.

9. Consider the differential equation $\frac{1}{y} dx - \frac{x}{y^2} dy = 0$.

(a) Let $M = \frac{1}{y}$ and $N = -\frac{x}{y^2}$. Then

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and $y \neq 0$, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = \frac{1}{y}$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \frac{1}{y} dx = \frac{x}{y} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\frac{x}{y} + B(y) \right] = -\frac{x}{y^2} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ -\frac{x}{y^2} + \frac{\partial B}{\partial y} &= -\frac{x}{y^2} \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = \frac{x}{y} + C$$

The general solution of the differential equation is

$$\boxed{\frac{x}{y} + C = 0}$$

where C is a constant.

10. Consider the differential equation $\frac{y}{x^2} dx - \frac{1}{x} dy = 0$.

(a) Let $M = \frac{y}{x^2}$ and $N = -\frac{1}{x}$. Then

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $x \neq 0$ and y , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = \frac{y}{x^2}$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \frac{y}{x^2} dx = -\frac{y}{x} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[-\frac{y}{x} + B(y) \right] = -\frac{1}{x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ -\frac{1}{x} + \frac{\partial B}{\partial y} &= -\frac{1}{x} \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = -\frac{y}{x} + C$$

The general solution of the differential equation is

$$\boxed{-\frac{y}{x} + C = 0}$$

where C is a constant.

11. Consider the differential equation $(x-1)^{-1}y dx + [\ln(2x-2) + y^{-1}] dy = 0$.

(a) Let $M = (x-1)^{-1}y$ and $N = \ln(2x-2) + y^{-1}$. Then

$$\frac{\partial M}{\partial y} = (x-1)^{-1} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{2}{2x-2} = \frac{1}{x-1}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $x > 1$ and $y \neq 0$, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = (x-1)^{-1}y$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (x-1)^{-1}y dx = y \int \frac{1}{x-1} dx = y \ln(x-1) + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [y \ln(x-1) + B(y)] = \ln(x-1) + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ \ln(x-1) + \frac{\partial B}{\partial y} &= \ln(2x-2) + y^{-1} \\ \ln(x-1) + \frac{\partial B}{\partial y} &= \ln(2(x-1)) + y^{-1} \\ \ln(x-1) + \frac{\partial B}{\partial y} &= \ln(2) + \ln(x-1) + y^{-1} \\ \frac{\partial B}{\partial y} &= \ln 2 + y^{-1} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = \ln(2) + y^{-1}$ with respect to y .

$$B(y) = \int (\ln(2) + y^{-1}) dy = y \ln(2) + \ln |y| + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = y \ln(x-1) + y \ln(2) + \ln |y| + C$$

The general solution of the differential equation is

$$\boxed{y \ln(x-1) + y \ln(2) + \ln |y| + C = 0}$$

where C is a constant.

12. Consider the differential equation $(2 \ln(5y) + x^{-1}) dx + 2xy^{-1} dy = 0$.

(a) Let $M = 2 \ln(5y) + x^{-1}$ and $N = 2xy^{-1}$. Then

$$\frac{\partial M}{\partial y} = 2 \frac{5}{5y} = \frac{2}{y} \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y^{-1}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $x \neq 0$ and $y > 0$, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = 2 \ln(5y) + x^{-1}$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (2 \ln(5y) + x^{-1}) dx = 2x \ln(5y) + \ln |x| + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [2x \ln(5y) + \ln |x| + B(y)] = \frac{2x}{y} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ \frac{2x}{y} + \frac{\partial B}{\partial y} &= 2xy^{-1} \\ \frac{\partial B}{\partial y} &= 0\end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = 2x \ln(5y) + \ln|x| + C$$

The general solution of the differential equation is

$$\boxed{2x \ln(5y) + \ln|x| + C = 0}$$

where C is a constant.

13. Consider the differential equation $(x+3)^{-1} \cos y \, dx - [\sin y \ln(5x+15) - y - 1] \, dy = 0$.

(a) Let $M = (x+3)^{-1} \cos y$ and $N = -[\sin y \ln(5x+15) - y - 1]$. Then

$$\frac{\partial M}{\partial y} = -\frac{\sin y}{x+3} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin y \frac{5}{5x+15} = -\sin y \frac{1}{x+3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $x > -3$ and y , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = (x+3)^{-1} \cos y$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (x+3)^{-1} \cos y \, dx = \cos y \int \frac{1}{x+3} \, dx = \ln(x+3) \cos y + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [\ln(x+3) \cos y + B(y)] = -\ln(x+3) \sin y + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ -\ln(x+3) \sin y + \frac{\partial B}{\partial y} &= -[\sin y \ln(5x+15) - y - 1] \\ -\ln(x+3) \sin y + \frac{\partial B}{\partial y} &= -\sin y \ln(5x+15) + y + 1 \\ -\ln(x+3) \sin y + \frac{\partial B}{\partial y} &= -\sin y \ln(5(x+3)) + y + 1 \\ -\ln(x+3) \sin y + \frac{\partial B}{\partial y} &= -\sin y \ln(5) - \sin y \ln(x+3) + y + 1 \\ \frac{\partial B}{\partial y} &= -\sin y \ln(5) + y + 1\end{aligned}$$

Now we integrate $\frac{dB}{dy} = -\sin y \ln(5) + y + 1$ with respect to y .

$$B(y) = \int (-\sin y \ln(5) + y + 1) \, dy = \ln(5) \cos y + \frac{1}{2}y^2 + y + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = \ln(x + 3) \cos y + \ln(5) \cos y + \frac{1}{2}y^2 + y + C$$

The general solution of the differential equation is

$$\ln(x + 3) \cos y + \ln(5) \cos y + \frac{1}{2}y^2 + y + C = 0$$

where C is a constant.

14. Consider the differential equation $p^2 \sec(2\theta) \tan(2\theta) d\theta + p[\sec(2\theta) + 2] dp = 0$.

(a) Let $M = p^2 \sec(2\theta) \tan(2\theta)$ and $N = p[\sec(2\theta) + 2]$. Then

$$\frac{\partial M}{\partial p} = 2p \sec(2\theta) \tan(2\theta) \quad \text{and} \quad \frac{\partial N}{\partial \theta} = 2p \sec(2\theta) \tan(2\theta)$$

Since $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial \theta}$ for all θ and p where defined, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial \theta} = M = p^2 \sec(2\theta) \tan(2\theta)$ partially with respect to θ (holding p constant). Then

$$f(\theta, p) = \int p^2 \sec(2\theta) \tan(2\theta) d\theta = \frac{1}{2}p^2 \sec(2\theta) + B(p)$$

where the constant of integration B is a function of p alone. Find $\frac{\partial f}{\partial p}$.

$$\frac{\partial f}{\partial p} = \frac{\partial}{\partial p} \left[\frac{1}{2}p^2 \sec(2\theta) + B(p) \right] = p \sec(2\theta) + \frac{\partial B}{\partial p}$$

The function f satisfies $\frac{\partial f}{\partial p} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial p} &= N \\ p \sec(2\theta) + \frac{\partial B}{\partial p} &= p[\sec(2\theta) + 2] \\ \frac{\partial B}{\partial p} &= 2p \end{aligned}$$

Now we integrate $\frac{dB}{dp} = 2p$ with respect to p .

$$B(p) = \int 2p dp = p^2 + C$$

Substituting for $B(p)$ in the function $f(\theta, p)$, we obtain

$$f(\theta, p) = \frac{1}{2}p^2 \sec(2\theta) + p^2 + C$$

The general solution of the differential equation is

$$\frac{1}{2}p^2 \sec(2\theta) + p^2 + C = 0$$

where C is a constant.

15. Consider the differential equation $\cos(x + y^2) dx + 2y \cos(x + y^2) dy = 0$.

(a) Let $M = \cos(x + y^2)$ and $N = 2y \cos(x + y^2)$. Then

$$\frac{\partial M}{\partial y} = -2y \sin(x + y^2) \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y \sin(x + y^2)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = \cos(x + y^2)$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \cos(x + y^2) dx = \sin(x + y^2) + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [\sin(x + y^2) + B(y)] = 2y \cos(x + y^2) + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 2y \cos(x + y^2) + \frac{\partial B}{\partial y} &= 2y \cos(x + y^2) \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = \sin(x + y^2) + C$$

The general solution of the differential equation is

$$\boxed{\sin(x + y^2) + C = 0}$$

where C is a constant.

16. Consider the differential equation $[\sin(2\theta) - 2p \cos(2\theta)] dp + [2p \cos(2\theta) + 2p^2 \sin(2\theta)] d\theta = 0$.

(a) Let $M = [\sin(2\theta) - 2p \cos(2\theta)]$ and $N = 2p \cos(2\theta) + 2p^2 \sin(2\theta)$. Then

$$\frac{\partial M}{\partial \theta} = 2 \cos(2\theta) + 4p \sin(2\theta) \quad \text{and} \quad \frac{\partial N}{\partial p} = 2 \cos(2\theta) + 4p \sin(2\theta)$$

Since $\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial p}$ for all p and θ , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial p} = M = [\sin(2\theta) - 2p \cos(2\theta)]$ partially with respect to p (holding θ constant). Then

$$f(p, \theta) = \int [\sin(2\theta) - 2p \cos(2\theta)] dp = p \sin(2\theta) - p^2 \cos(2\theta) + B(\theta)$$

where the constant of integration B is a function of θ alone. Find $\frac{\partial f}{\partial \theta}$.

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} [p \sin(2\theta) - p^2 \cos(2\theta) + B(\theta)] = 2p \cos(2\theta) + 2p^2 \sin(2\theta) + \frac{\partial B}{\partial \theta}$$

The function f satisfies $\frac{\partial f}{\partial \theta} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= N \\ 2p \cos(2\theta) + 2p^2 \sin(2\theta) + \frac{\partial B}{\partial \theta} &= 2p \cos(2\theta) + 2p^2 \sin(2\theta) \\ \frac{\partial B}{\partial \theta} &= 0\end{aligned}$$

$B(\theta)$ must equal a constant. Let $B(\theta) = C$ and substitute it in the function $f(p, \theta)$, we obtain

$$f(p, \theta) = p \sin(2\theta) - p^2 \cos(2\theta) + C$$

The general solution of the differential equation is

$$\boxed{p \sin(2\theta) - p^2 \cos(2\theta) + C = 0}$$

where C is a constant.

17. Consider the differential equation $(2ye^{2x} - x^2) dx + e^{2x} dy = 0$.

(a) Let $M = 2ye^{2x} - x^2$ and $N = e^{2x}$. Then

$$\frac{\partial M}{\partial y} = 2e^{2x} \quad \text{and} \quad \frac{\partial N}{\partial x} = 2e^{2x}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = 2ye^{2x} - x^2$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (2ye^{2x} - x^2) dx = ye^{2x} - \frac{1}{3}x^3 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[ye^{2x} - \frac{1}{3}x^3 + B(y) \right] = e^{2x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ e^{2x} + \frac{\partial B}{\partial y} &= e^{2x} \\ \frac{\partial B}{\partial y} &= 0\end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = ye^{2x} - \frac{1}{3}x^3 + C$$

The general solution of the differential equation is

$$\boxed{ye^{2x} - \frac{1}{3}x^3 + C = 0}$$

where C is a constant.

18. Consider the differential equation $(2xye^{x^2} - 3x^2) dx + e^{x^2} dy = 0$.

(a) Let $M = 2xye^{x^2} - 3x^2$ and $N = e^{x^2}$. Then

$$\frac{\partial M}{\partial y} = 2xe^{x^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xe^{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = 2xye^{x^2} - 3x^2$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int (2xye^{x^2} - 3x^2) dx = ye^{x^2} - x^3 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [ye^{x^2} - x^3 + B(y)] = e^{x^2} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ e^{x^2} + \frac{\partial B}{\partial y} &= e^{x^2} \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = ye^{x^2} - x^3 + C$$

The general solution of the differential equation is

$$\boxed{ye^{x^2} - x^3 + C = 0}$$

where C is a constant.

19. Consider the differential equation $\left[\frac{1}{x+y} + y^2 \right] dx + \left[\frac{1}{x+y} + 2xy \right] dy = 0$.

(a) Let $M = \frac{1}{x+y} + y^2$ and $N = \frac{1}{x+y} + 2xy$. Then

$$\frac{\partial M}{\partial y} = -\frac{1}{(x+y)^2} + 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{(x+y)^2} + 2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ when $x+y \neq 0$, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = \frac{1}{x+y} + y^2$ partially with respect to x (holding y constant).
Then

$$f(x, y) = \int \left[\frac{1}{x+y} + y^2 \right] dx = \ln|x+y| + xy^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [\ln |x + y| + xy^2 + B(y)] = \frac{1}{x + y} + 2xy + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ \frac{1}{x + y} + 2xy + \frac{\partial B}{\partial y} &= \frac{1}{x + y} + 2xy \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = \ln |x + y| + xy^2 + C$$

The general solution of the differential equation is

$$\boxed{\ln |x + y| + xy^2 + C = 0}$$

where C is a constant.

20. Consider the differential equation $\frac{y^2 - 2x^2}{xy^2 - x^3} dx + \frac{2y^2 - x^2}{y^3 - x^2y} dy = 0$.

(a) Let $M = \frac{y^2 - 2x^2}{xy^2 - x^3}$ and $N = \frac{2y^2 - x^2}{y^3 - x^2y}$. Then

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{y^2 - 2x^2}{xy^2 - x^3} \right) \\ &= \frac{(xy^2 - x^3)(2y) - (y^2 - 2x^2)(2xy)}{(xy^2 - x^3)^2} \\ &= \frac{2x^3y}{(xy^2 - x^3)^2} \\ &= \frac{2x^3y}{x^2(y^2 - x^2)^2} \\ &= \frac{2xy}{(y^2 - x^2)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{2y^2 - x^2}{y^3 - x^2y} \right) \\ &= \frac{(y^3 - x^2y)(-2x) - (2y^2 - x^2)(-2xy)}{(y^3 - x^2y)^2} \\ &= \frac{2xy^3}{(y^3 - x^2y)^2} \\ &= \frac{2xy^3}{y^2(y^2 - x^2)^2} \\ &= \frac{2xy}{(y^2 - x^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ when $x \neq 0$, $y \neq 0$, $x \neq y$ and $x \neq -y$, the differential equation is exact.

(b) Find f by integrating $\frac{\partial f}{\partial x} = M = \frac{y^2 - 2x^2}{xy^2 - x^3}$ partially with respect to x (holding y constant). Then

$$\begin{aligned} f(x, y) &= \int \frac{y^2 - 2x^2}{xy^2 - x^3} dx \\ &= \int \left(\frac{1}{x} - \frac{1}{2} \frac{1}{y-x} + \frac{1}{2} \frac{1}{y+x} \right) dx \\ &= \ln|x| + \frac{1}{2} \ln|y-x| + \frac{1}{2} \ln|y+x| + B(y) \end{aligned}$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\ln|x| + \frac{1}{2} \ln|y-x| + \frac{1}{2} \ln|y+x| + B(y) \right] = \frac{1}{2(y-x)} + \frac{1}{2(y+x)} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ \frac{1}{2(y-x)} + \frac{1}{2(y+x)} + \frac{\partial B}{\partial y} &= \frac{2y^2 - x^2}{y^3 - x^2y} \\ \frac{y}{y^2 - x^2} + \frac{\partial B}{\partial y} &= \frac{2y^2 - x^2}{y^3 - x^2y} \\ \frac{y^2}{y^3 - x^2y} + \frac{\partial B}{\partial y} &= \frac{2y^2 - x^2}{y^3 - x^2y} \\ \frac{y^2}{y^3 - x^2y} + \frac{\partial B}{\partial y} &= \frac{y^2}{y^3 - x^2y} + \frac{y^2 - x^2}{y^3 - x^2y} \\ \frac{y^2}{y^3 - x^2y} + \frac{\partial B}{\partial y} &= \frac{y^2}{y^3 - x^2y} + \frac{y^2 - x^2}{y(y^2 - x^2)} \\ \frac{y^2}{y^3 - x^2y} + \frac{\partial B}{\partial y} &= \frac{y^2}{y^3 - x^2y} + \frac{1}{y} \\ \frac{\partial B}{\partial y} &= \frac{1}{y} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = \frac{1}{y}$ with respect to y .

$$B(y) = \int \frac{1}{y} dy = \ln|y| + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = \ln|x| + \frac{1}{2} \ln|y-x| + \frac{1}{2} \ln|y+x| + \ln|y| + C$$

The general solution of the differential equation is

$$\ln|x| + \frac{1}{2} \ln|y-x| + \frac{1}{2} \ln|y+x| + \ln|y| + C = 0$$

where C is a constant.

21. Consider the differential equation $2y^3 \sin(2x) dx - 3y^2 \cos(2x) dy = 0$.

- (a) Let $M = 2y^3 \sin(2x)$ and $N = -3y^2 \cos(2x)$. Then

$$\frac{\partial M}{\partial y} = 6y^2 \sin(2x) \quad \text{and} \quad \frac{\partial N}{\partial x} = 6y^2 \sin(2x)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = 2y^3 \sin(2x)$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int 2y^3 \sin(2x) dx = -y^3 \cos(2x) + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [-y^3 \cos(2x) + B(y)] = -3y^2 \cos(2x) + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ -3y^2 \cos(2x) + \frac{\partial B}{\partial y} &= -3y^2 \cos(2x) \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substitute it in the function $f(x, y)$, we obtain

$$f(x, y) = -y^3 \cos(2x) + C$$

The general solution of the differential equation is

$$\boxed{-y^3 \cos(2x) + C = 0}$$

where C is a constant.

22. Consider the differential equation $\frac{3y^2}{x^2 + 3x} dx + \left(2y \ln \frac{5x}{x+3} + 3 \sin y \right) dy = 0$.

- (a) Let $M = \frac{3y^2}{x^2 + 3x}$ and $N = 2y \ln \frac{5x}{x+3} + 3 \sin y$. Then

$$\frac{\partial M}{\partial y} = \frac{6y}{x^2 + 3x} \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y \frac{1}{\frac{5x}{x+3}} \cdot \frac{(x+3)(5) - 5x}{(x+3)^2} = 2y \frac{x+3}{5x} \cdot \frac{15}{(x+3)^2} = \frac{6y}{x(x+3)}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on the domain $\{(x, y) | x < -3 \text{ or } x > 0\}$, the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M = \frac{3y^2}{x^2 + 3x}$ partially with respect to x (holding y constant). Then

$$\begin{aligned} f(x, y) &= \int \frac{3y^2}{x^2 + 3x} dx \\ &= y^2 \int \left(\frac{1}{x} - \frac{1}{x+3} \right) dx \\ &= y^2 (\ln |x| - \ln |x+3|) + B(y) \\ &= y^2 \ln \left| \frac{x}{x+3} \right| + B(y) \\ &= y^2 \ln \frac{x}{x+3} + B(y) \end{aligned}$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[y^2 \ln \frac{x}{x+3} + B(y) \right] = 2y \ln \frac{x}{x+3} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 2y \ln \frac{x}{x+3} + \frac{\partial B}{\partial y} &= 2y \ln \frac{5x}{x+3} + 3 \sin y \\ 2y \ln \frac{x}{x+3} + \frac{\partial B}{\partial y} &= 2y \ln \frac{x}{x+3} + 2y \ln(5) + 3 \sin y \\ \frac{\partial B}{\partial y} &= 2y \ln(5) + 3 \sin y \end{aligned}$$

Now we integrate $\frac{dB}{dy} = 2y \ln(5) + 3 \sin y$ with respect to y .

$$B(y) = \int (2y \ln(5) + 3 \sin y) dy = y^2 \ln(5) - 3 \cos y + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$\begin{aligned} f(x, y) &= y^2 \ln \frac{x}{x+3} + y^2 \ln(5) - 3 \cos y + C \\ &= y^2 \left(\ln \frac{x}{x+3} + \ln(5) \right) - 3 \cos y + C \\ &= y^2 \ln \frac{5x}{x+3} - 3 \cos y + C \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y^2 \ln \frac{5x}{x+3} - 3 \cos y + C = 0}$$

where C is a constant.

23. Consider the differential equation $(1 + y^2 + xy^2) dx + (x^2y + y + 2xy) dy = 0$, with $y(1) = 1$. Let $M = 1 + y^2 + xy^2$ and $N = x^2y + y + 2xy$. Find f by integrating $\frac{\partial f}{\partial x} = M = 1 + y^2 + xy^2$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (1 + y^2 + xy^2) dx = x + xy^2 + \frac{1}{2}x^2y^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[x + xy^2 + \frac{1}{2}x^2y^2 + B(y) \right] = 2xy + x^2y + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 2xy + x^2y + \frac{\partial B}{\partial y} &= x^2y + y + 2xy \\ \frac{\partial B}{\partial y} &= y \end{aligned}$$

Now we integrate $\frac{dB}{dy} = y$ with respect to y .

$$B(y) = \int y \, dy = \frac{1}{2}y^2 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x + xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + C$$

The general solution of the differential equation is $x + xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + C = 0$, where C is a constant. To find the particular solution, substitute $x = 1$ and $y = 1$ and solve for C .

$$\begin{aligned} x + xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + C &= 0 \\ 1 + 1 + \frac{1}{2} + \frac{1}{2} + C &= 0 \\ C &= -3 \end{aligned}$$

The particular solution is $\boxed{x + xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 - 3 = 0}$.

24. Consider the differential equation $(3x^2y^{-1} + 2x) \, dx + (y^2 - x^3y^{-2}) \, dy = 0$, with $y(3) = 3$. Let $M = 3x^2y^{-1} + 2x$ and $N = y^2 - x^3y^{-2}$. Find f by integrating $\frac{\partial f}{\partial x} = M = 3x^2y^{-1} + 2x$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (3x^2y^{-1} + 2x) \, dx = x^3y^{-1} + x^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^3y^{-1} + x^2 + B(y)] = -x^3y^{-2} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ -x^3y^{-2} + \frac{\partial B}{\partial y} &= y^2 - x^3y^{-2} \\ \frac{\partial B}{\partial y} &= y^2 \end{aligned}$$

Now we integrate $\frac{dB}{dy} = y^2$ with respect to y .

$$B(y) = \int y^2 \, dy = \frac{1}{3}y^3 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x^3y^{-1} + x^2 + \frac{1}{3}y^3 + C$$

The general solution of the differential equation is $x^3y^{-1} + x^2 + \frac{1}{3}y^3 + C = 0$, where C is a constant. To find the particular solution, substitute $x = 3$ and $y = 3$ and solve for C .

$$\begin{aligned}x^3y^{-1} + x^2 + \frac{1}{3}y^3 + C &= 0 \\ \frac{3^3}{3} + 3^2 + \frac{1}{3}3^3 + C &= 0 \\ C &= -27\end{aligned}$$

The particular solution is $\boxed{x^3y^{-1} + x^2 + \frac{1}{3}y^3 - 27 = 0}$.

25. Consider the differential equation $(2xy - \sin x) dx + (x^2 - 2y) dy = 0$, with $y(0) = 1$. Let $M = 2xy - \sin x$ and $N = x^2 - 2y$. Find f by integrating $\frac{\partial f}{\partial x} = M = 2xy - \sin x$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (2xy - \sin x) dx = x^2y + \cos x + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^2y + \cos x + B(y)] = x^2 + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ x^2 + \frac{\partial B}{\partial y} &= x^2 - 2y \\ \frac{\partial B}{\partial y} &= -2y\end{aligned}$$

Now we integrate $\frac{dB}{dy} = -2y$ with respect to y .

$$B(y) = - \int 2y dy = -y^2 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x^2y + \cos x - y^2 + C$$

The general solution of the differential equation is $x^2y + \cos x - y^2 + C = 0$, where C is a constant. To find the particular solution, substitute $x = 0$ and $y = 1$ and solve for C .

$$\begin{aligned}x^2y + \cos x - y^2 + C &= 0 \\ 0 + \cos 0 - 1 + C &= 0 \\ C &= 0\end{aligned}$$

The particular solution is $\boxed{x^2y + \cos x - y^2 = 0}$.

26. Consider the differential equation $y[y + \sin x] dx - \left[\cos x - 2xy + \frac{1}{1+y^2} \right] dy = 0$, with $y(0) = 1$. Let $M = y[y + \sin x]$ and $N = 2xy - \cos x - \frac{1}{1+y^2}$. Find f by integrating $\frac{\partial f}{\partial x} = M = y[y + \sin x]$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int y[y + \sin x] dx = \int [y^2 + y \sin x] dx = xy^2 - y \cos x + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [xy^2 - y \cos x + B(y)] = 2xy - \cos x + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 2xy - \cos x + \frac{\partial B}{\partial y} &= 2xy - \cos x - \frac{1}{1+y^2} \\ \frac{\partial B}{\partial y} &= -\frac{1}{1+y^2} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = -\frac{1}{1+y^2}$ with respect to y .

$$B(y) = -\int \frac{1}{1+y^2} dy = -\tan^{-1} y + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = xy^2 - y \cos x - \tan^{-1} y + C$$

The general solution of the differential equation is $xy^2 - y \cos x - \tan^{-1} y + C = 0$, where C is a constant. To find the particular solution, substitute $x = 0$ and $y = 1$ and solve for C .

$$\begin{aligned} xy^2 - y \cos x - \tan^{-1} y + C &= 0 \\ 0 - \cos 0 - \tan^{-1}(1) + C &= 0 \\ C &= 1 + \frac{\pi}{4} \end{aligned}$$

The particular solution is $\boxed{xy^2 - y \cos x - \tan^{-1} y + 1 + \frac{\pi}{4} = 0}$.

Applications and Extensions

27. Suppose the equation $M(x, y) dx + N(x, y) dy = 0$ has the property that

$$g(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

is a function of y only. Let $u(y) = e^{\int g(y) dy}$ so that

$$\begin{aligned} \frac{du}{dy} &= \frac{d}{dy} \left[e^{\int g(y) dy} \right] \\ &= e^{\int g(y) dy} \frac{d}{dy} \left[\int g(y) dy \right] \\ &= e^{\int g(y) dy} g(y) \\ &= u(y)g(y) \end{aligned}$$

To show that $u(y)[M(x, y) dx + N(x, y) dy] = 0$ is exact, let $M_1(x, y) = u(y)M(x, y)$ and $N_1(x, y) = u(y)N(x, y)$ so that

$$u(y)[M(x, y) dx + N(x, y) dy] = M_1(x, y) dx + N_1(x, y) dy = 0$$

Then

$$\begin{aligned}
 \frac{\partial M_1}{\partial y} &= \frac{\partial}{\partial y} [u(y)M(x, y)] \\
 &= \frac{\partial u}{\partial y} M(x, y) + u(y) \frac{\partial M}{\partial y} \\
 &= \frac{du}{dy} M(x, y) + u(y) \frac{\partial M}{\partial y} \\
 &= u(y)g(y)M(x, y) + u(y) \frac{\partial M}{\partial y} \\
 &= u(y) \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} M(x, y) + u(y) \frac{\partial M}{\partial y} \\
 &= u(y) \frac{\partial N}{\partial x} - u(y) \frac{\partial M}{\partial y} + u(y) \frac{\partial M}{\partial y} \\
 &= u(y) \frac{\partial N}{\partial x}
 \end{aligned}$$

and, because $u(y)$ is a function of y only,

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x} [u(y)N(x, y)] = u(y) \frac{\partial N}{\partial x}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ for all x and y , the differential equation $u(y)[M(x, y) dx + N(x, y) dy] = 0$ is exact.

28. Suppose the equation $M(x, y) dx + N(x, y) dy = 0$ has the property that

$$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

is a function of x only. Let $u(x) = e^{\int g(x) dx}$ so that

$$\begin{aligned}
 \frac{du}{dx} &= \frac{d}{dx} \left[e^{\int g(x) dx} \right] \\
 &= e^{\int g(x) dx} \frac{d}{dx} \left[\int g(x) dx \right] \\
 &= e^{\int g(x) dx} g(x) \\
 &= u(x)g(x)
 \end{aligned}$$

To show that $u(x)[M(x, y) dx + N(x, y) dy] = 0$ is exact, let $M_1(x, y) = u(x)M(x, y)$ and $N_1(x, y) = u(x)N(x, y)$ so that

$$u(x)[M(x, y) dx + N(x, y) dy] = M_1(x, y) dx + N_1(x, y) dy = 0$$

Then because $u(x)$ is a function of x only,

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y} [u(x)M(x, y)] = u(x) \frac{\partial M}{\partial y}$$

and

$$\begin{aligned}
 \frac{\partial N_1}{\partial x} &= \frac{\partial}{\partial x} [u(x)N(x, y)] \\
 &= \frac{\partial u}{\partial x} N(x, y) + u(x) \frac{\partial N}{\partial x} \\
 &= \frac{du}{dx} N(x, y) + u(x) \frac{\partial N}{\partial x} \\
 &= u(x)g(x)N(x, y) + u(x) \frac{\partial N}{\partial x} \\
 &= u(x) \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} N(x, y) + u(x) \frac{\partial N}{\partial x} \\
 &= u(x) \frac{\partial M}{\partial y} - u(x) \frac{\partial N}{\partial x} + u(x) \frac{\partial N}{\partial x} \\
 &= u(x) \frac{\partial M}{\partial y}
 \end{aligned}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ for all x and y , the differential equation $u(x)[M(x, y) dx + N(x, y) dy] = 0$ is exact.

29. Consider the differential equation $(4x^2 + y^2 + 1) dx + (x^2 - 2xy) dy = 0$.

(a) Let $M = 4x^2 + y^2 + 1$ and $N = x^2 - 2xy$. Since $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x - 2y$,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y - 2x}{x^2 - 2xy} = \frac{2(2y - x)}{x(x - 2y)} = -\frac{2}{x}$$

Let

$$u(x) = e^{\left[-\int \frac{2}{x} dx\right]} = e^{-2 \ln(x)} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$$

Then the equation

$$\left(4 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx + \left(1 - \frac{2y}{x}\right) dy = 0$$

is exact.

(b) Let $M_1 = 4 + \frac{y^2}{x^2} + \frac{1}{x^2}$ and $N_1 = \frac{x - 2y}{x}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \left(4 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) dx = 4x - \frac{y^2}{x} - \frac{1}{x} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[4x - \frac{y^2}{x} - \frac{1}{x} + B(y)\right] = -\frac{2y}{x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned}
 \frac{\partial f}{\partial y} &= N_1 \\
 -\frac{2y}{x} + \frac{\partial B}{\partial y} &= \frac{x - 2y}{x} \\
 -\frac{2y}{x} + \frac{\partial B}{\partial y} &= 1 - \frac{2y}{x} \\
 \frac{\partial B}{\partial y} &= 1
 \end{aligned}$$

Now we integrate $\frac{dB}{dy} = 1$ with respect to y .

$$B(y) = \int 1 \, dy = y + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = 4x - \frac{y^2}{x} - \frac{1}{x} + y + C$$

The general solution of the differential equation is

$$\boxed{4x - \frac{y^2}{x} - \frac{1}{x} + y + C = 0}$$

where C is a constant.

- (c) Differentiate $4x - \frac{y^2}{x} - \frac{1}{x} + y + C = 0$ with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx} \left(4x - \frac{y^2}{x} - \frac{1}{x} + y + C \right) &= 0 \\ 4 - \frac{(x)(2yy') - y^2}{x^2} + \frac{1}{x^2} + y' &= 0 \\ 4x^2 - 2xyy' + y^2 + 1 + y' &= 0 \\ 4x^2 + y^2 + 1 + (1 - 2xy)y' &= 0 \\ 4x^2 + y^2 + 1 + (1 - 2xy)\frac{dy}{dx} &= 0 \end{aligned}$$

The equivalent differential form is

$$(4x^2 + y^2 + 1) \, dx + (x^2 - 2xy) \, dy = 0$$

so $4x - \frac{y^2}{x} - \frac{1}{x} + y + C = 0$ is a solution this differential equation as well.

30. Consider the differential equation $4x^2y \, dx + (x^3 + y) \, dy = 0$.

- (a) Let $M = 4x^2y$ and $N = x^3 + y$. Since $\frac{\partial M}{\partial y} = 4x^2$ and $\frac{\partial N}{\partial x} = 3x^2$,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-x^2}{4x^2y} = -\frac{1}{4y}$$

Let

$$u(y) = e^{\left[-\int \frac{1}{4y} \, dy\right]} = e^{-\frac{1}{4} \ln(y)} = e^{\ln(y^{-1/4})} = y^{-1/4}$$

Then the equation

$$\boxed{4x^2y^{3/4} \, dx + (x^3y^{-1/4} + y^{3/4}) \, dy = 0}$$

is exact.

- (b) Let $M_1 = 4x^2y^{3/4}$ and $N_1 = x^3y^{-1/4} + y^{3/4}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1 = 4x^2y^{3/4}$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int 4x^2y^{3/4} \, dx = \frac{4}{3}x^3y^{3/4} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\frac{4}{3}x^3y^{3/4} + B(y) \right] = x^3y^{-1/4} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N_1 \\ x^3y^{-1/4} + \frac{\partial B}{\partial y} &= x^3y^{-1/4} + y^{3/4} \\ \frac{\partial B}{\partial y} &= y^{3/4} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = y^{3/4}$ with respect to y .

$$B(y) = \int y^{3/4} dy = \frac{4}{7}y^{7/4} + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = \frac{4}{3}x^3y^{3/4} + \frac{4}{7}y^{7/4} + C$$

The general solution of the differential equation is

$$\boxed{\frac{4}{3}x^3y^{3/4} + \frac{4}{7}y^{7/4} + C = 0}$$

where C is a constant.

(c) Differentiate $\frac{4}{3}x^3y^{3/4} + \frac{4}{7}y^{7/4} + C = 0$ with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx} \left(\frac{4}{3}x^3y^{3/4} + \frac{4}{7}y^{7/4} + C \right) &= 0 \\ 4x^2y^{3/4} + x^3y^{-1/4}\frac{dy}{dx} + y^{3/4}\frac{dy}{dx} &= 0 \\ 4x^2y + x^3\frac{dy}{dx} + y\frac{dy}{dx} &= 0 \quad (\text{multiply by } y^{1/4}) \\ 4x^2y + (x^3 + y)\frac{dy}{dx} &= 0 \end{aligned}$$

The equivalent differential form is

$$4x^2y dx + (x^3 + y) dy = 0$$

so $\frac{4}{3}x^3y^{3/4} + \frac{4}{7}y^{7/4} + C = 0$ is a solution this differential equation as well.

31. Consider the differential equation $y dx + (x^2y - x) dy = 0$.

(a) Let $M = y$ and $N = x^2y - x$. Since $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 2xy - 1$,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}$$

Let

$$u(x) = e^{\left[-\int \frac{2}{x} dx\right]} = e^{-2 \ln(x)} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$$

Then the equation

$$\boxed{\frac{y}{x^2} dx + \left(y - \frac{1}{x}\right) dy = 0}$$

is exact.

- (b) Let $M_1 = \frac{y}{x^2}$ and $N_1 = y - \frac{1}{x}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \frac{y}{x^2} dx = -\frac{y}{x} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[-\frac{y}{x} + B(y)\right] = -\frac{1}{x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N_1 \\ -\frac{1}{x} + \frac{\partial B}{\partial y} &= y - \frac{1}{x} \\ \frac{\partial B}{\partial y} &= y \end{aligned}$$

Now we integrate $\frac{dB}{dy} = y$ with respect to y .

$$B(y) = \int y dy = \frac{1}{2}y^2 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = -\frac{y}{x} + \frac{1}{2}y^2 + C$$

The general solution of the differential equation is

$$\boxed{-\frac{y}{x} + \frac{1}{2}y^2 + C = 0}$$

where C is a constant.

- (c) Differentiate $-\frac{y}{x} + \frac{1}{2}y^2 + C = 0$ with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx} \left(-\frac{y}{x} + \frac{1}{2}y^2 + C \right) &= 0 \\ -\frac{xy' - y}{x^2} + yy' &= 0 \\ -xy' + y + x^2yy' &= 0 \\ y + (x^2y - x)y' &= 0 \\ y + (x^2y - x)\frac{dy}{dx} &= 0 \end{aligned}$$

The equivalent differential form is

$$y \, dx + (x^2 y - x) \, dy = 0$$

so $-\frac{y}{x} + \frac{1}{2}y^2 + C = 0$ is a solution this differential equation as well.

32. Consider the differential equation $(\cos y + x) \, dx + x \sin y \, dy = 0$.

(a) Let $M = \cos y + x$ and $N = x \sin y$. Since $\frac{\partial M}{\partial y} = -\sin y$ and $\frac{\partial N}{\partial x} = \sin y$,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2 \sin y}{x \sin y} = -\frac{2}{x}$$

Let

$$u(x) = e^{\left[-\int \frac{2}{x} \, dx\right]} = e^{-2 \ln(x)} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$$

Then the equation

$$\left(\frac{\cos y}{x^2} + \frac{1}{x} \right) dx + \frac{\sin y}{x} dy = 0$$

is exact.

(b) Let $M_1 = \frac{\cos y}{x^2} + \frac{1}{x}$ and $N_1 = \frac{\sin y}{x}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1 = \frac{\cos y}{x^2} + \frac{1}{x}$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \left(\frac{\cos y}{x^2} + \frac{1}{x} \right) dx = -\frac{\cos y}{x} + \ln |x| + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[-\frac{\cos y}{x} + \ln |x| + B(y) \right] = \frac{\sin y}{x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N_1 \\ \frac{\sin y}{x} + \frac{\partial B}{\partial y} &= \frac{\sin y}{x} \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substituting it in the function $f(x, y)$, we obtain

$$f(x, y) = -\frac{\cos y}{x} + \ln |x| + C$$

The general solution of the differential equation is

$$-\frac{\cos y}{x} + \ln |x| + C = 0$$

where C is a constant.

- (c) Differentiate $-\frac{\cos y}{x} + \ln|x| + C = 0$ with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{d}{dx} \left(-\frac{\cos y}{x} + \ln|x| + C \right) &= 0 \\ -\frac{x(-\sin y)y' - \cos y}{x^2} + \frac{1}{x} &= 0 \\ x \sin y y' + \cos y + x &= 0 \\ \cos y + x + x \sin y \frac{dy}{dx} &= 0\end{aligned}$$

The equivalent differential form is

$$(\cos y + x) dx + x \sin y dy = 0$$

so $-\frac{\cos y}{x} + \ln|x| + C = 0$ is a solution this differential equation as well.

33. Consider the differential equation $(x^2 - x \sin y) dx + x^2 \cos y dy = 0$.

- (a) Let $M = x^2 - x \sin y$ and $N = x^2 \cos y$. Since $\frac{\partial M}{\partial y} = -x \cos y$ and $\frac{\partial N}{\partial x} = 2x \cos y$,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-3x \cos y}{x^2 \cos y} = -\frac{3}{x}$$

Let

$$u(x) = e^{\left[-\int \frac{3}{x} dx\right]} = e^{-3 \ln(x)} = e^{\ln(x^{-3})} = x^{-3} = \frac{1}{x^3}$$

Then the equation

$$\left(\left(\frac{1}{x} - \frac{\sin y}{x^2} \right) dx + \frac{\cos y}{x} dy \right) = 0$$

is exact.

- (b) Let $M_1 = \frac{1}{x} - \frac{\sin y}{x^2}$ and $N_1 = \frac{\cos y}{x}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1 = \frac{1}{x} - \frac{\sin y}{x^2}$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \left(\frac{1}{x} - \frac{\sin y}{x^2} \right) dx = \ln|x| + \frac{\sin y}{x} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\ln|x| + \frac{\sin y}{x} + B(y) \right] = \frac{\cos y}{x} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N_1 \\ \frac{\cos y}{x} + \frac{\partial B}{\partial y} &= \frac{\cos y}{x} \\ \frac{\partial B}{\partial y} &= 0\end{aligned}$$

$B(y)$ must equal a constant. Let $B(y) = C$ and substituting it in the function $f(x, y)$, we obtain

$$f(x, y) = \ln|x| + \frac{\sin y}{x} + C$$

The general solution of the differential equation is

$$\ln |x| + \frac{\sin y}{x} + C = 0$$

where C is a constant.

- (c) Differentiate $\ln |x| + \frac{\sin y}{x} + C = 0$ with respect to x to find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx} \left(\ln |x| + \frac{\sin y}{x} + C \right) &= 0 \\ \frac{1}{x} + \frac{x \cos y y' - \sin y}{x^2} &= 0 \\ x^2 + x^2 \cos y y' - x \sin y &= 0 \quad (\text{multiplied by } x^3) \\ x^2 - x \sin y + x^2 \cos y \frac{dy}{dx} &= 0 \end{aligned}$$

The equivalent differential form is

$$(x^2 - x \sin y) dx + x^2 \cos y dy = 0$$

so $\ln |x| + \frac{\sin y}{x} + C = 0$ is a solution this differential equation as well.

Challenge Problem

34. Consider the differential equation

$$\frac{2 \frac{\beta-y}{100} - 2 \cos \frac{\beta-y}{100}}{2 \frac{\alpha-x}{100} + 4 \cos \frac{\alpha-x}{100}} \frac{dy}{dx} = 1$$

- (a) The equation is separable, so rewrite and integrate.

$$\begin{aligned} \frac{2 \frac{\beta-y}{100} - 2 \cos \frac{\beta-y}{100}}{2 \frac{\alpha-x}{100} + 4 \cos \frac{\alpha-x}{100}} \frac{dy}{dx} &= 1 \\ 2 \frac{\beta-y}{100} - 2 \cos \frac{\beta-y}{100} dy &= 2 \frac{\alpha-x}{100} + 4 \cos \frac{\alpha-x}{100} dx \\ \int \left(2 \frac{\beta-y}{100} - 2 \cos \frac{\beta-y}{100} \right) dy &= \int \left(2 \frac{\alpha-x}{100} + 4 \cos \frac{\alpha-x}{100} \right) dx \\ \frac{\beta y}{50} - \frac{y^2}{100} + 200 \sin \frac{\beta-y}{100} &= \frac{\alpha x}{50} - \frac{x^2}{100} - 400 \sin \frac{\alpha-x}{100} + C \end{aligned}$$

The general solution is

$$\frac{\beta y}{50} - \frac{y^2}{100} + 200 \sin \frac{\beta-y}{100} = \frac{\alpha x}{50} - \frac{x^2}{100} - 400 \sin \frac{\alpha-x}{100} + C$$

where C is a constant.

- (b) Express the general solution as

$$\frac{\beta y}{50} - \frac{y^2}{100} + 200 \sin \frac{\beta-y}{100} - \frac{\alpha x}{50} + \frac{x^2}{100} + 400 \sin \frac{\alpha-x}{100} - C = 0$$

Let $f(x, y)$ equal the expression on the left-hand side of the equation, so that

$$f(x, y) = \frac{\beta y}{50} - \frac{y^2}{100} + 200 \sin \frac{\beta-y}{100} - \frac{\alpha x}{50} + \frac{x^2}{100} + 400 \sin \frac{\alpha-x}{100} - C$$

Given $\alpha = 93.43$ and $\beta = 59.12$, the function becomes

$$f(x, y) = \frac{59.12y}{50} - \frac{y^2}{100} + 200 \sin \frac{59.12 - y}{100} - \frac{93.43x}{50} + \frac{x^2}{100} + 400 \sin \frac{93.43 - x}{100} - C$$

Use the initial condition $f(0, 0) = 1247.2$ to find C .

$$f(0, 0) = 1247.2 = 200 \sin(0.5912) + 400 \sin(0.9343) - C$$

Solve for C , $C = 200 \sin(0.5912) + 400 \sin(0.9343) - 1247.2 \approx -814.055$ and

$$f(x, y) = \frac{59.12y}{50} - \frac{y^2}{100} + 200 \sin \frac{59.12 - y}{100} - \frac{93.43x}{50} + \frac{x^2}{100} + 400 \sin \frac{93.43 - x}{100} + 814.055$$

To find the oil concentration at point B , evaluate

$$f(162.14, 250.64) \approx 0.152$$

The oil concentration at point B is 0.152 pounds per million gallons.

16.4 First-Order Linear Differential Equations: Bernoulli Differential Equations

Concepts and Vocabulary

1. True. First-order linear differential equations have the form $\frac{dy}{dx} + P(x)y = Q(x)$, where the functions P and Q are continuous on their domains.
2. True. The first-order linear equation $\frac{dy}{dx} + P(x)y = 0$ is separable, since its differential form is $\frac{1}{y} dy + P(x) dx = 0$.
3. False. Multiplying the first-order linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$, where $Q(x) \neq 0$, by the integrating factor $e^{\int P(x) dx}$ results in a separable differential equation.
4. False. To solve a Bernoulli equation, $\frac{dy}{dx} + P(x)y = Q(x)y^n$, $n \neq 0$, $n \neq 1$, the first step is to multiply by y^{-n} .

Skill Building

5. The differential equation $\frac{dy}{dx} + 2xy = 0$ is separable. Separate the variables and integrate.

$$\begin{aligned} \frac{1}{y} dy + 2x dx &= 0 \\ \ln |y| + x^2 &= C_1 \\ \ln |y| &= -x^2 + C_1 \\ |y| &= e^{(-x^2 + C_1)} \\ |y| &= e^{-x^2} e^{C_1} \\ y &= \pm e^{-x^2} e^{C_1} \end{aligned}$$

Since C_1 is a constant, we may write $C = \pm e^{C_1}$ where $C \neq 0$ is a constant. The solution is now $y = Ce^{-x^2}$, where $C \neq 0$ is a constant.

Notice that the function $y = 0$ also satisfies the differential equation $\frac{dy}{dx} + 2xy = 0$, so it too is a solution. This particular solution matches the general solution above when $C = 0$. So the general solution is $\boxed{y = Ce^{-x^2}}$, where C is any constant.

6. The differential equation $\frac{dy}{dx} + \frac{1}{x}y = 0$ is separable. Separate the variables and integrate.

$$\begin{aligned}\frac{1}{y} dy + \frac{1}{x} dx &= 0 \\ \ln|y| + \ln|x| &= C_1 \\ \ln|xy| &= C_1 \\ |xy| &= e^{C_1} \\ xy &= \pm e^{C_1} \\ y &= \pm \frac{e^{C_1}}{x}\end{aligned}$$

Since C_1 is a constant, we may write $C = \pm e^{C_1}$ where $C \neq 0$ is a constant. The general solution is $\boxed{y = \frac{C}{x}}$, where $C \neq 0$ is a constant.

7. The differential equation $\frac{dy}{dx} + \frac{1}{x}y = 3x$ is first-order linear. Let $P(x) = \frac{1}{x}$. The integrating factor is

$$\begin{aligned}e^{\int P(x) dx} &= e^{\int x^{-1} dx} \\ &= e^{\ln x} \\ &= x\end{aligned}$$

Now multiply the differential equation by x and obtain

$$x \frac{dy}{dx} + y = 3x^2$$

The left side of the above equation is the derivative of xy , so we can write

$$\frac{d}{dx}(xy) = 3x^2$$

Integrate the equation above.

$$\begin{aligned}xy &= \int 3x^2 dx \\ &= x^3 + C\end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = x^2 + \frac{C}{x}}$$

where C is a constant.

8. The differential equation $\frac{dr}{d\theta} + \frac{4r}{\theta} = \theta$ is first-order linear. Let $P(\theta) = \frac{4}{\theta}$. The integrating factor is

$$\begin{aligned}e^{\int P(\theta) d\theta} &= e^{\int 4\theta^{-1} d\theta} \\ &= e^{4 \ln \theta} \\ &= e^{\ln \theta^4} \\ &= \theta^4\end{aligned}$$

Now multiply the differential equation by θ^4 and obtain

$$\theta^4 \frac{dr}{d\theta} + 4\theta^3 r = \theta^5$$

The left side of the above equation is the derivative of $\theta^4 r$, so we can write

$$\frac{d}{d\theta} (\theta^4 r) = \theta^5$$

Integrate the equation above.

$$\begin{aligned} \theta^4 r &= \int \theta^5 d\theta \\ &= \frac{1}{6} \theta^6 + C \end{aligned}$$

Solve for r . The general solution of the differential equation is

$$r = \frac{1}{6} \theta^2 + \frac{C}{\theta^4}$$

where C is a constant.

9. The differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is first-order linear. Let $P(x) = \frac{1}{x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int x^{-1} dx} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

Now multiply the differential equation by x and obtain

$$x \frac{dy}{dx} + y = x^3$$

The left side of the above equation is the derivative of xy , so we can write

$$\frac{d}{dx} (xy) = x^3$$

Integrate the equation above.

$$\begin{aligned} xy &= \int x^3 dx \\ &= \frac{1}{4} x^4 + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{1}{4} x^3 + \frac{C}{x}$$

where C is a constant.

10. The differential equation $\frac{dy}{dx} - \frac{2y}{x+1} = 3(x+1)^2$ is first-order linear. Let $P(x) = -\frac{2}{x+1}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= \exp \left[\int -\frac{2}{x+1} dx \right] \\ &= \exp [-2 \ln(x+1)] \\ &= \exp [\ln ((x+1)^{-2})] \\ &= (x+1)^{-2} \end{aligned}$$

Now multiply the differential equation by $(x+1)^{-2}$ and obtain

$$(x+1)^{-2} \frac{dy}{dx} - 2(x+1)^{-1}y = 3$$

The left side of the above equation is the derivative of $(x+1)^{-2}y$, so we can write

$$\frac{d}{dx} ((x+1)^{-2}y) = 3$$

Integrate the equation above.

$$\begin{aligned} (x+1)^{-2}y &= \int 3 dx \\ &= 3x + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = (x+1)^2(3x+C)}$$

where C is a constant.

11. The differential equation $\frac{dy}{dx} - 2y = e^{-x}$ is first-order linear. Let $P(x) = -2$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{-\int 2 dx} \\ &= e^{-2x} \end{aligned}$$

Now multiply the differential equation by e^{-2x} and obtain

$$e^{-2x} \frac{dy}{dx} - 2e^{-2x}y = e^{-3x}$$

The left side of the above equation is the derivative of $e^{-2x}y$, so we can write

$$\frac{d}{dx} (e^{-2x}y) = e^{-3x}$$

Integrate the equation above.

$$\begin{aligned} e^{-2x}y &= \int e^{-3x} dx \\ &= -\frac{1}{3}e^{-3x} + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = -\frac{1}{3}e^{-x} + Ce^{2x}}$$

where C is a constant.

12. The differential equation $\frac{dy}{dx} - \frac{y}{x} = x^{3/2}$ is first-order linear. Let $P(x) = -\frac{1}{x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int -x^{-1} dx} \\ &= e^{-\ln x} \\ &= e^{\ln(x^{-1})} \\ &= x^{-1} \end{aligned}$$

Now multiply the differential equation by x^{-1} and obtain

$$x^{-1} \frac{dy}{dx} - x^{-2} y = x^{1/2}$$

The left side of the above equation is the derivative of $x^{-1}y$, so we can write

$$\frac{d}{dx} (x^{-1}y) = x^{1/2}$$

Integrate the equation above.

$$\begin{aligned} x^{-1}y &= \int x^{1/2} dx \\ &= \frac{2}{3}x^{3/2} + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{2}{3}x^{5/2} + Cx$$

where C is a constant.

13. The differential equation $\frac{dy}{dx} + \frac{2y}{x} = x^2 + 1$ is first-order linear. Let $P(x) = \frac{2}{x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int 2x^{-1} dx} \\ &= e^{2 \ln x} \\ &= e^{\ln x^2} \\ &= x^2 \end{aligned}$$

Now multiply the differential equation by x^2 and obtain

$$x^2 \frac{dy}{dx} + 2xy = x^4 + x^2$$

The left side of the above equation is the derivative of x^2y , so we can write

$$\frac{d}{dx} (x^2y) = x^4 + x^2$$

Integrate the equation above.

$$\begin{aligned} x^2y &= \int (x^4 + x^2) dx \\ &= \frac{1}{5}x^5 + \frac{1}{3}x^3 + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{1}{5}x^3 + \frac{1}{3}x + \frac{C}{x^2}$$

where C is a constant.

14. The differential equation $\frac{dy}{dx} + 2xy = 2x$ is first-order linear. Let $P(x) = 2x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int 2x dx} \\ &= e^{x^2} \end{aligned}$$

Now multiply the differential equation by e^{x^2} and obtain

$$e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$$

The left side of the above equation is the derivative of $e^{x^2}y$, so we can write

$$\frac{d}{dx} \left(e^{x^2}y \right) = 2xe^{x^2}$$

Integrate the equation above.

$$\begin{aligned} e^{x^2}y &= \int 2xe^{x^2} dx \quad \text{Let } u = x^2. \\ &= e^{x^2} + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = 1 + Ce^{-x^2}}$$

where C is a constant.

15. The differential equation $\frac{dy}{dx} + e^xy = e^x$ is first-order linear. Let $P(x) = e^x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int e^x dx} \\ &= e^{e^x} \end{aligned}$$

Now multiply the differential equation by e^{e^x} and obtain

$$e^{e^x} \frac{dy}{dx} + e^xe^{e^x}y = e^xe^{e^x}$$

The left side of the above equation is the derivative of $e^{e^x}y$, so we can write

$$\frac{d}{dx} \left(e^{e^x}y \right) = e^xe^{e^x}$$

Integrate the equation above.

$$\begin{aligned} e^{e^x}y &= \int e^xe^{e^x} dx \quad \text{Let } u = e^x. \\ &= e^{e^x} + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = 1 + Ce^{-e^x}}$$

where C is a constant.

16. The differential equation $\frac{dy}{dx} + e^{-x}y = e^{-x}$ is first-order linear. Let $P(x) = e^{-x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int e^{-x} dx} \\ &= e^{-e^{-x}} \end{aligned}$$

Now multiply the differential equation by $e^{-e^{-x}}$ and obtain

$$e^{-e^{-x}} \frac{dy}{dx} + e^{-x}e^{-e^{-x}}y = e^{-x}e^{-e^{-x}}$$

The left side of the above equation is the derivative of $e^{-e^{-x}}y$, so we can write

$$\frac{d}{dx} \left(e^{-e^{-x}} y \right) = e^{-x} e^{-e^{-x}}$$

Integrate the equation above.

$$\begin{aligned} e^{-e^{-x}} y &= \int e^{-x} e^{-e^{-x}} dx && \text{Let } u = -e^{-x}. \\ &= e^{-e^{-x}} + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = 1 + C e^{e^{-x}}}$$

where C is a constant.

17. The differential equation $\frac{dy}{dx} + y \tan x = \cos x$ is first-order linear. Let $P(x) = \tan x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int \tan x dx} \\ &= e^{\ln(\sec x)} \\ &= \sec x \end{aligned}$$

Now multiply the differential equation by $\sec x$ and obtain

$$\sec x \frac{dy}{dx} + y \sec x \tan x = \cos x \sec x = 1$$

The left side of the above equation is the derivative of $y \sec x$, so we can write

$$\frac{d}{dx} (y \sec x) = 1$$

Integrate the equation above.

$$\begin{aligned} y \sec x &= \int dx \\ &= x + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = x \cos x + C \cos x}$$

where C is a constant.

18. The differential equation $\frac{dy}{dx} - y \csc x = \sin 2x$ is first-order linear. Let $P(x) = -\csc x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{-\int \csc x dx} \\ &= e^{\ln(\csc x + \cot x)} \\ &= \csc x + \cot x \end{aligned}$$

(Note: There are several forms for the integral $\int \csc x dx$. It can be shown that $\ln |\csc x - \cot x| = -\ln |\csc x + \cot x|$. The form chosen makes the cancelation easiest.) Now multiply the differential

equation by $\csc x + \cot x$ and obtain

$$\begin{aligned}
 (\csc x + \cot x) \frac{dy}{dx} - y(\csc^2 x + \csc x \cot x) &= \sin 2x(\csc x + \cot x) \\
 &= 2 \sin x \cos x \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x} \right) \\
 &= 2 \cos x + 2 \cos^2 x \\
 &= 2 \cos x + 1 + \cos 2x
 \end{aligned}$$

The left side of the above equation is the derivative of $(\csc x + \cot x)y$, so we can write

$$\frac{d}{dx} ((\csc x + \cot x)y) = 2 \cos x + 1 + \cos 2x$$

Integrate the equation above.

$$\begin{aligned}
 (\csc x + \cot x)y &= \int (2 \cos x + 1 + \cos 2x) dx \\
 &= 2 \sin x + x + \frac{1}{2} \sin 2x + C \\
 &= 2 \sin x + x + \sin x \cos x + C
 \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{2 \sin x + x + \sin x \cos x + C}{\csc x + \cot x}$$

where C is a constant.

19. The differential equation $\frac{dy}{dx} + y \cot x = \csc^2 x$ is first-order linear. Let $P(x) = \cot x$. The integrating factor is

$$\begin{aligned}
 e^{\int P(x) dx} &= e^{\int \cot x dx} \\
 &= e^{\ln(\sin x)} \\
 &= \sin x
 \end{aligned}$$

Now multiply the differential equation by $\sin x$ and obtain

$$\sin x \frac{dy}{dx} + y \cos x = \csc x$$

The left side of the above equation is the derivative of $y \sin x$, so we can write

$$\frac{d}{dx} (y \sin x) = \csc x$$

Integrate the equation above.

$$\begin{aligned}
 y \sin x &= \int \csc x dx \\
 &= \ln |\csc x - \cot x| + C
 \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{\ln |\csc x - \cot x| + C}{\sin x}$$

where C is a constant.

20. The differential equation $\frac{dy}{dx} + y \tan x = \cos^2 x$ is first-order linear. Let $P(x) = \tan x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int \tan x dx} \\ &= e^{\ln(\sec x)} \\ &= \sec x \end{aligned}$$

Now multiply the differential equation by $\sec x$ and obtain

$$\sec x \frac{dy}{dx} + y \sec x \tan x = \cos x$$

The left side of the above equation is the derivative of $y \sec x$, so we can write

$$\frac{d}{dx} (y \sec x) = \cos x$$

Integrate the equation above.

$$\begin{aligned} y \sec x &= \int \cos x dx \\ &= \sin x + C \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = \sin x \cos x + C \cos x}$$

where C is a constant.

21. Given the Bernoulli equation $\frac{dy}{dx} + x^{-1}y = 3x^2y^3$, with $n = 3$, multiply the equation by y^{-3} to obtain

$$y^{-3} \frac{dy}{dx} + x^{-1}y^{-2} = 3x^2$$

Let $v = y^{-2}$, so $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2} \frac{dv}{dx} + x^{-1}v &= 3x^2 \\ \frac{dv}{dx} - 2x^{-1}v &= -6x^2 \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{-2 \int x^{-1} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}$$

Now multiply the differential equation by x^{-2} and obtain

$$x^{-2} \frac{dv}{dx} - 2x^{-3}v = -6$$

The left side of the above equation is the derivative of $x^{-2}v$, so we can write

$$\frac{d}{dx} (x^{-2}v) = -6$$

Integrate the equation above.

$$\begin{aligned} x^{-2}v &= -\int 6 dx \\ &= -6x + C \end{aligned}$$

Solve for v .

$$v = -6x^3 + Cx^2$$

Using $v = y^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} y^{-2} &= -6x^3 + Cx^2 \\ y^2 &= \frac{1}{-6x^3 + Cx^2} \\ y &= \pm \sqrt{\frac{1}{-6x^3 + Cx^2}} \\ y &= \pm \frac{1}{\sqrt{x^2(C - 6x)}} \\ y &= \pm \frac{1}{x\sqrt{C - 6x}} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \pm \frac{1}{x\sqrt{C - 6x}}}$$

22. Given the Bernoulli equation $\frac{dy}{dx} + x^{-1}y = \frac{2}{3}x^2y^4$, with $n = 4$, multiply the equation by y^{-4} to obtain

$$y^{-4}\frac{dy}{dx} + x^{-1}y^{-3} = \frac{2}{3}x^2$$

Let $v = y^{-3}$, so $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{3}\frac{dv}{dx} + x^{-1}v &= \frac{2}{3}x^2 \\ \frac{dv}{dx} - 3x^{-1}v &= -2x^2 \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{-3\int x^{-1}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

Now multiply the differential equation by x^{-3} and obtain

$$x^{-3}\frac{dv}{dx} - 3x^{-4}v = -2x^{-1}$$

The left side of the above equation is the derivative of $x^{-3}v$, so we can write

$$\frac{d}{dx}(x^{-3}v) = -2x^{-1}$$

Integrate the equation above.

$$\begin{aligned} x^{-3}v &= -2\int x^{-1}dx \\ x^{-3}v &= -2\ln|x| + C \\ v &= -2x^3\ln|x| + Cx^3 \end{aligned}$$

Using $v = y^{-3}$, rewrite the solution in terms of x and y , and solve for y .

$$\begin{aligned} y^{-3} &= -2x^3\ln|x| + Cx^3 \\ y^3 &= \frac{1}{-2x^3\ln|x| + Cx^3} \\ y &= \frac{1}{\sqrt[3]{-2x^3\ln|x| + Cx^3}} \end{aligned}$$

The general solution of the differential equation is

$$y = \frac{1}{\sqrt[3]{-2x^3 \ln |x| + Cx^3}}$$

23. Given the Bernoulli equation $\frac{dy}{dx} + 2xy = xy^4$, with $n = 4$, multiply the equation by y^{-4} to obtain

$$y^{-4} \frac{dy}{dx} + 2xy^{-3} = x$$

Let $v = y^{-3}$, so $\frac{dv}{dx} = -3y^{-4} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{3} \frac{dv}{dx} + 2xv &= x \\ \frac{dv}{dx} - 6xv &= -3x \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{-6 \int x dx} = e^{-3x^2}$$

Now multiply the differential equation by e^{-3x^2} and obtain

$$e^{-3x^2} \frac{dv}{dx} - 6xe^{-3x^2} v = -3xe^{-3x^2}$$

The left side of the above equation is the derivative of $e^{-3x^2} v$, so we can write

$$\frac{d}{dx} (e^{-3x^2} v) = -3xe^{-3x^2}$$

Integrate the equation above.

$$\begin{aligned} e^{-3x^2} v &= - \int 3xe^{-3x^2} dx \quad \text{Use } u = e^{-3x^2} \\ e^{-3x^2} v &= \frac{1}{2} e^{-3x^2} + C_1 \\ v &= \frac{1}{2} + C_1 e^{3x^2} \end{aligned}$$

Using $v = y^{-3}$, rewrite the solution in terms of x and y , and solve for y .

$$\begin{aligned} y^{-3} &= 0.5 + C_1 e^{3x^2} \\ y^3 &= \frac{1}{0.5 + C_1 e^{3x^2}} \\ y &= \sqrt[3]{\frac{1}{0.5 + C_1 e^{3x^2}}} \\ y &= \sqrt[3]{\frac{2}{1 + 2C_1 e^{3x^2}}} \\ y &= \sqrt[3]{\frac{2}{1 + C e^{3x^2}}} \end{aligned}$$

The general solution of the differential equation is

$$y = \sqrt[3]{\frac{2}{1 + C e^{3x^2}}}$$

24. Given the Bernoulli equation $\frac{dy}{dx} + \frac{1}{x}y = 3xy^3$, with $n = 3$, multiply the equation by y^{-3} to obtain

$$y^{-3} \frac{dy}{dx} + \frac{1}{x} y^{-2} = 3x$$

Let $v = y^{-2}$, so $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2} \frac{dv}{dx} + \frac{1}{x} v &= 3x \\ \frac{dv}{dx} - 2x^{-1} v &= -6x \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{-\int 2x^{-1} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}$$

Now multiply the differential equation by x^{-2} and obtain

$$x^{-2} \frac{dv}{dx} - 2x^{-3} v = -6x^{-1}$$

The left side of the above equation is the derivative of $x^{-2}v$, so we can write

$$\frac{d}{dx} (x^{-2}v) = -6x^{-1}$$

Integrate the equation above.

$$\begin{aligned} x^{-2}v &= -\int 6x^{-1} dx \\ x^{-2}v &= -6 \ln |x| + C \\ v &= -6x^2 \ln |x| + Cx^2 \end{aligned}$$

Using $v = y^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} y^{-2} &= -6x^2 \ln |x| + Cx^2 \\ y^2 &= \frac{1}{-6x^2 \ln |x| + Cx^2} \\ y &= \pm \frac{1}{\sqrt{-6x^2 \ln |x| + Cx^2}} \\ y &= \pm \frac{1}{\sqrt{x^2(C - 6 \ln |x|)}} \\ y &= \pm \frac{1}{x\sqrt{C - 6 \ln |x|}} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \pm \frac{1}{x\sqrt{C - 6 \ln |x|}}}$$

25. Given the Bernoulli equation $\frac{dy}{dx} + \frac{1}{x}y = 3xy^{1/3}$, with $n = \frac{1}{3}$, multiply the equation by $y^{-1/3}$ to obtain

$$y^{-1/3} \frac{dy}{dx} + \frac{1}{x} y^{2/3} = 3x$$

Let $v = y^{2/3}$, so $\frac{dv}{dx} = \frac{2}{3}y^{-1/3}\frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned}\frac{3}{2}\frac{dv}{dx} + \frac{1}{x}v &= 3x \\ \frac{dv}{dx} + \frac{2}{3x}v &= 2x\end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{\int (2/3) \frac{1}{x} dx} = e^{(2/3) \ln x} = e^{\ln x^{2/3}} = x^{2/3}$$

Now multiply the differential equation by $x^{2/3}$ and obtain

$$x^{2/3}\frac{dv}{dx} + \frac{2}{3}x^{-1/3}v = 2x^{5/3}$$

The left side of the above equation is the derivative of $x^{2/3}v$, so we can write

$$\frac{d}{dx}(x^{2/3}v) = 2x^{5/3}$$

Integrate the equation above.

$$\begin{aligned}x^{2/3}v &= \int 2x^{5/3} dx \\ x^{2/3}v &= \frac{3}{4}x^{8/3} + C \\ v &= \frac{3}{4}x^2 + Cx^{-2/3}\end{aligned}$$

Using $v = y^{2/3}$, rewrite the solution in terms of x and y . The general solution of the differential equation is

$$\boxed{y^{2/3} = \frac{3}{4}x^2 + Cx^{-2/3}}$$

26. Given the Bernoulli equation $\frac{dy}{dx} + \frac{4y}{x} = xy^{1/2}$, with $n = \frac{1}{2}$, multiply the equation by $y^{-1/2}$ to obtain

$$y^{-1/2}\frac{dy}{dx} + 4x^{-1}y^{1/2} = x$$

Let $v = y^{1/2}$, so $\frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned}2\frac{dv}{dx} + 4x^{-1}v &= x \\ \frac{dv}{dx} + 2x^{-1}v &= \frac{1}{2}x\end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{\int 2x^{-1} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

Now multiply the differential equation by x^2 and obtain

$$x^2\frac{dv}{dx} + 2xv = \frac{1}{2}x^3$$

The left side of the above equation is the derivative of x^2v , so we can write

$$\frac{d}{dx}(x^2v) = \frac{1}{2}x^3$$

Integrate the equation above.

$$\begin{aligned}x^2v &= \int \frac{1}{2}x^3 dx \\x^2v &= \frac{1}{8}x^4 + C \\v &= \frac{1}{8}x^2 + Cx^{-2}\end{aligned}$$

Using $v = y^{1/2}$, rewrite the solution in terms of x and y , and solve for y .

$$\begin{aligned}y^{1/2} &= \frac{1}{8}x^2 + Cx^{-2} \\y &= \left(\frac{1}{8}x^2 + Cx^{-2}\right)^2\end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \left(\frac{1}{8}x^2 + Cx^{-2}\right)^2}$$

Applications and Extensions

27. Write the differential equation $x \frac{dy}{dx} - y = x^2e^x$ in the form

$$\frac{dy}{dx} - \frac{y}{x} = xe^x$$

This is a first-order linear differential equation. Let $P(x) = -\frac{1}{x}$. The integrating factor is

$$\begin{aligned}e^{\int P(x) dx} &= \exp \left[-\int \frac{1}{x} dx \right] \\&= \exp [-\ln x] \\&= \exp [\ln x^{-1}] \\&= x^{-1}\end{aligned}$$

Now multiply the differential equation by x^{-1} and obtain

$$x^{-1} \frac{dy}{dx} - x^{-2}y = e^x$$

The left side of the above equation is the derivative of $x^{-1}y$, so we can write

$$\frac{d}{dx} (x^{-1}y) = e^x$$

Integrate the equation above.

$$\begin{aligned}x^{-1}y &= \int e^x dx \\&= e^x + C\end{aligned}$$

Solve for y . The general solution of the differential equation is

$$\boxed{y = x(e^x + C)}$$

where C is a constant.

28. Write the differential equation $\frac{dy}{dx} = \frac{y}{x - y^2}$ in the form

$$y \, dx + (y^2 - x) \, dy = 0$$

Let $M = y$ and $N = y^2 - x$. Since $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$, then

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{2}{y}$$

is a function of y only (see Section 3, Problem 27). Let

$$u(y) = e^{\int -\frac{2}{y} \, dy} = e^{-2 \ln(y)} = e^{\ln(y^{-2})} = y^{-2} = \frac{1}{y^2}$$

Multiply the differential equation by the integrating factor $u(y)$. Then the equation

$$\frac{1}{y} \, dx + \left(1 - \frac{x}{y^2}\right) \, dy = 0$$

is exact. Let $M_1 = \frac{1}{y}$ and $N_1 = 1 - \frac{x}{y^2}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int \frac{1}{y} \, dx = \frac{x}{y} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\frac{x}{y} + B(y) \right] = -\frac{x}{y^2} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N_1 \\ -\frac{x}{y^2} + \frac{\partial B}{\partial y} &= 1 - \frac{x}{y^2} \\ \frac{\partial B}{\partial y} &= 1 \end{aligned}$$

Now we integrate $\frac{dB}{dy} = 1$ with respect to y .

$$B(y) = \int 1 \, dy = y + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = \frac{x}{y} + y + C$$

The general solution of the differential equation is

$$\boxed{\frac{x}{y} + y + C = 0}$$

where C is a constant.

29. Given the differential equation $dx + (2x - y^2) dy = 0$, let $M = 1$ and $N = 2x - y^2$. Since $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 2$, then

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = 2$$

is a function of y only (see Section 3, Problem 27). Let

$$u(y) = e^{[2 dy]} = e^{2y}$$

Then the equation

$$e^{2y} dx + e^{2y}(2x - y^2) dy = 0$$

is exact. Let $M_1 = e^{2y}$ and $N_1 = e^{2y}(2x - y^2) = 2xe^{2y} - y^2e^{2y}$. Find f by integrating $\frac{\partial f}{\partial x} = M_1$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int e^{2y} dx = xe^{2y} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [xe^{2y} + B(y)] = 2xe^{2y} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N_1$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N_1 \\ 2xe^{2y} + \frac{\partial B}{\partial y} &= 2xe^{2y} - y^2e^{2y} \\ \frac{\partial B}{\partial y} &= -y^2e^{2y} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = -y^2e^{2y}$ with respect to y , using integration by parts twice.

$$B(y) = - \int y^2 e^{2y} dy = \left(-\frac{1}{2}y^2 + \frac{1}{2}y - \frac{1}{4} \right) e^{2y} + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = \left(x - \frac{1}{2}y^2 + \frac{1}{2}y - \frac{1}{4} \right) e^{2y} + C$$

The general solution of the differential equation is

$$\left(x - \frac{1}{2}y^2 + \frac{1}{2}y - \frac{1}{4} \right) e^{2y} + C = 0$$

or

$$\boxed{x - \frac{1}{2}y^2 + \frac{1}{2}y - \frac{1}{4} + Ce^{-2y} = 0}$$

where C is a constant.

30. Consider the differential equation $(2x + y) dx - x dy = 0$. Both $2x + y$ and $-x$ are homogeneous of degree 1. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}(2x + xv) dx - x(x dv + v dx) &= 0 \\ 2x dx - x^2 dv &= 0 \\ \frac{2}{x} dx - dv &= 0 \quad \text{when } x \neq 0 \\ \int \frac{2}{x} dx - \int dv &= 0 \\ 2 \ln |x| - v + C &= 0 \\ 2 \ln |x| - \frac{y}{x} + C &= 0 \\ 2x \ln |x| - y + Cx &= 0 \\ y &= 2x \ln |x| + Cx\end{aligned}$$

Then the general solution is $\boxed{y = 2x \ln |x| + Cx}$, where C is a constant.

31. Write the differential equation $\cos x \frac{dy}{dx} + y = \sec x$ in the form

$$\frac{dy}{dx} + y \sec x = \sec^2 x$$

The differential equation is first-order linear. Let $P(x) = \sec x$. The integrating factor is

$$\begin{aligned}e^{\int P(x) dx} &= e^{\int \sec x dx} \\ &= e^{\ln(\sec x + \tan x)} \\ &= \sec x + \tan x\end{aligned}$$

Now multiply the differential equation by $\sec x + \tan x$ and obtain

$$(\sec x + \tan x) \frac{dy}{dx} + y(\sec^2 x + \sec x \tan x) = \sec^3 x + \sec^2 x \tan x$$

The left side of the above equation is the derivative of $(\sec x + \tan x)y$, so we can write

$$\frac{d}{dx} ((\sec x + \tan x)y) = \sec^3 x + \sec^2 x \tan x$$

Integrate the equation above.

$$(\sec x + \tan x)y = \int (\sec^3 x + \sec^2 x \tan x) dx$$

The first integral can be done using integration by parts. Let $u = \sec x$ and $dv = \sec^2 x dx$. Then

$$\begin{aligned}\int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ 2 \int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \\ 2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| \\ \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|\end{aligned}$$

The second integral can be done using the substitution $u = \tan x$.

$$\begin{aligned}
 (\sec x + \tan x)y &= \int (\sec^3 x + \sec^2 x \tan x) dx \\
 &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + \frac{1}{2} \tan^2 x + C_1 \\
 &= \frac{\sec x \tan x + \ln |\sec x + \tan x| + \tan^2 x + C}{2} \quad \text{where } C = 2C_1
 \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{\sec x \tan x + \ln |\sec x + \tan x| + \tan^2 x + C}{2(\sec x + \tan x)}$$

where C is a constant.

32. Write the differential equation $(1 + x^2) \frac{dy}{dx} + xy = x^3$ in the form

$$\frac{dy}{dx} + \frac{x}{x^2 + 1}y = \frac{x^3}{x^2 + 1}$$

The differential equation is first-order linear. Let $P(x) = \frac{x}{x^2 + 1}$. The integrating factor is

$$\begin{aligned}
 e^{\int P(x) dx} &= \exp \left[\int \frac{x}{x^2 + 1} dx \right] \\
 &= \exp \left[\frac{1}{2} \ln(x^2 + 1) \right] \\
 &= \exp \left[\ln(\sqrt{x^2 + 1}) \right] \\
 &= \sqrt{x^2 + 1}
 \end{aligned}$$

Now multiply the differential equation by $\sqrt{x^2 + 1}$ and obtain

$$\sqrt{x^2 + 1} \frac{dy}{dx} + \frac{x}{\sqrt{x^2 + 1}}y = \frac{x^3}{\sqrt{x^2 + 1}}$$

The left side of the above equation is the derivative of $\sqrt{x^2 + 1}y$, so we can write

$$\frac{d}{dx} \left(\sqrt{x^2 + 1}y \right) = \frac{x^3}{\sqrt{x^2 + 1}}$$

Integrate the equation above.

$$\begin{aligned}
 \sqrt{x^2 + 1}y &= \int \frac{x^3}{\sqrt{x^2 + 1}} dx \quad \text{Let } u = x^2 + 1 \\
 &= \int \frac{u - 1}{2\sqrt{u}} du \\
 &= \int \left(\frac{1}{2}u^{1/2} - \frac{1}{2}u^{-1/2} \right) du \\
 &= \frac{1}{3}u^{3/2} - u^{1/2} + C \\
 &= \frac{1}{3}(x^2 + 1)^{3/2} - (x^2 + 1)^{1/2} + C
 \end{aligned}$$

Solve for y . The general solution of the differential equation is

$$y = \frac{1}{3}x^2 - \frac{2}{3} + \frac{C}{\sqrt{x^2 + 1}}$$

where C is a constant.

33. Write the differential equation $dy + y dx = 2xy^2 e^x dx$ in the form

$$\frac{dy}{dx} + y = 2xy^2 e^x$$

to recognize that it is a Bernoulli equation with $n = 2$. Multiply the equation by y^{-2} to obtain

$$y^{-2} \frac{dy}{dx} + y^{-1} = 2xe^x$$

Let $v = y^{-1}$, so $\frac{dv}{dx} = -y^{-2} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{dv}{dx} + v &= 2xe^x \\ \frac{dv}{dx} - v &= -2xe^x \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{\int (-1) dx} = e^{-x}$$

Now multiply the differential equation by e^{-x} and obtain

$$e^{-x} \frac{dv}{dx} - e^{-x} v = -2x$$

The left side of the above equation is the derivative of $e^{-x}v$, so we can write

$$\frac{d}{dx}(e^{-x}v) = -2x$$

Integrate the equation above.

$$\begin{aligned} e^{-x}v &= -\int 2x dx \\ e^{-x}v &= -x^2 + C \\ v &= -x^2 e^x + C e^x \end{aligned}$$

Using $v = y^{-1}$, rewrite the solution in terms of x and y , and solve for y .

$$\begin{aligned} y^{-1} &= -x^2 e^x + C e^x \\ y &= \frac{1}{-x^2 e^x + C e^x} \\ y &= \frac{1}{e^x(C - x^2)} \end{aligned}$$

The general solution of the differential equation is

$$y = \frac{1}{e^x(C - x^2)}$$

34. Write the differential equation $dx + 2xy^{-1} dy = 2x^2y^2 dy$ in the form

$$\frac{dx}{dy} + 2xy^{-1} = 2x^2y^2$$

to recognize that it is a Bernoulli equation with $n = 2$. Multiply the equation by x^{-2} to obtain

$$x^{-2} \frac{dx}{dy} + 2x^{-1}y^{-1} = 2y^2$$

Let $v = x^{-1}$, so $\frac{dv}{dy} = -x^{-2} \frac{dx}{dy}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{dv}{dy} + 2y^{-1}v &= 2y^2 \\ \frac{dv}{dy} - 2y^{-1}v &= -2y^2 \end{aligned}$$

This is a first-order linear differential equation in y and v . The integrating factor is

$$e^{-\int 2y^{-1} dy} = e^{-2 \ln y} = e^{\ln y^{-2}} = y^{-2}$$

Now multiply the differential equation by y^{-2} and obtain

$$y^{-2} \frac{dv}{dy} - 2y^{-3}v = -2$$

The left side of the above equation is the derivative of $y^{-2}v$, so we can write

$$\frac{d}{dy} (y^{-2}v) = -2$$

Integrate the equation above.

$$\begin{aligned} y^{-2}v &= \int (-2) dy \\ y^{-2}v &= -2y + C \\ v &= -2y^3 + Cy^2 \end{aligned}$$

Using $v = x^{-1}$, rewrite the solution in terms of x and y .

$$\begin{aligned} x^{-1} &= -2y^3 + Cy^2 \\ x &= \frac{1}{-2y^3 + Cy^2} \\ -2xy^3 + Cxy^2 &= 1 \end{aligned}$$

The general solution of the differential equation is

$$\boxed{2xy^3 - Cxy^2 + 1 = 0}$$

35. Write the differential equation $2 \frac{dy}{dx} - yx^{-1} = 5x^3y^3$ in the form

$$\frac{dy}{dx} - \frac{1}{2}yx^{-1} = \frac{5}{2}x^3y^3$$

to recognize that it is a Bernoulli equation with $n = 3$. Multiply the equation by y^{-3} to obtain

$$y^{-3} \frac{dy}{dx} - \frac{1}{2}x^{-1}y^{-2} = \frac{5}{2}x^3$$

Let $v = y^{-2}$, so $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2} \frac{dv}{dx} - \frac{1}{2} x^{-1} v &= \frac{5}{2} x^3 \\ \frac{dv}{dx} + x^{-1} v &= -5x^3 \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{\int x^{-1} dx} = e^{\ln x} = x$$

Now multiply the differential equation by x and obtain

$$x \frac{dv}{dx} + v = -5x^4$$

The left side of the above equation is the derivative of xv , so we can write

$$\frac{d}{dx}(xv) = -5x^4$$

Integrate the equation above.

$$\begin{aligned} xv &= -\int 5x^4 dx \\ xv &= -x^5 + C \\ v &= \frac{C - x^5}{x} \end{aligned}$$

Using $v = y^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} y^{-2} &= \frac{C - x^5}{x} \\ y^2 &= \frac{x}{C - x^5} \\ y &= \pm \sqrt{\frac{x}{C - x^5}} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \pm \sqrt{\frac{x}{C - x^5}}}$$

36. Write the differential equation $dx - 2xy dy = 6x^3 y^2 e^{-2y^2} dy$ in the form

$$\frac{dx}{dy} - 2xy = 6x^3 y^2 e^{-2y^2}$$

to recognize that it is a Bernoulli equation with $n = 3$. Multiply the equation by x^{-3} to obtain

$$x^{-3} \frac{dx}{dy} - 2x^{-2} y = 6y^2 e^{-2y^2}$$

Let $v = x^{-2}$, so $\frac{dv}{dy} = -2x^{-3} \frac{dx}{dy}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2} \frac{dv}{dy} - 2yv &= 6y^2 e^{-2y^2} \\ \frac{dv}{dy} + 4yv &= -12y^2 e^{-2y^2} \end{aligned}$$

This is a first-order linear differential equation in y and v . The integrating factor is

$$e^{\int 4y \, dy} = e^{2y^2}$$

Now multiply the differential equation by e^{2y^2} and obtain

$$e^{2y^2} \frac{dv}{dy} + 4ye^{2y^2}v = -12y^2$$

The left side of the above equation is the derivative of $e^{2y^2}v$, so we can write

$$\frac{d}{dy} (e^{2y^2}v) = -12y^2$$

Integrate the equation above.

$$\begin{aligned} e^{2y^2}v &= -\int 12y^2 \, dy \\ e^{2y^2}v &= -4y^3 + C \\ v &= (C - 4y^3)e^{-2y^2} \end{aligned}$$

Using $v = x^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} x^{-2} &= (C - 4y^3)e^{-2y^2} \\ x^2 &= \frac{e^{2y^2}}{C - 4y^3} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{x^2 = \frac{e^{2y^2}}{C - 4y^3}}$$

37. Write the differential equation $(x^2 + 2y^2)\frac{dx}{dy} + xy = 0$ in the form

$$(x^2 + 2y^2) \, dx + xy \, dy = 0$$

Both $x^2 + 2y^2$ and xy are homogeneous of degree 2. Let $y = xv$. Then $dy = x \, dv + v \, dx$. Substitute these into the differential equation:

$$\begin{aligned} (x^2 + 2x^2v^2) \, dx + x^2v(x \, dv + v \, dx) &= 0 \\ (x^2 + 3x^2v^2) \, dx + x^3v \, dv &= 0 \\ x^2(1 + 3v^2) \, dx + x^3v \, dv &= 0 \\ \frac{1}{x} \, dx + \frac{v}{1 + 3v^2} \, dv &= 0 \quad \text{Divide by } x^3(1 + 3v^2) \, (x \neq 0) \\ \int \frac{1}{x} \, dx + \int \frac{v}{1 + 3v^2} \, dv &= 0 \\ \ln x + \frac{1}{6} \ln(1 + 3v^2) &= C_1 \\ 6 \ln x + \ln(1 + 3v^2) &= 6C_1 \\ \ln x^6 + \ln(1 + 3v^2) &= 6C_1 \\ \ln(x^6(1 + 3v^2)) &= 6C_1 \\ x^6(1 + 3v^2) &= e^{6C_1} \quad \text{Let } C = e^{6C_1} \neq 0 \\ x^6 \left(1 + \frac{3y^2}{x^2}\right) &= C \\ x^6 + 3x^4y^2 &= C \end{aligned}$$

It can be shown that $x = 0$ is a solution, and that $x = 0$ corresponds to $C = 0$. So C can be any constant. Then the general solution is $\boxed{x^6 + 3x^4y^2 = C}$, where C is a constant.

38. Write the differential equation $(3x^2y^2 - e^yx^4y)\frac{dy}{dx} = 2xy^3$ in the form

$$\frac{dx}{dy} - \frac{3}{2}xy^{-1} = -\frac{1}{2}x^3y^{-2}e^y$$

to recognize that it is a Bernoulli equation with $n = 3$. Multiply the equation by x^{-3} to obtain

$$x^{-3}\frac{dx}{dy} - \frac{3}{2}x^{-2}y^{-1} = -\frac{1}{2}y^{-2}e^y$$

Let $v = x^{-2}$, so $\frac{dv}{dy} = -2x^{-3}\frac{dx}{dy}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2}\frac{dv}{dy} - \frac{3}{2}y^{-1}v &= -\frac{1}{2}y^{-2}e^y \\ \frac{dv}{dy} + 3y^{-1}v &= y^{-2}e^y \end{aligned}$$

This is a first-order linear differential equation in y and v . The integrating factor is

$$e^{\int 3y^{-1} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3$$

Now multiply the differential equation by y^3 and obtain

$$y^3\frac{dv}{dy} + 3y^2v = ye^y$$

The left side of the above equation is the derivative of y^3v , so we can write

$$\frac{d}{dy}(y^3v) = ye^y$$

Integrate the equation above using the technique of integration by parts.

$$\begin{aligned} y^3v &= \int ye^y dx \\ y^3v &= (y-1)e^y + C \\ v &= \frac{(y-1)e^y + C}{y^3} \end{aligned}$$

Using $v = x^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} x^{-2} &= \frac{(y-1)e^y + C}{y^3} \\ x^2 &= \frac{y^3}{(y-1)e^y + C} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{x^2 = \frac{y^3}{(y-1)e^y + C}}$$

39. The differential equation $\frac{dy}{dx} + y = e^{-x}$ is first-order linear. Let $P(x) = 1$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

Now multiply the differential equation by e^x and obtain

$$e^x \frac{dy}{dx} + e^x y = 1$$

The left side of the above equation is the derivative of $e^x y$, so we can write

$$\frac{d}{dx} (e^x y) = 1$$

Integrate the equation above.

$$\begin{aligned} e^x y &= \int 1 dx \\ &= x + C \end{aligned}$$

Substitute $x = 0$ and $y = 5$ to find $C = 5$. Solve for y . The particular solution of the differential equation is

$$\boxed{y = \frac{x + 5}{e^x}}$$

40. The differential equation $\frac{dy}{dx} + \frac{2y}{x} = \frac{4}{x}$ is first-order linear. Let $P(x) = \frac{2}{x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int 2x^{-1} dx} \\ &= e^{2 \ln x} \\ &= e^{\ln x^2} \\ &= x^2 \end{aligned}$$

Now multiply the differential equation by x^2 and obtain

$$x^2 \frac{dy}{dx} + 2xy = 4x$$

The left side of the above equation is the derivative of $x^2 y$, so we can write

$$\frac{d}{dx} (x^2 y) = 4x$$

Integrate the equation above.

$$\begin{aligned} x^2 y &= \int 4x dx \\ &= 2x^2 + C \end{aligned}$$

Substitute $x = 1$ and $y = 6$ to find $C = 4$. Solve for y . The particular solution of the differential equation is

$$\boxed{y = 2 + 4x^{-2}}$$

41. The differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$ is first-order linear. Let $P(x) = \frac{1}{x}$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int x^{-1} dx} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

Now multiply the differential equation by x and obtain

$$x \frac{dy}{dx} + y = xe^x$$

The left side of the above equation is the derivative of xy , so we can write

$$\frac{d}{dx}(xy) = xe^x$$

Integrate the equation above using the technique of integration by parts.

$$\begin{aligned} xy &= \int xe^x dx \\ &= xe^x - e^x + C \end{aligned}$$

Substitute $x = -1$ and $y = e^{-1}$ to find $C = e^{-1}$. Solve for y . The particular solution of the differential equation is

$$y = e^x - \frac{e^x}{x} + \frac{e^{-1}}{x}$$

42. The differential equation $\frac{dy}{dx} + y \cot x = 2 \cos x$ is first-order linear. Let $P(x) = \cot x$. The integrating factor is

$$\begin{aligned} e^{\int P(x) dx} &= e^{\int \cot x dx} \\ &= e^{\ln(\sin x)} \\ &= \sin x \end{aligned}$$

Now multiply the differential equation by $\sin x$ and obtain

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin x \cos x = \sin 2x$$

The left side of the above equation is the derivative of $y \sin x$, so we can write

$$\frac{d}{dx}(y \sin x) = \sin 2x$$

Integrate the equation above.

$$\begin{aligned} y \sin x &= \int \sin 2x dx \\ &= -\frac{1}{2} \cos 2x + C \end{aligned}$$

Substitute $x = \frac{\pi}{2}$ and $y = 3$ to find $C = \frac{5}{2}$. Solve for y . The particular solution of the differential equation is

$$y = -\frac{1}{2} \cos 2x \csc x + \frac{5}{2} \csc x$$

43. Consider the differential equation $\frac{dy}{dx} + \frac{3y}{x} = \sin x$.

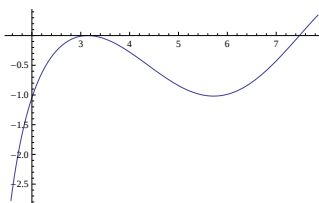
(a) The general solution of the differential equation is

$$y = \frac{-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x + C}{x^3}$$

(b) The particular solution that satisfies the boundary condition $y(\pi) = 0$ is

$$y = \frac{-x^3 \cos x + 3x^2 \sin x - 6 \sin x + 6x \cos x + 6\pi - \pi^3}{x^3}$$

(c) The graph of the particular solution over the interval $[\pi/2, 5\pi/2]$ is shown below.



44. Consider the differential equation $x \frac{dy}{dx} + 2y = \frac{5 \sin x}{x}$.

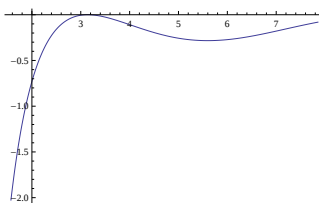
(a) The general solution of the differential equation is

$$y = \frac{C - 5 \cos x}{x^2}$$

(b) The particular solution that satisfies the boundary condition $y(\pi) = 0$ is

$$y = \frac{-5 - 5 \cos x}{x^2}$$

(c) The graph of the particular solution over the interval $[\pi/2, 5\pi/2]$ is shown below.



45. The derivative of position is velocity, that is

$$\frac{ds}{dt} = v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-(k/m)t}$$

Integrate to find the position.

$$\begin{aligned} s(t) &= \int \left(-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-(k/m)t}\right) dt \\ &= -\frac{mg}{k}t - \frac{m}{k} \left(v_0 + \frac{mg}{k}\right) e^{-(k/m)t} + C \\ &= -\frac{mg}{k}t - \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2}\right) e^{-(k/m)t} + C \end{aligned}$$

Use $s(0) = s_0$ to find C .

$$\begin{aligned}s_0 &= 0 - \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2} \right) e^0 + C \\s_0 &= - \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2} \right) + C \\s_0 + \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2} \right) &= C\end{aligned}$$

The position of a freely falling object at any time t is

$$s(t) = s_0 - \frac{mg}{k}t + \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2} \right) \left(1 - e^{-(k/m)t} \right)$$

46. The mass of the object in grams is $m = 2000$, the initial velocity in m/s is $v_0 = 0$ (because the object is dropped from rest), the initial height in meters is 2000, the acceleration due to gravity in m/s² is $g = 9.8$, and the constant of proportionality is $k = \frac{1}{2} = 0.5$.

(a) The velocity is

$$\begin{aligned}v(t) &= -\frac{mg}{k} + \left(v_0 + \frac{mg}{k} \right) e^{-(k/m)t} \\&= -\frac{(2000)(9.8)}{0.5} + \left(0 + \frac{(2000)(9.8)}{0.5} \right) e^{-(0.5/2000)t} \\&= \boxed{-39,200 + 39,200e^{-0.00025t}}\end{aligned}$$

Note that $v \leq 0$ when $t \geq 0$. The distance the object has fallen at time t is

$$s(t) = \int_0^t |v(x)| dx = \int_0^t -v(x) dx$$

not to be confused with the height of the object. The distance s that the object has fallen at time t is

$$\begin{aligned}s(t) &= \int_0^t |v(x)| dx = \int_0^t -v(x) dx \\&= \int_0^t 39,200 - 39,200e^{-0.00025x} dx \\&= 39,200x + 156,800,000e^{0.00025x} \Big|_0^t \\&= (39,200t + 156,800,000e^{0.00025t}) - (0 + 156,800,000e^0) \\&= \boxed{-156,800,000 + 39,200t + 156,800,000e^{-0.00025t}}\end{aligned}$$

Alternatively, let $\bar{s}(t)$ represent the height of the object. Then the distance the object has fallen is $s(t) = \bar{s}(0) - \bar{s}(t)$. Using the formula found in Problem 45, the distance that the object has fallen at time t is

$$\begin{aligned}s(t) &= 2000 - \left[\bar{s}_0 - \frac{mg}{k}t + \left(\frac{v_0 m}{k} + \frac{m^2 g}{k^2} \right) \left(1 - e^{-(k/m)t} \right) \right] \\&= 2000 - \left[2000 - \frac{(2000)(9.8)}{0.5}t + \left(0 + \frac{(2000)^2(9.8)}{(0.5)^2} \right) \left(1 - e^{-(0.5/2000)t} \right) \right] \\&= -156,800,000 + 39,200t + 156,800,000e^{-0.00025t}\end{aligned}$$

(b) The limiting velocity of the object is

$$\lim_{t \rightarrow \infty} (-39,200 + 39,200e^{-0.00025t}) = \boxed{-39,200 \text{ m/s}^2}$$

(c) The object will strike Earth when it has fallen 2000m.

$$\begin{aligned} 2000 &= -156,800,000 + 39,200t + 156,800,000e^{-0.00025t} \\ t &\approx 20.22007 \end{aligned}$$

The object strikes the Earth with a velocity of $v(20.22007) = \boxed{-197.657 \text{ m/s}^2}$.

47. Let $s(t)$ and $v(t)$ represent the position and velocity of the skydiver t seconds after the initial fall, for $0 \leq t \leq 5$. The only acceleration during the free fall is due to gravity, so $a = \frac{dv}{dt} = -32$ and for $0 \leq t \leq 5$

$$\begin{aligned} v(t) &= -\int 32 \, dt \\ &= -32t + C_1 \\ &= -32t \end{aligned}$$

since her initial velocity is $v(0) = 0$. The position is

$$\begin{aligned} s(t) &= -\int 32t \, dt \\ &= -16t^2 + C_2 \\ &= -16t^2 + 10,000 \end{aligned}$$

for $0 \leq t \leq 5$ since her initial height is 10,000ft. Her velocity when the parachute is deployed is $v(5) = -160$ and her position when the parachute is deployed is $s(5) = 9600$.

Let $\bar{s}(t)$ and $\bar{v}(t)$ represent the position and velocity of the skydiver t seconds after the parachute is deployed. Then $\bar{s}_0 = \bar{s}(0) = s(5) = 9600$ and $\bar{v}_0 = \bar{v}(0) = v(5) = -160$. The constant $k = 4$ is given and the skydiver's weight gives $mg = 160$ and $m = \frac{160}{32} = 5$. Then use the formulas in Problem 45 to find

$$\begin{aligned} \bar{v}(t) &= -\frac{mg}{k} + \left(\bar{v}_0 + \frac{mg}{k}\right)e^{-(k/m)t} \\ &= -\frac{160}{4} + \left(-160 + \frac{160}{4}\right)e^{-(4/5)t} \\ &= -40 - 120e^{-0.8t} \end{aligned}$$

for $t \geq 0$ and

$$\begin{aligned} \bar{s}(t) &= \bar{s}_0 - \frac{mg}{k}t + \left(\frac{\bar{v}_0 m}{k} + \frac{m^2 g}{k^2}\right)\left(1 - e^{-(k/m)t}\right) \\ &= 9600 - \frac{160}{4}t + \left(\frac{(-160)(5)}{4} + \frac{(5)(160)}{4^2}\right)\left(1 - e^{-(4/5)t}\right) \\ &= 9450 - 40t + 150e^{-0.8t} \end{aligned}$$

for $t \geq 0$.

(a) Since $\bar{v}(4) = -40 - 120e^{-0.8(4)} \approx -44.891$, the skydiver will be falling at a rate of $\boxed{44.891 \text{ ft/sec}}$ four seconds after the parachute opens.

- (b) The skydiver lands when $\bar{s}(t) = 0$. Solve for t .

$$\begin{aligned} 0 &= 9450 - 40t + 150e^{-0.8t} \\ t &\approx 236.250 \end{aligned}$$

The skydiver lands approximately 241.250 seconds after the initial fall or 236.250 seconds after the parachute opened.

- (c) Find $\bar{v}(236.250) = -40 - 120e^{-0.8(236.250)} = -40$. The skydiver's velocity is -40ft/sec when she lands.

48. Let M denote the population, R denote the number of people who initiated the rumor, and $y = y(t)$ denote the number of people who have heard the rumor after t days. Then $M - y$ is the number of people who have not heard the rumor after t days. Let k denote the constant of proportionality. Then the differential equation

$$\frac{dy}{dt} = ky(M - y)$$

models the rate at which the rumor spreads for $t \geq 0$, and

$$y(t) = \frac{RM}{R + (M - R)e^{-kMt}}$$

models the function for $t \geq 0$. Rewrite this equation using $M = 5000$ and $R = 100$:

$$\begin{aligned} y(t) &= \frac{(100)(5000)}{100 + (5000 - 100)e^{-5000kt}} \\ &= \frac{500,000}{100 + 4900e^{-5000kt}} \end{aligned}$$

The number of people who have heard the rumor after 3 days is 500. Use $y(3) = 500$ to find k .

$$\begin{aligned} 500 &= \frac{500,000}{100 + 4900e^{-5000k(3)}} \\ 500(100 + 4900e^{-15,000k}) &= 500,000 \\ 100 + 4900e^{-15,000k} &= 1000 \\ 4900e^{-15,000k} &= 900 \\ e^{-15,000k} &= \frac{9}{49} \\ -15,000k &= \ln\left(\frac{9}{49}\right) \\ k &= -\frac{\ln(9/49)}{15,000} \end{aligned}$$

- (a) The differential equation

$$\frac{dy}{dt} = -\frac{\ln(9/49)}{15,000}y(5000 - y)$$

models the rate of the spread of the rumor for $t \geq 0$.

- (b) The solution of the differential equation is

$$y(t) = \frac{500,000}{100 + 4900e^{t \ln(9/49)/3}}$$

for $t \geq 0$.

(c) The number of people who have heard the rumor after 8 days is

$$y(8) = \frac{500,000}{100 + 4900e^{8 \ln(9/49)/3}} \approx \boxed{3259}$$

(d) To find how long it will take for half of the people to hear the rumor, solve $y(t) = 2500$ for t .

$$\begin{aligned} 2500 &= \frac{500,000}{100 + 4900e^{t \ln(9/49)/3}} \\ 2500 \left(100 + 4900e^{t \ln(9/49)/3} \right) &= 500,000 \\ 100 + 4900e^{t \ln(9/49)/3} &= 200 \\ 4900e^{t \ln(9/49)/3} &= 100 \\ e^{t \ln(9/49)/3} &= \frac{1}{49} \\ t \frac{\ln(9/49)}{3} &= \ln\left(\frac{1}{49}\right) \\ t &= \frac{3 \ln(1/49)}{\ln(9/49)} \approx 7 \end{aligned}$$

It will take about $\boxed{7 \text{ days}}$ for half of the people to hear the rumor.

49. Let $y = y(t)$ be the amount of salt in the tank at time t . Salt enters the tank at the rate of $(3 \text{ kg/L})(8 \text{ L/min}) = 24 \text{ kg/min}$. The number of gallons of liquid remains a constant 100 L, so the concentration of salt is $y/100 \text{ kg/L}$. Then the rate at which the salt exits the tank is $(y/100 \text{ kg/L})(8 \text{ L/min}) = 2y/25 \text{ kg/min}$. The mixture problem is modeled by the differential equation

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 24 - \frac{2y}{25}$$

for $t \geq 0$. This equation is a separable differential equation and it is first-order linear differential equation. To solve it as a first-order linear differential equation, rewrite it as $\frac{dy}{dt} + \frac{2}{25}y = 24$. The integrating factor is

$$e^{\int P(t) dt} = e^{\int 2/25 dt} = e^{2t/25}$$

Multiply the differential equation by the integrating factor to obtain

$$\begin{aligned} e^{2t/25} \frac{dy}{dt} + \frac{2}{25} e^{2t/25} y &= 24e^{2t/25} \\ \frac{d}{dt} \left[e^{2t/25} y \right] &= 24e^{2t/25} \end{aligned}$$

Integrate both sides with respect to t .

$$\begin{aligned} e^{2t/25} y &= \int 24e^{2t/25} dt \\ e^{2t/25} y &= 300e^{2t/25} + C \\ y &= 300 + Ce^{-2t/25} \end{aligned}$$

The tank initially held pure water, so $y(0) = 0$, which we use to find C .

$$\begin{aligned} 0 &= 300 + Ce^0 \\ 0 &= 300 + C \\ -300 &= C \end{aligned}$$

The amount of salt in kg at time $t \geq 0$ is $y(t) = 300 - 300e^{-2t/25}$. After 5 minutes, there is $y(5) = 300 - 300e^{-2(5)/25} \approx \boxed{98.904 \text{ kg}}$ of salt in the tank. The limit of $y(t)$ as t approaches infinity is

$$\lim_{t \rightarrow \infty} (300 - 300e^{-2t/25}) = 300$$

so after a very long time there is approximately $\boxed{300 \text{ kg}}$ of salt in the tank, when the concentration of the salt in the liquid is nearly equal to the concentration of the brine flowing into the tank.

50. Let M denote the population, R denote the number of infected students initially, and $y = y(t)$ denote the number of infected students after t days. Then $M - y$ is the number of noninfected students after t days. Let k denote the constant of proportionality. Then the differential equation

$$\frac{dy}{dt} = ky(M - y)$$

models the rate at which the infection spreads for $t \geq 0$, and

$$y(t) = \frac{RM}{R + (M - R)e^{-kMt}}$$

models the function for $t \geq 0$. Rewrite this equation using $M = 10,000$ and $R = 10$:

$$\begin{aligned} y(t) &= \frac{(10)(10,000)}{10 + (10,000 - 10)e^{-10,000kt}} \\ &= \frac{100,000}{10 + 9990e^{-10,000kt}} \end{aligned}$$

The number of students infected after 10 days is 200. Use $y(10) = 200$ to find k .

$$\begin{aligned} 200 &= \frac{100,000}{10 + 9990e^{-10,000k(10)}} \\ 200(10 + 9990e^{-100,000k}) &= 100,000 \\ 10 + 9990e^{-100,000k} &= 500 \\ 9990e^{-100,000k} &= 490 \\ e^{-100,000k} &= \frac{49}{999} \\ -100,000k &= \ln\left(\frac{49}{999}\right) \\ k &= -\frac{\ln(49/999)}{100,000} \end{aligned}$$

- (a) The differential equation

$$\frac{dy}{dt} = -\frac{\ln(49/999)}{100,000}y(10,000 - y)$$

models the rate of the spread of the infection for $t \geq 0$.

- (b) The solution of the differential equation is

$$y(t) = \frac{100,000}{10 + 9990e^{t \ln(49/999)/10}}$$

for $t \geq 0$.

- (c) The number of students infected after 5 days is

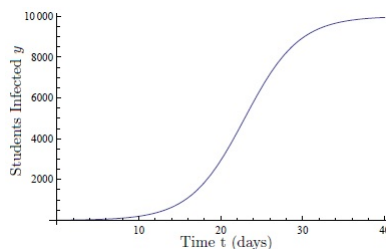
$$y(5) = \frac{100,000}{10 + 9990e^{5 \ln(49/999)/10}} \approx \boxed{45}$$

- (d) To find how long it will take for 75% of the students to be infected, solve $y(t) = 7500$ for t .

$$\begin{aligned}
 7500 &= \frac{100,000}{10 + 9990e^{t \ln(49/999)/10}} \\
 7500 \left(10 + 9990e^{t \ln(49/999)/10} \right) &= 100,000 \\
 10 + 9990e^{t \ln(49/999)/10} &= \frac{40}{3} \\
 9990e^{t \ln(49/999)/10} &= \frac{10}{3} \\
 e^{t \ln(49/999)/10} &= \frac{1}{2997} \\
 t \frac{\ln(49/999)}{10} &= \ln \left(\frac{1}{2997} \right) \\
 t &= \frac{10 \ln(1/2997)}{\ln(49/999)} \approx 27
 \end{aligned}$$

It will take about 27 days for 75% of the students to become infected.

- (e) The graph of $y = y(t)$ for the first 40 days of the flu epidemic is shown.



51. Rewrite the differential equation $\frac{dP}{dt} = P(10^{-2} - 10^{-6}P)$ in the form

$$\frac{dP}{dt} = 10^{-6}P(10^4 - P)$$

to recognize it as a logistic growth problem with $k = 10^{-6}$ and $M = 10^4$.

- (a) Given an initial population of $R = P(0) = 1500$, find the solution

$$\begin{aligned}
 P(t) &= \frac{RM}{R + (M - R)e^{-kMt}} \\
 &= \frac{(1500)(10^4)}{1500 + (10000 - 1500)e^{-(10^{-6})(10^4)t}} \\
 &= \frac{15,000,000}{1500 + 8500e^{-0.01t}} \\
 &= \frac{15,000,000}{500(3 + 17e^{-0.01t})} \\
 &= \frac{30,000}{3 + 17e^{-0.01t}}
 \end{aligned}$$

The population is modeled by the function

$$P(t) = \frac{30,000}{3 + 17e^{-0.01t}}$$

for $t \geq 0$.

(b) The limiting size of the population is $M = \boxed{10,000}$. It can be verified by the limit

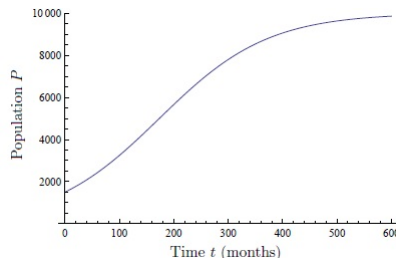
$$\lim_{t \rightarrow \infty} \frac{30,000}{3 + 17e^{-0.01t}} = \frac{30,000}{3} = 10,000$$

(c) To find when the population will equal one-half of the limiting value, solve $P(t) = 5000$.

$$\begin{aligned} 5000 &= \frac{30,000}{3 + 17e^{-0.01t}} \\ 5000(3 + 17e^{-0.01t}) &= 30,000 \\ 3 + 17e^{-0.01t} &= 6 \\ 17e^{-0.01t} &= 3 \\ e^{-0.01t} &= \frac{3}{17} \\ -0.01t &= \ln\left(\frac{3}{17}\right) \\ t &= -100 \ln\left(\frac{3}{17}\right) \approx 173 \end{aligned}$$

The population will be equal to one-half of the limiting value in approximately $\boxed{173 \text{ months}}$.

(d) The graph of the predicted population for $0 \leq t \leq 600$ (50 years) is shown.



52. Assuming E , R , and L are constants, the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

is a first-order linear differential equation. The integrating factor is

$$e^{\int P(t) dt} = e^{\int R/L dt} = e^{Rt/L}$$

Multiply the differential equation by the integrating factor.

$$\begin{aligned} e^{Rt/L} \frac{dI}{dt} + \frac{R}{L} e^{Rt/L} I &= \frac{E}{L} e^{Rt/L} \\ \frac{d}{dt} [e^{Rt/L} I] &= \frac{E}{L} e^{Rt/L} \end{aligned}$$

Integrate both sides of the differential equation with respect to t .

$$\begin{aligned} e^{Rt/L} I &= \int \frac{E}{L} e^{Rt/L} dt \\ e^{Rt/L} I &= \frac{E}{R} e^{Rt/L} + C \\ I &= \frac{E}{R} + C e^{-Rt/L} \end{aligned}$$

where C is a constant. Use the initial condition $I(0) = 0$ to find C .

$$\begin{aligned} 0 &= \frac{E}{R} + Ce^0 \\ 0 &= \frac{E}{R} + C \\ -\frac{E}{R} &= C \end{aligned}$$

The particular solution of the differential equation is

$$I(t) = \frac{E}{R} - \frac{E}{R}e^{-Rt/L}$$

53. Assuming E , R , and C are constants, the differential equation

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$$

is a first-order linear differential equation. The integrating factor is

$$e^{\int P(t) dt} = e^{\int 1/(RC) dt} = e^{t/(RC)}$$

Multiply the differential equation by the integrating factor.

$$\begin{aligned} e^{t/(RC)} \frac{dq}{dt} + \frac{1}{RC} e^{t/(RC)} q &= \frac{E}{R} e^{t/(RC)} \\ \frac{d}{dt} [e^{t/(RC)} q] &= \frac{E}{R} e^{t/(RC)} \end{aligned}$$

Integrate both sides of the differential equation with respect to t .

$$\begin{aligned} e^{t/(RC)} q &= \int \frac{E}{R} e^{t/(RC)} dt \\ e^{t/(RC)} q &= CE e^{t/(RC)} + K \\ q &= CE + K e^{-t/(RC)} \end{aligned}$$

where K is a constant. Use the initial condition $q(0) = 0$ to find K .

$$\begin{aligned} 0 &= CE + K e^0 \\ 0 &= CE + K \\ -CE &= K \end{aligned}$$

The particular solution of the differential equation is

$$q(t) = CE - CE e^{-t/(RC)} = CE \left(1 - e^{-t/(RC)} \right)$$

54. (a) The differential equation

$$\frac{dN}{dt} = kN(N_{\max} - N)$$

is a logistic differential equation so its solution is

$$N(t) = \frac{RN_{\max}}{R + (N_{\max} - R)e^{-kN_{\max}t}}$$

for $t \geq 0$.

- (b) Given an initial population of $R = 20$ and a maximum population of $N_{\max} = 400$, the logistic function becomes

$$N(t) = \frac{(20)(400)}{20 + (400 - 20)e^{-400kt}} = \frac{8000}{20 + 380e^{-400kt}} = \frac{400}{1 + 19e^{-400kt}}$$

The population doubles in 5 years, so use $N(5) = 40$ to find k .

$$\begin{aligned} 40 &= \frac{400}{1 + 19e^{-400k(5)}} \\ 40 &= \frac{400}{1 + 19e^{-2000k}} \\ 40(1 + 19e^{-2000k}) &= 400 \\ 1 + 19e^{-2000k} &= 10 \\ 19e^{-2000k} &= 9 \\ e^{-2000k} &= \frac{9}{19} \\ -2000k &= \ln\left(\frac{9}{19}\right) \\ k &= -\frac{\ln(9/19)}{2000} \end{aligned}$$

Now the logistic function is

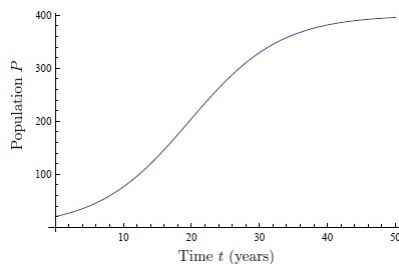
$$N(t) = \frac{400}{1 + 19e^{t \ln(9/19)/5}}$$

- (c) Since $N(43) = \frac{400}{1 + 19e^{43 \ln(9/19)/5}} \approx 388$, the population after 43 years as modeled will be approximately 388 fish.
- (d) The population is growing fastest when $\frac{dN}{dt}$ is at a maximum. Notice that by completing the square that the derivative equals

$$\begin{aligned} \frac{dN}{dt} &= kN(N_{\max} - N) \\ &= kNN_{\max} - kN^2 \\ &= -k\frac{N_{\max}^2}{4} + kNN_{\max} - kN^2 + k\frac{N_{\max}^2}{4} \\ &= -k\left(\frac{N_{\max}}{2} - N\right)^2 + k\frac{N_{\max}^2}{4} \end{aligned}$$

When $k > 0$ the maximum is when $\frac{N_{\max}}{2} - N = 0$ and when $k < 0$ the minimum is when $\frac{N_{\max}}{2} - N = 0$. Since $k > 0$, the maximum of the derivative and the moment when the population is growing fastest is when $N = \frac{N_{\max}}{2}$.

- (e) The graph of the predicted population $N = N(t)$ is shown.



55. (a) The differential equation

$$\frac{dN}{dt} = pN \ln \left(\frac{N_{\max}}{N} \right)$$

is separable. Separate the variables and integrate.

$$\begin{aligned} \frac{dN}{dt} &= pN \ln \left(\frac{N_{\max}}{N} \right) \\ \frac{dN}{dt} &= pN (\ln N_{\max} - \ln N) \\ \frac{1}{N (\ln N_{\max} - \ln N)} dN &= p dt \\ \int \frac{1}{N (\ln N_{\max} - \ln N)} dN &= \int p dt \quad u = \ln N_{\max} - \ln N, \text{ so } du = -\frac{1}{N} dN \\ - \int \frac{1}{u} du &= \int p dt \\ - \ln |u| &= pt + C_1 \\ - \ln |\ln N_{\max} - \ln N| &= pt + C_1 \\ \ln |\ln N_{\max} - \ln N| &= -pt - C_1 \\ |\ln N_{\max} - \ln N| &= e^{-pt - C_1} \\ |\ln N_{\max} - \ln N| &= e^{-pt} e^{-C_1} \\ \ln N_{\max} - \ln N &= \pm e^{-pt} e^{-C_1} \\ \ln N_{\max} - \ln N &= -C e^{-pt} \\ \ln \left(\frac{N_{\max}}{N} \right) &= -C e^{-pt} \\ \frac{N_{\max}}{N} &= \exp(-C e^{-pt}) \\ \frac{N}{N_{\max}} &= \exp(C e^{-pt}) \\ N &= N_{\max} \exp(C e^{-pt}) \end{aligned}$$

The solution to the differential equation is

$$\boxed{N(t) = N_{\max} \exp(C e^{-pt})}$$

for $t \geq 0$.

- (b) Given the maximum population of $N_{\max} = 400$, the function becomes

$$N(t) = N_{\max} \exp(C e^{-pt}) = 400 \exp(C e^{-pt})$$

Use the initial population $N(0) = 20$ to find C .

$$\begin{aligned} 20 &= 400 \exp(C e^0) \\ 20 &= 400 \exp(C) \\ 0.05 &= \exp(C) \\ \ln(0.05) &= C \end{aligned}$$

Now the function is $N(t) = 400 \exp(\ln(0.05)e^{-pt})$ The population doubles in 5 years, so use

$N(5) = 40$ to find p .

$$\begin{aligned}
 40 &= 400 \exp(\ln(0.05)e^{-5p}) \\
 0.1 &= \exp(\ln(0.05)e^{-5p}) \\
 \ln(0.1) &= \ln(0.05)e^{-5p} \\
 \frac{\ln(0.1)}{\ln(0.05)} &= e^{-5p} \\
 \ln\left(\frac{\ln(0.1)}{\ln(0.05)}\right) &= -5p \\
 -\frac{1}{5} \ln\left(\frac{\ln(0.1)}{\ln(0.05)}\right) &= p
 \end{aligned}$$

With $C \approx -2.996$ and $p \approx .0526$, the particular function is

$$N(t) = 400 \exp(-2.996e^{-0.0526t})$$

- (c) Since $N(43) = 400 \exp(-2.996e^{-0.0526(43)}) \approx 293$, the population after 43 years as modeled will be approximately 293 fish.
- (d) The population is growing fastest when $\frac{dN}{dt}$ is at a maximum, which is where its derivative is zero. Find the derivative of $\frac{dN}{dt}$.

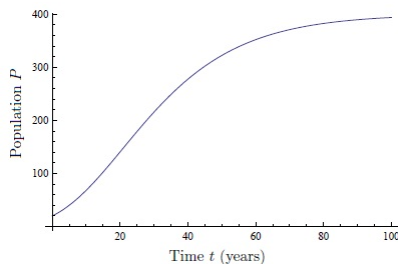
$$\begin{aligned}
 \frac{d}{dt} \left[\frac{dN}{dt} \right] &= \frac{d}{dt} \left[pN \ln\left(\frac{N_{\max}}{N}\right) \right] \\
 &= \frac{d}{dt} [pN (\ln N_{\max} - \ln N)] \\
 &= p \frac{dN}{dt} (\ln N_{\max} - \ln N) + pN \left(-\frac{1}{N} \frac{dN}{dt} \right) \\
 &= p \frac{dN}{dt} \ln\left(\frac{N_{\max}}{N}\right) - p \frac{dN}{dt} \\
 &= p \frac{dN}{dt} \left[\ln\left(\frac{N_{\max}}{N}\right) - 1 \right]
 \end{aligned}$$

Since neither p nor N' can equal zero, the derivative can only occur when

$$\ln\left(\frac{N_{\max}}{N}\right) = 1$$

Solve for N to find $N = \frac{N_{\max}}{e}$.

- (e) The graph of the predicted population $N = N(t)$ for $0 \leq t \leq 100$ is shown.



56.

57. The logistic differential equation can be rewritten by completing the square.

$$\begin{aligned}
\frac{dy}{dt} &= ky(M - y) \\
&= kMy - y^2 \\
&= -k\frac{M^2}{4} + kMy - ky^2 + k\frac{M^2}{4} \\
&= -\frac{k}{4}(M^2 - 4My + 4y^2) + k\frac{M^2}{4} \\
&= -k\left(\frac{M}{2} - y\right)^2 + k\frac{M^2}{4}
\end{aligned}$$

Recall that $k > 0$ so that $\frac{dy}{dt}$ is increasing when $y < \frac{M}{2}$ and decreasing when $y > \frac{M}{2}$. From this, it follows that the growth rate is a maximum when $y = \frac{M}{2}$.

Alternatively, find $\frac{d^2y}{dt^2}$.

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt} \left[\frac{dy}{dt} \right] \\
&= \frac{d}{dt} [ky(M - y)] \\
&= \frac{d}{dt} [kMy - ky^2] \\
&= kM\frac{dy}{dt} - 2ky\frac{dy}{dt} \\
&= k\frac{dy}{dt} (M - 2y) \\
&= k[ky(M - y)] (M - 2y) \\
&= k^2y(M - y)(M - 2y)
\end{aligned}$$

When $0 < y < \frac{M}{2}$, we have $M - y > \frac{M}{2} > 0$ and $M - 2y > 0$ so the derivative of $\frac{dy}{dt}$ is positive, which means $\frac{dy}{dt}$ is increasing. When $\frac{M}{2} < y < M$, we have $0 < M - y < \frac{M}{2}$ and $M - 2y < 0$ so the derivative of $\frac{dy}{dt}$ is negative, which means $\frac{dy}{dt}$ is decreasing. From this, it follows that the growth rate is a maximum when $y = \frac{M}{2}$.

58. Separate the variables in the differential equation

$$\frac{dy}{dt} = ky(M - y)$$

and integrate.

$$\begin{aligned}
 \frac{dy}{dt} &= ky(M-y) \\
 \frac{1}{y(M-y)} dy &= k dt \\
 \int \frac{1}{y(M-y)} dy &= \int k dt \\
 \frac{1}{M} \int \left(\frac{1}{y} + \frac{1}{M-y} \right) dy &= \int k dt \\
 \int \left(\frac{1}{y} + \frac{1}{M-y} \right) dy &= \int kM dt \\
 \ln |y| - \ln |M-y| &= kMt + C_1 \\
 \ln |M-y| - \ln |y| &= -kMt - C_1 \\
 \ln \left| \frac{M-y}{y} \right| &= -kMt - C_1 \\
 \left| \frac{M-y}{y} \right| &= e^{-kMt - C_1} \\
 \left| \frac{M-y}{y} \right| &= e^{-kMt} e^{-C_1} \\
 \frac{M-y}{y} &= \pm e^{-kMt} e^{-C_1} \\
 \frac{M-y}{y} &= Ce^{-kMt} \\
 \frac{M}{y} - 1 &= Ce^{-kMt} \\
 \frac{M}{y} &= 1 + Ce^{-kMt} \\
 \frac{y}{M} &= \frac{1}{1 + Ce^{-kMt}} \\
 y &= \frac{M}{1 + Ce^{-kMt}}
 \end{aligned}$$

If we assume that the initial condition is $y(0) = R$, $0 \leq R < M$, then $R = \frac{M}{1+C}$ or $C = \frac{M-R}{R}$ and

$$y(t) = \frac{M}{1 + Ce^{-kMt}} = \frac{M}{1 + \frac{M-R}{R}e^{-kMt}} = \frac{RM}{R + (M-R)e^{-kMt}}$$

This answer matches the answer obtained with the “Bernoulli solution”.

Challenge Problem

59. Let $y = y(t)$ be the volume of carbon dioxide in cubic-feet present in the room at time t in minutes. Carbon dioxide enters the room at the rate of $(0.04\%)(5000 \text{ ft}^3/\text{min}) = 2 \text{ ft}^3/\text{min}$. The volume of air in the room remains a constant $150,000 \text{ ft}^3/\text{min}$, so the percentage of carbon dioxide is $y/150,000\%$. Then the rate at which the carbon dioxide leaves the room is $(y/150,000\%)(5000 \text{ ft}^3/\text{min}) = y/30 \text{ ft}^3/\text{min}$. The mixture problem is modeled by the differential equation

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 2 - \frac{y}{30}$$

for $t \geq 0$. This equation is a separable differential equation and it is first-order linear differential equation. To solve it as a first-order linear differential equation, rewrite it as $\frac{dy}{dt} + \frac{1}{30}y = 2$. The

integrating factor is

$$e^{\int P(t) dt} = e^{\int 1/30 dt} = e^{t/30}$$

Multiply the differential equation by the integrating factor to obtain

$$\begin{aligned} e^{t/30} \frac{dy}{dt} + \frac{1}{30} e^{t/30} y &= 2e^{t/30} \\ \frac{d}{dt} [e^{t/30} y] &= 2e^{t/30} \end{aligned}$$

Integrate both sides with respect to t .

$$\begin{aligned} e^{t/30} y &= \int 2e^{t/30} dt \\ e^{t/30} y &= 60e^{t/30} + C \\ y &= 60 + Ce^{-t/30} \end{aligned}$$

Initially, 0.3% of the air in the room initially was carbon dioxide, so $y(0) = (0.003)(150,000) = 450$, which we use to find C .

$$\begin{aligned} 450 &= 60 + Ce^0 \\ 450 &= 60 + C \\ 390 &= C \end{aligned}$$

The volume of the carbon dioxide after t minutes is $y(t) = 60 + 390e^{-t/30}$. After 1 hour, there is $y(60) = 60 + 390e^{-60/30} = 60 + 390e^{-2}$ cubic-feet of carbon dioxide in the room, so the percentage is $\frac{60 + 390e^{-2}}{150,000} \approx \boxed{0.0752\%}$. After 2 hours, there is $y(120) = 60 + 390e^{-120/30} = 60 + 390e^{-4}$ cubic-feet of carbon dioxide in the room, so the percentage is $\frac{60 + 390e^{-4}}{150,000} \approx \boxed{0.0448\%}$.

60. Assume that the percentage of cash that comes into the bank each day that consists of new bills is the same as the percentage in the general circulation. Let $y = y(t)$ be the amount of new currency in circulation in millions of dollars t days after the new currency is first issued. Of the \$10 million that comes into the federal bank each day, a portion of it, namely $((y \text{ \$ million})/(\$3 \text{ billion}))(\$10 \text{ million}) = y/300$ million \$, consists of new currency. So the federal bank replaces $10 - y/300$ in \$ million of the old currency each day.

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 10 - \frac{y}{300}$$

for $t \geq 0$. This equation is a separable differential equation and it is first-order linear differential equation. To solve it as a first-order linear differential equation, rewrite it as $\frac{dy}{dt} + \frac{1}{300}y = 10$. The integrating factor is

$$e^{\int P(t) dt} = e^{\int 1/300 dt} = e^{t/300}$$

Multiply the differential equation by the integrating factor to obtain

$$\begin{aligned} e^{t/300} \frac{dy}{dt} + \frac{1}{300} e^{t/300} y &= 10e^{t/300} \\ \frac{d}{dt} [e^{t/300} y] &= 10e^{t/300} \end{aligned}$$

Integrate both sides with respect to t .

$$\begin{aligned} e^{t/300} y &= \int 10e^{t/300} dt \\ e^{t/300} y &= 3000e^{t/300} + C \\ y &= 3000 + Ce^{-t/300} \end{aligned}$$

When the new currency was first issued, there was none of the new currency in circulation, so $y(0) = 0$, which we use to find C .

$$\begin{aligned} 0 &= 3000 + Ce^0 \\ 0 &= 3000 + C \\ -3000 &= C \end{aligned}$$

The amount of new currency in circulation in millions of dollars after t days is $y(t) = 3000 - 3000e^{-t/300}$. To find how long it will take for the currency to be 95% new, solve $y(t) = (0.95)(3000) = 2850$.

$$\begin{aligned} 2850 &= 3000 - 3000e^{-t/300} \\ -150 &= -3000e^{-t/300} \\ 0.05 &= e^{-t/300} \\ \ln(0.05) &= -\frac{t}{300} \\ -300 \ln(0.05) &= t \\ 899 &\approx t \end{aligned}$$

It will take about 899 days for the currency in circulation to become 95% new.

61. Rewrite the differential equation

$$L \frac{dI}{dt} + RI = E_0 \sin(\omega t)$$

in the form

$$\frac{dI}{dt} + \frac{RI}{L} = \frac{E_0 \sin(\omega t)}{L}$$

to recognize it as a first-order linear differential equation. The integrating factor is

$$e^{\int P(t) dt} = e^{\int R/L dt} = e^{Rt/L}$$

Now multiply the differential equation by $e^{Rt/L}$ and obtain

$$e^{Rt/L} \frac{dI}{dt} + \frac{R}{L} e^{Rt/L} I = \frac{E_0}{L} \sin(\omega t) e^{Rt/L}$$

The left side of the above equation is the derivative of $e^{Rt/L} I$, so we can write

$$\frac{d}{dt} (e^{Rt/L} I) = \frac{E_0}{L} \sin(\omega t) e^{Rt/L}$$

Integrate the equation above.

$$\begin{aligned} e^{Rt/L} I &= \int \frac{E_0}{L} \sin(\omega t) e^{Rt/L} dt \\ e^{Rt/L} I &= \frac{E_0}{L} \frac{L (R \sin(\omega t) - \omega L \cos(\omega t)) e^{Rt/L}}{R^2 + \omega^2 L^2} + C \\ e^{Rt/L} I &= \frac{E_0 (R \sin(\omega t) - \omega L \cos(\omega t)) e^{Rt/L}}{R^2 + \omega^2 L^2} + C \\ I &= \frac{E_0 (R \sin(\omega t) - \omega L \cos(\omega t))}{R^2 + \omega^2 L^2} + C e^{-Rt/L} \end{aligned}$$

Use $I(0) = I_0$ to find C .

$$\begin{aligned} I_0 &= \frac{E_0 (R \sin(0) - \omega L \cos(0))}{R^2 + \omega^2 L^2} + C e^0 \\ I_0 &= \frac{-E_0 \omega L}{R^2 + \omega^2 L^2} + C \\ I_0 + \frac{E_0 \omega L}{R^2 + \omega^2 L^2} &= C \end{aligned}$$

The particular solution of the differential equation is

$$I = \frac{E_0 (R \sin(\omega t) - \omega L \cos(\omega t) + \omega L e^{-Rt/L})}{R^2 + \omega^2 L^2} + I_0 e^{-Rt/L}$$

16.5 Power Series Methods

Skill Building

1. Assume that the solution of the differential equation $y' + 3xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Since $y' = -3xy$, this leads to

$$\begin{aligned} \sum_{k=1}^{\infty} k a_k x^{k-1} &= -3x \sum_{k=0}^{\infty} a_k x^k \\ \sum_{k=1}^{\infty} k a_k x^{k-1} &= \sum_{k=0}^{\infty} (-3a_k x^{k+1}) \end{aligned}$$

To obtain relationships among the coefficients, write out the terms.

$$a_1 x^0 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = -3a_0 x - 3a_1 x^2 - 3a_2 x^3 - 3a_3 x^4 - \cdots$$

Because the coefficients of corresponding powers of x are equal, we have

$$a_1 = 0 \quad 3a_3 = -3a_1 = 0 \quad 5a_5 = -3a_3 = 0 \quad \dots \quad (2n+1)a_{2n+1} = -3a_{2n-1} = 0$$

and

$$2a_2 = -3a_0 \quad 4a_4 = -3a_2 \quad 6a_6 = -3a_4 \quad \dots \quad 2na_{2n} = -3a_{2n-2} \quad \dots$$

Express these relationships with even subscripts recursively as

$$\begin{aligned} a_2 &= \left(-\frac{3}{2}\right) a_0 \quad a_4 = \left(-\frac{3}{2}\right) \left(\frac{1}{2}\right) a_2 = \left(-\frac{3}{2}\right)^2 \left(\frac{1}{2}\right) a_0 \quad a_6 = \left(-\frac{3}{2}\right) \left(\frac{1}{3}\right) a_4 = \left(-\frac{3}{2}\right)^3 \left(\frac{1}{3!}\right) a_0 \\ a_8 &= \left(-\frac{3}{2}\right) \left(\frac{1}{4}\right) a_6 = \left(-\frac{3}{2}\right)^4 \left(\frac{1}{4!}\right) a_0 \quad \dots \quad a_{2n} = \left(-\frac{3}{2}\right) \left(\frac{1}{n}\right) a_{2n-2} = \left(-\frac{3}{2}\right)^n \left(\frac{1}{n!}\right) a_0 \end{aligned}$$

The power series takes on the form

$$y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} = \sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)^k \left(\frac{1}{k!}\right) a_0 x^{2k} = a_0 \sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)^k \frac{x^{2k}}{k!} = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{3x^2}{2}\right)^k$$

which we recognize as $y(x) = a_0 e^{-3x^2/2}$. Let $C = a_0$. Then the solution to the differential equation is

$$y(x) = a_0 \sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)^k \frac{x^{2k}}{k!} = C e^{-3x^2/2}$$

2. Assume that the solution of the differential equation $y' - x + 3xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Since $y' = x - 3xy$, this leads to

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = x - \sum_{k=0}^{\infty} 3 a_k x^{k+1}$$

To obtain relationships among the coefficients, write out the terms.

$$a_1 x^0 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 - 3a_0)x - 3a_1 x^2 - 3a_2 x^3 - 3a_3 x^4 - \cdots$$

Because the coefficients of corresponding powers of x are equal, we have

$$a_1 = 0 \quad 3a_3 = -3a_1 = 0 \quad 5a_5 = -3a_3 = 0 \quad \dots \quad (2n+1)a_{2n+1} = -3a_{2n-1} = 0$$

and

$$\begin{aligned} 2a_2 &= 1 - 3a_0 & \text{or equivalently} & \quad a_2 = \left(-\frac{3}{2}\right) a_0 + \frac{1}{2} = \left(-\frac{3}{2}\right) \left(a_0 - \frac{1}{3}\right) \\ 4a_4 &= -3a_2 & \text{or equivalently} & \quad a_4 = \left(-\frac{3}{2}\right) \left(\frac{1}{2}\right) a_2 = \left(-\frac{3}{2}\right)^2 \left(\frac{1}{2}\right) \left(a_0 - \frac{1}{3}\right) \\ 6a_6 &= -3a_4 & \text{or equivalently} & \quad a_6 = \left(-\frac{3}{2}\right) \left(\frac{1}{3}\right) a_4 = \left(-\frac{3}{2}\right)^3 \left(\frac{1}{3!}\right) \left(a_0 - \frac{1}{3}\right) \\ 8a_8 &= -3a_6 & \text{or equivalently} & \quad a_8 = \left(-\frac{3}{2}\right) \left(\frac{1}{4}\right) a_6 = \left(-\frac{3}{2}\right)^4 \left(\frac{1}{4!}\right) \left(a_0 - \frac{1}{3}\right) \end{aligned}$$

In general

$$a_{2n} = \left(-\frac{3}{2}\right)^n \left(\frac{1}{n!}\right) \left(a_0 - \frac{1}{3}\right)$$

for $n > 0$. The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_{2k} x^{2k} \\ &= a_0 + \sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^k \left(\frac{1}{k!}\right) \left(a_0 - \frac{1}{3}\right) x^{2k} \\ &= \frac{1}{3} + \left(a_0 - \frac{1}{3}\right) + \sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^k \left(\frac{1}{k!}\right) \left(a_0 - \frac{1}{3}\right) x^{2k} \\ &= \frac{1}{3} + \sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)^k \left(\frac{1}{k!}\right) \left(a_0 - \frac{1}{3}\right) x^{2k} \\ &= \frac{1}{3} + \left(a_0 - \frac{1}{3}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{3x^2}{2}\right)^k \end{aligned}$$

which we recognize as $y(x) = \frac{1}{3} + \left(a_0 - \frac{1}{3}\right) e^{-3x^2/2}$. Let $C = a_0 - \frac{1}{3}$. Then the solution to the differential equation is

$$y(x) = a_0 + \left(a_0 - \frac{1}{3}\right) \sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^k \frac{x^{2k}}{k!} = \frac{1}{3} + C e^{-3x^2/2}$$

3. Assume that the solution of the differential equation $y'' + y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + y = 0$, this leads to

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0$$

Change the index of summation in the power series on the right from k to $k-2$.

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + a_{k-2}] x^{k-2} &= 0 \end{aligned}$$

Then $n(n-1)a_n + a_{n-2} = 0$ for $n \geq 2$. Since the recursive formula skips a_{n-1} , split it up by even and odd indices: $(2n)(2n-1)a_{2n} + a_{2n-2} = 0$ and $(2n+1)(2n)a_{2n+1} + a_{2n-1} = 0$ so that

$$a_{2n} = -\frac{1}{(2n)(2n-1)} a_{2n-2} = \frac{1}{(2n)(2n-1)(2n-2)(2n-3)} a_{2n-4} = \cdots = (-1)^n \frac{1}{(2n)!} a_0$$

and $(2n)(2n-1)a_{2n} + a_{2n-2}$

$$a_{2n+1} = -\frac{1}{(2n+1)(2n)} a_{2n-1} = \frac{1}{(2n+1)(2n)(2n-1)(2n-2)} a_{2n-3} = \cdots = (-1)^n \frac{1}{(2n+1)!} a_1$$

for $n \geq 1$. The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} a_0 x^{2k} + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} a_1 x^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

which we recognize as

$$y(x) = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = a_0 \cos x + a_1 \sin x$$

4. Assume that the solution of the differential equation $y'' + xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + xy = 0$, this leads to

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+1} &= 0 \end{aligned}$$

Change the index of summation in the power series on the right from k to $k-3$.

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=3}^{\infty} a_{k-3} x^{k-2} &= 0 \\ 2a_2 x^0 + \sum_{k=3}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=3}^{\infty} a_{k-3} x^{k-2} &= 0 \\ 2a_2 x^0 + \sum_{k=3}^{\infty} [k(k-1) a_k + a_{k-3}] x^{k-2} &= 0 \end{aligned}$$

Then $a_2 = 0$ and $n(n-1)a_n + a_{n-3} = 0$ for $n \geq 3$, which can be written as

$$a_n = -\frac{1}{n(n-1)} a_{n-3}$$

Then

$$a_3 = -\frac{1}{2 \cdot 3} a_0 \quad a_6 = -\frac{1}{5 \cdot 6} a_3 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} a_0 \quad a_9 = -\frac{1}{8 \cdot 9} a_6 = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} a_0 \quad \dots$$

and

$$a_4 = -\frac{1}{3 \cdot 4} a_1 \quad a_7 = -\frac{1}{6 \cdot 7} a_4 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} a_1 \quad a_{10} = -\frac{1}{9 \cdot 10} a_7 = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} a_1 \quad \dots$$

and

$$a_5 = -\frac{1}{4 \cdot 5} a_2 = 0 \quad a_8 = -\frac{1}{7 \cdot 8} a_5 = 0 \quad a_{11} = -\frac{1}{10 \cdot 11} a_8 = 0 \quad \dots$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\ &= a_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots \right) \\ &\quad + a_1 \left(x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \dots \right) \end{aligned}$$

The power series representation of the general solution of the equation is

$$y(x) = a_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots \right) \\ + a_1 \left(x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \cdots \right)$$

5. Assume that the solution of the differential equation $y'' + x^2 y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + x^2 y = 0$, this leads to

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x^2 \sum_{k=0}^{\infty} a_k x^k = 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 0$$

Change the index of summation in the power series on the right from k to $k-4$.

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=4}^{\infty} a_{k-4} x^{k-2} = 0 \\ 2a_2 x^0 + 6a_3 x + \sum_{k=4}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=4}^{\infty} a_{k-4} x^{k-2} = 0 \\ 2a_2 x^0 + 6a_3 x + \sum_{k=4}^{\infty} [k(k-1) a_k + a_{k-4}] x^{k-2} = 0$$

Then $a_2 = 0$, $a_3 = 0$ and $n(n-1)a_n + a_{n-4} = 0$ for $n \geq 4$, which can be written as

$$a_n = -\frac{1}{n(n-1)} a_{n-4}$$

Then

$$a_4 = -\frac{1}{3 \cdot 4} a_0 \quad a_8 = -\frac{1}{7 \cdot 8} a_4 = \frac{1}{3 \cdot 4 \cdot 7 \cdot 8} a_0 \quad a_{12} = -\frac{1}{11 \cdot 12} a_8 = -\frac{1}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} a_0 \quad \cdots$$

and

$$a_5 = -\frac{1}{4 \cdot 5} a_1 \quad a_9 = -\frac{1}{8 \cdot 9} a_5 = \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} a_1 \quad a_{13} = -\frac{1}{12 \cdot 13} a_9 = -\frac{1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} a_1 \quad \cdots$$

and

$$a_6 = -\frac{1}{5 \cdot 6} a_2 = 0 \quad a_{10} = -\frac{1}{9 \cdot 10} a_6 = 0 \quad a_{14} = -\frac{1}{13 \cdot 14} a_{10} = 0 \quad \cdots$$

and

$$a_7 = -\frac{1}{6 \cdot 7} a_3 = 0 \quad a_{11} = -\frac{1}{10 \cdot 11} a_7 = 0 \quad a_{15} = -\frac{1}{14 \cdot 15} a_{11} = 0 \quad \cdots$$

The power series takes on the form

$$\begin{aligned}
 y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
 &= \sum_{k=0}^{\infty} a_{4k} x^{4k} + \sum_{k=0}^{\infty} a_{4k+1} x^{4k+1} + \sum_{k=0}^{\infty} a_{4k+2} x^{4k+2} + \sum_{k=0}^{\infty} a_{4k+3} x^{4k+3} \\
 &= a_0 \left(1 - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 7 \cdot 8} x^8 - \frac{1}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} x^{12} + \dots \right) \\
 &\quad + a_1 \left(x - \frac{1}{4 \cdot 5} x^5 + \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} x^9 - \frac{1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} x^{13} + \dots \right)
 \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned}
 y(x) &= a_0 \left(1 - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 7 \cdot 8} x^8 - \frac{1}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} x^{12} + \dots \right) \\
 &\quad + a_1 \left(x - \frac{1}{4 \cdot 5} x^5 + \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} x^9 - \frac{1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} x^{13} + \dots \right)
 \end{aligned}$$

6. Assume that the solution of the differential equation $y'' - 2xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + xy = 0$, this leads to

$$\begin{aligned}
 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - 2x \sum_{k=0}^{\infty} a_k x^k &= 0 \\
 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} (-2a_k x^{k+1}) &= 0
 \end{aligned}$$

Change the index of summation in the power series on the right from k to $k-3$.

$$\begin{aligned}
 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=3}^{\infty} (-2a_{k-3} x^{k-2}) &= 0 \\
 2a_2 x^0 + \sum_{k=3}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=3}^{\infty} (-2a_{k-3} x^{k-2}) &= 0 \\
 2a_2 x^0 + \sum_{k=3}^{\infty} [k(k-1) a_k - 2a_{k-3}] x^{k-2} &= 0
 \end{aligned}$$

Then $a_2 = 0$ and $n(n-1)a_n - 2a_{n-3} = 0$ for $n \geq 3$, which can be written as

$$a_n = \frac{2}{n(n-1)} a_{n-3}$$

Then

$$a_3 = \frac{2}{2 \cdot 3} a_0 \quad a_6 = \frac{2}{5 \cdot 6} a_3 = \frac{2^2}{2 \cdot 3 \cdot 5 \cdot 6} a_0 \quad a_9 = \frac{2}{8 \cdot 9} a_6 = \frac{2^3}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} a_0 \quad \dots$$

and

$$a_4 = \frac{2}{3 \cdot 4} a_1 \quad a_7 = \frac{2}{6 \cdot 7} a_4 = \frac{2^2}{3 \cdot 4 \cdot 6 \cdot 7} a_1 \quad a_{10} = \frac{2}{9 \cdot 10} a_7 = \frac{2^3}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} a_1 \quad \dots$$

and

$$a_5 = \frac{2}{4 \cdot 5} a_2 = 0 \quad a_8 = \frac{2}{7 \cdot 8} a_5 = 0 \quad a_{11} = \frac{2}{10 \cdot 11} a_8 = 0 \quad \dots$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\ &= a_0 \left(1 + \frac{2}{2 \cdot 3} x^3 + \frac{2^2}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{2^3}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots \right) \\ &\quad + a_1 \left(x + \frac{2}{3 \cdot 4} x^4 + \frac{2^2}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{2^3}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \dots \right) \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{2}{2 \cdot 3} x^3 + \frac{2^2}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{2^3}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots \right) \\ &\quad + a_1 \left(x + \frac{2}{3 \cdot 4} x^4 + \frac{2^2}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{2^3}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \dots \right) \end{aligned}$$

7. Assume that the solution of the differential equation $y'' + x^2 y' + xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + x^2 y' + xy = 0$, this leads to

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x^2 \sum_{k=1}^{\infty} k a_k x^{k-1} + x \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} &= 0 \end{aligned}$$

Change the indices of summation in the power series in the middle and on the right from k to $k-3$.

$$\begin{aligned}
 \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=4}^{\infty} (k-3)a_{k-3} x^{k-2} + \sum_{k=3}^{\infty} a_{k-3} x^{k-2} &= 0 \\
 2a_2 x^0 + \sum_{k=3}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=3}^{\infty} (k-3)a_{k-3} x^{k-2} + \sum_{k=3}^{\infty} a_{k-3} x^{k-2} &= 0 \\
 2a_2 x^0 + \sum_{k=3}^{\infty} [k(k-1)a_k + (k-3)a_{k-3} + a_{k-3}] x^{k-2} &= 0 \\
 2a_2 x^0 + \sum_{k=3}^{\infty} [k(k-1)a_k + (k-2)a_{k-3}] x^{k-2} &= 0
 \end{aligned}$$

(Notice that when $k=3$, the term $(k-3)a_{k-3}x^{k-2} = 0$, so the index of the middle sum can start at 3 rather than at 4.) Then $a_2 = 0$ and $n(n-1)a_n + (n-2)a_{n-3} = 0$ for $n \geq 3$, which can be written as

$$a_n = -\frac{n-2}{n(n-1)}a_{n-3}$$

Then

$$a_3 = -\frac{1}{2 \cdot 3}a_0 \quad a_6 = -\frac{4}{5 \cdot 6}a_3 = \frac{1 \cdot 4}{2 \cdot 3 \cdot 5 \cdot 6}a_0 \quad a_9 = -\frac{7}{8 \cdot 9}a_6 = -\frac{1 \cdot 4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}a_0 \quad \dots$$

and

$$a_4 = -\frac{2}{3 \cdot 4}a_1 \quad a_7 = -\frac{5}{6 \cdot 7}a_4 = \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7}a_1 \quad a_{10} = -\frac{8}{9 \cdot 10}a_7 = -\frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}a_1 \quad \dots$$

and

$$a_5 = -\frac{3}{4 \cdot 5}a_2 = 0 \quad a_8 = -\frac{6}{7 \cdot 8}a_5 = 0 \quad a_{11} = -\frac{9}{10 \cdot 11}a_8 = 0 \quad \dots$$

The power series takes on the form

$$\begin{aligned}
 y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
 &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\
 &= a_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 4}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1 \cdot 4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots \right) \\
 &\quad + a_1 \left(x - \frac{2}{3 \cdot 4} x^4 + \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \dots \right)
 \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned}
 y(x) &= a_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 4}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1 \cdot 4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots \right) \\
 &\quad + a_1 \left(x - \frac{2}{3 \cdot 4} x^4 + \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \dots \right)
 \end{aligned}$$

8. Assume that the solution of the differential equation $y'' + 3xy' + 3y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + 3xy' + 3y = 0$, this leads to

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + 3x \sum_{k=1}^{\infty} k a_k x^{k-1} + 3 \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} 3k a_k x^k + \sum_{k=0}^{\infty} 3a_k x^k &= 0 \end{aligned}$$

Change the indices of summation in the power series in the middle and on the right from k to $k-2$.

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=3}^{\infty} 3(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 3a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} 3(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 3a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + 3(k-2) a_{k-2} + 3a_{k-2}] x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + (3k-3) a_{k-2}] x^{k-2} &= 0 \end{aligned}$$

(Notice that when $k=2$, the term $3(k-2)a_{k-2}x^{k-2} = 0$, so the index of the middle sum can change from 3 to 2.) Then $n(n-1)a_n + (3n-3)a_{n-2} = 0$ for $n \geq 2$, which can be written as

$$a_n = -\frac{3n-3}{n(n-1)} a_{n-2} = -\frac{3}{n} a_{n-2}$$

Then

$$a_2 = -\frac{3}{2} a_0 \quad a_4 = -\frac{3}{4} a_2 = \frac{3^2}{2 \cdot 4} a_0 \quad a_6 = -\frac{3}{6} a_4 = -\frac{3^3}{2 \cdot 4 \cdot 6} a_0 \quad \cdots$$

and

$$a_3 = -\frac{3}{3} a_1 \quad a_5 = -\frac{3}{5} a_3 = \frac{3^2}{3 \cdot 5} a_1 \quad a_7 = -\frac{3}{7} a_5 = -\frac{3^3}{3 \cdot 5 \cdot 7} a_1 \quad \cdots$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= a_0 \left(1 - \frac{3}{2} x^2 + \frac{3^2}{2 \cdot 4} x^4 - \frac{3^3}{2 \cdot 4 \cdot 6} x^6 + \cdots \right) \\ &\quad + a_1 \left(x - \frac{3}{3 \cdot 4} x^3 + \frac{3^2}{3 \cdot 5} x^5 - \frac{3^3}{3 \cdot 5 \cdot 7} x^7 + \cdots \right) \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned} y(x) = & a_0 \left(1 - \frac{3}{2}x^2 + \frac{3^2}{2 \cdot 4}x^4 - \frac{3^3}{2 \cdot 4 \cdot 6}x^6 + \cdots \right) \\ & + a_1 \left(x - x^3 + \frac{3^2}{3 \cdot 5}x^5 - \frac{3^3}{3 \cdot 5 \cdot 7}x^7 + \cdots \right) \end{aligned}$$

9. Assume that the solution of the differential equation $y''' + y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

and

$$y'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3}$$

Since $y''' + y = 0$, this leads to

$$\sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} + \sum_{k=0}^{\infty} a_k x^k = 0$$

Change the indices of summation in the power series on the right from k to $k-3$.

$$\begin{aligned} \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} + \sum_{k=3}^{\infty} a_{k-3} x^{k-3} &= 0 \\ \sum_{k=3}^{\infty} [k(k-1)(k-2) a_k + a_{k-3}] x^{k-3} &= 0 \end{aligned}$$

Then $n(n-1)(n-2)a_n + a_{n-3} = 0$ for $n \geq 3$, which can be written as

$$a_n = -\frac{1}{n(n-1)(n-2)} a_{n-3}$$

Then

$$a_3 = -\frac{1}{1 \cdot 2 \cdot 3} a_0 = -\frac{1}{3!} a_0 \quad a_6 = -\frac{1}{4 \cdot 5 \cdot 6} a_3 = \frac{1}{6!} a_0 \quad a_9 = -\frac{1}{7 \cdot 8 \cdot 9} a_6 = -\frac{1}{9!} a_0 \quad \cdots$$

and

$$a_4 = -\frac{1}{2 \cdot 3 \cdot 4} a_1 = -\frac{1}{4!} a_1 \quad a_7 = -\frac{1}{5 \cdot 6 \cdot 7} a_4 = \frac{1}{7!} a_1 \quad a_{10} = -\frac{1}{8 \cdot 9 \cdot 10} a_7 = -\frac{1}{10!} a_1 \quad \cdots$$

and

$$a_5 = -\frac{1}{3 \cdot 4 \cdot 5} a_2 = -\frac{1}{5!} a_2 \quad a_8 = -\frac{1}{6 \cdot 7 \cdot 8} a_5 = \frac{1}{8!} a_2 \quad a_{11} = -\frac{1}{9 \cdot 10 \cdot 11} a_8 = -\frac{1}{11!} a_2 \quad \cdots$$

The power series takes on the form

$$\begin{aligned}
 y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
 &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\
 &= a_0 \left(1 - \frac{1}{3!} x^3 + \frac{1}{6!} x^6 - \frac{1}{9!} x^9 + \cdots \right) \\
 &\quad + a_1 \left(x - \frac{1}{4!} x^4 + \frac{1}{7!} x^7 - \frac{1}{10!} x^{10} + \cdots \right) \\
 &\quad + a_2 \left(x^2 - \frac{2}{5!} x^5 + \frac{2}{8!} x^8 - \frac{2}{11!} x^{11} + \cdots \right)
 \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned}
 y(x) &= a_0 \left(1 - \frac{1}{3!} x^3 + \frac{1}{6!} x^6 - \frac{1}{9!} x^9 + \cdots \right) \\
 &\quad + a_1 \left(x - \frac{1}{4!} x^4 + \frac{1}{7!} x^7 - \frac{1}{10!} x^{10} + \cdots \right) \\
 &\quad + a_2 \left(x^2 - \frac{2}{5!} x^5 + \frac{2}{8!} x^8 - \frac{2}{11!} x^{11} + \cdots \right)
 \end{aligned}$$

10. Assume that the solution of the differential equation $y''' - xy = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

and

$$y'''(x) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3}$$

Since $y''' - xy = 0$, this leads to

$$\begin{aligned}
 \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} - x \sum_{k=0}^{\infty} a_k x^k &= 0 \\
 \sum_{k=3}^{\infty} k(k-1)(k-2) a_k x^{k-3} - \sum_{k=0}^{\infty} a_k x^{k+1} &= 0
 \end{aligned}$$

Change the indices of summation in the power series on the right from k to $k-4$.

$$\begin{aligned} \sum_{k=3}^{\infty} k(k-1)(k-2)a_k x^{k-3} - \sum_{k=4}^{\infty} a_{k-4} x^{k-3} &= 0 \\ 6a_3 x^0 + \sum_{k=4}^{\infty} k(k-1)(k-2)a_k x^{k-3} - \sum_{k=4}^{\infty} a_{k-4} x^{k-3} &= 0 \\ 6a_3 x^0 + \sum_{k=3}^{\infty} [k(k-1)(k-2)a_k - a_{k-4}] x^{k-3} &= 0 \end{aligned}$$

Then $a_3 = 0$ and $n(n-1)(n-2)a_n - a_{n-4} = 0$ for $n \geq 4$, which can be written as

$$a_n = \frac{1}{n(n-1)(n-2)} a_{n-4}$$

Then

$$a_4 = \frac{1}{2 \cdot 3 \cdot 4} a_0 = \frac{1}{4!} a_0 \quad a_8 = \frac{1}{6 \cdot 7 \cdot 8} a_4 = \frac{5}{8!} a_0 \quad a_{12} = \frac{1}{10 \cdot 11 \cdot 12} a_8 = \frac{5 \cdot 9}{12!} a_0 \quad \dots$$

and

$$a_5 = \frac{1}{3 \cdot 4 \cdot 5} a_1 = \frac{2}{5!} a_1 \quad a_9 = \frac{1}{7 \cdot 8 \cdot 9} a_5 = \frac{2 \cdot 6}{9!} a_1 \quad a_{13} = \frac{1}{11 \cdot 12 \cdot 13} a_9 = \frac{2 \cdot 6 \cdot 10}{13!} a_1 \quad \dots$$

and

$$a_6 = \frac{1}{4 \cdot 5 \cdot 6} a_2 = \frac{2 \cdot 3}{6!} a_2 \quad a_{10} = \frac{1}{8 \cdot 9 \cdot 10} a_6 = \frac{2 \cdot 3 \cdot 7}{10!} a_2 \quad a_{14} = \frac{1}{12 \cdot 13 \cdot 14} a_{10} = \frac{2 \cdot 3 \cdot 7 \cdot 11}{14!} a_2 \quad \dots$$

and

$$a_7 = \frac{1}{4 \cdot 5 \cdot 6} a_3 = 0 \quad a_{11} = \frac{1}{8 \cdot 9 \cdot 10} a_7 = 0 \quad a_{15} = \frac{1}{12 \cdot 13 \cdot 14} a_{11} = 0 \quad \dots$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2} \\ &= a_0 \left(1 + \frac{1}{4!} x^4 + \frac{5}{8!} x^8 + \frac{5 \cdot 11}{12!} x^{12} + \dots \right) \\ &\quad + a_1 \left(x + \frac{2}{5!} x^5 + \frac{2 \cdot 6}{9!} x^9 + \frac{2 \cdot 6 \cdot 10}{13!} x^{13} + \dots \right) \\ &\quad + a_2 \left(x^2 + \frac{2 \cdot 3}{6!} x^6 + \frac{2 \cdot 3 \cdot 7}{10!} x^{10} + \frac{2 \cdot 3 \cdot 7 \cdot 11}{14!} x^{14} + \dots \right) \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{1}{4!} x^4 + \frac{5}{8!} x^8 + \frac{5 \cdot 11}{12!} x^{12} + \dots \right) \\ &\quad + a_1 \left(x + \frac{2}{5!} x^5 + \frac{2 \cdot 6}{9!} x^9 + \frac{2 \cdot 6 \cdot 10}{13!} x^{13} + \dots \right) \\ &\quad + a_2 \left(x^2 + \frac{2 \cdot 3}{6!} x^6 + \frac{2 \cdot 3 \cdot 7}{10!} x^{10} + \frac{2 \cdot 3 \cdot 7 \cdot 11}{14!} x^{14} + \dots \right) \end{aligned}$$

11. Assume that the solution of the differential equation $(1 + x^2)y'' - 4xy' + 6y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $(1 + x^2)y'' - 4xy' + 6y = 0$, this leads to

$$\begin{aligned} (1 + x^2) \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - 4x \sum_{k=1}^{\infty} k a_k x^{k-1} + 6 \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} k(k-1) a_k x^k - \sum_{k=1}^{\infty} 4k a_k x^k + \sum_{k=0}^{\infty} 6a_k x^k &= 0 \end{aligned}$$

Change the indices of summation in all the power series but the first from k to $k-2$.

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=4}^{\infty} (k-2)(k-3) a_{k-2} x^{k-2} - \sum_{k=3}^{\infty} 4(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 6a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} (k-2)(k-3) a_{k-2} x^{k-2} - \sum_{k=2}^{\infty} 4(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 6a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + (k-2)(k-3) a_{k-2} - 4(k-2) a_{k-2} + 6a_{k-2}] x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + (k-4)(k-5) a_{k-2}] x^{k-2} &= 0 \end{aligned}$$

(The factors $(k-2)$ and $(k-3)$ allow the middle two indices to begin at 2, since the additional terms are simply zero.) Then $n(n-1)a_n + (n-4)(n-5)a_{n-2} = 0$ for $n \geq 2$, which can be written as

$$a_n = -\frac{(n-4)(n-5)}{n(n-1)} a_{n-2}$$

Then

$$a_2 = -\frac{(-2) \cdot (-3)}{1 \cdot 2} a_0 = -3a_0 \quad a_4 = -\frac{(0)(-1)}{3 \cdot 4} a_2 = 0 \quad a_6 = -\frac{2 \cdot 1}{5 \cdot 6} a_4 = 0 \quad \dots$$

and

$$a_3 = -\frac{(-1)(-2)}{2 \cdot 3} a_1 = -\frac{1}{3} a_1 \quad a_5 = -\frac{0 \cdot 1}{4 \cdot 5} a_3 = 0 \quad a_7 = -\frac{2 \cdot 3}{6 \cdot 7} a_5 = 0 \quad \dots$$

The general solution of the equation is

$$\boxed{y(x) = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3 = a_0(1 - 3x^2) + a_1 \left(x - \frac{1}{3} x^3 \right)}$$

12. Assume that the solution of the differential equation $(x^2 + 2)y'' - 3xy' + 4y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $(x^2 + 2)y'' - 3xy' + 4y = 0$, this leads to

$$\begin{aligned} (x^2 + 2) \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - 3x \sum_{k=1}^{\infty} k a_k x^{k-1} + 4 \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=2}^{\infty} 2k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} k(k-1) a_k x^k - \sum_{k=1}^{\infty} 3k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k &= 0 \end{aligned}$$

Change the indices of summation in all the power series but the first from k to $k-2$.

$$\begin{aligned} \sum_{k=2}^{\infty} 2k(k-1) a_k x^{k-2} + \sum_{k=4}^{\infty} (k-2)(k-3) a_{k-2} x^{k-2} - \sum_{k=3}^{\infty} 3(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 4a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} 2k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} (k-2)(k-3) a_{k-2} x^{k-2} - \sum_{k=2}^{\infty} 3(k-2) a_{k-2} x^{k-2} + \sum_{k=2}^{\infty} 4a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [2k(k-1) a_k + (k-2)(k-3) a_{k-2} - 3(k-2) a_{k-2} + 4a_{k-2}] x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [2k(k-1) a_k + (k-4)^2 a_{k-2}] x^{k-2} &= 0 \end{aligned}$$

(The factors $(k-2)$ and $(k-3)$ allow the middle two indices to begin at 2, since the additional terms are simply zero.) Then $2n(n-1)a_n + (n-4)^2 a_{n-2} = 0$ for $n \geq 2$, which can be written as

$$a_n = -\frac{(n-4)^2}{2n(n-1)} a_{n-2}$$

Then

$$a_2 = -\frac{(-2)^2}{2 \cdot 1 \cdot 2} a_0 = -a_0 \quad a_4 = -\frac{(0)^2}{2 \cdot 3 \cdot 4} a_2 = 0 \quad a_6 = -\frac{2^2}{2 \cdot 5 \cdot 6} a_4 = 0 \quad \dots$$

and

$$a_3 = -\frac{(-1)^2}{2 \cdot 2 \cdot 3} a_1 = -\frac{1}{2(3!)} a_1 \quad a_5 = -\frac{1^2}{2 \cdot 4 \cdot 5} a_3 = \frac{1}{2^2(5!)} a_1 \quad a_7 = -\frac{3^2}{2 \cdot 6 \cdot 7} a_5 = -\frac{3^2}{2^3(7!)} a_1 \quad \dots$$

The general solution of the equation is

$$\boxed{y(x) = a_0(1 - x^2) + a_1 \left(x - \frac{1}{2(3!)} x^3 + \frac{1}{2^2(5!)} x^5 - \frac{3^2}{2^3(7!)} x^7 + \frac{3^2 5^2}{2^4(9!)} x^9 - \dots \right)}$$

13. (a) Assume that the solution of the differential equation $y'' = -xy' - y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, and $y'(0) = 0$ into the differential equation to find $y''(0)$.

$$y''(0) = -(0)y'(0) - y(0) = 0 - 1 = -1$$

Now differentiate $y'' = -xy' - y$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -xy'' - 2y' \\ y'''(0) &= -(0)(-1) - 2(0) = 0 \end{aligned}$$

Now differentiate $y''' = -xy'' - 2y'$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -xy''' - 3y'' \\ y^{(4)}(0) &= -(0)(0) - 3(-1) = 3 \end{aligned}$$

Now differentiate $y^{(4)} = -xy''' - 3y''$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -xy^{(4)} - 4y''' \\ y^{(5)}(0) &= -(0)(3) - 4(0) = 0 \end{aligned}$$

Now differentiate $y^{(5)} = -xy^{(4)} - 4y'''$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned} y^{(6)} &= -xy^{(5)} - 5y^{(4)} \\ y^{(6)}(0) &= -(0)(0) - 5(3) = -15 \end{aligned}$$

Now differentiate $y^{(6)} = -xy^{(5)} - 5y^{(4)}$ with respect to x to find $y^{(7)}(0)$.

$$\begin{aligned} y^{(7)} &= -xy^{(6)} - 6y^{(5)} \\ y^{(7)}(0) &= -(0)(-15) - 6(0) = 0 \end{aligned}$$

Now differentiate $y^{(7)} = -xy^{(6)} - 6y^{(5)}$ with respect to x to find $y^{(8)}(0)$.

$$\begin{aligned} y^{(8)} &= -xy^{(7)} - 7y^{(6)} \\ y^{(8)}(0) &= -(0)(0) - 7(-15) = 105 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{3}{4!}x^4 - \frac{15}{6!}x^6 + \frac{105}{8!}x^8 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	0.9950	0.9802	0.9560	0.9231	0.8825	0.8353	0.7827	0.7262	0.6671	0.6068

14. (a) Assume that the solution of the differential equation $y'' = 2xy' - y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 2$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = 2(0)y'(0) - y(0) = 0 - 2 = -2$$

Now differentiate $y'' = 2xy' - y$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= 2xy'' + 2y' - y' \\ y''' &= 2xy'' + y' \\ y'''(0) &= 2(0)(-2) + 1 = 1 \end{aligned}$$

Now differentiate $y''' = 2xy'' + y'$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned}y^{(4)} &= 2xy''' + 2y'' + y'' \\y^{(4)} &= 2xy''' + 3y'' \\y^{(4)}(0) &= 2(0)(1) + 3(-2) = -6\end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 2 + x - \frac{2}{2!}x^2 + \frac{1}{3!}x^3 - \frac{6}{4!}x^4 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	2.0	2.0901	2.1609	2.2125	2.2443	2.2552	2.2436	2.2071	2.1429	2.0475	1.9167

15. (a) Assume that the solution of the differential equation $y'' = y \sin x$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0) + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 0$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = y(0) \sin(0) = 0$$

Now differentiate $y'' = y \sin x$ with respect to x to find $y'''(0)$.

$$\begin{aligned}y''' &= y' \sin x + y \cos x \\y'''(0) &= (1) \sin(0) + (0) \cos(0) = 0\end{aligned}$$

Now differentiate $y''' = y' \sin x + y \cos x$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned}y^{(4)} &= y'' \sin x + y' \cos x + y' \cos x - y \sin x \\y^{(4)} &= y'' \sin x + 2y' \cos x - y \sin x \\y^{(4)}(0) &= (0) \sin(0) + 2(1) \cos(0) - (0) \sin(0) = 2\end{aligned}$$

Now differentiate $y^{(4)} = y'' \sin x + 2y' \cos x - y \sin x$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned}y^{(5)} &= y''' \sin x + y'' \cos x + 2y'' \cos x - 2y' \sin x - y' \sin x - y \cos x \\y^{(5)} &= y''' \sin x + 3y'' \cos x - 3y' \sin x - y \cos x \\y^{(5)}(0) &= 0\end{aligned}$$

Now differentiate $y^{(5)} = y''' \sin x + 3y'' \cos x - 3y' \sin x - y \cos x$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned}y^{(6)} &= y^{(4)} \sin x + y''' \cos x + 3y''' \cos x - 3y'' \sin x - 3y'' \sin x - 3y' \cos x - y' \cos x + y \sin x \\y^{(6)} &= y^{(4)} \sin x + 4y''' \cos x - 6y'' \sin x - 4y' \cos x + y \sin x \\y^{(6)}(0) &= -4\end{aligned}$$

Now differentiate $y^{(6)}$ with respect to x to find $y^{(7)}(0)$. (You may notice a pattern with the binomial coefficients.)

$$\begin{aligned}y^{(7)} &= y^{(5)} \sin x + 5y^{(4)} \cos x - 10y''' \sin x - 10y'' \cos x + 5y' \sin x + y \cos x \\y^{(7)}(0) &= 10\end{aligned}$$

Now differentiate $y^{(7)}$ with respect to x to find $y^{(8)}(0)$.

$$\begin{aligned} y^{(8)} &= y^{(6)} \sin x + 6y^{(5)} \cos x - 15y^{(4)} \sin x - 20y''' \cos x + 15y'' \sin x + 6y' \cos x - y \sin x \\ y^{(8)}(0) &= 6 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = x + \frac{2}{4!}x^4 - \frac{4}{6!}x^6 + \frac{10}{7!}x^7 + \frac{6}{8!}x^8 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.0	0.1000	0.2001	0.3007	0.4021	0.5051	0.6106	0.7195	0.8331	0.9527	1.0799

16. (a) Assume that the solution of the differential equation $y'' = -y \cos x$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 0$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = -y(0) \cos 0 = 0$$

Now differentiate $y'' = -y \cos x$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -y' \cos x + y \sin x \\ y'''(0) &= -(1) \cos(0) + (0) \sin(0) = -1 \end{aligned}$$

Now differentiate $y''' = -y' \cos x + y \sin x$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -y'' \cos x + y' \sin x + y' \sin x + y \cos x \\ y^{(4)} &= -y'' \cos x + 2y' \sin x + y \cos x \\ y^{(4)}(0) &= -y''(0) \cos(0) + 2y'(0) \sin(0) + y(0) \cos(0) = 0 \end{aligned}$$

Now differentiate $y^{(4)} = -y'' \cos x + 2y' \sin x + y \cos x$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -y''' \cos x + y'' \sin x + 2y'' \sin x + 2y' \cos x + y' \cos x - y \sin x \\ y^{(5)} &= -y''' \cos x + 3y'' \sin x + 3y' \cos x - y \sin x \\ y^{(5)}(0) &= -y'''(0) \cos(0) + 3y''(0) \sin(0) + 3y'(0) \cos(0) - y(0) \sin(0) = 4 \end{aligned}$$

Now differentiate $y^{(5)} = -y''' \cos x + 3y'' \sin x + 3y' \cos x - y \sin x$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned} y^{(6)} &= -y^{(4)} \cos x + y''' \sin x + 3y''' \sin x + 3y'' \cos x + 3y'' \cos x - 3y' \sin x - y' \sin x - y \cos x \\ y^{(6)} &= -y^{(4)} \cos x + 4y''' \sin x + 6y'' \cos x - 4y' \sin x - y \cos x \\ y^{(6)}(0) &= 0 \end{aligned}$$

Now differentiate $y^{(6)}$ with respect to x to find $y^{(7)}(0)$. (You may notice a pattern with the binomial coefficients.)

$$\begin{aligned} y^{(7)} &= -y^{(5)} \cos x + 5y^{(4)} \sin x + 10y''' \cos x - 10y'' \sin x - 5y' \cos x + y \sin x \\ y^{(7)}(0) &= -19 \end{aligned}$$

Now differentiate $y^{(7)}$ with respect to x to find $y^{(8)}(0)$.

$$\begin{aligned} y^{(8)} &= -y^{(6)} \cos x + 6y^{(5)} \sin x + 15y^{(4)} \cos x - 20y''' \sin x - 15y'' \cos x + 6y' \sin x + y \cos x \\ y^{(8)}(0) &= 0 \end{aligned}$$

Now differentiate $y^{(8)}$ with respect to x to find $y^{(9)}(0)$.

$$\begin{aligned} y^{(9)} &= -y^{(7)} \cos x + 7y^{(6)} \sin x + 21y^{(5)} \cos x - 35y^{(4)} \sin x - 35y''' \cos x + 21y'' \sin x + 7y' \cos x - y \sin x \\ y^{(9)}(0) &= 145 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = x - \frac{1}{3!}x^3 + \frac{4}{5!}x^5 - \frac{19}{7!}x^7 + \frac{145}{9!}x^9 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.0	0.0998	0.1987	0.2956	0.3897	0.4802	0.5665	0.6481	0.7249	0.7965	0.8633

17. (a) Assume that the solution of the differential equation $y'' = -y' - e^x y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 2$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = -y'(0) - e^0 y(0) = -3$$

Now differentiate $y'' = -y' - e^x y$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -y'' - e^x y' - e^x y \\ y'''(0) &= -(-3) - e^0(1) - e^0(2) = 0 \end{aligned}$$

Now differentiate $y''' = -y'' - e^x y' - e^x y$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -y''' - e^x y'' - e^x y' - e^x y \\ y^{(4)} &= -y''' - e^x y'' - 2e^x y' - e^x y \\ y^{(4)}(0) &= -0 - e^0(-3) - 2e^0(1) - e^0(2) = -1 \end{aligned}$$

Now differentiate $y^{(4)} = -y''' - e^x y'' - 2e^x y' - e^x y$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -y^{(4)} - e^x y''' - 2e^x y'' - 2e^x y' - e^x y \\ y^{(5)} &= -y^{(4)} - 3e^x y''' - 3e^x y'' - 3e^x y' - e^x y \\ y^{(5)}(0) &= -(-1) - 3e^0(0) - 3e^0(-3) - 3e^0(1) - e^0(2) = 5 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 2 + x - \frac{3}{2!}x^2 - \frac{1}{4!}x^4 + \frac{5}{5!}x^5 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	2.0	2.0850	2.1399	2.1648	2.1594	2.1237	2.0578	1.9620	1.8366	1.6823	1.5000

18. (a) Assume that the solution of the differential equation $y'' = -(3+x)y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, and $y'(0) = 0$ into the differential equation to find $y''(0)$.

$$y''(0) = -(3+0)(1) = -3$$

Now differentiate $y'' = -(3+x)y$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -(3+x)y' - y \\ y'''(0) &= -(3+0)(0) - 1 = -1 \end{aligned}$$

Now differentiate $y''' = -(3+x)y' - y$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -(3+x)y'' - y' - y' \\ y^{(4)} &= -(3+x)y'' - 2y' \\ y^{(4)}(0) &= -(3+0)(-3) - 2(0) = 9 \end{aligned}$$

Now differentiate $y^{(4)} = -(3+x)y'' - 2y'$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -(3+x)y''' - y'' - 2y'' \\ y^{(5)} &= -(3+x)y''' - 3y'' \\ y^{(5)}(0) &= -(3+0)(-1) - 3(-3) = 12 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 - \frac{3}{2!}x^2 - \frac{1}{3!}x^3 + \frac{9}{4!}x^4 + \frac{12}{5!}x^5 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	0.9849	0.9393	0.8638	0.7600	0.6307	0.4804	0.3147	0.1410	-0.0314	-0.1917

19. (a) Use the solution from Problem 5 with $y(0) = a_0 = 0$ and $y'(0) = a_1 = 2$. The Maclaurin series then becomes

$$y(x) = 2x - \frac{12}{5!}x^5 + \frac{504}{9!}x^9 - \frac{55,440}{13!}x^{13} + \frac{11,642,400}{17!}x^{17} + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.0	0.2000	0.4000	0.5998	0.7990	0.9969	1.1922	1.3832	1.5674	1.7415	1.9014

20. (a) Assume that the solution of the differential equation $y'' = 3x^2y' - 2xy$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = 3(0)^2 - 2(0)1 = 0$$

Now differentiate $y'' = 3x^2y' - 2xy$ with respect to x to find $y'''(0)$.

$$\begin{aligned}y''' &= 3x^2y'' + 6xy' - 2xy' - 2y \\y''' &= 3x^2y'' + 4xy' - 2y \\y'''(0) &= 3(0)^2(0) + 4(0)(1) - 2(1) = -2\end{aligned}$$

Now differentiate $y''' = 3x^2y'' + 4xy' - 2y$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned}y^{(4)} &= 3x^2y''' + 6xy'' + 4xy'' + 4y' - 2y' \\y^{(4)} &= 3x^2y''' + 10xy'' + 2y' \\y^{(4)}(0) &= 3(0)^2(-2) + 10(0)(0) + 2(1) = 2\end{aligned}$$

Now differentiate $y^{(4)} = 3x^2y''' + 10xy'' + 2y'$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned}y^{(5)} &= 3x^2y^{(4)} + 6xy''' + 10xy''' + 10y'' + 2y'' \\y^{(5)} &= 3x^2y^{(4)} + 16xy''' + 12y'' \\y^{(5)}(0) &= 3(0)^2(2) + 16(0)(-2) + 12(0) = 0\end{aligned}$$

Now differentiate $y^{(5)} = 3x^2y^{(4)} + 16xy''' + 12y''$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned}y^{(6)} &= 3x^2y^{(5)} + 6xy^{(4)} + 16xy^{(4)} + 16y''' + 12y'' \\y^{(6)} &= 3x^2y^{(5)} + 22xy^{(4)} + 28y''' \\y^{(6)}(0) &= 3(0)^2(0) + 22(0)(2) + 28(-2) = -56\end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 + x - \frac{2}{3!}x^3 + \frac{2}{4!}x^4 - \frac{56}{6!}x^6 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	1.0997	1.1975	1.2916	1.3805	1.4623	1.5352	1.5965	1.6431	1.6703	1.6722

21. (a) Assume that the solution of the differential equation $y^{(4)} = \ln(1+x)y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$, and $y'''(0) = 0$ into the differential equation to find $y^{(4)}(0)$.

$$y^{(4)}(0) = \ln(1)(1) = 0$$

Now differentiate $y^{(4)} = \ln(1+x)y$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned}y^{(5)} &= \ln(1+x)y' + (1+x)^{-1}y \\y^{(5)}(0) &= \ln(1)(1) + (1)^{-1}(1) = 1\end{aligned}$$

Now differentiate $y^{(5)} = \ln(1+x)y' + (1+x)^{-1}y$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned}y^{(6)} &= \ln(1+x)y'' + (1+x)^{-1}y' + (1+x)^{-1}y' - (1+x)^{-2}y \\y^{(6)} &= \ln(1+x)y'' + 2(1+x)^{-1}y' - (1+x)^{-2}y \\y^{(6)}(0) &= \ln(1)(0) + 2(1)^{-1}(1) - (1)^{-2}(1) = 1\end{aligned}$$

Now differentiate $y^{(6)} = \ln(1+x)y'' + 2(1+x)^{-1}y' - (1+x)^{-2}y$ with respect to x to find $y^{(7)}(0)$.

$$\begin{aligned}y^{(7)} &= \ln(1+x)y'''' + (1+x)^{-1}y''' + 2(1+x)^{-1}y'' - 2(1+x)^{-2}y' - (1+x)^{-2}y' + 2(1+x)^{-3}y \\y^{(7)} &= \ln(1+x)y'''' + 3(1+x)^{-1}y'' - 3(1+x)^{-2}y' + 2(1+x)^{-3}y \\y^{(7)}(0) &= \ln(1)(0) + 3(1)^{-1}(0) - 3(1)^{-2}(1) + 2(1)^{-3}(1) = -1\end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 + x + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	1.1000	1.2000	1.3000	1.4001	1.5003	1.6007	1.7015	1.8031	1.9056	2.0095

22. (a) Assume that the solution of the differential equation $y''' = -4y'' - 2y' + x^3y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, $y'(0) = 1$, and $y''(0) = 0$ into the differential equation to find $y'''(0)$.

$$y'''(0) = -4(0) - 2(1) + 0^3(1) = -2$$

Now differentiate $y''' = -4y'' - 2y' + x^3y$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned}y^{(4)} &= -4y''' - 2y'' + x^3y' + 3x^2y \\y^{(4)}(0) &= -4(-2) - 2(0) + 0^3(1) + 3(0)^2(1) = 8\end{aligned}$$

Now differentiate $y^{(4)} = -4y''' - 2y'' + x^3y' + 3x^2y$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned}y^{(5)} &= -4y^{(4)} - 2y''' + x^3y'' + 6x^2y' + 6xy \\y^{(5)}(0) &= -4(8) - 2(-2) + (0)^3(0) + 6(0)^2(1) + 6(0)(1) = -28\end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 + x - \frac{2}{3!}x^3 + \frac{8}{4!}x^4 - \frac{28}{5!}x^5 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	1.0997	1.1978	1.2931	1.3848	1.4719	1.5531	1.6265	1.6894	1.7379	1.7667

Applications and Extensions

23. Assume that the solution of the differential equation $y'' = -4y' - 4y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $y(0) = 1$ and $y'(0) = 2$ into the differential equation.

$$y''(0) = -4y'(0) - 4y(0) = -4(2) - 4(1) = -12$$

Now differentiate $y'' = -4y' - 4y$ with respect to x to find $y'''(0)$.

$$\begin{aligned}y''' &= -4y'' - 4y' \\y'''(0) &= -4(-12) - 4(2) = 40\end{aligned}$$

Now differentiate $y''' = -4y'' - 4y'$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned}y^{(4)} &= -4y''' - 4y'' \\y^{(4)} &= -4(40) - 4(-12) = -112\end{aligned}$$

Now differentiate $y^{(4)} = -4y''' - 4y''$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned}y^{(5)} &= -4y^{(4)} - 4y''' \\y^{(5)}(0) &= -4(-112) - 4(40) = 288\end{aligned}$$

Now differentiate $y^{(5)} = -4y^{(4)} - 4y'''$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned}y^{(6)} &= -4y^{(5)} - 4y^{(4)} \\y^{(6)}(0) &= -4(288) - 4(-112) = -704\end{aligned}$$

Now differentiate $y^{(6)} = -4y^{(5)} - 4y^{(4)}$ with respect to x to find $y^{(7)}(0)$.

$$\begin{aligned}y^{(7)} &= -4y^{(6)} - 4y^{(5)} \\y^{(7)}(0) &= -4(-704) - 4(288) = 1664\end{aligned}$$

and so on. The Maclaurin series then becomes

$$y(x) = 1 + 2x - \frac{12}{2!}x^2 + \frac{40}{3!}x^3 - \frac{112}{4!}x^4 + \frac{288}{5!}x^5 - \frac{704}{6!}x^6 + \frac{1664}{7!}x^7 - \dots$$

(a) Let

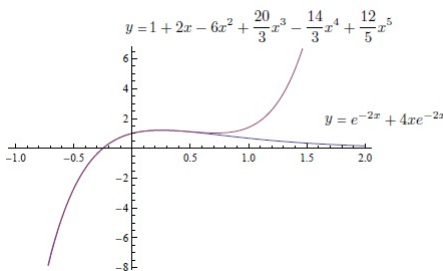
$$y(x) \approx 1 + 2x - 6x^2 + \frac{20}{3}x^3 - \frac{14}{3}x^4 + \frac{12}{5}x^5$$

approximate the solution to the differential equation.

(b) The particular solution of the differential equation is

$$y(x) = e^{-2x} (1 + 4x)$$

(c) The graph of the two functions from parts (a) and (b) are shown.



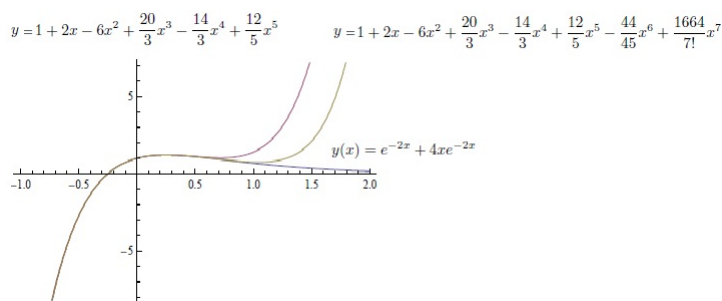
(d) Answers will vary. The approximation appears to be less accurate as x moves away from where the Maclaurin series is centered at 0.

(e) Let

$$y(x) \approx 1 + 2x - 6x^2 + \frac{20}{3}x^3 - \frac{14}{3}x^4 + \frac{12}{5}x^5 - \frac{44}{45}x^6 + \frac{104}{315}x^7$$

approximate the solution to the differential equation.

(f) The graph of the three functions from parts (a), (b), and (e) are shown.



Answers will vary. The approximation with more terms appears to be a better approximation to the solution than the approximation with fewer terms.

24. Assume that the solution of the differential equation $y' = xy(1 - y^2)$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

Use a Computer Algebra System and $y(0) = 2$ to find

$$y'(0) = 0 \quad y''(0) = -6 \quad y'''(0) = 0 \quad y^{(4)}(0) = 198 \quad y^{(5)}(0) = 0$$

$$y^{(6)}(0) = -17,370 \quad y^{(7)}(0) = 0 \quad y^{(8)}(0) = 2,970,450 \quad y^{(9)}(0) = 0 \quad y^{(10)}(0) = -562,897,755$$

The Maclaurin series then becomes

$$y(x) = 2 - \frac{6}{2!}x^2 + \frac{198}{4!}x^4 - \frac{17,370}{6!}x^6 + \frac{2,970,450}{8!}x^8 - \frac{562,897,755}{10!}x^{10} \dots$$

(a) Let

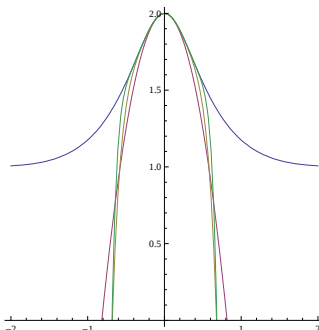
$$y(x) \approx 2 - \frac{6}{2!}x^2 + \frac{198}{4!}x^4 - \frac{17,370}{6!}x^6 + \frac{2,970,450}{8!}x^8 - \frac{562,897,755}{10!}x^{10}$$

approximate the solution to the differential equation.

(b) The particular solution to the differential equation is

$$y^2 \left(1 - \frac{3}{4}e^{-x^2} \right) = 1$$

(c) The graphs of the functions are shown.



Challenge Problem

25. (a) The differential equation $\frac{du}{dt} = -ku$ is separable. Separate the variables and integrate.

$$\begin{aligned}\frac{1}{u} du &= -k dt \\ \int \frac{1}{u} du &= - \int k dt \\ \ln|u| &= -kt + C_1 \\ |u| &= e^{-kt+C_1} \\ u &= \pm e^{kt} e^{C_1} \\ u &= C e^{kt}\end{aligned}$$

The initial condition $u(0) = u_0$ gives

$$u = u_0 e^{-kt}$$

and $l = u_0 - u$ gives

$$l = u_0 (1 - e^{-kt})$$

- (b) Solve for t in $r = \frac{l}{u}$.

$$\begin{aligned}r &= \frac{l}{u} \\ r &= \frac{u_0 (1 - e^{-kt})}{u_0 e^{-kt}} \\ r &= \frac{1 - e^{-kt}}{e^{-kt}} \\ r e^{-kt} &= 1 - e^{-kt} \\ e^{-kt} + r e^{-kt} &= 1 \\ e^{-kt} (1 + r) &= 1 \\ e^{-kt} &= \frac{1}{1 + r} \\ -kt &= \ln \left(\frac{1}{1 + r} \right) \\ -kt &= -\ln(1 + r) \\ t &= \frac{1}{k} \ln(1 + r)\end{aligned}$$

Now use the Maclaurin series for $f(x) = \ln(1 + x)$ to find $f(r) = \ln(1 + r)$.

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

when $-1 < x \leq 1$. Since $0 < r < 1$, we have

$$t = \frac{1}{k} \ln(1 + r) = \frac{1}{k} \left(r - \frac{r^2}{2} + \frac{r^3}{3} - \dots \right) \approx \frac{r}{k}$$

- (c) Use the half-life of uranium, 4.5×10^9 years, to find k .

$$\begin{aligned}\frac{u_0}{2} &= u_0 e^{-4.5 \times 10^9 k} \\ \frac{1}{2} &= e^{-4.5 \times 10^9 k} \\ \ln \left(\frac{1}{2} \right) &= -4.5 \times 10^9 k \\ -\ln(2) &= -4.5 \times 10^9 k \\ \frac{\ln(2)}{4.5 \times 10^9} &= k\end{aligned}$$

Use this k and $r \approx 0.054$ to find the lower bound.

$$\frac{r}{k} \approx \frac{0.054}{\frac{\ln(2)}{4.5 \times 10^9}} \approx 3.506 \times 10^8$$

The lower bound for the age of the Earth's crust is $\boxed{3.506 \times 10^8 \text{ years}}$.

Review Exercises

1. The differential equation $\frac{dy}{dx} + x^2y = \sin x$ is $\boxed{\text{first-order}}$ because the highest-order derivative is the first derivative. It has degree $\boxed{1}$ because the exponent of the first derivative is 1. Since $P_0(x) = 1$, $P_1(x) = x^2$, and $Q(x) = \sin x$ are all functions of x alone and no term contains more than one derivative, the differential equation is $\boxed{\text{linear}}$.
2. The differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = xe^x$ is $\boxed{\text{second-order}}$ because the highest-order derivative is the second derivative. It has degree $\boxed{1}$ because the exponent of the first derivative is 1. Since $P_0(x) = 1$, $P_1(x) = -5$, $P_2(x) = 3$, and $Q(x) = xe^x$ are all functions of x alone and no term contains more than one derivative, the differential equation is $\boxed{\text{linear}}$.
3. The differential equation $\frac{d^3y}{dx^3} + 4x\frac{d^2y}{dx^2} + (\sin^2 x)\frac{dy}{dx} + 3y = \sin x$ is $\boxed{\text{third-order}}$ because the highest-order derivative is the third derivative. It has degree $\boxed{1}$ because the exponent of the first derivative is 1. Since $P_0(x) = 1$, $P_1(x) = 4x$, $P_2(x) = \sin^2 x$, $P_3(x) = 3$, and $Q(x) = \sin x$ are all functions of x alone and no term contains more than one derivative, the differential equation is $\boxed{\text{linear}}$.
4. The differential equation $\left(\frac{dr}{ds}\right)^3 - r = 1$ is $\boxed{\text{first-order}}$ because the highest-order derivative is the first derivative. It has degree $\boxed{3}$ because the exponent of the first derivative is 3. Since the first derivative is not of degree 1, the differential equation is $\boxed{\text{nonlinear}}$.
5. For $y = e^x$, we have $\frac{dy}{dx} = e^x$ and $\frac{d^2y}{dx^2} = e^x$. Substitute these into the differential equation:

$$\begin{aligned}\frac{d^2y}{dx^2} - y &= 0 \\ e^x - e^x &= 0\end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

6. For $y = \frac{\ln x}{x^2}$, we have

$$\frac{dy}{dx} = \frac{x^2 \frac{1}{x} - \ln x (2x)}{x^4} = \frac{1 - 2 \ln x}{x^3}$$

and

$$\frac{d^2y}{dx^2} = \frac{x^3 \left(-2\frac{1}{x}\right) - (1 - 2 \ln x)(3x^2)}{x^6} = \frac{6 \ln x - 5}{x^4}$$

Substitute these into the differential equation:

$$\begin{aligned}
 \frac{d^2y}{dx^2} + \frac{5}{x} \frac{dy}{dx} + \frac{4}{x^2}y &= 0 \\
 \frac{6 \ln x - 5}{x^4} + \frac{5}{x} \frac{1 - 2 \ln x}{x^3} + \frac{4}{x^2} \frac{\ln x}{x^2} &= 0 \\
 \frac{6 \ln x - 5}{x^4} + \frac{5 - 10 \ln x}{x^4} + \frac{4 \ln x}{x^4} &= 0 \\
 \frac{6 \ln x - 5 + 5 - 10 \ln x + 4 \ln x}{x^4} &= 0 \\
 0 &= 0
 \end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

7. For $y = e^{-3x}$, we have $\frac{dy}{dx} = -3e^{-3x}$ and $\frac{d^2y}{dx^2} = 9e^{-3x}$. Substitute these into the differential equation:

$$\begin{aligned}
 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 3y &= 0 \\
 9e^{-3x} + 2(-3e^{-3x}) - 3e^{-3x} &= 0 \\
 9e^{-3x} - 6e^{-3x} - 3e^{-3x} &= 0 \\
 0 &= 0
 \end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

8. For $y = x^2 + 3x$, we have $\frac{dy}{dx} = 2x + 3$. Substitute these into the differential equation:

$$\begin{aligned}
 x \frac{dy}{dx} - y &= x^2 \\
 x(2x + 3) - (x^2 + 3x) &= x^2 \\
 2x^2 + 3x - x^2 - 3x &= x^2 \\
 x^2 &= x^2
 \end{aligned}$$

The function y satisfies the differential equation and so y is a solution.

9. $f(x, y) = x^3 + 2x^2y + 2xy^2 + y^3$ is a homogeneous function of degree 3, since

$$\begin{aligned}
 f(tx, ty) &= (tx)^3 + 2(tx)^2(ty) + 2(tx)(ty)^2 + (ty)^3 \\
 &= t^3x^3 + 2t^3x^2y + 2t^3xy^2 + t^3y^3 \\
 &= t^3(x^3 + 2x^2y + 2xy^2 + y^3) = t^3f(x, y) \quad \text{for all } t > 0
 \end{aligned}$$

10. $f(x, y) = x^2 - x^2y^2 + y^2$ is not a homogeneous function, since

$$f(tx, ty) = (tx)^2 - (tx)^2(ty)^2 + (ty)^2 = t^2(x^2 - t^2x^2y^2 + y^2) \neq t^k(x^2 - x^2y^2 + y^2)$$

for all $t > 0$ and some k .

11. $f(x, y) = xe^{y/x}$ is a homogeneous function of degree 1, since

$$\begin{aligned}
 f(tx, ty) &= (tx)e^{(ty)/(tx)} \\
 &= txe^{y/x} = tf(x, y) \quad \text{for all } t > 0
 \end{aligned}$$

12. $f(x, y) = \ln \frac{y}{x}$ is a homogeneous function of degree 0, since

$$\begin{aligned} f(tx, ty) &= \ln \frac{ty}{tx} \\ &= \ln \frac{y}{x} = t^0 f(x, y) \quad \text{for all } t > 0 \end{aligned}$$

13. (a) Express the differential equation

$$x(y^2 + 1) dx + y(x^2 - x) dy = 0$$

in the form

$$\begin{aligned} \frac{x}{x^2 - x} dx + \frac{y}{y^2 + 1} dy &= 0 \\ \frac{1}{x - 1} dx + \frac{y}{y^2 + 1} dy &= 0 \end{aligned}$$

The differential equation is separable.

- (b) Integrate the differential equation.

$$\begin{aligned} \int \frac{1}{x - 1} dx + \int \frac{y}{y^2 + 1} dy &= 0 \quad \text{Let } u = y^2 + 1. \\ \ln |x - 1| + \frac{1}{2} \ln(y^2 + 1) &= C_1 \\ 2 \ln |x - 1| + \ln(y^2 + 1) &= 2C_1 \\ \ln((x - 1)^2 + \ln(y^2 + 1)) &= 2C_1 \\ \ln[(x - 1)^2(y^2 + 1)] &= 2C_1 \\ (x - 1)^2(y^2 + 1) &= e^{2C_1} \\ (x - 1)^2(y^2 + 1) &= C \quad \text{Let } C = e^{2C_1} > 0. \\ y^2 + 1 &= \frac{C}{(x - 1)^2} \\ y^2 &= \frac{C}{(x - 1)^2} - 1 \\ y &= \pm \sqrt{\frac{C}{(x - 1)^2} - 1} \end{aligned}$$

where $C > 0$.

The general solution of the differential equation is

$$\boxed{y = \pm \sqrt{\frac{C}{(x - 1)^2} - 1}}$$

where $C > 0$ is a constant.

14. (a) Consider the differential equation $(2y + e^{2x}) dx + (2x + e^{2y}) dy = 0$. Let $M = 2y + e^{2x}$ and $N = 2x + e^{2y}$. Then

$$\frac{\partial M}{\partial y} = 2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (2y + e^{2x}) dx = 2xy + \frac{1}{2}e^{2x} + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[2xy + \frac{1}{2}e^{2x} + B(y) \right] = 2x + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 2x + \frac{\partial B}{\partial y} &= 2x + e^{2y} \\ \frac{\partial B}{\partial y} &= e^{2y} \end{aligned}$$

Now we integrate $\frac{dB}{dy} = e^{2y}$ with respect to y .

$$B(y) = \int e^{2y} dy = \frac{1}{2}e^{2y} + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = 2xy + \frac{1}{2}e^{2x} + \frac{1}{2}e^{2y} + C$$

The general solution of the differential equation is

$$\boxed{2xy + \frac{1}{2}e^{2x} + \frac{1}{2}e^{2y} + C = 0}$$

where C is a constant.

15. (a) Rewrite the differential equation $x \frac{dy}{dx} + y = x^6$ in the form

$$\frac{dy}{dx} + \frac{1}{x}y = x^5$$

to see that it is first-order linear. It is also exact.

- (b) Let $P(x) = \frac{1}{x}$. The integrating factor is

$$e^{\int P(x) dx} = e^{\int 1/x dx} = e^{\ln x} = x$$

Now multiply the differential equation by x and obtain

$$x \frac{dy}{dx} + y = x^6$$

The left side of the above equation is the derivative of xy , so we can write

$$\frac{d}{dx}(xy) = x^6$$

Integrate the equation above.

$$\begin{aligned} xy &= \int x^6 dx \\ xy &= \frac{1}{7}x^7 + C \\ y &= \frac{1}{7}x^6 + \frac{C}{x} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \frac{1}{7}x^6 + \frac{C}{x}}$$

where C is a constant.

16. (a) The differential equation

$$\frac{dy}{dx} - 2y = e^{3x}$$

is first-order linear.

- (b) Let $P(x) = -2$. The integrating factor is

$$e^{\int P(x) dx} = e^{-\int 2 dx} = e^{-2x}$$

Now multiply the differential equation by e^{-2x} and obtain

$$e^{-2x} \frac{dy}{dx} - 2e^{-2x}y = e^x$$

The left side of the above equation is the derivative of $e^{-2x}y$, so we can write

$$\frac{d}{dx} (e^{-2x}y) = e^x$$

Integrate the equation above.

$$\begin{aligned} e^{-2x}y &= \int e^x dx \\ e^{-2x}y &= e^x + C \\ y &= e^{3x} + Ce^{2x} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = e^{3x} + Ce^{2x}}$$

where C is a constant.

17. (a) Consider the differential equation $(2xy^3 + y^2 \cos x - 2x) dx + (3x^2y^2 + 2y \sin x) dy = 0$. Let $M = 2xy^3 + y^2 \cos x - 2x$ and $N = 3x^2y^2 + 2y \sin x$. Then

$$\frac{\partial M}{\partial y} = 6xy^2 + 2y \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy^2 + 2y \cos x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and y , the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (2xy^3 + y^2 \cos x - 2x) dx = x^2 y^3 + y^2 \sin x - x^2 + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^2 y^3 + y^2 \sin x - x^2 + B(y)] = 3x^2 y^2 + 2y \sin x + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned} \frac{\partial f}{\partial y} &= N \\ 3x^2 y^2 + 2y \sin x + \frac{\partial B}{\partial y} &= 3x^2 y^2 + 2y \sin x \\ \frac{\partial B}{\partial y} &= 0 \end{aligned}$$

and then $B(y) = C$ is a constant. Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x^2 y^3 + y^2 \sin x - x^2 + C$$

The general solution of the differential equation is

$$\boxed{x^2 y^3 + y^2 \sin x - x^2 + C = 0}$$

where C is a constant.

18. (a) Rewrite the differential equation $\frac{dy}{dx} + x^2 y - y = x^2 - 1$ in the form

$$\frac{dy}{dx} + (x^2 - 1)y = x^2 - 1$$

to see that it is first-order linear.

- (b) Let $P(x) = x^2 - 1$. The integrating factor is

$$e^{\int P(x) dx} = e^{\int x^2 - 1 dx} = e^{x^3/3 - x}$$

Now multiply the differential equation by $e^{x^3/3 - x}$ and obtain

$$e^{x^3/3 - x} \frac{dy}{dx} + (x^2 - 1)e^{x^3/3 - x} y = (x^2 - 1)e^{x^3/3 - x}$$

The left side of the above equation is the derivative of $e^{x^3/3 - x} y$, so we can write

$$\frac{d}{dx} (e^{x^3/3 - x} y) = (x^2 - 1)e^{x^3/3 - x}$$

Integrate the equation above.

$$\begin{aligned} e^{x^3/3 - x} y &= \int (x^2 - 1)e^{x^3/3 - x} dx \\ e^{x^3/3 - x} y &= e^{x^3/3 - x} + C \\ y &= 1 + C e^{-(x^3/3 - x)} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = 1 + C e^{-(x^3/3 - x)}}$$

where C is a constant.

19. (a) Express the differential equation

$$y \sin x \, dx - \cos x \, dy = 0$$

in the form

$$\begin{aligned}\frac{\sin x}{\cos x} \, dx - \frac{1}{y} \, dy &= 0 \\ \tan x \, dx - \frac{1}{y} \, dy &= 0\end{aligned}$$

The differential equation is separable. (It also happens to be exact.)

- (b) Integrate the differential equation.

$$\begin{aligned}\int \tan x \, dx - \int \frac{1}{y} \, dy &= 0 \\ \ln |\sec x| - \ln |y| &= C_1 \\ -\ln |\sec x| + \ln |y| &= -C_1 \\ \ln \left| \frac{y}{\sec x} \right| &= -C_1 \\ \left| \frac{y}{\sec x} \right| &= e^{-C_1} \\ \frac{y}{\sec x} &= \pm e^{-C_1} \\ \frac{y}{\sec x} &= C \quad \text{Let } C = \pm e^{-C_1} \\ y &= C \sec x\end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \frac{C}{\cos x}}$$

where C is a constant.

20. (a) Consider the differential equation $(2x \sin y - \ln y) \, dx + (x^2 \cos y - \frac{x}{y} + 3y^2) \, dy = 0$. Let $M = 2x \sin y - \ln y$ and $N = x^2 \cos y - \frac{x}{y} + 3y^2$. Then

$$\frac{\partial M}{\partial y} = 2x \cos y - \frac{1}{y} \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x \cos y - \frac{1}{y}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x and $y > 0$, the differential equation is exact.

- (b) Find f by integrating $\frac{\partial f}{\partial x} = M$ partially with respect to x (holding y constant). Then

$$f(x, y) = \int (2x \sin y - \ln y) \, dx = x^2 \sin y - x \ln y + B(y)$$

where the constant of integration B is a function of y alone. Find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^2 \sin y - x \ln y + B(y)] = x^2 \cos y - \frac{x}{y} + \frac{\partial B}{\partial y}$$

The function f satisfies $\frac{\partial f}{\partial y} = N$, so

$$\begin{aligned}\frac{\partial f}{\partial y} &= N \\ x^2 \cos y - \frac{x}{y} + \frac{\partial B}{\partial y} &= x^2 \cos y - \frac{x}{y} + 3y^2 \\ \frac{\partial B}{\partial y} &= 3y^2\end{aligned}$$

Now we integrate $\frac{dB}{dy} = 3y^2$ with respect to y .

$$B(y) = \int 3y^2 dy = y^3 + C$$

Substituting for $B(y)$ in the function $f(x, y)$, we obtain

$$f(x, y) = x^2 \sin y - x \ln y + y^3 + C$$

The general solution of the differential equation is

$$x^2 \sin y - x \ln y + y^3 + C = 0$$

where C is a constant.

21. (a) Both $x - y \tan \frac{y}{x}$ and $x \tan \frac{y}{x}$ are homogeneous of degree 1, so the differential equation is homogeneous.
 (b) Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}\left(x - y \tan \frac{y}{x}\right) dx + x \tan \frac{y}{x} dy &= 0 \\ \left(x - xv \tan \frac{xv}{x}\right) dx + x \tan \frac{xv}{x} (x dv + v dx) &= 0 \\ x dx + x^2 \tan v dv &= 0\end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^2 .

$$\begin{aligned}x dx + x^2 \tan v dv &= 0 \\ \frac{1}{x} dx + \tan v dv &= 0\end{aligned}$$

Integrate.

$$\begin{aligned}\int \frac{1}{x} dx + \int \tan v dv &= 0 \\ \ln |x| + \ln |\sec v| &= C_1 \\ \ln |x \sec v| &= C_1 \\ |x \sec v| &= e^{C_1} \\ x \sec v &= \pm e^{C_1} \\ \frac{x}{\cos v} &= \pm e^{C_1} \\ x &= \pm e^{C_1} \cos v \\ x &= \pm e^{C_1} \cos \left(\frac{y}{x}\right) \\ \pm \frac{x}{e^{C_1}} &= \cos \left(\frac{y}{x}\right) \\ Cx &= \cos \left(\frac{y}{x}\right) \quad \text{Let } C = \pm \frac{1}{e^{C_1}} \\ \cos^{-1}(Cx) &= \frac{y}{x} \\ x \cos^{-1}(Cx) &= y\end{aligned}$$

Then the general solution is $\boxed{y = x \cos^{-1}(Cx)}$, where C is a constant.

22. (a) Rewrite the differential equation $x \frac{dy}{dx} - 2y = x + 1$ in the form

$$\frac{dy}{dx} - \frac{2}{x}y = 1 + \frac{1}{x}$$

to see that it is $\boxed{\text{first-order linear}}$.

- (b) Let $P(x) = -\frac{2}{x}$. The integrating factor is

$$e^{\int P(x) dx} = e^{-\int 2/x dx} = e^{-2 \ln x} = e^{\ln(x^{-2})} = x^{-2}$$

Now multiply the differential equation by x^{-2} and obtain

$$x^{-2} \frac{dy}{dx} - 2x^{-3}y = x^{-2} + x^{-3}$$

The left side of the above equation is the derivative of $x^{-2}y$, so we can write

$$\frac{d}{dx} (x^{-2}y) = x^{-2} + x^{-3}$$

Integrate the equation above.

$$\begin{aligned} x^{-2}y &= \int (x^{-2} + x^{-3}) dx \\ x^{-2}y &= -x^{-1} - \frac{1}{2}x^{-2} + C \\ y &= -x - \frac{1}{2} + Cx^2 \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = Cx^2 - x - \frac{1}{2}}$$

where C is a constant.

23. (a) The differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^2}$$

is a $\boxed{\text{Bernoulli}}$ equation with $n = 3$.

- (b) Multiply the equation by y^{-3} to obtain

$$y^{-3} \frac{dy}{dx} + 2x^{-1}y^{-2} = x^{-2}$$

Let $v = y^{-2}$, so $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{1}{2} \frac{dv}{dx} + 2x^{-1}v &= x^{-2} \\ \frac{dv}{dx} - 4x^{-1}v &= -2x^{-2} \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{-4 \int x^{-1} dx} = e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4}$$

Now multiply the differential equation by x^{-4} and obtain

$$x^{-4} \frac{dv}{dx} - 4x^{-5}v = -2x^{-6}$$

The left side of the above equation is the derivative of $x^{-4}v$, so we can write

$$\frac{d}{dx}(x^{-4}v) = -2x^{-6}$$

Integrate the equation above.

$$\begin{aligned} x^{-4}v &= -\int 2x^{-6} dx \\ x^{-4}v &= \frac{2}{5}x^{-5} + C_1 \\ v &= \frac{2}{5}x^{-1} + C_1x^4 \end{aligned}$$

Using $v = y^{-2}$, rewrite the solution in terms of x and y .

$$\begin{aligned} y^{-2} &= \frac{2}{5}x^{-1} + C_1x^4 \\ y^2 &= \frac{1}{\frac{2}{5}x^{-1} + C_1x^4} \\ y^2 &= \frac{5x}{2 + 5C_1x^5} \\ y^2 &= \frac{5x}{2 + Cx^5} \quad \text{Let } C = 5C_1. \\ y &= \pm \sqrt{\frac{5x}{2 + Cx^5}} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \pm \sqrt{\frac{5x}{2 + Cx^5}}}$$

24. (a) The differential equation

$$\frac{dy}{dx} - 5y = -2y^2$$

is a Bernoulli equation with $n = 2$.

- (b) Multiply the equation by y^{-2} to obtain

$$y^{-2} \frac{dy}{dx} - 5y^{-1} = -2$$

Let $v = y^{-1}$, so $\frac{dv}{dx} = -y^{-2} \frac{dy}{dx}$, and substitute these into the equation above.

$$\begin{aligned} -\frac{dv}{dx} - 5v &= -2 \\ \frac{dv}{dx} + 5v &= 2 \end{aligned}$$

This is a first-order linear differential equation in x and v . The integrating factor is

$$e^{\int 5 dx} = e^{5x}$$

Now multiply the differential equation by e^{5x} and obtain

$$e^{5x} \frac{dv}{dx} + 5e^{5x} v = 2e^{5x}$$

The left side of the above equation is the derivative of $e^{5x}v$, so we can write

$$\frac{d}{dx} (e^{5x}v) = 2e^{5x}$$

Integrate the equation above.

$$\begin{aligned} e^{5x}v &= \int 2e^{5x} dx \\ e^{5x}v &= \frac{2}{5}e^{5x} + C \\ v &= \frac{2}{5} + Ce^{-5x} \end{aligned}$$

Using $v = y^{-1}$, rewrite the solution in terms of x and y .

$$\begin{aligned} y^{-1} &= \frac{2}{5} + Ce^{-5x} \\ y &= \frac{1}{\frac{2}{5} + Ce^{-5x}} \\ y &= \frac{5}{2 + Ce^{-5x}} \end{aligned}$$

The general solution of the differential equation is

$$\boxed{y = \frac{5}{2 + Ce^{-5x}}}$$

25. Rewrite the differential equation $x \frac{dy}{dx} + 2y = 6$ in the form

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{6}{x}$$

to see that it is first-order linear. Let $P(x) = \frac{2}{x}$. The integrating factor is

$$e^{\int P(x) dx} = e^{\int 2/x dx} = e^{2 \ln x} = e^{\ln(x^2)} = x^2$$

Now multiply the differential equation by x^2 and obtain

$$x^2 \frac{dy}{dx} + 2xy = 6x$$

The left side of the above equation is the derivative of x^2y , so we can write

$$\frac{d}{dx} (x^2y) = 6x$$

Integrate the equation above.

$$\begin{aligned} x^2y &= \int 6x dx \\ x^2y &= 3x^2 + C \\ y &= 3 + \frac{C}{x^2} \end{aligned}$$

The general solution of the differential equation is $y = 3 + \frac{C}{x^2}$ where C is a constant. Let $x = 2$ and $y = 8$ to find C .

$$\begin{aligned} y &= 3 + \frac{C}{x^2} \\ 8 &= 3 + \frac{C}{2^2} \\ 5 &= \frac{C}{4} \\ 20 &= C \end{aligned}$$

The particular solution is

$$\boxed{y = 3 + \frac{20}{x^2}}$$

26. Both xy and $y^2 + x\sqrt{x^2 + y^2}$ are homogeneous of degree 2. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} xy dy &= (y^2 + x\sqrt{x^2 + y^2}) dx \\ x^2 v(x dv + v dx) &= (x^2 v^2 + x\sqrt{x^2 + x^2 v^2}) dx \\ -x^2 \sqrt{1 + v^2} dx + x^3 v dv &= 0 \quad (\text{when } x > 0) \end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^3 and $\sqrt{1 + v^2}$.

$$\begin{aligned} -x^2 \sqrt{1 + v^2} dx + x^3 v dv &= 0 \\ -\frac{1}{x} dx + \frac{v}{\sqrt{1 + v^2}} dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned} -\int \frac{1}{x} dx + \int \frac{v}{\sqrt{1 + v^2}} dv &= 0 \quad \text{Let } u = 1 + v^2. \\ -\ln|x| + \sqrt{1 + v^2} &= C \\ -\ln|x| + \sqrt{1 + \frac{y^2}{x^2}} &= C \end{aligned}$$

Use $x = 1$ and $y = 1$ to find C :

$$-\ln|1| + \sqrt{1 + \frac{1}{1}} = \sqrt{2} = C$$

Replace C with $\sqrt{2}$ in the equation.

$$\begin{aligned} -\ln|x| + \sqrt{1 + \frac{y^2}{x^2}} &= \sqrt{2} \\ \sqrt{1 + \frac{y^2}{x^2}} &= \sqrt{2} + \ln|x| \\ 1 + \frac{y^2}{x^2} &= (\sqrt{2} + \ln|x|)^2 \\ \frac{y^2}{x^2} &= (\sqrt{2} + \ln|x|)^2 - 1 \\ y^2 &= x^2 (\sqrt{2} + \ln|x|)^2 - x^2 \end{aligned}$$

Then the particular solution is $\boxed{y^2 = x^2 (\sqrt{2} + \ln|x|)^2 - x^2}$.

27. The differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = 2$$

is first-order linear. Let $P(x) = \frac{1}{x}$. The integrating factor is

$$e^{\int P(x) dx} = e^{\int 1/x dx} = e^{\ln x} = x$$

Now multiply the differential equation by x and obtain

$$x \frac{dy}{dx} + y = 2x$$

The left side of the above equation is the derivative of xy , so we can write

$$\frac{d}{dx}(xy) = 2x$$

Integrate the equation above.

$$\begin{aligned} xy &= \int 2x dx \\ xy &= x^2 + C \\ y &= x + \frac{C}{x} \end{aligned}$$

Use $x = 1$ and $y = 0$ to find C : $0 = 1 + \frac{C}{1}$ so $C = -1$. The particular solution is

$$\boxed{y = x - \frac{1}{x}}$$

28. Both $x^2 + xy + y^2$ and $-x^2$ are homogeneous of degree 2, the differential equation $(x^2 + xy + y^2) dx - x^2 dy = 0$ is homogeneous. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned} (x^2 + xy + y^2) dx - x^2 dy &= 0 \\ (x^2 + x^2v + x^2v^2) dx - x^2(x dv + v dx) &= 0 \\ x^2(1 + v^2) dx - x^3 dv &= 0 \end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^3 and $1 + v^2$.

$$\begin{aligned} x^2(1 + v^2) dx - x^3 dv &= 0 \\ \frac{1}{x} dx - \frac{1}{1 + v^2} dv &= 0 \end{aligned}$$

Integrate.

$$\begin{aligned} \int \frac{1}{x} dx - \int \frac{1}{1 + v^2} dv &= 0 \\ \ln|x| - \tan^{-1} v &= C \\ \ln|x| - \tan^{-1} \left(\frac{y}{x} \right) &= C \end{aligned}$$

Use $x = 1$ and $y = 1$ to find C :

$$\ln|1| - \tan^{-1} \left(\frac{1}{1} \right) = C$$

so $C = -\frac{\pi}{4}$.

$$\begin{aligned}\ln|x| - \tan^{-1} \frac{y}{x} &= -\frac{\pi}{4} \\ \ln|x| + \frac{\pi}{4} &= \tan^{-1} \frac{y}{x} \\ \tan(\ln|x| + \frac{\pi}{4}) &= \frac{y}{x} \\ x \tan(\ln|x| + \frac{\pi}{4}) &= y\end{aligned}$$

Then the particular solution is $\boxed{y = x \tan(\ln|x| + \frac{\pi}{4})}$.

29. The differential equation is separable. Express the differential equation

$$\frac{dy}{dx} = \frac{y-1}{x+3}$$

in the form

$$\frac{1}{y-1} dy = \frac{1}{x+3} dx$$

Integrate the differential equation.

$$\begin{aligned}\int \frac{1}{y-1} dy &= \int \frac{1}{x+3} dx \\ \ln|y-1| &= \ln|x+3| + C_1 \\ |y-1| &= e^{\ln|x+3| + C_1} \\ |y-1| &= e^{\ln|x+3|} e^{C_1} \\ |y-1| &= |x+3| e^{C_1} \\ y-1 &= \pm(x+3)e^{C_1} \\ y-1 &= C(x+3) \quad \text{Let } C = \pm e^{C_1}. \\ y &= 1 + C(x+3)\end{aligned}$$

The general solution of the differential equation is $y = 1 + C(x+3)$. To find C , let $x = -1$ and $y = 0$. Then

$$\begin{aligned}y &= 1 + C(x+3) \\ 0 &= 1 + C(-1+3) \\ 0 &= 1 + 2C \\ -\frac{1}{2} &= C\end{aligned}$$

The particular solution is

$$\boxed{y = -\frac{1}{2}x - \frac{1}{2}}$$

30. The differential equation is separable. Express the differential equation

$$(1 - \cos x) \frac{dy}{dx} + y \sin x = 0$$

in the form

$$\frac{1}{y} dy + \frac{\sin x}{1 - \cos x} dx = 0$$

Integrate the differential equation.

$$\begin{aligned}
 \int \frac{1}{y} dy + \int \frac{\sin x}{1 - \cos x} dx &= 0 && \text{Let } u = 1 - \cos x. \\
 \ln |y| + \ln |1 - \cos x| &= C_1 \\
 \ln |y(1 - \cos x)| &= C_1 \\
 |y(1 - \cos x)| &= e^{C_1} \\
 y(1 - \cos x) &= \pm e^{C_1} \\
 y(1 - \cos x) &= C && \text{Let } C = \pm e^{C_1}. \\
 y &= \frac{C}{1 - \cos x}
 \end{aligned}$$

The general solution of the differential equation is $y = \frac{C}{1 - \cos x}$. To find C , let $x = \frac{\pi}{4}$ and $y = 1$. Then

$$\begin{aligned}
 y &= \frac{C}{1 - \cos x} \\
 1 &= \frac{C}{1 - \cos(\pi/4)} \\
 1 &= \frac{C}{1 - \frac{\sqrt{2}}{2}} \\
 1 - \frac{\sqrt{2}}{2} &= C
 \end{aligned}$$

The particular solution is

$$y = \frac{1 - \frac{\sqrt{2}}{2}}{1 - \cos x}$$

31. The differential equation

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$$

is first-order linear. Let $P(x) = -\frac{2}{x}$. The integrating factor is

$$e^{\int P(x) dx} = e^{-\int 2/x dx} = e^{-2 \ln x} = e^{\ln(x^{-2})} = x^{-2}$$

Now multiply the differential equation by x^{-2} and obtain

$$x^{-2} \frac{dy}{dx} - 2x^{-3}y = \cos x$$

The left side of the above equation is the derivative of $x^{-2}y$, so we can write

$$\frac{d}{dx} (x^{-2}y) = \cos x$$

Integrate the equation above.

$$\begin{aligned}
 x^{-2}y &= \int \cos x dx \\
 x^{-2}y &= \sin x + C \\
 y &= x^2(\sin x + C)
 \end{aligned}$$

The general solution of the differential equation is $y = x^2(\sin x + C)$ where C is a constant. Let $x = \frac{\pi}{2}$ and $y = 3$ to find C .

$$\begin{aligned} y &= x^2(\sin x + C) \\ 3 &= \frac{\pi^2}{4} \left(\sin \left(\frac{\pi}{2} \right) + C \right) \\ 3 &= \frac{\pi^2}{4} (1 + C) \\ \frac{12}{\pi^2} &= 1 + C \\ \frac{12}{\pi^2} - 1 &= C \end{aligned}$$

The particular solution is

$$y = x^2 \left(\sin x + \frac{12}{\pi^2} - 1 \right)$$

32. The differential equation is separable. Express the differential equation

$$\left(\frac{2x-1}{y} \right) dx + x \frac{1-x}{y^2} dy = 0$$

in the form

$$\left(\frac{2x-1}{x(1-x)} \right) dx + \frac{1}{y} dy = 0$$

Integrate the differential equation.

$$\begin{aligned} \int \left(\frac{2x-1}{x(1-x)} \right) dx + \int \frac{1}{y} dy &= 0 \\ \int \left(\frac{1}{1-x} - \frac{1}{x} \right) dx + \int \frac{1}{y} dy &= 0 \\ -\ln|1-x| - \ln|x| + \ln|y| &= C_1 \\ \ln \left| \frac{y}{x(1-x)} \right| &= C_1 \\ \left| \frac{y}{x(1-x)} \right| &= e^{C_1} \\ \frac{y}{x(1-x)} &= \pm e^{C_1} \\ \frac{y}{x(1-x)} &= C \quad \text{Let } C = \pm e^{C_1} \\ y &= Cx(1-x) \end{aligned}$$

The general solution of the differential equation is $y = Cx(1-x)$. To find C , let $x = -1$ and $y = 3$. Then

$$\begin{aligned} y &= Cx(1-x) \\ 3 &= C(-1)(1-(-1)) \\ 3 &= -2C \\ -\frac{3}{2} &= C \end{aligned}$$

The particular solution is

$$y = -\frac{3}{2}x(1-x)$$

33. Since $x^2 + y^2$ and $-xy$ are homogeneous of degree 2, the differential equation $(x^2 + y^2) dx - xy dy = 0$ is homogeneous. Let $y = xv$. Then $dy = x dv + v dx$. Substitute these into the differential equation:

$$\begin{aligned}(x^2 + y^2) dx - xy dy &= 0 \\(x^2 + x^2 v^2) dx - x^2 v(x dv + v dx) &= 0 \\x^2 dx - x^3 v dv &= 0\end{aligned}$$

For $x \neq 0$, we can separate the variables by dividing by x^3 .

$$\begin{aligned}x^2 dx - x^3 v dv &= 0 \\ \frac{1}{x} dx - v dv &= 0\end{aligned}$$

Integrate.

$$\begin{aligned}\int \frac{1}{x} dx - \int v dv &= 0 \\ \ln |x| - \frac{1}{2} v^2 &= C \\ \ln |x| - \frac{y^2}{2x^2} &= C \\ \ln |x| - \frac{y^2}{2x^2} &= -2 \quad (\text{since } y(1) = 2) \\ \ln |x| + 2 &= \frac{y^2}{2x^2} \\ 2x^2(\ln |x| + 2) &= y^2\end{aligned}$$

Then the particular solution is $y^2 = x^2(2 \ln |x| + 4)$, and when $x > 0$

$$\boxed{y = x\sqrt{4 + 2 \ln x}}$$

34. The differential equation is separable. Express the differential equation

$$y^2 \sin x dx + \left(\frac{1}{x} - \frac{y}{x} \right) dy = 0$$

in the form

$$x \sin x dx + \left(\frac{1}{y^2} - \frac{1}{y} \right) dy = 0$$

Integrate the differential equation.

$$\begin{aligned}\int x \sin x dx + \int \left(\frac{1}{y^2} - \frac{1}{y} \right) dy &= 0 \quad \text{Use } u = x \text{ and } dv = \sin x dx. \\ -x \cos x + \sin x - \frac{1}{y} - \ln |y| &= C\end{aligned}$$

The general solution of the differential equation is $-x \cos x + \sin x - \frac{1}{y} - \ln |y| = C$. To find C , let $x = \pi$ and $y = 1$. Then

$$\begin{aligned}-x \cos x + \sin x - \frac{1}{y} - \ln |y| &= C \\ -\pi \cos(\pi) + \sin(\pi) - \frac{1}{1} - \ln |1| &= C \\ \pi - 1 &= C\end{aligned}$$

The particular solution is

$$-x \cos x + \sin x - \frac{1}{y} - \ln |y| = \pi - 1$$

35. Assume that the solution of the differential equation $(x-2)y' + y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Since $(x-2)y' + y = 0$, this leads to

$$\begin{aligned} (x-2) \sum_{k=1}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k &= 0 \\ \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k &= 0 \end{aligned}$$

Change the index of summation in the power series on the first and third from k to $k-1$.

$$\begin{aligned} \sum_{k=2}^{\infty} (k-1) a_{k-1} x^{k-1} - \sum_{k=1}^{\infty} 2k a_k x^{k-1} + \sum_{k=1}^{\infty} a_{k-1} x^{k-1} &= 0 \\ \sum_{k=1}^{\infty} (k-1) a_{k-1} x^{k-1} - \sum_{k=1}^{\infty} 2k a_k x^{k-1} + \sum_{k=1}^{\infty} a_{k-1} x^{k-1} &= 0 \\ \sum_{k=1}^{\infty} [(k-1) a_{k-1} - 2k a_k + a_{k-1}] x^{k-1} &= 0 \\ \sum_{k=1}^{\infty} [k a_{k-1} - 2k a_k] x^{k-1} &= 0 \\ \sum_{k=1}^{\infty} [a_{k-1} - 2a_k] k x^{k-1} &= 0 \end{aligned}$$

(Notice that when $k=1$, the term $(k-1)a_{k-1}x^{k-1} = 0$ so the index of the first sum can change from 2 to 1.) Then $a_{n-1} - 2a_n = 0$ for $n \geq 1$, which can be written as

$$a_n = \frac{1}{2} a_{n-1}$$

Then

$$a_1 = \frac{1}{2} a_0 \quad a_2 = \frac{1}{2} a_1 = \frac{1}{2^2} a_0 \quad a_3 = \frac{1}{2} a_2 = \frac{1}{2^3} a_0 \quad a_4 = \frac{1}{2} a_3 = \frac{1}{2^4} a_0 \quad \cdots$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= a_0 \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k \end{aligned}$$

The power series representation of the general solution of the equation is

$$y(x) = a_0 \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$$

You may recognize the power series as geometric, in which case $y(x) = a_0 \frac{1}{1-x/2} = a_0 \frac{2}{2-x}$.

36. Assume that the solution of the differential equation $y'' + y' - 2y = 0$ can be expressed as the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Since $y'' + y' - 2y = 0$, this leads to

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Change the indices of summation in the second and third power series so that x is raised to the $k-2$ power in each.

$$\begin{aligned} \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=2}^{\infty} (k-1) a_{k-1} x^{k-2} - \sum_{k=2}^{\infty} 2 a_{k-2} x^{k-2} &= 0 \\ \sum_{k=2}^{\infty} [k(k-1) a_k + (k-1) a_{k-1} - 2 a_{k-2}] x^{k-2} &= 0 \end{aligned}$$

Then $n(n-1)a_n + (n-1)a_{n-1} - 2a_{n-2} = 0$ for $n \geq 2$, which can be written as

$$a_n = -\frac{n-1}{n(n-1)} a_{n-1} + \frac{2}{n(n-1)} a_{n-2} = -\frac{1}{n} a_{n-1} + \frac{2}{n(n-1)} a_{n-2}$$

$$\begin{aligned} a_2 &= -\frac{1}{2} a_1 + \frac{2}{1 \cdot 2} a_0 = -\frac{1}{2} a_1 + a_0 \\ a_3 &= -\frac{1}{3} a_2 + \frac{2}{2 \cdot 3} a_1 = -\frac{1}{3} \left(-\frac{1}{2} a_1 + a_0 \right) + \frac{1}{3} a_1 \\ &= \frac{1}{2} a_1 - \frac{1}{3} a_0 \\ a_4 &= -\frac{1}{4} a_3 + \frac{2}{3 \cdot 4} a_2 = -\frac{1}{4} \left(\frac{1}{2} a_1 - \frac{1}{3} a_0 \right) + \frac{1}{6} \left(-\frac{1}{2} a_1 + a_0 \right) \\ &= -\frac{5}{24} a_1 + \frac{1}{4} a_0 \\ a_5 &= -\frac{1}{5} a_4 + \frac{2}{4 \cdot 5} a_3 = -\frac{1}{5} \left(-\frac{5}{24} a_1 + \frac{1}{4} a_0 \right) + \frac{1}{10} \left(\frac{1}{2} a_1 - \frac{1}{3} a_0 \right) \\ &= \frac{11}{120} a_1 - \frac{1}{12} a_0 \end{aligned}$$

The power series takes on the form

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= a_0 \left(1 + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{12}x^5 + \cdots \right) \\ &\quad + a_1 \left(x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5 + \cdots \right) \end{aligned}$$

The power series representation of the general solution of the equation is

$$\begin{aligned} y(x) &= a_0 \left(1 + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{12}x^5 + \cdots \right) \\ &\quad + a_1 \left(x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5 + \cdots \right) \end{aligned}$$

37. (a) Assume that the solution of the differential equation $y'' = -y \sin x$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

Substitute $x = 0$, $y(0) = 1$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = y(0) \sin(0) = 0$$

Now differentiate $y'' = -y \sin x$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -y' \sin x - y \cos x \\ y'''(0) &= -(1) \sin(0) - (1) \cos(0) = -1 \end{aligned}$$

Now differentiate $y''' = -y' \sin x - y \cos x$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -y'' \sin x - y' \cos x - y' \cos x + y \sin x \\ y^{(4)} &= -y'' \sin x - 2y' \cos x + y \sin x \\ y^{(4)}(0) &= (0) \sin(0) - 2(1) \cos(0) + (1) \sin(0) = -2 \end{aligned}$$

Now differentiate $y^{(4)} = -y'' \sin x - 2y' \cos x + y \sin x$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -y''' \sin x - y'' \cos x - 2y'' \cos x + 2y' \sin x + y' \sin x + y \cos x \\ y^{(5)} &= -y''' \sin x - 3y'' \cos x + 3y' \sin x + y \cos x \\ y^{(5)}(0) &= 1 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 + x - \frac{1}{3!}x^3 - \frac{2}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	1.0998	1.1985	1.2948	1.3873	1.4742	1.5538	1.6242	1.6833	1.7287	1.7583

38. (a) Assume that the solution of the differential equation $y'' = -e^x y' - y$ is given by the Maclaurin series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

Substitute $x = 0$, $y(0) = 1$, and $y'(0) = 1$ into the differential equation to find $y''(0)$.

$$y''(0) = -e^0 y'(0) - y(0) = -2$$

Now differentiate $y'' = -e^x y' - y$ with respect to x to find $y'''(0)$.

$$\begin{aligned} y''' &= -e^x y'' - e^x y' - y' \\ y''' &= -e^x (y'' + y') - y' \\ y'''(0) &= -e^0 (-2 + 1) - 1 = 0 \end{aligned}$$

Now differentiate $y''' = -e^x (y'' + y') - y'$ with respect to x to find $y^{(4)}(0)$.

$$\begin{aligned} y^{(4)} &= -e^x (y''' + y'') - e^x (y'' + y') - y'' \\ y^{(4)} &= -e^x (y''' + 2y'' + y') - y'' \\ y^{(4)}(0) &= -e^0 (0 + 2(-2) + 1) - (-2) = 5 \end{aligned}$$

Now differentiate $y^{(4)} = -e^x (y''' + 2y'' + y') - y''$ with respect to x to find $y^{(5)}(0)$.

$$\begin{aligned} y^{(5)} &= -e^x (y^{(4)} + 2y''' + y'') - e^x (y''' + 2y'' + y') - y''' \\ y^{(5)} &= -e^x (y^{(4)} + 3y''' + 3y'' + y') - y''' \\ y^{(5)}(0) &= -e^0 (5 + 3(0) + 3(-2) + 1) - 0 = 0 \end{aligned}$$

Now differentiate $y^{(5)} = -e^x (y^{(4)} + 3y''' + 3y'' + y') - y'''$ with respect to x to find $y^{(6)}(0)$.

$$\begin{aligned} y^{(6)} &= -e^x (y^{(5)} + 3y^{(4)} + 3y''' + y'') - e^x (y^{(4)} + 3y''' + 3y'' + y') - y^{(4)} \\ y^{(6)} &= -e^x (y^{(5)} + 4y^{(4)} + 6y''' + 4y'' + y') - y^{(4)} \\ y^{(6)}(0) &= -e^0 (0 + 4(5) + 6(0) + 4(-2) + 1) - 5 = -18 \end{aligned}$$

The Maclaurin series then becomes

$$y(x) = 1 + x - \frac{2}{2!}x^2 + \frac{5}{4!}x^4 - \frac{18}{6!}x^6 + \dots$$

- (b) Construct a table that uses the first five nonzero terms of the series to approximate select values of y in the interval $0 \leq x \leq 1$.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0	1.0900	1.1603	1.2117	1.2452	1.2626	1.2658	1.2571	1.2388	1.2134	1.1833

39. Let $y(t)$ equal the number of bacteria at time t . The available food supply will support a maximum number M of bacteria, so the derivative $\frac{dy}{dt}$ is the rate of growth of the bacteria and is proportional to $M - y$, the difference between the maximum number and the number present. So

$$\frac{dy}{dt} = k(M - y)$$

where k is the constant of proportionality. Separate the variables and integrate the differential equation.

$$\begin{aligned}\frac{1}{M-y} dy &= k dt \\ \int \frac{1}{M-y} dy &= \int k dt \\ -\ln|M-y| &= kt + C_1 \\ \ln|M-y| &= -kt - C_1 \\ |M-y| &= e^{-kt-C_1} \\ |M-y| &= e^{-kt}e^{-C_1} \\ M-y &= \pm e^{-kt}e^{-C_1}\end{aligned}$$

Since C_1 is a constant, let $C = \pm e^{-C_1}$ where C is a constant. So $M-y = Ce^{-kt}$ and the number of bacteria as a function of time is

$$\boxed{y(t) = M - Ce^{-kt}}$$

40. Let $y = y(t)$ be the amount of chlorine in mg in the tank at time t . Chlorine enters the tank at the rate of $(2 \text{ mg/L})(100 \text{ L/min}) = 200 \text{ mg/min}$. The concentration of chlorine in mg/L is

$$\text{Concentration} = \frac{\text{amount of Cl}}{\text{number of L}} = \frac{y(t)}{1000 + (100 - 80)t} = \frac{y(t)}{1000 + 20t}$$

Then the rate at which the salt exits the tank is

$$\left(\frac{y(t)}{1000 + 20t} \text{ mg/L} \right) (80 \text{ L/min}) = \frac{80y(t)}{1000 + 20t} \text{ mg/min}$$

The mixture problem is modeled by the differential equation

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 200 - \frac{80y}{1000 + 20t}$$

or

$$\frac{dy}{dt} + \frac{4}{50+t} y = 200$$

for $t \geq 0$. This equation is a first-order linear differential equation. The integrating factor is

$$e^{\int P(t) dt} = e^{\int 4/(50+t) dt} = e^{4 \ln(50+t)} = e^{\ln((50+t)^4)} = (50+t)^4$$

Multiply the differential equation by the integrating factor to obtain

$$\begin{aligned}(50+t)^4 \frac{dy}{dt} + 4(50+t)^3 y &= 200(50+t)^4 \\ \frac{d}{dt} [(50+t)^4 y] &= 200(50+t)^4\end{aligned}$$

Integrate both sides with respect to t .

$$\begin{aligned}(50+t)^4 y &= \int 200(50+t)^4 dt \\ (50+t)^4 y &= 40(50+t)^5 + C \\ y &= 40(50+t) + C(50+t)^{-4}\end{aligned}$$

The tank initially held 500mg of chlorine, so $y(0) = 500$, which we use to find C .

$$\begin{aligned}y(0) = 500 &= 40(50) + C(50)^{-4} \\ -1500 &= \frac{C}{6,250,000} \\ -9,375,000,000 &= C\end{aligned}$$

The amount of chlorine in g at time $t \geq 0$ is $y(t) = 40(50 + t) - 9,375,000,000(50 + t)^{-4}$. After 1 hour (60 minutes), there is $y(60) = 40(110) - 9,375,000,000(110)^{-4} \approx \boxed{4336 \text{ mg}}$ of chlorine in the tank.

41. Find y' by differentiating $xy = c$ with respect to x .

$$\begin{aligned} xy' + y &= 0 \\ xy' &= -y \\ y' &= -\frac{y}{x} \end{aligned}$$

Replace y' with $-\frac{1}{y'}$ to obtain the differential equation for the family of orthogonal trajectories.

$$y' = \frac{x}{y}$$

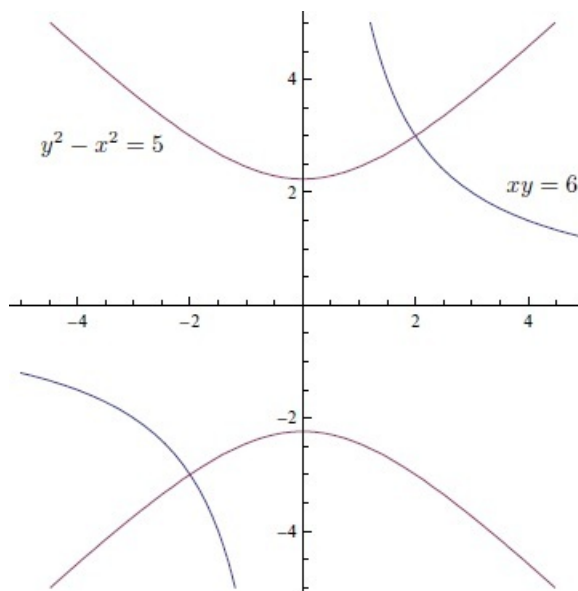
The differential equation $\frac{dy}{dx} = \frac{x}{y}$ is separable and can be written as

$$y \, dy = x \, dx$$

Integrate to find the general solution.

$$\begin{aligned} \int y \, dy &= \int x \, dx \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + K_1 \\ y^2 &= x^2 + 2K_1 \end{aligned}$$

Since K_1 is a constant, we may write $K = 2K_1$, where K is a constant. The orthogonal trajectories for the family of hyperbolas $xy = c$ is the family of hyperbolas $y^2 - x^2 = K$. To find the particular curves needed for the graph, substitute $x = 2$ and $y = 3$ into $xy = c$ and $y^2 = x^2 + K$ separately and solve for c and K . The graph of $xy = 6$ and $y^2 - x^2 = 5$ is shown.



42. The spread of the rumor is modeled by logistic growth, so

$$y(t) = \frac{RM}{R + (M - R)e^{-kMt}}$$

models the function for $t \geq 0$ in hours. Rewrite this equation using $M = 100,000$ and $R = 10$:

$$\begin{aligned} y(t) &= \frac{(10)(100,000)}{10 + (100,000 - 10)e^{-100,000kt}} \\ &= \frac{1,000,000}{10 + 99,990e^{-100,000kt}} \end{aligned}$$

The number of people who have heard the rumor after 1 hour is 100. Use $y(1) = 100$ to find k .

$$\begin{aligned} 100 &= \frac{1,000,000}{10 + 99,990e^{-100,000k}} \\ 100(10 + 99,990e^{-100,000k}) &= 1,000,000 \\ 10 + 99,990e^{-100,000k} &= 10,000 \\ 99,990e^{-100,000k} &= 9990 \\ e^{-100,000k} &= \frac{111}{1111} \\ -100,000k &= \ln\left(\frac{111}{1111}\right) \\ k &= -\frac{\ln(111/1111)}{100,000} \end{aligned}$$

The number of people who have heard the rumor after t hours is modeled by

$$y(t) = \frac{1,000,000}{10 + 99,990e^{\ln(111/1111)t}}$$

To find how long it will take for 25% of the people to hear the rumor, solve $y(t) = 25,000$ for t .

$$\begin{aligned} 25,000 &= \frac{1,000,000}{10 + 99,990e^{\ln(111/1111)t}} \\ 25,000(10 + 99,990e^{\ln(111/1111)t}) &= 1,000,000 \\ 10 + 99,990e^{\ln(111/1111)t} &= 40 \\ 99,990e^{\ln(111/1111)t} &= 30 \\ e^{\ln(111/1111)t} &= \frac{1}{3333} \\ t \ln\left(\frac{111}{1111}\right) &= \ln\left(\frac{1}{3333}\right) \\ t &= \frac{\ln(1/3333)}{\ln(111/1111)} \approx 3.521 \end{aligned}$$

It will take about 3.5 hours for 25% of the people to hear the rumor.